

# Parallel Graph Connectivity in Log Diameter Rounds

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## Abstract

Many modern parallel systems, such as MapReduce, Hadoop and Spark, can be modeled well by the MPC model. The MPC model captures well coarse-grained computation on large data — data is distributed to processors, each of which has a sublinear (in the input data) amount of memory and we alternate between rounds of computation and rounds of communication, where each machine can communicate an amount of data as large as the size of its memory. This model is stronger than the classical PRAM model, and it is an intriguing question to design algorithms whose running time is smaller than in the PRAM model.

One fundamental graph problem is connectivity. On an undirected graph with  $n$  nodes and  $m$  edges,  $O(\log n)$  round connectivity algorithms have been known for over 35 years. However, no algorithms with better complexity bounds were known. In this work, we give **fully scalable, faster algorithms for the connectivity problem**, by parameterizing the time complexity as a function of the *diameter* of the graph. Our main result is a  $O(\log D \log \log_{m/n} n)$  time connectivity algorithm for diameter- $D$  graphs, using  $\Theta(m)$  total memory. If our algorithm can use more memory, it can terminate in fewer rounds, and there is no lower bound on the memory per processor.

We extend our results to related graph problems such as spanning forest, finding a DFS sequence, exact/approximate minimum spanning forest, and bottleneck spanning forest. We also show that achieving similar bounds for reachability in *directed graphs* would imply faster boolean matrix multiplication algorithms.

We introduce several new algorithmic ideas. We describe a general technique called *double exponential speed problem size reduction* which roughly means that if we can use total memory  $N$  to reduce a problem from size  $n$  to  $n/k$ , for  $k = (N/n)^{\Theta(1)}$  in one phase, then we can solve the problem in  $O(\log \log_{N/n} n)$  phases. In order to achieve this fast reduction for graph connectivity, we use a multistep algorithm. One key step is a carefully constructed truncated broadcasting scheme where each node broadcasts neighbor sets to its neighbors in a way that limits the size of the resulting neighbor sets. Another key step is *random leader contraction*, where we choose a smaller set of leaders than many previous works do.

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# 1 Introduction

Recently, several parallel systems, including MapReduce [DG04, DG08], Hadoop [Whi12], Dryad [IBY+07], Spark [ZCF+10], and others, have become successful in practice. This success has sparked a renewed interest in algorithmic ideas for these parallel systems.

One important theoretical direction has been to develop good models of these modern systems and to relate them to classic models such as PRAM. The work of [FMS<sup>+</sup>10, KSV10, GSZ11, BKS13, ANOY14] have led to the model of *Massive Parallel Computing* (MPC) that balances accurate modeling with theoretical elegance. MPC is a variant of the Bulk Synchronous Parallel (BSP) model [Val90]. In particular, MPC allows  $N^\delta$  space per machine (processor), where  $\delta \in (0, 1)$  and  $N$  is the input size, with alternating rounds of unlimited local computation, and communication of up to  $N^\delta$  data per processor. An MPC algorithm can equivalently be seen as a small circuit, with arbitrary,  $N^\delta$ -fan-in gates; the depth of the circuit is the parallel time. Any PRAM algorithm can be simulated on MPC in the same parallel time [KSV10, GSZ11]. However, MPC is in fact more powerful than the PRAM: even computing the XOR of  $N$  bits requires near-logarithmic parallel-time on the most powerful CRCW PRAMs [BH89], whereas it takes constant,  $O(1/\delta)$ , parallel time on the MPC model.

The main algorithmic question of this area is then: for which problems can we design MPC algorithms that are *faster* than the best PRAM algorithms? Indeed, this question has been the focus of several recent papers, see, e.g., [KSV10, LMSV11, EIM11, ANOY14, AG18, AK17, IMS17, CLM<sup>+</sup>18]. Graph problems have been particularly well studied and one fundamental problem is *connectivity in a graph*. While this problem has a standard logarithmic time PRAM algorithm [SV82], we do not know whether we can solve it faster in the MPC model.

While we would like *fully scalable* algorithms—which work for any value of  $\delta > 0$ —there have been graph algorithms that use space close to the number of vertices  $n$  of the graph. In particular, the result of [LMSV11] showed a faster algorithm for the setting when the space per machine is polynomially larger than the number of vertices, i.e.,  $s \geq n^{1+\Omega(1)}$ , and hence the number of edges is necessarily  $m \geq n^{1+\Omega(1)}$ . In fact, similar space restrictions are pervasive for all known sub-logarithmic time graph algorithms, which require  $s = \Omega(\frac{n}{\log^{O(1)} n})$  [LMSV11, AG18, AK17, CLM<sup>+</sup>18] (the only exception is [ANOY14] who consider geometric graphs). We highlight the work of [CLM<sup>+</sup>18], who manage to obtain *slightly sublinear* space of  $n/\log^{\Omega(1)} n$  in  $\log^{O(1)} \log n$  parallel time, for the approximate matching problem and [ABB<sup>+</sup>17] who obtain *slightly sublinear* space of  $n/\log^{\Omega(1)} n$  in  $O(\log \log n)$  parallel time. We note that the space of  $\sim n$  also coincides with the space barrier of the semi-streaming model: essentially no graph problems are solvable in less than  $n$  space in the streaming model, unless we have many more passes; see e.g. the survey [McG09].

It remains a major open question whether there exist fully scalable connectivity MPC algorithms with sub-logarithmic time (e.g., for sparse graphs). There are strong indications that such algorithms do *not* exist: [BKS13] show logarithmic lower bounds for restricted algorithms. Alas, showing an unconditional lower bound may be hard to prove, as that would imply circuit lower bounds [RVW16].

In this work, we show **faster, fully scalable algorithms for the connectivity problem**, by parameterizing the time complexity as a function of the *diameter* of the graph. The diameter of the graph is the largest diameter of its connected components. Our main result is an  $O(\log D \log \log_{m/n} n)$  time connectivity algorithm for diameter- $D$  graphs with  $m$  edges. Parameterizing as a function of  $D$  is standard, say, in the distributed computing literature [PRS16, HHW18]. In fact, some previous MPC algorithms for connectivity in the applied communities have been *conjectured* to obtain  $O(\log D)$  time [RMCS13]; alas, we show in Section I the algorithm of [RMCS13] has a lower bound of  $\Omega(\log n)$  time.

Our algorithms exhibit a tradeoff between the total amount of memory available and the number of rounds of computation needed. For example, if the total space is  $\Omega(n^{1+\gamma'})$  for some constant  $\gamma' > 0$ , then our algorithms run in  $O(\log D)$  rounds only.

## 1.1 The MPC model

Before stating our full results, we briefly recall the MPC model [BKS13]. A detailed discussion appears in Section E, along with some core primitives implementable in the MPC model.

**Definition 1.1** ( $(\gamma, \delta)$  – MPC model). *Fix parameters  $\gamma, \delta > 0$ , and suppose  $N \geq 1$  is the input size. There are  $p \geq 1$  machines (processors) each with local memory size  $s = \Theta(N^\delta)$ , such that  $p \cdot s = O(N^{1+\gamma})$ . The space size is measured by words, each of  $\Theta(\log(s \cdot p))$  bits. The input is distributed on the local memory of  $\Theta(N/s)$  input machines. The computation proceeds in rounds. In each round, each machine performs computation on the data in its local memory, and sends messages to other machines at the end of the round. The total size of messages sent or received by a machine in a round is bounded by  $s$ . In the next round, each machine only holds the received messages in its local memory. At the end of the computation, the output is distributed on the output machines. Input/output machines and other machines are identical except that input/output machine can hold a part of the input/output. The parallel time of an algorithm is the number of rounds needed to finish the computation.*

In this model, the space per machine is sublinear in  $N$ , and the total space is only an  $O(N^\gamma)$  factor more than the input size  $N$ . In this paper, we consider the case when  $\delta$  is an arbitrary constant in  $(0, 1)$ . Our results are for both the most restrictive case of  $\gamma = 0$  (total space is linear in the input size), as well as  $\gamma > 0$  (for which our algorithms are a bit faster). The model from Definition 1.1 matches the model  $\text{MPC}(\epsilon)$  from [BKS13] with  $\epsilon = \gamma/(1 + \gamma - \delta)$  and the number of machines  $p = O(N^{1+\gamma-\delta})$ .

## 1.2 Our Results

While our main result is a  $\sim \log D$  time connectivity MPC algorithm, our techniques extend to related graph problems, such as spanning forest, finding a DFS sequence, and exact/approximate minimum spanning forest. We also prove a lower bound showing that, achieving similar bounds for reachability in *directed graphs* would imply faster boolean matrix multiplication algorithms.

We now state our results formally. For all results below, consider an input graph  $G = (V, E)$ , with  $n = |V|$ ,  $N = |V| + |E|$ , and  $D$  being the upper bound on the diameter of any connected component of  $G$ .

**Connectivity:** In the connectivity problem, the goal is to output the connected components of an input graph  $G$ , i.e. at the end of the computation,  $\forall v \in V$ , there is a unique tuple  $(x, y)$  with  $x = v$  stored on an output machine, where  $y$  is called the color of  $v$ . Any two vertices  $u, v$  have the same color if and only if they are in the same connected component.

**Theorem 1.2** (Connectivity in MPC, restatement of Theorem F.4). *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm (see Algorithm 3) which outputs the connected components of the graph  $G$  in  $O(\min(\log D \cdot \log \frac{\log n}{\log(N^{1+\gamma}/n)}, \log n))$  parallel time. The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

Notice that in the most restrictive case of  $\gamma = 0$  and  $m = n$ , we obtain  $O(\min(\log D \cdot \log \log n, \log n))$  time. When the total space is slightly larger, or the graph is slightly denser—i.e.  $\gamma > c$  or  $\log_n m > c$ , where  $c > 0$  is an arbitrarily small constant—then we obtain  $O(\log D)$  time.

**Remark 1.3.** We note the concurrent and independent work of [ASW18], who also give a connectivity algorithm in the MPC model but with different guarantees. In particular, their runtime is parameterized as a function of  $\lambda$ , which is a lower bound on the spectral gap<sup>1</sup> of the connected components of  $G$ . For a graph  $G$  with  $n$  vertices and  $m = \tilde{O}(n)$  edges, their algorithm runs in  $O(\log \log n + \log(1/\lambda))$  parallel time and uses  $\tilde{O}(n/\lambda^2)$  total space. In contrast, our algorithm has a runtime of  $O(\log D \cdot \log \log_{N/n} n)$ , where  $D$  is the largest diameter of a connected component of  $G$ , and  $N = \Omega(m)$  is the total space available. To compare the two runtimes, we note that: 1)  $D \leq O(\frac{\log n}{\lambda})$  for any undirected graph  $G$ ; and 2) there exist sparse graphs  $G^2$  with  $n$  vertices and  $O(n)$  edges such that  $\frac{1}{\lambda} \geq D \cdot n^{\Omega(1)}$  and  $D \leq O(\log n)$ . Thus, our results subsume [ASW18] in the case when total space is  $N = n^{1+\Omega(1)}$ , but are incomparable otherwise.

**Spanning forest problem:** In the spanning forest problem, the goal is to output a subset of edges of an input graph  $G$  such that the output edges together with the vertices of  $G$  form a spanning forest of the graph  $G$ . In the rooted spanning forest problem, in addition to the edges of the spanning forest, we are also required to orient the edge from child to parent, so that the parent-child pairs form a rooted spanning forest of the input graph  $G$ .

**Theorem 1.4** (Spanning Forest, restatement of Theorem F.14). *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$ -MPC algorithm (see Algorithm 11 and Algorithm 12) which outputs the rooted spanning forest of the graph  $G$  in  $O(\min(\log D \cdot \log \frac{\log n}{\log(N^{1+\gamma}/n)}, \log n))$  parallel time. The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

Our spanning forest algorithm can also output an approximation to the diameter, as follows.

**Theorem 1.5** (Diameter Estimator, restatement of Theorem F.15). *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$ -MPC algorithm which outputs a diameter estimator  $D'$  of the input graph  $G$  in  $O(\min(\log D \cdot \log \frac{\log n}{\log(N^{1+\gamma}/n)}, \log n))$  parallel time such that  $D \leq D' \leq D^{O(\log(1/\gamma'))}$ , where  $\gamma' = \frac{\log(N^{1+\gamma}/n)}{\log n}$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

**Depth-First-Search sequence:** If the input graph  $G$  is a tree, then we are able to output a Depth-First-Search sequence of that tree in  $O(\log D) + T$  parallel time, where  $T$  is parallel time to compute a rooted tree (see Theorem 1.4 for our upper bound of  $T$ ) for  $G$ . (See Section E.2 for a discussion how to represent a sequence in the MPC model.)

**Theorem 1.6** (DFS Sequence of a Tree, restatement of Theorem F.21). *Suppose the graph  $G$  is a tree. For any  $\gamma \in [\beta, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$ -MPC algorithm (Algorithm 17) that outputs a Depth-First-Search sequence for the input graph  $G$  in  $O(\min(\log D \cdot \log(1/\gamma), \log n))$  parallel time, where  $\beta = \Theta(\log \log n / \log n)$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

Applications of DFS sequence of a tree include lowest common ancestor, tree distance oracle, the size of every subtree, and others. See Section D.4 for a more detailed discussion of the DFS sequence of a tree.

**Minimum Spanning Forest:** In the minimum spanning forest problem, the goal is to compute the minimum spanning forest of a weighted graph  $G$ .

<sup>1</sup>The spectral gap of a graph  $G$  is the second smallest eigenvalue of the normalized Laplacian of  $G$ .

<sup>2</sup>We can construct  $G$  as the following: a bridge connects two 3-regular expanders where each expander has  $n/2$  vertices.

**Theorem 1.7** (Minimum Spanning Forest, restatement of Theorem G.3). *Consider a weighted graph  $G$  with weights  $w : E \rightarrow \mathbb{Z}$  such that  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ . For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which outputs a minimum spanning forest of  $G$  in  $O(\min(\log D_{MSF} \cdot \log(\frac{\log n}{1+\gamma \log n}), \log n) \cdot \frac{\log n}{1+\gamma \log n})$  parallel time, where  $D_{MSF}$  is the diameter (with respect to the number of edges/hops) of a minimum spanning forest of  $G$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

We note that we require the bounded weights condition merely to ensure that each weight is described by one word.

**Theorem 1.8** (Approximate Minimum Spanning Forest, restatement of Theorem G.4). *Consider a weighted graph  $G$  with weights  $w : E \rightarrow \mathbb{Z}_{\geq 0}$  such that  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ . For any  $\epsilon \in (0, 1)$ ,  $\gamma \in [\beta, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a  $(1 + \epsilon)$  approximate minimum spanning forest for  $G$  in  $O(\min(\log D_{MSF} \cdot \log(\frac{\log n}{\log(N^{1+\gamma}/(\epsilon^{-1}n \log n))}), \log n))$  parallel time, where  $\beta = \Theta(\log(\epsilon^{-1} \log n)/\log n)$ , and  $D_{MSF}$  is the diameter (with respect to the number of edges/hops) of a minimum spanning forest of  $G$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

**Theorem 1.9** (Bottleneck Spanning Forest, restatement of Theorem G.5). *Consider a weighted graph  $G$  with weights  $w : E \rightarrow \mathbb{Z}$  such that  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ . For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a bottleneck spanning forest for  $G$  in  $O(\min(\log D_{MSF} \cdot \log(\frac{\log n}{1+\gamma \log n}), \log n) \cdot \log(\frac{\log n}{1+\gamma \log n}))$  parallel time, where  $D_{MSF}$  is the diameter (with respect to the number of edges/hops) of a minimum spanning forest of  $G$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it returns FAIL.*

**Conditional hardness for directed reachability.** We also consider the reachability question in the directed graphs, for which we show similar to the above results are unlikely. In particular, we show that if there is a fully scalable multi-query directed reachability  $(0, \delta)$  – MPC algorithm with  $n^{o(1)}$  parallel time and polynomial local running time, then we can compute the Boolean Matrix Multiplication in  $n^{2+\epsilon+o(1)}$  time for arbitrarily small constant  $\epsilon > 0$ . We note that the equivalent problem for *undirected graphs* can be solved in  $O(\log D \log \log n)$  parallel time via Theorem 1.2.

**Theorem 1.10** (Directed Reachability vs. Boolean Matrix Multiplication, restatement of Theorem H.1). *Consider a directed graph  $G = (V, E)$ . If there is a polynomial local running time, fully scalable  $(\gamma, \delta)$  – MPC algorithm that can answer  $|V| + |E|$  pairs of reachability queries simultaneously for  $G$  in  $O(|V|^\alpha)$  parallel time, then there is a sequential algorithm which can compute the multiplication of two  $n \times n$  boolean matrices in  $O(n^2 \cdot n^{2\gamma+\alpha+\epsilon})$  time, where  $\epsilon > 0$  is a constant which can be arbitrarily small.*

Finally, in Section I we show hard instances for the algorithm [RMCS13].

### 1.3 Our Techniques

In this section, we give an overview of the various techniques that we use in our algorithms. More details, as well as some of the low level details of the implementation in the MPC model, are deferred to later sections.

Before getting into our techniques, we mention two standard tools to help us build our MPC subroutines. The first one is sorting: while in the PRAM model it takes  $\sim \log N$  parallel time, sorting takes only constant parallel time in the MPC model [Goo99, GSZ11]. The second tool is indexing/predecessor search [GSZ11], which also has a constant parallel time in MPC. Furthermore,



these two tools are fully scalable, and hence all the subroutines built on these two tools are also fully scalable. See Section E for how to use these two tools to implement the MPC operations needed for our algorithms.

**Graph Connectivity:** A natural approach to the graph connectivity problem is via the classic primitive of contracting to leaders: select a number of leader vertices, and contract every vertex (or most vertices) to a leader from its connected component (this is usually implemented by labeling the vertex by the corresponding leader). Indeed, many previous works (see e.g. [KSV10, RMCS13, KLM<sup>+</sup>14]) are based on this approach. There are two general questions to address in this approach: 1) how to choose leader vertices, and 2) how to label each vertex by its leader. For example, the algorithm in [KSV10] randomly chooses half of the vertices as leaders, and then contracts each non-leader vertex to one of its neighbor leader vertex. Thus, in each round of their algorithm, the number of vertices drops by a constant fraction. At the same time, half of the vertices are leaders, and hence their algorithm still needs at least  $\Omega(\log n)$  rounds to contract all the vertices to one leader. Note that a constant fraction of leaders is needed to ensure that there is a constant fraction of non-leader vertices who are adjacent to at least one leader vertex and hence are contracted. This leader selection method appears optimal for some graphs, e.g. path graphs.

To improve the runtime to  $\ll \log n$ , one would have to choose a much smaller fraction of the vertices to be leaders. Indeed, for a graph where every vertex has a large degree, say at least  $d \gg \log n$ , we can choose fewer leaders: namely, we can choose each vertex to be a leader with probability  $p = \Theta((\log n)/d)$ . Then the number of leaders will be about  $\tilde{O}(n/d)$ , while each non-leader vertex has at least one leader neighbor with high probability. After contracting non-leader vertices to leader vertices, the number of remaining vertices is only a  $1/d$  fraction of original number of vertices.

By the above discussion, the goal would now be to modify our input graph  $G$  so that every vertex has a uniformly large degree, without affecting the connectivity of the graph. An obvious such modification is to add edges between pairs of vertices that are already in the same connected component. In particular, if a vertex  $v$  learns of a large number of vertices which are in the same connected component as  $v$ , then we can add edges between  $v$  and those vertices to increase the degree of  $v$ . A naïve way to implement the latter is via broadcasting: each vertex  $v$  first initializes a set  $S_v$  which contains all the neighbors of  $v$ , and then, in each *round*, every vertex  $v$  updates the set  $S_v$  by adding the union of the sets  $S_u$  over all neighbors  $u$  of  $v$  (old and new). This approach takes log-diameter number of rounds, and each vertex learns all vertices which are in the same connected component at the end of the procedure. However, in a single round, the total communication needed may be as huge as  $\Omega(n^3)$  since each of  $n$  vertices may have  $\Omega(n)$  neighbors, each with a set of size  $\Omega(n)$ .

Since our goal of each vertex  $v$  is to learn only  $d$  vertices in the same component (not necessarily the entire component), we can therefore implement a “truncated” version of the above broadcasting procedure:

1. If  $S_v$  already had size  $d$ , then we do not need any further operation for  $S_v$ .
2. If  $u$  is in  $S_v$ , and  $S_u$  already has  $d$  vertices, then we can just put all the elements from  $S_u$  into  $S_v$  and thus  $S_v$  becomes of size  $d$ .
3. If  $|S_v| < d$ , and for every  $u \in S_v$ , the set  $S_u$  is also smaller than  $d$ , then we can implement one step of the broadcasting — add the union of  $S_u$ ’s, for all neighbors  $u \in S_v$ , to  $S_v$ .

In the above procedure, if the number of vertices in  $S_v$  is smaller than  $d$  after the  $i^{\text{th}}$  round, then we expect  $S_v$  to contain all the vertices whose distance to  $v$  is at most  $2^i$ . Thus, the above procedure



also takes at most log-diameter rounds. Furthermore, the total communication needed is at most  $O(n \cdot d^2)$ .

Our full graph connectivity algorithm implements the above “truncated broadcasting” procedure iteratively, for values  $d$  that follow a certain “schedule”, depending on the available space. At the beginning of the algorithm, we have an  $n$  vertex graph  $G$  with diameter  $D$ , and a total of  $\Omega(m)$  space. The algorithm proceeds in phases, where each phase takes  $O(\log D)$  rounds of communication. In the first phase, the starting number of vertices is  $n_1 = n$ . We implement a truncated broadcasting procedure where the target degree  $d$  is  $d_1 = (m/n_1)^{1/2}$ , using  $O(\log D)$  rounds and  $O(m)$  total space. Then we can randomly select  $\tilde{O}(n_1/d_1)$  leaders, and contract all the non-leader vertices to leader vertices. At the end of the first phase, the total number of remaining vertices is at most  $n_2 = \tilde{O}(n_1/d_1) = \tilde{O}(n_1^{1.5}/m^{0.5})$ . In general, suppose, at the beginning of the  $i^{\text{th}}$  phase, the number of remaining vertices is  $n_i$ . Then we use the truncated broadcasting procedure for value  $d$  set to  $d_i = (m/n_i)^{1/2}$ , thus making each vertex have degree at least  $d_i = (m/n_i)^{1/2}$  in  $O(\log D)$  number of communication rounds and  $O(m)$  total space. Then we choose  $\tilde{O}(n_i/d_i)$  leaders, and, after contracting non-leaders, the number  $n_{i+1}$  of remaining vertices is at most  $\tilde{O}(n_i^{1.5}/m^{0.5})$ . Let us look at the progress of the value  $d_i$ . We have that  $d_{i+1} = \tilde{\Omega}((m/n_{i+1})^{1/2}) = \tilde{\Omega}((m^{1.5}/n_i^{1.5})^{1/2}) = \tilde{\Omega}(d_i^{1.5})$ . Thus, we are making double exponential progress on  $d_i$ , which implies that the total number of phases needed is at most  $O(\log \log_{m/n} n)$ , and the total parallel time is thus  $O(\log D \cdot \log \log_{m/n} n)$ .

This technique of double-exponential progress is more general and extends to other problems beyond connectivity. In particular, for a problem, suppose its size is characterized by a parameter  $n$  (not necessarily the input size—e.g. in connectivity problem,  $n$  is the number of vertices). When  $n$  is a constant, the problem can be solved in  $O(1)$  parallel time. If there is a procedure that uses total space  $\Theta(m)$  to reduce the problem size to at most  $n/k$  for  $k = (m/n)^c$ ,  $c = \Omega(1)$ , then we can repeat the procedure  $O(\log \log_{m/n} n)$  times to solve the overall problem. In particular, after repeating the procedure  $i$  times, the problem size is  $n_i \leq n_{i-1}/(m/n_{i-1})^c \leq n \cdot (n/m)^{(1+c)^i - 1}$ . We call this technique *double-exponential speed problem size reduction*.

**Remark 1.11.** *For any problem characterized by a size parameter  $n$ , if we can use parallel time  $T$  and total space  $\Theta(m)$  to reduce the problem size such that the reduced problem size is  $n/k$  for  $k = (m/n)^{\Omega(1)}$ , then we can solve the problem in  $O(m)$  total space and  $O(T \cdot \log \log_{m/n} n)$  parallel time.*

**Spanning Forest and Diameter Estimator:** Extending a connectivity algorithm to a spanning forest algorithm is usually straightforward. For example, in [KSV10], they only contract a non-leader vertex to an adjacent leader vertex, thus their algorithm can also give a spanning forest, using the contracted edges. Here however, extending our connectivity algorithm to a spanning forest algorithm requires several new ideas. In our connectivity algorithm, because of the added edges, we only ensure that when a vertex  $u$  is contracted to a vertex  $v$ ,  $u$  and  $v$  must be in the same connected component; but  $u$  and  $v$  may not be adjacent in the original graph. Thus, we need to record more information to help us build a spanning forest.

We can represent a forest as a collection of parent pointers  $\text{par}(v)$ , one for each vertex  $v \in V$ . If  $v$  is a root in the forest, then we let  $\text{par}(v) = v$ . We use  $\text{dep}_{\text{par}}(v)$  to denote the depth of  $v$  in the forest, i.e.  $\text{dep}_{\text{par}}(v)$  is the distance from  $v$  to its root. Let  $\text{dist}_G(u, v)$  denote the distance between two vertices  $u$  and  $v$  in a graph  $G$ .

Our connectivity algorithm uses the “neighbor increment” procedure described above. We observed that if the set  $S_v$  has fewer than  $d$  vertices after the  $i^{\text{th}}$  round, then  $S_v$  should contain all the vertices with distance at most  $2^i$  to  $v$ . This motivates us to maintain a shortest path tree for

$S_v$ , with root  $v$ . In the  $i^{\text{th}}$  round, if we need to update  $S_v$  to be  $\bigcup_{u \in S_v} S_u$ , then we can update the shortest path tree of  $S_v$  in the following way:

1. For each  $x \in S_u$  for some  $u \in S_v$ , we can create a tuple  $(x, u)$ .
2. Then, for each  $x \in (\bigcup_{u \in S_v} S_u) \setminus S_v$ , we can sort all the tuples  $(x, u_1), (x, u_2), \dots, (x, u_k)$  such that  $u_1$  minimizes  $\min_{u \in S_v} \text{dist}_G(v, u) + \text{dist}_G(u, x)$ . Since  $u$  is in  $S_v$ ,  $x$  is in  $S_u$ , it is easy to get the value of  $\text{dist}_G(v, u), \text{dist}_G(u, x)$  by the information of shortest path tree for  $S_v$  and  $S_u$ . Then we set the new parent of  $x$  in the shortest path tree for  $S_v$  to be the parent of  $x$  in the shortest path tree for  $S_{u_1}$ .

Since  $S_v$  before the update contains all the vertices which have distance to  $v$  at most  $2^{i-1}$ , the union of the shortest path from  $x$  to  $u_1$  and the shortest path from  $u_1$  to  $v$  must be the shortest path from  $x$  to  $v$ . Then by induction, we can show that the parent of  $x$  in the shortest path tree for  $S_{u_1}$  is also the parent of  $x$  in the shortest path tree for updated  $S_v$ . Thus, this modified “neighbor increment” procedure can find  $n$  local shortest path trees where there is a tree with root  $v$  for each vertex  $v$ . Furthermore, the procedure still takes  $O(\log D)$  rounds. And we can still use  $O(nd^2)$  total space to make each shortest path tree have size at least  $d$ . Next, we show how to use these  $n$  local shortest path trees to construct a forest with the roots in the forest being the leaders.

As discussed in the connectivity algorithm, if every local shortest path tree has size at least  $d$ , we can choose each vertex as a leader with probability  $p = \Theta((\log n)/d)$  and then every tree will contain at least one leader with high probability. Let  $L$  be the set of sampled leaders, and let  $\text{dist}_G(v, L)$  be defined as  $\min_{u \in L} \text{dist}_G(v, u)$ . Let  $v$  be a non-leader vertex, i.e.  $v \in V \setminus L$ . According to the shortest path tree for  $S_v$ <sup>3</sup>, since  $L \cap S_v \neq \emptyset$ , we can find a child  $u$  of the root  $v$  such that  $\text{dist}_G(v, L) > \text{dist}_G(u, L)$ ; in this case we set  $\text{par}(v) = u$ . For vertex  $v \in L$ , we can set  $\text{par}(v) = v$ . We can see now that  $\text{par}$  denotes a rooted forest where the roots are sampled leaders. Furthermore, since  $\forall v \notin L, (v, \text{par}(v))$  is from the shortest path tree for  $S_v$ , we know that  $v$  and  $\text{par}(v)$  are adjacent in the original graph  $G$ . After doing the above for all nodes  $v$ , the forest denoted by the resulting vector  $\text{par}$  must be a subgraph of the spanning forest of  $G$ . We then apply the standard doubling algorithm to contract all the vertices to their leaders (roots), in  $O(\log D)$  rounds. Therefore, the problem is reduced to finding a spanning forest in the contracted graph. The number of vertices remaining in the contracted graph is at most  $\tilde{O}(n/d)$ , where  $d = (m/n)^{\Theta(1)}$ . By Remark 1.11, we can output a spanning forest in  $O(\log D \cdot \log \log_{m/n} n)$  parallel time.

Although the above algorithm can output the edges of a spanning forest, it cannot output a rooted spanning forest. To output a rooted spanning forest, we follow a top-down construction. Suppose now we have a rooted spanning forest of the contracted graph. Since we have all the information of how vertices were contracted, we know the contraction trees in the original graph. To merge these contraction trees into the rooted spanning forest of the contracted graph, we only need to change the root of each contraction tree to a proper vertex in that tree. This changing root operation can be implemented by the doubling algorithm via a divide-and-conquer approach.

Since the spanning forest algorithm needs  $O(\log \log_{m/n} n)$  phases to contract all vertices to a single vertex, the total parallel time to compute a rooted spanning forest is  $O(\log D \cdot \log \log_{m/n} n)$ . Furthermore, the depth of the rooted spanning forest will be at most  $O(D^{O(\log \log_{m/n} n)})$ . Thus, we can use the doubling algorithm to calculate the depth of the tree, and output this depth as an estimator of the diameter of the input graph.

<sup>3</sup> The construction of  $S_v$  for spanning forest algorithm is slightly different from that described in the connectivity algorithm.  $S_v$  in spanning forest algorithm has a stronger property:  $\forall u \in V \setminus S_v, \text{dist}_G(u, v)$  must be at least  $\text{dist}_G(u', v)$  for any  $u' \in S_v$ .

**Depth-First-Search Sequence:** Here, when the input graph  $G$  is a tree, our goal is to output a DFS sequence for this tree. Once we have this sequence, it is easy to output a rooted tree. Thus, computing a DFS sequence is at least as hard as computing a rooted tree, and all the previous algorithms need  $\Omega(\log n)$  parallel time to do so.

First of all, we use our spanning forest algorithm to compute a rooted tree, reducing the problem to computing a DFS sequence for a rooted tree. The idea is motivated by TeraSort [O'M08]. If the size of the tree is small enough such that it can be handled by a single machine, then we can just use a single machine to generate its DFS sequence. Otherwise, our algorithm can be roughly described as follows. (Recall that  $\delta$  is the parameter such that each machine has  $\Theta(n^\delta)$  local memory.)

1. Sample  $n^{\delta/2}$  leaves  $l_1, l_2, \dots, l_s$ .
2. Determine the order of sampled leaves in the DFS sequence.
3. Compute the DFS sequence  $\tilde{A}$  of the tree which only consists of sampled leaves and their ancestors.
4. Compute the DFS sequence  $A_v$  of every root- $v$  subtree which does not contain any sampled leaf.
5. Merge  $\tilde{A}$  and all the  $A_v$ .

The first and second steps go as follows. Since we only sample  $n^{\delta/2}$  leaves, we can send them to a single machine. We generate queries for every pair of sampled leaves where each query  $(l_i, l_j)$  queries the lowest common ancestor of  $(l_i, l_j)$ . We have  $n^\delta$  such queries in total. Since the input tree is rooted, we can use a doubling algorithm to preprocess a data structure in  $O(\log D)$  parallel time and answer all the queries simultaneously in  $O(\log D)$  parallel time. Thus, we know the lowest common ancestor of any pair of sampled leaves, and we can store this all on a single machine. Based on the information of lowest common ancestors of each pair of sampled leaves, we are able to determine the order of the leaves.

For the third step, suppose the sampled leaves have order  $l_1, l_2, \dots, l_s$ . Let  $v$  be the root of the tree. Then the DFS sequence  $\tilde{A}$  should be: the path from  $v$  to  $l_1$ , the path from  $l_1$  to the lowest common ancestor of  $(l_1, l_2)$ , the path from the lowest common ancestor of  $(l_1, l_2)$  to  $l_2$ , the path from  $l_2$  to the lowest common ancestor of  $(l_2, l_3)$ , ..., the path from  $l_s$  to  $v$ . We can find these paths simultaneously by a doubling algorithm together with a divide-and-conquer algorithm in  $O(\log D)$  parallel time.

In the fourth step, we apply the procedure recursively. Suppose the total number of leaves in the tree is  $q \leq n$ . Since we randomly sampled  $n^{\delta/2}$  number of leaves, with high probability, each subtree which does not contain a sampled leaf will have at most  $O(q/n^{\delta/2})$  number of leaves. Thus, the depth of the recursion will be at most a constant,  $O(1/\delta)$ .

**Minimum Spanning Forest and Bottleneck Spanning Forest.** Recall that the input is a graph  $G = (V, E = (e_1, e_2, \dots, e_m))$  together with a weight function  $w$  on  $E$ . Without loss of generality, we only consider the case when all the weights of edges are different, i.e.  $w(e_1) < w(e_2) < \dots < w(e_m)$ . Since the weights of edges are different, the minimum spanning forest of the graph is unique. By Kruskal's algorithm, the diameter of the graph induced by the first  $i$  edges for any  $i \in [m]$  is at most the depth of the minimum spanning forest. Now, let us use  $D$  to denote the depth of the minimum spanning forest.

We first discuss the minimum spanning forest algorithm. A crucial observation of Kruskal's algorithm is: if we want to determine which edges in  $e_i, e_{i+1}, \dots, e_j$  are in the minimum spanning forest, we can always contract the first  $i - 1$  edges to obtain a graph  $G'$ , run a minimum spanning forest algorithm on the contracted graph  $G'$ , and observe whether an edge is included in the spanning

forest of  $G'$ . Thus, if the total space is  $\Theta(m^{1+\gamma})$ , we can have  $m^\gamma$  copies of the graph, where the  $i^{\text{th}}$  copy contracts the first  $(i-1) \cdot m^{1-\gamma}$  edges. Thus, we are able to divide the edges into  $m^\gamma$  groups where each group has  $m^{1-\gamma}$  number of edges. We only need to solve the minimum spanning forest problem for each group. Then in the second phase, we can divide the edges into  $m^{2\gamma}$  groups where each group has  $m^{1-2\gamma}$  number of edges. Thus, the total number of phases needed is at most  $O(1/\gamma)$ . In each phase, we just need to run our connectivity algorithm to contract the graph.

For the approximate minimum spanning forest algorithm, we use a similar idea. If we want a  $(1+\epsilon)$  approximation, then we round each weight to the closest value  $(1+\epsilon)^i$  for some integer  $i$ . After rounding, there are only  $O(1/\epsilon \cdot \log n)$  edge groups. Since our total space is at least  $\Omega(m \log(n)/\epsilon)$ , we can make  $O(1/\epsilon \cdot \log n)$  copies of the graph. The  $i^{\text{th}}$  copy of the graph contracts all the edges in group  $1, 2, \dots, i-1$ . Then, we only need to run our spanning forest algorithm on each copy to determine which edges should be chosen in each group.

Another application of our *double exponential speed problem size reduction* technique is bottleneck spanning forest. For the bottleneck spanning forest, suppose we have  $\Theta(km)$  total space. We can have  $k$  copies of the graph where the  $i^{\text{th}}$  copy contracts the first  $(i-1) \cdot m/k$  number of edges. We can determine the group of  $O(m/k)$  edges which contains the bottleneck edge. Thus, we reduce the problem to  $O(m/k)$ . According to Remark 1.11, the number of phases is at most  $O(\log \log_k m)$ , and each phase needs  $T$  parallel time, where  $T$  is the parallel time for spanning forest.

**Directed Reachability vs. Boolean Matrix Multiplication** If there is a fully scalable multi-query directed reachability MPC algorithm with almost linear total space, we can simulate the algorithm in sequential model. Thus, it will imply a good sequential multi-query directed reachability algorithm which implies a good sequential Boolean Matrix Multiplication algorithm.

## 1.4 Roadmap

The rest of the paper contains the technical details of our algorithms. In Section 2, we described a simplified connectivity algorithm. In Section A, we described the notations. In Sections B, C, and D, we give the details of our main algorithms for connectivity, spanning forest and depth first search sequence. In these sections, we focus on the design of the algorithms and the analysis of the number of rounds. In Section E, we describe the MPC model in detail and discuss some known primitives in that model. In Section F, we discuss how to implement the details of our algorithms in the MPC model to achieve the bounds claimed in the previous sections. In Section G, we show how to apply our connectivity and spanning forest algorithm in minimum spanning forest and bottleneck spanning forest problems. In Section I, we show hard instances for the algorithm [RMCS13]. In Section J, we show an alternative approach for random leader selection.

## 2 A Simplified Batch Algorithm for Connectivity

In this section, we show a simplified version of our connectivity algorithm.

Firstly, let us describe the simplified version of truncated broadcasting procedure in the following. Since  $G'$  is obtained by adding edges between the vertices in the same component of  $G$ .  $G'$  will preserve the connectivity of  $G$ . The parallel time needed is at most  $O(\log D)$  where  $D$  is the diameter of  $G$ . The procedure takes at most  $O(nd^2 + m)$  total space.

Truncated Broadcasting for Neighbor Increment:

- **Input:**
  - A graph  $G = (V, E)$  with  $n = |V|$  vertices and  $m = |E|$  number of edges.
  - A parameter  $d$ .
- **Output:**
  - A graph  $G' = (V, E')$  such that  $\forall v \in V, |\Gamma(v)| \geq d$ .  $\triangleright \Gamma(v)$  denotes the neighbors of  $v$ .
- **While**  $\exists x \in V$  such that  $|\Gamma(x)| < d$  :
  - For each  $v \in V$  with  $|\Gamma(v)| < d$  :
    - \* If  $\exists u \in \Gamma(v)$  which has  $|\Gamma(u)| \geq d$ , then  $\Gamma(v) \leftarrow \Gamma(v) \cup \Gamma(u)$ .
    - \* Otherwise,  $\Gamma(v) \leftarrow \Gamma(v) \cup \bigcup_{u \in \Gamma(v)} \Gamma(u)$ .

We can apply the above procedure to make each vertex have a large degree. Next, let us briefly describe how to choose the leaders and implement the contraction operation for the graph where each vertex has a large degree. The following procedure just needs  $O(n + m)$  total space and  $O(1)$  parallel time. If every vertex has degree at least  $d$ , then in the following procedure we can reduce the number of vertices to  $\tilde{O}(n/d)$  by contracting all the vertices to  $\tilde{O}(n/d)$  number of leaders.

Random Leader Contraction:

- **Input:**
  - A graph  $G = (V, E)$  with  $n = |V|$  vertices where each vertex has degree at least  $d$ .
- **Output:**
  - A graph  $G' = (V', E')$  with  $\tilde{O}(n/d)$  vertices.
  - A mapping  $\text{par} : V \rightarrow V'$ , such that  $\text{par}(v)$  is the vertex that  $v$  contracts to.
- **Leader Selection:**
  - Let  $L$  denote the set of leaders.
  - For each  $v \in V$ , with probability at least  $\tilde{\Omega}(1/d)$ , choose  $v$  as a leader, i.e.  $L \leftarrow L \cup \{v\}$ .
- **Contraction:**
  - For each  $v \in L$ , let  $\text{par}(v) = v$ , and put  $v$  into  $V'$ .
  - For each  $v \in V \setminus L$ , choose  $u \in \Gamma(v) \cap L$ , and set  $\text{par}(u) = v$ .
  - For each  $(u, v) \in E$ , if  $\text{par}(u) \neq \text{par}(v)$ , put the edge  $(\text{par}(u), \text{par}(v))$  into  $E'$ .

Finally, we describe the simplified version of our connectivity algorithm in the following.

Connectivity:

- **Input:**

- A graph  $G = (V, E)$  with  $n = |V|$  vertices and  $m = |E|$  edges.
- Total space  $N$  which is  $\Theta(m)$ .

- **Output:**

- A mapping  $\text{col} : V \rightarrow V$  satisfies  $\forall u, v \in V, \text{col}(u) = \text{col}(v)$  if and only if  $u$  and  $v$  are connected.

- **Initialization:**

- Let  $G_0 \leftarrow G, n_0 \leftarrow n$ .

- **In phase  $i$ :**

- Compute  $G'_{i-1}$  : Increase the degree of every vertex in  $G_{i-1}$  to at least  $d_i = \Theta((N/n_{i-1})^{1/2})$ .
- Compute  $G_i$  : Select  $n_i = \tilde{O}(n_{i-1}/d_i)$  leaders in  $G'_{i-1}$  and contract all the vertices to the leaders.
- If  $v$  is contracted to  $u$ , record  $\text{par}(v) = u$ .
- If  $G_i$  does not have any edges, then for every vertex  $v$  in  $G_i$ , set  $\text{par}(v) = v$ , and exit the loop.

- **Finding the root leader:**

- For each  $v \in V$ , find the root of  $v$  in  $\text{par}$ , i.e. find  $u = \text{par}(\text{par}(\dots \text{par}(v)))$  such that  $\text{par}(u) = u$ .
- Set  $\text{col}(v) = u$ .

After phase  $i$ , the number of vertices survived is at most  $\tilde{O}(n_{i-1}/(N/n_{i-1})^{1/2})$ . By Remark 1.11, there will be at most  $O(\log \log_{N/n} n)$  phases. For phase  $i$ , we need  $O(\log D)$  parallel time to increase the neighbors of every vertex in  $G_{i-1}$ . The total parallel time is thus  $O(\log D \cdot \log \log_{N/n} n)$ . The total space used in phase  $i$  is at most  $O(m + (N/n_{i-1})^{1/2} \cdot n_{i-1}) = O(N)$ .

## A Notations

$[n]$  denotes the set  $\{1, 2, \dots, n\}$ . Let  $G$  be an undirected graph with vertex set  $V$  and edge set  $E$ . For  $v \in V$ ,  $\Gamma_G(v)$  denotes the set of neighbors of  $v$  in  $G$ , i.e.  $\Gamma_G(v) = \{u \in V \mid (v, u) \in E\}$ . For any  $u, v \in V$ ,  $\text{dist}_G(u, v)$  denotes the distance between  $u, v$  in graph  $G$ . If  $u, v$  are not in the same connected component, then  $\text{dist}_G(u, v) = \infty$ . If  $u, v$  are in the same connected component, then  $\text{dist}_G(u, v) < \infty$ . For  $v \in V$ ,  $\{u \in V \mid \text{dist}_G(u, v) < \infty\}$  is the set of all the vertices in the same connected component as  $v$ . The diameter  $\text{diam}(G)$  of  $G$  is the largest diameter of its components, i.e.  $\text{diam}(G) = \max_{u, v \in V: \text{dist}_G(u, v) < \infty} \text{dist}_G(u, v)$ .

## B Graph Connectivity

### B.1 Neighbor Increment Operation

In this section, we describe a procedure which can increase the number of neighbors of every vertex and preserve the connectivity at the same time. The input of the procedure is an undirected graph  $G = (V, E)$  and a parameter  $m$  which is larger than  $|V|$ . The output is a graph  $G' = (V, E')$  such that for each vertex  $v$ , either the connected component which contains  $v$  is a clique or  $v$  has at least  $\lceil (m/|V|)^{1/2} \rceil - 1$  neighbors. Furthermore,  $|E'| \leq |E| + m$ . We use  $\Gamma_G(v)$  to denote the neighbors of  $v$  in graph  $G$ , i.e.  $\Gamma_G(v) = \{u \in V \mid (u, v) \in E\}$ . Similarly, we let  $\Gamma_{G'}(v)$  be the neighbors of  $v$  in  $G'$ , i.e.  $\Gamma_{G'}(v) = \{u \in V \mid (u, v) \in E'\}$ .

**Lemma B.1.** *Let  $G = (V, E)$  be an undirected graph,  $m \in \mathbb{Z}_{\geq 0}$  which has  $m \geq 4|V|$ . Let  $n = |V|$ . Let  $r$  be the value at the end of the procedure  $\text{NEIGHBORINCREMENT}(m, G)$  (Algorithm 1.) Then  $\forall i \in \{0, 1, \dots, r\}, v \in V$ ,  $S_v^{(i)}$  satisfies the following properties:*

1.  $v \in S_v^{(i)}$ .
2.  $\forall u \in S_v^{(i)}, \text{dist}_G(u, v) < \infty$ .
3.  $|S_v^{(i)}| < \lceil (m/n)^{1/2} \rceil \Rightarrow S_v^{(i)} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^i\}$ .
4.  $|S_v^{(i)}| \leq m/n$ .

*Proof.* For property 1, we can prove it by induction. When  $i = 0$ , due to line 3, we know  $v \in S_v^{(0)}$ . Suppose property 1 holds for  $S_v^{(i-1)}$  for all  $v \in V$ . If  $S_v^{(i)}$  is updated by line 17, there are two cases: 1. if  $u = v$ , then  $v \in S_v^{(i-1)}$ , and the condition of line 17 does not hold, thus  $v \in S_v^{(i)}$ ; 2. if  $u \neq v$ , then after implementing line 17,  $v$  will not be removed, thus  $v \in S_v^{(i)}$ . If  $S_v^{(i)}$  is updated by line 20, then since  $v \in S_v^{(i-1)}$ ,  $v$  is also in the set  $S_v^{(i)}$ . Thus, property 1 holds for every  $S_v^{(i)}$ .

For property 2, we can prove it by induction. When  $i = 0$ , it is easy to see  $S_v^{(0)} \subseteq \Gamma_G(v) \cup \{v\}$ , thus property 2 holds for it. Suppose property 2 holds for  $S_v^{(i-1)}$  for all  $v \in V$ . If  $S_v^{(i)}$  is updated by line 17, then since  $u \in S_v^{(i-1)}$  and  $S_v^{(i)} \subseteq S_v^{(i-1)} \cup \{v\}$ , all the vertices from  $S_v^{(i)}$  are in the same connected component as  $u$  and  $u$  is in the same connected component as  $v$ . Thus, property 2 holds in this case. If  $S_v^{(i)}$  is updated by the line 20, then  $\forall p \in S_v^{(i)}$ , there exists  $u \in S_v^{(i-1)}$  such that  $p \in S_u^{(i-1)}$ . We have  $p$  is in the same connected component as  $u$ , and  $u$  is in the same connected component as  $v$ . Thus, property 2 also holds in this case.

