# Notes on 131AH

Fall 2017 - I. Neeman

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# Contents

# 1 Peano Axioms

Creating Natural numbers by incrementing from 0

**Axiom 2.1** - 0 is a natural number
**Axiom 2.2** - if $n$ is a natural number, then $n++$ is a natural number
**Axiom 2.3** - 0 is not the successor of any natural number (i.e. $\forall n, n++ \neq 0$).
**Axiom 2.4** - Different natural number have different successors (i.e. if $n \neq m$ then $n++ \neq m++$).
**Axiom 2.5** - (Principle of Induction) Let $A$ be a set of natural numbers. Suppose $0 \in A$ and $\forall$ natural number $n$, if $n \in A$ then $n++ \in A$ (closed under successor operation) Then $A = \mathbb{N}$.

# 2 Recursion

Suppose we have a function $g$ on natural numbers and a natural number c, then $\exists!$ function for natural numbers so that

- (a) $f(0) = c$ starting on point $c$
- (b) $\forall n, f(n++) = g(f(n))$ what to do with successors

**proof:** (will only prove uniqueness, assume existence)
suppose$f_1, f_2$ both work, WTS $\forall n f_1(n) = f_2(n)$.
Using induction, have to show

- $f_1(0) = f_2(0)$, is true by (a)
- if $f_1(n) = f_2(n)$, then $f_1(n++) = f_2(n++)$ is true by (b)

Assume $f_1(n) = f_2(n)$
Then $f_1(n++) = g(f_1(n))$ by (b)
$= g(f_2(n))$ by induction hypothesis
$= f_2(n++)$
Use induction to prove recursion.

# 3 Addition

(defined by recursion)
fix $m$, Define $n+m$ by recursion on $n$:

- $0+m := m$ (starting value)
- $(n++)+m := (n+m)++$ (step function)

(formally, use proposition of definition by recursion with $c=m$, $g(i)=i++$)

**Lemma:** $\forall n$, $n+0=n$

**Proof:** verify by induction on $n$
want to show

- (a) $0+0=0$, true by definition for $m=0$
- (b) if $n+0=0$ then $(n++)+0=n++$
  i.e. $(n++)+0=(n+0)++=n++$, by def of addition with $m=0$ and by induction hypothesis.

**Lemma:** $\forall n, m\ n+(m++)=(n+m)++$

**Proof:** by induction on n, definition of addition for $m, n++$
want to show

- (a) $0+(m++)=(0+m)++$
  $0+(m++)=m++=(0+m)++$ by definition
- (b) if $n+(m++)=(n+m)++$, then $(n++)+(m++)=((n++)+m)++$
  $(n++)+(m++)=(n+(m++))++$ by definition
  by induction hypothesis $=((n+m)++)++$
  by definition of addition $=((n++)+m)++$

### 3.1 Commutativity, associativity, cancellation

**Proposition: commutativity**
$\forall a, b \in \mathbb{N}$
$a + b = b + a$ proof by induction on $a$, keeping $b$ constant

**Proposition: associativity**
$\forall a, b, c \in \mathbb{N}$
$(a + b) + c = a + (b + c)$

**Proposition: cancellation** $\forall a, b, c \in \mathbb{N}$
if $a + b = a + c$ then $b = c$
(proof by induction on a)

## 4 Ordering

**Definition** $n \in \mathbb{N}$ is called positive if $n \neq 0$

**Proposition** If $a$ is positive and $b \in \mathbb{N}$, then $a + b$ is positive. (Proof by induction)

**Corollary** If $a + b = 0$ , then $a = 0$, immediate by last proposition,
and $b = 0$ since $0 + b = b = 0$

**Definition of ordering**: for $n, m \in \mathbb{N}$
$n \leq m$ iff $\exists a\ n + a = m$, $n < m$ iff $n \leq m$ and $n \neq m$

**Properties of Ordering**

- $a \geq a$
- $a \geq b$ and $b \geq c \rightarrow a \geq c$ proof by associativity
- $\forall c\ a \geq b$ iff $a + c \geq b + c$
- $a < b$ iff $a{+}{+} \leq b$
- $a < b$ iff $b = a + d$ for some positive $d$

**Proposition:** $\forall n, m$ exactly one of the following hold

- $a < b$
- $a = b$
- $a > b$

## 4.1 Strong principle of induction

suppose $0 \in A$, regular induction wants to show $n \in A \to n++ \in A$ to show $A = \mathbb{N}$
with strong induction, still would want to show $n++ \in A$, but

- $(\forall m < n++)\ m \in A \to n++ \in A$
- then $A = \mathbb{N}$

Can now combine the two statements to get

**Strong induction**: suppose $A \subseteq \mathbb{N}$ is such that
$\forall k$ if $(\forall m < k)\ m \in A$ then $k \in A$
then $A = \mathbb{N}$

Use regular induction on a different property

**From Tao, 2.2.14**
let $m_0 \in \mathbb{N}$, $P(m)$ for $m \in \mathbb{N}$
suppose $\forall m \geq m_0$ if $P(m')\ \forall m_0 \leq m' < m \to P(m)$
(in particular $P(m_0)$ true)
$\to P(m)\ \forall m \geq m_0\ m \in \mathbb{N}$

Define $Q(n) = P(m)\ \forall m_0 \leq m < n$
prove $P(m')\ m' > m$

$Q(0)$ vacuously true
suppose $Q(n) = P(m)\ \forall m_0 \leq m < n$
prove $Q(n++) = P(m)\ \forall m_0 \leq m < n++$

# 5 Multiplication

**Definition:** let $m \in \mathbb{N}$ $0 * m := 0$
Suppose $n * m$, $(n++) * m := (n * m) + m$.
Commutative: let $m, n \in \mathbb{N}$ $m * n = n * m$
Induct on $n$: let $n = 0$ $0 * m := 0$ by definition So know $0 * m := 0$, now have to prove $m * 0 = 0$

# 6 Integers

equivalence class of formal expressions

## 6.1 equivalence classes and integers

**Definition:** An **equivalence relation** on a set $a$ is a relation $\equiv$ such that

- (1) reflexivity $\forall a \in A\ a \equiv a$
- (2) symmetry $\forall a, b \in A\ a \equiv b$ iff $b \equiv a$
- (3) transitivity $\forall a, b, c \in A$ if $a \equiv b$ and $b \equiv c$ then $c \equiv a$

**Definition:** define the **equivalence class** of $a \in A$ to be $[a] = \{a' | a' \equiv a\}$

**Proposition:** Assuming equivalence relation $\equiv$, $[a] = [b]$ iff $a \equiv b$

> **proof:** suppose $[a] = [b]$ show elements of $[a]$ are in $[b]$ and vice versa
> $\Rightarrow$ use transitivity and symmetry, show equivalence classes are literally equal
> $\Leftarrow$ use reflexivity

**Definition:** An integer is the equivalence class of an expression of $a - -b$, where $a, b \in \mathbb{N}$ under the equivalence relation $a - -b \equiv c - -d$ iff $a + d = b + c$

**Proposition:** $\equiv$ above is an equivalence relation

> **proof:** (for transitivity) suppose $a_1 - -b_1 = a_2 - -b_2$ and $a_2 - -b_2 = a_3 - -b_3$, then $a_1 + b_2 = b_1 + a_2$, $a_2 + b_3 = b_2 + a_3$.
> want to show $a_1 + b_3 = b_1 + a_3$
> $a_1 + a_2 + b_2 + b_3 = a_2 + a_3 + b_1 + b_2 \implies a_1 + b_3 = b_1 + a_3$ by cancellation, associativity and commutativity

## 6.2 addition, subtraction, multiplication

**Defining addition of integers** Need operation that respects equivalence relation.
Let $\equiv$ be an equivalence relation on $A$. A function $f$ on tuples from $A$ into $A$ respects $\equiv$ if when $a_1 \equiv b_1 \equiv ...a_k \equiv b_k$, then $f(a_1...a_k) \equiv f(b_1...b_k)$. If $f$ respects $\equiv$, then it induces a function on the equivalence classes. Formally, $f([a]) = f([b])$.

**Defining addition of integers, continued** The sum of integers $a - -b$, $c - -d$ is $(a + c) - -(b + d)$. (Need to check + respects $\equiv$)

**proof: (half of it)**
suppose $a_1 - -b_1 \equiv a_2 - -b_2$
will show $(a_1 - -b_1) + (c - -d) \equiv (a_2 - -b_2) + (c - -d)$
(formal proof needs different $c$, $d$, values)
$\Rightarrow$ show $(a_1+c)--(b_1+d) \equiv (a_2+c)--(b_2+d) \Rightarrow$ show $a_1+c+b_2+d = a_2+c+b_1+d$
enough to show $a_1 + b_2 = a_2 + b_1$
then add $c + d$ on both sides
$\Rightarrow$ have $a_1 + b_2 = a_2 + b_1$ since $a_1 - -b_1 \equiv a_2 - -b_2$

**Defining negation of integers** (Have to check equivalence relation) For an integer $a--b$ define $-(a - -b)$ to be $b - -a$. If $a_1 - -b_1 \equiv a_2 - -b_2$ then $-(a_1 - -b_1) \equiv -(a_2 - -b_2)$. Clear if $a_2 + b_2 = a_2 + b_1$, then $b_2 + a_1 = b_1 + a_2$ so $b_1 - -a_1 \equiv b_2 - -a_1$

**Lemma, dichotomy of integers**
For every integer $a - -b$, exactly one of the following holds
(1) $a - -b$ is 0
(2) $a - -b$ is $n$ for a positive number $n \in \mathbb{N}$
(3) $a - -b$ is $-n$ for a positive natural number $n$

**Defining subtraction** For integers $x, y$ define $x - y := x + (-y)$ equivalently $(a - -b) - (c - -d) := (a - -b) + (d - -c)$

**Defining multiplication** $(a - -b)(c - -d) = (ac + bd) - -(bc + ad)$
$\rightarrow$ check if respects integer equivalence relation $\equiv$ to view it as operation on $\equiv$ classes
$\rightarrow$ on non-negatives, this operation agrees with the multiplication defined on $\mathbb{N}$ defined recursively from (+)

## 6.3 zero division, cancellation, and ordering

**Proposition: No zero divisions** if $ab = 0$, then either $a = 0$ or $b = 0$

**Proposition: Cancellation** if $ac = bc$ and $c$ is nonzero, then $a = b$

**Definition:** for integers $a, b$ define $a \leq b$ iff there is a nonnegative natural $n$ such that $a + n = b$

**Definition:** $a < b$ iff $a \leq b$ and $a \neq b$

**Properties of order**

- $a \leq b$ iff $b - a$ is a natural number
- $a < b$ iff $b - a$ is positive
- if $a \leq b$ then $(\forall c)$ $a + c \leq b + c$
- if $a < b$ then $(\forall c)$ $a + c < b_c$ (strict order in the book)
- if $a < b$ and $c$ is positive, then $ac < bc$
- if $a < b$ and $c$ is negative, then $ac > bc$
- $a < b$ and $b < c \Rightarrow a < c$
- for any $a, b$ exactly one of the following holds $a < b, a = b, b < a$

