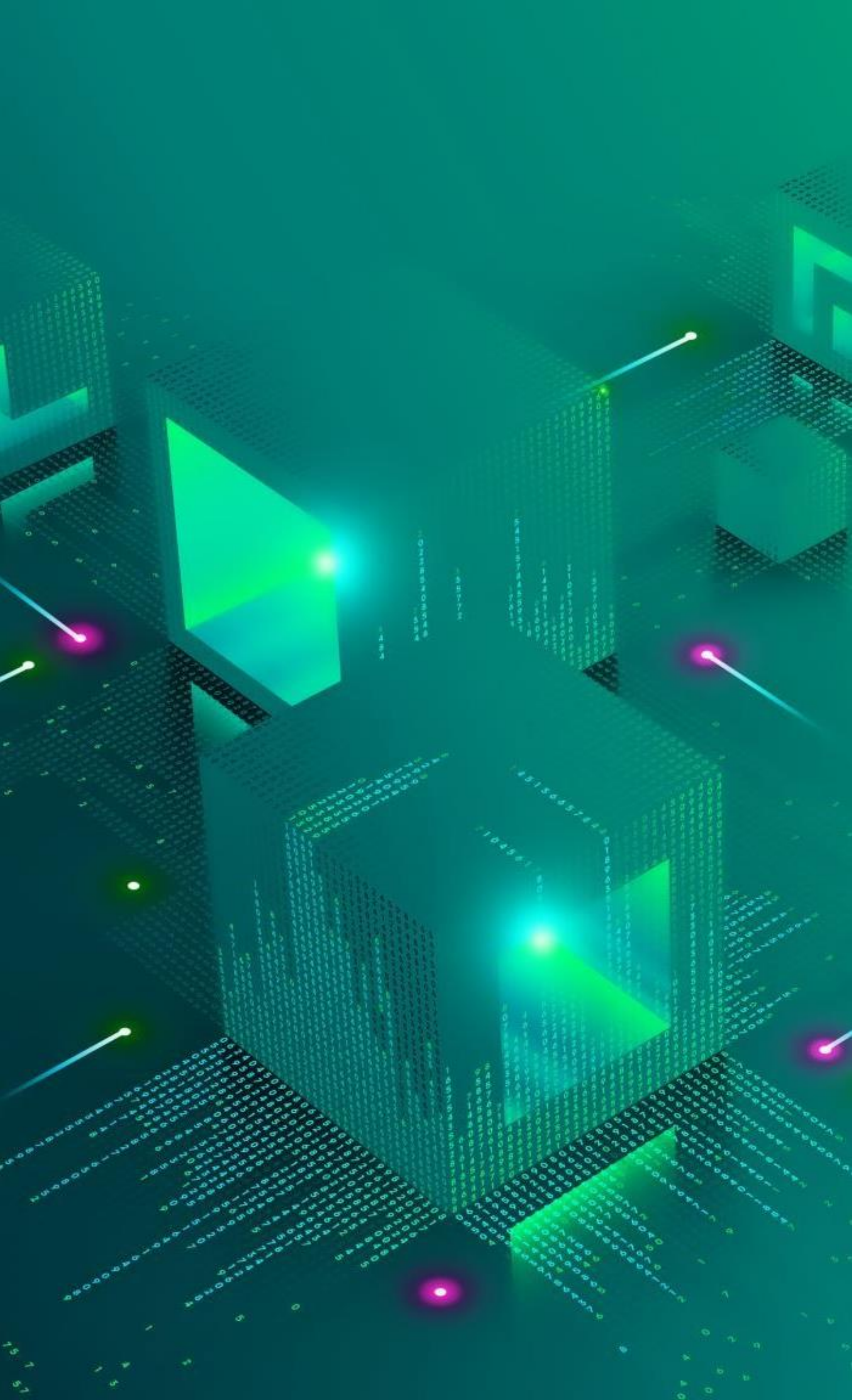

QUANTUM COMPUTING ASSIGNMENT 2

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CONTENTS

QUANTUM TELEPORTATION

- In ***quantum teleportation***, the properties of ***quantum entanglement*** are used to send a spin state (qubit) between observers without physically moving the involved particle.
- The particles themselves are not really teleported, but the state of one particle is destroyed on one side and extracted on the other side, so the information that the state encodes is communicated.
- The process is not instantaneous, because information must be communicated classically between observers as part of the process.
- The usefulness of quantum teleportation lies in its ability to send quantum information arbitrarily far distances without exposing quantum states to thermal decoherence from the environment or other adverse effects.