For property 3, we can prove it by induction. When  $i = 0$ , due to line 7, we have  $|S_v^{(0)}| < \lceil (m/n)^{1/2} \rceil \rightarrow S_v^{(0)} = \Gamma_G(v) \cup \{v\} = \{u \in V \mid \text{dist}_G(u, v) \leq 1\}$ . Suppose property 3 holds for



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**Algorithm 1** Neighbor Increment Operation

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1: procedure NEIGHBORINCREMENT( $m, G = (V, E)$ ) ▷ Lemma B.1, Lemma B.3
2: ▷ Output:  $G' = (V, E')$ 
3:   Initially,  $n = |V|$ ,  $E' = \emptyset$  and let  $S_v^{(0)} = \{v\}$  for all  $v \in V$ .
4:   for  $v \in V$  do ▷ Initially, let  $S_v^{(0)}$  be the set (or subset) of direct neighbors
5:     for  $u \in \Gamma_G(v)$  do
6:       if  $|S_v^{(0)}| < \lceil (m/n)^{1/2} \rceil$  then
7:          $S_v^{(0)} \leftarrow S_v^{(0)} \cup \{u\}$ .
8:       end if
9:     end for
10:  end for
11:   $r \leftarrow 1$ .
12:  for true do
13:    for  $v \in V$  do
14:      if  $\exists u \in S_v^{(r-1)}, |S_u^{(r-1)}| \geq \lceil (m/n)^{1/2} \rceil$  then ▷ neighbor  $u$  has many neighbors
15:         $S_v^{(r)} = S_u^{(r-1)} \cup \{v\}$ .
16:        if  $|S_v^{(r)}| > |S_u^{(r-1)}|$  then
17:           $S_v^{(r)} \leftarrow S_v^{(r)} \setminus \{u\}$ .
18:        end if
19:      else
20:         $S_v^{(r)} = \bigcup_{u \in S_v^{(r-1)}} S_u^{(r-1)}$ . ▷ neighbors of  $v$ 's neighbors are  $v$ 's new neighbors.
21:      end if
22:    end for
23:    ▷  $S_v^{(r)}$  is large or is a component
24:    if  $\forall v \in V$ , either  $|S_v^{(r)}| \geq \lceil (m/n)^{1/2} \rceil$  or  $|S_v^{(r)}| = |S_v^{(r-1)}|$  then
25:      Let  $E' = E \cup \bigcup_{v \in V} \{(v, u) \mid v \in S_u^{(r)} \text{ or } u \in S_v^{(r)}, u \neq v\}$ .
26:      return  $G' = (V, E')$ 
27:    else
28:       $r \leftarrow r + 1$ .
29:    end if
30:  end for
31: end procedure
```

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$S_v^{(i-1)}$  for all  $v \in V$ . Since if  $|S_v^{(i)}| < \lceil (m/n)^{1/2} \rceil$ , then  $S_v^{(i)}$  can only be updated by line 20, and  $\forall u \in S_v^{(i-1)}$ , it has  $|S_u^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ . Thus,  $S_v^{(i)} = \bigcup_{u \in S_v^{(i-1)}} S_u^{(i-1)} = \bigcup_{u \in V, \text{dist}_G(u, v) \leq 2^{i-1}} \{p \in V \mid \text{dist}_G(p, u) \leq 2^{i-1}\} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^i\}$ . Thus, property 3 holds.

For property 4, we can prove it by induction. When  $i = 0$ , due to line 7,  $\forall v \in V$ , we have  $|S_v^{(0)}| \leq \lceil (m/n)^{1/2} \rceil \leq m/n$ , where the last inequality follows by  $m/n \geq 4$ . Now suppose property 4 holds for  $S_v^{(i-1)}$  for all  $v \in V$ . If  $S_v^{(i)}$  is updated by line 17, then  $|S_v^{(i)}| = |S_u^{(i-1)}| \leq m/n$ . If  $S_v^{(i)}$  is updated by line 20, we know  $\forall u \in S_v^{(i-1)}, |S_u^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ . Notice that by property 1,  $v \in S_v^{(i-1)}$ , so  $|S_v^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ . Thus,  $|S_v^{(i)}| = |\bigcup_{u \in S_v^{(i-1)}} S_u^{(i-1)}| \leq (m/n)^{1/2} \cdot (m/n)^{1/2} \leq m/n$ .  $\square$

The following definition defines the number of iterations of Algorithm 1.

**Definition B.2.** Given an undirected graph  $G = (V, E)$  and a parameter  $m \in \mathbb{Z}_{\geq 0}$ ,  $m \geq 4|V|$ , the

number of iterations of  $\text{NEIGHBORINCREMENT}(m, G)$  (Algorithm 1) is the value of  $r$  at the end of the procedure.

In the following lemma, we characterize the properties of Algorithm 1.

**Lemma B.3.** *Let  $G = (V, E)$  be an undirected graph,  $m \in \mathbb{Z}_{\geq 0}$  which has  $m \geq 4|V|$ . Let  $G' = (V, E')$  be the output of  $\text{NEIGHBORINCREMENT}(m, G)$ . We have:*

1. *The number of iterations (Definition B.2),  $r \leq \min(\lceil \log(\text{diam}(G)) \rceil, \lceil \log(m/n) \rceil) + 1$ .*
2. *For all  $u, v \in V$ ,  $\text{dist}_G(u, v) < \infty \Leftrightarrow \text{dist}_{G'}(u, v) < \infty$ .*
3.  *$\forall v \in V$ , if  $|\Gamma_{G'}(v)| < \lceil (m/n)^{1/2} \rceil - 1$ , then the connected component in  $G'$  which contains  $v$  is a clique. It also implies that  $\forall u, v \in V$ , if  $|\Gamma_{G'}(v)| < \lceil (m/n)^{1/2} \rceil - 1$  and  $|\Gamma_{G'}(u)| \geq \lceil (m/n)^{1/2} \rceil - 1$ , then  $\text{dist}_{G'}(u, v) = \infty$ .*
4.  *$E \subseteq E', |E'| \leq |E| + m$ .*

*Proof.* For property 1, if  $r > \lceil \log(\text{diam}(G)) \rceil + 1$ , then let  $i = \lceil \log(\text{diam}(G)) \rceil + 1$ . Let  $v \in V$ . By property 3 of Lemma B.1, if  $|S_v^{(i)}| < \lceil (m/n)^{1/2} \rceil$ , then  $S_v^{(i)} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^i\} = \{u \in V \mid \text{dist}_G(u, v) \leq 2 \cdot 2^{\lceil \log(\text{diam}(G)) \rceil}\} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ . Furthermore, if  $|S_v^{(i)}| < \lceil (m/n)^{1/2} \rceil$ , then  $|S_v^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ , which means that  $S_v^{(i-1)} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^{i-1}\} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^{\lceil \log(\text{diam}(G)) \rceil}\} = \{u \in V \mid \text{dist}_G(u, v) < \infty\} = S_v^{(i)}$ . Then due to the condition in line 24, it will end the procedure in this round, which contradicts to  $r > \lceil \log(\text{diam}(G)) \rceil + 1 = i$ . Similarly, if  $r > \lceil \log(m/n) \rceil + 1$ , then let  $i = \lceil \log(m/n) \rceil + 1$ . Furthermore, if  $|S_v^{(i)}| < \lceil (m/n)^{1/2} \rceil$ , then  $|S_v^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ , which means that  $S_v^{(i-1)} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^{i-1}\} \neq \{u \in V \mid \text{dist}_G(u, v) \leq 2^i\} = S_v^{(i)}$ . Thus, there exists  $u \in S_v^{(i)}$  such that  $|S_v^{(i)}| > \text{dist}_G(u, v) > 2^{i-1} > m/n \geq \lceil (m/n)^{1/2} \rceil$  which leads to a contradiction.

For property 2, if  $u, v$  are in the same connected component in  $G$ , then since  $E \subseteq E'$ ,  $u, v$  are in the same connected component in  $G'$ . If  $u, v$  are in the same connected component in  $G'$ , then there should be a path  $u = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_p = v$  in  $G'$ , i.e.  $\forall j \in [p-1], (u_j, u_{j+1}) \in E'$ .  $(u_j, u_{j+1}) \in E'$  implies that either  $(u_j, u_{j+1}) \in E$  or  $u_j \in S_{u_{j+1}}^{(r)}$  or  $u_{j+1} \in S_{u_j}^{(r)}$ . By property 2 of Lemma B.1, we know that  $\forall j \in [p-1]$ ,  $u_j$  and  $u_{j+1}$  are in the same connected component in  $G$ . Thus,  $u$  and  $v$  are in the same connected component in  $G$ .

For property 3, due to line 25, if  $|\Gamma_{G'}(v)| < \lceil (m/n)^{1/2} \rceil - 1$ , then we have  $|S_v^{(r)}| < \lceil (m/n)^{1/2} \rceil$ . By property 3 of Lemma B.1, and the condition in line 24, we know  $\{u \in V \mid \text{dist}_G(u, v) \leq 2^r\} = S_v^{(r)} = S_v^{(r-1)} = \{u \in V \mid \text{dist}_G(u, v) \leq 2^{r-1}\}$ . Thus,  $S_v^{(r)} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ . Due to property 2, we have  $\Gamma_{G'}(v) \cup \{v\} \subseteq \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ . Notice that  $S_v^{(r)} \subseteq \Gamma_{G'}(v) \cup \{v\}$ , thus, we have  $\Gamma_{G'}(v) \cup \{v\} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ . Let  $v' \in \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ , then due to property 2,  $\Gamma_{G'}(v') \cup \{v'\} \subseteq \{u \in V \mid \text{dist}_G(u, v') < \infty\} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ , then we have  $|\Gamma_{G'}(v') \cup \{v'\}| < \lceil (m/n)^{1/2} \rceil$ . Thus,  $|S_{v'}^{(r)}| < \lceil (m/n)^{1/2} \rceil$ . By property 3 of Lemma B.1, and the condition in line 24, we know  $\{u \in V \mid \text{dist}_G(u, v') \leq 2^r\} = S_{v'}^{(r)} = S_{v'}^{(r-1)} = \{u \in V \mid \text{dist}_G(u, v') \leq 2^{r-1}\}$ . Thus,  $S_{v'}^{(r)} = \{u \in V \mid \text{dist}_G(u, v') < \infty\}$ . Thus,  $\Gamma_{G'}(v') \cup \{v'\} = \{u \in V \mid \text{dist}_G(u, v') < \infty\} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ . Thus,  $\forall p, q \in \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ , we have  $(p, q) \in E'$ , which means that  $\{u \in V \mid \text{dist}_G(u, v) < \infty\}$  is a clique in  $G'$ .

Now consider two vertices  $u, v \in V$ . Suppose  $|\Gamma_{G'}(v)| < \lceil (m/n)^{1/2} \rceil - 1$ , then we have that  $\{p \in V \mid \text{dist}_G(p, v) < \infty\}$  is a clique in  $G'$ . Thus,  $\forall q \in \{p \in V \mid \text{dist}_G(p, v) < \infty\}$ , we have  $|\Gamma_{G'}(q)| = |\Gamma_{G'}(v)| < \lceil (m/n)^{1/2} \rceil - 1$ . If  $|\Gamma_{G'}(u)| \geq \lceil (m/n)^{1/2} \rceil - 1$ , then  $\text{dist}_{G'}(u, v) = \infty$ .

For property 4, by line 25, we have  $E \subseteq E'$  and  $|E'| \leq |E| + \sum_{v \in V} |S_v^{(r)}| \leq |E| + n \cdot m/n = |E| + m$  where the last inequality follows by the property 4 of Lemma B.1.  $\square$

## B.2 Random Leader Selection

Given an undirected graph  $G = (V, E)$ , to design a connected component algorithm, a natural way is constantly contracting the vertices in the same component. One way to do the contraction is that we randomly choose some vertices as leaders, then contract non-leader vertices to the neighbor leader vertices.

In this section, we show that if  $\forall v \in V$ , the number of neighbors of  $v$  is large enough, then we can just sample a small number of leaders such that for each non-leader vertex  $v \in V$ , there is at least one neighbor of  $v$  which is chosen as a leader. A more generalized statement is stated in the following lemma.

**Lemma B.4.** *Let  $V$  be a vertex set with  $n$  vertices. Let  $0 < \gamma \leq n, \delta \in (0, 1)$ . For each  $v \in V$ , let  $S_v$  be a subset of  $V \setminus \{v\}$  with size at least  $\gamma - 1$ . Let  $l : V \rightarrow \{0, 1\}$  be a random hash function such that  $\forall v \in V, l(v)$  are i.i.d. Bernoulli random variables, i.e.*

$$l(v) = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{otherwise.} \end{cases}$$

If  $p \geq \min((10 \log(2n/\delta))/\gamma, 1)$ , then, with probability at least  $1 - \delta$ ,

1.  $\sum_{v \in V} l(v) \leq \frac{3}{2}pn$ ;
2.  $\forall v \in V, \exists u \in S_v \cup \{v\}$  such that  $l(u) = 1$ .

*Proof.* For a fixed vertex  $v \in V$ , we have

$$\begin{aligned} & \Pr \left( \sum_{u \in S_v \cup \{v\}} (\mathbf{E}(l(u)) - l(u)) > \frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) \right) \\ & \leq \exp \left( - \frac{\frac{1}{2} \left( \frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) \right)^2}{\sum_{u \in S_v \cup \{v\}} \mathbf{Var}(l(u)) + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u))} \right) \\ & \leq \exp \left( - \frac{\frac{1}{2} \left( \frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) \right)^2}{\sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u))} \right) \\ & = \exp \left( - \frac{3}{28} \cdot \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) \right) = \exp \left( - \frac{3}{28} \cdot p \cdot |S_v \cup \{v\}| \right) \leq \frac{\delta}{2n}, \end{aligned}$$

where the first inequality follows by Bernstein inequality and  $|l(u) - \mathbf{E}(l(u))| \leq 1$ , the second inequality follows by  $\mathbf{Var}(l(u)) \leq \mathbf{E}(l^2(u)) = \mathbf{E}(l(u))$ . The last inequality follows by  $|S_v \cup \{v\}| \geq \gamma$ , and  $p \geq \min((10 \log(2n/\delta))/\gamma, 1)$ . Since  $\frac{1}{2} \sum_{u \in S_v \cup \{v\}} \mathbf{E}(l(u)) \geq 1$ , with probability at least  $1 - \delta/(2n)$ ,  $\sum_{u \in S_v \cup \{v\}} l(u) \geq 1$ . By taking union bound over all  $S_v$ , with probability at least  $1 - \delta/2$ ,  $\forall v \in V, \exists u \in S_v \cup \{v\}, l(u) = 1$ .

Similarly, we have

$$\begin{aligned}
& \Pr \left( \sum_{u \in V} (l(u) - \mathbf{E}(l(u))) > \frac{1}{2} \sum_{u \in V} \mathbf{E}(l(u)) \right) \\
& \leq \exp \left( - \frac{\frac{1}{2} \left( \frac{1}{2} \sum_{u \in V} \mathbf{E}(l(u)) \right)^2}{\sum_{u \in V} \mathbf{Var}(l(u)) + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \sum_{u \in V} \mathbf{E}(l(u))} \right) \\
& \leq \exp \left( - \frac{\frac{1}{2} \left( \frac{1}{2} \sum_{u \in V} \mathbf{E}(l(u)) \right)^2}{\sum_{u \in V} \mathbf{E}(l(u)) + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \sum_{u \in V} \mathbf{E}(l(u))} \right) \\
& = \exp \left( - \frac{3}{28} \cdot \sum_{u \in V} \mathbf{E}(l(u)) \right) \\
& = \exp \left( - \frac{3}{28} \cdot p \cdot |V| \right) \leq \frac{\delta}{2n} \leq \frac{\delta}{2}.
\end{aligned}$$

Since  $\sum_{u \in V} \mathbf{E}(l(u)) = p \cdot n$ , with probability at least  $1 - \delta/2$ ,  $\sum_{u \in V} l(u) \leq 1.5pn$ .

By taking union bound, with probability at least  $1 - \delta$ ,  $\sum_{u \in V} l(u) \leq 1.5pn$  and  $\forall v \in V, \exists u \in S_v \cup \{v\}, l(u) = 1$ .  $\square$

If the number of neighbors of each vertex is not large, then we can still have a constant fraction of vertices which can contract to a leader.

**Lemma B.5.** *Let  $V$  be a vertex set with  $n$  vertices. Let  $S_v$  be a subset of  $V \setminus \{v\}$  with size at least 1. Let  $l : V \rightarrow \{0, 1\}$  be a random hash function such that  $\forall v \in V, l(v)$  are i.i.d. Bernoulli random variables, i.e.*

$$l(v) = \begin{cases} 1 & \text{with probability } \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Let  $L = \{v \in V \mid l(v) = 1\} \cup \{v \in V \mid \forall u \in S_v \cup \{v\}, l(u) = 0\}$ .  $\mathbf{E}(L) \leq 0.75n$ .

*Proof.* For  $v \in V$ ,  $\Pr(l(v) = 1) = \frac{1}{2}$ . Let  $u \in S_v$ . Then  $\Pr(\forall x \in S_v \cup \{v\}, l(x) = 0) \leq \Pr(l(v) = 0, l(u) = 0) = 0.25$ .  $\mathbf{E}(|L|) = \sum_{v \in V} \Pr(v \in L) \leq 0.75n$ .  $\square$

### B.3 Tree Contraction Operation

In this section, we introduce the contraction operation. Firstly, let us introduce the concept of the parent pointers which can define a rooted forest.

**Definition B.6.** *Given a set of vertices  $V$ , let  $\text{par} : V \rightarrow V$  satisfy that  $\forall v \in V, \exists i > 0$  such that  $\text{par}^{(i)}(v) = \text{par}^{(i+1)}(v)$ , where  $\forall v \in V, j > 0, \text{par}^{(j)}(v)$  is defined as  $\text{par}(\text{par}^{(j-1)}(v))$ , and  $\text{par}^{(0)}(v) = v$ . Then, we call such  $\text{par}$  a set of parent pointers on  $V$ . For  $v \in V$ , if  $\text{par}(v) = v$ , then we say  $v$  is a root of  $\text{par}$ .  $\text{par}$  can have more than one root. The depth of  $v \in V$ ,  $\text{dep}_{\text{par}}(v)$  is the smallest  $i \in \mathbb{Z}_{\geq 0}$  such that  $\text{par}^{(i)}(v) = \text{par}^{(i+1)}(v)$ . The root of  $v \in V$ ,  $\text{par}^{(\infty)}(v)$  is defined as  $\text{par}^{(\text{dep}_{\text{par}}(v))}(v)$ . The depth of  $\text{par}$ ,  $\text{dep}(\text{par})$  is defined as  $\max_{v \in V} \text{dep}_{\text{par}}(v)$ .*

It is easy to see that a set of parent pointers  $\text{par}$  on  $V$  formed a rooted forest on  $V$ . For a vertex  $v \in V$ , if  $\text{par}(v) = v$ , then  $v$  is a root in the forest. Otherwise  $\text{par}(v)$  is the parent of  $v$  in the forest.

In the following, we define the union operation of several sets of parent pointers.

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**Algorithm 2** Tree Contraction Operation

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```
1: procedure TREECONTRACTION( $G = (V, E)$ ,  $\text{par} : V \rightarrow V$ )  $\triangleright$  Lemma B.10, Corollary B.13
2:  $\triangleright$  Output:  $G' = (V', E')$ ,  $\text{par}^{(\infty)}(v)$  for all  $v \in V$ 
3:   Initially, for each  $v \in V$  let  $g^{(0)}(v) \leftarrow \text{par}(v)$ . Let  $V' = \emptyset, E' = \emptyset$ .
4:    $l \leftarrow 0$ .
5:   for  $\exists v \in V, \text{par}(g^{(l)}(v)) \neq g^{(l)}(v)$  do
6:      $l \leftarrow l + 1$ .
7:     For each  $v \in V$ , compute  $g^{(l)}(v) = g^{(l-1)}(g^{(l-1)}(v))$ .  $\triangleright g^{(l)}$  is  $\text{par}^{(2^l)}$ 
8:   end for
9:    $r \leftarrow l$ .  $\triangleright r$  is the number of iterations, and is used in the analysis.
10:  For  $v \in V$ , if  $\text{par}(v) = v$ , let  $V' \leftarrow V' \cup \{v\}$ .
11:  For  $(u, v) \in E$ , if  $g^{(r)}(u) \neq g^{(r)}(v)$ , let  $E' \leftarrow E' \cup \{(g^{(r)}(u), g^{(r)}(v))\}$ .
    $\triangleright \forall v \in V$ , contract  $v$  to  $\text{par}^{(\infty)}(v)$ 
12:  return  $g^{(r)}(v)$  as  $\text{par}^{(\infty)}(v)$  for all  $v \in V$ , and  $G' = (V', E')$ 
13: end procedure
```

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**Definition B.7.** Let  $\text{par}_1 : V_1 \rightarrow V_1, \text{par}_2 : V_2 \rightarrow V_2, \dots, \text{par}_k : V_k \rightarrow V_k$  be  $k$  sets of parent pointers on vertex sets  $V_1, V_2, \dots, V_k$  respectively, where  $\forall i \neq j \in [k], V_i \cap V_j = \emptyset$ . Then  $\text{par} = \text{par}_1 \cup \text{par}_2 \cup \dots \cup \text{par}_k$  is a set of parent pointers on the vertex set  $V_1 \cup V_2 \cup \dots \cup V_k$  such that  $\forall i \in [k], v \in V_k, \text{par}(v) = \text{par}_i(v)$ .

Now we focus on the parent pointers which can preserve the connectivity of the graph.

**Definition B.8.** Given a graph  $G = (V, E)$  and a set of parent pointers  $\text{par}$  on  $V$ , if  $\forall v \in V$ , we have  $\text{dist}_G(v, \text{par}(v)) < \infty$ , then  $\text{par}$  is compatible with  $G$ .

It is easy to show the following fact:

**Fact B.9.** Given a graph  $G = (V, E)$  and a set of parent pointers  $\text{par}$  which is compatible with  $G$ , then  $\forall u, v \in V$  with  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , we have  $\text{dist}_G(u, v) < \infty$ .

*Proof.* By the definition of compatible,  $\forall v \in V, \text{dist}_G(v, \text{par}(v)) < \infty$ . By induction,  $\forall l \in \mathbb{Z}_{>0}, v \in V$ , we have  $\text{dist}_G(v, \text{par}^{(l)}(v)) \leq \text{dist}_G(v, \text{par}^{(l-1)}(v)) + \text{dist}_G(\text{par}^{(l-1)}(v), \text{par}^{(l)}(v)) < \infty$ . Thus, for any pair of vertices  $u, v \in V$ , if  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , then  $\text{dist}_G(u, v) \leq \text{dist}_G(u, \text{par}^{(\infty)}(u)) + \text{dist}_G(\text{par}^{(\infty)}(u), v) < \infty$ .  $\square$

In this section, we describe a procedure which can be used to reduce the number of vertices. The input of the procedure is an undirected graph  $G = (V, E)$  and a set of parent pointers  $\text{par} : V \rightarrow V$ , where  $\text{par}$  is compatible with  $G$ . The output of the procedure will be the root of each vertex in  $V$  and an undirected graph  $G' = (V', E')$  which satisfies  $V' = \{v \in V \mid \text{par}(v) = v\}, E' = \{(u, v) \in V' \times V' \mid u \neq v, \exists (p, q) \in E, \text{par}^{(\infty)}(p) = u, \text{par}^{(\infty)}(q) = v\}$ . Notice that  $V'$  only contains all the roots in the forest induced by  $\text{par}$ , and  $|E'| \leq |E|$ .

**Lemma B.10.** Let  $G = (V, E)$  be an undirected graph,  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6). Then TREECONTRACTION( $G, \text{par}$ ) (See Algorithm 2) will output output  $(G', g^{(r)})$  with  $r \leq \lceil \log \text{dep}(\text{par}) \rceil$  satisfies the following properties:

1.  $\forall v \in V, g^{(r)}(v) = \text{par}^{(\infty)}(v)$ .
2.  $V' = \{v \in V \mid \text{par}(v) = v\}$ .

$$3. E' = \{(u, v) \in V' \times V' \mid u \neq v, \exists(p, q) \in E, \text{par}^{(\infty)}(p) = u, \text{par}^{(\infty)}(q) = v\}.$$

*Proof.* One crucial observation is the following claim.

**Claim B.11.**  $\forall l \in \{0, 1, \dots, r\}, v \in V$ , we have  $g^{(l)}(v) = \text{par}^{(2^l)}(v)$ .

*Proof.* The proof is by induction. When  $l = 0$ ,  $\forall v \in V, g^{(0)}(v) = \text{par}(v) = \text{par}^{(1)}(v)$ , the claim is true. Suppose for  $l - 1$ , we have  $\forall v \in V, g^{(l-1)}(v) = \text{par}^{(2^{l-1})}(v)$ , then  $\forall v \in V, g^{(l)}(v) = g^{(l-1)}(g^{(l-1)}(v)) = \text{par}^{(2^{l-1})}(\text{par}^{(2^{l-1})}(v)) = \text{par}^{(2^l)}(v)$ . So the claim is true.  $\square$

If  $r > \lceil \log \text{dep}(\text{par}) \rceil$ , then  $r - 1 \geq \lceil \log \text{dep}(\text{par}) \rceil$ . Due to claim B.11, we have  $\forall v \in V, g^{(r-1)}(v) = \text{par}^{(2^{r-1})}(v) = \text{par}^{(\infty)}(v)$ . Due to the condition in line 5, the loop will stop when  $l \leq r - 1$  which leads to a contradiction to line 9. Thus, at the end of the algorithm,  $r$  should be at most  $\lceil \log \text{dep}(\text{par}) \rceil$ .

Since we have  $\forall v \in V, \text{par}(g^{(r)}(v)) = g^{(r)}(v)$  at the end of the Algorithm 2,  $\forall v \in V, g^{(r)}(v)$  must be  $\text{par}^{(\infty)}(v)$ . Then due to line 10 and line 11, we have  $V' = \{v \in V \mid \text{par}(v) = v\}, E' = \{(u, v) \in V' \times V' \mid u \neq v, \exists(p, q) \in E, \text{par}^{(\infty)}(p) = u, \text{par}^{(\infty)}(q) = v\}$ .  $\square$

**Definition B.12.** Let  $G = (V, E)$  be an undirected graph,  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6). Then the number of iteration of TREECONTRACTION( $G, \text{par}$ ) is defined as the value of  $r$  at the end of the procedure.

**Corollary B.13** (Preserved connectivity and diameter). Let  $G = (V, E)$  be an undirected graph,  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) which is compatible (See Definition B.8) with  $G$ . Then at the end of the Algorithm 2,  $r \leq \lceil \log \text{dep}(\text{par}) \rceil$  and the output  $(G', g^{(r)})$  will satisfy the following properties:

1.  $\text{diam}(G') \leq \text{diam}(G)$ .
2.  $\forall u, v \in V, \text{dist}_G(u, v) < \infty \Rightarrow \text{dist}_{G'}(\text{par}^{(\infty)}(u), \text{par}^{(\infty)}(v)) < \infty$ .
3.  $\forall u, v \in V, \text{dist}_G(u, v) < \infty \Leftarrow \text{dist}_{G'}(\text{par}^{(\infty)}(u), \text{par}^{(\infty)}(v)) < \infty$ .

*Proof.* By Lemma B.10, we have  $r \leq \lceil \log \text{dep}(\text{par}) \rceil$ ,  $V' = \{v \in V \mid \text{par}(v) = v\}$  and  $E' = \{(u, v) \in V' \times V' \mid u \neq v, \exists(p, q) \in E, \text{par}^{(\infty)}(p) = u, \text{par}^{(\infty)}(q) = v\}$ .

For any two vertices  $u, v \in V$  which are in the same connected component in  $G$ , then there should be a path  $u = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_p = v$  in graph  $G$ . So  $\forall i \in [p - 1], (u_i, u_{i+1}) \in E$  which means that either  $\text{par}^{(\infty)}(u_i) = \text{par}^{(\infty)}(u_{i+1})$  or  $(\text{par}^{(\infty)}(u_i), \text{par}^{(\infty)}(u_{i+1})) \in E'$ . Thus,  $\text{par}^{(\infty)}(u_1) \rightarrow \text{par}^{(\infty)}(u_2) \rightarrow \dots \rightarrow \text{par}^{(\infty)}(u_p)$  is a valid path in  $G'$ , and the length of this path in  $G'$  is at most  $p$ . Thus, the properties 1 and 2 are true.

For any two vertices  $u, v \in V$  which are not in the same connected component in  $G$ , but there is a path  $\text{par}^{(\infty)}(u) = u'_1 \rightarrow u'_2 \rightarrow \dots \rightarrow u'_p = \text{par}^{(\infty)}(v)$  in  $G'$ , then it means that there exists vertices  $u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}, \dots, u_{p,1}, u_{p,2} \in V$  which satisfies

- (a)  $\forall i \in [p - 1], (u_{i,2}, u_{i+1,1}) \in E, \text{par}^{(\infty)}(u_{i,2}) = u'_i, \text{par}^{(\infty)}(u_{i+1,1}) = u'_{i+1}$ .
- (b)  $u_{1,1} = u, u_{p,2} = v$ .
- (c)  $\forall i \in [p], \text{par}^{(\infty)}(u_{i,1}) = \text{par}^{(\infty)}(u_{i,2})$ . By Fact B.9, we have  $\text{dist}_G(u_{i,1}, u_{i,2}) < \infty$ .

Thus, there exists a path from  $u$  to  $v$ . This contradicts to that  $u, v$  are not in the same connected component. Therefore, property 3 is also true.  $\square$

## B.4 Connectivity Algorithm

In this section, we described a batch algorithm for graph connectivity/connected components problem. The input is an undirected graph  $G = (V, E)$ , a space/rounds trade-off parameter  $m$ , and the rounds parameter  $r \leq |V|$ . The output is a function  $\text{col} : V \rightarrow V$  such that  $\forall u, v \in V, \text{dist}_G(u, v) < \infty \Leftrightarrow \text{col}(u) = \text{col}(v)$ .

The algorithm is described in Algorithm 3. The following theorem shows the correctness of Algorithm 3.

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### Algorithm 3 Graph Connectivity

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1: procedure CONNECTIVITY( $G = (V, E), m, r$ ) ▷ Theorem B.14, Theorem B.21
2:   Output: FAIL or  $\text{col} : V \rightarrow V$ .
3:    $n \leftarrow |V|$ 
4:    $\forall v \in V, h_0(v) \leftarrow \text{null}$ .
5:    $G_0 = (V_0, E_0) = G$ , i.e.  $V_0 = V, E_0 = E$ .
6:    $n_0 = n$ .
7:   for  $i = 1 \rightarrow r$  do
8:      $\forall v \in V, h_i(v) \leftarrow \text{null}$ . ▷  $h_i(v)$  is the vertex that  $v$  contracts to
9:      $G'_i = (V'_i, E'_i) = \text{NEIGHBORINCREMENT}(m, G_{i-1})$ . ▷ Algorithm 1
10:    Compute  $V''_i = \{v \in V'_i \mid |\Gamma_{G'_i}(v)| \geq \lceil (m/n_{i-1})^{1/2} \rceil - 1\}$ .
11:    Compute  $E''_i = \{(u, v) \in E_{i-1} \mid u \in V''_i, v \in V''_i\}$ .
12:     $G''_i = (V''_i, E''_i)$ . ▷  $G''_i$  is obtained by removing all the small components of  $G_i$ 
13:    Let  $\gamma_i = \lceil (m/n_{i-1})^{1/2} \rceil, p_i = \min((30 \log(n) + 100)/\gamma_i, 1/2)$ .
14:    Let  $l_i : V''_i \rightarrow \{0, 1\}$  be a random hash function such that  $\forall v \in V''_i, l_i(v)$  are i.i.d. Bernoulli
    random variables, and  $\Pr(l_i(v) = 1) = p_i$ .
15:    Let  $L_i = \{v \in V''_i \mid l_i(v) = 1\} \cup \{v \in V''_i \mid \forall u \in \Gamma_{G'_i}(v) \cup \{v\}, l_i(u) = 0\}$ . ▷  $L_i$  are leaders
16:     $\forall v \in V''_i$  with  $v \in L_i$ , let  $\text{par}_i(v) = v$ .
17:     $\forall v \in V''_i$  with  $v \notin L_i$ , let  $\text{par}_i(v) = \min_{u \in L_i \cap (\Gamma_{G'_i}(v) \cup \{v\})} u$ . ▷ Non-leader finds a leader.
18:    Let  $((V_i, E_i), g_i^{(r'_i)}) = \text{TREECONTRACTION}(G''_i, \text{par}_i)$ . ▷ Algorithm 2
19:     $G_i = (V_i, E_i)$ .
20:     $n_i = |V_i|$ .
21:    For each  $v \in V'_i \setminus V''_i$ , let  $h_i(v) \leftarrow \min_{u \in \Gamma_{G'_i}(v) \cup \{v\}} u$ . ▷ Contract small component to one vertex
22:    For each  $v \in V''_i \setminus V_i$ , let  $h_i(v) \leftarrow g_i^{(r'_i)}(v)$ . ▷ Contract non-leader to leader
23:    For each  $v \in V$ , if  $h_{i-1}(v) \neq \text{null}$ , then let  $h_i(v) = h_{i-1}(v)$ .
24:  end for
25:  If  $n_r \neq 0$ , return FAIL.
26:   $((\tilde{V}, \tilde{E}), \text{col}) = \text{TREECONTRACTION}(G, h_r)$ . ▷ Algorithm 2
27:  return col.
28: end procedure

```

---

**Theorem B.14** (Correctness of Algorithm 3). *Let  $G = (V, E)$  be an undirected graph,  $m \geq 4|V|$ , and  $r \leq |V|$  be the rounds parameter. If  $\text{CONNECTIVITY}(G, m, r)$  (Algorithm 3) does not output FAIL, then  $\forall u, v \in V$ , we have  $\text{dist}_G(u, v) < \infty \Leftrightarrow \text{col}(u) = \text{col}(v)$ .*

*Proof.* Firstly, we show that the input of line 18 is valid.

**Claim B.15.**  $\forall i \in [r]$ ,  $\text{par}_i$  is a set of parent pointers on  $V''_i$ , (See Definition B.6) and is compatible (See Definition B.8) with  $G''_i$ .

*Proof.*  $\forall v \in V''_i$ , if  $v \in L_i$ , then  $\text{par}_i(v) = v$ . For  $v \in V''_i \setminus L_i$ , due to property 3 of Lemma B.3, we have  $\text{par}_i(v) \in V''_i$ . Since  $\text{par}_i(v) \in L_i$ , we have  $\text{par}_i(\text{par}_i(v)) = \text{par}_i(v)$ . Thus,  $\text{par}_i : V''_i \rightarrow V''_i$  is a



set of parent pointers on  $V_i''$ . Due to property 2 of Lemma B.3 and  $\text{dist}_{G_i'}(\text{par}_i(v), v) < \infty$ , we know that  $\text{dist}_{G_{i-1}}(\text{par}_i(v), v) < \infty$ . Thus,  $\text{dist}_{G_i''}(\text{par}_i(v), v) < \infty$ . It implies that  $\text{par}_i$  is compatible with  $G_i''$ .  $\square$

The following claim shows that the number of the remaining vertices cannot increase after each round.

**Claim B.16.** *If  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL, then  $\forall i \in [r], V_i \subseteq V_i'' \subseteq V_i' = V_{i-1}$ .*

*Proof.* Let  $i \in [r]$ . Due to Claim B.15, the input of line 18 is valid. Then, we can apply property 2 of Lemma B.10 to get  $V_i \subseteq V_i''$ . By the construction of  $V_i''$  we have  $V_i'' \subseteq V_i'$ . Since the procedure  $\text{NEIGHBORINCREMENT}(m, G_{i-1})$  (Algorithm 1) does not change the vertex set, we have  $V_i' = V_{i-1}$ .  $\square$

Now, we show that  $\forall u, v \in V_i, \text{dist}_{G_i}(u, v) < \infty \Leftrightarrow \text{dist}_G(u, v) < \infty$ .

**Claim B.17.** *If  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL, then  $\forall i \in [r], \forall u, v \in V_i$ , we have  $\text{dist}_{G_i}(u, v) < \infty \Leftrightarrow \text{dist}_G(u, v) < \infty$ .*

*Proof.* The proof is by induction. Suppose  $\forall u, v \in V_{i-1}, \text{dist}_{G_{i-1}}(u, v) < \infty \Leftrightarrow \text{dist}_G(u, v) < \infty$ .  $\forall w, z \in V_i$ , according to Claim B.16,  $w, z \in V_i''$ . By property 2,3 of Lemma B.13, and property 2 of Lemma B.10,  $\text{dist}_{G_i}(w, z) < \infty \Leftrightarrow \text{dist}_{G_i''}(w, z) < \infty$ . Due to property 2,3 of Lemma B.3, there is no edge in  $E_{i-1}$  between  $V_i''$  and  $V_i' \setminus V_i''$ . According to Claim B.16,  $w, z \in V_{i-1}$ . Thus,  $\text{dist}_{G_i''}(w, z) < \infty \Leftrightarrow \text{dist}_{G_{i-1}}(w, z) < \infty$ . By induction hypothesis, we have  $\forall w, z \in V_i, \text{dist}_{G_i}(w, z) < \infty \Leftrightarrow \text{dist}_G(w, z) < \infty$ .  $\square$

The following claim states that once a vertex  $v \in V$  is contracted to an another vertex, it will never be operated.

**Claim B.18.** *Suppose  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL.  $\forall i \in \{0, 1, \dots, r\}, v \in V$ , we have  $h_i(v) = \text{null} \Leftrightarrow v \in V_i$ . Furthermore,  $\forall v \in V, \exists j \in [r]$  such that  $h_0(v) = h_1(v) = \dots = h_{j-1}(v) = \text{null}$  and  $h_j(v) = h_{j+1}(v) = \dots = h_r(v) \neq \text{null}, \text{dist}_G(v, h_r(v)) < \infty$ .*

*Proof.* When  $i = 0$ ,  $\forall v \in V, h_0(v) = \text{null}, v \in V_0 = V$ . Suppose it is true that  $\forall v \in V, h_{i-1}(v) = \text{null} \Leftrightarrow v \in V_{i-1}$ . If  $v \notin V_i$ , according to Claim B.16, there are three cases:  $v \in V_i'' \setminus V_i, v \in V_i' \setminus V_i'', v \notin V_{i-1}$ . In the first case, due to line 22,  $h_i(v) \neq \text{null}$ . In the second case, due to line 21,  $h_i(v) \neq \text{null}$ . In the third case, due to line 23,  $h_i(v) \neq \text{null}$ . If  $h_i(v) = \text{null}$ , then  $h_i(v)$  cannot be updated by line 21, line 22 or line 23 which implies that  $v \in V_{i-1}, v \notin V_i' \setminus V_i'', v \notin V_i'' \setminus V_i$ . Thus,  $v \in V_i$ .

Since the procedure does not FAIL, we have  $n_r = 0$  which means that  $\forall v \in V, h_r(v) \neq \text{null}$ . Notice that by line 23, if  $h_{i-1}(v) \neq \text{null}$ , then  $h_i(v) = h_{i-1}(v)$ . Thus,  $\forall v \in V, \exists j \in [r]$  such that  $h_0(v) = h_1(v) = \dots = h_{j-1}(v) = \text{null}$  and  $h_j(v) = h_{j+1}(v) = \dots = h_r(v) \neq \text{null}$ .

For  $v \in V$ , if  $h_j(v) \neq \text{null}$  and  $h_{j-1}(v) = \text{null}$ , then  $h_j(v)$  can only be updated by 21 or line 22. In both cases,  $\text{dist}_{G_{j-1}}(v, h_j(v)) < \infty$ . By Claim B.17, we have that  $\text{dist}_G(v, h_j(v)) < \infty$ .  $\square$

In the following, we show that  $h_r$  is a rooted tree such that  $\text{dist}_G(u, v) < \infty \Leftrightarrow u, v$  have the same root. Due to Claim B.18, if  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL, then  $n_r = 0$  which implies that  $\forall v \in V, h_r(v) \neq \text{null}$ . Thus, we can define  $h_r^{(k)}(v)$  for  $k \in \mathbb{Z}_{>0}$  as applying  $h_r$  on  $v$   $k$  times.  $\forall v \in V$ , by Claim B.18, let  $j \in [r]$  satisfy that  $h_j(v) \neq \text{null}$  and  $h_{j-1}(v) = \text{null}$ . If  $h_j(v)$  is updated by line 22, then  $h_j(h_j(v)) = \text{null}$ . If  $h_j(v)$  is updated by line 21, then  $h_j(h_j(v)) = h_j(v)$ .

In both cases,  $h_j$  cannot create a cycle. Thus, we can define  $h_r^{(\infty)}(v) = h_r^{(k)}(v)$  for some  $k$  which satisfies  $h_r(h_r^{(k)}(v)) = h_r^{(k)}(v)$ .

**Claim B.19.** *Suppose  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL. Then  $\forall u, v \in V$ , we have  $\text{dist}_G(u, v) < \infty \Leftrightarrow h_r^{(\infty)}(u) = h_r^{(\infty)}(v)$ .*

*Proof.* Let  $u, v \in V$ . By Claim B.18, if  $h_r^{(\infty)}(u) = h_r^{(\infty)}(v)$  we have  $\text{dist}_G(u, v) < \infty$ .

If  $\text{dist}_G(u, v) < \infty$ , then let  $u' = h_r^{(\infty)}(u), v' = h_r^{(\infty)}(v)$ . By Claim B.18,  $\text{dist}_G(u', v') \leq \text{dist}_G(u, u') + \text{dist}_G(u, v) + \text{dist}_G(v, v') < \infty$ , and we can find  $j \in [r]$  such that  $h_j(u') \neq \text{null}, h_{j-1}(u') = \text{null}$ . Without loss of generality, we can assume  $h_{j-1}(v') = \text{null}$  (otherwise we can swap  $u'$  and  $v'$ ). Due to Claim B.18,  $u', v' \in V_{i-1}$ . Since  $h_j(u') = h_r(u') = u'$ ,  $h_j(u')$  can be only updated by line 21, and  $u' \in V_j' \setminus V_j''$ . Then due to property 3 of Lemma B.10,  $v'$  should be in  $\Gamma_{G_i'}(u) \cup \{u\}$ . Since  $h_j(v') = h_r(v') = v'$ , we can conclude that  $u' = v'$ .  $\square$

If  $\text{CONNECTIVITY}(G, m, r)$  does not output FAIL, then in line 26,  $\text{col}$  is exactly  $h_r^{(\infty)}$ . By Claim B.19, we have  $\forall u, v \in V, \text{dist}_G(u, v) < \infty \Leftrightarrow \text{col}(u) = \text{col}(v)$ .  $\square$

Now let us consider the number of iterations of Algorithm 3 and the success probability.

**Definition B.20** (Total iterations). *Let  $G = (V, E)$  be an undirected graph,  $\text{poly}(n) \geq m > 4n$ , and  $r \leq n$  be the rounds parameter where  $n$  is the number of vertices in  $G$ . The total number of iterations of  $\text{CONNECTIVITY}(G, m, r)$  (Algorithm 3) is defined as  $\sum_{i=1}^r (k_i + r_i')$ , where  $k_i$  denotes the number of iterations (See Definition B.2) of  $\text{NEIGHBORINCREMENT}(m, G_{i-1})$  (see line 9), and  $r_i'$  denotes the number of iterations (See Definition B.12) of  $\text{TREECONTRACTION}(G_i'', \text{par}_i)$  (see line 18).*

**Theorem B.21** (Success probability and total iterations). *Let  $G = (V, E)$  be an undirected graph,  $\text{poly}(n) \geq m > 4n$ , and  $r \leq n$  be the rounds parameter where  $n = |V|$ . Let  $c > 0$  be a sufficiently large constant. If  $r \geq c \log \log_{m/n}(n)$ , then with probability at least 0.98,  $\text{CONNECTIVITY}(G, m, r)$  (Algorithm 3) will not return FAIL. If  $\text{CONNECTIVITY}(G, m, r)$  succeeds, let  $k_i$  denote the number of iterations (See Definition B.2) of  $\text{NEIGHBORINCREMENT}(m, G_{i-1})$  (see line 9), and let  $r_i'$  denote the number of iterations of (See Definition B.12) of  $\text{TREECONTRACTION}(G_i'', \text{par}_i)$  (see line 18), then*

1.  $\forall i \in [r], r_i' = 0$ .
2.  $\forall i \in [r], k_i$  is at most  $\lceil \log(\text{diam}(G)) \rceil + 1$ .
3. The number of iterations of line 26 is at most  $\lceil \log r \rceil$ .
4.  $\sum_{i=1}^r k_i \leq O(r \log(\text{diam}(G)))$ .

Let  $c_1 > 0$  be a sufficiently large constant. If  $m \geq c_1 n \log^4 n$ , then with probability at least 0.99,  $\sum_{i=1}^r k_i \leq O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n))$ . If  $m < c_1 n \log^4 n$ , then with probability at least 0.98,  $\sum_{i=1}^r k_i \leq O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n) + (\log \log(n))^2)$ .

*Proof.* Suppose  $\text{CONNECTIVITY}(G, m, r)$  succeeds. Property 1 follows by  $\forall v \in V_i'', \text{par}_i(\text{par}_i(v)) = \text{par}_i(v)$  and Lemma B.10. Property 2 follows by  $\text{diam}(G_r) \leq \text{diam}(G_r'') \leq \text{diam}(G_r') \leq \text{diam}(G_{r-1}) \leq \text{diam}(G_{r-1}'') \leq \text{diam}(G_{r-1}') \leq \dots \leq \text{diam}(G_0) = \text{diam}(G)$  and property 1 of Lemma B.3. Property 3 follows by the depth of  $h_r$  is at most  $r$  and Lemma B.10. Property 4 follows by property 2.

Now let us prove the success probability. Let  $i \in [r]$ . If  $p_i < 0.5$ , then we can apply Lemma B.4 on vertex set  $V_i''$ , parameter  $\gamma_i$ , and hash function  $l_i$ . Notice that the set  $S_v$  in the statement of

Lemma B.4 is  $\Gamma_{G'_i}(v)$  in the algorithm. Notice that  $|V''_i| \leq n$ . Then in the  $i^{\text{th}}$  round, if  $p_i < 0.5$ , then with probability at most  $1/(100n^2)$ ,  $L_i$  will be  $\{v \in V''_i \mid l_i(v) = 1\}$ , and  $n_i = |L_i| \leq 1.5p_i n_{i-1}$ . By taking union bound over all  $i \in [r]$ , we have that with probability at least 0.99, event  $\mathcal{E}$  happens: for all  $i \in [r]$ , if  $p_i < 0.5$ , then  $n_i \leq 1.5p_i n_{i-1} \leq 0.75n_{i-1}$ . Suppose  $\mathcal{E}$  happens. For  $i \in [r]$ ,  $p_i = 0.5$ , if we apply Lemma B.5, then condition on  $n_{i-1}$ , we have  $\mathbf{E}(n_i) \leq 0.75n_{i-1}$ . Thus, we know  $\forall i \in [r]$ ,  $\mathbf{E}(n_i) \leq 0.75 \mathbf{E}(n_{i-1}) \leq 0.75^i n$ .

Next, we discuss the case for  $p_0 = 0.5$  and the case for  $p_0 < 0.5$  separately.

If  $p_0 = 0.5$ , then  $m \leq n \cdot (600 \log n)^4$ . By Markov's inequality, when  $i^* \geq 4 \log_{4/3}(6000 \log n)$ , with probability at least 0.99,  $n_{i^*} \leq n/(600 \log n)^4$  and thus  $p_{i^*} < 0.5$ . Condition on this event and  $\mathcal{E}$ , we have

$$\begin{aligned}
n_r &\leq \frac{\left( \frac{\left( \frac{n_{i^*}^{1.5}}{m^{0.5}} (45 \log n + 150) \right)^{1.5}}{m^{0.5}} (45 \log n + 150) \right)^{\dots}}{\dots} && \text{(Apply } r' = r - i^* \text{ times)} \\
&= \frac{n_{i^*}^{1.5^{r'}}}{m^{1.5^{r'} - 1}} (45 \log n + 150)^{2 \cdot (1.5^{r'} - 1)} \\
&= n_{i^*} / (m/n_{i^*})^{1.5^{r'} - 1} \cdot (45 \log n + 150)^{2 \cdot (1.5^{r'} - 1)} \\
&\leq n / (m / (n_{i^*} (45 \log n + 150)^2))^{1.5^{r'} - 1} \\
&\leq n / (m / (n_{i^*} (45 \log n + 150)^2))^{1.5^{r'/2}} \\
&\leq n / (m/n)^{1.5^{r'/2}} \leq \frac{1}{2},
\end{aligned}$$

where the second inequality follows by  $n_{i^*} \leq n$ , the third inequality follows by  $r' \geq 5$ , the forth inequality follows by  $n_{i^*} \leq n/(600 \log n)^4$ , and the last inequality follows by  $r' \geq \frac{2}{\log 1.5} \log \log_{m/n}(2n)$ . Since  $4n \leq m \leq n \cdot (600 \log n)^4$ ,  $\log \log_{m/n} n = \Theta(\log \log n)$ . Let  $c > 0$  be a sufficiently large constant. Thus, when  $r \geq c \log \log_{m/n} n \geq i^* + r' = 4 \log(6000 \log n) / \log(4/3) + \frac{2}{\log 1.5} \log \log_{m/n}(2n)$ , with probability at least 0.98,  $\text{CONNECTIVITY}(G, m, r)$  will not fail.

Since property 1 of Lemma B.3, we have  $k_i \leq O(\log(\min(m/n_{i-1}, \text{diam}(G))))$ . Thus,

$$\begin{aligned}
\sum_{i=1}^r k_i &= \sum_{i=1}^{i^*} k_i + \sum_{i=i^*+1}^r k_i \leq O((\log \log n)^2) + \sum_{i=i^*+1}^r k_i \\
&\leq O((\log \log n)^2) + \sum_{i: i \geq i^*+1, m/n_{i-1} \leq \text{diam}(G)} k_i + \sum_{i: i \leq r, m/n_{i-1} > \text{diam}(G)} k_i \\
&\leq O((\log \log n)^2) + O\left( \sum_{i=0}^{\lceil \log_{1.25} \log_2(\text{diam}(G)) \rceil} \log(2^{1.25^i}) \right) + O\left( \sum_{i=0}^{\lceil \log_{1.25} \log_{\text{diam}(G)}(m) \rceil} \log(\text{diam}(G)) \right) \\
&\leq O((\log \log n)^2) + O(\log(\text{diam}(G))) + O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n)) \\
&\leq O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n) + (\log \log(n))^2),
\end{aligned}$$

where the first inequality follows by  $i^* = O(\log \log n)$  and  $\forall i \leq [i^*]$ ,  $m/n_{i-1} \leq \text{poly}(\log n)$ , the third inequality follows by  $m/n_{i+1} \geq (m/n_i)^{1.5}/(45 \log n + 150) \geq (m/n_i)^{1.25}$ .

If  $m > n \cdot (600 \log n)^4$ , then  $\forall i \in \{0\} \cup [r-1]$ , we have  $p_i < 0.5$ . Since  $\mathcal{E}$  happens. We have:

$$\begin{aligned}
n_r &\leq \frac{\left( \frac{\left( \frac{n^{1.5}}{m^{0.5}} (45 \log n + 150) \right)^{1.5}}{m^{0.5}} (45 \log n + 150) \right)^{\dots}}{\dots} && \text{(Apply } r \text{ times)} \\
&= \frac{n^{1.5^r}}{m^{1.5^r - 1}} (45 \log n + 150)^{2 \cdot (1.5^r - 1)} \\
&= n / (m/n)^{1.5^r - 1} \cdot (45 \log n + 150)^{2 \cdot (1.5^r - 1)} \\
&= n / \left( m / \left( n (45 \log n + 150)^2 \right) \right)^{1.5^r - 1} \\
&\leq n / \left( m / \left( n (45 \log n + 150)^2 \right) \right)^{1.5^{r/2}} \\
&\leq n / \left( m / \left( n (200 \log n)^2 \right) \right)^{1.5^{r/2}} \\
&\leq \frac{1}{2},
\end{aligned}$$

where the second inequality follows by  $r \geq 5$ , the third inequality follows by  $45 \log n + 150 \leq 200 \log n$ , and the last inequality follows by

$$r \geq c \log \log_{m/n} n \geq 2 \log_{1.5} \log_{(m/n)^{1/2}} 2n \geq 2 \log_{1.5} \log_{m/(n(200 \log n)^2)} 2n$$

for a sufficiently large constant  $c > 0$ .

By property 1 of Lemma B.3, we have  $k_i \leq O(\log(\min(m/n_{i-1}, \text{diam}(G))))$ . Thus,

$$\begin{aligned}
\sum_{i=1}^r k_i &\leq \sum_{m/n_{i-1} \leq \text{diam}(G)} k_i + \sum_{m/n_{i-1} > \text{diam}(G)} k_i \\
&\leq O \left( \sum_{i=0}^{\lceil \log_{1.25} \log_2(\text{diam}(G)) \rceil} \log(2^{1.25^i}) \right) + O \left( \sum_{i=0}^{\lceil \log_{1.25} \log_{\text{diam}(G)}(m) \rceil} \log(\text{diam}(G)) \right) \\
&\leq O(\log(\text{diam}(G))) + O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n)),
\end{aligned}$$

where the first inequality follows by  $m/n_{i+1} \geq (m/n_i)^{1.5}/(45 \log n + 150) \geq (m/n_i)^{1.25}$ .

Since  $n_r$  is an integer,  $n_r$  must be 0 when  $n_r \leq 1/2$ . Let  $c > 0$  be a sufficiently large constant. For all  $m \geq 4n$ , if  $r \geq c \log \log_{m/n} n$  then  $\text{CONNECTIVITY}(G, m, r)$  will succeed with probability at least 0.98. □

## C Spanning Forest

### C.1 Local Shortest Path Tree

In this section, we introduce an important procedure which will be used in the spanning tree algorithm. Roughly speaking, our procedure can merge several local shortest path trees into a larger local shortest path tree. Before we describe the details of the procedure, let us look at some concepts.

**Definition C.1** (Local shortest path tree (LSPT)). *Let  $V'$  be a set of vertices,  $v$  be a vertex in  $V'$ , and  $\text{par} : V' \rightarrow V'$  be a set of parent pointers (See Definition B.6) on  $V'$  which satisfies that  $v$  is the only root of  $\text{par}$ . Let  $T = (V', \text{par})$ . Given an undirected graph  $G = (V, E)$ , if  $V' \subseteq V$  and  $\forall u \in V' \setminus \{v\}, (u, \text{par}(u)) \in E, \text{dep}_{\text{par}}(u) = \text{dist}_G(u, v)$ , then we say  $T$  is a local shortest path tree*

(LSPT) in  $G$ , and  $T$  has root  $v$ . The vertex set ( $V'$  in the above) in  $T$  is denoted as  $V_T$ . The set of parent pointers (par in the above) in  $T$  is denoted as  $\text{par}_T$ . For short,  $\text{dep}_{\text{par}_T}$  is denoted as  $\text{dep}_T$ , and  $\text{dep}(\text{par}(T))$  is denoted as  $\text{dep}(T)$ .

**Definition C.2.** Given an undirected graph  $G = (V, E)$ , a vertex  $v \in V$ , and  $s \in \mathbb{Z}_{\geq 0}$ , we define the ball centered at  $v$  with radius  $s$  as the set  $B_{G,s}(v) = \{u \in V \mid \text{dist}_G(u, v) \leq s\}$ .

If in the context graph  $G$  is clear, then we use  $B_s(v)$  to denote  $B_{G,s}(v)$ .

**Definition C.3** (Local complete shortest path tree (LCSPT)). Given an undirected graph  $G = (V, E)$ ,  $s \in \mathbb{Z}_{\geq 0}$  and a local shortest path tree  $T = (V_T, \text{par}_T)$  in  $G$  where  $T$  has root  $v \in V$ . If  $V_T = B_{G,s}(v)$ , then we call  $T$  a local complete shortest path tree (LCSPT) in  $G$ . The root of  $T$  is  $v$ . The radius of  $T$  is  $s$ .

Let  $\tilde{T} = (V_{\tilde{T}}, \text{par}_{\tilde{T}})$  with radius  $s_1 \in \mathbb{Z}_{\geq 0}$  and root  $v$  be a local complete shortest path tree in some graph  $G = (V, E)$ . For  $s_2 \in \mathbb{Z}_{\geq 0}$ , if for every  $u \in V_{\tilde{T}}$ , we have a local complete shortest path tree  $T(u) = (V_{T(u)}, \text{par}_{T(u)})$  with root  $u$  and radius  $s_2$ , then we can compute a larger local complete shortest path tree  $\hat{T}$  with root  $v$  and radius  $s_1 + s_2$ . The procedure is described in Algorithm 4.

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**Algorithm 4** Local Complete Shortest Path Tree Expansion

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1: procedure TREEEXPANSION( $\tilde{T}, \text{dep}_{\tilde{T}}, \{T(u) \mid u \in V_{\tilde{T}}\}, \{\text{dep}_{T(u)} \mid u \in V_{\tilde{T}}\}$ ) ▷ Lemma C.4
    ▷  $\tilde{T} = (V_{\tilde{T}}, \text{par}_{\tilde{T}})$  with root  $v$  and radius  $s_1$  is a LCSPT in graph  $G = (V, E)$ .
    ▷  $\text{dep}_{\tilde{T}} : V_{\tilde{T}} \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $\tilde{T}$ .
    ▷  $\forall u \in V_{\tilde{T}}, T(u) = (V_{T(u)}, \text{par}_{T(u)})$  with root  $u$  and radius  $s_2$  is a LCSPT in  $G$ .
    ▷  $\forall u \in V_{\tilde{T}}, \text{dep}_{T(u)} : V_{T(u)} \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $T(u)$ .

2:   Output:  $\hat{T} = (V_{\hat{T}}, \text{par}_{\hat{T}}, \text{dep}_{\hat{T}})$ .
3:   Let  $V_{\hat{T}} = \bigcup_{u \in V_{\tilde{T}}} V_{T(u)}$ .
4:    $\forall x \in V_{\hat{T}}, \text{par}_{\hat{T}}(x) \leftarrow \text{par}_{\tilde{T}}(x)$ .
5:    $\forall x \in V_{\hat{T}}, h(x) \leftarrow \text{dep}_{\tilde{T}}(x)$ .
6:    $\forall x \in V_{\hat{T}} \setminus V_{\tilde{T}}, u_x \leftarrow \arg \min_{u: u \in V_{\tilde{T}}, x \in V_{T(u)}} \text{dep}_{\tilde{T}}(u) + \text{dep}_{T(u)}(x), \text{par}_{\hat{T}}(x) \leftarrow \text{par}_{T(u_x)}(x)$ .
▷  $u_x$  is on the shortest path from  $x$  to  $v$ .

7:    $\forall x \in V_{\hat{T}} \setminus V_{\tilde{T}}, h(x) \leftarrow \text{dep}_{\tilde{T}}(u_x) + \text{dep}_{T(u_x)}(x)$ .
8:   return  $\hat{T} = (V_{\hat{T}}, \text{par}_{\hat{T}})$ , and return  $h : V_{\hat{T}} \rightarrow \mathbb{Z}_{\geq 0}$  as  $\text{dep}_{\hat{T}}$ .
9: end procedure

```

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**Lemma C.4.** Let  $G = (V, E)$  be an undirected graph,  $s_1, s_2 \in \mathbb{Z}_{\geq 0}$ , and  $v \in V$ . Let  $\tilde{T} = (V_{\tilde{T}}, \text{par}_{\tilde{T}})$  with root  $v$  and radius  $s_1$  be a local complete shortest path tree in  $G$ , and  $\text{dep}_{\tilde{T}} : V_{\tilde{T}} \rightarrow \mathbb{Z}_{\geq 0}$  be the depth of every vertex in  $\tilde{T}$ .  $\forall u \in V_{\tilde{T}}$ , let  $T(u)$  with root  $u$  and radius  $s_2$  be a local complete shortest path tree in  $G$ , and  $\text{dep}_{T(u)} : V_{T(u)} \rightarrow \mathbb{Z}_{\geq 0}$  be the depth of every vertex in  $T(u)$ . Let  $(\hat{T} = (V_{\hat{T}}, \text{par}_{\hat{T}}, \text{dep}_{\hat{T}}) = \text{TREEEXPANSION}(\tilde{T}, \text{dep}_{\tilde{T}}, \{T(u) \mid u \in V_{\tilde{T}}\}, \{\text{dep}_{T(u)} \mid u \in V_{\tilde{T}}\}))$  (Algorithm 4), then  $\hat{T}$  is a local complete shortest path tree with root  $v$  and radius  $s_1 + s_2$  in  $G$ . In addition,  $\text{dep}_{\hat{T}}$  records the depth of every vertex in  $\hat{T}$ .