# 7 Rationals

**Definition: Rational numbers** are the equivalence classes of expressions $a//b$ for integers $a$ and non-zero integers $b$, under the equivalence relation $a//b \equiv c//d$ iff $ad = bc$
$\Rightarrow$ nonzero because cancellation law of ints is restricted to non-zero integers

**check equivalence**

- reflexivity: since $ab = ab \Rightarrow a//b = a//b$
- symmetry: clear
- transitivity: suppose $a//b \equiv c//d$ and $c//d \equiv e//f$ want $a//b \equiv e//f$. $adf = bcf$ and $bcf = bed$, so $adf = bed$, know by definition that $d$ is nonnegative. $\Rightarrow$ since $d$ is nonzero, can apply cancellation law, get $af = be$ so $a//b \equiv e//f$

**Definition of sum** $(a//b) + (c//d) := (ad + cb)//bd$

**Definition of multiplication** $(a//b)(c//d) := (ac)//bd$

**check:**

- $bd \neq 0$
- respects $\equiv$
- works same way as would with integers

Identify $\mathbb{Z}$ with the subset $\{a//1 | a \in \mathbb{Z}\}$ of $\mathbb{Q}$
($\rightarrow$ identify integer $a$ with rational $a//1$)
check all operations on $\mathbb{Z}$ agree with new operations on $\mathbb{Q}$, then can start proving properties

**Definition** for integer $a//b$ with $a \neq 0$, define $(a//b)^{-1} := b//a$
Sequence of rationals without limits

### 7.1 backwards induction

**Backwards induction proof**

$\Rightarrow$ Know $f(n++) \to f(n)$, assume $P(n++) \to P(n)$
$\Rightarrow$ Let $Q(m)$ be [if $P(m)$ is true, then $(\forall k) \leq m, k = 0, P(k)$ true]
$\Rightarrow$ Want to prove $Q(m)$ by induction on $m$
$\Rightarrow$ Base: $Q(0)$ [if $P(0) \Rightarrow (\forall k) \leq 0, k = 0, P(k)$ true]
$\Rightarrow$ Suppose $Q(m)$, prove $Q(m++)$
$\Rightarrow$ WTS $Q(m++)$ :[if $P(m++)$ true, then $(\forall k) \leq m++$, $P(k)$ true]
if $k \leq m++$
(1) $\Rightarrow$ know $P(k)$ true for $k < m++$
(2) $\Rightarrow$ $P(k)$ true for $k = m++$

## 8 Dedekind cuts

– By Chernikov, refer to Neeman's notes up until cardinalities

## 9 Real Numbers

– See Neeman's notes

## 10 Cardinalities

**Definition:** Sets A, B have the same cardinality with $A \sim B$ if $\exists$ a bijection $f : A \to B$

Set $I_n = \{0, ..., n-1\}$

- $A$ is finite if $\exists n \sim \mathbb{N}$ where $A \sim I_n$
- $A$ is countable if $A \sim \mathbb{N}$
- $A$ is uncountable if A is not finite and not countable

  note:

- if A is uncountable and $B \sim A$ then B is countable
- if A is infinite and $B \sim A$ then B is infinite

**Proposition:** If $A$ is finite and $B \subseteq A$ then $B$ is finite.

**Proof:** use $\forall n \in \mathbb{N} \forall u \in I_n$ there is $k \in \mathbb{N}$ such that $u \sim I_k$.
$\Rightarrow$ Clear that $\mathbb{N}$ is countable
$\Rightarrow$ $\mathbb{Z}$ is also a countable equation using this bijection:

$$n \mapsto \begin{cases} \dfrac{n}{2} & : n \rightarrow even \\ -\dfrac{n+1}{2} & : n \rightarrow odd \end{cases} \quad \text{from } \mathbb{N} \text{ to } \mathbb{Z}$$

## 10.1 countable union of countable sets, and props that follow

**Theorem:** let $A_n$ $n \in \mathbb{N}$ each be countable. Then $\bigcup_{i=1}^{\infty} A_i$ is countable.
(The countable union of countable sets is countable)

**proof:** $f_n$ is a bijection $f_n : \mathbb{N} \rightarrow A_n$
$f_0(0)$ $f_0(1)$ $f_0(2)$ $f_0(3)$ ... $A_0$ define surjection onto this square
$f_1(0)$ $f_1(1)$ $f_1(2)$ ... $A_1$
$f_2(0)$ $f_2(1)$ ... $A_2$
$f_3(0)$ ... $A_3$

Define $g : \mathbb{N} \rightarrow \bigcup_{n \in \mathbb{N}} A_n$ to enumerate the union in the order indicated.

By recursion define these sequences:
$g(0)$ $g(2)$ $g(5)$ $g(9)$ define surjection onto this square
$g(1)$ $g(4)$ $g(8)$
$g(3)$ $g(7)$
$g(6)$
Then $g : \mathbb{N} \rightarrow \bigcup_{i=1}^{\infty} A_i$ is onto. By previous claim, $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

**Corollary:** $\mathbb{Q}$ is countable
**proof:** for $m \in \mathbb{N}$ greater than 0 have $\{\frac{n}{m} | n \in \mathbb{Z}\} \sim \mathbb{Z}$, so countable.
Have $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{\frac{n}{m} | n \in \mathbb{Z}\}$
So $\mathbb{Q}$ is a countable union of other countable sets, and by the last property, $\mathbb{Q}$ is countable.

**Theorem:** Let $A_1, A_2$ be countable.
Then $A_1 \times A_2 := \{< a_1, a_2 > | a_1 \in A_1, a_2 \in A_2\}$ is countable.
**Proof:** Have $A_1 \times A_2 = \bigcup_{u \in A_1} \{< u, a_2 > | a_2 \in A_2\}$ (countable $\sim A_2$)
So $A_1 \times A_2$ is a countable union of countable sets.

**Theorem:** let $A$ be countable. For $n \in \mathbb{N}$ greater than or equal to 1, let $B_n$ be the set of n-tuples from $A$.
Then $B_n$ is countable.
**Proof:** By induction on $n$
$\Rightarrow B_1 \sim A$ (Bijection on $< a > \mapsto a$), so $B_1$ is countable.
$\Rightarrow B_{n+1} \sim A \times B_n$ ($< a_1, a_2, ..., a_n, a_{n+1} > \mapsto < a_1, < a_2, ..., a_n >>$)
countable by induction using previous theorem.

## 10.2 Cantor... and uncountable things

**Theorem:** Let $C$ be the set of functions $e : \mathbb{N} \to \{0, 1\}$. (Think of $e$ as an infinite sequence $e(0), e(1), e(2)...$ of 0s and 1s). Then C is uncountable.

**Theorem: (my boi Cantor's diagonal argument)** let $C$ be the set of functions from $\mathbb{N} \to \{0, 1\}$ (infinite sequence of $0, 1$s, such as $e(0), e(1), e(2), e(3)...$)
Then $C$ is not countable.

**Proof:** Suppose for contradiction that $C$ is countable. Say $f : \mathbb{N} \to C$ is a bijection.
Write $e_i$ for each $f(i)$:

$e_0(0)$ $e_0(1)$ $e_0(2)$ $e_0(3)$ ... $e_0$
$e_1(0)$ $e_1(1)$ $e_1(2)$ $e_1(3)$ ... $e_1$
$e_2(0)$ $e_2(1)$ $e_2(2)$ $e_2(3)$ ... $e_2$
$e_3(0)$ $e_3(1)$ $e_3(2)$ $e_3(3)$ ... $e_3$

**WTS** $f$ is not a bijection
Define $s : \mathbb{N} \to \{0, 1\}$ (element of $C$) by

$$s(n) = \begin{cases} 1 & \text{if } e_n(n) = 0 \\ 0 & \text{if } e_n(n) = 1 \end{cases}$$

So $s \in C$ and for every $i$, $s \neq e_i$ (because by definition of $s, s(i) \neq e(i)$).
So we found an element $s \in C$ which is not in the range of $f$, so $f$ is not onto.

## 10.3 Schroder-Bernstein

**Definition:** $A$ is of smaller or equal cardinality than $B$ (write $A \preceq B$) iff there's an injection $f : A \to B$

**Proposition:** If $A \preceq B$ and $A$ is uncountable, then $B$ is uncountable.

> **Proof(ish):** Suppose by contradiction $A \to B \to \mathbb{N}$. Then $A \to \bigcup \mathbb{N}$ (some subset of $\mathbb{N}$), contradiction.

**Theorem (Schroder-Bernstein):** If $A \preceq B$ and $B \preceq A$ then $A \sim B$
(i.e. if $\exists f : A \to B, g : B \to A$ both one-to-one, then there is a bijection $h : A \to B$).
(used to prove two sets are the same cardinality.)

Note: Let $C$ be the set of functions from $\mathbb{N} \to \{0, 1\}$.
Will prove later that $C \preceq \mathbb{R}$ (if an uncountable set injects into $\mathbb{R}$, then $\mathbb{R}$ is uncountable.)

In the next proposition we will see $\mathbb{R} \preceq C$
So by Schroder-Bernstein (plus $C \preceq \mathbb{R}$), $C \sim \mathbb{R}$

**Proposition:** $\mathbb{R} \preceq C$

**Proof:** Fix a bijection $h : \mathbb{N} \to \mathbb{Q}$ for $x \in \mathbb{R}$ set $e_x : \mathbb{N} \to \{0, 1\}$
map real to rationals below it, convert set of rationals into a sequence of 0s and 1s.

$$e_x(n) = \begin{cases} 1 \text{ if } h(n) < x & \text{(if nth rational is in the set)} \\ 0 \text{ if } h(n) \geq x & \text{(if is not in the set)} \end{cases}$$

Check that $x \mapsto e_x$ is one-to-one (from $\mathbb{R}$ into $C$)
To see that, fix $x \neq y$, say $x < y$ (wlog)
By density, have $q \in \mathbb{Q}$ such that $x < q < y$.
So you'd have $n$ such that $q = h(n)$.
Since $e_x(n) = 0$ because $q > x$ and $e_y(n) = 1$ so $e_x \neq e_y$, and we have an injection.

# 11 Pigeonhole Principle

(a) if for $n \in \mathbb{N}$ $f : I_n \to I_n$ is 1-1 then $f$ is onto.
(b) if $A$ is finite and $B \subsetneq A$ then $B \nsim A$

**Prop:** suppose $A$ is infinite and $f : \mathbb{N} \to A$ is onto, then $A$ is countable.

**proof:** let $U = \{n \in \mathbb{N} | (\forall k < n) f(k) \neq f(n)\}$
($f(n)$ is a "new" element of $A$)
then $f \upharpoonright U : U \to A$ is a bijection, so $u \sim A$, enough to show $u$ is countable and $A$ is also countable.