Process of Quantum Teleportation

Suppose Alice has state C, which she wants to send to Bob. To achieve this, Alice and Bob should follow the sequence of steps:

1) Generate an entangled pair of electrons with spin states A and B, in a particular Bell state:

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B).$$

2) Alice measures the Bell state of AC, entangling A and C while disentangling B. The process of measuring the Bell state projects a non-entangled state into an entangled state, since all four Bell states are entangled.

Expanding Alice's full original state, she starts with:

$$|\Psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B) \otimes (c_1|\uparrow\rangle_C + c_2|\downarrow\rangle_C)$$

Multiplying out the states and changing to the Bell basis of A and C, this state can be rewritten:

$$\begin{aligned}|\Psi\rangle_{ABC} &= \frac{1}{2}|\Phi_0\rangle_{AC} \otimes |\Psi\rangle_B + \frac{1}{2}|\Phi_1\rangle_{AC} \otimes \sigma_1|\Psi\rangle_B + \frac{1}{2}|\Phi_2\rangle_{AC} \otimes \sigma_2|\Psi\rangle_B + \frac{1}{2}|\Phi_3\rangle_{AC} \otimes \sigma_3|\Psi\rangle_B \\ &= \sum_{i=0}^{3} \frac{1}{2}|\Phi_i\rangle_{AC} \otimes \sigma_i|\Psi\rangle_B\end{aligned}$$

When Alice measures the Bell state of A and C, she will find one of $|\Phi_0\rangle_{AC}$, $|\Phi_1\rangle_{AC}$, $|\Phi_2\rangle_{AC}$, $|\Phi_3\rangle_{AC}$, each with probability $\frac{1}{4}$. Whichever $|\Phi_i\rangle_{AC}$ she measures, the state of particle B will be $\sigma_i|\Psi\rangle_B$ after measurement.

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3) To send Bob the state of particle C, therefore, Alice does not need to send Bob the possibly infinite amount of information contained in the coefficients c_1 and c_2 which may be real numbers out to arbitrary precision. She needs only to send the integer i of the Bell state of A and C, which is a maximum of two bits of information. Alice can send this information to Bob in whatever classical way she likes.

4) Bob receives the integer i from Alice that labels the Bell state $|\Phi_i\rangle_{AC}$ that she measured. After Alice's measurement, the overall state of the system is:

$$|\Psi\rangle_{ABC} = |\Phi_i\rangle_{AC} \otimes \sigma_i |\Psi\rangle_B.$$

Bob therefore applies σ_i to the disentangled $|\Psi\rangle_B$ state on his end, by measuring the spin along axis i. Since $\sigma_i^2 = I$ for all i, Bob is left with the overall state:

$$|\Psi\rangle_{ABC} = |\Phi_i\rangle_{AC} \otimes |\Psi\rangle_B$$

Bob has therefore changed the spin state of particle B to:

$$|\Psi\rangle_B = c_1|\uparrow\rangle_B + c_2|\downarrow\rangle_B,$$

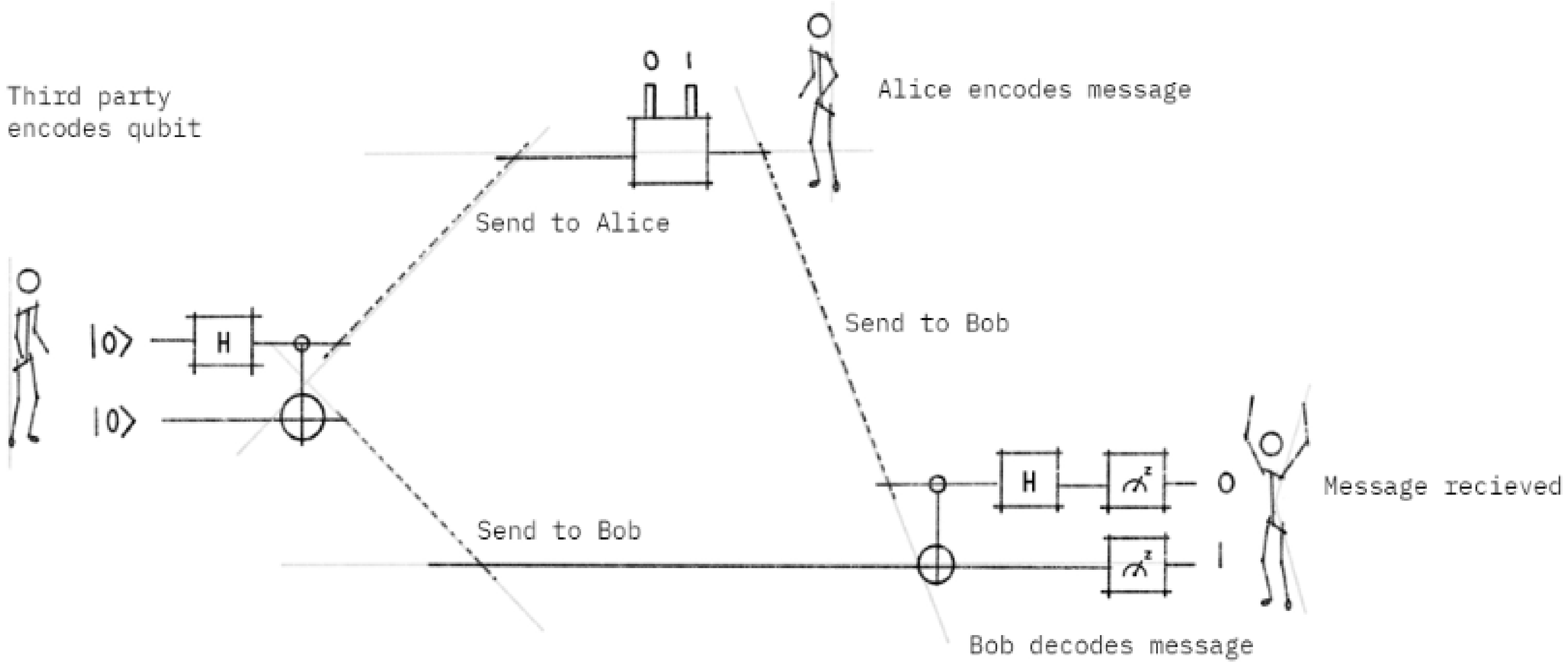
which is identical to the original state of particle C that Alice wanted to send. The information in state C has been "teleported" to Bob's state: the final spin state of B looks like C's original state. Note, however, that the particles involved never change between observers: Alice always has A and C, and Bob always has B.

SUPERDENSE CODING

Superdense coding is a procedure that allows us to send two classical bits to another party using just a single qubit during communication. In a simplified way, this protocol replaces two classical bits with one qubit.

Steps Involved:

- Preparation: Prepare (by convention) state; split the two pair among sender and receiver.
- Transport: Sender physically moves away with h(is/er) pair of entangled qubit.
- Encoding: Sender encodes classical bits of information to h(is/er) pair of qubit.
- Transmission: Sender transmits h(is/er) qubit to receiver.
- Decoding: Receiver decodes and measures h(is/er) qubit to retrieve sender's classical state.


Third party
encodes qubit
0 1
Alice encodes message
Send to Alice
|0⟩
H
|0⟩
Send to Bob
Send to Bob
H
0
1
Message recieved
Bob decodes message

- **Step 1 : Preparation**

It starts with a third party (say, Charlie) who prepares a pair of entangled states at $|0\rangle$ Applying Hadamard gates to both the initialized qubits a CX gate is applied in order to create an entangled state. (First one as control and second one as the target).

- **Step 2 : Separation**

Sender takes q0 and receiver takes q1, they travel far apart.

- **Step 3 : Encoding**

➢ Charlie sends the first qubit to sender and the second qubit to receiver. The goal of the protocol is for sender to send 2 classical bits of information to receiver using his/her qubit.

➢ But before he/she does, sender needs to apply a set of quantum gates to his/her qubit depending on the 2 bits of information sender wants to send.

➢ Thus if the sender wants to send a 00, he/she does nothing to the qubit (apply the identity (I) gate). If wants to send a 01, then applies the X gate.

➢ Depending on what the sender wants to send, the sender applies the appropriate gate, then sends the qubit to the receiver for the final step in the process.