*Proof.* If  $x \in B_{s_1+s_2}(v)$ , then there must exist  $u \in V$  such that  $\text{dist}_G(v, u) \leq s_1$  and  $\text{dist}_G(u, x) \leq s_2$ . Thus,  $V_{\hat{T}} = \bigcup_{u \in V_{\tilde{T}}} V_{T(u)} = \bigcup_{u \in B_{s_1}(v)} B_{s_2}(u) = B_{s_1+s_2}(v)$ .

Now we want to prove that  $\text{par}_{\hat{T}} : V_{\hat{T}} \rightarrow V_{\hat{T}}$  also satisfies the condition that  $\hat{T}$  is a local shortest path tree. We can prove it by induction. If  $\text{dist}_G(u, v) = 0$ , then it means  $u = v$ . In this case,  $\text{par}_{\hat{T}}(u) = \text{par}_{\tilde{T}}(u) = v$ , and  $h(u) = \text{dep}_{\tilde{T}}(u) = 0$ . Let  $s \in [s_1 + s_2]$ . Suppose  $\forall x \in B_{s-1}(v)$ , we have

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**Algorithm 5** Doubling Algorithm for Local Complete Shortest Path Trees
 

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1: procedure MULTIRADIUSLCSPT( $G = (V, E), m$ ) ▷ Lemma C.6
2:   Output:  $r, \{T_i(v) \mid i \in \{0\} \cup [r], v \in V\}, \{\text{dep}_{T_i(v)} \mid i \in \{0\} \cup [r], v \in V, T_i(v) \neq \text{null}\}$ 
3:   Initialization:
4:    $\forall v \in V$ , if  $|\{v\} \cup \Gamma_G(v)| < \lceil (m/n)^{1/4} \rceil$ , then let  $T_0(v) \leftarrow (\{v\} \cup \Gamma_G(v), \text{par}_{T_0(v)})$ ,
5:   where  $\text{par}_{T_0(v)} : \{v\} \cup \Gamma_G(v) \rightarrow \{v\} \cup \Gamma_G(v)$ , and  $\forall u \in \{v\} \cup \Gamma_G(v), \text{par}_{T_0(v)}(u) = v$ .
6:    $\forall v \in V$ , if  $|\{v\} \cup \Gamma_G(v)| \geq \lceil (m/n)^{1/4} \rceil$ , then let  $T_0(v) \leftarrow \text{null}$ .
7:    $\forall v \in V$ , if  $T_0(v) \neq \text{null}$ , let  $\text{dep}_{T_0(v)} : V_{T_0(v)} \rightarrow \mathbb{Z}_{\geq 0}$  s.t.  $\text{dep}_{T_0(v)}(v) = 0, \forall u \in \Gamma_G(v), \text{dep}_{T_0(v)}(u) = 1$ .
8:    $r = 1$ .
9:   Main Loop:
10:  for true do
11:    for  $v \in V$  do ▷ If  $T_r(v) \neq \text{null}$ ,  $T_r(v)$  is a local complete shortest path tree with radius  $2^r$ .
12:      if  $T_{r-1}(v)$  is null then  $T_r(v) \leftarrow \text{null}$ .
13:      else if  $\exists u \in V_{T_{r-1}(v)}, T_{r-1}(u)$  is null then  $T_r(v) \leftarrow \text{null}$ .
14:      else
15:         $(T_r(v), \text{dep}_{T_r(v)}) = \text{TREEEXPANSION} \left( T_{r-1}(v), \text{dep}_{T_{r-1}(v)}, \bigcup_{u \in V_{T_{r-1}(v)}} \{T_{r-1}(u)\}, \bigcup_{u \in V_{T_{r-1}(v)}} \{\text{dep}_{T_{r-1}(u)}\} \right)$ . ▷ Algorithm 4
16:        If  $|V_{T_r(v)}| \geq \lceil (m/n)^{1/4} \rceil$ , let  $T_r(v) \leftarrow \text{null}$ .
17:      end if
18:    end for
19:    if  $\forall v \in V$  either  $T_r(v) = \text{null}$  or  $|V_{T_r(v)}| = |V_{T_{r-1}(v)}|$  then
20:      return  $r, \{T_i(v) \mid i \in \{0\} \cup [r], v \in V\}, \{\text{dep}_{T_i(v)} \mid i \in \{0\} \cup [r], v \in V, T_i(v) \neq \text{null}\}$ 
21:    end if
22:     $r \leftarrow r + 1$ .
23:  end for
24: end procedure

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$h(x) = \text{dep}_{\hat{T}}(x) = \text{dist}_G(x, v)$ . If  $B_s(v) = B_{s-1}(v)$ , then we are already done. Otherwise, let  $x$  be the vertex which has  $\text{dist}_G(x, v) = s$ . If  $x \in B_{s_1}(v)$ , then  $h(x) = \text{dep}_{\hat{T}}(x) = \text{dist}_G(x, v)$ . Additionally, we have  $\text{par}_{\hat{T}}(x) = \text{par}_{\hat{T}}(x)$ . Therefore,  $\text{dep}_{\hat{T}}(x) = \text{dep}_{\hat{T}}(\text{par}_{\hat{T}}(x)) + 1 = \text{dist}_G(v, \text{par}_{\hat{T}}(x)) + 1 = \text{dist}_G(v, x)$ . If  $x \in B_{s_2}(v) \setminus B_{s_1}(v)$ , then  $h(x) = \min_{u: \text{dist}_G(v, u) \leq s_1, \text{dist}_G(u, x) \leq s_2} \text{dep}_{\hat{T}}(u) + \text{dep}_{T(u)}(x) = \min_{u: \text{dist}_G(v, u) \leq s_1, \text{dist}_G(u, x) \leq s_2} \text{dist}_G(v, u) + \text{dist}_G(u, x) = \text{dist}_G(v, x) = s$ . And we have  $\text{dist}_G(v, x) = \text{dist}_G(v, u_x) + \text{dist}_G(u_x, x)$ . Notice that  $\text{dist}_G(v, \text{par}_{T(u_x)}(x)) = \text{dist}_G(v, u_x) + \text{dist}_G(u_x, \text{par}_{T(u_x)}(x)) = \text{dist}_G(v, x) - 1 = s - 1$ . Thus,

$$\text{dep}_{\hat{T}}(x) = \text{dep}_{\hat{T}}(\text{par}_{\hat{T}}(x)) + 1 = \text{dep}_{\hat{T}}(\text{par}_{T(u_x)}(x)) + 1 = s.$$

To conclude,  $\hat{T}$  is a local complete shortest path tree with root  $v$  and radius  $s_1 + s_2$  in  $G$ . In addition,  $\text{dep}_{\hat{T}}$  records the depth of every vertex in  $\hat{T}$ .  $\square$

## C.2 Multiple Local Shortest Path Trees

In this section, we show a procedure which is a generalization of neighbor increment procedure shown in Section B.1. The input of the procedure is an undirected graph  $G = (V, E)$  and a parameter  $m$  which is larger than  $|V| = n$ . The output will be  $n$  local shortest path trees (See Definition C.1) such that  $\forall v \in V$ , there is a shortest path tree with root  $v$ . Furthermore, the size of each shortest path tree is at least  $\lceil (m/|V|)^{1/4} \rceil$  and at most  $\lceil (m/|V|)^{1/2} \rceil$ . The algorithm is described in Algorithm 6. The high level idea is that we firstly use doubling technique and the algorithm described in Section C.1 to get local complete shortest path trees rooted at every vertex with multiple radius, and then use



these LCSPTs to find large enough local shortest path trees rooted at every vertex. The doubling algorithm is described in Algorithm 5.

**Definition C.5.** Given a graph  $G = (V, E)$  and a parameter  $m \in \mathbb{Z}_{\geq 0}, m \geq |V|$ , the number of iterations of  $\text{MULTIRADIUSLCSPT}(G, m)$  (Algorithm 5) is the value of  $r$  at the end of the procedure.

**Lemma C.6.** Let  $G = (V, E)$  be an undirected graph, and  $m$  be a parameter which is at least  $|V|$ . Let  $(r, \{T_i(v) \mid i \in \{0\} \cup [r], v \in V\}, \{\text{dep}_{T_i(v)} \mid i \in \{0\} \cup [r], v \in V, T_i(v) \neq \text{null}\}) = \text{MULTIRADIUSLCSPT}(G, m)$  (Algorithm 5). We have following properties.

1.  $\forall i \in \{0\} \cup [r], v \in V$ , if  $T_i(v) \neq \text{null}$ , then  $T_i(v)$  is a LCSPT (See Definition C.3) with root  $v$  and radius  $2^i$  in  $G$ . Furthermore,  $\text{dep}_{T_i(v)}$  records the depth of every vertex in  $T_i(v)$ .
2.  $\forall i \in \{0\} \cup [r], v \in V$ ,  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil \Leftrightarrow T_i(v) = \text{null}$ .
3. For  $v \in V$ , if  $T_r(v) \neq \text{null}$ , then  $V_{T_r(v)} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ .
4. The number of iterations (see Definition C.5)  $r \leq \min(\lceil \log(\text{diam}(G)) \rceil, \lceil \log(m/n) \rceil) + 1$ .

*Proof.* For property 1, we can prove it by induction. If  $i = 0$ , the property holds by line 4, line 5 and line 7. Now suppose  $\forall v \in V$ , if  $T_{i-1}(v)$  is not null, then  $T_{i-1}(v)$  is a LCSPT with root  $v$  and radius  $2^{i-1}$  in  $G$ , and  $\text{dep}_{T_{i-1}(v)}$  records the depth of every vertex in  $T_{i-1}(v)$ . For  $v \in V$ , notice that the only place that will make  $T_i(v)$  not null is line 15, and if the procedure run line 15, any of  $T_{i-1}(v)$  and  $T_{i-1}(u)$  with  $u \in V_{T_{i-1}(v)}$  cannot be null. By Lemma C.4, since the radius of  $T_{i-1}(v)$  is  $2^{i-1}$ , and  $\forall u \in V_{T_{i-1}(v)}, T_{i-1}(u)$  has radius  $2^{i-1}$ ,  $T_i(v)$  is a LCSPT with root  $v$  and radius  $2^i$ . Furthermore  $\text{dep}_{T_i(v)}$  records the depth of every vertex in  $T_i(v)$ .

For property 2, if  $i = 0$ , then this property holds by line 4 to line 7. For  $i \in [r]$ , our proof is by induction. Suppose the property holds for  $i - 1$ . Now consider  $T_i(v)$  for  $v \in V$ . The only way to make  $T_i(v)$  not null is line 15. If the procedure invokes line 15, then any of  $T_{i-1}(v)$  and  $T_{i-1}(u)$  with  $u \in V_{T_{i-1}(v)}$  cannot be null. By property 1 and Lemma C.4,  $T_i(v)$  will be a LCSPT with root  $v$  and radius  $2^i$  in line 15. If  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil$ , then  $T_i(v)$  is set to be null in line 16. Thus, we already got  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil \Rightarrow T_i(v) = \text{null}$ . Now we want to show  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil \Leftrightarrow T_i(v) = \text{null}$ . If  $T_i(v) = \text{null}$ , then there are three cases. The first case is that  $T_i(v)$  is set at line 12. In this case,  $T_{i-1}(v) = \text{null}$  implies  $|B_{G, 2^i}(v)| \geq |B_{G, 2^{i-1}}(v)| \geq \lceil (m/n)^{1/4} \rceil$ . The second case is that  $T_i(v)$  is set at line 13. In this case,  $\exists u \in V_{T_{i-1}(v)} = B_{G, 2^{i-1}}(v)$  such that  $|B_{G, 2^{i-1}}(u)| \geq \lceil (m/n)^{1/4} \rceil$  which implies  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil$ . In the final case,  $T_i(v)$  is set at line 16, and thus,  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil$ .

For property 3, if  $T_r(v) \neq \text{null}$ , then by property 1, we know  $V_{T_r(v)} = B_{G, 2^r}(v)$ . By the condition in line 19, we know  $V_{T_r(v)} = V_{T_{r-1}(v)}$  which implies  $B_{G, 2^r}(v) = B_{G, 2^{r-1}}(v)$ . Thus,  $V_{T_r(v)} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ .

For property 4, we can prove it by contradiction. If  $r > \lceil \log(\text{diam}(G)) \rceil + 1$ , then let  $i = \lceil \log(\text{diam}(G)) \rceil + 1$ . By the condition in line 19, we know there is a vertex  $v \in V$  such that  $T_i(v) \neq \text{null}$  and  $V_{T_i(v)} \neq V_{T_{i-1}(v)}$ . It means that  $B_{G, 2^i}(v) \neq B_{G, 2^{i-1}}(v)$ , i.e.  $\exists u \in V, \text{dist}_G(v, u) > 2^{i-1}$ . But this contradicts to  $i = \lceil \log(\text{diam}(G)) \rceil + 1$ . Similarly, if  $r > \lceil \log(m/n) \rceil + 1$ , then let  $i = \lceil \log(m/n) \rceil + 1$ . By the condition in line 19, we know there is a vertex  $v \in V$  such that  $T_i(v) \neq \text{null}$  and  $V_{T_i(v)} \neq V_{T_{i-1}(v)}$ . If  $2^{i-1} \leq \text{diam}(G)$ , then we have  $V_{T_i(v)} \neq V_{T_{i-1}(v)}$  which leads to a contradiction. If  $2^{i-1} \geq \text{diam}(G)$ , then  $|B_{G, 2^{i-1}}(v)| \geq 2^{i-1} \geq m/n \geq \lceil (m/n)^{1/4} \rceil$  which contradicts to property 2.  $\square$



Next, we show how to use Algorithm 5 to design an algorithm which can output  $|V|$  number of local shortest path trees rooted at every vertex in  $V$ . The details of the algorithm is described in Algorithm 6, and the guarantees of the algorithm is stated in the following lemma.

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**Algorithm 6** Large Local Shortest Path Trees

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1: procedure MULTIPLELARGETREES( $G = (V, E), m$ ) ▷ Lemma C.7, Lemma C.11
2:   Output:  $\{\tilde{T}(v) \mid v \in V\}, \{\text{dep}_{\tilde{T}(v)} \mid v \in V\}$ .
3:    $(r, \{T_i(v) \mid i \in \{0\} \cup [r], v \in V\}, \{\text{dep}_{T_i(v)} \mid i \in \{0\} \cup [r], v \in V, T_i(v) \neq \text{null}\}) = \text{MULTIRADIUSLCSPT}(G, m)$ . ▷ Algorithm 5
4:    $\forall v \in V$  with  $T_r(v) \neq \text{null}$  let  $\tilde{T}(v) = T_r(v)$  and  $\text{dep}_{\tilde{T}(v)} \leftarrow \text{dep}_{T_r(v)}$ .
5:    $\forall v \in V$  with  $T_r(v) = \text{null}$ , let  $\tilde{T}_0(v) = (\{v\}, \text{par}_{\tilde{T}_0(v)}, s_0(v) = 0, \text{and } \text{dep}_{\tilde{T}_0(v)} : \{v\} \rightarrow \mathbb{Z}_{\geq 0}$ ,
6:   where  $\text{par}_{\tilde{T}_0(v)} : \{v\} \rightarrow \{v\}$  satisfies  $\text{par}_{\tilde{T}_0(v)}(v) = v$ , and  $\text{dep}_{\tilde{T}_0(v)}(v) = 0$ .
7:   for  $i = 1 \rightarrow r$  do
8:     for  $v \in \{u \in V \mid T_r(u) = \text{null}\}$  do
9:       if  $\forall u \in V_{\tilde{T}_{i-1}(v)}, T_{r-i}(u) \neq \text{null}$  then
10:          $(\tilde{T}_i(v), \text{dep}_{\tilde{T}_i(v)}) = \text{TREEEXPANSION} \left( \tilde{T}_{i-1}(v), \text{dep}_{\tilde{T}_{i-1}(v)}, \bigcup_{u \in V_{\tilde{T}_{i-1}(v)}} \{T_{r-i}(u)\}, \bigcup_{u \in V_{\tilde{T}_{i-1}(v)}} \{\text{dep}_{T_{r-i}(u)}\} \right)$ . ▷ Algorithm 4
11:         If  $|V_{\tilde{T}_i(v)}| < \lceil (m/n)^{1/4} \rceil$ , then let  $s_i(v) = s_{i-1}(v) + 2^{r-i}$ .
12:         Otherwise, let  $s_i(v) = s_{i-1}(v)$ ,  $\tilde{T}_i(v) \leftarrow \tilde{T}_{i-1}(v)$ ,  $\text{dep}_{\tilde{T}_i(v)} \leftarrow \text{dep}_{\tilde{T}_{i-1}(v)}$ .
13:       else
14:         Let  $s_i(v) = s_{i-1}(v)$ ,  $\tilde{T}_i(v) = \tilde{T}_{i-1}(v)$ ,  $\text{dep}_{\tilde{T}_i(v)} \leftarrow \text{dep}_{\tilde{T}_{i-1}(v)}$ .
15:       end if
16:     end for
17:   end for ▷  $\tilde{T}_r(v)$  is a LCSPT with root  $v$  and the largest radius s.t.  $|V_{\tilde{T}_r(v)}| < \lceil (m/n)^{1/4} \rceil$ .
18:    $\forall v \in V$ , if  $|\Gamma_G(v) \cup \{v\}| \leq \lceil (m/n)^{1/4} \rceil$ , then let  $N(v) = \Gamma_G(v) \cup \{v\}$ .
19:   Otherwise arbitrarily choose  $N(v) \subseteq \Gamma_G(v) \cup \{v\}$  with  $|N(v)| = \lceil (m/n)^{1/4} \rceil$ .
20:   for  $v \in \{u \in V \mid T_r(u) = \text{null}\}$  do ▷ Expand  $\tilde{T}_r(v)$  a little bit to get large enough  $\tilde{T}$ .
21:     if  $\forall u \in V_{\tilde{T}_r(v)}, T_0(u) \neq \text{null}$  then
22:        $(\tilde{T}(v), \text{dep}_{\tilde{T}(v)}) = \text{TREEEXPANSION} \left( \tilde{T}_r(v), \text{dep}_{\tilde{T}_r(v)}, \bigcup_{u \in V_{\tilde{T}_r(v)}} \{T_0(u)\}, \bigcup_{u \in V_{\tilde{T}_r(v)}} \{\text{dep}_{T_0(u)}\} \right)$ . ▷ Algorithm 4
23:     else
24:       Select an arbitrary  $u_v \in V_{\tilde{T}_r(v)}$  with  $T_0(u_v) = \text{null}$ .
25:       Let  $V_{\tilde{T}(v)} = N(u_v) \cup V_{\tilde{T}_r(v)}$ .
26:        $\forall x \in V_{\tilde{T}(v)}$ , let  $\text{par}_{\tilde{T}(v)}(x) = \text{par}_{\tilde{T}_r(v)}(x)$ ,  $\text{dep}_{\tilde{T}(v)}(x) = \text{dep}_{\tilde{T}_r(v)}(x)$ .
27:        $\forall x \in N(u_v), x \notin V_{\tilde{T}_r(v)}$ , let  $\text{par}_{\tilde{T}(v)}(x) = u_v$ ,  $\text{dep}_{\tilde{T}(v)}(x) = \text{dep}_{\tilde{T}_r(v)}(u_v) + 1$ .
28:       Let  $\tilde{T}(v) = (V_{\tilde{T}(v)}, \text{par}_{\tilde{T}(v)})$ .
29:     end if
30:   end for
31:   return  $\{\tilde{T}(v) \mid v \in V\}, \{\text{dep}_{\tilde{T}(v)} \mid v \in V\}$ .
32: end procedure

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**Lemma C.7.** Let  $G = (V, E)$  be an undirected graph, and  $m$  be a parameter which is at least  $16|V|$ . Let  $(\{\tilde{T}(v) \mid v \in V\}, \{\text{dep}_{\tilde{T}(v)} \mid v \in V\}) = \text{MULTIPLELARGETREES}(G, m)$ . (Algorithm 6) Then, the output satisfies the following properties.

1.  $\forall v \in V$ ,  $\tilde{T}(v)$  is a LSPT (See Definition C.1) with root  $v$ , and  $\text{dep}_{\tilde{T}(v)}$  records the depth of every vertex in  $\tilde{T}(v)$ .
2.  $\forall v \in V, u \in V_{\tilde{T}(v)}, w \in V \setminus V_{\tilde{T}(v)}$ , it satisfies  $\text{dist}_G(v, u) \leq \text{dist}_G(v, w)$ .

3.  $\forall v \in V$ , either  $|V_{\tilde{T}(v)}| \geq \lceil (m/n)^{1/4} \rceil$  or  $V_{\tilde{T}(v)} = \{u \in V \mid \text{dist}_G(u, v) < \infty\}$ .

4.  $\forall v \in V$ ,  $|V_{\tilde{T}(v)}| \leq \lfloor (m/n)^{1/2} \rfloor$ .

*Proof.* Before we prove above properties, we first show some crucial observations.

**Claim C.8.**  $\forall v \in V, i \in \{0\} \cup [r]$ , if  $T_i(v) \neq \text{null}$ , then  $T_i(v)$  is a LCSPT with root  $v$  and radius  $2^i$  in graph  $G$ . Furthermore,  $\text{dep}_{T_i(v)} : V_{T_i(v)} \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $T_i(v)$ . If  $T_i(v) = \text{null}$ , then  $|B_{G, 2^i}(v)| \geq \lceil (m/n)^{1/4} \rceil$ .

*Proof.* Follows by property 1 and property 2 of Lemma C.6 directly.  $\square$

**Claim C.9.** Let  $v \in V$  be a vertex with  $T_r(v) = \text{null}$ . Then,  $\forall i \in \{0\} \cup [r]$ ,  $\tilde{T}_i(v)$  is a LCSPT (See Definition C.3) with root  $v$  and radius  $s_i(v)$  in  $G$ , and  $\text{dep}_{\tilde{T}_i(v)}$  records the depth of every vertex in  $\tilde{T}_i(v)$ . Furthermore, we have  $|B_{G, s_i(v)}(v)| < \lceil (m/n)^{1/4} \rceil$ ,  $|B_{G, s_i(v)+2^{r-i}}(v)| \geq \lceil (m/n)^{1/4} \rceil$ .

*Proof.* Let  $v \in V$  be a vertex with  $T_r(v) = \text{null}$ . When  $i = 0$ , then due to line 5 and line 6,  $\tilde{T}_0(v)$  is a LCSPT (See Definition C.3) with root  $v$  and radius  $0 = s_0(v)$  in  $G$ . According to property 2 of Lemma C.6, since  $T_r(v) = \text{null}$ , we know  $|B_{G, 0+2^r}(v)| \geq \lceil (m/n)^{1/4} \rceil$ .

For  $i \in [r]$ , we prove it by induction. Suppose the claim is true for  $i - 1$ . By Claim C.8, Lemma C.4 and the condition in line 9, if the procedure executes line 10, then we know  $\tilde{T}_i(v)$  is a LCSPT with root  $v$  and radius  $s_{i-1}(v) + 2^{r-i}$  in  $G$  at the end of the execution of line 10, and  $\text{dep}_{\tilde{T}_i(v)}$  records the depth of every vertex in  $\tilde{T}_i(v)$ . If  $|V_{\tilde{T}_i(v)}| < \lceil (m/n)^{1/4} \rceil$ , then  $|B_{G, s_{i-1}(v)+2^{r-i}}(v)| < \lceil (m/n)^{1/4} \rceil$ . The procedure will execute line 11, and thus  $s_i(v)$  is the radius of  $\tilde{T}_i(v)$ . In addition, since  $s_i(v) = s_{i-1}(v) + 2^{r-i}$  and  $|B_{G, s_{i-1}(v)+2^{r-i+1}}(v)| \geq \lceil (m/n)^{1/4} \rceil$ , we have  $|B_{G, s_i(v)+2^{r-i}}(v)| \geq \lceil (m/n)^{1/4} \rceil$ . If at the end of line 10,  $|V_{\tilde{T}_i(v)}| \geq \lceil (m/n)^{1/4} \rceil$ , then we know  $|B_{G, s_{i-1}(v)+2^{r-i}}(v)| \geq \lceil (m/n)^{1/4} \rceil$ . In this case,  $\tilde{T}_i(v)$ ,  $s_i(v)$  and  $\text{dep}_{\tilde{T}_i(v)}$  will be set to be  $\tilde{T}_{i-1}(v)$ ,  $s_{i-1}(v)$  and  $\text{dep}_{\tilde{T}_{i-1}(v)}$  respectively, and thus  $|B_{G, s_i(v)}(v)| < \lceil (m/n)^{1/4} \rceil$ . If the condition in line 9 does not hold, then we know  $|B_{G, s_{i-1}(v)+2^{r-i}}(v)| \geq \lceil (m/n)^{1/4} \rceil$  by claim C.8. In this case,  $\tilde{T}_i(v)$ ,  $s_i(v)$  and  $\text{dep}_{\tilde{T}_i(v)}$  will also be set to be  $\tilde{T}_{i-1}(v)$ ,  $s_{i-1}(v)$  and  $\text{dep}_{\tilde{T}_{i-1}(v)}$  respectively, and thus  $|B_{G, s_i(v)}(v)| < \lceil (m/n)^{1/4} \rceil$ .  $\square$

Claim C.9 shows that for each vertex  $v \in V$ , we know  $\tilde{T}_r(v)$  is a LCSPT with root  $v$  and radius  $s_r(v)$  in  $G$  such that  $|B_{G, s_r(v)}(v)| < \lceil (m/n)^{1/4} \rceil$  and  $|B_{G, s_r(v)+1}(v)| \geq \lceil (m/n)^{1/4} \rceil$ .

Now, let us prove property 1 and property 2. For  $v \in V$ , if  $T_r(v) \neq \text{null}$ , then  $\tilde{T}(v)$ ,  $\text{dep}_{\tilde{T}(v)}$  will be set to be  $T_r(v)$ ,  $\text{dep}_{T_r(v)}$  respectively. By Claim C.8, the properties holds. Let  $v$  be a vertex in  $V$  with  $T_r(v) = \text{null}$ . If  $\tilde{T}(v)$  is assigned at line 22, then by Lemma C.4, we know  $\tilde{T}(v)$  is a LCSPT with root  $v$ , and  $\text{dep}_{\tilde{T}(v)}$  records the depth of every vertex in  $\tilde{T}(v)$ . Thus, both properties hold. If  $\tilde{T}(v)$  is assigned at line 28, then there are two cases for the vertices in  $V_{\tilde{T}(v)}$ :

1. If  $x$  is in  $V_{\tilde{T}_r(v)}$ , then since Claim C.9 shows  $\tilde{T}_r(v)$  is a LCSPT with root  $v$ , it is easy to show  $\text{dep}_{\tilde{T}(v)}(x) = \text{dep}_{\tilde{T}_r(v)}(x) = \text{dist}_G(v, x)$ , and  $\text{par}_{\tilde{T}(v)}(x) = \text{par}_{\tilde{T}_r(v)}(x) \in E$ .
2. if  $x$  is in  $N(u_v)$  but not in  $V_{\tilde{T}_r(v)}$ , then since  $\tilde{T}_r(v)$  is a LCSPT with root  $v$  and radius  $s_r(v)$ ,  $\text{dist}_G(v, x) \geq s_r(v) + 1$ . Also notice that  $\text{dist}_G(v, x) \leq \text{dist}_G(v, u_v) + \text{dist}_G(u_v, x) \leq s_r(v) + 1$ . Therefore,  $\text{dist}_G(v, x) = s_r(v) + 1$ ,  $\text{dep}_{\tilde{T}(v)}(x) = \text{dep}_{\tilde{T}(v)}(u_v) + 1 = \text{dist}_G(v, x) = s_r(v) + 1$ . Since  $x \in N(u_v)$ ,  $(\text{par}_{\tilde{T}(v)}(x), x) = (u_v, x) \in E$ .

Thus,  $\tilde{T}$  is a LSPT with root  $v$ , and it proves property 1. Due to above both cases, we know  $B_{G,s_r(v)}(v) \subseteq V_{\tilde{T}}$ , and  $\forall x \in V_{\tilde{T}(v)}$ ,  $\text{dist}_G(v, x) \leq s_r(v) + 1$ . Thus, property 2 holds.

For property 3 and property 4, we have two cases. The first case is when  $v$  satisfies  $T_r(v) \neq \text{null}$ . In this case  $\tilde{T}(v) = T_r(v)$ , due to property 3, 4 of Lemma C.6, we have  $V_{\tilde{T}(v)} = \{u \in V \mid \text{dist}_G(v, u) < \infty\}$ , and  $|V_{\tilde{T}(v)}| < \lceil (m/n)^{1/4} \rceil \leq \lfloor (m/n)^{1/2} \rfloor$ . The second case is  $T_r(v) = \text{null}$ . In this case, if  $\tilde{T}(v)$  is assigned at line 22, then  $V_{\tilde{T}(v)} = B_{G,s_r(v)+1}(v)$ . Then by Claim C.9, we can get  $|V_{\tilde{T}(v)}| \geq \lceil (m/n)^{1/4} \rceil$ . Because  $|V_{\tilde{T}(v)}| < \lceil (m/n)^{1/4} \rceil$  and  $\forall u \in V_{\tilde{T}(v)}$ ,  $|V_{T_0(u)}| < \lceil (m/n)^{1/4} \rceil$ , we know  $|V_{\tilde{T}(v)}| \leq \lfloor (m/n)^{1/2} \rfloor$ . If  $\tilde{T}(v)$  is assigned at line 28, then  $|V_{\tilde{T}(v)}| \geq |N(u_v)| \geq \lceil (m/n)^{1/4} \rceil$ , and  $|V_{\tilde{T}(v)}| \leq |V_{\tilde{T}(v)}| + |N(u_v)| < 2 \cdot \lceil (m/n)^{1/4} \rceil \leq \lfloor (m/n)^{1/2} \rfloor$ .  $\square$

**Definition C.10.** Let graph  $G = (V, E)$ , and let  $m$  be a parameter which is at least  $16|V|$ . The number of iterations of  $(\{\tilde{T}(v) \mid v \in V\}, \{\text{dep}_{\tilde{T}(v)} \mid v \in V\}) = \text{MULTIPLELARGETREES}(G, m)$  (Algorithm 6) is defined as the value of  $r$  in the procedure.

**Lemma C.11** (Number of iterations of Algorithm 6). Let  $G = (V, E)$  be an undirected graph, and let  $m$  be a parameter which is at least  $16|V|$ . The number of iterations (see Definition C.10) of  $(\{\tilde{T}(v) \mid v \in V\}, \{\text{dep}_{\tilde{T}(v)} \mid v \in V\}) = \text{MULTIPLELARGETREES}(G, m)$  (Algorithm 6) is at most  $\min(\lceil \log(\text{diam}(G)) \rceil, \lceil \log(m/n) \rceil) + 1$ .

*Proof.* It follows by property 4 of Lemma C.6 directly.  $\square$

### C.3 Path Generation and Root Changing

In this section, we show a procedure which can output a path from a certain vertex to the root in a rooted tree. Then we show how to use the procedure to change the root of a rooted tree to a certain vertex in the tree. To output the vertex-root path, we have two stages. The first stage is using doubling method to compute the depth and the  $2^i$ th (for all  $i \in \{0, 1, \dots, \log(\text{dep})\}$ ) ancestor of each vertex. The second stage is using divide-and-conquer technique to split the path into segments, and recursively find the path for each segment. Once we have the procedure to find the vertex-root path, then we can use it to implement root-changing. The idea is very simple, if we want to change the root to a certain vertex, we just need to find the path from that vertex to the root, and reverse the parent pointers of every vertex on the path. The path finding procedure is described in Algorithm 8. The root changing procedure is described in Algorithm 9.

**Definition C.12.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . The number of iterations of  $\text{FINDANCESTORS}(\text{par})$  is defined as the value of  $r$  at the end of the procedure.

**Lemma C.13.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$  (Algorithm 7). Then the number of iterations (see Definition C.12)  $r$  should be at most  $\lceil \log(\text{dep}(\text{par}) + 1) \rceil$ ,  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$ , and  $\forall i \in \{0\} \cup [r], v \in V$   $g_i(v) = \text{par}^{(2^i)}(v)$ .

*Proof.*  $h_l$  and  $g_l$  will satisfies the properties in the following claim.

**Claim C.14.**  $\forall i \in \{0\} \cup [r], v \in V$   $g_i(v) = \text{par}^{(2^i)}(v)$ , and if  $\text{dep}_{\text{par}}(v) \leq 2^i - 1$  then  $h_i(v) = \text{dep}_{\text{par}}(v)$ . Otherwise  $\text{dep}_{\text{par}}(v) = \text{null}$ .

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**Algorithm 7** Depth and Ancestors of Every Vertex

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1: procedure FINDANCESTORS ( $\text{par} : V \rightarrow V$ ) ▷ Lemma C.13
2:   For  $v \in V$  let  $g_0(v) = \text{par}(v)$ . If  $\text{par}(v) = v$ , let  $h_0(v) = 0$ . Otherwise, let  $h_0(v) = \text{null}$ .
3:   Let  $l = 0$ .
4:   for  $\exists v \in V, h_l(v) = \text{null}$  do
5:      $l \leftarrow l + 1$ .
6:     for  $v \in V$  do
7:       Let  $g_l(v) = g_{l-1}(g_{l-1}(v))$ . ▷  $g_l$  is  $\text{par}^{(2^l)}$ .
8:       if  $h_{l-1}(v) \neq \text{null}$  then  $h_l(v) = h_{l-1}(v)$ .
9:       else if  $h_{l-1}(g_{l-1}(v)) \neq \text{null}$  then  $h_l(v) = h_{l-1}(g_{l-1}(v)) + 2^{l-1}$ .
10:      else  $h_l(v) = \text{null}$ .
11:      end if
12:    end for
13:  end for
14:  Let  $r = l, \text{dep}_{\text{par}} \leftarrow h_r$ .
15:  return  $r, \text{dep}_{\text{par}}, \{g_i : V \rightarrow V \mid i \in \{0\} \cup [r]\}$ . ▷  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$ 
16: end procedure
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*Proof.* The proof is by induction. The claim is obviously true when  $i = 0$ . Suppose the claim is true for  $i - 1$ . We have  $g_i(v) = g_{i-1}(g_{i-1}(v)) = \text{par}^{(2^{i-1})}(\text{par}^{(2^{i-1})}(v)) = \text{par}^{(2^i)}(v)$ . If  $h_i(v) \neq \text{null}$ , then there are two cases. In the first case, we have  $h_i(v) = h_{i-1}(v)$ . By induction we know  $h_i(v) = \text{dep}_{\text{par}}(v)$ . In the second case, we have  $h_i(v) = h_{i-1}(g_{i-1}(v)) + 2^{i-1} = \text{dep}_{\text{par}}(\text{par}^{(2^{i-1})}(v)) + 2^{i-1}$ . Notice that in this case  $h_{i-1}(v) = \text{null}$ , thus by the induction,  $\text{dep}_{\text{par}}(v) \geq 2^{i-1}$ . Therefore,  $\text{dep}_{\text{par}}(v) = \text{dep}_{\text{par}}(\text{par}^{(2^{i-1})}(v)) + 2^{i-1} = h_i(v)$ . If  $h_i(v) = \text{null}$ , then it means that  $h_{i-1}(\text{par}^{(2^{i-1})}(v)) = \text{null}$  which implies that  $\text{dep}_{\text{par}}(v) \geq 2^i$ .  $\square$

Due to the above claim, we know that if  $i \geq \lceil \log(\text{dep}_{\text{par}}(v) + 1) \rceil$  then  $h_i(v) \neq \text{null}$ . Thus, we have  $r \leq \lceil \log(\text{dep}(\text{par}) + 1) \rceil$ . Since the procedure returns  $h_r$  as  $\text{dep}_{\text{par}}$ , the returned  $\text{dep}_{\text{par}}$  is correct.  $\square$

**Lemma C.15.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $q$  be a vertex in  $V$ . Let  $(\text{dep}_{\text{par}}, P, w) = \text{FINDPATH}(\text{par}, q)$  (Algorithm 8). Then  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$  and  $P \subseteq V$  is the set of all vertices on the path from  $q$  to the root of  $q$ , i.e.  $P = \{v \in V \mid \exists k \geq 0, v = \text{par}^{(k)}(q)\}$ . If  $\text{dep}_{\text{par}}(q) \geq 1$ , then  $w = \text{par}^{(\text{dep}_{\text{par}}(q)-1)}(q)$ . Furthermore,  $k$  should be at most  $\lceil \log(\text{dep}(\text{par})) \rceil$ .

*Proof.* By Lemma C.13, since  $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$ , we know  $r$  should be at most  $\lceil \log(\text{dep}(\text{par}) + 1) \rceil$ ,  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$ , and  $\forall i \in \{0\} \cup [r], v \in V, g_i(v) = \text{par}^{(2^i)}(v)$ . Thus  $k = \lceil \log(\text{dep}_{\text{par}}(q)) \rceil \leq \lceil \log(\text{dep}(\text{par}) + 1) \rceil$

Now let us prove that  $P$  is the vertex set of all the vertices on the path from  $q$  to the root of  $q$ . We use divide-and-conquer to get  $P$ . The following claim shows that  $S_i$  is a set of segments which is a partition of the path, and each segment has length at most  $2^{k-i}$ .

**Claim C.16.**  $\forall i \in \{0\} \cup [k]$ ,  $S_i$  satisfies the following properties:

1.  $\exists (x, y) \in S_i$  such that  $x = q$ .
2.  $\exists (x, y) \in S_i$  such that  $y = g_r(q)$ .

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**Algorithm 8** Path in a Tree
 

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1: procedure FINDPATH ( $\text{par} : V \rightarrow V, q \in V$ ) ▷ Lemma C.15
2:   Output:  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}, P \subseteq V, w \in V \cup \{\text{null}\}$ .
3:    $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$  ▷ Algorithm 7
4:   Let  $S_0 = \{(q, g_r(q))\}, k = \lceil \log(\text{dep}_{\text{par}}(q)) \rceil$ . ▷  $S_0$  contains  $(q, \text{the root of } q)$ 
5:   for  $i = 1 \rightarrow k$  do ▷  $S_i$  is a set of segments partitioned the path from  $q$  to the root of  $q$ 
6:     Let  $S_i \leftarrow \emptyset$ .
7:     for  $(x, y) \in S_{i-1}$  do
8:       if  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(y) > 2^{k-i}$  then  $S_i \leftarrow S_i \cup \{(x, g_{k-i}(x)), (g_{k-i}(x), y)\}$ .
9:       else  $S_i \leftarrow S_i \cup \{(x, y)\}$ .
10:      end if
11:    end for
12:  end for ▷  $S_k$  only contains segments with length at most 1
13:  Let  $P \leftarrow \{q\}$ 
14:  for  $(x, y) \in S_k$  do
15:    Let  $P \leftarrow P \cup \{y\}$ 
16:  end for
17:  Find  $w \in P$  with  $\text{dep}_{\text{par}}(w) = 1$ . If  $w$  does not exist, let  $w \leftarrow \text{null}$ .
18:  return  $(\text{dep}_{\text{par}}, P, w)$ 
19: end procedure

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3.  $\forall (x, y) \in S_i, \text{dep}_{\text{par}}(y) - \text{dep}_{\text{par}}(x) \leq 2^{k-i}$ .
4.  $\forall (x, y) \in S_i$ , if  $y \neq g_r(q)$ , then  $\exists (x', y') \in S_i, x' = y$ .
5.  $\forall (x, y) \in S_i, \exists j \in \mathbb{Z}_{\geq 0}, \text{par}^{(j)}(x) = y$ .

*Proof.* Our proof is by induction. According to line 4, all the properties hold when  $i = 0$ . Suppose all the properties hold for  $i - 1$ . For property 1, by induction we know there exists  $(x, y) \in S_{i-1}$  such that  $x = q$ . Then by line 8 and line 9, there must be an  $(x, y')$  in  $S_i$ . For property 2, by induction we know there exists  $(x, y) \in S_{i-1}$  such that  $y = g_r(q)$ . Thus, there must be an  $(x', y)$  in  $S_i$ . For property 3, if  $(x, y)$  is added into  $S_i$  by line 9, then  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(y) \leq 2^{k-i}$ . Otherwise, in line 8, we have  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(g_{k-i}(x)) \leq 2^{k-i}$ ,  $\text{dep}_{\text{par}}(g_{k-i}(x)) - \text{dep}_{\text{par}}(y) \leq 2^{k-i+1} - 2^{k-i} = 2^{k-i}$ . For property 4, if  $(x, y)$  is added into  $S_i$  by line 9, then by induction there is  $(y, y') \in S_{i-1}$ , and thus by line 9 and line 8, there must be  $(y, y'') \in S_i$ . Otherwise, in line 8 will generate two pairs  $(x, g_{k-i}(x)), (g_{k-i}(x), y)$ . For  $(x, g_{k-i}(x))$ , the property holds. For  $(g_{k-i}(x), y)$ , there must be  $(y, y') \in S_{i-1}$  and thus there should be  $(y, y'') \in S_i$ . For property 5, since  $g_{k-i}(x) = \text{par}^{(k-i)}(x)$ , for all pairs generated by line 8 and line 9, the property holds.  $\square$

By Claim C.16, we know

$$S_k = \{(q, \text{par}(q)), (\text{par}(q), \text{par}^{(2)}(q)), (\text{par}^{(2)}(q), \text{par}^{(3)}(q)), \dots, (\text{par}^{(\text{dep}_{\text{par}}(q)-1)}(q), \text{par}^{(\text{dep}_{\text{par}}(q))}(q))\}.$$

Thus,  $P$  is the set of all the vertices on the path from  $q$  to the root of  $q$ . And  $w = \text{par}^{(\text{dep}_{\text{par}}(q)-1)}(q)$  when  $\text{dep}_{\text{par}}(q) \geq 1$ .  $\square$

**Lemma C.17.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $q$  be a vertex in  $V$ . Let  $\widehat{\text{par}} = \text{ROOTCHANGE}(\text{par}, q)$  (Algorithm 9). Then  $\widehat{\text{par}} : V \rightarrow V$

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**Algorithm 9** Root Changing

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1: procedure ROOTCHANGE( $\text{par} : V \rightarrow V, q \in V$ ) ▷ Lemma C.17
2:   Output:  $\widehat{\text{par}} : V \rightarrow V$ .
3:    $(\text{dep}_{\text{par}}, P, w) = \text{FINDPATH}(\text{par}, q)$ . ▷ Algorithm 8
4:    $\forall v \in V \setminus P$ , let  $\widehat{\text{par}}(v) = \text{par}(v)$ .
5:   Let  $\widehat{\text{par}}(q) = q$ .
6:   Let  $h : \{0\} \cup [\text{dep}_{\text{par}}(q)] \rightarrow P$  such that  $\forall i \in \{0\} \cup [\text{dep}_{\text{par}}(q)], h(i) = x$  where  $\text{dep}_{\text{par}}(x) = i$ .
7:   for  $v \in P \setminus \{q\}$  do ▷ Reverse par of all the vertices on the path from  $q$  to the root of  $q$ .
8:     Let  $\widehat{\text{par}}(v) = h(\text{dep}_{\text{par}}(v) + 1)$ .
9:   end for
10:  return  $\widehat{\text{par}}$ .
11: end procedure
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is still a set of parent pointers (See Definition B.6) on  $V$ .  $\forall v \in V$ , if  $\text{par}^{(\infty)}(v) = \text{par}^{(\infty)}(q)$  then  $\widehat{\text{par}}^{(\infty)}(v) = q$ . Otherwise  $\widehat{\text{par}}^{(\infty)}(v) = \text{par}^{(\infty)}(v)$ .  $\forall u \neq v \in V, \text{par}(v) = u \Leftrightarrow$  either  $\widehat{\text{par}}(v) = u$  or  $\widehat{\text{par}}(u) = v$ . Furthermore,  $\text{dep}(\widehat{\text{par}}) \leq 2 \text{dep}(\text{par})$ .

*Proof.* For a vertex  $v \in V$ , if  $\{u \mid i \in \mathbb{Z}_{\geq 0}, u = \text{par}^{(i)}(v)\} \cap P = \emptyset$ , then we have  $\forall i \in \mathbb{Z}_{\geq 0}, \text{par}^{(i)}(v) = \widehat{\text{par}}^{(i)}(v)$ . According to Lemma C.15,  $P = \{u \in V \mid i \in \mathbb{Z}_{\geq 0}, \text{par}^{(i)}(q) = u\}$ . Then for all  $v \in P \setminus \{q\}$ , if  $\text{par}(u) = v$  then  $\widehat{\text{par}}(v) = u$ . Thus,  $\forall u \in P, \widehat{\text{par}}^{(\infty)}(u) = q$ . Let  $i^*$  be the smallest number such that  $\text{par}^{(i^*)}(v) \in P$ . Then  $\widehat{\text{par}}^{(i^*)}(v) \in P$ . Thus,  $\widehat{\text{par}}^{(\infty)}(v) = \widehat{\text{par}}^{(\infty)}(\widehat{\text{par}}^{(i^*)}(v)) = q$ . Furthermore, we have  $\forall v \in V, \text{dep}(\widehat{\text{par}}) \leq \text{dep}(\text{par}) + \text{dep}_{\text{par}}(q) \leq 2 \text{dep}(\text{par})$ .  $\square$

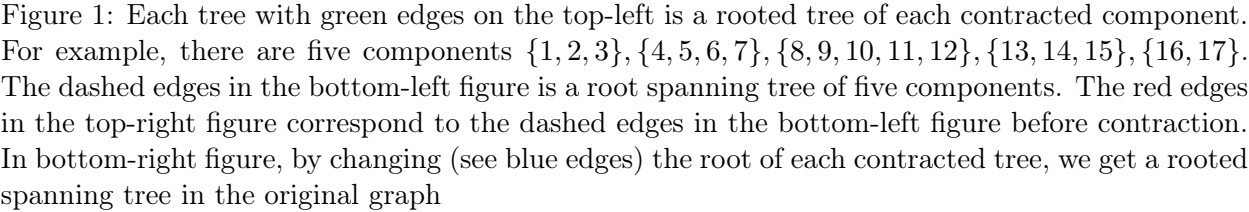
## C.4 Spanning Forest Expansion

In this section, we give the definition of spanning forest. If we are given a spanning forest of a contracted graph and spanning trees of each contracted component, then we show a procedure which can merge them to get a spanning forest of the original graph. Before go to the details, let us formally define the spanning forest.

**Definition C.18** (Rooted Spanning Forest). *Let  $G = (V, E)$  be an undirected graph. Let  $\text{par} : V \rightarrow V$  be a set of parent pointers which is compatible (Definition B.8) with  $G$ . If  $\forall u, v \in V, \text{dist}_G(u, v) < \infty \Rightarrow \text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , and  $\forall v \in V, \text{par}(v) \neq v \Rightarrow (v, \text{par}(v)) \in E$ , then we call  $\text{par}$  a rooted spanning forest of  $G$ .*

The Algorithm 10 shows how to combine the spanning forest in the contracted graph with local spanning trees to get a spanning forest in the graph before contraction. Figure 1 shows an example.