Note: $U \neq 0$, $0 \in U$, and $U$ is infinite:
(suppose by contradiction that $U$ is finite, at some point you reach some $l$ where you stop adding elements to $U$)

**WTS:** $U \sim \mathbb{N}$ using these notes and (knowing) the fact that $U \subseteq \mathbb{N}$
by recursion define $h : \mathbb{N} \to U$ as follow:
$h(0) = min(U)$ (exists since $U$ nonempty)
$h(n+1) = least(k \in U)$ such that $k > (h(n))$ (exists since $U$ is infinite and hence $U \nsubseteq \{h(0), h(1), ...h(n)\}$)
Can check $n : \mathbb{N} \to U$ is a bijection.

## 11.1 Chris's proof of pigeonhole

Recall $J_n := \{k \in \mathbb{N} | 1 \leq k \leq n\}$ or $I_n$
Define a set $A$ is finite if $A \sim J_n$ for some $n$
i.e. $\exists$ bijection $A \to J_n$

**Proposition 1– pigeonhole principle:** if $J_n \to J_n$ is injective, then $f$ is surjective.

**Lemma:** if $n \geq 2$ then for any $k \in J_n, \exists$ injection $\tau : J_n \backslash \{k\} \to J_{n-1}$

**proof**: Define $\tau$ by

$$\tau(j) = \begin{cases} j \text{ if } j < k \\ j - 1 \text{ if } j > k \end{cases}$$

suppose we have $\tau(j_1) = \tau(j_2)$, prove $j_1 = j_2$
have four cases in relation to $k$

- case 1: $j_1, j_2 < k$ then $j_1 = j_2$

- case 2: $j_1, j_2 > k$ then $j_1 - 1 = j_2 - 1 \Rightarrow j_1 = j_2$
- case 3: $j_1 < k, j_2 > k$ then $j_1 = j_2 - 1 \Rightarrow j_1 + 1 = j_2$
  $\Rightarrow$ by assumption on $j_2$, $j_1 + 1 > k \Rightarrow j_1 \geq k \Rightarrow$ contradicts assumption that it's less than $k$, $(j_1 < k)$
- case 4: same, (contradiction)

**Proof of proposition 1:** Induct on $n$ base case $n = 1$
$J_1 = \{1\}$ so automatically injective ($f : J_1 \Rightarrow J_1 =$Id), and surjective

**Inductive step:** Assume that any injection $J_{n-1} \Rightarrow J_{n-1}$ is surjective. Let $f : J_n \Rightarrow J_n$ injective.

**Define:** Let $k = f(n)$ consider restriction $f \restriction_{J-1}: J_{n-1} \Rightarrow J_n \backslash \{k\}$.
The codomain is ok, by injectivity.
Then $f \restriction_{J_n}$ is injective and by lemma 2, $\exists$ injection $\tau : J_n \backslash \{k\} \to J_{n-1}$
So $\tau \circ f \restriction_{J_{n-1}}: J_{n-1} \to J_{n-1}$ is injective and therefore surjective by inductive hypothesis.
**Note** $g \circ h$ surjective $\Rightarrow g$ is surjective $\Rightarrow h$ is surjective
e.g. $h(n) : \mathbb{N} \to 2\mathbb{N}, g(n) : 2\mathbb{N} \to \mathbb{N}$

In particular $\tau$ is surjective so it's also bijective.
Therefore $f \restriction J_{n-1} = \tau^{-1} \circ (\tau \circ f \restriction_{J_{n-1}})$ is surjective.
$range(f) \supseteq range(f \restriction_{J_{n-1}}) \supseteq J_n \backslash \{n\}$
But by definition of $k$ $k \in range(f)$ so $range(f) \supseteq J_n$ i.e. $f$ is surjective, using restriction argument.

**Corollary** if $B \subsetneq A$ and $A$ is finite, then $\nexists$ injection $f : A \to B$

**proof** can assume that $A = J_n$ (check)
Suppose $\exists$ injection $f : A \to B$ (by contradiction)
Case $b = \emptyset$ is clear
Let $k \in J_n \backslash B$ and let $\tau : J_n \backslash \{k\} \to J_{n-1}$ (lemma 2)
Then we have: $J_{n-1} \longmapsto_{identity} J_n \longmapsto_f B \longmapsto_{\tau \restriction_B} J_{n-1}$
Note that $J_{n-1} \longmapsto J_{n-1}$ is an injection, surjection by pigeonhole (prop 1)
In particular, $B \sim J_{n-1}$ so $J_{n-1} \sim J_{n-1}$ (we're still doing a contradiction)
Let $g : I_n \to I_{n-1}$ (and $I_n \sim I_{n-1}$ by above)
So $I_n \mapsto_g I_{n-1}$ is a surjection by the pigeonhole principle
However the restriction $I_{n-1} \mapsto I_{n-1}$ is surjective, so the whole thing can't be injective.

# 12 Metric Spaces

**Definition:** Let $Y$ be a set.

A function $d : Y \times Y \to \{x \in \mathbb{R} | x > 0\}$ is a **metric** on $Y$ iff
(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$
(2) $d(x, y) = d(y, x)$
(3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle)

Example: $d(x, y) = |x - y|$ on $y \in \mathbb{R}$ is a metric. A metric space is a pair $(Y, d)$ where $d$ is a metric on $Y$.

**Definition:** let $(Y, d)$ be a metric space. Let $E \subseteq Y$

- for $x \in Y, r > 0$ define $N_r(x) = \{u \in Y | d(x, y) < r\}$
(ball centered at $x$, with radius $r$)
- $x$ is a **limit point** of $E$ iff $(\forall r > 0(E - \{x\}) \cap (N_r(x) \neq \emptyset))$
find something in $E$ close to $x$, for any small $r$.
- $x$ is an **isolated point** of $E$ if $x \in E$ and $x$ is not a limit point of $E$
- $E$ is **closed** if every limit point of $E$ belongs to $E$
- $E$ is **perfect** if $E$ is closed and has no isolated points.
- $x$ is an **interior point** of $E$ if $\exists(r > 0)$ such that $N_r(x) \subseteq E$
(an entire ball of $x$ is contained in $E$)
- $E$ is **open** if every $x \in E$ is an interior point of $E$
- $E$ is **dense** if $(\forall x \in Y)(\forall r > 0)N_r(x) \cap E \neq \emptyset$
- $E^c$ denotes $Y \backslash E$
- $E$ is **bounded** if $\exists x \in Y, \exists M \in \mathbb{R}$ such that $E \subseteq N_m(x)$.

## 12.1 neighborhoods are open, and open sets

**Theorem:** Each $N_r(x)$ (neighborhood) is open

**Proof (using triangle inequality)** Fix $u \in N_r(x)$
Need $\epsilon > 0$ such that $N_\epsilon(u) \subseteq N_r(x)$
Let $h = d(x, u)$ and let $\epsilon = r - h > 0$
we claim $N_\epsilon(u) \subseteq N_r(x)$
To prove this fix $y \in N_\epsilon(u)$
By triangle inequality have $d(x, y) \leq d(x, u) + d(u, y) < r$ $(d(x, u) = h, d(u, y) < \epsilon)$

**Theorem:** $\{u | d(x, u) > r\}$ is open

**Proof:** similar to previous theorem. From now on, call $N_r(x)$ the open ball of radius $r$ at $x$

## 12.2 any neighborhood around a limit point contains infinitely many points of the set

**Proposition:** If $p$ is a limit point of $E$, then for every $r > 0, E \cap N_r(x)$ is infinite. (Know from definition it has at least one point $\neq x$)

**proof:** suppose not, say $E \cup N_r(x) = \{y_1, ..., y_n\}$ for $n \in \mathbb{N}$
So $E \cap N_r(x) \subseteq y_1, ..., y_n \cup \{x\}$ for $n \in \mathbb{N}$ where $y_1 \neq x$
Let $s_i = d(x, y_i) > 0$ and let $s = min(s_1, ..., s_n) > 0$ (small claim: finite sets have a minimum, by induction on $n$)
Then for each $i$, $y_i \notin N_s(x) \cap E$
Have $s < r$, so $N_s(x) \subseteq N_r(x)$, so $E \cap N_s(r) \subseteq E \cap N_r(x) \subseteq \{y_1, ..., y_n\} \cup \{x\}$
Since $y_i \notin N_s(x)$, have in fact $E \cap N_s(x) \subseteq \{x\}$
This contradicts the fact that $x$ is a limit point of $E$.

## 12.3 complements of open sets are closed

**Theorem** $E$ is open if $E^c$ is closed

**Proof:**
($\Rightarrow$) Suppose $E$ is open, let $x$ be a limit point of $E^c$
Suppose for contradiction that $x \notin E * c$ i.e. $x \in E$
Since $E$ is open, have $r > 0$ such that $N_r(x) \subseteq E$, i.e. $N_r(x) \cup E^c = \emptyset$
This contradicts the assumption that $x$ is a limit point of $E^c$
($\Leftarrow$) Fix $x \in E$. Need $r > 0$ such that $N_r(x) \subseteq E$
Suppose there is no such $r$, so for every $r > 0$ have $y \in N_r(x)$ such that $y \notin E$ (i.e. $y \in E^c$)
Note $y \neq x$ since $x \in E$ and $y \notin E$. This shows $x$ is a limit point of $E^c$
$E^c$ is closed (by assumption) since $E^c$ is closed. This implies $x \in E^c$, a contradiction.

**corollary:** $F$ is closed if $F^c$ is open

saw before that $\{u|d(x,u) > r\}$ is open, so $\{u|d(x,u) \leq r\}$ is closed.
- Refer to this set as the **closed ball** of radius $r$ at $x$
- Refer to the open ball $N_r(x)$ as the basic open neighborhoods of $x$
- Refer to any open $D$ such that $x \in D$ as an open neighborhood of $x$

## 12.4 union of open sets are open, and cor. about closed sets

**Theorem:** Every union of open sets is open (immediate from definition)
Finite intersections of open sets is open.

**Proof:** Say $A_1 ... A_n$ are open.
Fix a point in the intersection $x \in \bigcap_{i=1}^{n} A_i$ (finite)
Then for each $i$ we have $r_i > 0$ such that $N_{r_i}(x) \subseteq A_i$
(since $x \in A_i$ and $A_i$ is open)
let $r = min(r_1, ..., r_n) > 0$
check $N_r(x) \subseteq \bigcap_{i=1}^{n} A_i$

**Corollary:** Finite union of closed sets is closed (complement)

**Proof (sketch):** If $f_i$ are closed, look at their complements
$A_i = F_i^c$ which are open, and use that $\cap F_i = (\cup A_i)^c$

## 12.5 closure definition and operations

**Definition:** Let $E \subseteq Y$. Define $E'$ to be the set of limit points of $E$.
Define $\bar{E} = E \cup E'$

**Proposition:** if $F \supseteq E$ and $F$ is closed, then $F \supseteq \bar{E} \subseteq$
**Proof:** Every $x \in E'$ is a limit point of $F$ so in $F$.
Also every $x \in E$ is in $F$ since $F \supseteq E$.