Intended Message	Corresponding Encoding Alice Applies	Resulting State
00	I (does nothing)	$\frac{1}{\sqrt{2}}\left(\lvert 00\rangle + \lvert 11\rangle\right) = \lvert\Phi^+\rangle$
10	X	$\frac{1}{\sqrt{2}}\left(\lvert 01\rangle + \lvert 10\rangle\right) = \lvert\Psi^+\rangle$
01	Z	$\frac{1}{\sqrt{2}}\left(\lvert 00\rangle - \lvert 11\rangle\right) = \lvert\Phi^-\rangle$
11	$X \cdot Z$	$\frac{1}{\sqrt{2}}\left(-\lvert 01\rangle + \lvert 10\rangle\right) = -\lvert\Psi^-\rangle$

- **Step 4 : Transmission**

Sender sends h(is/er) encoded qubit physically to receiver.

- **Step 5 : Decoding**

➢ The receiver receives the sender's qubit (leftmost qubit) and uses his qubit to decode the sent message. Notice that he/she does not need to have knowledge of the state in order to decode it – he/she simply uses the restoration operation.

➢ The receiver applies a CNOT gate using the leftmost qubit as control and the rightmost as target.

➢ Then he applies a Hadamard gate and finally performs a measurement on both qubits to extract transmitted message.

Receiver Receives	After CNOT gate	After H gate	Classical bits after measurement
$\lvert 00\rangle + \lvert 11\rangle$	$\lvert 00\rangle + \lvert 01\rangle$	$\lvert 00\rangle$	00
$\lvert 01\rangle + \lvert 10\rangle$	$\lvert 11\rangle + \lvert 10\rangle$	$\lvert 10\rangle$	10
$\lvert 00\rangle - \lvert 11\rangle$	$\lvert 00\rangle - \lvert 01\rangle$	$\lvert 01\rangle$	01
$-\lvert 01\rangle + \lvert 10\rangle$	$-\lvert 11\rangle + \lvert 10\rangle$	$-\lvert 11\rangle$	11

The Complete Protocol in a Simulator

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Enter your intended message from 00, 01, 10 and 11 :   11
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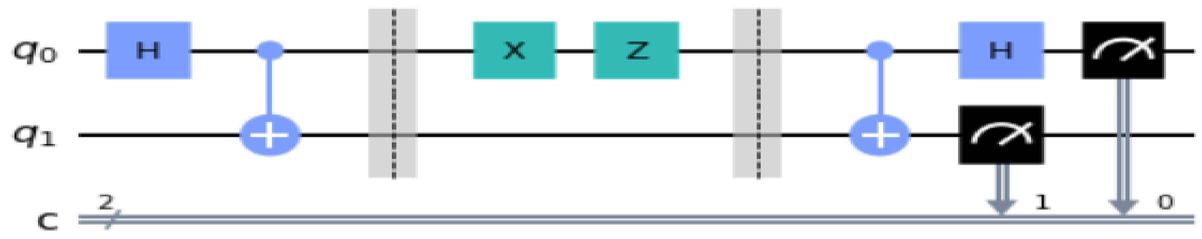


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Message received: 11
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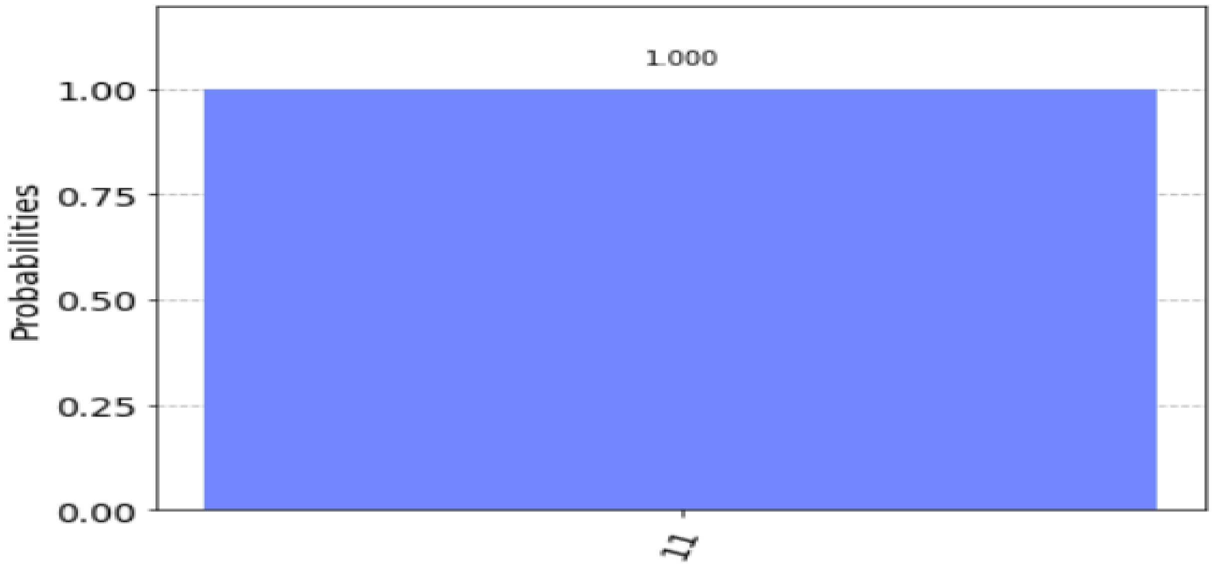