**Lemma C.19.** *Let  $G_2 = (V_2, E_2)$  be an undirected graph. Let  $\widetilde{\text{par}} : V_2 \rightarrow V_2$  be a set of parent pointers (See Definition B.6) which satisfies that  $\forall v \in V_2$  with  $\widetilde{\text{par}}(v) \neq v$ ,  $(v, \widetilde{\text{par}}(v))$  must be in  $E_2$ . Let  $G_1 = (V_1, E_1)$  be an undirected graph satisfies  $V_1 = \{v \in V_2 \mid \widetilde{\text{par}}(v) = v\}, E_1 = \{(u, v) \in V_1 \times V_1 \mid u \neq v, \exists (x, y) \in E_2, \widetilde{\text{par}}^{(\infty)}(x) = u, \widetilde{\text{par}}^{(\infty)}(y) = v\}$ . Let  $\text{par} : V_1 \rightarrow V_1$  be a rooted spanning forest (See Definition C.18) of  $G_1$ . Let  $f : V_1 \times V_1 \rightarrow \{\text{null}\} \cup (V_2 \times V_2)$  satisfy the following property: for  $u \neq v \in V_1$ , if  $\text{par}(u) = v$ , then  $f(u, v) \in \{(x, y) \in E_2 \mid \widetilde{\text{par}}^{(\infty)}(x) = u, \widetilde{\text{par}}^{(\infty)}(y) = v\}$ , and  $f(v, u) \in \{(x, y) \in E_2 \mid \widetilde{\text{par}}^{(\infty)}(x) = v, \widetilde{\text{par}}^{(\infty)}(y) = u\}$ . Let  $\widehat{\text{par}} = \text{FORESTEXPANSION}(\text{par}, \widetilde{\text{par}}, f)$ . Then  $\widehat{\text{par}} : V_2 \rightarrow V_2$  is a rooted spanning forest of  $G_2$ . In addition,  $\text{dep}(\widehat{\text{par}}) \leq (2 \cdot \text{dep}(\widetilde{\text{par}}) + 1)(\text{dep}(\text{par}) + 1)$ .*





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**Algorithm 10** Spanning Forest Expansion

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1: procedure FORESTEXPANSION( $\text{par} : V_1 \rightarrow V_1, \widehat{\text{par}} : V_2 \rightarrow V_2, f : V_1 \times V_1 \rightarrow \{\text{null}\} \cup (V_2 \times V_2)$ )
   ▷ Lemma C.19
2:   Output:  $\widehat{\text{par}} : V_2 \rightarrow V_2$ .
3:    $((V'_2, \emptyset), \widehat{\text{par}}^{(\infty)}) = \text{TREECONTRACTION}((V_2, \emptyset), \widehat{\text{par}})$ . ▷ Algorithm 2
4:   for  $v \in V_1$  do
5:     Let  $V_2(v) = \{u \in V_2 \mid \widehat{\text{par}}^{(\infty)}(u) = v\}$ .
6:     Let  $\widehat{\text{par}}_v : V_2(v) \rightarrow V_2(v)$  such that  $\forall u \in V_2(v), \widehat{\text{par}}_v(u) = \widehat{\text{par}}(u)$ .
7:     if  $\text{par}(v) \neq v$  then
8:       Let  $(x_v, y_v) = f(v, \text{par}(v))$ .
9:        $\widehat{\text{par}}_v = \text{ROOTCHANGE}(\widehat{\text{par}}_v, x_v)$ . ▷ Algorithm 9
10:      Let  $\widehat{\text{par}}(x_v) = y_v$ , and  $\forall u \in V_2(v) \setminus \{x_v\}, \widehat{\text{par}}(u) = \widehat{\text{par}}_v(u)$ .
11:     else  $\forall u \in V_2(v)$  let  $\widehat{\text{par}}(u) = \widehat{\text{par}}_v(u)$ .
12:     end if
13:   end for
14:   return  $\widehat{\text{par}}$ .
15: end procedure

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*Proof.* Let  $x, y \in V_2, u = \widehat{\text{par}}^{(\infty)}(x), v = \widehat{\text{par}}^{(\infty)}(y) \in V_1$  if  $\text{dist}_{G_2}(x, y) < \infty$ , then since  $E_1 = \{(u', v') \in V_1 \times V_1 \mid u' \neq v', \exists (x', y') \in E_2, \widehat{\text{par}}^{(\infty)}(x') = u', \widehat{\text{par}}^{(\infty)}(y') = v'\}$ , it must be true that  $\text{dist}_{G_1}(u, v) < \infty$ . Since  $\text{par}$  is a spanning forest of  $G_1$ , we have  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ . It suffices to say  $\forall x \in V_2, \widehat{\text{par}}^{(\infty)}(x) = \text{par}^{(\infty)}(\widehat{\text{par}}^{(\infty)}(x))$ . We can prove it by induction on  $\text{dep}_{\text{par}}(\widehat{\text{par}}^{(\infty)}(x))$ . Let  $u = \widehat{\text{par}}^{(\infty)}(x)$ . If  $\text{dep}_{\text{par}}(u) = 0$ , then  $\text{par}^{(\infty)}(u) = u$ . In this case, we have  $\widehat{\text{par}}^{(\infty)}(x) = \widehat{\text{par}}^{(\infty)}(x) = u = \text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(\widehat{\text{par}}^{(\infty)}(x))$ , and also we have  $\text{dep}_{\widehat{\text{par}}}(x) = \text{dep}_{\widehat{\text{par}}}(x)$ . Now suppose for all  $x \in V_2$  with  $\text{dep}_{\text{par}}(\widehat{\text{par}}^{(\infty)}(x)) \leq i - 1$ , it has  $\widehat{\text{par}}^{(\infty)}(x) = \text{par}^{(\infty)}(\widehat{\text{par}}^{(\infty)}(x))$  and  $\text{dep}_{\widehat{\text{par}}}(x) \leq i \cdot (2 \text{dep}(\widehat{\text{par}}) + 1)$ . Let  $y \in V_2$  satisfy  $\text{dep}_{\text{par}}(\widehat{\text{par}}^{(\infty)}(y)) = i$ . Let  $v = \widehat{\text{par}}^{(\infty)}(y)$ . By line 8 and the properties of  $f$ , we know  $\widehat{\text{par}}^{(\infty)}(x_v) = v$ , and  $\widehat{\text{par}}^{(\infty)}(y_v) = \text{par}(v)$ . By line 9, line 10 and Lemma C.17, we have  $\widehat{\text{par}}_v^{(\infty)}(y) = x_v, \widehat{\text{par}}(x_v) = y_v$ . Thus, there must be  $k \leq 2 \text{dep}_{\widehat{\text{par}}}(y)$  such that  $\widehat{\text{par}}^{(k)}(y) = x_v$ . Since  $\widehat{\text{par}}^{(\infty)}(y_v) = \text{par}^{(\infty)}(v)$  and  $\text{dep}_{\widehat{\text{par}}}(y_v) \leq i \cdot (2 \text{dep}(\widehat{\text{par}}) + 1)$ , we have  $\widehat{\text{par}}^{(\infty)}(y) = \text{par}^{(\infty)}(v) = \text{par}^{(\infty)}(\widehat{\text{par}}^{(\infty)}(y))$  and  $\text{dep}_{\widehat{\text{par}}}(y) \leq (i + 1) \cdot (2 \text{dep}(\widehat{\text{par}}) + 1)$ .

In addition, by the properties of  $f$  and Lemma C.17,  $\forall v \in V_2$  with  $\widehat{\text{par}}(v) \neq v$ , we have  $(v, \widehat{\text{par}}(v)) \in E_2$ . To conclude,  $\widehat{\text{par}} : V_2 \rightarrow V_2$  is a spanning forest of  $G_2$ , and  $\text{dep}(\widehat{\text{par}}) \leq (\text{dep}(\text{par}) + 1)(2 \text{dep}(\widehat{\text{par}}) + 1)$ .  $\square$

## C.5 Spanning Forest Algorithm

In this section, we show how to apply the ideas shown in connectivity algorithm to get an spanning forest algorithm. Algorithm 11 can output a spanning forest of a graph  $G$ , but the edges are not orientated. Then in the Algorithm 12, we assign each forest edge an direction thus it is a rooted spanning forest.

Before we prove the correctness of the algorithms, let us briefly introduce the meaning of each variables appeared in the algorithms.

In Algorithm 11,  $G_0$  is the original input graph, for  $i \in \{0\} \cup [r - 1]$ ,  $G'_i$  is obtained by deleting all the small size connected components in  $G_i$ , and  $G_{i+1}$  is obtained by contracting some vertices of  $G'_i$ . For a vertex  $v$  in graph  $G_i$ , if  $h_i(v) = \text{null}$ , then it means that the connected component which contains  $v$  is deleted when obtaining  $G'_i$ . If  $h_i(v) \neq \text{null}$ , it means that the vertex  $v$  is contracted

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**Algorithm 11** Undirected Graph Spanning Forest
 

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1: procedure SPANNINGFOREST( $G = (V, E), m, r$ ) ▷ Corollary C.24, Theorem C.29
2:   Output: FAIL or  $\{V_i \subseteq V \mid i \in \{0\} \cup [r]\}, \{\text{par}_i : V_i \rightarrow V_i \mid i \in \{0\} \cup [r-1]\}, \{h_i : V_i \rightarrow V_{i+1} \cup \{\text{null}\} \mid i \in \{0\} \cup [r-1]\}, F \subseteq E$ .
3:    $n_0 = n = |V|, G_0 = (V_0, E_0) = (V, E)$ .
4:   Let  $g_0 : E_0 \rightarrow E$  be an identity map.
5:   Let  $n'_0 = n_0$ .
6:   for  $i = 0 \rightarrow r - 1$  do
7:      $D_i \leftarrow \emptyset$ .
8:      $(\{\tilde{T}_i(v) \mid v \in V_i\}, \{\text{dep}_{\tilde{T}_i(v)} \mid v \in V_i\}) = \text{MULTIPLELARGETREES}(G_i, m)$ . ▷ Algorithm 6
9:     Let  $V'_i = \{v \in V_i \mid |V_{\tilde{T}_i(v)}| \geq \lceil (m/n_i)^{1/4} \rceil\}, E'_i = \{(u, v) \in E_i \mid u, v \in V'_i\}, G'_i = (V'_i, E'_i)$ .
10:     $\forall v \in V_i \setminus V'_i$ , let  $h_i(v) = \text{null}, u_v = \min_{u \in V_{\tilde{T}_i(v)}} u$ . Let  $\text{par}_i(v) = \text{par}_{\tilde{T}_i(u_v)}(v)$ .
11:     $\forall v \in V_i \setminus V'_i$ , if  $\text{par}_i(v) \neq v$ , then  $D_i \leftarrow D_i \cup \{g_i(\text{par}_i(v), v), g_i(v, \text{par}_i(v))\}$ .
12:    Let  $\gamma_i = \lceil (m/n_i)^{1/4} \rceil, p_i = \min((30 \log(n) + 100)/\gamma_i, 1/2)$ .
13:    Let  $l_i : V'_i \rightarrow \{0, 1\}$  be chosen randomly s.t.  $\forall v \in V'_i, l_i(v)$  are i.i.d. Bernoulli random variables with  $\Pr(l_i(v) = 1) = p_i$ .
14:    Let  $L_i = \{v \in V'_i \mid l_i(v) = 1\} \cup \{v \in V'_i \mid \forall u \in V_{\tilde{T}_i(v)}, l_i(u) = 0\}$ .
15:    For  $v \in V'_i$ , let  $z_i(v) = \arg \min_{u \in L_i \cap V_{\tilde{T}_i(v)}} \text{dep}_{\tilde{T}_i(v)}(u)$ . If  $z_i(v) = v$ , let  $\text{par}_i(v) = v$ .
16:    Otherwise,  $(\text{dep}_{\tilde{T}_i(v)}, P_i(v), w_i(v)) = \text{FINDPATH}(\text{par}_{\tilde{T}_i(v)}, z_i(v))$ , and let  $\text{par}_i(v) = w_i(v)$ . ▷ Algorithm 8
17:    Let  $((V_{i+1}, E_{i+1}), \text{par}_i^{(\infty)}) = \text{TREECONTRACTION}(G'_i, \text{par}_i : V'_i \rightarrow V'_i)$ . ▷ Algorithm 2
18:     $G_{i+1} = (V_{i+1}, E_{i+1}), n_{i+1} = |V_{i+1}|$ .
19:     $\forall v \in V'_i, h_i(v) = \text{par}_i^{(\infty)}(v)$ . If  $\text{par}_i(v) \neq v$ , then  $D_i \leftarrow D_i \cup \{g_i(\text{par}_i(v), v), g_i(v, \text{par}_i(v))\}$ .
20:    Let  $g_{i+1} : E_{i+1} \rightarrow E$  satisfy  $g_{i+1}(u, v) = \min_{(x, y) \in E_i, h_i(x)=u, h_i(y)=v} g_i(x, y)$ .
21:    Let  $n'_{i+1} = n'_i + n_{i+1}$ . If  $n'_{i+1} > 40n$ , then return FAIL.
22:  end for
23:  If  $n_r \neq 0$ , return FAIL.
24:  Let  $F = \bigcup_{i \in \{0\} \cup [r-1]} D_i$ .
25:  return  $\{V_i \mid i \in \{0\} \cup [r]\}, \{\text{par}_i \mid i \in \{0\} \cup [r-1]\}, \{h_i \mid i \in \{0\} \cup [r-1]\}, F$ .
26: end procedure

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to the vertex  $h_i(v)$  when obtaining  $G_{i+1}$ .  $\text{par}_i$  is a rooted forest (may not be spanning) in graph  $G_i$ , if a tree from the forest is spanning in  $G_i$ , then all the vertex in that tree will be deleted when obtaining  $G'_i$ . Otherwise all the vertices in that tree will be contracted to the root, and the root will be one of the vertex in  $G_{i+1}$ . Since each connected component in  $G_{i+1}$  is obtained by contraction of some vertices in a connected component in  $G_i$ , each edge in  $G_{i+1}$  must correspond to an edge in  $G_i$  where the end vertices of the edge are contracted to different vertices. Thus, each edge in  $G_i$  should correspond to an edge in  $G$ , and  $g_i : E_i \rightarrow E$  records the such correspondence.  $D_i$  records the edges added to the spanning forest  $F$  in the  $i^{\text{th}}$  round. For each vertex  $v$  in graph  $G_i$ ,  $\tilde{T}_i(v)$  is a local shortest path tree (See definition C.1) which is either with a large size or is a spanning tree in the component of  $v$ .  $L_i$  is a set of random leaders in  $G'_i$  such that in each local shortest path tree  $\tilde{T}_i(v)$ , there is at least one leader shown in the tree. The following lemmas formally state the properties of the algorithm.

**Lemma C.20.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\text{diam}(G) = \text{diam}(G_0) \geq \text{diam}(G'_0) \geq \text{diam}(G_1) \geq \text{diam}(G'_1) \geq \dots \geq \text{diam}(G_r)$ .*

*Proof.* By property 3 of Lemma C.7,  $\forall [i] \in \{0\} \cup [r-1]$ , there is no edge between  $V_i \setminus V'_i$  and  $V'_i$ . Thus,  $\text{diam}(G'_i) \leq \text{diam}(G_i)$ . Then due to property 1 of Corollary B.13, we have  $\text{diam}(G_{i+1}) \leq$

$\text{diam}(G'_i)$ . □

**Lemma C.21.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\forall i \in \{0\} \cup [r-1]$ ,  $\text{dep}(\text{par}_i) \leq \min(\text{diam}(G), \lfloor (m/n_i)^{1/2} \rfloor)$ .*

*Proof.* Let  $v \in V_i$ . If  $v \in V_i \setminus V'_i$ , then due to property 3 of Lemma C.7, we have  $V_{\tilde{T}_i(v)} = V_{\tilde{T}_i(u_v)}$ . Due to Lemma C.20 and Lemma C.7, we have  $\text{dep}_{\text{par}_i}(v) \leq \text{dep}(\tilde{T}_i(u_v)) \leq \min(\text{diam}(G), \lfloor (m/n_i)^{1/2} \rfloor)$ . For  $v \in V_i$ , we define  $\text{dist}_{G_i}(v, L_i) = \min_{u \in L_i} \text{dist}_{G_i}(v, u)$ . By Lemma C.7, we know  $\text{dist}_{G_i}(v, L_i) = \text{dist}_{G_i}(v, z_i(v))$ . Since  $\tilde{T}_i(v)$  is a LSPT (See Definition C.1), by applying Lemma C.15, we know  $\text{dist}_{G_i}(v, L_i) = \text{dist}_{G_i}(w_i(v), L_i) + 1$ , and  $(v, w_i(v)) \in E_i$ . Thus, by induction on  $\text{dist}_{G_i}(v, L_i)$ , we can get  $\text{dep}_{\text{par}_i}(v) \leq \text{dist}_{G_i}(v, L_i)$ . By Lemma C.20 and Lemma C.7, we can conclude  $\text{dep}(\text{par}_i)(v) \leq \min(\text{diam}(G), \lfloor (m/n_i)^{1/2} \rfloor)$ . □

**Lemma C.22.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\forall i \in \{-1, 0\} \cup [r-1]$ , we can define*

$$\forall v \in V, h^{(i)}(v) = \begin{cases} v & i = -1; \\ h_i(h^{(i-1)}(v)) & h^{(i-1)}(v) \neq \text{null}; \\ \text{null} & \text{otherwise} . \end{cases}$$

*Then we have following properties:*

1. If  $h^{(i)}(v) \neq \text{null}$ , then  $h^{(i)}(v) \in V_{i+1}$ .
2.  $\forall u, v \in V, h^{(i)}(u) \neq h^{(i)}(v), (u, v) \in E$ , we have  $(h^{(i)}(u), h^{(i)}(v)) \in E_{i+1}$ .
3.  $\forall (x, y) \in E_{i+1}, (u, v) = g_{i+1}(x, y)$ , we have  $(u, v) \in E, h^{(i)}(u) = x, h^{(i)}(v) = y$ .

*Proof.* For property 1, we can prove it by induction. It is true for  $i = -1$ . If  $h_0(v) \neq \text{null}$ , we know  $h_0(v)$  must be assigned at line 19. Due to property 2 of Lemma B.10,  $h_0(v) \in V_1$ . Suppose  $\forall v \in V, h^{(i-1)}(v) \neq \text{null}$ , we have  $h^{(i-1)}(v) \in V_i$ . For a vertex  $v$  with  $h^{(i)}(v) \neq \text{null}$ , according to the definition of  $h^{(i)}(v)$ , we know  $h^{(i-1)}(v) \neq \text{null}$ . Let  $u = h^{(i-1)}(v)$ .  $u$  must be a vertex in  $G_i$  by the induction hypothesis. Since  $h^{(i)}(v) \neq \text{null}$ , we know  $h_i(u) \neq \text{null}$ . Thus,  $h_i(u)$  must be assigned at line 19. Due to property 2 of Lemma B.10,  $h_i(u)$  must be in  $G_{i+1}$ , which implies  $h^{(i)}(v) \in V_{i+1}$ .

For property 2, we can also prove it by induction. It is true for  $i = -1$ . If  $(u, v) \in E$ , then due to property 3 of Lemma C.7, either both  $u, v$  are in  $V'_0$  or both  $u, v$  are in  $V_0 \setminus V'_0$ . If both  $u, v$  are in  $V_0 \setminus V'_0$ , then  $h_0(u) = h_0(v) = \text{null}$ . Otherwise, if  $h_0(u) \neq h_0(v)$ , then due to property 3 of Lemma B.10,  $(h_0(u), h_0(v)) \in E_1$ . Now suppose we have  $\forall u, v \in V$ , if  $h^{(i-1)}(u) \neq h^{(i-1)}(v), (u, v) \in E$ , then  $(h^{(i-1)}(u), h^{(i-1)}(v)) \in E_i$ . Let  $(u, v) \in E, h^{(i)}(u) \neq h^{(i)}(v)$ . Let  $x = h^{(i-1)}(u), y = h^{(i-1)}(v)$ . Due to property 3 of Lemma C.7, either both  $x, y$  are in  $V'_i$  or both are in  $V_i \setminus V'_i$ . If  $x, y \in V_i \setminus V'_i$ , then  $h_i(x) = h_i(y) = \text{null}$  which contradicts to  $h^{(i)}(u) \neq h^{(i)}(v)$ . Thus, both of  $x, y \in V'_i$ . Then due to property 3 of Lemma B.10,  $(h_i(x), h_i(y)) \in E_{i+1}$ . Thus,  $(h^{(i)}(u), h^{(i)}(v)) \in E_{i+1}$ .

For property 3, we can prove it by induction. It is true for  $i = -1$ . Let us consider the case when  $i = 0$ . Due to property 3 of Lemma B.10 and the definition of  $g_0, g_1$ , we have  $\forall (x, y) \in E_1, (u, v) = g_1(x, y), h_0(u) = x, h_0(v) = y, (u, v) \in E$ . Now suppose the property holds for  $i - 1$ . Let  $(x, y) \in E_{i+1}$ . Then  $g_{i+1}(x, y) = g_i(x', y')$  for some  $(x', y') \in E_i, h_i(x') = x, h_i(y') = y$ . Let  $(u, v) = g_i(x', y')$ . By the induction hypothesis  $(u, v) \in E, h^{(i-1)}(u) = x', h^{(i-1)}(v) = y'$ . Thus,  $h^{(i)}(u) = x, h^{(i)}(v) = y$ . □

**Lemma C.23.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\forall i \in \{-1, 0\} \cup [r-1]$ , we can define*

$$\forall v \in V, h^{(i)}(v) = \begin{cases} v & i = -1 \\ h_i(h^{(i-1)}(v)) & h^{(i-1)}(v) \neq \text{null} \\ \text{null} & \text{otherwise.} \end{cases}$$

*Let  $\forall i \in \{0\} \cup [r]$ ,  $\widehat{G}_i = (V_i, \widehat{E}_i = \{(x, y) \mid (u, v) \in \bigcup_{j=i}^{r-1} D_j, h^{(j-1)}(u) = x, h^{(j-1)}(v) = y\})$ . Then  $\widehat{G}_i$  is a spanning forest of  $G_i$ .*

*Proof.* The proof is by induction. When  $i = r$ , since  $V_r = \emptyset$ ,  $\widehat{G}_r = (\emptyset, \emptyset)$  is a spanning forest of  $G_r$ . Now suppose  $\widehat{G}_{i+1}$  is a spanning forest of  $G_{i+1}$ . Let  $u, v \in V_i$ . By property 2, 3 of Lemma C.22, we have  $\widehat{E}_i \subseteq E_i$ . Thus, if  $\text{dist}_{G_i}(u, v) = \infty$ , then  $\text{dist}_{\widehat{G}_i}(u, v) = \infty$ . If  $\text{dist}_{G_i}(u, v) < \infty$ , there are several cases:

1. If  $h_i(u) = h_i(v) = \text{null}$ , then due to line 10, we know  $u_u = u_v$ , and  $\widetilde{T}_i(u_v)$  is a spanning tree of the component which contains  $u, v$ . Thus,  $\widehat{G}_i$  has a spanning tree of the component which contains  $u, v$ .
2. If  $h_i(u) = h_i(v) \neq \text{null}$ , then  $\text{par}_i : \{x \in V_i \mid h_i(x) = h_i(v)\} \rightarrow \{x \in V_i \mid h_i(x) = h_i(v)\}$  is a tree, and  $\forall y \in \{x \in V_i \mid h_i(x) = h_i(v)\}$ , if  $\text{par}_i(y) \neq y$ , then  $(y, \text{par}_i(y)) \in \widehat{E}_i$ . Since  $\widehat{G}_{i+1}$  does not have any cycle, there is a unique path from  $u$  to  $v$  in  $\widehat{G}_i$ .
3. If  $h_i(u) \neq h_i(v)$ , then neither of them can be null. Since  $\widehat{G}_{i+1}$  is a spanning forest on  $G_{i+1}$ , there must be a unique path from  $h_i(u)$  to  $h_i(v)$  in  $\widehat{G}_{i+1}$ . Suppose the path in  $\widehat{G}_{i+1}$  is  $h_i(u) = p_1 - p_2 - \dots - p_t = h_i(v)$ . Then there must be a sequence of vertices in  $G_i$ ,  $u = p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, \dots, p_{t,1}, p_{t,2} = v$  such that  $h_i(p_{j,1}) = h_i(p_{j,2}) = p_j$  and  $(p_{j-1,2}, p_{j,1}) \in \widehat{E}_i$ . Thus, there is a unique path from  $u$  to  $v$ .

Thus,  $\widehat{G}_i$  is a spanning forest of  $G_i$ . □

**Corollary C.24** (Correctness of Algorithm 11). *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\widehat{G}_0 = (V, F)$  is a spanning forest of  $G$ .*

*Proof.* Just apply Lemma C.23 for  $i = 0$  case. □

**Lemma C.25.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then  $\forall i \in \{-1, 0\} \cup [r-1]$ , we can define*

$$\forall v \in V, h^{(i)}(v) = \begin{cases} v & i = -1 \\ h_i(h^{(i-1)}(v)) & h^{(i-1)}(v) \neq \text{null} \\ \text{null} & \text{otherwise.} \end{cases}$$

*$\forall i \in \{0\} \cup [r-1], v \in V_i$  with  $\text{par}_i(v) \neq v$ , there exists  $(x, y) \in F$  such that  $h^{(i-1)}(x) = v, h^{(i-1)}(y) = \text{par}_i(v)$ .*

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**Algorithm 12** Rooted Spanning Forest

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1: procedure ORIENTATE( $\{V_i \mid i \in \{0\} \cup [r]\}$ ,  $\{\text{par}_i \mid i \in \{0\} \cup [r-1]\}$ ,  $\{h_i \mid i \in \{0\} \cup [r-1]\}$ ,  $F$ )
     $\triangleright$  Takes the output of Algorithm 11 as input.
     $\triangleright$  Theorem C.26

2:   Output:  $\text{par} : V_0 \rightarrow V_0$ .
3:   Let  $F_0 = F$ .
4:   for  $i = 0 \rightarrow r-1$  do
5:     Initialize  $F_{i+1} \leftarrow \emptyset$ ,  $f_{i+1} : V_{i+1} \times V_{i+1} \rightarrow \{\text{null}\}$ .
6:      $\forall (u, v) \in F_i$ ,  $h_i(u) \neq h_i(v)$ , let  $F_{i+1} \leftarrow F_{i+1} \cup \{(h_i(u), h_i(v))\}$ ,  $f_{i+1}(h_i(u), h_i(v)) \leftarrow (u, v)$ .
7:   end for
8:    $\widehat{\text{par}}_r : \emptyset \rightarrow \emptyset$ .
9:   for  $i = r \rightarrow 1$  do  $\triangleright \widehat{\text{par}}_i$  is the spanning forest of  $G_i$ .
10:    Let  $\tilde{V}_i = V_i \cup \{v \in V_{i-1} \mid h_{i-1}(v) = \text{null}, \text{par}_{i-1}(v) = v\}$ .
11:    Let  $\widetilde{\text{par}}_i : \tilde{V}_i \rightarrow \tilde{V}_i$  satisfy  $\forall v \in V_i$ ,  $\widetilde{\text{par}}_i(v) = \widehat{\text{par}}_i(v)$ , and  $\forall v \in \tilde{V}_i \setminus V_i$ ,  $\widetilde{\text{par}}_i(v) = v$ .
12:    Let  $\widehat{\text{par}}_{i-1} = \text{FORESTEXPANSION}(\widetilde{\text{par}}_i, \text{par}_{i-1}, f_i)$ .  $\triangleright$  Algorithm 10
13:  end for
14:  Return  $\widehat{\text{par}}_0$  as  $\text{par}$ .
15: end procedure

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*Proof.* By line 11, line 19,  $\forall i \in \{0\} \cup [r-1]$ ,  $v \in V_i$ ,  $\text{par}_i(v) \neq v$ , we have  $g_i(v, \text{par}_i(v))$ ,  $g_i(\text{par}_i(v), v) \in D_i \subseteq F$ . Since  $(\text{par}_i(v), v) \in E_i$ , by property 3 of Lemma C.22,  $(x, y) = g_i(v, \text{par}_i(v))$  satisfies  $h^{(i-1)}(x) = v$ ,  $h^{(i-1)}(y) = \text{par}_i(v)$ .  $\square$

**Theorem C.26** (Correctness of Algorithm 12). *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then let the output be the input of  $\text{ORIENTATE}(\cdot)$ , (Algorithm 12) and the output  $\text{par} : V \rightarrow V$  of  $\text{ORIENTATE}(\cdot)$  will be a rooted spanning forest (See Definition C.18) of  $G$ . Furthermore,  $\text{dep}(\text{par}) \leq O(\text{diam}(G))^r$ .*

*Proof.* The proof is by induction. We want to show  $\widehat{\text{par}}_i$  is a rooted spanning forest of  $G_i$ . When  $i = r$ , since  $V_r = \emptyset$ , the claim is true. Now suppose we have  $\widehat{\text{par}}_{i+1}$  is a spanning forest of  $G_{i+1}$ . Let  $\tilde{G}_{i+1} = (\tilde{V}_{i+1}, E_{i+1})$ . It is easy to see  $\widetilde{\text{par}}_{i+1} : \tilde{V}_{i+1} \rightarrow \tilde{V}_{i+1}$  is a spanning forest of  $\tilde{G}_{i+1}$ . An observation is  $\tilde{V}_{i+1} = \{v \in V_i \mid \text{par}_i(v) = v\}$ . Thus,  $\widetilde{\text{par}}_{i+1}, \text{par}_i$  satisfies the condition in Lemma C.19 when invoking  $\text{FORESTEXPANSION}(\widetilde{\text{par}}_{i+1}, \text{par}_i, f_{i+1})$ . By Lemma C.25, we know  $f_{i+1}$  also satisfies the condition in Lemma C.19 when we invoke  $\text{FORESTEXPANSION}(\widetilde{\text{par}}_{i+1}, \text{par}_i, f_{i+1})$ . Thus,  $\widehat{\text{par}}_i$  is a rooted spanning forest of  $G_i$  due to Lemma C.19.

By Lemma C.19, we have  $\text{dep}(\widehat{\text{par}}_i) \leq 16 \text{dep}(\widehat{\text{par}}_{i+1}) \text{diam}(G)$ . By induction, we have  $\text{dep}(\text{par}) \leq O(\text{diam}(G))^r$ .  $\square$

**Lemma C.27.** *Let  $G = (V, E)$  be an undirected graph,  $m$  be a parameter which is at least  $16|V|$ , and  $r \leq n$  be a round parameter. If  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL, then with probability at least 0.89,  $\sum_{i=0}^r n_i \leq 40n$ .*

*Proof.* Since  $V_r \subseteq V_{r-1} \subseteq \dots \subseteq V_0 = V$ , we have  $n_r \leq n_{r-1} \leq n_{r-2} \leq \dots \leq n$ . Due to line 14, line 15 and line 17, we know  $\forall i \in \{0\} \cup [r-1]$ ,  $V_{i+1} = L_i$ . If  $p_i < 1/2$ , we know  $p_i = (30 \log(n) + 100)/\gamma_i$ . Since  $|V_{\tilde{T}_i}(v)| \geq \gamma_i$ , we can apply Lemma B.4 to get  $\Pr(|L_i| \leq 1.5p_i n_i) \geq \Pr(|L_i| \leq 0.75n_i) \geq 1 - 1/(100n)$ . By taking union bound over all  $i \in \{0\} \cup [r-1]$ , with probability at least 0.99, if  $p_i < 0.5$ , then

$n_{i+1} \leq 0.75n_i$ . By applying Lemma B.5, condition on  $n_i$  and  $p_i = \frac{1}{2}$ , we have  $\mathbf{E}(n_{i+1}) \leq 0.75n_i$ . By Markov's inequality, with probability at 0.89, we have  $\sum_{i=0}^r n_i \leq 40n$ .  $\square$

Now let us define the total iterations of Algorithm 11 as the following:

**Definition C.28** (Total iterations). *Let graph  $G = (V, E)$ ,  $m \leq \text{poly}(|V|)$  be a parameter which is at least  $16|V|$ , and  $r$  be a rounds parameter. The total number of iterations of  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) is defined as  $\sum_{i=0}^{r-1} (k_i + k'_i)$ , where  $\forall i \in \{0\} \cup [r-1]$ ,  $k_i$  denotes the number of iterations (See Definition C.10) of  $\text{MULTIPLELARGETREES}(G_i, m)$  (see line 8), and  $k'_i$  denotes the number of iterations (See Definition B.12) of  $\text{TREECONTRACTION}(G'_i, \text{par}_i)$  (see line 17).*

**Theorem C.29** (Success probability of Algorithm 11). *Let  $G = (V, E)$  be an undirected graph. Let  $m \leq \text{poly}(n)$  and  $m \geq 16|V|$ . Let  $r$  be a rounds parameter. Let  $c > 0$  be a sufficiently large constant. If  $r \geq c \log \log_{m/n} n$ , then with probability at least 0.79,  $\text{SPANNINGFOREST}(G, m, r)$  (Algorithm 11) does not return FAIL. Furthermore, let  $\forall i \in \{0\} \cup [r-1]$ ,  $k_i$  be the number of iterations (See Definition C.10) of  $\text{MULTIPLELARGETREES}(G_i, m)$  and  $k'_i$  be the number of iterations (See Definition B.12) of  $\text{TREECONTRACTION}(G'_i, \text{par}_i : V'_i \rightarrow V'_i)$ . Let  $c_1 > 0$  be a sufficiently large constant. If  $m \geq c_1 n \log^8 n$ , then with probability at least 0.99,  $\sum_{i=0}^{r-1} k_i + k'_i \leq O(\min(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n), r \log(\text{diam}(G))))$ . If  $m < c_1 n \log^8 n$ , then with probability at least 0.98,  $\sum_{i=0}^{r-1} k'_i + k_i \leq O(\min(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n) + (\log \log n)^2, r \log(\text{diam}(G))))$ .*

*Proof.* Due to Lemma C.27, with probability at last 0.89, we have  $\forall i \in [r]$ ,  $n'_i \leq 40n$ . Thus, we can condition on that  $\text{SPANNINGFOREST}(G, m, r)$  will not fail on line 21.

Due to Lemma C.11,  $k_i \leq O(\log(\text{diam}(G_i))) \leq O(\log(\text{diam}(G)))$ . Due to Corollary B.13 and Lemma C.21,  $k'_i \leq O(\log(\text{diam}(G)))$ . Thus,  $\sum_{i \in \{0\} \cup [r-1]} k'_i + k_i \leq O(r \log(\text{diam}(G)))$ .

Since  $V_r \subseteq V_{r-1} \subseteq V_{r-2} \subseteq \dots \subseteq V_0 = V$ , we have  $n_r \leq n_{r-1} \leq n_{r-2} \leq \dots \leq n$ . Due to line 14, line 15 and line 17, we know  $\forall i \in \{0\} \cup [r-1]$ ,  $V_{i+1} = L_i$ . If  $p_i < 1/2$ , we know  $p_i = (30 \log(n) + 100)/\gamma_i$ . Since  $|V_{\tilde{T}_i}(v)| \geq \gamma_i$ , we can apply Lemma B.4 to get  $\Pr(|L_i| \leq 1.5p_i n_i) \geq 1 - 1/(100n)$ . By taking union bound over all  $i \in \{0\} \cup [r-1]$ , with probability at least 0.99, if  $p_i < 0.5$ , then  $n_{i+1} \leq 1.5p_i n_i \leq 0.75n_i$ . Let  $\mathcal{E}$  be the event that  $\forall i \in \{0\} \cup [r-1]$ , if  $p_i < 0.5$ , then  $n_{i+1} \leq 1.5p_i n_i$ . Now, we suppose  $\mathcal{E}$  happens.

If  $p_0 = 0.5$ , then  $m \leq n \cdot (600 \log n)^8$ . By applying Lemma B.5,  $\mathbf{E}(n_{i+1}) = \mathbf{E}(|L_i|) \leq 0.75 \mathbf{E}(n_i) \leq \dots \leq 0.75^{i+1} n$ . By Markov's inequality, when  $i^* \geq 8 \log(6000 \log n) / \log(4/3)$ , with probability at least 0.99,  $n_{i^*} \leq n / (600 \log n)^8$  and thus  $p_{i^*} < 0.5$ . Condition on this event and  $\mathcal{E}$ , we have

$$\begin{aligned}
n_r &\leq \frac{\left( \frac{\left( \frac{n_{i^*}^{1.25}}{m^{0.25}} (45 \log n + 150) \right)^{1.25}}{m^{0.25}} (45 \log n + 150) \right)^{\dots}}{\dots} && \text{(Apply } r' = r - i^* \text{ times)} \\
&= n_{i^*} / (m/n_{i^*})^{1.25^{r'} - 1} \cdot (45 \log n + 150)^{4 \cdot (1.25^{r'} - 1)} \\
&\leq n / (m / (n_{i^*} (45 \log n + 150)^4))^{1.25^{r'} - 1} \\
&\leq n / (m / (n_{i^*} (45 \log n + 150)^4))^{1.25^{r'/2}} \leq n / (m/n)^{1.25^{r'/2}} \leq 1/2,
\end{aligned}$$

where the second inequality follows by  $n_{i^*} \leq n$ , the third inequality follows by  $r' \geq 5$ , the forth inequality follows by  $n_{i^*} \leq n / (600 \log n)^8$ , and the last inequality follows by  $r' \geq \frac{2}{\log 1.25} \log \log_{m/n}(2n)$ . Since  $16n \leq m \leq n \cdot (600 \log n)^8$ ,  $\log \log_{m/n} n = \Theta(\log \log n)$ . Let  $c > 0$  be a sufficiently large constant. Thus, when  $r \geq c \log \log_{m/n} n \geq i^* + r' = 8 \log(6000 \log n) / \log(4/3) + \frac{2}{\log 1.25} \log \log_{m/n}(2n)$ , with probability at least 0.98,  $n_r = 0$  implies that  $\text{SPANNINGFOREST}(G, m, r)$  will not fail. Due to



Lemma C.11, we have  $k_i \leq O(\min(\log(m/n_i), \log(\text{diam}(G))))$ . Thus,

$$\begin{aligned}
& \sum_{i=0}^{r-1} k_i \\
&= \sum_{i=0}^{i^*} k_i + \sum_{i=i^*+1}^{r-1} k_i \\
&\leq O((\log \log n)^2) + \sum_{i=i^*+1}^{r-1} k_i \\
&\leq O((\log \log n)^2) + \sum_{i: i \geq i^*+1, m/n_i \leq \text{diam}(G)} k_i + \sum_{i: i \leq r, m/n_i > \text{diam}(G)} k_i \\
&\leq O((\log \log n)^2) + O\left(\sum_{i=0}^{\lceil \log_{1.25} \log_2(\text{diam}(G)) \rceil} \log(2^{1.25^i})\right) + O\left(\sum_{i=0}^{\lceil \log_{1.25} \log_{\text{diam}(G)}(m) \rceil} \log(\text{diam}(G))\right) \\
&\leq O((\log \log n)^2) + O(\log(\text{diam}(G))) + O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n)) \\
&\leq O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n) + (\log \log(n))^2),
\end{aligned}$$

where the first inequality follows by  $i^* = O(\log \log n)$  and  $\forall i \leq [i^*], m/n_i \leq \text{poly}(\log n)$ , the third inequality follows by  $m/n_{i+1} \geq (m/n_i)^{1.25}/(45 \log n + 100) \geq (m/n_i)^{1.125}$ . Due to Corollary B.13 and Lemma C.21, we also have  $k'_i \leq O(\min(\log(m/n_i), \log(\text{diam}(G))))$ . Then, by the same argument, we have  $\sum_{i=0}^{r-1} k'_i = O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n) + (\log \log(n))^2)$ .

If  $m > n \cdot (600 \log n)^8$ , then  $\forall i \in \{0\} \cup [r-1]$ , we have  $p_i < 0.5$ . Since  $\mathcal{E}$  happens. We have:

$$\begin{aligned}
n_r &\leq \frac{\left(\frac{\left(\frac{n^{1.25}}{m^{0.25}}(45 \log n + 150)\right)^{1.25}}{m^{0.25}}(45 \log n + 150)\right)^{\dots}}{\dots} && \text{(Apply } r \text{ times)} \\
&= \frac{n^{1.25^r}}{m^{1.25^r-1}} (45 \log n + 150)^{4 \cdot (1.25^r-1)} \\
&= n / (m/n)^{1.25^r-1} \cdot (45 \log n + 150)^{4 \cdot (1.25^r-1)} \\
&= n / \left(m / (n(45 \log n + 150)^4)\right)^{1.25^r-1} \\
&\leq n / \left(m / (n(45 \log n + 150)^4)\right)^{1.25^{r/2}} \\
&\leq n / \left(m / (n(200 \log n)^4)\right)^{1.25^{r/2}} \\
&\leq \frac{1}{2},
\end{aligned}$$

where the second inequality follows by  $r \geq 5$ , the third inequality follows by  $45 \log n + 150 \leq 200 \log n$ , and the last inequality follows by

$$r \geq c \log \log_{m/n} n \geq 2 \log_{1.25} \log_{(m/n)^{1/2}} 2n \geq 2 \log_{1.25} \log_{m/(n(200 \log n)^4)} 2n,$$

for a sufficiently large constant  $c > 0$ . Since  $n_r$  is an integer,  $n_r$  must be 0 when  $n_r \leq 1/2$ . SPANNINGFOREST( $G, m, r$ ) will succeed with probability at least 0.99. Due to Lemma C.11, we



have  $k_i \leq O(\min(\log(m/n_i), \log(\text{diam}(G))))$ . Thus,

$$\begin{aligned} \sum_{i=0}^{r-1} k_i &\leq \sum_{m/n_i \leq \text{diam}(G)} k_i + \sum_{m/n_i > \text{diam}(G)} k_i \\ &\leq O\left(\sum_{i=0}^{\lceil \log_{1.25} \log_2(\text{diam}(G)) \rceil} \log(2^{1.25^i})\right) + O\left(\sum_{i=0}^{\lceil \log_{1.25} \log_{\text{diam}(G)}(m) \rceil} \log(\text{diam}(G))\right) \\ &\leq O(\log(\text{diam}(G))) + O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n)), \end{aligned}$$

where the second inequality follows by  $m/n_{i+1} \geq (m/n_i)^{1.25}/(45 \log n + 100) \geq (m/n_i)^{1.125}$ . Due to Corollary B.13 and Lemma C.21, we also have  $k'_i \leq O(\min(\log(m/n_i), \log(\text{diam}(G))))$ . Then, by the same argument, we have  $\sum_{i=0}^{r-1} k'_i = O(\log(\text{diam}(G))) + O(\log(\text{diam}(G)) \log \log_{\text{diam}(G)}(n))$ .  $\square$

## D Depth-First-Search Sequence for Tree and Applications

### D.1 Lowest Common Ancestor and Multi-Paths Generation

Given a rooted forest induced by  $\text{par} : V \rightarrow V$  which is a set of parent pointers (See Definition B.6) on  $V$ , and a set of  $q$  queries  $Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q) \mid u_i, v_i \in V\}$ , we show an algorithm which can return a mapping  $\text{lca} : Q \rightarrow (V \cup \{\text{null}\}) \times (V \cup \{\text{null}\}) \times (V \cup \{\text{null}\})$  such that  $\forall (u_i, v_i) \in Q, (p, p_{u_i}, p_{v_i}) = \text{lca}(u_i, v_i)$  satisfies the following properties:

1. If  $\text{par}^{(\infty)}(u_i) = \text{par}^{(\infty)}(v_i)$ , then  $p$  is the lowest ancestor of  $u_i$  and  $v_i$ . Otherwise  $p = p_{u_i} = p_{v_i} = \text{null}$ .
2. Suppose  $p \neq \text{null}$ . If  $p \neq u_i$ , then  $p_{u_i}$  is an ancestor of  $u_i$  and  $\text{par}(p_{u_i}) = p$ . Otherwise,  $p_{u_i} = \text{null}$ .
3. Suppose  $p \neq \text{null}$ . If  $p \neq v_i$ , then  $p_{v_i}$  is an ancestor of  $v_i$  and  $\text{par}(p_{v_i}) = p$ . Otherwise,  $p_{v_i} = \text{null}$ .

Before we describe the algorithms, let us formally define *ancestor* and the *lowest common ancestor*.

**Definition D.1** (Ancestor). Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . For  $u, v \in V$ , if  $\exists k \in \mathbb{Z}_{\geq 0}$  such that  $u = \text{par}^{(k)}(v)$ , then  $u$  is an ancestor of  $v$ .

**Definition D.2** (Common ancestor and the lowest common ancestor).  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . For  $u, v \in V$ , if  $w$  is an ancestor of  $u$  and is also an ancestor of  $v$ , then  $w$  is a common ancestor of  $(u, v)$ . If a common ancestor  $w$  of  $(u, v)$  satisfies  $\text{dep}_{\text{par}}(w) \geq \text{dep}_{\text{par}}(x)$  for any common ancestor  $x$  of  $(u, v)$ , then  $w$  is the lowest common ancestor (LCA) of  $(u, v)$ .

**Definition D.3** (Path between two vertices).  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . For  $u, v \in V$ , if  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , then the path from  $u$  to  $v$  is a sequence  $(x_1, x_2, \dots, x_j, x_{j+1}, \dots, x_k)$  such that  $\forall i \neq i' \in [k], x_i \neq x_{i'}, x_1 = u, x_k = v, x_j$  is the lowest common ancestor of  $(u, v)$ ,  $\forall i \in [j-1], \text{par}(x_i) = x_{i+1}$ , and  $\forall i \in \{j+1, j+2, \dots, k\}, \text{par}(x_i) = x_{i-1}$ .

The algorithm which can compute the lowest common ancestor is described in Algorithm 13.

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**Algorithm 13** Lowest Common Ancestor
 

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1: procedure LCA( $\text{par} : V \rightarrow V, Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\}$ ) ▷ Lemma D.4
2:   Output:  $\text{lca} : Q \rightarrow (V \cup \{\text{null}\}) \times (V \cup \{\text{null}\}) \times (V \cup \{\text{null}\})$ 
3:    $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$ . ▷ Algorithm 7
4:    $\forall (u, v) \in Q$ , if  $u = v$  then let  $\text{lca}(u, v) = (u, \text{null}, \text{null})$ ,  $Q \leftarrow Q \setminus \{(u, v)\}$ .
5:    $\forall (u, v) \in Q$ ,  $g_r(u) \neq g_r(v)$ , let  $\text{lca}(u, v) = (\text{null}, \text{null}, \text{null})$ .
6:   Let  $Q' = \emptyset$ .
7:    $\forall (u, v) \in Q$ ,  $g_r(u) = g_r(v)$ , if  $\text{dep}_{\text{par}}(u) \geq \text{dep}_{\text{par}}(v)$ , then let  $Q' \leftarrow Q' \cup \{(u, v)\}$ ; Otherwise let
    $Q' \leftarrow Q' \cup \{(v, u)\}$ .
8:   Let  $h_r : Q' \rightarrow Q'$  be an identity mapping.
9:   for  $i = r - 1 \rightarrow 0$  do ▷ Move  $u$  to the almost same depth as  $v$ .
10:    For each  $(u, v) \in Q'$ , let  $(x, v) = h_{i+1}(u, v)$ . If  $\text{dep}_{\text{par}}(x) - 2^i > \text{dep}_{\text{par}}(v)$ , then let  $h_i(u, v) =$ 
     $(g_i(x), v)$ ; Otherwise let  $h_i(u, v) = (x, v)$ .
11:  end for
12:  For each  $(u, v) \in Q'$  with  $\text{par}(h_0(u)) = v$ , if  $(u, v) \in Q$ , then let  $\text{lca}(u, v) = (v, h_0(u), \text{null})$ ; Otherwise
   $\text{lca}(v, u) = (v, \text{null}, h_0(u))$ .
13:  Let  $Q'' = \emptyset$ .
14:  For each  $(u, v) \in Q'$  with  $\text{par}(h_0(u)) \neq v$ ,  $\text{dep}_{\text{par}}(h_0(u)) > \text{dep}_{\text{par}}(v)$  let  $Q'' \leftarrow Q'' \cup \{(u, v)\}$ ,
   $h'_r(u, v) \leftarrow (\text{par}(h_0(u)), v)$ .
15:  For each  $(u, v) \in Q'$  with  $\text{dep}_{\text{par}}(u) = \text{dep}_{\text{par}}(v)$  let  $Q'' \leftarrow Q'' \cup \{(u, v)\}$ ,  $h'_r(u, v) \leftarrow (u, v)$ .
16:  for  $i = r - 1 \rightarrow 0$  do ▷ Move  $u, v$  to the lowest common ancestor.
17:    For each  $(u, v) \in Q''$ , let  $(x, y) = h'_{i+1}(u, v)$ . If  $g_i(x) \neq g_i(y)$ , then let  $h'_i(u, v) = (g_i(x), g_i(y))$ ;
    Otherwise let  $h'_i(u, v) = (x, y)$ .
18:  end for
19:  For each  $(u, v) \in Q''$ , if  $(u, v) \in Q$ , then let  $\text{lca}(u, v) = (\text{par}(h'_0(u)), h'_0(u), h'_0(v))$ ; Otherwise
   $\text{lca}(v, u) = (\text{par}(h'_0(v)), h'_0(v), h'_0(u))$ .
20:  return  $\text{lca}$ .
21: end procedure

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**Lemma D.4.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\}$  be a set of  $q$  pairs of vertices, and  $\forall i \in [q], u_i \neq v_i$ . Let  $\text{lca} = \text{LCA}(\text{par}, Q)$  (Algorithm 13). Then for any  $(u, v) \in Q$ ,  $(p, p_u, p_v) = \text{lca}(u, v)$  satisfies the following properties:

1. If  $\text{par}^{(\infty)}(u) \neq \text{par}^{(\infty)}(v)$ , then  $p = p_u = p_v = \text{null}$ .
2. If  $u$  (or  $v$ ) is the lowest common ancestor of  $(u, v)$ , then  $p = u, p_u = \text{null}, p_v \neq u$  is an ancestor of  $v$  such that  $\text{par}(p_v) = u$  (or  $p = v, p_v = \text{null}, p_u \neq v$  is an ancestor of  $u$  such that  $\text{par}(p_u) = v$ .)
3. If neither  $u$  nor  $v$  is the lowest common ancestor of  $(u, v)$  and  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , then  $p$  is the lowest common ancestor of  $(u, v)$ ,  $p_u \neq p$  is an ancestor of  $u$ ,  $p_v \neq p$  is an ancestor of  $v$ , and  $\text{par}(p_u) = \text{par}(p_v) = p$ .

*Proof.* According to Lemma C.13,  $r$  should be at most  $\lceil \log(\text{dep}(\text{par}) + 1) \rceil$ ,  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$ , and  $\forall i \in \{0\} \cup [r], v \in V$   $g_i(v) = \text{par}^{(2^i)}(v)$ . Then property 1 follows by line 5 directly.

Then for all  $(u, v) \in Q$  with  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , either  $(u, v) \in Q'$  or  $(v, u) \in Q'$ . For each  $(u, v) \in Q'$ , we have  $\text{dep}_{\text{par}}(u) \geq \text{dep}_{\text{par}}(v)$ . For all  $(u, v) \in Q'$ , with  $\text{dep}_{\text{par}}(u) > \text{dep}_{\text{par}}(v)$ , by induction we can prove that  $\forall i \in \{0\} \cup [r - 1], (x, y) = h_i(u, v)$  satisfies that  $x$  is an ancestor of  $u$ ,  $y = v$ ,  $\text{dep}_{\text{par}}(x) > \text{dep}_{\text{par}}(v)$  and  $\text{par}^{(2^i)}(x)$  is an ancestor of  $v$ . Thus, for  $(p, p_u, p_v) = \text{lca}(u, v)$ , if  $v$

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**Algorithm 14** Multiple Paths
 

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1: procedure MULTIPATH( $\text{par} : V \rightarrow V, Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\}$ )  $\triangleright$  Lemma D.5
2:   Output:  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}, \{P_i \subseteq V \mid i \in [q]\}$ .
3:    $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$ .  $\triangleright$  Algorithm 7
4:    $\forall j \in [q]$ , let  $S_j^{(0)} = \{(u_j, v_j) \mid (u_j, v_j) \in Q\}$ .
5:   for  $i = 1 \rightarrow r$  do
6:     for  $j = 1 \rightarrow q$  do  $\triangleright S_j^{(i)}$  is a set of segments partitioned the path from  $u_j$  to  $v_j$ .
7:       Let  $S_j^{(i)} \leftarrow \emptyset$ .
8:       for  $(x, y) \in S_j^{(i-1)}$  do
9:         if  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(y) > 2^{r-i}$  then  $S_j^{(i)} \leftarrow S_j^{(i)} \cup \{(x, g_{r-i}(x)), (g_{r-i}(x), y)\}$ .
10:        else  $S_j^{(i)} \leftarrow S_j^{(i)} \cup \{(x, y)\}$ .
11:        end if
12:      end for
13:    end for
14:  end for  $\triangleright S_j^{(r)}$  only contains segments with length 1
15:  Let  $\forall j \in [q], P_j \leftarrow \{u_j\}$ .
16:  for  $j = 1 \rightarrow q$  do
17:    for  $(x, y) \in S_j^{(r)}$  do
18:      Let  $P_j \leftarrow P_j \cup \{y\}$ .
19:    end for
20:  end for
21: end procedure

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is the lowest common ancestor of  $u$ , then we have  $p = v, p_u = h_0(u), p_v = \text{null}$ . In this case,  $h_0(u)$  is an ancestor of  $u$ , and  $\text{dep}_{\text{par}}(u) = \text{dep}_{\text{par}}(v) + 1, \text{par}(h_0(u)) = v$ . Thus, property 2 holds.

For all  $(u, v) \in Q$  with  $\text{par}^{(\infty)}(u) = \text{par}^{(\infty)}(v)$ , if neither  $u$  nor  $v$  is the lowest common ancestor of  $(u, v)$ , then we know either  $(u, v)$  or  $(v, u)$  is in  $Q''$ . Now let  $(u, v) \in Q''$ . We have  $\text{dep}_{\text{par}}(h'_r(u)) = \text{dep}_{\text{par}}(h'_r(v)), h'_r(u) \neq h'_r(v)$ , and  $h'_r(u), h'_r(v)$  are ancestors of  $u, v$  respectively. We can prove by induction to get  $\forall i \in \{0\} \cup [r], h'_i(u) \neq h'_i(v)$  and  $\text{par}^{(2^i)}(h'_i(u)) = \text{par}^{(2^i)}(h'_i(v))$  is a common ancestor of  $(u, v)$ . Thus,  $p = \text{par}(h'_0(u)) = \text{par}(h'_0(v))$  is the lowest common ancestor of  $(u, v)$ , and  $\text{dep}_{\text{par}}(h'_0(u)) = \text{dep}_{\text{par}}(h'_0(v)) = \text{dep}_{\text{par}}(p) + 1$ . Since  $p_u = h'_0(u), p_v = h'_0(v)$ , property 3 holds.  $\square$

In Algorithm 14, we show a generalization of Algorithm 8 such that we can find multiple vertex-to-ancestor paths simultaneously.