**Theorem:** $\bar{E}$ is closed.
**Corollary:** $\bar{E}$ is the smallest closed set containing $E$.
**Proof (of theorem):** Show $\bar{E}^c$ is open.
Fix $y \in \bar{E}^c$ i.e. $y \notin \bar{E}$.
Then $y \notin E$ and $y \notin E'$
i.e. $y$ is not a limit point of $E$
So there is a $r > 0$ such that $N_r(y) \cap E = \emptyset$
Since $N_r(y)$ is disjoint from $E$, it's contained in $E^c$.
**WTS** it's contained in $\bar{E}^c$, so **WTS** it's disjoint from $\bar{E}$
(i.e. show $N_r(y) \cap \bar{E} = \emptyset$) Showing $N_r(y) \subseteq \bar{E}^c$ is what we need to see if $\bar{E}^c$ is open, so what's left is to show $N_r(y) \cap E' = \emptyset$. Suppose for contradiction we have $u \in N_r(y) \cap E'$.
Since $u$ is a limit point, need to find some radius so that this open ball is contained in the bigger open ball.
Since $N_r(y)$ is open, have $\epsilon > 0$ such that $N_\epsilon(u) \subseteq N_r(y)$.
Since $u \in E'$ must have $N_\epsilon(u) \cap E \neq \emptyset$.
This is a contradiction since in fact $N_r(y) \cap E = \emptyset$.

# 13 Compactness

**Definition** Let $E \subseteq Y$. An **open cover** of $E$ is a collection of open sets $G_\alpha$, $\alpha \in J$ such that $\bigcup_{\alpha \in J} G_\alpha \supseteq E$ ($J$ is the index set, can be anything). Write the open cover as $\{G_\alpha\}_{\alpha \in J}$ or $\{G_\alpha | \alpha \in J\}$

**Definition** $E \subseteq Y$ is **compact** if for every open cover $\{G_\alpha\}_{\alpha \in J}$ of $E$ there is a finite $\{a_1, ..., a_n\} \subseteq J$ such that $G_{\alpha_1} \cup ... \cup G_{\alpha_n} \supseteq E$. Then $\{G_{\alpha_i}\}$ $i \in \{1, ..., n\}$ is an open cover of $E$. Refer to it as the (finite) subcover of $\{G_\alpha\}_{\alpha \in J}$

Aim to prove there are other compact sets, by working in $\mathbb{R}$. By an **open interval**, means any $(a, b) = \{x \in \mathbb{R} | a < x < b\}$ (open because it's an open ball of radius $\frac{b-a}{2}$ centered at $\frac{b+a}{2}$). By a **closed interval**, means any $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ (equal to a closed ball)

## 13.1 cantor's intersection theorem (for real numbers)

**Proposition:** Let $I_n = [a_n, b_n]$ be closed and bounded intervals (in $\mathbb{R}$)

Suppose $I_{n+1} \subseteq I_n$. Then $\bigcap_{n \in \mathbb{N}} I_n \neq 0$.
**Proof:** Let $A = \{a_n | n \in \mathbb{N}\}$. $A$ is bounded above, in fact every $b_k$ is an upper bound of all $a_n$. ($A$ is nonempty, bounded above). Let $z = sup(A)$.
Have $z \leq b_k$ for each $k$, since $b_k$ is a bound for $A$.
Also $z \geq a_n$ for each $n$, since $a_n \in A$
So $(\forall n) a_n \leq z \leq b_n$ so $z \in \bigcap_{n \in \mathbb{N}} I_n$.

## 13.2 proving Heine-Borel

**Theorem:** (in $\mathbb{R}$) let $[a,b]$ be a (bounded) closed interval. Then $[a,b]$ is compact.

**Proof:** Show every open cover has a finite subcover.
Let $\{G_\alpha\}_{\alpha\in J}$ be an open cover for $[a.b]$. Suppose for contradiction there's no finite $F \subseteq J$ such that $\bigcup\limits_{\alpha\in F} G_\alpha \supseteq [a,b]$. ($\leftarrow$ find something in interval not in union)
We work by recursion to define a closed bounded interval $I_n = [a_n, b_n]$ such that

- (1) It's contained in open cover $\bigcup\limits_{\alpha\in J} G_\alpha \supseteq I_n$
- (2) There's no finite $F \subseteq J$ such that $\bigcup\limits_{a\in F} G_\alpha \supseteq I_n$
- (3) Recursively define $I_{n+1} \supseteq I_n$ (*)
- (4) Each interval divides in half $b_n - a_n \leq \frac{b-a}{2^n}$

(*) To define the recursion, set $I_0 = [a,b]$. Suppse $I_n$ has been defined, Let $c = \frac{a_n+b_n}{2}$, $I_{left} = [a_n, c]$, $I_{right} = [c, b_n]$.

Have $\bigcup\limits_{\alpha\in J} G_\alpha \supseteq I_{left}$, $\bigcup\limits_{\alpha\in J} G_\alpha \supseteq I_{right}$.

**Claim:** At least one of $I_{left}$, $I_{right}$ cannot be covered by $\bigcup\limits_{a\in F} G_\alpha$ for any finite $F$.
(why? if $F_{left}$, $F_{right}$ are finite such that $\bigcup\limits_{a\in F_{left}} G_\alpha \supseteq I_{left}$ and $\bigcup\limits_{a\in F_{right}} G_\alpha \supseteq I_{right}$, then $\bigcup\limits_{a\in F_{left}\cup F_{right}} G_\alpha \supseteq I_n$, contradicting (2) because $F_{left} \cup F_{right}$ would be finite.)

**Proof of claim:** Take $I_{n+1}$ to be the interval which cannot be covered by $\bigcup\limits_{a\in F} G_\alpha$ for a finite $F$. By last property, $\bigcap\limits_{n\in\mathbb{N}} I_n \neq \emptyset$.
Fix $z \in \bigcap\limits_{n\in\mathbb{N}} I_n$.
Since $z \in I_0 = [a,b]$ and $\bigcup\limits_{\alpha\in J} G_\alpha \supseteq [a,b]$ must have some $\beta$ such that $z \in G_\beta$.
Since $G_\beta$ s open, have some $\epsilon > 0$ such that $N_\epsilon(z) \subseteq G_\beta$.
Fix $n$ large enough that $\frac{b-a}{2^n} < \frac{\epsilon}{2}$.
Have $a_n \leq z \leq b_n$. So $b_n - z$ is at most $\frac{\epsilon}{2}$ and $b_n - z \leq \frac{\epsilon}{2} < \epsilon$.
$z - a_n$ is at most $\frac{\epsilon}{2}$ and $z - a_n \leq \frac{\epsilon}{2} < \epsilon$.
So $a_n, b_n \in N_\epsilon(z)$ so $[a_n, b_n] \subseteq N_\epsilon(z)$.
So $I_n = [a_n, b_n] \subseteq G_\beta$.
Then, $F = \{\beta\}$ gives a finite subcover of $I_n$, contradicting (2). Therefore, there's no

open cover with no finite subcover, so every open cover has a finite subcover.

Rudin showed closed **cells** are compact (cutting the "desert" by $2^k$).

**Proposition:** Compact sets are closed and bounded.
show the compliment a compact set is open

**Proof:** Let $K$ be compact, let $x$ be a limit point of $K$.
Suppose for contradiction $x \notin K$ (suppose not closed).
For $\epsilon > 0$ let $G_\epsilon = \{u | d(x,u) > \epsilon\}$.
Then $\bigcup_{\epsilon>0} G_\epsilon$ is open and $\bigcup_{\epsilon>0} G_\epsilon \supseteq \{x\}^c \supseteq K$.
By compactness there is a finite $\{\epsilon_1, ..., \epsilon_n\}$ such that $G_{\epsilon_1}, ..., G_{\epsilon_n} \supseteq K$.
Let $r = min\{\epsilon_1, ..., \epsilon_n\}$. Then $K \subseteq G_r$.
So $N_r(x) \cap K = \emptyset$ (K is covered by complement).
Thus $x$ is not a limit point of $K$, contradiction.
$\Rightarrow$ Compact sets are also bounded, left for exercise.

## 13.3 closed subsets of compact sets are compact

**Proposition:** If $K$ is compact and $E \subseteq K$ is closed, then $E$ is compact.

**Proof:** Fix an open cover $\{G_\alpha\}_{\alpha \in J}$ of $E$.
Let $G* = E^c$, which is open.
Then $\{G_\alpha\}_{\alpha \in J} \cup \{G*\}$ is an open cover of $K$.
By compactness of $K$, must have a finite $\{\alpha_1, ..., \alpha_n\} \subseteq J$
such that $G_{a_1} \cup ... \cup G_{a_n} \cup G* \supseteq K$.
Then, $G_{a_1} \cup ... \cup G_{a_n} \cup G* \supseteq K \backslash G* \supseteq E$.

## 13.4 proving Bolzano-Weierstrass

**Corollary:** In $\mathbb{R}$ if $E$ is closed and bounded (both above and below), then $E$ is compact.

**Proof:** By boundedness, have some $n$ such that $E \subseteq [-n, n]$.
So $E$ is a closed subset of compact set $[-n, n]$, hence $E$ is compact.

**Proposition:** If $K$ is a compact set then every infinite subset in $K$ has a limit in $K$

**Proof:** Let $A \subseteq K$ be infinite and suppose $A$ has no limit in $K$.
Then for each $x \in K$, we have some $\epsilon > 0$ such that $A \cap (N_\epsilon(x) \backslash \{x\}) = \emptyset$.
i.e. $A \cap N_\epsilon(x) \subseteq \{x\}$,
Then, $\{N_{\epsilon_x}(x)\}_{x \in K}$ is an open cover of $K$.
By compactness, the open cover has a finite subcover, and thus has a finite $x_1, ..., x_n \in K$ such that $N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n) \supseteq K \supseteq A$.
$A$ is contained in the union of these neighborhoods, and since the only elements in the neighborhoods are $x_i$'s for $i = 1, ..., n$. But then since $N_{\epsilon_{x_i}}(x_i) \cap A \subseteq \{x_i\}$, so $A \subseteq \{x_1, ..., x_n\}$ so $A$ is finite, which is a contradiction.

**Theorem:** The following are equivalent on $\mathbb{R}$:
(1) $E$ is closed and bounded
(2) $E$ is compact
(3) every infinite subset of $E$ has a limit in $E$

**Proof:** already proved (1)⇒(2) and (2)⇒(3), remains to prove (3)⇒(1).
Fix $E$. Suppose (3), will prove (1).
⇒ Proving $E$ is bounded by proving it's bounded above and below.
If $E$ is not bounded above, then for every $n \in \mathbb{N}$ can find $x_n \in E$ such that $x_n > n$.
Then $\{x_n | n \in \mathbb{N}\}$ has no limit point, a contradiction.
Similarly, $E$ has to be bounded below. ⇒ Now to prove $E$ is closed.
Fix $x \in E'$ (pick some limit point of $E$) and prove $x \in E$.
For each $n > 0$ fix some $x_n \in E \cap (N_frac1n(x) \backslash \{x\})$, constructing a sequence.
Can check that $x$ is the only limit point of $\{x_n | n > 0\}$.
By assumption (3), the set $\{x_n | n > 0\}$ has a limit point in $E$, so $x \in E$.

**Corollary:** Every bounded infinite subset of $\mathbb{R}$ has a limit in $\mathbb{R}$.

Now go to the general metric spaces.