Bipartite states and Schmidt decomposition

Bipartite states are one of the basic objects in Quantum Information Theory and will be defined as follows:

Pure States

Definition

- Let $H = H_A \otimes H_B$ be a Hilbert space defined as a tensor product of two Hilbert spaces H_A and H_B. We call some pure state $|\psi\rangle_{AB}$ on the composite system $A \cup B$ **Bipartite**, if it is written with respect to the partition AB, which means $|\psi\rangle AB = \sum ij\ \chi ij\ |i\rangle A \otimes |j\rangle B$ where $|i\rangle_A$ and $|j\rangle_B$ are bases in H_A and H_B respectively.

Schmidt Theorem (Schmidt Decomposition)

There is a statement in linear algebra, according to which for every $|\psi\rangle_{AB}$ there exist bases $|u_i\rangle_A$ and $|v_j\rangle_B$ such that $|\psi\rangle_{AB} = \sum_{i=1}^{n} \sqrt{\tilde{\chi}_i}\, |u_i\rangle_A \otimes |v_i\rangle_B$, where $n = \min(dim(H_A), dim(H_B))$ and $\sum_{i=1}^{n} \tilde{\chi}_i = 1$.

The Schmidt Decomposition is useful for the separability characterization of pure states:

1. The state $|\psi\rangle_{AB}$ is separable if and only if there is only one non-zero Schmidt coefficient $\tilde{\chi}_i = 1$,
   $\tilde{\chi}_j = 0 \quad \forall j \neq i$;
2. If more than one Schmidt coefficients are non-zero, then the state is entangled;
3. If all the Schmidt coefficients are non-zero and equal, then the state is said to be *maximally entangled*.

Mixed States

- Definition

Let ρ_{AB} be a mixed state on a composite system $A \cup B$. Then we say that ρ_{AB} is a bipartite mixed state on $H_A \otimes H_B$ and write $\rho_{AB} = \sum_{ij} \lambda_{ij} G_i^A \otimes G_j^B$.

Schmidt Decomposition

The Decomposition can be also written for operators: $\rho_{AB} = \sum_{i}^{\bar{\lambda}} \lambda_i \tilde{G}_i^A \otimes \tilde{G}_i^B$, where $\tilde{\lambda}$ = max(dim(H_A)^2, dim(H_B)^2) are Schmidt numbers, which can be connected to the separability question of a bipartite state.

Generalization to multipartite states

Since the interest in entanglement theory is also shifting to the multipartite case, i.e. to systems composed of \; n>2 subsystems, the question of a *generalized* Schmidt Decomposition arises naturally.

Definition: For a pure state $|\psi\rangle le_{A_1 \dots A_n}$ belonging to a Hilbert space $H = H_1 \otimes \dots \otimes H_n$ we can define the **generalized Schmidt Decomposition**

$$|\psi\rangle le_{A_1\dots A_n} = \Sigma_{i\,=\,1}^{min\{d_{A_1},\,\dots,\,d_{A_n}\}} a_i|e_{A_1}\rangle le \otimes \dots\rangle le|e_{A_n}\rangle le.$$

In the multipartite setting, pure states admit a generalized Schmidt Decomposition only if, tracing out any subsystem, the rest is in a fully separable state.