The following lemma claims the properties of the outputs of Algorithm 14. And the proof is similar to the proof of Lemma C.15.

**Lemma D.5.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\} \subseteq V \times V$  satisfy  $\forall j \in [q], v_j$  is an ancestor (See Definition D.1) of  $u_j$  in  $\text{par}$ . Let  $(\text{dep}_{\text{par}}, \{P_j \mid j \in [q]\}) = \text{MULTIPATH}(\text{par}, Q)$  (Algorithm 14). Then  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$  and  $\forall j \in [q], P_j \subseteq V$  is the set of all vertices on the path from  $u_j$  to  $v_j$ , i.e.  $P_j = \{v \in V \mid \exists k_1, k_2 \in \mathbb{Z}_{\geq 0}, v = \text{par}^{(k_1)}(u_j), v_j = \text{par}^{(k_2)}(v)\}$ . Furthermore,  $r$  should be at most  $\lceil \log(\text{dep}(\text{par}) + 1) \rceil$ .*

*Proof.* By Lemma C.13, since  $(r, \text{dep}_{\text{par}}, \{g_i \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$ , we know  $r$  should be at most  $\lceil \log(\text{dep}(\text{par}) + 1) \rceil$ ,  $\text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}$  records the depth of every vertex in  $V$ ,

and  $\forall i \in \{0\} \cup [r], v \in V \ g_i(v) = \text{par}^{(2^i)}(v)$ .

For  $j \in [q]$ , let us prove that  $P_j$  is the vertex set of all the vertices on the path from  $u_j$  to its ancestor  $v_j$ . We use divide-and-conquer to get  $P_j$ . The following claim shows that  $S_j^{(i)}$  is a set of segments which is a partition of the path from  $u_j$  to  $v_j$ , and each segment has length at most  $2^{r-i}$ .

**Claim D.6.**  $\forall i \in \{0\} \cup [r], j \in [q] \ S_j^{(i)}$  satisfies the following properties:

1.  $\exists (x, y) \in S_j^{(i)}$  such that  $x = u_j$ .
2.  $\exists (x, y) \in S_j^{(i)}$  such that  $y = v_j$ .
3.  $\forall (x, y) \in S_j^{(i)}, \text{dep}_{\text{par}}(y) - \text{dep}_{\text{par}}(x) \leq 2^{r-i}$ .
4.  $\forall (x, y) \in S_j^{(i)}$ , if  $y \neq v_j$ , then  $\exists (x', y') \in S_j^{(i)}, x' = y$ .
5.  $\forall (x, y) \in S_j^{(i)}, \exists k \in \mathbb{Z}_{\geq 0}, \text{par}^{(k)}(x) = y$ .

*Proof.* We fix a  $j \in [q]$ . Our proof is by induction. According to line 4, all the properties hold when  $i = 0$ . Suppose all the properties hold for  $i - 1$ . For property 1, by induction we know there exists  $(x, y) \in S_j^{(i-1)}$  such that  $x = u_j$ . Then by line 9 and line 10, there must be an  $(x, y')$  in  $S_j^{(i)}$ . For property 2, by induction we know there exists  $(x, y) \in S_j^{(i-1)}$  such that  $y = v_j$ . Thus, there must be an  $(x', y)$  in  $S_j^{(i)}$ . For property 3, if  $(x, y)$  is added into  $S_j^{(i)}$  by line 10, then  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(y) \leq 2^{r-i}$ . Otherwise, in line 9, we have  $\text{dep}_{\text{par}}(x) - \text{dep}_{\text{par}}(g_{r-i}(x)) \leq 2^{r-i}, \text{dep}_{\text{par}}(g_{r-i}(x)) - \text{dep}_{\text{par}}(y) \leq 2^{r-i+1} - 2^{r-i} = 2^{r-i}$ . For property 4, if  $(x, y)$  is added into  $S_j^{(i)}$  by line 10, then by induction there is  $(y, y') \in S_j^{(i-1)}$ , and thus by line 10 and line 9, there must be  $(y, y'') \in S_j^{(i)}$ . Otherwise, in line 9 will generate two pairs  $(x, g_{r-i}(x)), (g_{r-i}(x), y)$ . For  $(x, g_{r-i}(x))$ , the property holds. For  $(g_{r-i}(x), y)$ , there must be  $(y, y') \in S_{i-1}$  and thus there should be  $(y, y'') \in S^{(i)}$ . For property 5, since  $g_{r-i}(x) = \text{par}^{(r-i)}(x)$ , for all pairs generated by line 9 and line 10, the property holds.  $\square$

By Claim D.6, we know

$$S_j^{(r)} = \{ \begin{aligned} & (u_j, \text{par}(u_j)), \\ & (\text{par}(u_j), \text{par}^{(2)}(u_j)), \\ & (\text{par}^{(2)}(u_j), \text{par}^{(3)}(u_j)), \\ & \dots, \\ & (\text{par}^{(\text{dep}_{\text{par}}(u_j) - \text{dep}_{\text{par}}(v_j) - 1)}(u_j), \text{par}^{(\text{dep}_{\text{par}}(u_j) - \text{dep}_{\text{par}}(v_j))}(u_j)) \end{aligned} \}.$$

Thus,  $P_j$  is the set of all the vertices on the path from  $u_j$  to an ancestor  $v_j$ .  $\square$

## D.2 Depth-First-Search Sequence for a Tree

Since we can use our spanning tree algorithm to get a rooted tree, in this section, we only consider how to get a Depth-First-Search (DFS) sequence for a rooted tree. Before we go to the details, let us firstly give formal definitions of some useful concepts.

**Definition D.7** (Children in the forest). *Given a set of parent pointers (See Definition B.6)  $\text{par} : V \rightarrow V$  on a vertex set  $V$ .  $\forall u, v \in V, u \neq v$  if  $\text{par}(u) = v$ , then we say  $u$  is a child of  $v$ .  $\forall v \in V$ , we can define  $\text{child}_{\text{par}}(v)$  as the set of all children of  $v$ , i.e.  $\text{child}_{\text{par}}(v) = \{u \in V \mid u \neq v, \text{par}(u) = v\}$ . Furthermore, if  $u$  is the  $k^{\text{th}}$  smallest vertex in the children set  $\text{child}_{\text{par}}(v)$ , then we say  $\text{rank}_{\text{par}}(u) = k$ , or  $u$  is the  $k^{\text{th}}$  child of  $v$ . If  $\text{par}(v) = v$ , then  $\text{rank}_{\text{par}}(v) = 1$ . We use  $\text{child}_{\text{par}}(v, k)$  to denote the  $k^{\text{th}}$  child of  $v$ .*

For simplicity of the notation, if  $\text{par} : V \rightarrow V$  is clear in the context, we just use  $\text{child}(v)$ ,  $\text{rank}(v)$  and  $\text{child}(v, k)$  to denote  $\text{child}_{\text{par}}(v)$ ,  $\text{rank}_{\text{par}}(v)$  and  $\text{child}_{\text{par}}(v, k)$  respectively.

**Definition D.8** (Leaves in the forest). *Given a set of parent pointers (See Definition B.6)  $\text{par} : V \rightarrow V$  on a vertex set  $V$ . If  $\text{child}_{\text{par}}(v) = \emptyset$ , then  $v$  is called a leaf. The set of all the leaves of  $\text{par}$  is defined as  $\text{leaves}(\text{par}) = \{v \mid \text{child}_{\text{par}}(v) = \emptyset\}$ .*

**Definition D.9** (Subtree). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $v \in V, V' = \{u \in V \mid v \text{ is an ancestor (Definition D.1) of } u\}$ . Let  $\text{par}' : V' \rightarrow V'$  be a set of parent pointers on  $V'$ . If  $\forall u \in V' \setminus \{v\}, \text{par}'(u) = \text{par}(u)$ , and  $\text{par}'(v) = v$ , then we say  $\text{par}'$  is the subtree of  $v$  in  $\text{par}$ . For  $u \in V'$ , we say  $u$  is in the subtree of  $v$ .*

**Definition D.10** (Depth-First-Search (DFS) sequence). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $v$  be a vertex in  $V$ . If  $v$  is a leaf (See Definition D.8) in  $\text{par}$ , then the DFS sequence of the subtree (See Definition D.9) of  $v$  is  $(v)$ . Otherwise the DFS sequence of the subtree of  $v$  in  $\text{par}$  is recursively defined as*

$$(v, a_{1,1}, a_{1,2}, \dots, a_{1,n_1}, v, a_{2,1}, a_{2,2}, \dots, a_{2,n_2}, v, \dots, a_{k,1}, a_{k,2}, \dots, a_{k,n_k}, v),$$

where  $k = |\text{child}(v)|$  is the number of children (See Definition D.7) of  $v$ , and  $\forall i \in [k], (a_{i,1}, \dots, a_{i,n_i})$  is the DFS sequence of the subtree of  $\text{child}(v, i)$ , i.e. the  $i^{\text{th}}$  child of  $v$ .

If  $\forall u \in V, \text{par}^{(\infty)}(u) = v$ , then the subtree of  $v$  is exactly  $\text{par}$ , and thus the DFS sequence of the subtree of  $v$  is also called the DFS sequence of  $\text{par}$ .

Here are some useful facts of the above defined DFS sequence.

**Fact D.11.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $A = (a_1, a_2, \dots, a_m)$  be the DFS sequence (See Definition D.10) of  $\text{par}$ . Then,  $A$  satisfies the following properties:*

1.  $\forall v \in V$ ,  $v$  appears exactly  $|\text{child}(v)| + 1$  times in  $A$ .
2. If  $a_i$  is the  $k^{\text{th}}$  time that  $v$  appears, and  $a_j$  is the  $(k+1)^{\text{th}}$  time that  $v$  appears. Then  $(a_{i+1}, a_{i+2}, \dots, a_{j-1})$  is the DFS sequence of the subtree of  $\text{child}(v, k)$  (See Definition D.7), the  $k^{\text{th}}$  child of  $v$ . Furthermore,  $a_{i+1}$  is the first time that  $\text{child}(v, k)$  appears, and  $a_{j-1}$  is the last time of  $\text{child}(v, k)$  appears.
3. If  $a_i$  is the first time that  $v$  appears, and  $a_j$  is the last time that  $v$  appears. Then  $(a_i, a_{i+1}, \dots, a_j)$  is the DFS sequence of the subtree of  $v$ .

4.  $m = 2|V| - 1$ .

*Proof.* The property 1, 2, 3 directly follows by Definition D.10.

For property 4, notice that  $\forall u \in V, \text{par}(u) \neq u$ ,  $u$  can only be a child of  $\text{par}(u)$ . Thus,  $\sum_{v \in V} (|\text{child}(v)| + 1) = |V| - 1 + |V| = 2|V| - 1$ .  $\square$

Due to the above fact, if  $v$  is a leaf in  $\text{par}$ , then it will only once in the DFS sequence. Thus, we are able to determine the order of all the leaves.

**Definition D.12** (The order of the leaves). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $A = (a_1, a_2, \dots, a_m)$  be the DFS sequence (See Definition D.10) of  $\text{par}$ . Let  $u, v$  be two leaves (See definition D.8) of  $\text{par}$ . If  $u$  appears before  $v$  in  $A$ , then we say  $u <_{\text{par}} v$ .*

### D.2.1 Leaf Sampling

Given a set of rooted trees, our goal is to sample a set of leaves for each tree, and to give an order of those sampled leaves. The algorithm is shown in Algorithm 15.

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#### Algorithm 15 Leaf Sampling

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1: procedure LEAFSAMPLING( $\text{par} : V \rightarrow V, m, \delta$ ) ▷ Lemma D.13
2:   Output:  $A = (a_1, a_2, \dots, a_s)$ .
3:   Let  $t = \lceil m^{1/3} \rceil$ .
4:   Compute  $L = \text{leaves}(\text{par})$ .
5:   Compute  $\text{rank} : V \rightarrow \mathbb{Z}_{\geq 0}$  such that  $\forall v \in V, \text{rank}(v) = \text{rank}_{\text{par}}(v)$ . ▷ Definition D.7
6:   If  $|V| \leq m$ , let  $\{a_1, a_2, \dots, a_s\} = L$ , and return  $A = (a_1, a_2, \dots, a_s)$  which satisfies  $a_1 <_{\text{par}}$ 
    $a_2 <_{\text{par}} \dots <_{\text{par}} a_s$ . ▷  $<_{\text{par}}$  follows Definition D.12
7:   If  $|L| \leq 8t$ , let  $S = L$ .
8:   Let  $p = \min(1, 640(1 + \log(m)/\delta)t/|L|)$ .
9:   If  $|L| > t$ , sample each  $v \in L$  with probability  $p$  independently. let  $S$  be the set of samples.
10:  Compute  $\text{par}' : V \rightarrow V$  such that  $\forall v \in V$ , if  $\text{child}_{\text{par}}(v) \neq \emptyset$ , then  $\text{par}'(v) = \text{child}_{\text{par}}(v, 1)$ ;
   Otherwise let  $\text{par}'(v) = v$ . ▷  $\text{par}'(v)$  points to  $v$ 's first child in  $\text{par}$ .
11:   $(r', \text{dep}_{\text{par}'} : V \rightarrow \mathbb{Z}_{\geq 0}, \{g'_i : V \rightarrow V \mid i \in \{0\} \cup [r']\}) = \text{FINDANCESTORS}(\text{par}')$ .
12:  Find  $w \in V$  with  $\text{par}(w) = w$ . ▷ Find the root.
13:  Let  $a_1 = g'_{r'}(w)$ ,  $S \leftarrow S \cup \{a_1\}$ . ▷ Find the first leaf.
14:  Let  $Q = \{(u, v) \mid (u, v) \in S \times S, u \neq v\}$ .
15:  Let  $\text{lca} = \text{LCA}(\text{par}, Q)$ . ▷ Algorithm 13
16:  Let  $s = |S|$ .
17:   $(r, \text{dep}_{\text{par}} : V \rightarrow \mathbb{Z}_{\geq 0}, \{g_i : V \rightarrow V \mid i \in \{0\} \cup [r]\}) = \text{FINDANCESTORS}(\text{par})$ .
18:  for  $i = 2 \rightarrow s$  do ▷ Determine the order of sampled leaves.
19:    For all  $x, y \in S \setminus \{a_1, a_2, \dots, a_{i-1}\}$ , let  $(p_{x,y}, p_{xy,x}, p_{xy,y}) = \text{lca}(x, y)$ .
20:    Find  $x^* \in S \setminus \{a_1, a_2, \dots, a_{i-1}\}$  s.t.  $\forall y \in S \setminus \{a_1, a_2, \dots, a_{i-1}, x^*\}$ ,  $\text{rank}(p_{x^*y,y}) <$ 
     $\text{rank}(p_{x^*y,y})$ .
21:    Let  $a_i = x^*$ .
22:  end for
23:  return  $A = (a_1, a_2, \dots, a_s)$ .
24: end procedure

```

---

**Lemma D.13.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $m > 0, \delta \in (0, 1)$  be parameters, and let  $|V| \leq m^{1/\delta}$ . Let  $(a_1, a_2, \dots, a_s) = \text{LEAFSAMPLING}(\text{par}, m, \delta)$  (Algorithm 15). Then it has following properties:

1.  $a_1 <_{\text{par}} a_2 <_{\text{par}} \dots <_{\text{par}} a_s$ .
2. If  $|V| \leq m$  or  $|\text{leaves}(\text{par})| \leq 8\lceil m^{1/3} \rceil$ , then  $\{a_1, a_2, \dots, a_s\} = \text{leaves}(\text{par})$ . Otherwise, with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $\forall v \in \text{leaves}(\text{par}) \setminus \{a_1\}$ , there is a vertex  $w \in \{a_1, a_2, \dots, a_s\}$  such that  $w <_{\text{par}} v$  and the number of leaves between  $w$  and  $v$  is at most  $|\text{leaves}(\text{par})|/\lceil m^{1/3} \rceil$ , i.e.  $|\{u \in \text{leaves}(\text{par}) \mid w <_{\text{par}} u <_{\text{par}} v\}| \leq |\text{leaves}(\text{par})|/\lceil m^{1/3} \rceil$ .
3. If  $|V| > m$  and  $|\text{leaves}(\text{par})| > 8\lceil m^{1/3} \rceil$ , then with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $s = |S| = |\{a_1, a_2, \dots, a_s\}| \leq 960\lceil m^{1/3} \rceil(1 + \log(m)/\delta)$ .

*Proof.* Firstly, let us focus on property 1. According to line 11 to line 13 and Lemma C.13, we know  $\forall k \in \mathbb{Z}_{\geq 0} \text{rank}_{\text{par}}(\text{par}^{(k)}(a_1)) = 1$ , and  $\text{par}'(a_1) = a_1$  which implies that  $a_1$  is a leaf. Due to the definition of D.10, we know that  $a_1$  must be the first leaf appeared in the DFS sequence of  $\text{par}$ . We can prove the property by induction. Suppose we already have  $a_1 <_{\text{par}} a_2 <_{\text{par}} \dots <_{\text{par}} a_{i-1}$ . According to line 19 and Lemma D.4,  $p_{a_{i-1}, a_i}$  is the LCA of  $(a_{i-1}, a_i)$ .  $p_{a_{i-1}, a_i}$  is a child of  $p_{a_{i-1}, a_i}$  and is an ancestor of  $a_{i-1}$ .  $p_{a_{i-1}, a_i}$  is a child of  $p_{a_{i-1}, a_i}$  and is an ancestor of  $a_i$ . By Fact D.11, since  $\text{rank}(p_{a_{i-1}, a_i, a_{i-1}}) < \text{rank}(p_{a_{i-1}, a_i, a_i})$ , we have  $a_{i-1} <_{\text{par}} a_i$ . To conclude, we have  $a_1 <_{\text{par}} a_2 <_{\text{par}} \dots <_{\text{par}} a_s$ .

For property 2, if  $|V| \leq m$  or  $|\text{leaves}(\text{par})| \leq 8\lceil m^{1/3} \rceil$ , then by line 6 and line 7, we know  $\{a_1, a_2, \dots, a_s\} = \text{leaves}(\text{par})$ .

Now consider the case when  $|V| > m$  and  $|\text{leaves}(\text{par})| > 8\lceil m^{1/3} \rceil$ . Let  $t = \lceil m^{1/3} \rceil$ . Let  $\text{leaves}(\text{par}) = \{u_1, u_2, \dots, u_q\}$ , and let  $u_1 <_{\text{par}} u_2 <_{\text{par}} \dots <_{\text{par}} u_q$ . Let us partition  $u_1, \dots, u_q$  into  $4 \cdot t$  groups  $G_1 = \{u_1, u_2, \dots, u_{\lfloor q/(4t) \rfloor}\}, G_2 = \{u_{\lfloor q/(4t) \rfloor + 1}, u_{\lfloor q/(4t) \rfloor + 2}, \dots, u_{2 \cdot \lfloor q/(4t) \rfloor}\}, \dots, G_{4t} = \{u_{(4t-1) \cdot \lfloor q/(4t) \rfloor + 1}, u_{(4t-1) \cdot \lfloor q/(4t) \rfloor + 2}, \dots, u_q\}$ . Then each group has size at least  $q/(8t)$  and at most  $q/(2t)$ . For a certain  $G_i$ , by Chernoff bound, we have

$$\begin{aligned} & \Pr \left( |G_i \cap S| \leq \frac{1}{2} \cdot \frac{q}{8t} \cdot p \right) \\ & \leq \exp \left( -\frac{1}{8} \cdot \frac{q}{8t} \cdot p \right) \\ & \leq 1/(100m^{10/\delta}) \end{aligned}$$

where the last inequality follows by  $p = \min(1, (10 + 10 \log(m)/\delta) \cdot 64t/q)$ . Notice that  $q \leq |V| \leq m^{1/\delta}$ . We can take union bound over all  $G_i$ . Then with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $\forall i \in [4t], G_i \cap S \neq \emptyset$ . Thus,  $\forall v \in \text{leaves}(\text{par})$ , there is a vertex  $w \in \{a_1, a_2, \dots, a_s\}$  such that  $w <_{\text{par}} v$  and the number of leaves between  $w$  and  $v$  is at most  $|\text{leaves}(\text{par})|/\lceil m^{1/3} \rceil$ , i.e.  $|\{u \in \text{leaves}(\text{par}) \mid w <_{\text{par}} u <_{\text{par}} v\}| \leq |\text{leaves}(\text{par})|/\lceil m^{1/3} \rceil$ .

For property 3, by applying Chernoff bound, we have

$$\begin{aligned} & \Pr \left( |S \cap \text{leaves}(\text{par})| \geq \frac{3}{2} |\text{leaves}(\text{par})| \cdot p \right) \\ & \leq \exp \left( -\frac{1}{12} \cdot |\text{leaves}(\text{par})| \cdot p \right) \\ & \leq 1/(100m^{10/\delta}) \end{aligned}$$

where the last inequality follows by  $p = \min(1, (10 + 10 \log(m)/\delta) \cdot 64t/|\text{leaves}(\text{par})|)$ .

Since  $\frac{3}{2} |\text{leaves}(\text{par})| \cdot p \leq 960\lceil m^{1/3} \rceil(1 + \log(m)/\delta)$ , we complete the proof.  $\square$



### D.2.2 DFS Subsequence

Let  $\text{par} : V \rightarrow V$  be a set of parent pointers on a vertex set  $V$ , and  $\text{par}$  has a unique root  $v$ . Let  $\{u_1, u_2, \dots, u_q\} = \text{leaves}(\text{par})$ , and  $u_1 <_{\text{par}} u_2 <_{\text{par}} \dots <_{\text{par}} u_q$ . One observation is that the DFS sequence of  $\text{par}$  can be generated in the following way:

1. The first part of the DFS sequence is the path from  $v$  to  $u_1$ .
2. Then it follows by the path from  $\text{par}(u_1)$  to the LCA of  $(u_1, u_2)$ , the path from one of the child of the LCA of  $(u_1, u_2)$  to  $u_2$ , the path from  $\text{par}(u_2)$  to the LCA of  $(u_2, u_3)$ , the path from one of the child of the LCA of  $(u_2, u_3)$  to  $u_3, \dots$ , the path from one of the child of the LCA of  $(u_{q-1}, u_q)$  to  $u_q$ .
3. The last part of the DFS sequence is a path from  $\text{par}(u_q)$  to  $v$ .

**Fact D.14.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root  $v$ . Let  $\{u_1, u_2, \dots, u_q\} = \text{leaves}(\text{par})$  (See Definition D.8), and  $u_1 <_{\text{par}} u_2 <_{\text{par}} \dots <_{\text{par}} u_q$ . Let  $A = (a_1, a_2, \dots, a_m)$  be the DFS sequence (See Definition D.10) of  $\text{par}$ . Then,

1. If  $u_1$  appears at  $a_i$ , then  $(a_1, a_2, \dots, a_i)$  is the path from  $v$  to  $u_1$ .
2.  $\forall i \in [q-1]$ , if  $u_i$  appears at  $a_j$ , and  $u_{i+1}$  appears at  $a_k$ , then  $\exists j < t < k$  such that  $a_t$  is the LCA of  $(u_i, u_{i+1})$ . In addition,  $(a_j, a_{j+1}, \dots, a_t)$  is the path from  $a_j$  to  $a_t$ , and  $(a_t, a_{t+1}, \dots, a_k)$  is the path from  $a_t$  to  $a_k$ .
3. If  $u_q$  appears at  $a_i$ , then  $(a_i, a_{i+1}, \dots, a_m)$  is the path from  $u_q$  to  $v$ .

*Proof.* Property 1, 3 follows by the definition of DFS sequence (See Definition D.10) and a simple induction.

Now consider property 2. Since  $A$  is a DFS sequence,  $\forall l \in \{j, j+1, \dots, k-1\}$ , either  $\text{par}(a_l) = a_{l+1}$  or  $\text{par}(a_{l+1}) = a_l$ . Thus, the path between  $u_i$  and  $u_{i+1}$  is a subsequence of  $(a_j, a_{j+1}, \dots, a_k)$ . If  $\text{par}(a_{l+1}) = a_l$  but  $a_{l+1}$  is not on the path between  $u_i$  and  $u_{i+1}$ , then there must be a leaf  $x$  in the subtree of  $a_{l+1}$  which implies  $u_i <_{\text{par}} x <_{\text{par}} u_{i+1}$ , and thus leads to a contradiction. If  $\text{par}(a_l) = a_{l+1}$  but  $a_{l+1}$  is not on the path between  $u_i$  and  $u_{i+1}$ , then both  $u_i$  and  $u_{i+1}$  should be in the subtree of  $a_l$ , and both of  $u_i$  and  $u_{i+1}$  should be in the DFS sequence of the subtree of  $a_l$ . But we know  $a_{l+1}$  cannot be in the DFS sequence of the subtree of  $a_l$ . This leads to a contradiction.  $\square$

**Lemma D.15.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , with a unique root. Let  $v \in V$ . Let  $V' = V \setminus \{u \in V \mid v \text{ is an ancestor (See Definition D.1) of } u\}$ . Let  $\text{par}' : V' \rightarrow V'$  satisfy  $\forall u \in V', \text{par}'(u) = \text{par}(u)$ . Then the DFS sequence (See Definition D.10) of  $\text{par}'$  is a subsequence of the DFS sequence of  $\text{par}$ .

*Proof.* The proof follows by the property 3 of Fact D.11 directly.  $\square$

**Corollary D.16.** Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $v_1, v_2, \dots, v_t$  be  $t$  vertices in  $V$ . Let  $V' = V \setminus \{u \in V \mid \exists v \in \{v_1, \dots, v_t\}, v \text{ is an ancestor (See Definition D.1) of } u\}$ . Let  $\text{par}' : V' \rightarrow V'$  satisfy  $\forall u \in V', \text{par}'(u) = \text{par}(u)$ . Then the DFS sequence (See Definition D.10) of  $\text{par}'$  is a subsequence of the DFS sequence of  $\text{par}$ .

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**Algorithm 16** DFS subsequence

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```
1: procedure SUBDFS( $\text{par} : V \rightarrow V, m, \delta$ ) ▷ Lemma D.17, Lemma D.18
2:   Output:  $V' \subseteq V, A = (a_1, a_2, \dots, a_s)$ .
3:   If  $V = \{v\}$ , return  $V' = V, A = (v)$ .
4:   Let  $v$  be the root in  $\text{par}$ , i.e.  $\text{par}(v) = v$ .
5:    $L = (l_1, l_2, \dots, l_t) = \text{LEAFSAMPLING}(\text{par}, m, \delta)$ . ▷ Algorithm 15
6:    $Q = \{(l_i, l_{i+1}) \mid i \in [t-1]\}$ .
7:    $\text{lca} = \text{LCA}(\text{par}, Q)$ . ▷ Algorithm 13
8:    $\forall i \in [t-1], (p_{l_i, l_{i+1}}, p_{i, l_i}, p_{i, l_{i+1}}) = \text{lca}(l_i, l_{i+1})$ .
9:    $Q' = \{(l_1, v), (\text{par}(l_1), p_{l_1, l_2}), (l_2, p_{l_1, l_2}), (\text{par}(l_2), p_{l_2, l_3}), (l_3, p_{l_2, l_3}), \dots, (l_t, p_{l_{t-1}, l_t}), (\text{par}(l_t), v)\}$ .
10:   $(\text{dep}_{\text{par}}, \{P_i \mid i \in [2t]\}) = \text{MULTIPATH}(\text{par}, Q')$ . ▷ Algorithm 14
11:   $V' = \bigcup_{i=1}^{2t} P_i$ .
12:  Let  $\text{par}' : V' \rightarrow V'$  satisfy  $\forall v \in V', \text{par}'(v) = \text{par}(v)$ .
13:  for  $i \in \{1, 3, 5, \dots, 2t-1\}$  do
14:    Compute  $A'_i = (u_1, u_2, \dots, u_{|P_i|})$  such that  $\{u_1, u_2, \dots, u_{|P_i|}\} = P_i$  and  $\text{dep}_{\text{par}}(u_1) < \text{dep}_{\text{par}}(u_2) < \dots < \text{dep}_{\text{par}}(u_{|P_i|})$ 
15:  end for
16:  for  $i \in \{2, 4, 6, \dots, 2t\}$  do
17:    Compute  $A'_i = (u_1, u_2, \dots, u_{|P_i|})$  such that  $\{u_1, u_2, \dots, u_{|P_i|}\} = P_i$  and  $\text{dep}_{\text{par}}(u_1) > \text{dep}_{\text{par}}(u_2) > \dots > \text{dep}_{\text{par}}(u_{|P_i|})$ 
18:  end for
19:  Let  $A' = A'_1 A'_2 \dots A'_{2t}$ . ▷  $A'$  is the concatenation of  $A'_1, A'_2, \dots, A'_{2t}$ .
20:   $\forall u \in V'$ , compute  $\text{rank}_{\text{par}}(u)$  and  $\text{rank}_{\text{par}'}(u)$ .
21:   $\forall u \in V', i \in [| \text{child}_{\text{par}'}(u) | + 1]$  compute  $\text{pos}(u, i) = j$  such that the  $j^{\text{th}}$  element in  $A'$  is the  $i^{\text{th}}$  time that  $u$  appears.
22:  Let  $b$  be the length of  $A'$ .
23:  Initialize  $c : [b] \rightarrow \mathbb{Z}_{\geq 0}$ . ▷  $c$  determine the number of copies needed for each element in  $A'$ 
24:  for  $u \in V' \setminus \{v\}$  do
25:    If  $u \in \text{leaves}(\text{par}')$ , let  $c(\text{pos}(u, 1)) = 1$ . ▷ A leaf should only have one copy.
26:    If  $\text{rank}_{\text{par}'}(u) = 1$ , let  $c(\text{pos}(\text{par}'(u), 1)) = \text{rank}_{\text{par}}(u)$ .
27:    If  $\text{rank}_{\text{par}'}(u) = | \text{child}_{\text{par}'}(\text{par}'(u)) |$ , let  $c(\text{pos}(\text{par}'(u), \text{rank}_{\text{par}'}(u) + 1)) = | \text{child}_{\text{par}}(\text{par}(u)) | + 1 - \text{rank}_{\text{par}}(u)$ .
28:    If  $1 \leq \text{rank}_{\text{par}'}(u) < | \text{child}_{\text{par}'}(\text{par}'(u)) |$ , let  $c(\text{pos}(\text{par}'(u), \text{rank}_{\text{par}'}(u) + 1)) = \text{rank}_{\text{par}}(\text{child}_{\text{par}}(\text{par}'(u), \text{rank}_{\text{par}'}(u) + 1)) - \text{rank}_{\text{par}}(u)$ .
29:  end for
30:  For each  $j \in [b]$ , duplicate the  $j^{\text{th}}$  element of  $A'$   $c(j)$  times. Let  $A$  be the obtained sequence.
31:  return  $V', A$ .
32: end procedure
```

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*Proof.* The proof is by induction on  $t$ . When  $t = 1$ , then the statement is true by Lemma D.15. Suppose the statement is true for  $t-1$ . Let  $V'' = V \setminus \{u \in V \mid \exists v \in \{v_1, \dots, v_{t-1}\}, v \text{ is an ancestor of } u\}$ , and let  $\text{par}'' : V'' \rightarrow V''$  satisfy  $\forall v \in V'', \text{par}''(v) = \text{par}(v)$ . By induction hypothesis, the DFS sequence of  $\text{par}''$  is a subsequence of the DFS sequence of  $\text{par}$ . If one of the  $v_1, \dots, v_{t-1}$  is an ancestor of  $v_t$ , then  $\text{par}' = \text{par}''$ , thus, the DFS sequence of  $\text{par}'$  is a subsequence of the DFS sequence of  $\text{par}$ . Otherwise, we have  $V' = V'' \setminus \{u \in V'' \mid v_t \text{ is an ancestor of } u\}$ . By Lemma D.15, the DFS sequence of  $\text{par}'$  is a subsequence of the DFS sequence of  $\text{par}''$ . Thus, the DFS sequence of  $\text{par}'$  is

a subsequence of the DFS sequence of  $\text{par}$ .  $\square$

**Lemma D.17** (Removing several subtrees). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $m > 0, \delta \in (0, 1)$  be parameters, and let  $|V| \leq m^{1/\delta}$ . Let  $(V', A) = \text{SUBDFS}(\text{par}, m, \delta)$  (Algorithm 16). Then  $\forall u \in V'$ , we have  $\text{par}(u) \in V'$ . Furthermore, with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $\forall u \in V \setminus V'$ , the number of leaves (See Definition D.8) in the subtree (See Definition D.9) of  $u$  is at most  $\lfloor |\text{leaves}(\text{par})| / \lceil m^{1/3} \rceil \rfloor$ .*

*Proof.* By Lemma D.13, we know  $L \subseteq \text{leaves}(\text{par})$ , and  $l_1 <_{\text{par}} l_2 <_{\text{par}} \dots <_{\text{par}} l_t$ .

We first prove  $\forall u \in V', \text{par}(u) \in V'$ . Our proof is by induction on the leaf  $l_i$ . By Lemma D.4, we have that  $\forall i \in [t-1], p_{l_i, l_{i+1}}$  is the LCA of  $(l_i, l_{i+1})$ ,  $p_{l_i, l_{i+1}}$  is an ancestor of  $l_{i+1}$ , and  $p_{l_i, l_{i+1}} \neq p_{l_{i+1}, l_{i+2}}, \text{par}(p_{l_i, l_{i+1}}) = p_{l_i, l_{i+1}}$ . By Lemma 14,  $P_1$  contains all the vertices on the path from  $l_1$  to the root  $v$ .  $P_1$  is the set of all the ancestors of  $l_1$ . Thus, every ancestor  $u$  of  $l_1$  is in  $P_1$  and satisfies  $\text{par}(u) \in V'$ .  $P_2$  contains all the vertices on the path from  $l_1$  to an ancestor of  $l_1$ . Thus,  $P_2 \subseteq P_1$ . Suppose now  $P_1 \cup P_2 \cup \dots \cup P_{2i-2} = \{u \in V \mid \exists j \in [i-1], u \text{ is an ancestor of } l_j\}$ . Notice that  $P_{2i-1}$  contains all the vertices on the path from  $l_i$  to the ancestor  $p_{i-1, l_i}$ . Since  $\text{par}(p_{i-1, l_i}) = p_{l_{i-1}, l_i}$  is also an ancestor of  $l_{i-1}$ , we have  $P_1 \cup P_2 \cup \dots \cup P_{2i-2} \cup P_{2i-1} = \{u \in V \mid \exists j \in [i], u \text{ is an ancestor of } l_j\}$ . Since  $P_{2i}$  contains all the vertices on the path from  $l_i$  to an ancestor of  $l_i$ , we have  $P_{2i} \subseteq P_1 \cup P_2 \cup \dots \cup P_{2i-2} \cup P_{2i-1}$ . To conclude, we have  $V' = P_1 \cup P_2 \cup \dots \cup P_{2t} = \{u \in V \mid \exists j \in [t], u \text{ is an ancestor of } l_j\}$ . Thus,  $\forall u \in V'$ , we have  $\text{par}(u) \in V'$ .

By Lemma D.13, with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $\forall u \in \text{leaves}(\text{par}) \setminus L$ , there exists  $w \in L, w <_{\text{par}} u$  such that  $|\{x \in \text{leaves}(\text{par}) \mid w <_{\text{par}} x <_{\text{par}} u\}| \leq \lfloor |\text{leaves}(\text{par})| / \lceil m^{1/3} \rceil \rfloor$ . In the following, we condition on the above event happens. Let  $u \in V \setminus V'$ . Due to Fact D.11, the DFS sequence of the subtree of  $u$  in  $\text{par}$  must be a consecutive subsequence of the DFS sequence of  $\text{par}$ . Thus,  $\exists x, y \in \text{leaves}(\text{par})$ , the leaves in the subtree of  $u$  in  $\text{par}$  is the set  $\{z \in \text{leaves}(\text{par}) \mid x <_{\text{par}} z <_{\text{par}} y\} \cup \{x\} \cup \{y\}$ . If the number of leaves in the subtree of  $u$  is more than  $\lfloor |\text{leaves}(\text{par})| / \lceil m^{1/3} \rceil \rfloor$ , then  $\exists l_i \in L, u$  is an ancestor of leaf  $l_i$ . But  $l_i \in V'$  contradicts to  $u \notin V'$ . Thus, the number of leaves in the subtree of  $u$  is at most  $\lfloor |\text{leaves}(\text{par})| / \lceil m^{1/3} \rceil \rfloor$ .  $\square$

**Lemma D.18** ( $A$  is a subsequence). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $m > 0, \delta \in (0, 1)$  be parameters, and let  $|V| \leq m^{1/\delta}$ . Let  $(V', A) = \text{SUBDFS}(\text{par}, m, \delta)$  (Algorithm 16). Then  $A$  is a subsequence of the DFS sequence of  $\text{par}$ . Furthermore,  $\forall u \in V', u$  appears in  $A$  exactly  $|\text{child}_{\text{par}}(u) + 1|$  times, and  $\forall u \notin V', u$  does not appear in  $A$ .*

*Proof.* We first show that  $A'$  is the DFS sequence of  $\text{par}'$ .

**Claim D.19.**  $A'$  is the DFS sequence of  $\text{par}' : V' \rightarrow V'$ .

*Proof.* By Lemma D.13, we know  $\{l_1, l_2, \dots, l_t\} = L \subseteq \text{leaves}(\text{par})$ , and  $l_1 <_{\text{par}} l_2 <_{\text{par}} \dots <_{\text{par}} l_t$ . By Lemma D.4, we have that  $\forall i \in [t-1], p_{l_i, l_{i+1}}$  is the LCA of  $(l_i, l_{i+1})$ ,  $p_{l_i, l_{i+1}}$  is an ancestor of  $l_{i+1}$ , and  $p_{l_i, l_{i+1}} \neq p_{l_{i+1}, l_{i+2}}, \text{par}(p_{l_i, l_{i+1}}) = p_{l_i, l_{i+1}}$ . By Lemma 14,  $\forall i \in [t]$ ,  $P_{2i-1}$  and  $P_{2i}$  only contains some ancestors of  $l_i$ . Thus,  $\text{leaves}(\text{par}') = L$ .

According to Lemma D.17 and Corollary D.16, the DFS sequence of  $\text{par}'$  is a subsequence of the DFS sequence of  $\text{par}$ . Thus, we still have  $l_1 <_{\text{par}'} l_2 <_{\text{par}'} \dots <_{\text{par}'} l_t$ . Due to Lemma 14,  $P_1$  contains all the vertices on the path from  $l_1$  to the root  $v$ ,  $P_{2t}$  contains all the vertices on the path from  $l_t$  to the root  $v$ ,  $\forall i \in [t-1]$ ,  $P_{2i}$  contains all the vertices on the path from  $\text{par}'(l_i)$  to the LCA of  $(l_i, l_{i+1})$ , and  $P_{2i+1}$  contains all the vertices on the path from  $l_{i+1}$  to  $p_{l_i, l_{i+1}}$ . Thus,  $A'_1$  is the path from the root  $v$  to leaf  $l_1$ ,  $A'_{2t}$  is the path from  $l_t$  to the root  $v$ ,  $\forall i \in [t-1]$ ,  $A'_{2i} A'_{2i+1}$  is the path from  $\text{par}'(l_i)$  to  $l_{i+1}$ . Due to Fact D.14,  $A' = A'_1 A'_2 \dots A'_{2t}$  is the DFS sequence of  $\text{par}'$ .  $\square$

Let us define some notations. Let  $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_s\}$  be the DFS sequence of  $\text{par}$ .  $\forall u \in V$ , let  $\text{st}_{\tilde{A}}(u) = j$  such that  $\tilde{a}_j$  is the first time that  $u$  appears in  $\tilde{A}$ . We define  $\text{ed}_{\tilde{A}}(u)$  be the position such that  $\tilde{a}_{\text{ed}_{\tilde{A}}(u)}$  is the last time that  $u$  appears in  $\tilde{A}$ . Similarly,  $\forall u \in V'$ , we can define  $\text{st}_{A'}(u), \text{st}_A(u), \text{ed}_{A'}(u), \text{ed}_A(u)$  to be the positions of the first time  $u$  appears in  $A'$ , the first time  $u$  appears in  $A$ , the last time  $u$  appears in  $A'$ , and the last time  $u$  appears in  $A$  respectively.

Since  $v$  is the root (in both  $\text{par}$  and  $\text{par}'$ ), it suffices to prove that  $(a_{\text{st}_A(v)}, a_{\text{st}_A(v)+1}, \dots, a_{\text{ed}_A(v)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(v)}, \tilde{a}_{\text{st}_{\tilde{A}}(v)+1}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(v)})$ . Our proof is by induction on  $\text{dep}_{\text{par}}(u)$  for  $u \in V'$ . If  $\text{dep}_{\text{par}}(u) = \text{dep}(\text{par})$ , then  $u$  must be a leaf in  $\text{par}'$  (or  $\text{par}$ , since  $\text{par}'$  and  $\text{par}$  are the same on  $V'$ ). In this case,  $\text{st}_A(u) = \text{ed}_A(u)$ ,  $\text{st}_{\tilde{A}}(u) = \text{ed}_{\tilde{A}}(u)$ , and  $(a_{\text{st}_A(u)}) = (\tilde{a}_{\text{st}_{\tilde{A}}(u)}) = (u)$ . Suppose for all  $u \in V'$  with  $\text{dep}_{\text{par}}(u) > d$ , we have that  $(a_{\text{st}_A(u)}, \dots, a_{\text{ed}_A(u)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(u)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(u)})$ . Let  $u$  be a vertex in  $V'$  with  $\text{dep}_{\text{par}}(u) = d$ . If  $u$  is a leaf, then it is the same as the previous argument. Now let us consider the case when  $u$  is not a leaf. According to Claim D.19,  $A'$  is the DFS sequence of  $\text{par}'$ . Due to line 30,  $A$  is obtained by duplicating each element of  $A'$  several times. Let  $w_1, w_2, \dots, w_k$  be the children of  $u$  in  $\text{par}'$ , and  $\text{rank}_{\text{par}'}(w_1) = 1, \text{rank}_{\text{par}'}(w_2) = 2, \dots, \text{rank}_{\text{par}'}(w_k) = |\text{child}_{\text{par}'}(u)|$ . Then, according to Fact D.11,  $(a_{\text{st}_A(u)}, \dots, a_{\text{ed}_A(u)})$  should look like:

$$(u, \dots, u, a_{\text{st}_A(w_1)}, \dots, a_{\text{ed}_A(w_1)}, u, \dots, u, a_{\text{st}_A(w_2)}, \dots, a_{\text{ed}_A(w_2)}, \dots, a_{\text{st}_A(w_k)}, \dots, a_{\text{ed}_A(w_k)}, u, \dots, u)$$

where the number of  $u$  before  $a_{\text{st}_A(w_1)}$  is  $\text{rank}_{\text{par}}(w_1)$  (see line 26), the number of  $u$  before  $a_{\text{st}_A(w_i)}$  for  $i \in [k] \setminus \{1\}$  is  $\text{rank}_{\text{par}}(w_i) - \text{rank}_{\text{par}}(w_{i-1})$  (see line 28), and the number of  $u$  after  $a_{\text{ed}_A(w_k)}$  is  $|\text{child}_{\text{par}}(u)| - \text{rank}_{\text{par}}(w_k) + 1$  (see line 27). Since  $\tilde{A}$  is the DFS sequence of  $\text{par}$ , according to Fact D.11, the number of  $u$  in  $\tilde{A}$  before  $\tilde{a}_{\text{st}_{\tilde{A}}(w_1)}$  is  $\text{rank}_{\text{par}}(w_1)$ . By our induction hypothesis,  $(a_{\text{st}_A(w_1)}, \dots, a_{\text{ed}_A(w_1)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(w_1)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(w_1)})$ . Thus,  $(a_{\text{st}_A(u)}, \dots, a_{\text{ed}_A(w_1)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(u)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(w_1)})$ . According to Fact D.11,  $\forall i \in [k] \setminus \{1\}$ , the number of  $u$  in  $\tilde{A}$  between  $\tilde{a}_{\text{ed}_{\tilde{A}}(w_{i-1})}$  and  $\tilde{a}_{\text{st}_{\tilde{A}}(w_i)}$  is  $\text{rank}_{\text{par}}(w_i) - \text{rank}_{\text{par}}(w_{i-1})$ . By our induction hypothesis, for all  $i \in [k] \setminus \{1\}$ ,  $(a_{\text{st}_A(w_i)}, \dots, a_{\text{ed}_A(w_i)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(w_i)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(w_i)})$ . Thus,  $(a_{\text{st}_A(u)}, \dots, a_{\text{ed}_A(w_k)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(u)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(w_k)})$ . According to Fact D.11, the number of  $u$  in  $\tilde{A}$  after  $\tilde{a}_{\text{ed}_{\tilde{A}}(w_k)}$  is  $|\text{child}_{\text{par}}(u)| - \text{rank}_{\text{par}}(w_k) + 1$ . Thus,  $(a_{\text{st}_A(u)}, \dots, a_{\text{ed}_A(u)})$  is a subsequence of  $(\tilde{a}_{\text{st}_{\tilde{A}}(u)}, \dots, \tilde{a}_{\text{ed}_{\tilde{A}}(u)})$ . Furthermore, the number of  $u$  appears in  $A$  is  $|\text{child}_{\text{par}}(u)| - \text{rank}_{\text{par}}(w_k) + 1 + \text{rank}_{\text{par}}(w_1) + \sum_{i=2}^k \text{rank}_{\text{par}}(w_i) - \text{rank}_{\text{par}}(w_{i-1}) = |\text{child}_{\text{par}}(u)| + 1$ .

Since  $A'$  is the DFS sequence of  $\text{par}'$ ,  $\forall u \notin V'$ ,  $u$  does not appear in  $A'$ . Thus,  $\forall u \notin V'$ ,  $u$  does not appear in  $A$ . □

### D.2.3 DFS Sequence

In this section, we show how to use Algorithm 16 as a subroutine to output a DFS sequence. The high level idea is that we use Algorithm 16 to generate subsequences of the DFS sequence in each iteration, and we ensure that the miss part of the DFS sequence must be the DFS sequences of many subtrees. After the  $i^{\text{th}}$  iteration, we should ensure that the number of leaves of each subtree which has missing DFS sequence is at most  $n/m^i$ , where  $m$  is some parameter depends on some computational resources (e.g. memory size of a machine). The description of the algorithm is shown in Algorithm 17. Figure 2 shows one step in our algorithm.

**Theorem D.20** (Correctness of DFS sequence). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $n = |V|, m = n^\delta$  for some constant*

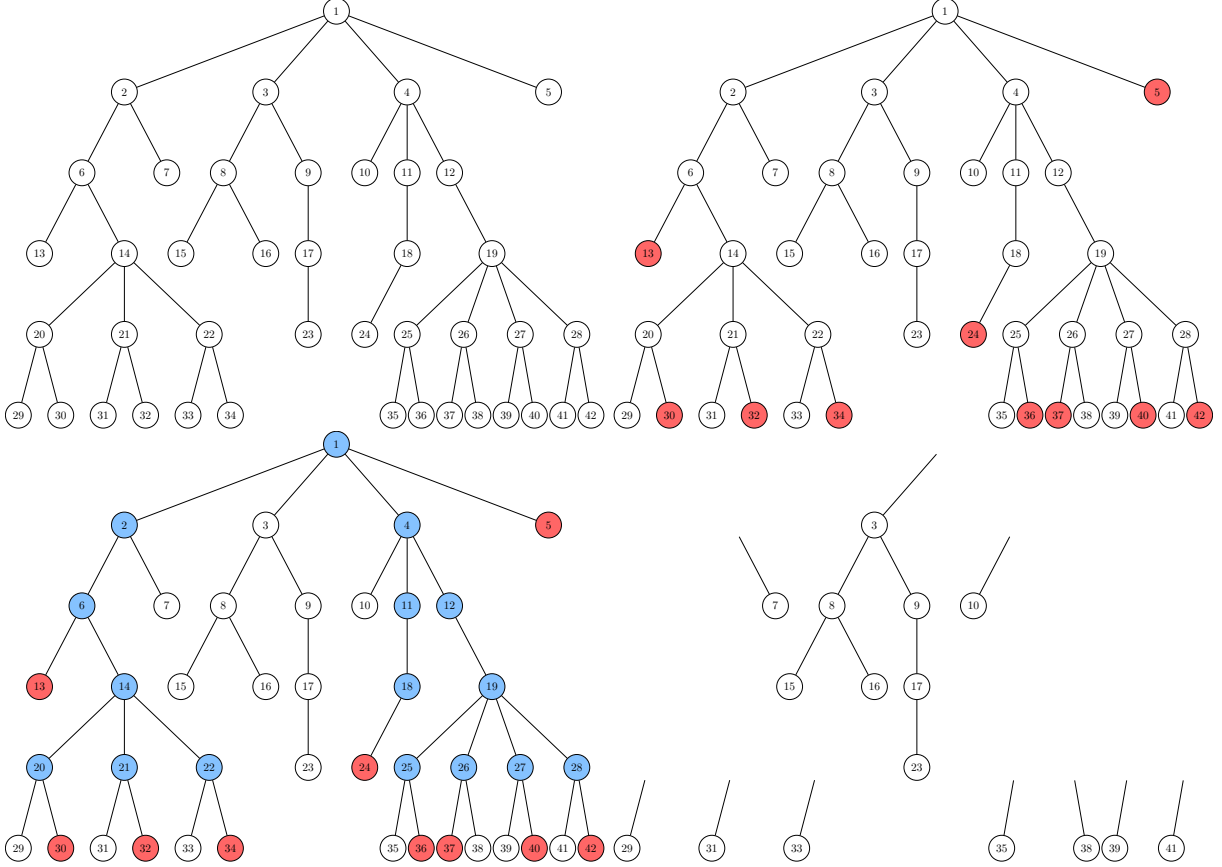


Figure 2: Given a tree that has 42 vertices (top-left), we label all the vertices from 1 to 42. Firstly, we sample some leaves (red vertices, i.e.  $\{5, 13, 24, 30, 32, 34, 36, 37, 40, 42\}$ ) in the tree (top-right tree). Then we find a DFS sequence of the tree (the tree formed by all the blue and red vertices in the bottom-left tree) which only contains all the sampled leaves and their ancestors. Finally, we recursively find the DFS sequences of remaining subtrees (bottom-right).

$\delta \in (0, 1)$ . If  $A = \text{DFS}(\text{par}, m)$  (Algorithm 17) does not output FAIL, then  $A$  is the DFS sequence of  $\text{par}$ .

*Proof.* It suffices to prove the following claim.

**Claim D.21.** Let  $i \in \{0\} \cup [r]$ .  $A_i$  is a subsequence of the DFS sequence of  $\text{par}$ .  $\forall v \in V_i$ ,  $\text{par}(v) \in V_i$ . Furthermore,  $\forall v \in V_i$ ,  $v$  appears in  $A_i$  exactly  $|\text{child}_{\text{par}}(v)| + 1$  times, and  $\forall v \notin V_i$ ,  $v$  does not appear in  $A_i$ .

*Proof.* Our proof is by induction on  $i$ . If  $i = 0$ , then by Lemma D.18,  $A_0$  is a subsequence of the DFS sequence of  $\text{par}$ ,  $\forall v \in V_0$ ,  $v$  appears in  $A_0$  exactly  $|\text{child}_{\text{par}}(v)| + 1$  times, and  $\forall v \notin V_0$ ,  $v$  does not appear in  $A_0$ . By Lemma D.17, we have  $\forall v \in V_0$ ,  $\text{par}(v) \in V_0$ .

Suppose the claim is true for  $i - 1$ . Let  $u \in V_i$ .

If  $u \in V_{i-1}$ , then since  $V_{i-1} \subseteq V_i$ ,  $\text{par}(u) \in V_i$ . Otherwise  $u \in V_{i,v}$  for some  $v$  with  $\text{par}_i(v) = v$ . If  $u = v$ , then  $\text{par}(v) \in V_{i-1} \subseteq V_i$ . Otherwise, by Lemma D.17,  $\text{par}(u) \in V_{i,v} \subseteq V_i$ .