## 13.5 finite intersection prop. and Cantor's intersection theorem

**Theorem:** Let $K_\alpha$ $\alpha \in J$ be a collection of compact sets (in some metric space Y) such that for every finite $\{\alpha_1, ..., \alpha_n\} \in J$, $K_{\alpha_1} \cap ... \cap K_{\alpha_n} \neq \emptyset$. Then $\bigcap\limits_{\alpha \in J} K_\alpha \neq \emptyset$.

**Corollary:** If $K_0 \supseteq K_1 \supseteq K_2 \supseteq ...$ are compact and nonempty, then $\bigcap\limits_{n=0}^{\alpha} K_n \neq \emptyset$.

> **Proof:** Suppose for cells $\bigcap\limits_{\alpha \in J} K_\alpha = \emptyset$.
> Let $G_\alpha = K_\alpha^c$ (open).
> Then $\bigcup\limits_{\alpha \in J} G_\alpha = Y$. Pick $\beta \in J$.
> Then $\bigcup\limits_{\alpha \in J} G_\alpha \supseteq K_\beta$.
> Since $K_\beta$ is compact, we have finite $\{\alpha_1, ..., \alpha_n\} \in J$ such that $G_{\alpha_1} \cup ... \cup G_{\alpha_n} \supseteq K_\beta$.
> i.e. $K_{\alpha_1}^c \cup ... \cup K_{\alpha_n}^c \supseteq K_\beta$
> So $K_{\alpha_1} \cup ... \cup K_{\alpha_n} \cup K_\beta = \emptyset$, a contradiction.

For the proof above, basically take a bunch of nested compact sets. Assume by contradiction the intersection is empty. Then its complement, the union of the complement of each compact set, would be the whole metric space, which would make an open cover for each of the compact sets. Fix a random compact set. By compactness, there exists a finite subcover that covers the fixed compact set. The complement of each open cover will be a compact set (by its definition). So the intersection of each of those (complementary) compact sets and the originally fixed compact set will be empty, a contradiction to the theorem above the corollary.

# 14 Perfect Sets

*"(The following) is on the list of the most complicated things we've done so far"*
*- Neeman during Week 5*

**Theorem:** Let $P$ be a perfect subset of a compact set $K$. Then $P$ is uncountable.

**Proof:** Let $C$ be the set of functions $e : \mathbb{N} \to \{0, 1\}$.
Recall $C$ is uncountable, so we will inject $C$ into $P$ to show it's uncountable,
by constructing a one-to-one $h : C \to P$.

⇒ Find a $K - empty$ or $K_\emptyset$ compact subset of $P$.
⇒ Then divide $K_\emptyset$ into two compact subsets, $K_0$ $K_1$.
⇒ Divide the compact subset, $(K_{00}, K_{01})$, $(K_{10}, K_{11})$.
⇒ Divide again, get $K_{0000}, K_{0001}$ . . . building a decreasing compact set that we can follow. We know the intersection is not empty.

For every finite set $S$ of 0s and 1s, will assign a compact nonempty set $K_s \subseteq P$.
We will make sure that
(1) $K_{s\wedge 0}, K_{s\wedge 1} \subseteq K_s$ (split into subsets)
(2) $K_{s\wedge 0} \cap K_{s\wedge 1} = \emptyset$

Then, by the last theorem, for every $e : \mathbb{N} \to \{0, 1\}$
(the intersection of a decreasing sequence of nonempty compact sets is nonempty)

$\bigcap_{l\in\mathbb{N}} K_{e\restriction l} \neq \emptyset$ $(e(0), e(1), e(2)...$ restricted to l)

Let $h(e)$ be some element of this intersection.

*Note $h(e) \in K_\emptyset \subseteq P$ so $h$ maps $C$ into $P$.

**Claim:** $h : C \to P$ is one-to-one

**Proof:** Let $e, d \in C$, $e \neq d$
Let $k$ be least such that $e(k) \neq d(k)$.
Set $s = e \restriction k = d \restriction k$. Suppose WLOG $e(k) = 0, d(k) = 1$
So $e \restriction k + 1 = s \wedge 0$, $e \restriction k + 1 = s \wedge 1$.
Then $h(e) \in K_{s\wedge 0}$ and $h(d) \in K_{s\wedge 1}$, will then become disjoint sets.
By (2), $h(e) \neq h(d)$.

**Original proof, continued:** $\Rightarrow$ Remains to construct $K$s.

Will make sure to have

(3) $K_s = P \cap \overline{N_{r_s}(u_s)}$ for some $r_s > 0, u_s \in P$

Construct by recursion on length of $s$.

To start off, pick some $u_\emptyset \in P$, some $r_\emptyset$ and set $K_\emptyset = P \cap \overline{N_{r_\emptyset}(u_\emptyset)}$.

(Make sure to construct compact, nonempty sets.

This is a closed subset of a compact set, so it's compact)

Have $K_\emptyset \neq \emptyset$ since $u_\emptyset \in K_\emptyset$.

$K_\emptyset$ is an intersection of closed sets, hence closed.

Also $K_\emptyset \subseteq P$ and $P$ is a subset of a compact set. So $K_\emptyset$ is compact.

Suppose $K_s$ has been constructed, need to construct $K_{s\wedge 0}, K_{s\wedge 1}$.

Since $u_s \in P$ and $P$ has no isolated points (they also have isolated neighborhoods entirely contained in $B(u_s, r_s)$), can find distinct $u_{s\wedge 0}, u_{s\wedge 1} \in P \subseteq N_{r_s}(u_s)$.

Let $\frac{d(u_{s\wedge 0}, u_{s\wedge 1})}{3} = r$.

Set $K_{s\wedge 0} = P \cap \overline{N_r(u_{s\wedge 0})}$, $K_{s\wedge 1} = P \cap \overline{N_r(u_{s\wedge 1})}$

(pick $r$ small enough so they're in the original neighborhood)

Check closed subsets are compact, nonempty.

must make some neighborhood $B$ entirely contained in $B(u_s, r_s)$.

We can use this to prove that in $\mathbb{R}$, $[0, 1]$ is uncountable. Similarly any ball $[a, b]$ in $\mathbb{R}$ is uncountable.

# 15 Cantor Set - example of a perfect set.

**Definition of a cantor set**

$\Rightarrow$ Start off with a compact set, split intervals
$E_0 = [0, 1]$
$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
$\Rightarrow$ each $E_n$ is the union of $2^n$ interals of length $\frac{1}{3^n}$
$\Rightarrow$ construct $E_{n+1}$ by dropping the middle third of each interval of $E_n$

**Claim-ish:**

Each $E_n$ is closed, compact. The intersection is nonempty.
Set $P = \bigcap\limits_{n \in \mathbb{N}} E_n$. Then $P \neq \emptyset$.

**Proof-ish:**

Note $P$ is closed because it's the intersection of closed sets, in fact every endpoint of every interval of $E_n$ belongs to $P$.
Every $x \in P$, every $\epsilon > 0$, every $n$ large enough that $\frac{1}{3^n} < \epsilon$, $x$ has to be in some interval of $E_n$. So the end points of this interval are two elements of $P$ within distance of $< \epsilon$ of $x$. At least one of them is not $x$, so $x$ is not isolated in $P$. Thus $P$ is closed with no isolated points, and thus is **perfect**.

**Note:** For any two points of distance $> \frac{1}{3^n}$, no interval of $E_n$ contains both. So $\exists$ a point between them outside $E_n$, meaning $\exists$ a point between them outside $P$. In particular, $P$ contains no intervals (but has plenty of points).

# 16 Sequences

Write $\{x_n\}$ or $\{x_n\}_{n=1,2,3...}$ for the sequence $x_1, x_2, x_3, ...$ (strictly speaking this is the function $n \mapsto x_n$, set might be finite even though the sequence repeats infinitely)

**Definition:** A sequence $\{x_n\}$ **converges** to $x*$ in some metric space $(Y, d)$ iff $(\forall \epsilon > 0)$ for all large enough $n$, $d(x_n, x*) < \epsilon$. (i.e. $\exists N\ \forall n \geq N\ l(x_n, x*) < \epsilon$).
$\Rightarrow$ Write $\lim_{n\to\infty} x_n = x*$ if this is the case, or $x_n \to x*$ as $n \to \infty$
$\Rightarrow$ If $\{x_n\}$ does not converge, we say it **diverges**

**Notes:**

- if $x_n \to x*$ then $x*$ is a limit point of $\{x_n | n \in \mathbb{N}\}$, except possibly if $x_n = x*$ for all large enough $n$.
- No $z \neq x*$ can be a limit point (if $d(z, x*) > 0$, let $\epsilon = \frac{d(z,x*)}{2}$. For large enough $n$, $d(x_n, x*) < \epsilon$, so $d(x_n, z) > \epsilon$, using the triangle inequality).
- If $\{x_n\}$ converges, then it has a unique limit.
- If $\{x_n\}$ converges, then it is **bounded**,
  i.e. $\exists R > 0, \exists u$ such that $(\forall n) x_n \in N_R(u)$.

  **Proof:** Let $x* = \lim_{n\to\infty} x_n$ using $\epsilon = 1$, get $N$ such that $\forall n \geq N$, $d(x_n, x*) < \epsilon$. Let $R = max(1, d(x_1, x*), ..., d(x_N, x*)) + \frac{1}{17}$ $\leftarrow$want something greater than max. Then, $(\forall n) x_n \in N_R(x*)$
- if $z$ is a limit point of $E$, then there is a sequence $\{x_n\}$ of points $x_n \in E$ such that $\lim x_n = z$.

## 16.1 properties of limits

**Theorem:** (In $\mathbb{R}$) If $\lim x_n = x*$ $\lim y_n = y*$
then $\{x_n + y_n\}$ converges and $\lim(x_n + y_n) = x * + y*$

**Proof:** Fix $\epsilon > 0$.
Need $N$ such that $(\forall n \geq N)$ $d(x_n + y_n, x * + y*) < \epsilon$.
Since $x_n \to x*$, have $N_1$ such that $n \geq N_1 \to d(x_n, x*) < \frac{\epsilon}{2}$
Since $y_n \to y*$, have $N_2$ such that $n \geq N_2 \to d(y_n, y*) < \frac{\epsilon}{2}$
Let $N = max(N_1, N_2)$.
For $n \geq N$, have

$$|x_n + y_n - (x * + y*)| = |(x_n - x*) + (y_n - y*)| \leq |x_n - x * | + |y_n + y * | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Similarly,

- $\lim c * x_n = c * \lim x_n$
- $\lim x_n * y_n = \lim x_n * \lim y_n$
- $\lim \frac{1}{x_n} = \frac{1}{\lim x_n}$, assuming $x_n$ and limit shouldn't be 0

## 16.2 subsequences and compactness

**Definition:** Let $\{x_n\}$ be a sequence. Given $n_1 < n_2 < n_3 < n_4 ... n_k ...$, we call $\{x_{n_k}\}_{k=1,2,...}$ a **subsequence** of $\{x_n\}$.

**Proposition:** If $\lim x_n = x*$, then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x*$. (fix epsilon, make sure $x_{n_k}$ within epsilon of $x*$).

**Theorem:** if $K$ is compact (in a metric space (Y, d)) and $\{x_n\}$ is a sequence of points in $K$, then $\{x_n\}$ has a subsequence which converges to some $x* \in K$ (compact$\Rightarrow$ sequentially compact)

**Proof:** Look at $E = \{x_n | n \in \mathbb{N}\}$.
Case 1: if $E$ is finite, some points get repeated and there must be some point $x* \in E$ which is repeated infinitely many times in $\{x_n\}$. Let $n_1 < n_2 < n_3 < ...$ be such that $x_{n_k} = x*$, then $\{x_{n_k}\}$ converges to $x*$.
Case 2: If $E$ is infinite, by an earlier theorem (any infinite set in a compact set has a limit point), have $x* \in k$ as a limit point of $E$. Now, define $n_k$ by recursion: (defining subsequence that converges to limit point)

- $n_1 = 1$
- Take $n_{k+1} > n_k$ such that $d(x_{n_{k+1}}, x*) < \frac{1}{k}$ (possible, since $E \cap N_{\frac{1}{k}}(x^k)$ is infinite)

## 16.3 Cauchy sequences

**Definition:**
$\{a_n\}$ is a **Cauchy sequence** if $(\forall \epsilon > n)\ \exists N, \forall n, m \geq N$ such that $d(a_n, a_m) < \epsilon$.

Note: If $\{a_n\}$ is Cauchy, then it's bounded.
If $\{a_n\}$ converges, then it's Cauchy.

**Proposition:** If $\{a_n\}$ is Cauchy and $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ which converges to $u$, then $\{a_n\}$ converges to $u$.

**Proof:** Fix $\epsilon > 0$.
Then find $N$ such that $(\forall n \geq N)\ d(a_n, a_m) < \epsilon$.
Since $\{a_n\}$ is Cauchy $\exists N$ such that $\forall n, m \geq N\ d(a_n, a_m) < \frac{\epsilon}{2}$.
Since $a_{n_k} \to u$, have arbitrarily large $k$ such that $d(a_{n_k}, u) < \frac{\epsilon}{2}$.
In particular, can find a $k$ such that $n_k > N$.
Then $\forall n \geq N\ d(a_n, u) \leq d(a_n, a_{n_k}) + d(a_{n_k}, u) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

**Corollary:** If $K$ is compact and $\{a_n\}$ is Cauchy with $a_n \in K$, then $\{a_n\}$ converges to a point in $K$.

**Proof:** (proved differently in poppa Rudin)
Define $diam(E) = sup\{d(p, q) | p, q \in E\}$
If $a_n$ is Cauchy then setting $\epsilon_n = \{a_n, a_{n+1}, a_{n+2}, ...\}$, the diameter goes to 0 (immediate from definition).
Prove $diam(\overline{E}) = diam(E)$, so $diam(\overline{E_n}) \to 0$. $\overline{E_n}$ is closed, contained in compact set so it's compact.
The intersection of the set of diameters is nonempty,
since it goes to 0 the intersection is one point.
If $\{a_n\} \subseteq K$ is compact, $\overline{E_n}$ is compact, $S_n \cap \overline{E_n} \neq \emptyset$.
Using $diam(\overline{E_n}) \to 0$ show $\cap \overline{E_n}$ is a single point. Show $\{a_n\}$ converges to this point.

# 17 Completeness

**Definition:** A metric space is **complete** if every Cauchy sequence has a limit point in the space.

**Example:** $\mathbb{R}$ (*"our favorite space"*) is complete

**Proof:** If $\{a_n\} \subseteq \mathbb{R}$ is Cauchy, then
(a) It's bounded, so $\{a_n\} \subseteq [a, b]$ for some bounded interval $[a, b]$
(b) $\{a_n\}$ has a convergent subsequence
(c) $\{a_n\}$ converges

**Example:** $\mathbb{Q}$ (*"our second favorite space"*) is not complete
$\Rightarrow$ finding limits $a_1, a_2, a_3, ...$, use sups and infs to find subsequence. The sups keep decreasing, and take the inf of all the sups.

## 17.1 metric space completion

See Neeman's notes

# 18 Limsup and Liminf

**Definition:** We say $a_n \mapsto +\infty$
if $\forall x < \mathbb{R}, \exists N_1$ such that $\forall n \geq N, a_n > x_1$ (similar for $a_n \mapsto -\infty$)
if $\epsilon$ is not bounded above $sup(\epsilon) = +\infty$
(similar to $a_n \mapsto -\infty$)

**Definition:** $S_n$ is monotone decreasing if $S_n \leq S_{n+1}$ similarly for increasing.

**Proposition:** If $\{S_n\}$ is monotone increasing, $\lim S_n = \sup S_n$ (in generalized sense, may be $+\infty$), similar for monotone decreasing.

- See Neeman's notes for the rest - defined limsups and liminfs and proved the next two theorems

**Theorem:** $\limsup a_n$ is a subsequent limit point of $\{a_n\}$ and the largest such

**Theorem:** $\liminf a_n$ is a subsequent limit point of $\{a_n\}$ and the smallest one

**Proposition:** $\liminf a_n \leq \limsup a_n$

**Proof:** (Directly) *"Clearly,"* $\inf\{a_n, a_{n+1}, ...\} \leq \sup\{a_n, a_{n+1}, ...\}$.
In our previous terminology $t_n \leq s_n$.
(Previously, we defined for sequence $\{a_n\}$,
$t_n = \inf\{a_n, a_{n+1}, a_{n+2}\}$, and $\liminf\limits_{n\to\infty} a_n := \sup t_n$
$s_n = \sup\{a_n, a_{n+1}, a_{n+2}\}$, and $\limsup\limits_{n\to\infty} a_n := \sup t_n$)
For $n \leq m$, have $t_n \leq t_m$ and $s_n \leq s_n$.
So $t_n \leq t_m \leq s_m \leq s_n$. In particular, for $n < m, t_m \leq s_n$.
Letting $m$ vary, get $\sup_n t_m \leq s_n$. Now letting $n$ vary, get $\sup_m t_m \leq \inf_n s_n$.

**Proposition:** If $a_n \leq b_n, (\forall n)$, then $\liminf a_n \leq \liminf b_n$ and $\limsup a_n \leq \limsup b_n$.

# 19 Series

Working in $\mathbb{R}$, can also be done in $\mathbb{C}, \mathbb{R}^k$.

**Definition:** Let $\{a_n\}$ be a sequence. We say $\sum_{n=1}^{\infty} a_n$ converges to $s$ and write $\sum_{n=1}^{\infty} a_n = s$, iff the sequence (of partial sums) $S_n = \sum_{i=1}^{n} a_i$ converges to $s$. Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

**Example:** The series $\sum_{n=0}^{\infty} x^n$ converges $\frac{1}{1-x}$ if $0 \leq x < 1$ and diverges if $x \geq 1$.

**Proof:** The partial sum is

$$S_n = \sum_{i=1}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

if $0 \leq x < 1$ then $\frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x}$
if $x > 1$ then $\frac{1-x^{n+1}}{1-x}$ diverges.
if $x = 1$ then $\sum_{n=1}^{\infty} x^n \Rightarrow \sum_{n=1}^{\infty} 1$ diverges.

## 19.1 cauchy criterion

**Proposition:** $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \epsilon > 0, \exists N, \forall m > n \geq N \mid \sum_{i=n+1}^{m} a_n| < \epsilon$

**Proof:** ($S_n$'s converge iff $S_n$ is a Cauchy sequence)
*"Trivial"*, $\sum_{i=n+1}^{m} a_i$ is exactly $S_m - S_n$, so the statement of the prop is equivalent to saying $S_n$ converges iff $\forall \epsilon > 0 \exists N such that \forall m > n \geq N |S_m - S_n| \leq \epsilon$, i.e. iff $S_n$ is Cauchy. This statement holds (in $\mathbb{R}$) since in $\mathbb{R}$, a sequence converges iff it is Cauchy.

**Corollary:** If $\sum a_n$ converge, then $a_n \to 0$
converse is false: e.g. $\sum \frac{1}{n}$ but $\frac{1}{n} \to 0$

## 19.2 sequence of partial sums has to be bounded

**Proposition:** If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges iff the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$ is bounded

Note: If $\sum a_n$ converge then $\forall c \in \mathbb{R}, \sum c * a_n converges$

## 19.3 comparison tests

- If $|a_n| \leq b_n$ and $\sum b_n$ converge, then $\sum a_n$ converge, i.e. $|\sum_{i=n+1}^{m} a_i| \leq \sum_{j<n+1}^{m} |a_i| \leq \sum_{j=n+1}^{m} b_i$
- If $0 \leq d_n \leq a_n$ and $\sum d_n$ diverges, then $\sum a_n$ diverges. contrapositive of pervious.

## 19.4 dyadic criterion

Suppose $a_1 \geq a_2 \geq ... \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converge iff $\sum_{k=0}^{\infty} 2^k * a_{2^k}$ converges.

**Proof:** The terms above are $\geq 0$, so convergence for each series is equivalent to the partial sums being bounded. Let $S_n = \sum_{i=1}^{n} a_i$ and $t_k = \sum_{i=0}^{k} 2^i a_{2^i}$.
Show $\{S_n\}$ is bounded iff $\{t_n\}$ is bounded. Rewrite $t_k = a_1 + 2a_2 + 4a_4 + ... + 2^k a_{2^k}$ as $a_1 + a_2 + a_2 + ...a_{2^k} + ... + a_{2^k}$ with $2^{k+1} - 1$ terms.
By the monotonicity of the sequence, this is $\geq a_1 + a_2 + a_3 + ... + a_{2^{k+1}-1} - 1 = S_{2^{k+1}-1}$
$\Rightarrow t_k > S_{2^{k+1}-1} \geq S_n \forall n \leq 2^{k+1} - 1$. So $t_k$ bounded $\rightarrow S_n$ bounded.
For $n \geq 2^k$,

$S_n \geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ... + a_{2^{k-1}+1} + ... + a_{2^k}$
$\geq a_1 + a_2 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + ... + a_{2^k}$
$= a_1 + a_2 + 2a_4 + 4a_8 + ... + 2^{k-1} a_{2^k}$
$= \frac{1}{2} a_1 + \frac{1}{2} t_k \geq \frac{1}{2} t_k$

**Corollary:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

**Proof:** Case 1: $p > 0$ so $\frac{1}{n^p}$ decreases. By dyadic criterion, enough to show

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}}$$

converges iff $p > 1$
$\Rightarrow \sum_{k=0}^{\infty} (2^{(1-p)k})$
This converges if $2^{1-p} < 1$ iff $1 - p < 0$ iff $p > 1$

Case 2: $p \leq 0$. Then $\frac{1}{n^p} \nrightarrow 0$ as $n \to \infty$ so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

## 19.5 root test

Let $\sum a_n$ be a series. Set $d = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$
(a) If $\alpha < 1$ $\sum a_n$ converges
(b) If $\alpha > 1$ $\sum a_n$ diverges
(c) If $\alpha = 1$ either is possible, e.g. $\sum \frac{1}{n}$ vs $\sum \frac{1}{n^2}$

**Proof:**
(a) suppose $\inf_{n_0} \sup_{n \geq n_0} |\alpha_n|^{\frac{1}{n}} < 1$
Then there is some $n_0$ such that $\sup_{n \geq n_0} |\alpha_n|^{\frac{1}{n}} < 1$
Fix some $\beta$ between the *sup* and 1
Then $\beta < 1, \forall n \geq n_0, |\alpha_n|^{\frac{1}{n}} < \beta$ so $|\alpha_n| < \beta^n$
Since $\beta < 1$, $\sum_{n=1}^{\infty} \beta^n$ converges. By comparison test, $\sum_{n=n_0}^{\infty} \alpha_n$ converges.