Now consider the property of  $A_i$ . If  $u \in V_{i-1}$ , then since  $A_{i-1}$  is a subsequence of  $A_i$ , and by Lemma D.18  $u$  cannot appear in any  $A_{i,v}$ ,  $u$  must appear in  $A_i$  exactly  $|\text{child}_{\text{par}}(u)| + 1$  times. Otherwise  $u \in V_{i,v}$  for some  $v$  with  $\text{par}_i(v) = v$ . By Lemma D.18,  $u$  must appear in

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**Algorithm 17** DFS sequence

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1: procedure DFS( $\text{par} : V \rightarrow V, m$ ) ▷ Theorem D.20, Theorem D.22
2:   Output: FAIL or  $A = (a_1, a_2, \dots, a_{2|V|-1})$ .
3:    $n = |V|, \delta = 1/\log_m n$ .
4:   Let  $\text{par}_0 = \text{par}$ .
5:    $(V_0, A_0) = \text{SUBDFS}(\text{par}_0, m, \delta)$ . ▷ Algorithm 16
6:   Let  $r = \lceil 3/\delta \rceil + 2$ .
7:   for  $i = 1 \rightarrow r$  do ▷  $v \in V_i \Leftrightarrow v$  appears in  $A_i$ 
▷ If  $v \in V_i$ , then  $v$  appears  $|\text{child}_{\text{par}}(v)| + 1$  times in  $A_i$ 
8:     Let  $V'_i = V \setminus V_{i-1}$ .
9:     Initialize  $\text{par}_i : V'_i \rightarrow V'_i$ .
10:    For  $v \in V'_i$ , if  $\text{par}(v) \in V_{i-1}$ , let  $\text{par}_i(v) = v$ ; Otherwise, let  $\text{par}_i(v) = \text{par}(v)$ .
11:     $((V''_i, \emptyset), \text{par}_i^{(\infty)}) = \text{TREECONTRACTION}((V'_i, \emptyset), \text{par}_i)$ . ▷ Algorithm 2
12:     $V_i \leftarrow V_{i-1}$ .
13:     $A_i \leftarrow A_{i-1}$ .
14:    for  $v \in V'_i, \text{par}_i(v) = v$  do ▷ The DFS sequence of the subtree of  $v$  in  $\text{par}$  is missing.
15:      Let  $V'_i(v) = \{u \in V'_i \mid \text{par}_i^{(\infty)}(u) = v\}$ .
16:      Let  $\text{par}_{i,v} : V'_i(v) \rightarrow V'_i(v)$  satisfy  $\forall u \in V'_i(v), \text{par}_{i,v}(u) = \text{par}_i(u)$ .
17:      Let  $(V_{i,v}, A_{i,v}) = \text{SUBDFS}(\text{par}_{i,v}, m, \delta)$ . ▷ Algorithm 16
18:       $V_i \leftarrow V_i \cup V_{i,v}$ .
19:      Insert  $A_{i,v}$  after the  $\text{rank}_{\text{par}}(v)^{\text{th}}$  time appearance of  $v$  in  $A_i$ .
20:    end for
21:  end for
22:  If  $V_r = V$ , return  $A_r$  as  $A$ . Otherwise, return FAIL.
23: end procedure
```

---

$A_{i,v} \mid \text{child}_{\text{par}_{i,v}}(u) + 1 = |\text{child}_{\text{par}}(u)| + 1$  times. Since  $u$  cannot appear in  $A_{i-1}$ ,  $u$  must appear in  $A_i$  exactly  $|\text{child}_{\text{par}}(u)| + 1$  times. For  $v \in V'_i$  with  $\text{par}_i(v) = v$ , according to Fact D.11 and  $\forall w \in \{x \in V \mid v \text{ is an ancestor of } x\}$ ,  $w$  cannot be in  $V_{i-1}$ , the  $\text{rank}_{\text{par}}(v)^{\text{th}}$  time appearance of  $v$  and the  $(\text{rank}_{\text{par}}(v) + 1)^{\text{th}}$  time appearance of  $v$  should be adjacent in  $A_{i-1}$ . Due to Lemma D.18,  $A_{i,v}$  is a subsequence of the DFS sequence of the subtree of  $v$  in  $\text{par}$ . Due to Fact D.11,  $A_i$  is still a subsequence of the DFS sequence of  $\text{par}$  after insertion of the sequence  $A_{i,v}$ .

For any  $x \notin V_i$ , by Lemma D.18,  $x$  cannot be in any  $A_{i,v}$ . By our induction hypothesis,  $x$  cannot be in  $A_{i-1}$ . Thus,  $x$  cannot be in  $A_i$ .  $\square$

If the procedure does not output FAIL, then according to the above Claim D.21,  $\forall v \in V_r = V$ ,  $v$  appears in  $A_r = A$  exactly  $|\text{child}_{\text{par}}(v)| + 1$  times, and  $A_r = A$  is a subsequence of the DFS sequence of  $\text{par}$ . Due to Fact D.11,  $A = A_r$  is the DFS sequence of  $\text{par}$ .  $\square$

The following lemma claims the success probability of Algorithm 17.

**Theorem D.22** (Success probability). *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $n = |V|, m = n^\delta$  for some constant  $\delta \in (0, 1)$ . With probability at least  $1 - 1/(100n^4)$ ,  $A = \text{DFS}(\text{par}, m)$  (Algorithm 17) does not output FAIL.*

*Proof.*  $\forall i \in [r], v \in V'_i$  with  $\text{par}_i(v) = v$ , let  $\mathcal{E}_{i,v}$  be the event that  $\forall u \in V'_i(v) \setminus V_{i,v}$ , the number of leaves in the subtree of  $u$  in  $\text{par}$  is at most  $|\text{leaves}(\text{par}_{i,v})|/n^{\delta/3}$ . Notice that due to Lemma D.17,



if  $\text{par}_i(v) = v$ , then  $v$  will be in  $V_i$ . Thus, we use  $\mathcal{E}_v$  to denote the event  $\mathcal{E}_{i,v}$ . By Lemma D.17,  $\mathcal{E}_v$  happens with probability at least  $1 - 1/(100n^5)$ . By taking union bound over all  $v$ , with probability at least  $1 - 1/(100n^4)$ , all the events  $\mathcal{E}_v$  will happen.

**Claim D.23.** *Condition on all the events  $\mathcal{E}_v$  happen.  $\forall i \in [r], v \in V'_i$  with  $\text{par}_i(v) = v$ , we have  $|\text{leaves}(\text{par}_{i,v})| \leq n/n^{(i-1)\delta/3}$ .*

*Proof.* When  $i = 1$ , the claim is obviously true, since  $|\text{leaves}(\text{par}_{i,v})| \leq n$ . Suppose the claim holds for  $i - 1$ . Let  $v \in V'_i$  with  $\text{par}_i(v) = v$ . There must be  $v' \in V'_{i-1}$  with  $\text{par}_{i-1}(v') = v'$ , and  $v \in V'_{i-1}(v') \setminus V_{i-1,v'}$ . Since  $\mathcal{E}_{v'}$  happens, the number of leaves in the subtree of  $v$  in  $\text{par}$  is at most  $|\text{leaves}(\text{par}_{i-1,v'})|/n^{\delta/3} \leq n/n^{(i-1)\delta/3}$ .  $\square$

If  $V'_r \neq \emptyset$ , then  $\exists v \in V'_r$  with  $\text{par}_i(v) = v$  and  $|\text{leaves}(\text{par}_{i,v})| \geq 1$ . If all the events  $\mathcal{E}_v$  happens, it will contradict to Claim D.23. Thus, if all the events  $\mathcal{E}_v$  happens,  $V'_r$  must be  $\emptyset$ , and thus  $V_r = V$  which implies that the procedure will not fail.  $\square$

### D.3 Range Minimum Query

Range Minimum Query (RMQ) problem is defined as following: given a sequence of  $n$  numbers  $a_1, a_2, \dots, a_n$ , the goal is to preprocess the sequence  $a$  to get a data structure such that for any query  $(p, q)$ ,  $(p < q)$  we can efficiently find the element which is the minimum in  $a_p, a_{p+1}, \dots, a_q$ . A classic method is to preprocess a sparse table  $f$  in  $\log(n)$  number of iterations such that  $\forall i \in [n], j \in [\lceil \log n \rceil] \cup \{0\}$ ,  $f_{i,j} = \arg \min_{i \leq i' \leq \min(n, i+2^j-1)} a_{i'}$ . To answer query for  $(p, q)$ , it just needs to return  $\arg \min_{i \in \{f_{p,j^*}, f_{q-2^{j^*}+1, j^*}\}} a_i$  for  $j^* = \lfloor \log(q - p + 1) \rfloor$ . In this section, we firstly show a modified data structure. We will compute  $\hat{f}_{i,j} = \arg \min_{i \leq i' \leq \min(n, i+\lceil n^\delta \rceil^j-1)} a_{i'}$ . The Algorithm is shown in Algorithm 18. Then we show how to use  $\hat{f}$  to compute  $f$  in Algorithm 19.

---

#### Algorithm 18 A Sparser Table for RMQ

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```

1: procedure SPARSETABLE+( $a_1, a_2, \dots, a_n, \delta$ ) ▷ Lemma D.24
2: ▷ Output:  $\hat{f}_{i,j}$  for  $i \in [n], j \in \{0\} \cup \lceil [1/\delta] \rceil$ 
3:   Initially, for all  $i \in [n]$  let  $\hat{f}_{i,0} = i$ .  $\forall i > n, j \in \mathbb{Z}$ , let  $\hat{f}_{i,j} = 0$ , and let  $a_0 = \infty$ . Let  $m = \lceil n^\delta \rceil$ .
4:   For  $t \in \lceil [1/\delta] \rceil$ , let  $S_t = \{x \mid \exists y \in [m-1], x = y \cdot m^t\}$ .
5:   for  $l = 1 \rightarrow \lceil [1/\delta] \rceil$  do
6:     for  $j = 0 \rightarrow \lceil n/m \rceil$  do
7:        $i \leftarrow j \cdot m + 1$ .
8:        $z_{j,l}^* \leftarrow \arg \min_{z: t \in [l-1], x \in S_t, z = \hat{f}_{j \cdot m + 1 + x, t}} a_z$ .
9:       for  $i' = 0 \rightarrow \min(m-1, n-i)$  do
10:         $T \leftarrow \{x \in \mathbb{Z} \mid i+i' \leq x \leq i+m-1\} \cup \{x \in \mathbb{Z} \mid i+m^l \leq x \leq i+m^l+i'-1\} \cup \{z_{j,l}^*\}$ 
11:         $\hat{f}_{i+i',l} = \arg \min_{z \in T} a_z$ .
12:       end for
13:     end for
14:      $l \leftarrow l + 1$ .
15:   end for
16:   return  $\hat{f}_{i,j}$  for  $i \in [n], j \in \{0\} \cup \lceil [1/\delta] \rceil$ .
17: end procedure

```

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**Algorithm 19** A Sparse Table for RMQ
 