(b) Suppose $\inf_{n_0} \sup_{n \geq n_0} |\alpha_n|^{\frac{1}{n}} > 1$
So for each $n_0$, $\sup_{n \geq n_0} |\alpha_n|^{\frac{1}{n}} > 1$
So $\forall n_0, \exists n \geq n_0$ such that $|\alpha_n|^{\frac{1}{n}} > 1 \Rightarrow |a_n| > 1$
So $|a_n| \nrightarrow 0$ so $\sum a_n$ diverges.

(c) ...left as exercise?

## 19.6 ratio test

Let $\sum a_n$ be a series

(a) If $\limsup |\frac{a_{n+1}}{a_n}| < 1$ then $\sum a_n$ converges

(b) If $|\frac{a_{n+1}}{a_n}| \geq 1$ for a tail end of $n$ then $\sum a_n$ diverges.

**Proof:**

(b) Fix $n_0$ such that $\forall n \geq n_0, |\frac{a_{n+1}}{a_n}| \geq 1$.

By induction for $n \geq n_0, |a_n| \geq |a_{n_0}|$ so $a_n \nrightarrow 0$ so $\sum a_n$ diverges.

(a) suppose $\limsup |\frac{a_{n+1}}{a_n}| < 1$ i.e. $\inf_{n_0} \sup_{n \geq n_0} |\frac{a_{n+1}}{a_n}| < 1$

So $\exists n_0$ such that $\sup_{n \geq n_0} |\frac{a_{n+1}}{a_n}| < 1$.

Let $\beta < 1$ be this sup, so $\forall n \geq n_0$, $|\frac{a_{n+1}}{a_n}| \leq \beta$ i.e. $|a_{n+1}| \leq \beta |a_n|$.

By induction, for $n \geq n_0$ $|a_n| \leq \beta^{n-n_0}|a_{n_0}|$, the series $\sum\limits_{n=n_0}^{\infty} \beta^{n-n_0}|a_{n_0}|$ converges.

By the comparison test, then $\sum\limits_{n=1} \infty a_n$ converges.

**Theorem:** for any sequence $a_n$ of nonnegative numbers, $\limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$

Can get similar result for liminf, reversing inequality (and root test for convergence is more sensitive than ratio test)

**Proof:** Let $\alpha = \limsup \frac{a_{n+1}}{a_n}$.

May assume $\alpha < +\infty$ (for $\alpha = +\infty$, theorem holds trivially)

Fix $\beta > \alpha$.

Since $\inf_{n_0} \sup_{n \geq n_0} \frac{\alpha_{n+1}}{\alpha_n} < \beta$, then have $n_0$ such that $\sup_{n \geq n_0} \frac{\alpha_{n+1}}{\alpha_n} < \beta$,

i.e. $\forall n \geq n_0$ $\alpha_{n+1} < \beta\alpha_n$

So for all $n \geq n_0$ $\alpha_n \leq \beta^{n-n_0}\alpha_{n_0} = \beta^n \frac{\alpha_{n_0}}{\beta^{n_0}}$

So $(\alpha_n)^{\frac{1}{n}} \leq \beta(\frac{\alpha_{n_0}}{\beta^{n_0}})^{\frac{1}{n}}$

So $\lim\limits_{n\to\infty} \sup(\alpha_n)^{\frac{1}{n}} \leq \lim\limits_{n\to\infty} \sup \beta(\frac{\alpha_{n_0}}{\beta^{n_0}})^{\frac{1}{n}} = \lim \beta(\frac{\alpha_{n_0}}{\beta^{n_0}})^{\frac{1}{n}} = \beta$

So $\limsup(\alpha_n)^{\frac{1}{n}} \leq \alpha$

## 19.7 alternating series:

Suppose $|c_1| \geq |c_2| \geq |c_3| \geq ...$
$|c_n| \to 0$ as $n \to 0$, $c_n \leq 0$, $c_{n+1} \geq 0$, then $\sum c_n$ converges.

**Example:** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

**Sketch of proof:** $S_n = \sum_{i=1}^{n} c_n$. Show $S_2 \leq S_4 \leq S_6 \leq ...$
$S_1 \geq S_3 \geq S_5 \geq ...$
$S_{2n} \leq S_{2n+1}$
so $S_2 \leq S_4 \leq S_6 \leq ... \leq S_5 \leq S_3 \leq S_1$
Show $S_{2m+2} - S_{2m+1} \to 0$
$\sup S_{2m} = \inf S_{2m+1} \to 0$ so $S_n$ converges to $\sup S_{2m} = \inf S_{2m+1}$
So $S_n$ converges to $\sup S_{2m} = \inf S_{2m+1}$

## 19.8 summation by parts

Let $\{a_n\}\{b_n\}$be two sequences. Aim to find a formula for $\sum_{n=p}^{q} a_n b_n$.

Let $A_n = \sum_{k=0}^{n} a_k (A_{-1} = 0)$. Note: $a_n = A_n - A_{n-1}$.

**Lemma:** $\sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} a_n b_n = A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n)$

**Proof:** $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$

$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} = \sum_{n=p}^{q-1} A_n b_n + A_q b_q - \sum_{n=p-1}^{q-1} A_n b_{n+1}$

$= A_q b_q + \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p = A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q-1} (A_n b_n - A_n b_{n+1})$

$= A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n)$

**Theorem:** Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $A_n = \sum_{k=0}^{\infty} a_k$.
Suppose (1) $\{A_n\}$ is bounded, (2) $b_0 \geq b_1 \geq b_2 \geq ...$, (3) $\lim b_n \to 0$,
then $\sum a_n b_n$ converges.

**Proof:** Fix $M$ such that $(\forall n)|A_n| = M$. Fix $\epsilon > 0$.
Have to find $N$ such that $q \geq p \geq N \to |\sum_{n=p}^{q} a_n b_n| < \epsilon$.
Let $N$ be large enough that $b_N < \frac{\epsilon}{2M}$.
Then $|\sum_{n=p}^{q} a_n b_n| = |A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q} A_n(b_n - b_{n+1})|$
$\leq |A_q b_q| + |A_{p-1} b_p| + \sum_{n=p}^{q-1} |A_n|(b_n - b_{n+1})$ (This inequality is because $b_n - b_{n+1} \geq 0$).
$\leq M(b_q + b_p + \sum_{n=p}^{q-1}(b_n - b_{n+1})) = M(b_q + b_p + b_p - b_q) = M2b_p \leq M2b_N < \epsilon$

**Example:** previous result on alternating series. $C_n$ of alternating signs.
$(C_{2m+1} \geq 0, C_{2m} \leq 0)$ $k_n$ decreases as $c_n \to 0$.
Then $\sum c_n$ converges. Can get this from last theorem by taking $b = |c_n|$ $a_n = (-1)^{n+1}$.

## 19.9 absolute convergence

**Definition:** $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

**Note:** if $\sum |a_n|$ converges then $\sum a_n$ converges by the comparison test. There are series that converge, but not absolutely, e.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

**Definition:** Let $\sum_{n=0}^{\infty} a_n$ be a series.

An arrangement of $\sum a_n$ is a series $\sum_{k=0}^{\infty} a_{\sigma(k)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ (bijection).

## 19.10 rearrangements

**Theorem:** Suppose $\sum_{n=0}^{\infty} a_n$ converges, but not absolutely. Then for every $u \in \mathbb{R} \cup \{-\infty, +\infty\}$ there is a rearrangement $\sum_{k=0}^{\infty} a_{\sigma(k)}$ of $\sum a_n$ such that $\sum_{k=0}^{\infty} a_{\sigma(k)} = u$

**Part of proof:** Will prove the case $u = +\infty$. The general proof has similar ideas.
Let $A_n = \{n | a_n \geq 0\}, B_n = \{n | a_n < 0\}$

**Claim:** At least one of $\sum_{n=0, n\in A}^{\infty} a_n, \sum_{n=0, n\in B}^{\infty} -a_n$ diverges to $+\infty$.

**Proof:** (Will first show at least one of them diverges, then show both diverge)
Suppose not. Then both series converge. So $\forall \epsilon > 0$ have $N$ such that $\forall n_1, n_2 \geq N$
$|\sum_{i=n_1, i\in A}^{n_2} a_i| < \frac{\epsilon}{2}$ and $|\sum_{i=n_1, i\in B}^{n_2} -a_i| < \frac{\epsilon}{2}$
Since the terms $\geq 0$, have $\sum_{i=n_1, i\in A}^{n_2} a_i < \frac{\epsilon}{2}$ and $\sum_{i=n_1, i\in B}^{n_2} -a_i < \frac{\epsilon}{2}$
So $\sum_{i=n_1}^{n_2} |a_i| = \sum_{i=n_1, i\in A}^{n_2} |a_i| + \sum_{i=n_1, i\in B}^{n_2} |a_i| = \sum_{i=n_1, i\in A}^{n_2} a_i + \sum_{i=n_1, i\in A}^{n_2} -a_i < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
But this shows $\sum |a_n|$ converges, a contradiction.

**Claim:** Both $\sum_{n=0, n\in A}^{\infty} a_n$ and $\sum_{n=0, n\in B}^{\infty} -a_n$ diverges to $+\infty$

**Proof:** Suppose for contradiction one of the series is bounded.
WLOG say $\sum_{n=0, n\in A}^{\infty} a_n$ is bounded by $\mathbb{R}$.
By theorem assumption $\sum a_n$ converges say to $z \in \mathbb{R}$.
Fix $N$ large enough so that $n \geq N$ $-|z| - \frac{1}{2} < \sum_{k=0}^{n} a_k < |z| + \frac{1}{2}$.
Can separate between $A, B$ $-|z| - \frac{1}{2} < \sum_{k=0, k\in A}^{n} a_k + \sum_{k=0, k\in B}^{n} a_k < |z| + \frac{1}{2}$.
Since $\sum_{k=0, k\in A}^{n} a_k < R$, must then have $-|z| - \frac{1}{2} - R < \sum_{k=0, k\in B}^{n} a_k < |z| + \frac{1}{2} + R$.
This shows $\sum_{n=0, n\in B}^{\infty} a_n$ is bounded, so both $\sum_{n=0, n\in A}^{\infty} a_n$ $\sum_{n=0, n\in B}^{\infty} a_n$ are bounded, contradicting the previous claim.

We can now define $\sigma$ so that $\sum_{k=0}^{\infty} a_{\sigma(k)} = +\infty$.