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```

1: procedure SPARSETABLE( $a_1, a_2, \dots, a_n, \delta$ ) ▷ Lemma D.25
2: ▷ Output:  $f_{i,j}$  for  $i \in [n], j \in \{0\} \cup [\lceil \log n \rceil]$ .
3:   Initially, for all  $i \in [n]$  let  $f_{i,0} = i$ .  $\forall i > n, j \in \mathbb{Z}$ , let  $f_{i,j} = 0$ , and let  $a_0 = \infty$ . Let  $m = \lceil n^\delta \rceil$ .
4:   Let  $\{\widehat{f}_{p,q} \mid p \in [n], q \in \{0\} \cup \lceil 1/\delta \rceil\} = \text{SPARSETABLE}^+(a_1, a_2, \dots, a_n, \delta)$ . ▷ Algorithm 18
5:   Let all undefined  $\widehat{f}_{p,q}$  be 0.
6:   for  $t \in [\lceil \log n \rceil]$  do
7:     if  $2^t \leq m$  then
8:        $k_t \leftarrow -1$ 
9:        $S_t \leftarrow \emptyset$ 
10:    else
11:       $k_t \leftarrow \lfloor \log_m(2^t - m) \rfloor$ 
12:       $S_t \leftarrow \{x \mid x \in [2^t - m - m^{k_t} + 1] \text{ s.t. } x \equiv 1 \pmod{m^{k_t}} \text{ or } (2^t - m - x) \equiv -1 \pmod{m^{k_t}}\}$ 
13:    end if
14:  end for
15:  for  $j = 0 \rightarrow \lceil n/m \rceil$  do
16:    for  $t = 0 \rightarrow \lceil \log n \rceil$  do
17:       $i \leftarrow j \cdot m + 1$ .
18:       $z_{j,t}^* \leftarrow \arg \min_{z: x \in S_t, z = \widehat{f}_{j \cdot m + m + x, k_t}} a_z$ .
19:      for  $i' = 0 \rightarrow \min(m - 1, n - i)$  do
20:         $T_1 \leftarrow \{x \in \mathbb{Z} \mid i + i' \leq x \leq \min(i + m - 1, i + i' + 2^t - 1)\}$ 
21:         $T_2 \leftarrow \{x \in \mathbb{Z} \mid \max(i + 2^t, i + i') \leq x \leq i + 2^t + i' - 1\}$ 
22:         $T \leftarrow T_1 \cup T_2 \cup \{z_{j,t}^*\}$ 
23:         $f_{i+i',t} = \arg \min_{z \in T} a_z$ .
24:      end for
25:    end for
26:  end for
27:  return  $f_{i,j}$  for  $i \in [n], j \in \{0\} \cup [\lceil \log n \rceil]$ .
28: end procedure

```

---

**Lemma D.24.** Let  $a_1, a_2, \dots, a_n$  be a sequence of numbers, and  $\delta \in (0, 1)$ . Let  $\{\widehat{f}_{p,q}\}$  be the output of  $\text{SPARSETABLE}^+(a_1, a_2, \dots, a_n, \delta)$  (Algorithm 18). Then  $\forall p \in [n], q \in \{0\} \cup \lceil 1/\delta \rceil$ ,  $\widehat{f}_{p,q} = \arg \min_{p \leq p' \leq \min(n, i + \lceil n^\delta \rceil^q - 1)} a_{p'}$ .

*Proof.* The proof is by induction on  $q$ . When  $q = 0$ , the statement obviously holds for all  $\widehat{f}_{p,0}$ . Suppose all  $p \in [n]$ ,  $\widehat{f}_{p,0}, \widehat{f}_{p,1}, \dots, \widehat{f}_{p,q-1}$  satisfy the property. The first observation is that the value of  $\widehat{f}_{p,q}$  will be assigned in the procedure when  $l = q, j = \lfloor (p-1)/m \rfloor, i' = (p-1) \bmod m$ . Then by line 8,  $z_{j,l}^*$  will be the position of the minimum value in  $a_{j \cdot m + m}, a_{j \cdot m + m + 2}, \dots, a_{j \cdot m + m^l - 1}$  by our induction hypothesis. Then by line 11,  $\widehat{f}_{i+i',l}$  will be the position of the minimum value in  $a_{j \cdot m + i' + 1}, a_{j \cdot m + i' + 2}, \dots, a_{j \cdot m + i' + m^l}$ . Thus, Since  $j \cdot m + i' + 1 = p$ ,  $\widehat{f}_{p,q}$  satisfies the property.  $\square$

**Lemma D.25.** Let  $a_1, a_2, \dots, a_n$  be a sequence of numbers, and  $\delta \in (0, 1)$ . Let  $\{f_{p,q}\}$  be the output of  $\text{SPARSETABLE}(a_1, a_2, \dots, a_n, \delta)$  (Algorithm 19). Then  $\forall p \in [n], q \in \{0\} \cup \lceil \log n \rceil$ ,  $f_{p,q} = \arg \min_{p \leq p' \leq \min(n, i + 2^q - 1)} a_{p'}$ .

*Proof.* Let  $m = \lceil n^\delta \rceil$ . By Lemma D.24,  $\forall x \in [n], y \in \{0\} \cup \lceil 1/\delta \rceil$ ,  $\widehat{f}_{x,y} = \arg \min_{x \leq x' \leq \min(n, i + m^y - 1)} a_{x'}$ . Thus, by the definition of  $S_t$ , we know  $z_{j,t}^* = \arg \min_{j \cdot m + m + 1 \leq z \leq j \cdot m + 2^t} a_z$ . An observation is that

the value of  $f_{p,q}$  will be assigned in the procedure when  $t = q, j = \lfloor (p-1)/m \rfloor, i' = (p-1) \bmod m$ . By line 23, we know

$$f_{p,q} = f_{i+i',t} = \arg \min_{z: i+i'+1 \leq z \leq i+i'+2^t} a_z = \arg \min_{p \leq p' \leq \min(n, i+2^q-1)} a_{p'}.$$

□

## D.4 Applications of DFS Sequence

In this section, we briefly discuss some applications of the DFS sequence of a tree.

Since the DFS sequence of a subtree should be a continuous subsequence of the DFS sequence of the tree, one direct application of the DFS sequence is to compute the size of each subtree, i.e. for each subtree with root  $v$ , we can find the first place  $v$  appeared and the last place  $v$  appeared, and then calculate the vertices between those two appearances.

Another application of the DFS sequence and the range minimum query is to output a data structure which can answer any LCA query in  $O(1)$  time (for both sequential and parallel). This is better than the data structure provided by Section D.1 which needs  $O(\log D)$  time (for both sequential and parallel) to answer the query.

Since it is easy to output a data structure which can answer the depth of each vertex in  $O(1)$  time (in both sequential and parallel), together with the lowest common ancestor data structure, we can answer the query of the tree distance between any two vertices in  $O(1)$  time (for both sequential and parallel).

## E The MPC Model

In this section, let us introduce the computational model studied in this paper. Suppose we have  $p$  machines indexed from 1 to  $p$  each with memory size  $s$  words, where  $n$  is the number of words of the input and  $p \cdot s = O(n^{1+\gamma}), s = \Theta(n^\delta)$ . Here  $\delta \in (0, 1)$  is a constant,  $\gamma \in \mathbb{R}_{\geq 0}$ , and a word has  $\Theta(\log(s \cdot p))$  bits. Thus, the total space in the system is only  $O(n^\gamma)$  factor more than the input size  $n$ , and each machine has local memory size sublinear in  $n$ . When  $0 \leq \gamma \leq O(1/\log n)$ , the total space is just linear in the input size. The computation proceeds in rounds. At the beginning of the computation, the input is distributed on the local memory of  $\Theta(n/s)$  input machines. Input machines and other machines are identical except that input machine can hold a part of the input in its local memory at the beginning of the computation while each of other machines initially holds nothing. In each round, each machine performs computation on the data in its local memory, and sends messages to other machines (including the sender itself when it wants to keep the data) at the end of the round. Although any two machines can communicate directly in any round, the total size of messages (including the self-sent messages) sent or received of a machine in a round is bounded by  $s$ , its local memory size. In the next round, each machine only holds the received messages in its local memory. At the end of the computation, the output is distributed on the output machines. Output machines and other machines are identical except that output machine can hold a part of the output in its local memory at the end of the computation while each of other machines holds nothing. We call the above model  $(\gamma, \delta)$ -MPC model. The model is exactly the same as the model  $\text{MPC}(\epsilon)$  defined by [BKS13] with  $\epsilon = \gamma/(1 + \gamma - \delta)$  and the number of machines  $p = O(n^{1+\gamma-\delta})$ . Since we care more about the total space used by the algorithm, we use  $(\gamma, \delta)$  to characterize the model, while in [BKS13] they use parameter  $\epsilon$  to characterize the repetition of the data. The main complexity measure is the number of rounds  $R$  required to solve the problem.

## E.1 Basic MPC Algorithms

**Sorting** One of the most important algorithms in MPC model is sorting. The following theorem shows that there is an efficient sorting algorithm.

**Theorem E.1** ([GSZ11, Goo99]). *Sorting can be solved in  $c/\delta$  rounds in  $(0, \delta)$  – MPC model for any constant  $\delta \in (0, 1)$ , where  $c \geq 0$  is a universal constant. Precisely, there is an algorithm  $\mathcal{A}$  in  $(0, \delta)$  – MPC model such that for any set  $S$  of  $n$  comparable items stored  $O(n^\delta)$  per machine on input machines,  $\mathcal{A}$  can run in  $c/\delta$  rounds and leave the  $n$  items sorted on the output machines, i.e. the output machine with smaller index holds a smaller part of  $O(n^\delta)$  items.*

Notice that for any  $\delta' \geq \delta$ ,  $O(1)$  number of machines with  $\Theta(n^{\delta'})$  memory can always simulate the computation of  $O(n^{\delta'-\delta})$  number of machines with  $\Theta(n^\delta)$  memory. Thus, if an algorithm  $\mathcal{A}$  can solve a problem in  $(\gamma, \delta)$  – MPC model in  $R(n)$  rounds, then  $\mathcal{A}$  can be simulated in  $(\gamma', \delta')$  – MPC model still in  $R(n)$  rounds with all  $\gamma' \geq \gamma, \delta' \geq \delta$ .

**Indexing** In the indexing problem, a set  $S = \{x_1, x_2, \dots, x_n\}$  of  $n$  items are stored  $O(n^\delta)$  per machine on input machines. The output is

$$S' = \{(x, y) \mid x \in S, y - 1 \text{ is the number of items before } x\}$$

of  $n$  pairs stored  $O(n^\delta)$  per machine on output machines. Here, “an item  $x' \in S$  is before  $x \in S$ ” means that  $x'$  is held by a input machine with a smaller index, or  $x', x$  are stored in the same input machine but  $x'$  has a smaller local memory address.

**Prefix sum** In the prefix sum problem, a set  $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$  of  $n$  (item, number) pairs are stored  $O(n^\delta)$  per machine on input machines. The output is

$$S' = \left\{ (x, y') \mid (x, y) \in S, y' - y = \sum_{(\tilde{x}, \tilde{y}) \text{ is before } (x, y)} \tilde{y} \right\}$$

of  $n$  pairs stored  $O(n^\delta)$  per machine on output machines. Here, “an pair  $(\tilde{x}, \tilde{y}) \in S$  is before  $(x, y) \in S$ ” means that  $(\tilde{x}, \tilde{y})$  is held by a input machine with a smaller index, or  $(\tilde{x}, \tilde{y}), (x, y)$  are stored in the same input machine but  $(\tilde{x}, \tilde{y})$  has a smaller local memory address. Notice that indexing problem is a special case of prefix sum problem.

**Theorem E.2** ([GSZ11]). *Indexing/prefix sum problem can be solved in  $c/\delta$  rounds in  $(0, \delta)$  – MPC model for any constant  $\delta \in (0, 1)$ , where  $c \geq 0$  is a universal constant.*

Once each item has an index, it is able to reallocate them onto the machines.

**Load balance** Sometimes, local computations of a machine may generate new data. When some machines are not able to keep the new data generated, we need to do loading balance. Fortunately, this operation can be done in constant number of rounds of computations.

For arbitrary constant  $\delta \in (0, 1)$ , we are able to spend constant number of rounds to reallocate the data in  $(0, \delta)$  – MPC model such that if a machine is not empty, the size of its local data is at least  $n^\delta/k$  and is at most  $2n^\delta/k$  where  $k > 1$  is an arbitrary constant. The method is very simple, we can use the algorithm mentioned in Theorem E.2 to index each data item, and then send them to the corresponding machine.

**Predecessor** In the predecessor problem, a set  $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$  of  $n$  (item, 0/1) pairs are stored  $O(n^\delta)$  per machine on input machines. The output machines are all input machines. If an input (also output) machine holds a tuple  $(x_i, y_i) \in S$  at the beginning of the computation, then at the end of the computation, that machine should still hold the tuple  $(x_i, y_i)$ . In addition, if an input (also output) machine holds a tuple  $(x, 0) \in S$  at the beginning of the computation, then at the end of the computation, that machine should hold a tuple  $(x, x')$  such that  $(x', 1) \in S$ , and  $(x', 1)$  is the last tuple occurred before  $(x, 0)$ . Here, “ $(x', 1)$  is before  $(x, 0)$ ” means that  $(x', 1)$  is held by a input machine with a smaller index, or  $(x', 1), (x, 0)$  are stored in the same input machine but  $(x', 1)$  has a smaller local memory address.

**Theorem E.3** ([GSZ11]). *Predecessor problem can be solved in  $c/\delta$  rounds in  $(0, \delta)$  – MPC model for any constant  $\delta \in (0, 1)$ , where  $c \geq 0$  is a universal constant.*

Roughly speaking the algorithm is as the following: firstly, build a  $\Theta(n^\delta)$  branching tree on the machines, then follows by bottom-up stages to collect the last  $(x_l, 1)$  tuple in each large interval and then follows by top-down stages to compute the predecessors of every prefix. For completeness, we describe the algorithm for predecessor problem in the following:

Predecessor Algorithm:

- **Setups:**

- There are  $2p = \Theta(n^\delta)$  machines indexed from 1 to  $2p$  each with local memory size  $s = \Theta(n^\delta)$ . The machine with index from  $p + 1$  to  $2p$  are input/output machines.
- $(x_1, y_1), \dots, (x_n, y_n)$  are stored on input/output machine  $p + 1$  to  $2p$ , where  $\forall i \in [n], y_i \in \{0, 1\}$ .
- The goal: If an input machine holds a tuple  $(x, y)$  with  $y = 0$ , then it will create a tuple  $(x, x')$  at the end of the computation, where  $(x', y')$  is the last tuple with  $y' = 1$  stored before  $(x, y)$ .

- **Bottom-up stage ( $O(1/\delta)$  constant rounds):**

- Let  $d = s/10$  be the branching factor.
- In the  $i^{\text{th}}$  round, each machine  $j$  with  $j$  in the range  $\lfloor p/d^{i-1} \rfloor + 1$  to  $\lfloor (2p - 1)/d^{i-1} \rfloor + 1$  sends the last  $(x_l, y_l)$  tuple with  $y_l = 1$  in its local memory to machine  $\lfloor (j - 1)/d \rfloor + 1$ . If machine  $j$  does not have any tuple with  $y_l = 1$ , it just sends an arbitrary tuple to machine  $\lfloor (j - 1)/d \rfloor + 1$ .
- Until the end of the computation, machine  $j$  sends itself messages to keep the data. The stage ends when machine 1 receives messages.

- **Top-down stage ( $O(1/\delta)$  constant rounds):**

- Let  $d = s/10$  be the branching factor.
- In the  $i^{\text{th}}$  round, each machine  $j$  with  $j$  in the range  $\lfloor d^{i-2} \rfloor + 1$  to  $\min(d^{i-1}, p)$  sends to each machine  $h$  in the range  $(j - 1)d + 1$  to  $\min(j \cdot d, 2p)$  a tuple  $(x_l, y_l)$  which is the last tuple with  $y_l = 1$  appeared before machine  $h$ .
- The stage ends when machine  $2p$  receives messages.

- **The last round:**

- Machine  $i \in \{p + 1, \dots, 2p\}$  scans its local memory, for each tuple  $(x, y)$  with  $y = 0$ , create a tuple  $(x, x')$  where  $(x', y')$  is the last tuple stored before  $(x, y)$  with  $y' = 1$ .

## E.2 Data Organization

In this section, we introduce the method to organize the data in the system.

**Set** Let  $S = \{x_1, x_2, \dots, x_m\}$  be a set of  $m$  items, and each item  $x_i$  can be described by  $O(1)$  number of words. If  $x \in S$  is equivalent to that there is a unique machine which holds a pair  $(“S”, x)$  in its local memory, then we say that  $S$  is stored in the system. Here “ $S$ ” is the name of the set  $S$  and can be described by  $O(1)$  number of words.

Let  $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$  be a set of  $m$  sets, where  $\forall i \in [m]$ ,  $S_i$  is stored in the system, and the name “ $S_i$ ” of each set  $S_i$  can be described by  $O(1)$  number of words. If  $S \in \mathcal{S}$  is equivalent to that there is a unique machine which holds a pair  $(“\mathcal{S}”, “S”)$  in its local memory, then we say  $\mathcal{S}$  is stored in the system. Here “ $\mathcal{S}$ ” is the name of  $\mathcal{S}$  and can be described by  $O(1)$  number of words.

Let  $S$  be a set stored in the system. If machine  $i$  has a pair  $(“S”, x)$ , then we say that the element  $x$  of  $S$  is held by the machine  $i$ . If every element of  $S$  is held by a machine with index in  $\{i, i+1, \dots, j\}$ , then we say  $S$  is stored on the machine  $i$  to the machine  $j$ .

The total space needed to store  $S$  is  $\Theta(m)$ .

**Mapping** Let  $f : U \rightarrow H$  be a mapping from a finite set  $U$  to a set  $H$ . In the following, we show how to use a set to represent a mapping.

**Definition E.4** (Set representation of a mapping). *Let  $f : U \rightarrow H$  be a mapping from a finite set  $U$  to a set  $H$ . Let  $S = \{(x, y) \mid x \in U, y = f(x)\}$ . then the set  $S$  is a set representation of the mapping  $f$ .*

Let  $U$  be a finite set where each element of  $U$  can be described by  $O(1)$  number of words. Let  $S$  be a set representation of the mapping  $f : U \rightarrow H$ . If  $S$  is stored in the system, then we say  $f$  is stored in the system. If  $S$  is stored on the machine  $i$  to the machine  $j$ , then  $f$  is stored on the machine  $i$  to the machine  $j$ . At any time of the system, there can be at most one set representation  $S$  of  $f$  stored in the system. Furthermore, the name of  $S$  is “ $f$ ” which is the same as the name of mapping  $f$ , and can be described by  $O(1)$  number of words.

The total space needed to store  $f$  is the total space needed to store  $S$ , and thus is  $\Theta(|U|)$ .

**Sequence** Let  $A = (a_1, a_2, \dots, a_m)$  be a sequence of  $m$  elements. In the following, we show how to use a set to represent a sequence.

**Definition E.5** (Set representation of a sequence). *Let  $A = (a_1, a_2, \dots, a_m)$  be a sequence of  $n$  elements. If a set  $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\} \subseteq \mathbb{R} \times \{a_1, a_2, \dots, a_m\}$  satisfies  $x_1 < x_2 < \dots < x_m, y_1 = a_1, y_2 = a_2, \dots, y_m = a_m$ , then the set  $S$  is a set representation of the sequence  $A$ . Furthermore, if  $x_1 = 1, x_2 = 2, \dots, x_m = m$ , then  $S$  is a standard set representation of  $A$ .*

Let  $A$  be a sequence of elements where each element can be described by  $O(1)$  number of words. Let  $S$  be a set representation of the sequence  $A$ . If  $S$  is stored in the system, then we say  $A$  is stored in the system. If  $S$  is stored on the machine  $i$  to the machine  $j$ , then  $A$  is stored on the machine  $i$  to the machine  $j$ . At any time of the system, there can be at most one set representation  $S$  of  $A$  stored in the system. Furthermore, the name of  $S$  is “ $A$ ” which is the same as the name of sequence  $A$ , and can be described by  $O(1)$  number of words.

The total space needed to store  $A$  is the total space needed to store  $S$ , and thus is  $\Theta(m)$ .

### E.3 Set Operations

In this section, we introduce some MPC model operations for sets.

**Duplicates removing** There are  $n$  tuples stored in the machines. But there are some duplicates of them. The goal is to remove all the duplicates. To achieve this, we can just sort all the tuples. After sorting, if a tuple is different from its previous tuple, then we keep it. Otherwise, we remove the tuple.

**Sizes of sets** Suppose we have  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. Our goal is to get the sizes of all the sets. We can firstly sort all the tuples such that the tuples from the same set are consecutive. Then we can calculate the index of each tuple. Every machine can scan all the tuples in its local memory, if  $x$  is an element of set  $S_i$  and has the smallest/largest index  $y$ , then create a pair ("boundary of ' $S_i$ ' ",  $y$ ). Then we sort all the created pairs, then for each set  $S_i$ , there are two pairs ("boundary of ' $S_i$ ' ",  $y_1$ ), ("boundary of ' $S_i$ ' ",  $y_2$ ) stored on the same machine. Each machine can store its local memory. For each pair of tuples ("boundary of ' $S_i$ ' ",  $y_1$ ), ("boundary of ' $S_i$ ' ",  $y_2$ ) with  $y_1 < y_2$ , the machine can generate a new tuple (" $f$ ", (" $S_i$ ",  $y_2 - y_1 + 1$ )). Finally, there will be a mapping  $f$  stored in the system, where  $f(S_i) = |S_i|$ . Thus, the total number of rounds is a constant.

**Copies of sets** Suppose we have  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. Let  $s_1, s_2, \dots, s_k \in \mathbb{Z}_{\geq 1}$ . If a machine holds an element  $x \in S_i$ , then the machine knows the value of  $s_i$ . Our goal is to create sets  $S_{1,1}, S_{1,2}, \dots, S_{1,s_1}, S_{2,1}, S_{2,2}, \dots, S_{2,s_2}, \dots, S_{k,s_k}$  and make them stored in the system, where  $S_{i,j}$  is a copy of  $S_i$ .

The idea is very simple: for an element  $x \in S_i$ , we need to make  $s_i$  copies (" $S_{i,1}$ ",  $x$ ), (" $S_{i,2}$ ",  $x$ ),  $\dots$ , (" $S_{i,s_i}$ ",  $x$ ) of tuple (" $S_i$ ",  $x$ ). But the issue is that  $s_i$  may be very large such that it is not able to generate all the copies of a tuple on a single machine. For the above reason, we implement it in three steps: firstly we compute the new "position" of each original tuple among all the copies, then send the original tuples to their new "positions", and finally filling the gap by generating copies between any two adjacent original tuples. Precisely, each machine can scan its local memory, and assign each tuple (" $S_i$ ",  $x$ ) a weight  $s_i$ . Then we can use prefix sum algorithm (See Theorem E.2) to compute the prefix sum of each tuple (" $S_i$ ",  $x$ ). The prefix sum  $\text{pos}("S_i", x)$  of a tuple (" $S_i$ ",  $x$ ) denotes the new "position" of the last copy of this tuple when all the copies are generated. Let  $n = \sum_{i=1}^k s_i \cdot |S_i|$ . Let machine 1 to  $t$  be  $t$  empty machines each maintains  $s/10$  "positions", i.e. machine 1 has "positions" 1 to  $s/10$ , machine 2 has "positions"  $s/10 + 1$  to  $2s/10$ , and so on. Let  $t \cdot s/10 = \Theta(n)$ . The machine which holds tuple (" $S_i$ ",  $x$ ) sends the tuple (" $S_i$ ",  $x$ ) to the "position"  $\text{pos}("S_i", x) - s_i + 1$ , and sends the tuple (" $S_{i,s_i}$ ",  $x$ ) to the "position"  $\text{pos}("S_i", x)$ . Then each machine  $i \in [t]$  scans its "positions". If a "position" received a tuple, the machine marks that "position" as "1". Otherwise, the machine marks that position as "0". Now we can apply the predecessor algorithm (See Theorem E.3) such that each empty "position" learns its predecessor tuple. If the predecessor tuple of an empty "position"  $l$  is (" $S_i$ ",  $x$ ), and the predecessor tuple is at "position"  $l'$ , then create a tuple (" $S_{i,l-l'}$ ",  $x$ ) at this empty position. Thus, at the end of all the computations,  $S_{1,1}, S_{1,2}, \dots, S_{1,s_1}, S_{2,1}, S_{2,2}, \dots, S_{2,s_2}, \dots, S_{k,s_k}$  are stored on the system.

**Indexing elements in sets** Suppose we have  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. The goal is to compute a mapping  $f$  such that  $\forall i \in [k], x \in S_i$ ,  $x$  is the  $f(S_i, x)^{\text{th}}$  element of  $S_i$ .

To achieve this goal, we can sort (See Theorem E.1) all the tuples such that the elements from the same set are stored consecutively on several machines. Then we can run indexing algorithm (See Theorem E.2) to compute the global index of each tuple. In the next, each machine scans its local data. If (" $S_i$ ",  $x$ ) is in the local memory, and  $x$  is the first element of  $S_i$ , then the machine marks this tuple as "1". For other tuples in the local memory, the machine marks them as "0".

Then we can invoke predecessor algorithm (See Theorem E.3) on all the tuples. At the end of the computation, each machine scans its all tuples. For a tuple  $(“S_i”, x)$  with global index  $l$ , the machine determine the index of  $x$  in  $S_i$  based on the global index  $l'$  of its predecessor  $(“S_i”, x)$ . Precisely, the machine creates a tuple  $(“f”, ((“S_i”, x), l - l' + 1))$  stored in the memory. Thus at the end of the computation, the desired mapping  $f$  is stored in the system.

**Set merging** Suppose we need to merge several sets  $S_1, S_2, \dots, S_k$  stored on the system, i.e. create a new set  $S = \bigcup_{i=1}^k S_i$ . To implement this operation, each machine scans its local memory. If there is a tuple  $(“S_i”, x)$  in its memory, then it creates a tuple  $(“S”, x)$ . Finally, we just need to remove all the duplicates.

**Set membership** Suppose we have  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. There is an another set  $Q = \{(x_1, y_1), \dots, (x_q, y_q)\}$  also stored in the system where  $x_i$  is the name of a set  $S$ , and  $y_i$  is an item. The goal is to answer whether  $y_i$  is in  $S$ .

To achieve this, we can firstly sort all the tuples. For tuple with form  $(“S_i”, x)$ , the first key is  $S_i$ , the second key is  $x$ , and the third key is  $-\infty$  which has the highest priority. For tuple with form  $(“Q”, (x, y))$ , the first key is  $x$ , the second key is  $y$ , and the third key is  $\infty$  which has the lowest priority. The comparison in the sorting procedure firstly compare the first key, then the second key, and finally the third key. After sorting, for each tuple with form  $(“S_i”, x)$ , we mark it as “1”. For each tuple with form  $(“Q”, (x, y))$ , we mark it as “0”. Now we can apply the predecessor algorithm (See Theorem E.3). For each tuple  $(“Q”, (x, y))$ , if its predecessor is  $(“S”, y)$  where  $x$  is the name of “S”, then we create a tuple  $(“f”, ((x, y), 1))$ ; Otherwise, we create a tuple  $(“f”, ((x, y), 0))$ . Thus, at the end of the computation, there is a mapping  $f$  stored on the system such that for each  $(x, y) \in Q$ , if  $x$  is the name of some set  $S_i$ , and  $y \in S_i$ , then  $f(x, y) = 1$ ; Otherwise  $f(x, y) = 0$ .

## E.4 Mapping Operations

In this section, we introduce some MPC model operations for mapping. The most important operation is called Multiple queries.

**Multiple queries** We have  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. Without loss of generality,  $S_1, S_2, \dots, S_t$  ( $t \leq k$ ) are sets representations of mappings (See Definition E.4)  $f_1 : U_1 \rightarrow H_1, f_2 : U_2 \rightarrow H_2, \dots, f_t : U_t \rightarrow H_t$  respectively. When a machine does local computation, it may need to query some values which are in the form  $f_i(u)$  for some  $u \in U_i$ . The following lemma shows that we can answer all the such queries simultaneously in constant number of rounds in  $(0, \delta)$  – MPC model for all constant  $\delta \in (0, 1)$ . It means that we can use constant number of rounds to simulate concurrent read operations on a shared memory where  $S_1, \dots, S_k$  are stored in the shared memory.

**Lemma E.6** (Multiple queries). *Let  $\delta \in (0, 1)$  be an arbitrary constant. There is a constant number of rounds algorithm  $\mathcal{A}$  in  $(0, \delta)$  – MPC model which satisfies the following properties. The input of  $\mathcal{A}$  contains two parts. The first part are  $k$  sets  $S_1, S_2, \dots, S_k$  stored (See Section E.2 for data organization of sets) on the input machines, where  $S_1, S_2, \dots, S_t$  ( $t \leq k$ ) are sets representations of mappings (See Definition E.4)  $f_1 : U_1 \rightarrow H_1, f_2 : U_2 \rightarrow H_2, \dots, f_t : U_t \rightarrow H_t$  respectively. The second part is a set  $Q = \{(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_q, y_q, z_q)\}$  stored on the input machines, where  $\forall (x, y, z) \in Q$ ,  $x$  is the name “ $f_i$ ” of the mapping  $f_i$  for some  $i \in [t]$ ,  $y$  is an element in  $U_i$ , and  $z$  is the index of the input machine which holds the element  $(x, y, z)$  of  $Q$ . The total input size  $n = |Q| + \sum_{i=1}^k |S_i|$ . The output machines are all the input machines.  $\forall i \in [k], x \in S_i$ , if the element  $x$  of  $S_i$  is held by the input (also output) machine  $j$ , then at the end of the computation,*



the element  $x$  of  $S_i$  should still be held by the output (also input) machine  $j$ . Let  $Q'$  be the set  $\{(x, y, z, w) \mid \exists (x, y, z) \in Q, w = f_i(y), \text{ where } x \text{ is the name of } f_i\}$ . At the end of the computation,  $Q'$  is stored on the output (also input) machines such that  $\forall (x, y, z, w) \in Q'$ , the element  $(x, y, z, w)$  of  $Q'$  is held by the machine  $z$ .

*Proof.* The idea is that we can firstly use sorting (See Theorem E.1) to make queries and the corresponding values be stored consecutively in several machines. The issue remaining is that there may be many queries queried the same position such that some queries may not be stored in the machine which holds the corresponding value. In this case, we need to find the predecessor by invoking the algorithm shown in Theorem E.3.  $\square$

The Multiple queries algorithm is shown as the following:

Multiple Queries Algorithm:

- **Setups:**
  - There are  $3p = \Theta(n^\delta)$  machines indexed from 1 to  $3p$  each with local memory size  $s = \Theta(n^\delta)$ .
  - The machine with index from  $2p + 1$  to  $3p$  are input/output machines.
  - Sets  $S_1, S_2, \dots, S_k, Q$  are stored on machine  $2p + 1$  to  $3p$ .  $\triangleright$  Corresponding to Lemma E.6
- **The first round:**
  - Machine  $i \in \{2p + 1, \dots, 3p\}$  scans its local memory, and send all the tuples with form  $(“f_j”, (x, y))$  or  $(“Q”, (x, y, z))$  to machine  $i - p$ , where  $“f_j”$  is the name of  $f_j$  (also  $S_j$ ) for  $j \in [t]$ . Until the end of the computation, machine  $i$  sends itself messages to keep its local data.
- **Using constant number ( $O(1/\delta)$ ) of rounds to sort:**
  - Use machine 1 to  $2p$  to sort all the tuples stored on machine  $p + 1$  to  $2p$ , and thus at the end of this stage, machine  $p + 1$  to  $2p$  holds sorted tuples. For tuple with the form  $(“f_j”, (x, y))$ , the first key value is  $“f_j”$ , the second key value is  $x$  and the third key value is  $-\infty$  which is the highest priority. For tuple with form  $(“Q”, (x, y, z))$ , the first key value is  $x$ , the second key value is  $y$ , and the third key value is  $\infty$  which is the lowest priority. The comparison in the sorting is: Firstly compare the first key. If they are the same, then compare the second key. If they are still the same, compare the third key.
- **Using constant number ( $O(1/\delta)$ ) of rounds to find predecessors:**
  - Machine  $p + 1$  to  $2p$  scans its local memory. For a tuple in the form  $(“f_j”, (x, y))$ , the machine marked it as “1”. For a tuple in the form  $(“Q”, (x, y, z))$ , the machine marked it as “0”.
  - Machine 1 to  $2p$  together invoke the Predecessor algorithm (Theorem E.3), where the input is on machine  $p + 1$  to machine  $2p$ .
- **The last round:**
  - Machine  $p + 1$  to  $2p$  scans its local memory. For each tuple with form  $(“Q”, (x, y, z))$ , it sends machine  $z$  a tuple  $(“Q'”, (x, y, z, w))$ , where  $x$  is the name of  $f_j$ , and  $w = f_j(y)$ .

## E.5 Sequence Operations

In this section, we introduce some MPC model operations for sequence.

**Sequence standardizing** Suppose there is a sequence  $A$ , and one of its set representation (see Definition E.5)  $S$  is stored in the system. The goal is to modify the set  $S$  such that  $S$  is a standard set representation of  $A$ .

We can compute the index (see **Indexing elements in sets** in Section E.3) of elements in  $S$ . Then for each element  $(x, y) \in S$ , we can query (see **Multiple queries** in Section E.4) the index of  $(x, y)$  in  $S$ . Suppose the index is  $i$ , we modify the tuple  $(“S”, (x, y))$  to  $(“S”, (i, y))$ .

**Sequence duplicating** Suppose there is a sequence  $A = (a_1, a_2, \dots, a_s)$ , and one of its set representation (see Definition E.5)  $S$  is stored in the system. Furthermore, there is a mapping  $f : [s] \rightarrow \mathbb{Z}_{\geq 0}$  which is also stored in the system. The goal is to get a set  $S'$  stored in the system such that  $S'$  is a set representation of the sequence:

$$\underbrace{(a_1, a_1, \dots, a_1)}_{f(1) \text{ times}}, \underbrace{(a_2, a_2, \dots, a_2)}_{f(2) \text{ times}}, \dots, \underbrace{(a_s, a_s, \dots, a_s)}_{f(s) \text{ times}}.$$

Firstly, we can standardize (see the above paragraph **Sequence standardizing**) the set  $S$ . Then for each tuple  $(“S”, (i, a_i))$ , we create a tuple  $(“S_i”, a_i)$ , and we can query (see **Multiple queries** in Section E.4) the value of  $f(i)$ . Then we can copy (see **Copies of sets** in Section E.3) set  $S_i$   $f(i)$  times. For each tuple  $(“S_{i,j}”, a_i)$ , we create a tuple  $(“S'”, ((i, j), a_i))$ . Then we can compute the index (see **Indexing elements in sets** in Section E.3) of each element in  $S'$ . For each tuple  $(“S'”, ((i, j), a_i))$ , we can query (see **Multiple queries** in Section E.4) the index  $i'$  of it, and then modify the tuple as  $(“S'”, (i', a_i))$ .

**Sequence insertion** Suppose there are  $k + 1$  sequences  $A = (a_1, a_2, \dots, a_s), A_1, \dots, A_k$  which have sets representations (see Definition E.5)  $S, S_1, \dots, S_k$  respectively and stored on the system. There is also a mapping  $f : [k] \rightarrow \{0\} \cup [s]$  stored on the system where  $\forall i \neq j \in [k], f(i) \neq f(j)$ . The goal is to insert each sequence  $A_i$  into the sequence  $A$ , and  $A_i$  should be between the element  $a_{f(i)}$  and  $a_{f(i)+1}$ .

Firstly, we can standardize (see **Sequence standardizing** in Section E.3)  $S$ . Then we can compute the total size (see **Sizes of sets** in Section E.3)  $N = |S| + |S_1| + \dots + |S_k| + 1$ . For each tuple  $(“S”, (i, a_i))$ , we can modify it as  $(“S”, (i \cdot N, a_i))$ . For each tuple  $(“S_i”, (j, a_{ij}))$ , we query (see **Multiple queries** in Section E.4) the value of  $f(i)$ , then create a tuple  $(“S”, (f(i) \cdot N + j, a_{ij}))$ .

## E.6 Multiple Tasks

In this section, we show that if the entire computational tasks consist of some independent small computational tasks, then we are able to schedule the machines such that the small computational tasks can be computed simultaneously.

**Task and multiple tasks problem** A computational task here is running a specific algorithm on specific input data.

There are  $k$  sets  $S_1, S_2, \dots, S_k$  stored in the system. Let  $n = \sum_{i=1}^k |S_i|$  be the total input size. There are  $h$  independent computational tasks  $T_1, T_2, \dots, T_h$ . Each task  $T_i$  needs to take some sets  $\mathcal{S}_i \subseteq \{S_1, S_2, \dots, S_k\}$  as its input, and is running a  $(\gamma_i, \delta_i)$  – MPC algorithm in  $r_i$  rounds where  $\gamma_i \in \mathbb{R}_{\geq 0}$ , constant  $\delta_i \in (0, 1)$ .  $\forall i \in [h]$ , let  $n_i = \sum_{S \in \mathcal{S}_i} |S|$  be the input size of task  $T_i$ . Without loss of generality, we can assume that the input of different tasks are disjoint. Otherwise we can use sets copying technique (See Section E.3) to generate different copies of input sets for the tasks shared the same input set. The goal here is to use the small number of rounds to finish all the tasks. Since we can always use sorting and indexing to extract the desired input data. The most naive way is to compute the tasks one-by-one. This can be trivially done in  $r = O(\sum_{i=1}^h r_i)$  rounds in  $(\gamma, \delta)$  – MPC model for  $\gamma = \log_n(h) + \max_{i \in [h]} \gamma_i, \delta = \max_{i \in [h]} \delta_i$ . Here we show how

to compute all the tasks simultaneously in  $r = O(\max_{i \in [h]} r_i)$  rounds in  $(\gamma, \delta)$  – MPC model for  $\gamma = \log_n(m) - 1, \delta = \max_{i \in [h]} \delta_i$ , where  $m = \Theta(n + \sum_{i=1}^h n_i^{1+\gamma_i})$ .

Each machine scans its local memory. If the machine holds a tuple  $(“S_i”, x)$ , and  $S_i$  is a part of input of task  $T_j$ , then it creates a tuple  $(“W_j”, (“S_i”, x))$ . Thus, at the end of this step, there are additional  $h$  sets  $W_1, W_2, \dots, W_h$  stored in the system. Here  $W_i, i \in [h]$  contains all the information of input data of task  $T_i$ . Then we can compute a mapping  $f$  such that  $\forall i \in [h], f(W_i) = |W_i|$  (see Section E.3). Thus, we know the input size of each task. Then each machine scans its local memory. If the machine holds a tuple  $(“f”, (“W_i”, |W_i|))$ , then it creates a tuple  $(“H_i”, |W_i|)$ , i.e. a set  $H_i = \{|W_i|\}$ . Then for each set  $H_i = \{|W_i|\}, i \in [h]$ , we can copy (see Section E.3) it  $s_i = c \cdot |W_i|^{1+\gamma_i}$  times for a sufficiently large  $c$  to get sets  $H_{i,1} = H_{i,2} = \dots = H_{i,s_i} = |W_i|$ . Each set  $H_{i,j}$  is just a placeholder of one unit working space of the task  $T_i$ . Thus, the number of copies of the set  $H_i$  is the total space needed for the task  $T_i$ . We can sort all the tuples  $(“H_{i,j}”, |W_i|)$  on machines with index in  $I = \{2, 5, 8, 11, \dots, 3p - 1\}$ , where local memory  $s = \Theta(n^\delta)$ , total required memory  $m = \Theta(n + \sum_{i=1}^h n_i^{1+\gamma_i})$ , and  $p = \Theta(m/s)$ . For each machine with index  $q \in I$ , the tuples on that machine must be in the following form

$$\begin{aligned} & (“H_{i,j}”, |W_i|), (“H_{i,j+1}”, |W_i|), \dots, (“H_{i,s_i}”, |W_i|), (“H_{i+1,1}”, |W_{i+1}|), \dots, (“H_{i+1,s_{i+1}}”, |W_{i+1}|), \\ & (“H_{i+2,1}”, |W_{i+2}|), \dots, (“H_{i+2,s_{i+2}}”, |W_{i+2}|), \dots, (“H_{i',1}”, |W_{i'}|), \dots, (“H_{i',j'}”, |W_{i'}|). \end{aligned}$$

Then machine  $q$  just sends all the tuples  $(“H_{i,j}”, |W_i|), (“H_{i,j+1}”, |W_i|), \dots, (“H_{i,s_i}”, |W_i|)$  to machine  $q - 1$ , and sends all the tuples  $(“H_{i',1}”, |W_{i'}|), (“H_{i',2}”, |W_{i'}|), \dots, (“H_{i',j'}”, |W_{i'}|)$  to machine  $q + 1$ . Thus,  $\forall i \in [h]$ ,

1. either all the  $H_{i,1}, H_{i,2}, \dots, H_{i,s_i}$  are stored on consecutive machines, machine  $q$  to machine  $q'$ , and any of machine  $q$  to machine  $q'$  does not hold other tuples,
2. or there is a unique machine  $q$  which holds all the sets  $H_{i,1}, H_{i,2}, \dots, H_{i,s_i}$ .

For each machine  $q \in [3p]$ , if  $H_{i,1}$  is held by machine  $q$ , then it creates a tuple  $(“st”, (“T_i”, q))$ . If  $H_{i,s_i}$  is held by machine  $q$ , then it creates a tuple  $(“ed”, (“T_i”, q))$ . The mapping  $st, ed$  then are stored in the system, where  $st(T_i)$  is the index of the first machine assigned to task  $T_i$ , and  $ed(T_i)$  is the index of the last machine assigned to task  $T_i$ . Recall that  $W_i$  contains all the information of the input data to task  $T_i$ . The remaining task is to move the input data of task  $T_i$  to the machines with index from  $st(T_i)$  to  $ed(T_i)$ . According to Section E.3, we can compute a mapping  $f'$ , such that  $f'(W_i, x)$  records the index of  $x \in W_i$  in set  $W_i$ . Now, each machine scans its local memory. For each tuple  $(“W_j”, (“S_i”, x))$ , the machine needs to query the value of  $f'(W_j, (“S_i”, x))$ , the value of  $st(T_j)$  and the value of  $ed(T_j)$ . By Lemma E.6, these queries can be handled simultaneously in constant number of rounds. Then the machine can send the tuple  $(“S_i”, x)$  to the corresponding machine based on the value of  $f'(W_j, (“S_i”, x))$ ,  $st(T_j)$ , and  $ed(T_j)$ . Finally,  $\forall i \in [h]$ , since  $\delta \geq \delta_i$  and  $(ed(T_i) - st(T_i) + 1) \cdot s = \Theta(n_i^{1+\gamma_i})$ , the machines with index from  $st(T_i)$  to  $ed(T_i)$  can simulate task  $T_i$  in  $r_i$  number of rounds.

## F Implementations in MPC Model

In this section, we show how to implement all the previous batch algorithms in MPC model.

### F.1 Neighbor Increment Operation

**Lemma F.1.** *Let graph  $G = (V, E), n = |V|, N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ . NEIGHBORINCREMENT( $m, G$ ) (Algorithm 1) can be implemented in  $(\gamma, \delta)$  – MPC model*

for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition B.2) of NEIGHBORINCREMENT( $m, G$ ).

*Proof.* To implement line 7, we can create a tuple  $(“S_v^{(0)}”, u)$  for each tuple  $(“E”, (v, u))$ . Then for each  $(“S_v^{(0)}”, u)$  we can compute the index (see **Indexing elements in sets** and **Multiple queries**) of  $u$  in set  $S_v^{(0)}$ . If the index of  $u$  in set  $S_v^{(0)}$  is at least  $\lceil (m/n)^{1/2} \rceil$ , then delete  $u$  from  $S_v^{(0)}$ , i.e. delete the tuple  $(“S_v^{(0)}”, u)$ .

Now let us discuss how to implement line 14 and line 17 in the  $i^{\text{th}}$  iteration. Firstly, we can compute the size of every set stored in the system (see **Sizes of sets**). Then for each tuple  $(“S_v^{(i-1)}”, u)$ , the corresponding machine queries (see **Multiple queries**) the size of  $S_u^{(i-1)}$ . If  $|S_u^{(i-1)}| \geq \lceil (m/n)^{1/2} \rceil$ , then create a tuple  $(“\text{temp}_v^i”, u)$ . We can index (see **Indexing elements in sets**) all the elements in set  $\text{temp}_v^i$ , and only keep the element with index 1. Thus,  $\text{temp}_v^i$  has a only element  $u$ , and we need to create a set  $S_v^{(i)} = S_u^{(i-1)}$ . Notice that there may be many  $v \in V$  which needs need to implement  $S_v^{(i)} = S_u^{(i-1)}$ . Thus, for each tuple  $(“\text{temp}_v^i”, u)$ , we create a tuple  $(“\text{target}_u^i”, v)$ .  $v \in \text{target}_u^i$  means that  $S_v^{(i)}$  needs a copy of  $S_u^{(i-1)}$ . Thus,  $|\text{target}_u^i|$  means that  $S_u^{(i-1)}$  needs to copy  $|\text{target}_u^i|$  times. For each tuple  $(“S_u^{(i-1)}”, x)$ , the machine queries (see **Multiple queries**) the size of  $\text{target}_u^i$ . Then each set  $S_u^{(i-1)}$  can be copied (see **Copies of sets**)  $|\text{target}_u^i|$  times. For each tuple  $(“\text{target}_u^i”, v)$ , we query (see **Multiple queries**) the index (see **Indexing elements in sets**) of  $v$  in set  $\text{target}_u^i$ , and then create a tuple  $(“f^i”, ((“\text{target}_u^i”, x), v))$ , where  $x$  is the index of  $v$  in  $\text{target}_u^i$ . Thus  $f^i$  is a mapping such that  $f^i(\text{target}_u^i, x)$  is the  $x^{\text{th}}$  element in  $\text{target}_u^i$ . For each tuple  $(“S_{u,j}^{(i-1)}”, x)$ , we query (see **Multiple queries**) the value  $v = f^i(\text{target}_u^i, j)$ , and then create a tuple  $(“S_v^{(i)}”, x)$ , and a tuple  $(“S_v^{(i)}”, v)$ . We then remove the duplicates (see **Duplicates removing**) of elements of for every set  $S_v^{(i)}$ . For each tuple  $(“\text{temp}_v^i”, u)$ , query (see **Multiple queries**) the size (see **Sizes of sets**) of  $S_v^{(i)}$  and  $S_u^{(i-1)}$ . If  $|S_v^{(i)}| > |S_u^{(i-1)}|$ , then we create a tuple  $(“g^i”, (v, (u, “delete”)))$ ; Otherwise, create a tuple  $(“g^i”, (v, (u, “keep”)))$ . Finally, for each tuple  $(S_v^{(i)}, x)$ , we query (see **Multiple queries**)  $(u, o) = g^i(v)$ , if  $u = x$  and  $o = “delete”$ , the machine deletes the tuple  $(S_v^{(i)}, x)$ .

Next, let us discuss how to implement line 20. Similar as before, we can compute the size of every set stored in the system (see **Sizes of sets**). Then for each tuple  $(“S_v^{(i-1)}”, u)$ , the corresponding machine queries (see **Multiple queries**) the size of  $S_u^{(i-1)}$ . If  $|S_u^{(i-1)}| \geq \lceil (m/n)^{1/2} \rceil$ , then create a tuple  $(“\text{temp}_v^i”, u)$ . For each tuple  $(“V”, v)$ , we can create a tuple  $(“\text{temp}_v^i”, \text{null})$ . Then for each tuple  $(“V”, v)$  we can query (see **Multiple queries**) the size (see **Sizes of sets**) of  $\text{temp}_v^i$ . If  $|\text{temp}_v^i| = 1$ , then we create a tuple  $(“f^i”, 1)$ ; Otherwise, we create a tuple  $(“f^i”, 0)$ . Thus, mapping  $f^i$  is stored in the system, and  $f^i(v) = 1$  if and only if  $\forall u \in S_v^{(i-1)}, |S_u^{(i-1)}| < \lceil (m/n)^{1/2} \rceil$ . For each tuple  $(“S_v^{(i-1)}”, u)$ , we query (see **Multiple queries**) the value  $f^i(v)$ . If  $f^i(v) = 1$ , we create a tuple  $(“\text{target}_u^i”, v)$ . Thus,  $v \in \text{target}_u^i$  means that  $S_u^{(i-1)}$  should be a part of  $S_v^{(i)}$ .  $|\text{target}_u^i|$  means that  $S_u^{(i-1)}$  needs to copy  $|\text{target}_u^i|$  times. For each tuple  $(“S_u^{(i-1)}”, v)$ , we query (see **Multiple queries**) the size (see **Sizes of sets**) of  $\text{target}_u^i$ . Then we can copy (see **Copies of sets**) each set  $S_u^{(i-1)}$   $|\text{target}_u^i|$  times. Then for each tuple  $(“\text{target}_u^i”, v)$ , we can query (see **Multiple queries**) the index  $x$  (see **Indexing elements in sets**) of  $v$  in set  $\text{target}_u^i$ , and then create a tuple  $(“f^i”, ((“\text{target}_u^i”, x), v))$  which means that the  $x^{\text{th}}$  element of  $\text{target}_u^i$  is  $f^i(\text{target}_u^i, x) = v$ . For each tuple  $(“S_{u,j}^{(i-1)}”, x)$ , we query (see **Multiple queries**) the value  $v = f^i(\text{target}_u^i, j)$ , and then create a tuple  $(“S_v^{(i)}”, x)$ . We then remove the duplicates (see **Duplicates removing**) of elements of for every set  $S_v^{(i)}$ .

Finally, let us consider how to implement line 24. It is very simple, we only need to query the sizes of sets. For each tuple  $(“V”, v)$ , query (see **Multiple queries**) the size (see **Sizes of sets**) of  $S_v^{(i)}$  and  $S_v^{(i-1)}$ , if  $v$  satisfies the condition, create a tuple  $(“Done”, v)$ . Every machine queries (see **Multiple queries**) the size (see **Sizes of sets**) of Done. If it is  $|V|$ , then all the machines know that they finish the loop. In the end, for each tuple  $(S_v^{(r)}, u)$  we create tuples  $(“E'”, (u, v)), (“E'”, (v, u))$ , and for each tuple  $(“E”, (u, v))$  we create tuple  $(“E'”, (u, v))$ . Then we then remove the duplicates (see **Duplicates removing**) of elements of  $E$ .

In the  $i^{\text{th}}$  iteration, we only need to maintain sets  $V, E, S_v^{(i-1)}$ . Since all the copy operation will create at most  $n \cdot (m/n)^{1/2} \cdot (m/n)^{1/2} = m$  tuples, the total space needed is  $\Theta(m)$  plus the space needed to maintain  $V, E, S_v^{(i)}$ . By Property 4 of Lemma B.1,  $|S_v^{(i)}| \leq m/n$ . Thus, the total space is  $\Theta(m) + |V| + |E| + \sum_{v \in V} |S_v^{(i)}| = \Theta(m) + N = \Theta(m)$ .

The above implementation shows that the parallel time is  $O(r)$ , where  $r$  is the number of iterations (see Definition B.2).  $\square$

## F.2 Tree Contraction Operation

In this section, we show how to implement Algorithm 2 in MPC model.

**Lemma F.2.** *Let graph  $G = (V, E)$  and  $\text{par} : V \rightarrow V$  be a set of parent points (see Definition B.6) on the vertex set  $V$ . TREECONTRACTION( $G, \text{par}$ ) (Algorithm 2) can be implemented in  $(0, \delta)$ -MPC model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition B.12) of TREECONTRACTION( $G, \text{par}$ ).*

*Proof.* Let  $N = |V| + |E|$ . Then the total space is  $\Theta(N)$ .

Initially, each machine scans its local memory. If there is a tuple  $(“V”, v)$ , then it queries the value of  $\text{par}(v)$ . It needs  $O(1)$  parallel time to answer all the queries (see Multiple queries in Lemma E.6). Then the machine creates a tuple  $(“g^{(0)}”, (v, \text{par}(v)))$ . Thus, in the initialization stage, mapping  $g^{(0)}, \text{par}$ , set  $V, E$  are stored in the system.

In the  $l^{\text{th}}$  iteration, Each machine scans its local memory. If there is a tuple  $(“V”, v)$ , then it queries the value of  $g^{(l-1)}(v)$ . This can be done by Multiple queries. Then it queries the value of  $\text{par}(g^{(l-1)}(v))$ . This can also be done by Multiple queries. If  $\text{par}(g^{(l-1)}(v)) = g^{(l-1)}(v)$ , it creates a tuple  $(“Done”, v)$ . Then the machines can compute the sizes (see Section E.3) of  $V$  and Done. Each machine queries the size of  $V$  and Done. This can be done by Multiple queries. Then if  $|V| = |\text{Done}|$ , every machine knows that the iterations are finished. Otherwise, the machine which holds  $(“V”, v)$  queries the value of  $g^{(l-1)}(g^{(l-1)}(v))$ . This can be done by Multiple queries. And then it creates a tuple  $(“g^{(l)}”, (v, g^{(l-1)}(g^{(l-1)}(v))))$ .

At the end, if a machine holds a tuple  $(“V”, v)$ , then the queries  $\text{par}(v)$ . If  $v = \text{par}(v)$ , it creates a tuple  $(“V'”, v)$ . If a machine holds a tuple  $(“E”, (u, v))$ , then it queries  $g^{(r)}(u), g^{(r)}(v)$ , and creates a tuple  $(“E'”, (g^{(r)}(u), g^{(r)}(v)))$ .

Since at the end of each iteration  $l$ , the system only stores mappings  $\text{par} : V \rightarrow V, g^{(r)} : V \rightarrow V$ , and sets  $V, E$ , the total space used is at most  $O(N)$ . Thus, we can implement the algorithm in  $(0, \delta)$ -MPC model.

The total parallel time is  $O(r)$ . By Corollary B.13,  $r = O(\text{dep}(\text{par}))$ . Thus, the total parallel time is  $O(\text{dep}(\text{par}))$ .  $\square$

## F.3 Graph Connectivity

**Theorem F.3.** *Let graph  $G = (V, E), n = |V|, N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ . Let  $r > 0$  be a round parameter. CONNECTIVITY( $G, m, r$ ) (Algorithm 3) can be implemented*



in  $(\gamma, \delta)$  – MPC model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(R)$ , where  $R$  is the total number of iterations (see Definition B.20) of  $\text{CONNECTIVITY}(G, m, r)$ .

*Proof.* Initially, we store sets  $V_0, E_0, V, E$  and mapping  $h_0$  in the system. Now consider the  $i^{\text{th}}$  round. Due to Lemma F.1, line 9 can be implemented in total space  $\Theta(m)$  and with  $O(k_i)$  parallel time, where  $k_i$  is the number of iterations (See Definition B.2) of  $\text{NEIGHBORINCREMENT}(m, G_{i-1})$ . To store  $V'_i$  and  $E'_i$ , we need total space  $\Theta(m)$ . Line 10 can be implemented by operations described in **Sizes of sets** and **Multiple queries** (see Section E). Line 11 can be implemented by the operations described in **Set membership** and **Multiple queries**. To implement line 14, for each tuple  $(“V''_i”, v)$ , we can create a tuple  $(“l_i”, (v, x))$  where  $x = 1$  with probability  $p_i$ ,  $x = 0$  with probability  $1 - p_i$ . To calculate  $p_i$ , the machine only needs to know  $n_{i-1}$ . This can be done by the operations described in **Sizes of sets** and **Multiple queries**. Line 15 and line 16 can be implemented by operations described in **Set membership** and **Multiple queries**. For line 17, set  $L_i \cap (\Gamma_{G'_i}(v) \cup \{v\})$  can be computed by operations described in **Set membership** and **Multiple queries**. Then, by operations in **Indexing elements in sets** and **Multiple queries**, we can get  $\min_{u \in L_i \cap (\Gamma_{G'_i}(v) \cup \{v\})} u$ . Finally, by operation described in **Multiple queries**,  $\forall v \in V''_i$  with  $v \notin L_i$ , the tuple  $(“par_i”, (v, x))$  can be created, where  $x = \min_{u \in L_i \cap (\Gamma_{G'_i}(v) \cup \{v\})} u$ . Due to Lemma F.2, line 18 can be implemented in total  $\Theta(m)$  space and  $O(r'_i)$  parallel running time, where  $r'_i$  is the number of iterations (see Definition B.12) of  $\text{TreeContraction}(G''_i, \text{par}_i)$ . Line 21 can be implemented by operations in **Set membership**, **Indexing elements in sets** and **Multiple queries**. Line 22 can be implemented by operations in **Set membership** and **Multiple queries**. Line 23 can be implemented by **Multiple queries**. For other  $v \in V$  with  $h_i(v) = \text{null}$  assigned by line 8, we can use the operations in **Set membership** and **Multiple queries** to find those  $v$ , and create a tuple  $(“h_i”, v, \text{null})$ .

Thus, in the  $i^{\text{th}}$  round, the parallel time needed is  $O(k_i + r'_i)$ . At the end of the  $i^{\text{th}}$  round, we only need to keep sets  $V_i, E_i, V, E$  and mapping  $h_i$  in the system. It will take total space at most  $O(m)$ .

Due to Lemma F.2, line 26 can be implemented in at most  $O(m)$  total space and  $O(\log r)$  parallel time.

Thus, the total parallel time is  $O(\log r + \sum_{i=1}^r (k_i + r'_i)) = O(\sum_{i=1}^r (k_i + r'_i))$ . By definition B.20, the total parallel time is  $O(R)$ , where  $R$  is the total number of iterations of  $\text{CONNECTIVITY}(G, m, r)$ . The total space in the computation is always at most  $\Theta(m)$ .  $\square$

Here, we are able to conclude the following theorem for graph connectivity problem.

**Theorem F.4.** *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm (see Algorithm 3) which can output the connected components for any graph  $G = (V, E)$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time, where  $D$  is the diameter of  $G$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $\gamma' = (1 + \gamma) \log_n \frac{2N}{n^{1/(1+\gamma)}}$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* The implementation of Algorithm 3 in MPC model is shown by Theorem F.3. The correctness of Algorithm 3 is proved by Theorem B.14. The total parallel time of Algorithm 3 is proved by Theorem B.21.  $\square$

## F.4 Algorithms for Local Shortest Path Trees

In this section, we mainly explained how to implement local shortest path tree algorithms described in Section C.1 and Section C.2.

**Lemma F.5.** Let  $G = (V, E)$  be an undirected graph,  $s_1, s_2 \in \mathbb{Z}_{\geq 0}$ , and  $v \in V$ . Let  $\tilde{T} = (V_{\tilde{T}}, \text{par}_{\tilde{T}})$  with root  $v$  and radius  $s_1$  be a local complete shortest path tree (see Definition C.3) in  $G$ , and  $\text{dep}_{\tilde{T}} : V_{\tilde{T}} \rightarrow \mathbb{Z}_{\geq 0}$  be the depth of every vertex in  $\tilde{T}$ .  $\forall u \in V_{\tilde{T}}$ , let  $T(u)$  with root  $u$  and radius  $s_2$  be a local complete shortest path tree in  $G$ , and  $\text{dep}_{T(u)} : V_{T(u)} \rightarrow \mathbb{Z}_{\geq 0}$  be the depth of every vertex in  $T(u)$ . Then  $\text{TREEEXPANSION}(\tilde{T}, \text{dep}_{\tilde{T}}, \{T(u) \mid u \in V_{\tilde{T}}\}, \{\text{dep}_{T(u)} \mid u \in V_{\tilde{T}}\})$  (Algorithm 4) can be implemented in  $(0, \delta) - \text{MPC}$  model for any constant  $\delta \in (0, 1)$  in  $O(1)$  parallel time.

*Proof.* For line 3, we apply operation shown in **Copies of sets** to copy each  $V_{T(u)}$ , then we can merge (see **Set merging**) all the copies to get  $V_{\tilde{T}}$ . To implement line 4 and line 5, we only need to apply the operation shown in **Multiple queries**. To implement line 6, for each tuple  $(“V_{T(u)}”, x)$ , we can firstly check whether  $x \in V_{\tilde{T}} \setminus V_{\tilde{T}}$  by operations described in **Set membership** and **Multiple queries**. If  $x \in V_{\tilde{T}} \setminus V_{\tilde{T}}$ , then we can query the values of  $\text{dep}_{\tilde{T}}(u)$  and  $\text{dep}_{T(u)}(x)$  by operations shown in **Multiple queries**. Then we create a tuple  $(“\text{temp}_x”, (\text{dep}_{\tilde{T}}(u) + \text{dep}_{T(u)}(x), u))$ . By **Indexing elements in sets** and **Multiple queries**, we can find the element with the smallest index in set  $\text{temp}_x$ , and thus that element is  $(\text{dep}_{\tilde{T}}(u_x) + \text{dep}_{T(u_x)}(x), u_x)$ . Finally, the remaining things in line 6 and line 7 can be done by the operations described by **Multiple queries**.

For all the operations, the total space is always linear. The parallel time needed for the above operations is also a constant.  $\square$

**Lemma F.6.** Let graph  $G = (V, E)$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ .  $\text{MULTIRADIUSLCSPT}(G, m)$  (Algorithm 5) can be implemented in  $(\gamma, \delta) - \text{MPC}$  model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition C.5) of  $\text{MULTIRADIUSLCSPT}(G, m)$ .

*Proof.* To implement line 4 to line 6, we can scan all the tuples  $(“E”, (u, v))$ , then query the size of  $\{v\} \cup \Gamma_G(v)$  and the size of  $\{u\} \cup \Gamma_G(u)$ , where these operations are described in **Sizes of sets** and **Multiple queries**. Then based on the sizes, we decide whether we need to create the corresponding tuples for  $V_{T_0(v)}, V_{T_0(u)}, \text{par}_{T_0(u)}, \text{par}(T_0(v))$ .

Now consider the main loop. We focus on the  $i^{\text{th}}$  round. Line 12 can be implemented by the operation described in **Multiple queries**. To implement line 13, for each tuple  $(“V_{T_{i-1}(v)}”, u)$ , we can query (see **Multiple queries**) whether  $T_{i-1}(u)$  is null. If  $T_{i-1}(u)$  is null, then we create a tuple  $(“\text{temp}_v”, u)$ . Then for each tuple  $(“V”, v)$ , we can query the size of  $\text{temp}_v$  by operations described in **Sizes of sets** and **Multiple queries**. If the size is not 0, then  $T_i(v)$  must be null. Line 15 can be implemented by coping input for different tasks and running tasks in parallel, where it only needs operations shown in **Copies of sets**, **Multiple queries** and **Multiple Tasks** (see Section E.6). According to Lemma F.5, it only needs  $O(1)$  parallel time. Line 16 and line 19 only need the operation shown in **Multiple queries**.

Thus, the total parallel time is  $O(r)$  where  $r$  is the number of iterations (see Definition C.5) of  $\text{MULTIRADIUSLCSPT}(G, m)$ . For the total space, we stored the sets  $V_{T_i(v)}$  for all  $i \in [r], v \in V$  and mappings  $\text{par}_{T_i(v)}, \text{dep}_{T_i(v)}$  for all  $i \in [r], v \in V$ . By Lemma C.6, the total space to store all of them is at most  $O(r \cdot n \cdot (m/n)^{1/4}) = O(m)$ . In the  $i^{\text{th}}$  round of the main loop, line 15 may make copies of the set. By Lemma C.6, the input size of each task will be at most  $O((m/n)^{1/4} \cdot (m/n)^{1/4})$ . Since there are at most  $n$  tasks, the total space needed is at most  $O(m)$ .  $\square$

**Lemma F.7.** Let graph  $G = (V, E)$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ .  $\text{MULTIPLELARGETREES}(G, m)$  (Algorithm 6) can be implemented in  $(\gamma, \delta) - \text{MPC}$  model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition C.10) of  $\text{MULTIPLELARGETREES}(G, m)$ .



*Proof.* By Lemma F.6, line 3 can be implemented in total space  $m$  and  $O(r)$  parallel time where  $r$  is the number of iterations (see Definition C.5) of  $\text{MULTIRADIUSLCSPT}(G, m)$ . Line 4 to line 6 can be implemented by the operation described by **Multiple queries**. The implementation of line 7 to line 17 is similar as the implementation of the main loop of Algorithm 5 (See Lemma F.6 for details of the implementation). The implementation of line 18 and line 19 only needs the operation described in **Indexing elements in sets** and **Multiple queries**. Line 22 can be implemented by copying input sets for different tasks and running multiple tasks in parallel, where the operations needed are described in **Copies of sets**, **Multiple queries** and **Multiple Tasks** (see Section E.6). Line 24 to line 28 can be implemented by the operations described in **Copies of sets**, **Set membership**, **Indexing elements in sets**, and **Multiple queries**.

The total parallel time of the first loop is  $O(r)$  since it has  $r$  rounds. The second loop can be done in one round. Thus the parallel time of the second loop is  $O(1)$ . Then the total parallel time is  $O(r)$ . Due to Lemma F.6 and Lemma C.11,  $r$  is the number of iterations (see Definition C.10) of  $\text{MULTIPLELARGETREES}(G, m)$ .

We stored all the  $V_{T_i(v)}$ ,  $V_{\tilde{T}_i(v)}$ ,  $\text{par}_{T_i(v)}$ ,  $\text{par}_{\tilde{T}_i(v)}$ ,  $\text{dep}_{T_i(v)}$ ,  $\text{dep}_{\tilde{T}_i(v)}$  in the system. By Lemma C.7 and Lemma C.6, the total space needed to store them is at most  $O(m)$ . Furthermore, at any round, the size of all the input copies for multiple tasks is at most  $n \cdot (m/n)^{1/4} \cdot (m/n)^{1/4} = O(m)$ . Thus, the total space needed is  $O(m)$ .  $\square$

## F.5 Path Generation and Root Changing

**Lemma F.8.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $n = |V|$ .  $\text{FINDANCESTORS}(\text{par})$  (Algorithm 7) can be implemented in  $(\gamma, \delta) - \text{MPC}$  model for any  $\gamma \geq \frac{\log n}{\log \log n}$  and any constant  $\delta \in (0, 1)$ . The parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition C.12) of  $\text{FINDANCESTORS}(\text{par})$ .*

*Proof.* The structure of the whole algorithm is the same as the Algorithm 2 (see Lemma F.2). All the steps can be done by operation described in **Multiple queries**.

Since the number of rounds needed is  $r$ , the parallel time is  $O(r)$ . For the total space, we need to store all the mappings  $g_1, \dots, g_r$ . At the end of the  $i^{\text{th}}$  round, we need to store mapping  $h_i$ . According to Lemma C.13,  $r = O(\log n)$ . Thus, the total space is  $O(rn) = O(n \log n)$ .  $\square$

**Lemma F.9.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $q$  be a vertex in  $V$ , and  $n = |V|$ .  $\text{FINDPATH}(\text{par}, q)$  (Algorithm 8) can be implemented in  $(\gamma, \delta) - \text{MPC}$  model for any  $\gamma \geq \frac{\log n}{\log \log n}$  and any constant  $\delta \in (0, 1)$ . The parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition C.12) of  $\text{FINDANCESTORS}(\text{par})$  (Algorithm 7).*

*Proof.* By Lemma F.8,  $\text{FINDANCESTORS}(\text{par})$  can be implemented in  $(\gamma, \delta) - \text{MPC}$  model for  $\gamma \geq \frac{\log n}{\log \log n}$  and any constant  $\delta \in (0, 1)$ . All the other steps in the algorithm can be done by operation described in **Multiple queries**. Notice that, after each round, we need to do load balancing which can be done by operation described in **Load balance**.

The number of rounds must be smaller than  $O(r)$ , where  $r$  should be the number of iterations of  $\text{FINDANCESTORS}(\text{par})$  according to Lemma F.8.

We store all the mappings  $g_i, \text{dep}_{\text{par}}$  in the system. They need  $O(n \log n)$  total space. In the  $i^{\text{th}}$  round, we only need to additionally store set  $S_i$  which has size at most  $O(n)$ . Thus, the total space needed is at most  $O(n \log n)$ .  $\square$

**Lemma F.10.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $q$  be a vertex in  $V$ .  $\text{ROOTCHANGE}(\text{par}, q)$  (Algorithm 9) can be implemented in  $(\gamma, \delta) - \text{MPC}$*

model for any  $\gamma \geq \frac{\log n}{\log \log n}$  and any constant  $\delta \in (0, 1)$ . The parallel running time is  $O(r)$ , where  $r$  is the number of iterations (see Definition C.12) of FINDANCESTORS( $\text{par}$ ) (Algorithm 7).

*Proof.* By Lemma F.9, FINDPATH( $\text{par}, q$ ) can be implemented in  $(\gamma, \delta)$ -MPC model. The remaining steps in the procedure can be implemented by the operation described by **Multiple queries**, and has  $O(1)$  parallel running time.

The total space needed is the total space needed for FINDPATH( $\text{par}, q$ ) plus the space needed to store mapping  $h, \widehat{\text{par}}$ . Thus the total space needed is  $O(n \log n) + O(n) = O(n \log n)$ .

The parallel running time is linear in the parallel running time of FINDPATH( $\text{par}, q$ ). Then, by Lemma F.9, the parallel running time is  $O(r)$  where  $r$  is the number of iterations (see Definition C.12) of FINDANCESTORS( $\text{par}$ ).  $\square$

## F.6 Spanning Forest Algorithm

**Lemma F.11.** Let  $G_2 = (V_2, E_2)$  be an undirected graph. Let  $\widehat{\text{par}} : V_2 \rightarrow V_2$  be a set of parent pointers (See Definition B.6) which satisfies that  $\forall v \in V_2$  with  $\widehat{\text{par}}(v) \neq v$ ,  $(v, \widehat{\text{par}}(v))$  must be in  $E_2$ . Let  $G_1 = (V_1, E_1)$  be an undirected graph satisfies  $V_1 = \{v \in V_2 \mid \widehat{\text{par}}(v) = v\}$ ,  $E_1 = \{(u, v) \in V_1 \times V_1 \mid u \neq v, \exists (x, y) \in E_2, \widehat{\text{par}}^{(\infty)}(x) = u, \widehat{\text{par}}^{(\infty)}(y) = v\}$ . Let  $\text{par} : V_1 \rightarrow V_1$  be a rooted spanning forest (See Definition C.18) of  $G_1$ . Let  $f : V_1 \times V_1 \rightarrow \{\text{null}\} \cup (V_2 \times V_2)$  satisfy the following property: for  $u \neq v \in V_1$ , if  $\text{par}(u) = v$ , then  $f(u, v) \in \{(x, y) \in E_2 \mid \widehat{\text{par}}^{(\infty)}(x) = u, \widehat{\text{par}}^{(\infty)}(y) = v\}$ , and  $f(v, u) \in \{(x, y) \in E_2 \mid \widehat{\text{par}}^{(\infty)}(x) = v, \widehat{\text{par}}^{(\infty)}(y) = u\}$ . Let  $n = |V_2|$ . Then FORESTEXPANSION( $\text{par}, \widehat{\text{par}}, f$ ) (Algorithm 10) can be implemented in  $(\gamma, \delta)$ -MPC model for any  $\gamma \geq \log n / \log \log n$  and any constant  $\delta \in (0, 1)$  in parallel running time  $O(R)$ , where  $R = \log(\text{dep}(\widehat{\text{par}}))$ .

*Proof.* Due to Lemma F.2, line 3 can be done in  $O(R)$  parallel time for  $R = \log(\text{dep}(\widehat{\text{par}}))$ . Line 9 corresponds to multiple tasks, we can implement them parallelly by operations described in **Multiple queries**, and **Multiple Tasks** (see Section E.6). By Lemma F.10, the total space needed is at most  $O(n \log n)$  and the parallel running time is at most  $O(R)$  where  $R = \log(\text{dep}(\widehat{\text{par}}))$ .  $\square$

**Theorem F.12.** Let graph  $G = (V, E)$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ . Let  $r > 0$  be a round parameter. SPANNINGFOREST( $G, m, r$ ) (Algorithm 11) can be implemented in  $(\gamma, \delta)$ -MPC model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(R)$ , where  $R$  is the total number of iterations (see Definition C.28) of SPANNINGFOREST( $G, m, r$ ).

*Proof.* At the beginning of the algorithm, we just store sets  $V, E, V_0, E_0$  and mapping  $g_0$  in the system.

Consider the  $i^{\text{th}}$  round of the loop. By Lemma F.7, line 8 can be implemented in total space  $\Theta(m)$  and in parallel running time  $O(k_i)$  where  $k_i$  is the number of iterations (see Definition C.10) of MULTIPLELARGETREES( $G_i, m$ ). Line 9 can be implemented by operations described in **Sizes of sets**, **Set membership**, and **Multiple queries**. Line 10 can be implemented by operations described in **Indexing elements in sets**, **Set membership**, and **Multiple queries**. In line 12, to calculate  $\gamma_i$ , we need to query  $n_i$ , this can be done by operations described in **Sizes of sets** and **Multiple queries**. In line 14, to compute  $L_i$ , we only need operations described in **Set membership** and **Multiple queries**. Line 15 can be implemented by operations shown in **Set membership**, **Indexing elements in sets** and **Multiple queries**. By Lemma F.9, for line 16, there are multiple tasks each can be implemented in  $O(|V_{\widetilde{T}_i(v)}| \log |V_{\widetilde{T}_i(v)}|)$  total space, and  $O(k_i)$  parallel time. We can schedule these multiple tasks (see Section E.6) such that we can finish them in parallel in  $O(k_i)$  parallel time. According to Lemma F.2, for line 17, we can implement it in

$O(n_i) = O(n)$  total space, and in  $O(k'_i)$  parallel time, where  $k'_i$  is the number of iterations (see Definition B.12) of TREECONTRACTION( $G'_i, \text{par}_i$ ). Line 19 can be done by the operation described in **Multiple queries**. Line 20 can be done by the operation described in **Indexing elements in sets** and **Multiple queries**.

Thus, the parallel time is  $O(R)$ , where  $R = \sum_{i=0}^{r-1} (k_i + k'_i)$ . By definition of the total number of iterations (see Definition C.28) of SPANNINGFOREST( $G, m, r$ ).  $R$  is the total number of iterations of SPANNINGFOREST( $G, m, r$ ).

For the space, we store all the sets  $V, E, V_i, D_i$  and mappings  $\text{par}_i, h_i$  in all the rounds. Notice that  $\sum_{i=0}^r |V_i| \leq 40|V|$ . Thus this part takes only  $O(N)$  space. In the  $i^{\text{th}}$  round, we additionally store all the sets  $V_{\tilde{T}_i(v)}, V'_i, E'_i, L_i$  and all the mappings  $\text{par}_{\tilde{T}_i(v)}, \text{dep}_{\tilde{T}_i(v)}, l_i, z_i$ . The total space for this part is at most  $O(m)$ . For line 16, it creates multiple tasks. The input of each task is at most  $|V_{\tilde{T}_i(v)}| \leq (m/n_i)^{1/2}$ . There are at most  $n_i$  tasks, and by Lemma F.9, each task will need space at most  $O(|V_{\tilde{T}_i(v)}| \log |V_{\tilde{T}_i(v)}|)$ . Thus, the space for this part is at most  $O(m)$ . To conclude, the total space needed is at most  $O(m)$ .  $\square$

**Theorem F.13.** *Let graph  $G = (V, E), n = |V|, N = |V| + |E|$  and  $m = \Theta(N^\gamma)$  for some arbitrary  $\gamma \in [0, 2]$ . Let  $r > 0$  be a round parameter. If SPANNINGFOREST( $G, m, r$ ) (Algorithm 11) does not return FAIL, then let the output be the input of ORIENTATE( $\cdot$ ) (Algorithm 12), and ORIENTATE( $\cdot$ ) can be implemented in  $(\gamma, \delta)$  – MPC model for any constant  $\delta \in (0, 1)$ . Furthermore, the parallel running time is  $O(R)$ , where  $R$  is the total number of iterations (see Definition C.28) of SPANNINGFOREST( $G, m, r$ ).*

*Proof.* Line 4 to line 7 can be implemented by operations described in **Multiple queries**. Notice that there is a trick here, if  $f_i(u, v) = \text{null}$ , we do not need to store the tuple  $(“f_i”, ((u, v), \text{null}))$  in the system. The total space needed to store all the mappings  $f_i$  and all the sets  $F_i$  for  $i \in \{0\} \cup [r]$  is at most  $\sum_{i=0}^r |V_i| = O(m)$ .

Line 10 and line 11 can be implemented by operations described in **Set membership** and **Multiple queries**.

We now look at the second loop, and focus on round  $i$ . Line 12 can be implemented by Lemma F.11. The total space needed is at most  $O(|V_i| \cdot (m/|V_i|)^{1/2} \cdot \log(m/|V_i|)) = O(m)$ . The parallel running time needed is at most  $O(k_i)$ , where  $k_i$  is the number of iterations (see Definition C.10) of MULTIPLELARGETREES( $G_i, m$ ),  $G_i$  is the intermediate graph in the procedure SPANNINGFOREST( $G, m, r$ ).

Thus, the parallel running time is  $O(R)$ , where  $R$  is the total number of iterations (see Definition C.28) of SPANNINGFOREST( $G, m, r$ ). The total space needed is  $O(m)$ .  $\square$

Now, we are able to conclude the following theorem for spanning forest problem.

**Theorem F.14.** *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm (see Algorithm 11 and Algorithm 12) which can output the rooted spanning forest for any graph  $G = (V, E)$  in  $O(\min(\log D \cdot \log \frac{1}{\gamma'}, \log n))$  parallel time, where  $D$  is the diameter of  $G$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $\gamma' = (1 + \gamma) \log_n \frac{2N}{n^{1/(1+\gamma)}}$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* Algorithm 11 outputs all the edges in the spanning forest and all the contraction information. Algorithm 12 takes the output of Algorithm 11 as its input, and outputs a rooted spanning forest.

The implementation of Algorithm 11 and Algorithm 12 in MPC model is shown by Theorem F.12 and Theorem F.13 respectively. The correctness of Algorithm 11 and Algorithm 12 is proved by Corollary C.24 and Theorem C.26 respectively. The parallel time of Algorithm 11 and Algorithm 12 is proved by Theorem C.29.  $\square$

A byproduct of our spanning forest algorithm is an estimator of the diameter of the graph.

**Theorem F.15.** *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a diameter estimator  $D'$  for any graph  $G = (V, E)$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time such that  $D \leq D' \leq D^{O(\log(1/\gamma'))}$ , where  $D$  is the diameter of  $G$ ,  $n = |V|$ ,  $N = |V| + |E|$  and  $\gamma' = (1 + \gamma) \log_n \frac{2N}{n^{1/(1+\gamma)}}$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* By Theorem F.14, we can find a rooted spanning forest. By Theorem C.26, the depth of that rooted spanning forest is at most  $D^{O(\log(1/\gamma'))}$ . Then we can implement a doubling algorithm (e.g. Modified Lemma F.8, Algorithm 7 without maintaining useless  $g_l$ ) with log in depth parallel time to output the depth of that spanning forest.  $\square$

## F.7 Lowest Common Ancestor and Multi-Paths Generation

**Lemma F.16.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\}$  be a set of  $q$  pairs of vertices, and  $\forall i \in [q], u_i \neq v_i$ . Let  $n = |V|$ ,  $N = n + q$ .  $\text{LCA}(\text{par}, Q)$  (Algorithm 13) can be implemented in  $(\gamma, \delta)$  – MPC model for any  $\gamma \geq \log \log N / \log N$  and any constant  $\delta \in (0, 1)$  in  $O(\log(\text{dep}(\text{par})))$  parallel running time.*

*Proof.* By Lemma F.8, line 3 can be implemented in space  $O(N \log N)$  and  $O(\log(\text{dep}(\text{par})))$  parallel running time. It is easy to see that all the other steps in the procedure can be done by the operations shown in **Multiple queries**.

Thus, the total space needed is  $O(N \log N)$  and the parallel running time is  $O(\log(\text{dep}(\text{par})))$ .  $\square$

**Lemma F.17.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ . Let  $Q = \{(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)\} \subseteq V \times V$  satisfy  $\forall j \in [q], v_j$  is an ancestor (See Definition D.1) of  $u_j$  in  $\text{par}$ . Let  $n = |V|$ ,  $N = n + q$ .  $\text{MULTIPATH}(\text{par}, Q)$  (Algorithm 14) can be implemented in  $(\gamma, \delta)$  – MPC model for any  $\gamma$  with  $N \log N + \sum_{i=1}^q (\text{dep}_{\text{par}}(u_i) - \text{dep}_{\text{par}}(v_i) + 1) = O(N^\gamma)$  and any constant  $\delta \in (0, 1)$  in  $O(\text{dep}(\text{par}))$  parallel running time.*

*Proof.* By Lemma F.8, line 3 can be implemented in space  $O(N \log N)$  and  $O(\log(\text{dep}(\text{par})))$  parallel running time. It is easy to see that all the other steps in the procedure can be done by the operations shown in **Multiple queries**. Notice that after each round, we need to do load balancing (see **Load balance**) to make each machine have large enough available local memory. The total space needed is to store all the pathes and the output of line 3. Notice that in round  $i$ , we do not need to keep  $S_j^{(i')}$  for  $i' < i - 1$ , thus, the space to keep  $S_j^{(i)}$  for all  $j \in [q]$  only needs  $O(\sum_{j=1}^q (\text{dep}_{\text{par}}(u_j) - \text{dep}_{\text{par}}(v_j) + 1))$  space.

Thus, the total space needed is at most  $O(N \log N + \sum_{i=1}^q (\text{dep}_{\text{par}}(u_i) - \text{dep}_{\text{par}}(v_i) + 1)) = O(N^\gamma)$ . The parallel running time is then  $O(\text{dep}(\text{par}))$ .  $\square$

## F.8 Leaf Sampling

**Lemma F.18.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $n = |V|$ . Let  $\delta$  be an arbitrary constant in  $(0, 1)$ , and let  $m = \lceil n^\delta \rceil$ . Then  $\text{LEAFSAMPLING}(\text{par}, m, \delta)$  (Algorithm 15) can be implemented in  $(\gamma, \delta)$ -MPC model for any  $\gamma \geq \log \log n / \log n$ . Furthermore, with probability at least  $1 - 1/(100m^{5/\delta})$ , the parallel running time is at most  $O(\log \text{dep}(\text{par}))$ .*

*Proof.* To implement line 4, for each  $v \in V$ , we can add  $\text{par}(v)$  to a temporary set  $X$ . Then each  $v$  can check whether  $v$  is a leaf by checking whether  $v$  is in  $X$ , and this can be done by the operations shown in **Set membership** and **Multiple queries**.

To implement line 5, for each  $v \in V$ , we can add  $v$  to the set  $\text{child}_{\text{par}}(\text{par}(v))$ . Then  $\text{rank}$  can be computed by the operations shown in **Indexing elements in sets** and **Multiple queries**. For line 6, we can implement it on a single machine, since a single machine has local memory  $\Theta(m)$ . For line 7 to line 9, for each  $x \in L$ , we add  $x$  into  $S$  with probability  $p$ , where  $p$  can be computed by querying the size of  $L$  (see **Sizes of sets** and **Multiple queries**). Line 10 can be implemented by operation described in **Indexing elements in sets**, **Set membership**, and **Multiple queries**. By Lemma F.2, line 11 can be implemented in total space  $O(N \log N)$  and  $O(\log \text{dep}(\text{par}))$  parallel time. By Property 3 of Lemma D.13, with probability at least  $1 - 1/(100m^{5/\delta})$ ,  $|S|^2 = O(m)$ . Thus,  $Q$  can be stored on a single machine. By Lemma F.16, line 15 can be implemented in total space  $O(n \log n + |Q|) = O(n \log n)$  and in  $O(\log \text{dep}(\text{par}))$  parallel time. By Lemma F.8, line 17 can be implemented in total space  $O(n \log n)$  and in  $O(\log \text{dep}(\text{par}))$  parallel time. Then line 18 to line 22 can be implemented on a single machine.

Thus, the total space needed is at most  $O(n \log n)$ . The parallel time is at most  $O(\log \text{dep}(\text{par}))$   $\square$

## F.9 DFS Sequence

**Lemma F.19.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $n = |V|$ . Let  $\delta$  be an arbitrary constant in  $(0, 1)$ , and let  $m = \lceil n^\delta \rceil$ .  $\text{SUBDFS}(\text{par}, m, \delta)$  (Algorithm 16) can be implemented in  $(\gamma, \delta)$ -MPC model for any  $\gamma \geq \log \log n / \log n$ . Furthermore, with probability at least  $1 - 1/(100m^{5/\delta})$ , the parallel running time is at most  $O(\log \text{dep}(\text{par}))$ .*

*Proof.* By Lemma F.18, line 5 can be implemented in total space  $O(n \log n)$  and with probability at least  $1 - 1/(100m^{5/\delta})$  has parallel running time  $O(\log \text{dep}(\text{par}))$ . By Lemma F.16, line 7 can be implemented in total space  $O(n \log n)$  and in parallel running time  $O(\log \text{dep}(\text{par}))$ . Line 9 can be implemented by operation shown in **Multiple queries**. By Lemma F.17, since all the pathes are disjoint (except the first path and the last path intersecting on the root) and  $V$  has  $n$  vertices, line 10 can be implemented in  $O(n \log n)$  total space and in  $O(\log \text{dep}(\text{par}))$  parallel running time. Loop in line 13 and Loop in line 16 can be implemented in parallel, and can be implemented by operations shown in **Indexing elements in sets** and **Multiple queries**. Line 20 can be implemented by operations shown in **Indexing elements in sets** and **Multiple queries**. Now we describe the implementation of line 21. Firstly, we can standardize (see **Sequence standardizing**) the sequence  $A'$ . For each tuple  $(A', (j, u))$ , create a tuple  $(\text{temp}_u, j)$ . Thus,  $\text{temp}_u$  is a set which contains all the positions that  $u$  appeared. For each tuple  $(\text{temp}_u, j)$ , we query (see **Multiple queries**) the index  $i$  (see **Indexing elements in sets**) of  $j$  in set  $(\text{temp}_u, j)$ , and create a tuple  $(\text{pos}, ((u, i), j))$ . Thus, the desired mapping  $\text{pos}$  is stored in the system. The loop in line 24 is implemented in parallel. Line 25 can be implemented by the operations shown in **Set**



**membership** and **Multiple queries**. Line 26 to line 28 can be implemented by the operation shown in **Multiple queries**. Finally, line 30 can be implemented by **Multiple queries** and **Sequence duplicating**.

The total space used in the procedure is at most  $O(n \log n)$ . The parallel running time is  $O(\log \text{dep}(\text{par}))$ .  $\square$

**Theorem F.20.** *Let  $\text{par} : V \rightarrow V$  be a set of parent pointers (See Definition B.6) on a vertex set  $V$ , and  $\text{par}$  has a unique root. Let  $n = |V|, m = n^\delta$  for some arbitrary constant  $\delta \in (0, 1)$ .  $\text{DFS}(\text{par}, m)$  (Algorithm 17) can be implemented in  $(\gamma, \delta)$ -MPC model for any  $\gamma \geq \log \log n / \log n$ . With probability at least 0.99, the parallel running time is  $O(\log(\text{dep}(\text{par})))$ .*

*Proof.* By Lemma F.19, line 5 can be implemented in total space  $O(n \log n)$ . With probability at least  $1 - 1/(100n^5)$ , the parallel running time is  $O(\log(\text{dep}(\text{par})))$ . Line 8 to line 10 can be implemented by operations shown in **Set membership** and **Multiple queries**. By Lemma F.2, line 11 can be implemented in  $O(n)$  total space, and  $O(\log \text{dep}(\text{par}))$  parallel running time. The loop in line 14 contains multiple tasks (see Section E.6 **Multiple Tasks**), thus we can implement those tasks in parallel. By Lemma F.19, line 17 can be implemented in total space  $O(|V'_i(v)| \log |V'_i(v)|)$ . Furthermore, with probability at least  $1 - 1/(100n^5)$ , the parallel running time is  $O(\log(\text{dep}(\text{par})))$ . Thus, the total space needed for those tasks is at most  $O(n \log n)$ . Line 19 can be implemented by operations shown in **Indexing elements in sets**, **Sequence insertion** and **Multiple queries**.

Thus, the total space needed is  $O(n \log n)$ . By taking union bound over all the task SUBDFS, with probability at least 0.99, the parallel running time is  $O(\log \text{dep}(\text{par}))$ .  $\square$

Now we are able to conclude the following theorem.

**Theorem F.21.** *For any  $\gamma \in [\beta, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$ -MPC algorithm (Algorithm 17) which can output a Depth-First-Search sequence for any tree graph  $G = (V, E)$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time, where  $n = |V|$ ,  $\beta = \Theta(\log \log n / \log n)$ ,  $D$  is the diameter of  $G$ , and  $\gamma' = \gamma + \Theta(1/\log n)$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* Firstly, by Theorem F.14, we can find a rooted tree. Algorithm 17 can output the DFS sequence for a rooted tree.

The implementation and parallel time of Algorithm 17 is shown by Theorem F.20. The correctness of Algorithm 17 is proved by Theorem D.20. The success probability of Algorithm 17 is proved by Theorem D.22.  $\square$

## F.10 Range Minimum Query

**Lemma F.22.** *Let  $A = (a_1, a_2, \dots, a_n)$  be a sequence of numbers. Let  $\delta$  be an arbitrary constant in  $(0, 1)$ .  $\text{SPARSETABLE}^+(a_1, a_2, \dots, a_n, \delta)$  (Algorithm 18) can be implemented in  $(0, \delta)$ -MPC model with  $O(1)$  parallel running time.*

*Proof.* Let  $A$  be the sequence  $(a_1, a_2, \dots, a_n)$ . The algorithm takes  $O(1/\delta)$  rounds.  $m$  is the local space of a machine. There are  $\Theta(n/m)$  machines each holds a consecutive  $\Theta(m)$  elements of sequence  $A$ . Now consider the round  $l$ . Machine  $j \in \{0\} \cup \lceil [n/m] \rceil$  needs to compute  $\hat{f}_{j \cdot m+1, l}, \hat{f}_{j \cdot m+2, l}, \dots, \hat{f}_{j \cdot m+m-1, l}$ . The number of queries machine  $j$  made in line 8 and line 11 is at most  $\sum_{t=1}^{\lceil 1/\delta \rceil} |S_t| + 2m \leq O(m/\delta) = O(m)$ . Thus, there are total  $O(n)$  queries. These queries can be answered simultaneously by operation shown in **Multiple queries**.

Thus, the total space needed is  $O(n)$ , and the parallel running time is  $O(1)$ .  $\square$

**Lemma F.23.** *Let  $a_1, a_2, \dots, a_n$  be a sequence of numbers. Let  $\delta$  be an arbitrary constant in  $(0, 1)$ .  $\text{SPARSETABLE}(a_1, a_2, \dots, a_n, \delta)$  (Algorithm 19) can be implemented in  $(\gamma, \delta)$ -MPC model for any  $\gamma \geq \log \log n / \log n$  in  $O(1)$  parallel time.*

*Proof.* By Lemma F.22, line 4 can be implemented in  $O(n)$  total space and  $O(1)$  parallel time. The loop in line 15 is similar to Algorithm 18. Each machine  $j$  needs to compute  $f_{j \cdot m+1, t}, \dots, f_{j \cdot m+m-1, t}$  for all  $t \in [\lceil \log n \rceil] \cup \{0\}$ . The difference from Algorithm 18 is that, it can compute for all  $t$  at the same time since it only depends on the value of  $\hat{f}$ . The number of queries made by each machine is  $O(m \log n)$ . Thus, the total number of queries is at most  $O(n \log n)$ . These queries can be answered simultaneously by operation shown in **Multiple queries**.

Thus, the total space needed is  $O(n \log n)$ , and the parallel running time is  $O(1)$ .  $\square$

## G Minimum Spanning Forest

In this section, we discuss how to apply our connectivity/spanning forest algorithm to the Minimum Spanning Forest (MSF) and Bottleneck Spanning Forest (BSF) problem.

The input of MSF/BSF problem is an undirected graph  $G = (V, E)$  together with a weight function  $w : E \rightarrow \mathbb{Z}$ , where  $E$  contains  $m$  edges  $e_1, e_2, \dots, e_m$  with  $w(e_1) \leq w(e_2) \leq \dots \leq w(e_m)$ . The goal of MSF is to output a spanning forest such that the sum of weights of the edges in the forest is minimized. The goal of BSF is to output a spanning forest such that the maximum weight of the edges in the forest is minimized.  $D$  is the diameter of the minimum spanning forest. If there are multiple choices of the minimum spanning forest, then let  $D$  be the minimum diameter among all the minimum spanning forests.

For simplicity, in all of our proofs, we only discuss the case when all the edges have different weights, i.e.  $w(e_1) < w(e_2) < \dots < w(e_m)$ . In this case, the minimum spanning forest is unique. It is easy to extend our algorithms to the case when there are edges with the same weight. We omit the proof for this fact.

Firstly, we show that  $D$  is an upper bound of the diameter of  $G'$  where the vertex set of  $G'$  is the vertex set of  $G$ , and the edge set of  $G'$  is  $\{e_1, e_2, \dots, e_i\}$  for some arbitrary  $i \in [m]$ .

**Lemma G.1.** *Given a graph  $G = (V, E)$  for  $E = \{e_1, e_2, \dots, e_m\}$  together with a weight function  $w$  which satisfies  $w(e_1) < w(e_2) < \dots < w(e_m)$ , then the diameter of  $G' = (V, E')$  is at most  $D$ , where  $D$  is the diameter of the minimum spanning forest of  $G$ , and  $E'$  only contains the first  $i$  edges of  $E$ , i.e.  $e_1, e_2, \dots, e_i$  for some arbitrary  $i \in [m]$ .*

*Proof.* The proof follows by Kruskal's algorithm directly.  $\square$

Our algorithms is based on the following simple but useful Lemma.

**Lemma G.2.** *Given a graph  $G = (V, E)$  for  $E = \{e_1, e_2, \dots, e_m\}$  together with a weight function  $w$  which satisfies  $w(e_1) \leq w(e_2) \leq \dots \leq w(e_m)$ ,  $\forall 1 \leq i < j \leq m$ , an edge  $e$  from  $\{e_i, e_{i+1}, \dots, e_j\}$  is in the minimum spanning forest of  $G$  if and only if  $e'$  from  $\{e'_i, e'_{i+1}, \dots, e'_j\}$  is in the minimum spanning forest of  $G'$ , where the vertices of  $G'$  is obtained by contracting all the edges  $e_1, e_2, \dots, e_{i-1}$  of  $G$ , and  $e', e'_i, e'_{i+1}, \dots, e'_j$  are the edges (or vertices) in  $G'$  which corresponds to the edges  $e, e_i, e_{i+1}, \dots, e_j$  before contraction.*

*Proof.* The proof follows by Kruskal's algorithm directly.  $\square$

A natural way to apply Lemma G.2 to parallel minimum spanning forest algorithm is that we can divide the edges into several groups, and recursively solve the minimum spanning forest for



each group of edges. More precisely, suppose we have total space  $\Theta(km)$ , we can divide  $E$  into  $k$  groups  $E_1, E_2, \dots, E_k$ , where  $E_i = \{e_{(i-1) \cdot m/k+1}, e_{(i-1) \cdot m/k+2}, \dots, e_{i \cdot m/k}\}$ . We can compute graph  $G_1, G_2, \dots, G_k$  where the vertices of  $G_i$  is obtained by contracting all the edges from  $e_1$  to  $e_{(i-1) \cdot m/k}$ , the edges of  $G_i$  are corresponding to the edges in  $E_i$ . Then by Lemma G.2, we can obtain the whole minimum spanning forest by solving these  $k$  size  $O(m/k)$  minimum spanning forest problems. For each sub-problem, we can assign it  $\Theta(m)$  working space, thus each sub-problem still has  $\Theta(k)$  factor more total space. Therefore, we can recursively apply the above argument.

**Theorem G.3.** *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a minimum spanning forest for any weighted graph  $G = (V, E)$  with weights  $w : E \rightarrow \mathbb{Z}$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n) \cdot 1/\gamma')$  parallel time, where  $n = |V|$ ,  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ ,  $D$  is the diameter of a minimum spanning forest of  $G$ , and  $\gamma' = \gamma/2 + \Theta(1/\log n)$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* Let  $n = |V|, m = |E|$ . Let  $E = \{e_1, \dots, e_m\}$  with  $w(e_1) \leq w(e_2) \leq \dots \leq w(e_m)$ . The total space in the system is  $\Theta(m^{1+\gamma})$ . Let  $k = \Theta(m^{\gamma/2})$ . By our previous discussion, we can divide  $E$  into  $k$  groups  $E_1, E_2, \dots, E_k$ , where  $E_i = \{e_{(i-1) \cdot m/k+1}, e_{(i-1) \cdot m/k+2}, \dots, e_{i \cdot m/k}\}$ . By Lemma G.1 and Theorem F.4, we can use  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time and  $\Theta(km^{1+\gamma/2})$  total space to compute graph  $G_1, G_2, \dots, G_k$  where the vertices of  $G_i$  is obtained by contracting all the edges from  $e_1$  to  $e_{(i-1) \cdot m/k}$ , the edges of  $G_i$  are corresponding to the edges in  $E_i$  after contraction.

By Lemma G.2, it suffices to recursively solve the minimum spanning forest problem for each group  $G_i$ . Since each time, we split the edges into  $k$  groups, the recursion will have at most  $O(1/\gamma')$  levels. At the end of the recursion, we are able to determine for every edge  $e$  whether  $e$  is in the minimum spanning forest.

Now let us consider the success probability. Although Theorem F.4 is a randomized algorithm, the parallel time is always bounded by  $\min(\log D \cdot \log(1/\gamma'), \log n)$ . If we repeat the algorithm until it succeeds, the expectation of number of trials is a constant. Furthermore, for each level of the recursion, we can regard the graphs in all the tasks composed one large graph. Thus, in real implementation, in each level of the recursion, we will only invoke one connectivity procedure. Thus in expectation, the total parallel time is  $O(\min(\log D \cdot \log(1/\gamma'), \log n) \cdot 1/\gamma')$ . By applying Markov's inequality, we complete the proof.  $\square$

In the following theorem, we show that Lemma G.2 can also be applied in approximate minimum spanning forest problem.

**Theorem G.4.** *For any  $\gamma \in [\beta, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a  $(1 + \epsilon)$  approximate minimum spanning forest for any weighted graph  $G = (V, E)$  with weights  $w : E \rightarrow \mathbb{Z}_{\geq 0}$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time, where  $n = |V|$ ,  $N = |V| + |E|$ ,  $\beta = \Theta(\log(\epsilon^{-1} \log n) / \log n)$ ,  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ ,  $D$  is the diameter of a minimum spanning forest of  $G$ , and  $\gamma' = (1 + \gamma - \beta) \log_n \frac{2N}{n^{1/(1+\gamma-\beta)}}$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* For each edge  $e \in E$ , we can round  $w(e)$  to  $w'(e)$  such that  $w'(e) = 0$  when  $w(e) = 0$ , and  $w'(e) = (1 + \epsilon)^i$  when  $w(e) \neq 0$ , and  $i$  is the smallest integer such that  $w(e) \leq (1 + \epsilon)^i$ .

Since  $|w(e)| \leq \text{poly}(n)$  for all  $e \in E$ , there are only  $k = O(\log(n)/\epsilon)$  different values of  $w'(e)$ . We can divide  $E$  into  $k$  groups, where the  $i^{\text{th}}$  group  $E_i$  contains all edges with the  $i^{\text{th}}$  largest weight in  $w'$ . By Lemma G.1 and Theorem F.4, we can use  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time and  $\Theta(kN^{1+\gamma-\beta}) = \Theta(N^{1+\gamma})$  total space to compute graph  $G_1, G_2, \dots, G_k$  where the vertices of

$G_i$  is obtained by contracting all the edges from  $E_1$  to  $E_{i-1}$ , the edges of  $G_i$  are corresponding to the edges in  $E_i$  after contraction.

Then, for each  $G_i$ , since all the edges have the same  $w'$  weight, any spanning forest of  $G_i$  is a minimum spanning forest of  $G_i$ . By Theorem F.14, we can use  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time and  $\Theta(kN^{1+\gamma-\beta}) = \Theta(N^{1+\gamma})$  total space to compute the spanning forest for each graph  $G_1, G_2, \dots, G_k$ . By Lemma G.2, the union of all the minimum spanning forest with respect to  $w'$  must be the minimum spanning forest of  $G$  with respect to  $w'$ . Since all the weights  $w$  are nonnegative integers,  $w'$  is a  $(1 + \epsilon)$  approximation to  $w$ . Therefore, our output minimum spanning forest with respect to  $w'$  is a  $(1 + \epsilon)$  approximation to the minimum spanning forest with respect to  $w$ .

For the success probability, we can apply the similar argument made in the proof of Theorem G.3 to prove that the success probability is at least 0.98.  $\square$

In the following, we show that if we only need to find the largest edge in the minimum spanning tree, then we are able to get a better parallel time. It is an another application of our

**Theorem G.5.** *For any  $\gamma \in [0, 2]$  and any constant  $\delta \in (0, 1)$ , there is a randomized  $(\gamma, \delta)$  – MPC algorithm which can output a bottleneck spanning forest for any weighted graph  $G = (V, E)$  with weights  $w : E \rightarrow \mathbb{Z}$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n) \cdot \log(1/\gamma'))$  parallel time, where  $n = |V|$ ,  $\forall e \in E, |w(e)| \leq \text{poly}(n)$ ,  $D$  is the diameter of a minimum spanning forest of  $G$ , and  $\gamma' = \gamma/2 + \Theta(1/\log n)$ . The success probability is at least 0.98. In addition, if the algorithm fails, then it will return FAIL.*

*Proof.* Let  $n = |V|, m = |E|$ . Let  $E = \{e_1, \dots, e_m\}$  with  $w(e_1) \leq w(e_2) \leq \dots \leq w(e_m)$ . The total space in the system is  $\Theta(m^{1+\gamma})$ . Let  $k = \Theta(m^{\gamma/2})$ . By our previous discussion, we can divide  $E$  into  $k$  groups  $E_1, E_2, \dots, E_k$ , where  $E_i = \{e_{(i-1) \cdot m/k + 1}, e_{(i-1) \cdot m/k + 2}, \dots, e_{i \cdot m/k}\}$ . By Lemma G.1 and Theorem F.4, we can use  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time and  $\Theta(km^{1+\gamma/2})$  total space to compute graph  $G_1, G_2, \dots, G_k$  where the vertices of  $G_i$  is obtained by contracting all the edges from  $e_1$  to  $e_{(i-1) \cdot m/k}$ , the edges of  $G_i$  are corresponding to the edges in  $E_i$  after contraction.

By Lemma G.2, the edge with largest weight must be in the group  $E_i$  for some  $i$  with  $G_{i+1} = G_{i+2}$ . Thus, we reduce the problem size to  $m/k$ . By Remark 1.11, we can finish the recursion in  $O(\log(1/\gamma'))$  phases.

Suppose the bottleneck is  $e_i$ , then by Theorem F.14, we can find a spanning forest by only using edges from  $\{e_1, \dots, e_i\}$  in  $O(\min(\log D \cdot \log(1/\gamma'), \log n))$  parallel time and in  $\Theta(m^{1+\gamma/2})$  total space. Thus, the resulting spanning forest is a bottleneck spanning forest.

For the success probability, we can apply the similar argument made in the proof of Theorem G.3 to prove that the success probability is at least 0.98.  $\square$

## H Directed Reachability vs. Boolean Matrix Multiplication

In this section, we discuss the directed graph reachability problem which is a directed graph problem highly related to the undirected graph connectivity. In the all-pair directed graph reachability problem, we are given a directed graph  $G = (V, E)$ , the goal is to answer for every pair  $(u, v) \in V \times V$  whether there is a directed path from  $u$  to  $v$ . There is a simple standard way to reduce Boolean Matrix Multiplication to all-pair directed graph reachability problem. In the Boolean Matrix Multiplication problem, we are given two boolean matrices  $A, B \in \{0, 1\}^{n \times n}$ , the goal is to compute  $C = A \cdot B$ , where  $\forall i, j \in [n], C_{i,j} = \bigvee_{k \in [n]} A_{i,k} \wedge B_{k,j}$ . The reduction is as the following. We create  $3n$  vertices  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n$ . For every  $i, j \in [n]$ , if  $A_{i,j} = 1$ ,

then we add an edge from  $u_i$  to  $v_j$ , and if  $B_{i,j} = 1$ , then we add an edge from  $v_i$  to  $w_j$ . Thus,  $C_{i,j} = 1$  is equivalent to there is a path from  $u_i$  to  $w_j$ . Thus, if we can solve all-pair directed graph reachability problem in  $O(T)$  sequential time, then we can solve Boolean Matrix Multiplication in  $O(T)$  time. For the current status of sequential running time of Boolean Matrix Multiplication problem, we refer readers to [LG14] and the references therein.

Now, consider the multi-query directed graph reachability problem. In this problem, we are given a directed graph  $G = (V, E)$  together with  $|V| + |E|$  queries where each query queries the reachability from vertex  $u$  to vertex  $v$ . The goal is to answer all these queries. A similar problem in the undirected graph is called multi-query undirected graph connectivity problem. In this problem, we are given an undirected graph  $G = (V, E)$  together with  $|V| + |E|$  queries where each query queries the connectivity between vertex  $u$  and vertex  $v$ .

According to Theorem F.4 and Lemma E.6, there is a polynomial local running time fully scalable  $\sim \log D$  parallel time  $(0, \delta)$ -MPC algorithm for multi-query undirected graph connectivity problem. Here polynomial local running time means that there is a constant  $c > 0$  (independent from  $\delta$ ) such that every machine in one round can only have  $O((n^\delta)^c)$  local computation.

For multi-query directed graph reachability problem, we show that if there is a polynomial local running time fully scalable  $(\gamma, \delta)$ -MPC algorithm which can solve multi-query reachability problem in  $O(n^\alpha)$  parallel time, then we can solve all-pair directed graph reachability problem in  $O(n^2 \cdot n^{2\gamma+\alpha+\epsilon})$  sequential running time for any arbitrarily small constant  $\epsilon > 0$ . Especially, if the algorithm is in  $(0, \delta)$ -MPC model, and the parallel time is  $n^{o(1)}$ , then we will have an  $O(n^{2+\epsilon+o(1)})$  sequential running time algorithm for Boolean Matrix Multiplication which implies a break through in this field.

Suppose we have a such MPC algorithm. Let the input size be  $\Theta(m)$ , i.e. the number of edges is  $\Theta(m)$ , and the number of queries is also  $\Theta(m)$ . Then the total space is  $\Theta(m^{1+\gamma})$ . Let  $\delta = \epsilon/(c-2)$ . Then the number of machines is  $\Theta(m^{1+\gamma-\delta})$ . Now we just simulate this  $(\gamma, \delta)$ -MPC algorithm sequentially, the total running time is  $O(m^{1+\gamma-\delta} \cdot m^{c\delta} \cdot n^\alpha) = O(m \cdot n^{2\gamma+\epsilon+\alpha})$ . To answer reachability for all pairs, we need total  $O(n^2 \cdot m \cdot n^{2\gamma+\epsilon+\alpha}/m) = O(n^2 \cdot n^{2\gamma+\alpha+\epsilon})$  time. Therefore, we can use this algorithm to solve Boolean Matrix Multiplication in  $O(n^2 \cdot n^{2\gamma+\alpha+\epsilon})$  time.

**Theorem H.1.** *If there is a polynomial local running time fully scalable  $(\gamma, \delta)$ -MPC algorithm which can answer  $|V| + |E|$  pairs of reachability queries simultaneously for any directed graph  $G = (V, E)$  in  $O(|V|^\alpha)$  parallel time, then there is a sequential algorithm which can compute the multiplication of two  $n \times n$  boolean matrices in  $O(n^2 \cdot n^{2\gamma+\alpha+\epsilon})$  time, where  $\epsilon > 0$  is a constant which can be arbitrarily small.*

*Proof.* See above discussions. □

## I Discussion on a Previous Conjectured Fast Algorithm

In this section, we discuss the hard example for the algorithm described by [RMCS13]. In [RMCS13], they conjectured that their Hash-to-Min connectivity algorithm can finish in  $O(\log D)$  rounds. The description of their algorithm is as the following:

1. The input graph is  $G = (V, E)$ .
2. For each vertex  $v \in V$ , initialize a set  $S_v^{(0)} = v$ .
3. in round  $i$ :
  - (a) Each vertex  $v$  find  $u \in S_v^{(i-1)}$  which has the minimum label, i.e.  $u = \min_{x \in S_v^{(i-1)}} x$ .

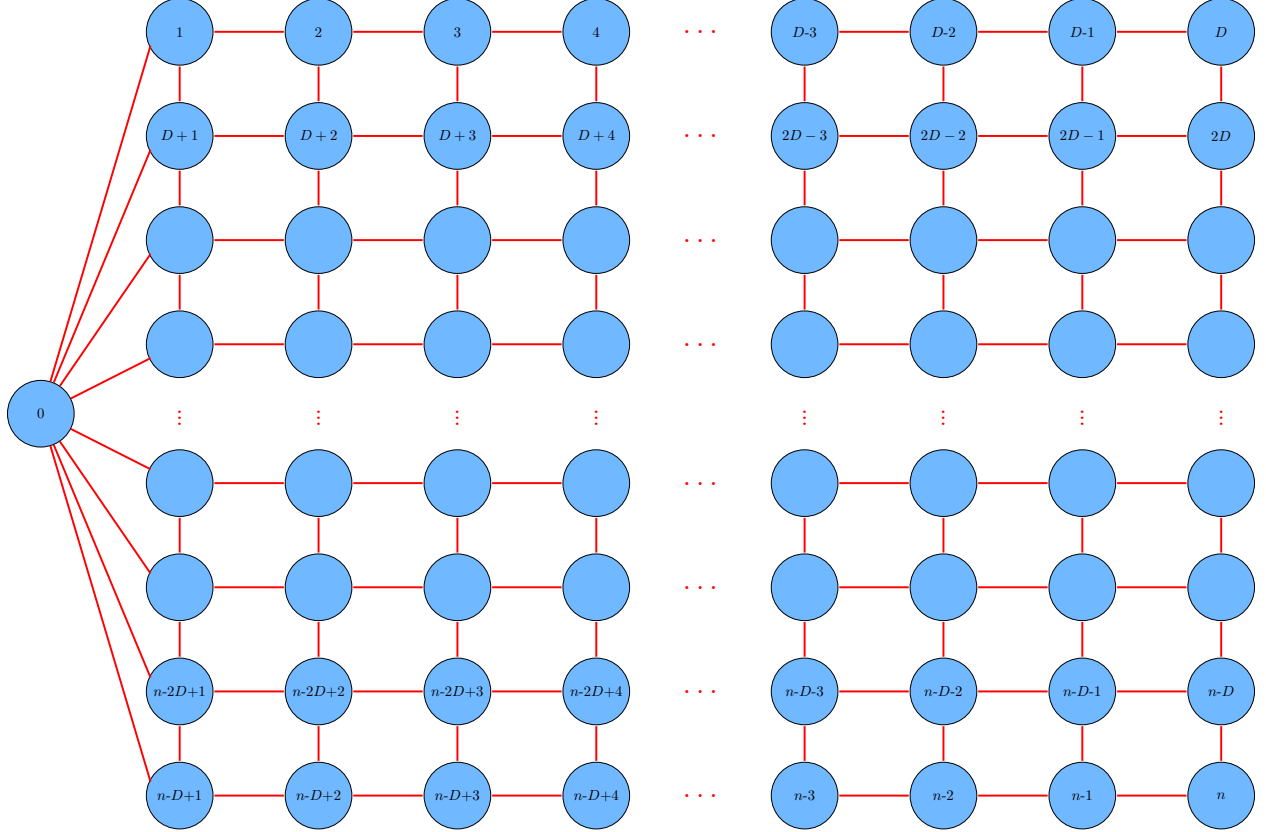


Figure 3: A hard example for [RMCS13]. For each  $i \in \{2, 3, \dots, n/D - 1\}$  and  $j \in \{1, 2, \dots, D - 1\}$ , node  $(i - 1) \cdot D + j$  has degree 4. For node  $D$  and  $n$ , they have degree 2. Node 0 has degree  $D$ . All the other nodes have degree 3.

- (b)  $v$  sends  $u$  the all the vertices in  $S_v^{(i-1)}$ .
- (c)  $v$  sends every  $x \in S_v^{((i-1))} \setminus \{u\}$  the vertex  $u$ .
- (d) Let  $S_v^{(i)}$  be  $\{v\}$  union the set of all the vertices received.
- (e) If for all  $v$ ,  $S_v^{(i)}$  is the same as  $S_v^{(i-1)}$ , then finish the procedure.

The above procedure can be seen as the modification of the graph: in each round, all the vertices together create a new graph. For each vertex  $v$ , let  $u$  be the neighbor of  $v$  with the minimum label, and if  $x$  is a neighbor of  $v$ , then add an edge between  $x$  and  $u$  in the new graph. So in each round, each vertex just communicates with its neighbors to update the new minimum neighbor it learned. At the end of the algorithm, it is obvious that the minimum vertex in each component will have all the other vertices in that component, and for each non minimum vertex, it will have the minimum vertex in the same component.

A hard example for this algorithm is shown by Figure 3. The example is a thin and tall grid graph with a vertex connected to all the vertices in the first column. The total number of vertices is  $n$ . The grid graph has  $D = \frac{1}{2} \log n$  columns and  $n/D$  rows. We index each column from left to right by 1 to  $D$ . We index each row from top to down by 1 to  $n/D$ . The single large degree vertex has label 0. The  $i^{\text{th}}$  row has the vertices with label  $(i - 1) \cdot D + 1$  to  $i \cdot D$  from the first column to the  $D^{\text{th}}$  column. We claim that if vertex  $v$  is the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column, then before round  $k$  for  $2^k < i, k < j$ , the neighbors of  $v$  will only in column  $j - 1$ , column  $j$  and column  $j + 1$ .

Furthermore, the minimum neighbor of  $v$  in column  $j - 1$  will be  $v - (2^{k-1} - 1) \cdot D - 1$ . The minimum neighbor of  $v$  in column  $j$  will be  $v - 2^{k-1} \cdot D$ . The minimum neighbor of  $v$  in column  $j + 1$  will be  $v - D \cdot (2^{k-1} - 1) + 1$ . This claim is true when  $k = 1$ . Then by induction, we can prove the claim. Thus, it will take at least  $\Theta(D)$  rounds to finish the procedure where  $D = \Theta(\log n)$ .

If we randomly label the vertices at the beginning, then consider the case we copy that hard structure at least  $n^{n+2}$  times, then with high probability, there is a component which has the labels with the order as the same as described above. In this case, the procedure needs  $\Omega(\log \log N)$  rounds, where  $N = n^{n+3}$  is the total number of the vertices.

Also notice that, even we give more total space to this algorithm, this algorithm will not perform better. In our connectivity algorithm, if we have  $\Omega(n^{1+\epsilon})$  total space for some arbitrary constant  $\epsilon > 0$ , then our parallel running time is  $O(\log D)$ .

## J Alternative Approach for Leader Selection

In this section, we show that there is a different way to select leaders (see Section B.2). The number of leaders selected by this approach will depend on the sum of inverse degrees of all the vertices. Let us first introduce the concept of *Min Parent Forest*.

### J.1 Min Parent Forest

Let  $G = (V, E)$  be an undirected graph where  $V$  denotes the vertex set of  $G$ , and  $E$  denotes the edge set of  $G$ . Each vertex  $v \in V$  has a weight  $w(v) \in \mathbb{R}$ , and it also has a unique label from  $\mathbb{Z}$ . For convenience, for each vertex  $v \in V$ , we also use  $v$  to denote its label. Let  $\Gamma_G(v)$  denote the set of neighbors of  $v$ , i.e.  $\Gamma_G(v) = \{u \in V \mid (u, v) \in E\}$ . If  $G$  is clear in the context, we just use  $\Gamma(v)$  to denote  $\Gamma_G(v)$ . The size of  $\Gamma(v)$ ,  $|\Gamma(v)|$ , is called the degree of  $v$ . Let  $f_{G,w} : V \rightarrow V$  be the “min-weight-parent” function defined as the following:

1. If  $w(v) = \min_{u \in \Gamma(v) \cup \{v\}} w(u)$ , then  $f_{G,w}(v) = v$ .
2. Otherwise, let  $u^* \in \Gamma(v)$  be the vertex which has the smallest weight, i.e.  $w(u^*) = \min_{u \in \Gamma(v)} w(u)$ . If there is more than one choice of  $u^*$ , let  $u^*$  be the one with the smallest label. And  $f_{G,w}(v)$  is defined to be  $u^*$ .

We call  $(V, f_{G,w})$  the *min-parent-forest* of graph  $G$  with vertex weights  $w$ . We can then define  $i$ -step “min-weight-parent” function. For  $v \in V$ , we define  $f_{G,w}^{(0)}(v) = v$ . For  $i \in \mathbb{Z}_{>0}$ , we can define  $f_{G,w}^{(i)}$  as the following:

$$\forall v \in V, f_{G,w}^{(i)}(v) = f_{G,w}(f_{G,w}^{(i-1)}(v)).$$

In the following, we define the concept of roots in the *min-parent-forest*.

**Definition J.1** (Roots in the forest). *Let  $v \in V$ , and let  $(V, f_{G,w})$  be the min-parent-forest of graph  $G = (V, E)$  with vertex weights  $w$ . If  $f_{G,w}(v) = v$ , then  $v$  is a root in the forest  $(V, f_{G,w})$ .*

The depth of a vertex  $v$  is defined as the distance on the tree between  $v$  and the corresponding root in the forest.

**Definition J.2** (The depth of  $v$ ). *Let  $v \in V$ , and let  $(V, f_{G,w})$  be the min-parent-forest of graph  $G = (V, E)$  with vertex weights  $w$ . The depth of  $v$  in the forest  $(V, f_{G,w})$  is the smallest  $i \in \mathbb{Z}_{\geq 0}$  such that  $f_{G,w}^{(i)}(v) = f_{G,w}^{(i+1)}(v)$ . We use  $\text{dep}_{G,w}(v)$  to denote the depth of  $v$  in  $(V, f_{G,w})$ . We call*

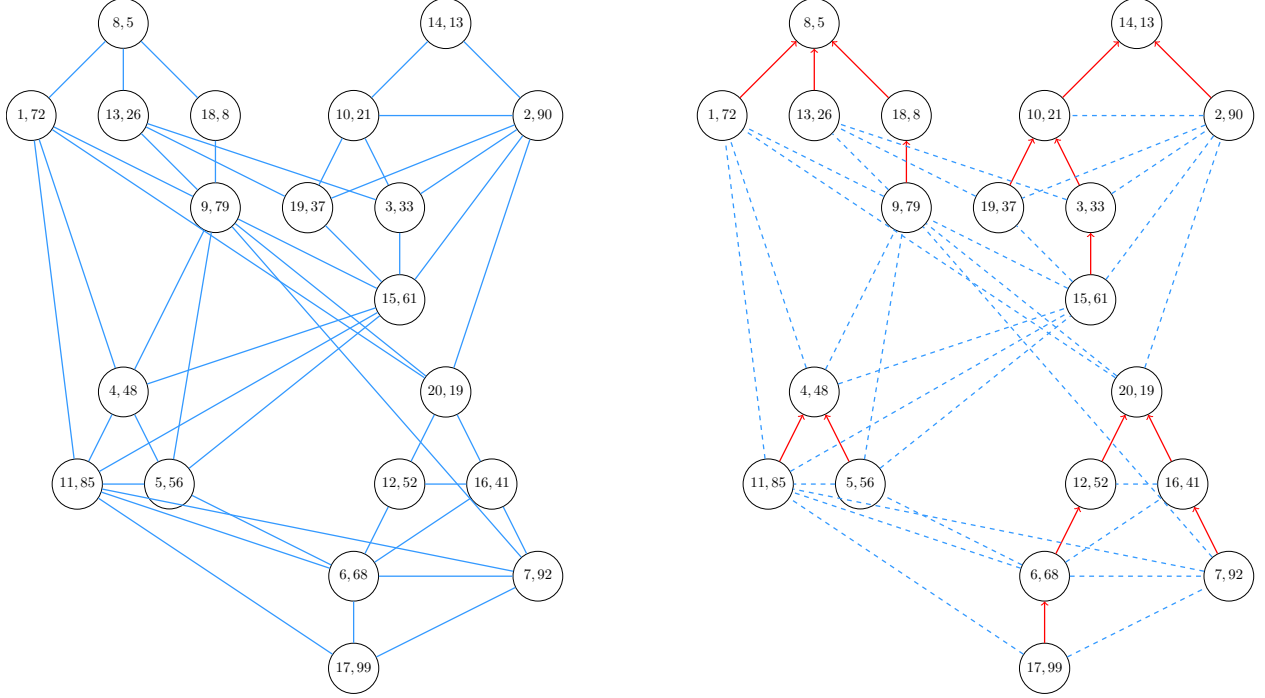


Figure 4: An example where  $\#roots \approx \sum_{i=1}^{20} 1/(d(v_i) + 1)$ . For each node, it has two numbers, the first number is the ID, and the second number is weight.  $\sum_{i=1}^{20} 1/(d(v_i) + 1) = 1/4 + 1/3 + 1/3 + 1/6 + 1/5 + 1/5 + 1/5 + 1/5 + 1/5 + 1/6 + 1/4 + 1/6 + 1/8 + 1/7 + 1/6 + 1/9 + 1/8 + 1/7 + 1/6 + 1/4 \approx 3.89$  and  $\#roots = 4$ .

$f_{G,w}^{(\text{dep}_{G,w}(v))}(v)$  the root of  $v$ . For the simplicity of the notation, we also use  $f_{G,w}^{(\infty)}(v)$  to denote the root of  $v$ .

The above definition is well defined since if  $f_{G,w}^{(i+1)}(v) \neq f_{G,w}^{(i)}(v)$  then  $w(f_{G,w}^{(i+1)}(v))$  should be strictly smaller than  $w(f_{G,w}^{(i)}(v))$  by the definition of  $f_{G,w}$  and  $f_{G,w}^{(j)}$  for all  $j \in \mathbb{Z}_{\geq 0}$ . Therefore, there must exist  $i$  such that  $f_{G,w}^{(i)}(v) = f_{G,w}^{(i+1)}(v)$ .

The depth of the forest is the largest depth among all the vertices.

**Definition J.3** (The depth of the *min-parent-forest*). The depth  $\text{dep}(G, w)$  of the forest  $(V, f_{G,w})$  is defined as:

$$\text{dep}(G, w) = \max_{v \in V} \text{dep}_{G,w}(v).$$

If the weights  $w$  of vertices of  $G$  are some i.i.d. random variables, then with high probability, the depth of  $(V, f_{G,w})$  is only  $O(\log |V|)$ . Precisely, we have the following Lemma.

**Lemma J.4** (The depth of the random *min-parent-forest*). Let  $G = (V, E)$  be an undirected graph with  $n$  vertices where  $V = \{v_1, v_2, \dots, v_n\}$ , and the labels satisfies  $v_1 < v_2 < \dots < v_n$ . Let  $w(v_1), w(v_2), \dots, w(v_n)$  be  $n$  i.i.d. random variables drawn uniformly from  $[N]$ . If  $N > n^2/\delta$  for some  $\delta \in (0, 1)$ , then for any  $t \geq 60 \log n$ ,

$$\Pr_{w \sim [N]^n} (\text{dep}(G, w) \leq t) \geq 1 - \delta - e^{-\frac{1}{2}t}.$$



*Proof.* Let  $w(v_1), w(v_2), \dots, w(v_n)$  be  $n$  i.i.d. random variables drawn uniformly from  $[N]$ . Let  $(V, f_{G,w})$  be the *min-parent-forest* of  $(G, w)$ . For a fixed  $s \in V$ , we create a set of random variables  $z_1, z_2, \dots, z_n$  by the following deterministic procedure:

1. Let  $z_1 = w(s), k = 0, S_k = \{s\}, u_k = s, i = 2, \text{pos}(s) \leftarrow 1$ .
2. Let  $S_{k+1} = S_k$ .
3. For  $j = 1 \rightarrow n$ ,  
if  $v_j \in \Gamma(u_k)$  and  $v_j \notin S_k$  then let  $\text{pos}(v_j) \leftarrow i, S_{k+1} \leftarrow S_k \cup \{v_j\}, z_i = w(v_j), i \leftarrow i + 1$ .
4. If  $f_{G,w}(u_k) \neq u_k$ , then let  $u_{k+1} = f_{G,w}(u_k), k \leftarrow k + 1$  and go to step 2.
5. Otherwise, for  $j = 1 \rightarrow n$ ,  
if  $v_j \notin S_{k+1}$  then let  $\text{pos}(v_j) \leftarrow i, z_i = w(v_j), i \leftarrow i + 1$ .

It is easy to observe that  $k$  is exactly  $\text{dep}_{G,w}(s)$  at the end of the above procedure. The reason is that  $u_0 = s = f_{G,w}^{(0)}(s), \forall j \in [k], u_j = f_{G,w}(u_{j-1}) = f_{G,w}^{(j)}(s)$  and  $f_{G,w}(u_k) = u_k$ .

**Fact J.5.**  $\forall v \in V, w(v) = z_{\text{pos}(v)}$ , where  $\text{pos} : [V] \rightarrow [n]$  and  $\text{pos}^{-1} : [n] \rightarrow [V]$ .

**Claim J.6.**  $\forall j \in \{0, 1, \dots, k+1\}, S_j = \{u_0\} \cup \bigcup_{p=0}^{j-1} \Gamma(u_p)$ .

*Proof.* We can prove this by induction. The statement is obviously true for  $S_0$  since  $S_0 = \{u_0\}$ . Now suppose the claim is true for  $S_{j-1}$ . Then according to the step 3 of the procedure  $S_j = S_{j-1} \cup (\Gamma(u_{j-1}) \setminus S_{j-1}) = S_{j-1} \cup \Gamma(u_{j-1}) = \{u_0\} \cup \bigcup_{p=0}^{j-1} \Gamma(u_p)$ .  $\square$

**Claim J.7.**  $\forall j \in \{0, 1, \dots, k\}, w(u_j) = \min_{v \in S_j} w(v)$ .

*Proof.* Since  $\forall j \in [k], u_j = f_{G,w}(u_{j-1})$ , we have  $w(u_j) = \min_{v \in \Gamma(u_{j-1}) \cup \{u_{j-1}\}} w(v)$ . Then we have  $w(u_j) = \min_{v \in \{u_0\} \cup \bigcup_{p=0}^{j-1} \Gamma(u_p)} w(v) = \min_{v \in S_j} w(v)$ , where the last equality follows by Claim J.6.  $\square$

We use  $\text{pos}^{-1}(i)$  to denote vertex  $v$  which satisfies  $\text{pos}(v) = i$ . According to the step 3, it is easy to see  $\forall j \in \{0, 1, \dots, k+1\}$ , we have  $\{\text{pos}^{-1}(i) \mid i \in [|S_j|]\} = S_j$ .

**Claim J.8.**  $\forall j \in \{0, 1, \dots, k\}, z_{\text{pos}(u_j)} = \min_{p \in [\text{pos}(u_j)]} z_p$ .

*Proof.*  $z_{\text{pos}(u_j)} = w(u_j) = \min_{v \in S_j} w(v) = \min_{v \in S_j} z_{\text{pos}(v)} = \min_{p \in [|S_j|]} z_p = \min_{p \in [\text{pos}(u_j)]} z_p$ , where the second equality follows by Claim J.7, and the last equality follows by  $u_j \in S_j$ , so  $\text{pos}(u_j) \leq |S_j|$ .  $\square$

Now we define another set of random variables  $y_1, y_2, \dots, y_n$ , where  $\forall i \in [n], y_i \in \{0, 1\}$  and  $y_i = 1$  if and only if  $z_i = \min_{j \in [i]} z_j$ . According to Claim J.8, we have that  $\forall i \in \{0, 1, \dots, k\}, y_{\text{pos}(u_i)} = 1$ . Thus,  $\text{dep}_{G,w}(s) = k \leq \sum_{i=1}^n y_i$ . To upper bound  $\text{dep}_{G,w}(s)$ , it suffices to upper bound  $\sum_{i=1}^n y_i$ .

Before we look at  $y_1, \dots, y_n$ , we firstly focus on the properties of  $z_1, \dots, z_n$ :

**Claim J.9.**  $z_1, z_2, \dots, z_n$  are  $n$  i.i.d random variables drawn uniformly from  $[N]$ .

*Proof.* A key observation is that if  $z_1, z_2, \dots, z_n$  are given, then we can recover  $w(v_1), w(v_2), \dots, w(v_n)$  exactly by the following deterministic procedure:

1. Let  $w(s) = z_1, k = 0, S_k = \{s\}, u_k = s, i = 2$ .
2. Let  $S_{k+1} = S_k$ .



3. For  $j = 1 \rightarrow n$ ,  
if  $v_j \in \Gamma(u_k)$  and  $v_j \notin S_k$  then let  $S_{k+1} \leftarrow S_k \cup \{v_j\}$ ,  $w(v_j) = z_i$ ,  $i \leftarrow i + 1$ .
4. If  $f_{G,w}(u_k) \neq u_k$ , then let  $u_{k+1} = f_{G,w}(u_k)$ ,  $k \leftarrow k + 1$  and go to step 2.
5. Otherwise, for  $j = 1 \rightarrow n$ ,  
if  $v_j \notin S_{k+1}$  then let  $\text{pos}(v_j) \leftarrow i$ ,  $w(v_j) = z_i$ ,  $i \leftarrow i + 1$ .

Notice that after step 3,  $\forall v \in \Gamma(u_k) \cup \{u_k\}$ ,  $w(v)$  is already recovered, thus we can implement step 4. Thus, the above procedure is a valid procedure. Since  $z_1, \dots, z_n$  are generated by  $w(v_1), \dots, w(v_n)$ , we can also know  $z_1, \dots, z_n$  by given  $w(v_1), \dots, w(v_n)$ . This means that

$$H(z_1, z_2, \dots, z_n \mid w(v_1), w(v_2), \dots, w(v_n)) = H(w(v_1), w(v_2), \dots, w(v_n) \mid z_1, z_2, \dots, z_n) = 0,$$

where  $H(\cdot)$  is the information entropy. Notice that

$$\begin{aligned} & I(z_1, z_2, \dots, z_n; w(v_1), w(v_2), \dots, w(v_n)) \\ &= H(z_1, z_2, \dots, z_n) - H(z_1, z_2, \dots, z_n \mid w(v_1), w(v_2), \dots, w(v_n)) \\ &= H(w(v_1), w(v_2), \dots, w(v_n)) - H(w(v_1), w(v_2), \dots, w(v_n) \mid z_1, z_2, \dots, z_n), \end{aligned}$$

where  $I(\cdot)$  is the mutual information. Thus,  $H(z_1, z_2, \dots, z_n) = H(w(v_1), w(v_2), \dots, w(v_n)) = n \log N$ . For  $i \in [n]$ , since the size of the support of  $z_i$  is at most  $N$ ,  $H(z_i) \leq \log N$  where the equality holds if and only if  $z_i$  is uniformly distributed on  $[N]$ . Also notice that  $H(z_1, z_2, \dots, z_n) \leq \sum_{i=1}^n H(z_i)$ , where the equality holds if and only if  $z_i$  are independent. Since  $\sum_{i=1}^n H(z_i) \leq n \log N$ , we have  $H(z_1, z_2, \dots, z_n) = \sum_{i=1}^n H(z_i)$ , and for each  $i \in [n]$ ,  $H(z_i) = \log N$ . Thus,  $z_1, z_2, \dots, z_n$  are i.i.d. random variables drawn uniformly from  $[N]$ .  $\square$

**Claim J.10.** *If  $N > n^2/\delta$  for some  $\delta \in (0, 1)$ , then with probability at least  $1 - \delta$ ,  $\forall i \neq j \in [n]$ , we have  $w(v_i) \neq w(v_j)$ .*

*Proof.* Recall that  $w(v_1), w(v_2), \dots, w(v_n)$  are  $n$  i.i.d. random variables drawn uniformly from  $N$ . For any  $i \neq j \in [n]$ , the  $\Pr(w(v_i) \neq w(v_j)) = 1/N$ , thus  $\mathbf{E}(|\{(i, j) \in [n] \times [n] \mid i \neq j, w(v_i) \neq w(v_j)\}|) \leq n^2/N$ . By Markov's inequality,

$$\Pr(|\{(i, j) \in [n] \times [n] \mid i \neq j, w(v_i) \neq w(v_j)\}| \geq 1) \leq n^2/N \leq \delta.$$

Thus,

$$\Pr(\forall i \neq j \in [n], z_i \neq z_j) \geq 1 - \delta.$$

$\square$

**Claim J.11.** *Let  $\mathcal{E}$  be the event that  $\forall i \neq j \in [n], w(v_i) \neq w(v_j)$ . Then, for any  $t \geq 3 \sum_{i=1}^n \frac{1}{i}$ , we have*

$$\Pr_{w \sim [N]^n} \left( \sum_{i=1}^n y_i \geq t + \sum_{i=1}^n \frac{1}{i} \mid \mathcal{E} \right) \leq e^{-\frac{3}{4}t}.$$

*Proof.* Note that  $\mathcal{E}$  happened if and only if we have  $\forall i \neq j \in [n], z_i \neq z_j$ . Due to Claim J.9,  $z_1, z_2, \dots, z_n$  are i.i.d. random variables drawn uniformly from  $[N]$ , then conditioned on  $\mathcal{E}$ ,  $y_1, y_2, \dots, y_n$  are independent, and the probability that  $y_i = 1$  is  $1/i$ . Thus, we have:

$$\begin{aligned}
& \Pr \left( \sum_{i=1}^n y_i \geq \sum_{i=1}^n \frac{1}{i} + t \mid \mathcal{E} \right) \\
&= \Pr \left( \sum_{i=1}^n (y_i - \mathbf{E}(y_i \mid \mathcal{E})) \geq t \mid \mathcal{E} \right) \\
&\leq \exp \left( -\frac{\frac{1}{2}t^2}{\sum_{i=1}^n \mathbf{Var}(y_i \mid \mathcal{E}) + \frac{1}{3}t} \right) \\
&\leq \exp \left( -\frac{\frac{1}{2}t^2}{\sum_{i=1}^n \frac{1}{i} + \frac{1}{3}t} \right) \\
&\leq \exp \left( -\frac{\frac{1}{2}t^2}{\frac{2}{3}t} \right) \\
&= \exp \left( -\frac{3}{4}t \right),
\end{aligned}$$

where the first equality follows by  $\mathbf{E}(y_i \mid \mathcal{E}) = 1/i$ . The first inequality follows by Bernstein inequality. The second inequality follows by

$$\sum_{i=1}^n \mathbf{Var}(y_i \mid \mathcal{E}) \leq \sum_{i=1}^n \mathbf{E}(y_i^2 \mid \mathcal{E}) = \sum_{i=1}^n \mathbf{E}(y_i \mid \mathcal{E}) = \sum_{i=1}^n \frac{1}{i}.$$

The third inequality follows by  $\sum_{i=1}^n \frac{1}{i} \leq \frac{1}{3}t$ . □

For a fixed vertex  $s \in V$ , due to Claim J.11, for any  $t \geq 3 \sum_{i=1}^n 1/i$ , we have

$$\Pr \left( \text{dep}_{G,w}(s) \geq \sum_{i=1}^n 1/i + t \mid \mathcal{E} \right) \leq e^{-\frac{3}{4}t}. \tag{1}$$

Thus, for any  $t \geq 60 \log n$ ,

$$\begin{aligned}
& \Pr_{w \sim [N]^n} (\exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq t) \\
& \leq \Pr \left( \exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \right) \\
& = \Pr \left( \exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \mid \mathcal{E} \right) \Pr(\mathcal{E}) \\
& \quad + \Pr \left( \exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \mid \neg \mathcal{E} \right) \Pr(\neg \mathcal{E}) \\
& \leq \Pr \left( \exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \mid \mathcal{E} \right) + \Pr(\neg \mathcal{E}) \\
& \leq \Pr \left( \exists s \in V \text{ s.t. } \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \mid \mathcal{E} \right) + \delta \\
& \leq \sum_{s \in V} \Pr \left( \text{dep}_{G,w}(s) \geq 5t/6 + \sum_{i=1}^n 1/i \mid \mathcal{E} \right) + \delta \\
& \leq ne^{-\frac{5}{8}t} + \delta \\
& \leq e^{-\frac{1}{2}t} + \delta
\end{aligned}$$

where the first inequality follows by  $\frac{1}{6}t \geq 10 \log n \geq \sum_{i=1}^n 1/i$ . The third inequality follows by Claim J.10. The fourth inequality follows by union bound. The fifth inequality follows by Equation (1). The sixth inequality follows by  $e^{-\frac{1}{8}t} \leq \frac{1}{n}$ .

Thus, we can conclude that for any  $t \geq 60 \log n$ , we have  $\Pr(\text{dep}(G, w) \leq t) \geq 1 - \delta - e^{-\frac{1}{2}t}$ .  $\square$

**Lemma J.12** (The number of roots of the random *min-parent-forest*). *Let  $G = (V, E)$  be an undirected graph with  $n$  vertices where  $V = \{v_1, v_2, \dots, v_n\}$ , and the labels satisfies  $v_1 < v_2 < \dots < v_n$ . Let  $w(v_1), w(v_2), \dots, w(v_n)$  be  $n$  i.i.d. random variables drawn uniformly from  $[N]$ . Let  $\delta \in (0, 1)$ . If  $N > n^3$ , then*

$$\Pr_{w \sim [N]^n} \left( |\{v \in V \mid f_{G,w}(v) = v\}| \geq \frac{2}{\delta} \sum_{v \in V} \frac{1}{|\Gamma(v)| + 1} \right) \leq \delta.$$

*Proof.* Let  $w(v_1), w(v_2), \dots, w(v_n)$  be  $n$  i.i.d. random variables drawn uniformly from  $[N]$ . Let  $\mathcal{E}$  be the event that  $\forall i \neq j \in [n], w(v_i) \neq w(v_j)$ . Notice that for  $i \neq j$ , the probability that  $w(v_i) = w(v_j)$  is  $1/N$ . Thus,  $\mathbf{E}(|\{(i, j) \in [n] \times [n] \mid i \neq j, w(v_i) = w(v_j)\}|) \leq n^2/N$ . Thus, if  $N > n^2$ , then  $\Pr(\neg \mathcal{E}) = \Pr(|\{(i, j) \in [n] \times [n] \mid i \neq j, w(v_i) = w(v_j)\}| \geq 1) \leq n^2/N \leq \frac{1}{n}$ . Now, we fix a vertex

$v \in V$ ,

$$\begin{aligned}
& \Pr(f_{G,w}(v) = v) \\
&= \Pr(f_{G,w}(v) = v \mid \mathcal{E}) \Pr(\mathcal{E}) + \Pr(f_{G,w}(v) = v \mid \neg \mathcal{E}) \Pr(\neg \mathcal{E}) \\
&\leq \Pr(f_{G,w}(v) = v \mid \mathcal{E}) + \Pr(\neg \mathcal{E}) \\
&\leq \Pr\left(w(v) = \min_{u \in \{v\} \cup \Gamma(v)} w(u) \mid \mathcal{E}\right) + \frac{1}{n} \\
&\leq \frac{1}{|\Gamma(v)| + 1} + \frac{1}{n} \\
&\leq \frac{2}{|\Gamma(v)| + 1}
\end{aligned}$$

where the third inequality follows by the symmetry of all the variables  $w(u)$  for  $u \in \{v\} \cup \Gamma(v)$  so condition on all the  $w$  are different, with probability  $\frac{1}{1+|\Gamma(v)|}$ ,  $w(v)$  is the smallest one. The last inequality follows by  $|\Gamma(v)| + 1 \leq |V| = n$ .

Thus,  $\mathbf{E}(|\{v \in V \mid f_{G,w}(v) = v\}|) \leq \sum_{v \in V} \frac{2}{|\Gamma(v)| + 1}$ . Let  $\delta \in (0, 1)$ , then by Markov's inequality,

$$\Pr\left(|\{v \in V \mid f_{G,w}(v) = v\}| \geq \frac{2}{\delta} \sum_{v \in V} \frac{1}{|\Gamma(v)| + 1}\right) \leq \delta.$$

□

## J.2 Leader Selection via Min Parent Forest

Given a graph, we can randomly assign each vertex a weight, thus we have a *min-parent-forest*, then we select those roots in the *min-parent-forest* as leaders, and try to contract all the vertices to the leaders. If we replace line 13 to line 17 of Algorithm 3 by Algorithm 20. We can get a new algorithm with the following guarantees.

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### Algorithm 20 Leader Selection via Min Parent Forest

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- 1: Let  $N = 100rn^{10}$ .
  - 2:  $\forall v \in V'_i$ , let  $w_i(v)$  be i.i.d. random variables drawn uniformly from  $[N]$ .
  - 3:  $\forall v \in V''_i$ , let  $\text{par}_i(v) = f_{G'_i, w_i}(v)$ .  $\triangleright (V'_i, f_{G'_i, w_i})$  is a *min-parent-forest* of  $G'_i$  with  $w_i$ .
- 

**Theorem J.13.** *Suppose we replace line 13 to line 17 of Algorithm 3 by Algorithm 20.*

*Let  $G = (V, E)$  be an undirected graph,  $m = \Omega(n)$ , and  $r \leq n$  be the rounds parameter where  $n$  is the number of vertices in  $G$ . Let  $c > 0$  be a sufficiently large constant. If  $r \geq c \log \log_{m/n}(n)$ , then with probability at least  $2/3$ , the modified  $\text{CONNECTIVITY}(G, m, r)$  (Algorithm 3) will not return *FAIL*, and the total number of iterations (see Definition B.20) of the modified  $\text{CONNECTIVITY}(G, m, r)$  is at most  $O(r \cdot (\log D + \log \log n))$ , where  $D = \text{diam}(G)$ .*

*Proof.* Let  $k_i$  denote the number of iterations (see Definition B.2) of  $\text{NEIGHBORINCREMENT}(m, G_{i-1})$ . By Lemma B.3, we have  $k_i \leq O(\log D)$ . Thus,  $\sum_{i=1}^r k_i = O(r \cdot \log D)$ .

According to Lemma J.4, with probability at least  $1 - \frac{2}{100r}$ ,  $\text{dep}(G''_i, w_i) \leq O(\log n)$ . By Lemma B.10, with probability at least  $1 - \frac{2}{100r}$ , the number of iteration of  $\text{TREECONTRACTION}(G''_i, \text{par}_i)$  (see Definition B.12)  $r'_i \leq O(\log \log n)$ . By taking union bound over all  $i \in [r]$ , then with probability at least  $1 - \frac{1}{50}$ ,  $\sum_{i=1}^r r'_i \leq O(r \cdot \log \log n)$ .

Due to the Property 3 of Lemma B.3,  $\forall i \in [r], \forall v \in V_i'', u \in \Gamma_{G_i'}(v)$ , we have  $u \in V_i''$  which means that  $u \in \Gamma_{G_i''}(v)$ . Thus,  $|\Gamma_{G_i''}(v)| \geq \lceil (m/n_{i-1})^{1/2} \rceil - 1$ . Then due to Lemma J.12, we have that with probability at most  $\frac{1}{8}$ ,  $n_i \geq 16n_{i-1}^{3/2}/m^{1/2}$ . Since  $m/n \geq m/n_i \geq 1024$ , we have that with probability at most  $\frac{1}{8}$ ,  $n_i \geq n_{i-1}^{11/10}/m^{1/10}$ . Let  $y_1, y_2, \dots, y_r$  be  $r$  random variables. If  $n_i \geq n_{i-1}^{11/10}/m^{1/10}$ , then  $y_i = 1$ , otherwise  $y_i = 0$ . We have  $\mathbf{E}(\sum_{i=1}^r y_i) \leq \frac{r}{8}$ . By Markov's inequality, we have  $\Pr(\sum_{i=1}^r y_i \geq \frac{r}{2}) \leq \frac{1}{4}$ . Thus, with probability at least  $\frac{3}{4}$ ,  $\sum_{i=1}^r y_i \leq \frac{r}{2}$ . Notice that when  $y_i = 0$ , then  $n_i \leq n_{i-1}^{11/10}/m^{1/10}$ , and when  $y_i = 1$ , we have  $n_i \leq n_{i-1}$ . So if there are at least  $\frac{r}{2}$  number of  $y_i$ s which are 0, then

$$\begin{aligned} n_r &\leq \frac{\left( \frac{\left( \frac{n^{1.1}}{m^{0.1}} \right)^{1.1}}{m^{0.1}} \right) \cdots}{\dots} && \text{(Apply } r/2 \text{ times)} \\ &= \frac{n^{1.1^{r/2}}}{m^{1.1^{r/2}-1}} \\ &= n/(m/n)^{1.1^{r/2}-1} \\ &\leq n/(m/n)^{1.1^{r/4}} \\ &\leq \frac{1}{2} \end{aligned}$$

where the last inequality follows by  $r \geq \frac{4}{\log 1.1} (\log \log_{m/n}(2n))$ . Since  $n_r$  is an integer, when  $n_r \leq \frac{1}{2}$ ,  $n_r = 0$ . Thus, we can conclude that if  $r \geq c \cdot \log \log_{m/n} n$  for a sufficiently large constant  $c > 0$ , then with probability at least  $\frac{3}{4} - \frac{1}{50} \geq \frac{2}{3}$ , the modified CONNECTIVITY( $G, m, r$ ) will not output FAIL.  $\square$

Notice that though the theoretical guarantees of the *min-parent-forest* leader selection method is worse than the random leader sampling, the merit of *min-parent-forest* leader selection method is that it can have an “early start”.

Consider the case when the total space size  $m$  is  $\Theta(n)$ . In this case, random leader sampling will always sample a half of the vertices as the leaders until the total space  $m$  is  $\text{poly}(\log n)$  larger than the number of vertices. However, *min-parent-forest* leader selection method can make a large progress at the beginning, it will choose the number of leaders to be about the sum of inverse degrees. Furthermore, the depth of the *min-parent-forest* may not always have  $\log n$  depth. Thus, it is an interesting question which leader selection approach has better performance in practice.

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