Will define by recursion on $l \in \mathbb{N}$:

- Define $n_l$
- Define $\sigma(0)\sigma(1)...\sigma(n_{l-1})$
- Set $n_0 = 0$

Suppose $n_l$ $\sigma(0)\sigma(1)...\sigma(n_l)$ have been defined. Let $j$ be the least in $B-\{\sigma(0)\sigma(1)...\sigma(n-1)\}$
(first negative term not yet covered)
Let $p$ be least in $A - \{\sigma(0)\sigma(1)...\sigma(n-1)\}$
Let $p* > p$ be large enough that $\sum\limits_{n=p,n\in A}^{p*} a_n > |a_j| + 1$
(possible since $\sum\limits_{n=0,n\in A}^{\infty} a_n$ diverges to $+\infty$) Let $\sigma(n_k)\sigma(n_{k+1})...\sigma(n_{k+r})$.
List all elements of $A$ from $p$ to $p*$.
Let $\sigma(n_{l+r+1}) = j$ let $n_{l+1} = n_{l+r+2}$.
Then $\sum\limits_{i=0}^{n_{l+1-1}} a_{\sigma(i)} \geq \sum\limits_{i=0}^{n_{l-1}} a_{\sigma(i)+1}$ for each $h$ between $n_l$ and $n_{l+1-1}$.
$\sum\limits_{i=0}^{n} a_{\sigma(i)} \geq \sum\limits_{i=0}^{n_{l-1}} a_{\sigma(i)}$ from this it follows $\sum\limits_{i=0}^{\infty} a_{\sigma(i)}$ diverges to $+\infty$.

**Theorem:** Suppose $\sum a_n$ converges absolutely. Let $u = \sum\limits_{n=1}^{\infty} a_n$. Then every rearrangement of $\sum a_n$ converges to $u$.

**Proof:** Fix $\epsilon > 0$. Need $N$ such that $\forall n > N$. Fix rearrangement $\sum a_{\sigma(n)}$ $|\sum\limits_{k=1}^{\infty} a_{\sigma(k)} - u| < \sigma$.
Fix $M_1$ such that $\forall n \geq M_1$ $|\sum\limits_{k=1}^{n} a_k - u| < \frac{\epsilon}{2}$.
Fix $M_2$ such that $\forall n_1, n_2 \geq M_2$ $\sum\limits_{k=n_1}^{n_2} |a_k| < \frac{\epsilon}{2}$
Let $N$ be large enough that $\{\sigma(0)\sigma(1)...\sigma(N)\} \supseteq \{1, 2, 3...max\{M_1, M_2\}\}$. Let $M = max\{M_1, M_2\}$
Now for $n \geq N : |u - \sum\limits_{k=1}^{n} a_{\sigma(k)}| = |u - \sum\limits_{k=1}^{M} a_k + \sum\limits_{k=1}^{M} a_k - \sum\limits_{k=1}^{n} a_{\sigma(k)}|$
$\leq |u - \sum\limits_{k=1}^{M} a_k| + |\sum\limits_{k=1}^{n} a_{\sigma(k)} - \sum\limits_{k=1}^{M} a_k|$
$\leq \frac{\epsilon}{2} + |\sum\limits_{k=1,\sigma(k)>M}^{n} a_{\sigma(k)}| \leq \frac{\epsilon}{2} + \sum\limits_{k=1,\sigma(k)>M}^{n} |a_{\sigma(k)}| \leq \frac{\epsilon}{2} + \sum\limits_{i=M+1}^{max(\sigma(k)|k\leq n)} |a_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

# 20 Number e

**Definition:** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, converges by ratio test.

**Theorem:** $\lim n \to \infty (1 + \frac{1}{n})^n$ exists and equals $e$.

**Proof:** Let $S_n = \sum k = 0^n \frac{1}{k!}, t_n = (1 + \frac{1}{n})^n$. Show $\liminf t_n \geq S_n, \forall n$.
Will show (1) $t_n \leq S_n$ (2) $\liminf\limits_{k\to\infty} t_k \geq S_n$ (for each $n$).
From (1) get $\limsup\limits_{n\to\infty} t_n \leq \limsup\limits_{n\to\infty} S_n = \lim S_n = e$
From (2) get $\liminf\limits_{k\to\infty} t_k \geq \sup S_n = \lim S_n = e$. $\liminf t_n = \limsup t_n = e$, so $t_n \to e$.
To prove (1), $t_n = (1 + \frac{1}{n})^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(1)^{n-j}\frac{1}{n^j}$ (binomial theorem)
$\sum_{j=0}^{n} \frac{1}{j!}\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}...\frac{n-j+1}{n} \geq \sum_{j=0}^{n} \frac{1}{j!}\frac{k}{k}\frac{k-1}{k}...\frac{k-j+1}{k}$ (hold $j$ fixed and $k \to \infty$ then it goes to 1)
$\frac{k}{k}\frac{k-1}{k}...\frac{k-j+1}{k} \to 1$.
Hence for each fixed $n$ $\sum_{j=0} n\frac{1}{j!}\frac{k}{k}\frac{k-1}{k}...\frac{k-j+1}{k} \to \sum_{j=0}^{n}\frac{1}{j!}$ (fine as long as $n$ is finite).
So $\liminf\limits_{k\to\infty} \sum_{j=0}^{n} \frac{1}{j!}\frac{k}{k}\frac{k-1}{k}...\frac{k-j+1}{k} = S_n$.
So $\liminf\limits_{k\to\infty} t_k \geq S_n$.

## 20.1 some inequalities

Will use properties of function $x \mapsto e^x$.

(1) for each $y \in (0, +\infty)$ there is $x$ such that $e^x = y$ (will prove later). Since $x \mapsto e^x$ is monotone increasing, then $x$ is unique. It is denoted by $\log y$

(2) $x \mapsto e^x$ is convex, this means $\forall x_1, x_2$, the line between $(x_1, e^{x_1})$ $(x_2, e^{x_2})$ is above the function $e^x$. ($x \in [x_1, x_2]$) with equality only at $x_1, x_2$. Any point in the middle is some kind of average $tx_1 + (1-t)x_2$.

# 21 Young, Holder, Minkowski's Inequalities, and $l^p$ properties

- see Neeman's notes

# 22 Continuous Functions

**Definition:** Let $X, Y$ be metric spaces, $E \subseteq X$ and $f : E \to Y$.
$f$ is **continuous** at a point $p \in E$ if $\forall \delta > -0$ such that $d(x,p) < \delta \to d(f(x), f(p)) < \epsilon$

- $f$ is continuous on $E$ if $f$ is continuous on each $p \in E$
  i.e. $\forall p \in E\ \forall \epsilon > 0\ \exists \delta > 0\ d(x,p) < \delta \to d(f(x), f(p)) < \epsilon$
- $f$ is uniformly continuous on $E$ if $\delta$ can be found independently of $p$
  i.e. $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall p \in E\ d(x,p) < \delta \to d(f(x), f(p)) < \epsilon$ (choice of $\delta$ does not depend on $p$)

## 22.1 behavior on dense set determines general behavior

**only applies to continuous functions***

**Claim:** Let $f : E \to \mathbb{R}$ be continuous on $E$. Suppose $D$ is dense in $E$ (i.e. $\overline{D} \supseteq E$) and $(\forall x \in D)\ f(x) \leq 0$. Then $\forall p < E\ f(p) \leq 0$.

**Proof:** Fix $p \in E$. Suppose by contradiction $f(p) > 0$.
Let $\epsilon$ be small enough that $f(p) - \epsilon > 0$.
Then $|y - f(p)| < \epsilon \to y > 0$.
By continuity have $\delta > 0$ such that $d(x,p) < \delta \to |f(x), f(p)| < \epsilon$.
Since $f(x) > 0$, then $\forall p \in E\ f(p) \leq 0$.
Rewriting, we have $x \in B_\delta(p) \to f(x) > 0$.
By density on $D$, can find $x \in B_\delta(p) \cap D$. But then $f(x) \leq 0$, a contradiction.

## 22.2 open and closed sets on continuous functions

**Theorem:** $f : X \to Y$ is continuous on $X \iff \forall$ open $V \subseteq Y$ $f^{-1}(V)$ is open in $X$.

**Note:** Since $V \subseteq Y$ is open $\iff Y \backslash V$ is closed
and $f^{-1}(V)$ is open $\iff f^{-1}(V \backslash Y) = X \backslash f^{-1}(V)$ is closed.

**Theorem implies:** $f$ is continuous on $X$ iff $\forall$ closed $C \subseteq Y$ $f^{-1}(C)$ is closed on $X$

**Proof of theorem:**

($\Rightarrow$) prove pre-image is open.
Fix $V \subseteq Y$ open, **WTS** $f^{-1}(V)$ is open in $X$.
Fix $p \in f^{-1}(V)$. Have to find $r > 0$ such that $B_r(p) \subseteq f^{-1}(V)$.
So $f(p) \in V$. Since $V$ is open $\exists \epsilon > 0$ such that $B_\epsilon(f(p)) \subseteq V$
i.e. $d(y, f(p)) < \epsilon \to y \in V$.
By continuity, have $\delta > 0$ such that $d(x,p) < \delta \to d(f(x), f(p)) < \epsilon$
i.e. $x \in B_\delta(p) \to f(x) \in B_\epsilon(f(p))$.
by above, $f(x) \in V$ and $x \in f^{-1}(V)$.

($\Leftarrow$) prove continuity given open image and pre-image.
Fix $p \in X$. Have to show $f$ is continuous at $p$.
Fix $\epsilon > 0$. $V = B_\epsilon(f(p))$ is open in $Y$. Hence $f^{-1}(V)$ is open in $X$.
Have $p \in f^{-1}(V)$ since $f(p) \in B_r(f(p)) \in V$.
Since $f^{-1}(V)$ is open, there is $\delta > 0$ such that $B_\delta(p) \subseteq f^{-1}(V)$.
Then $x \in B_\delta(p) \to f(x) \in V = B_\epsilon(f(p))$, i.e. $d(x,p) < \delta \to d(f(x), f(p)) < \epsilon$.

## 22.3 Intermediate Value Theorem

**Theorem:** Let $f[a,b] \to \mathbb{R}$ be continuous. Let $f(a) \leq c \leq f(b)$.
Then $\exists x \in (a,b)$ such that $f(x) = c$.

**Proof:** $x \in (a,b)$ with metric on $\mathbb{R}$. Then $f : X \to \mathbb{R}$ is continuous.
Let $A = f^{-1}(-\infty, c)$. Let $B = f^{-1}(c, \infty)$.
By last theorem, $A, B$ are open in $X$. Have $a \in A$ $b \in B$ so $A, B$ are nonempty.
Also $A, B$ disjoint. If $\forall x$ $f(x) \neq c$ then $A \cup B = X$ (open in $X$).
Since $[a,b]$ is connected, this is impossible. So there is $x \in X = [a,b]$ such that $f(x) = c$.
Since $f(a) \neq c$ $f(b) \neq c$, must have $x \in (a,b)$.

Recall why $[a,b]$ is connected:

Suppose by contradiction $[a, b] = A \cup B$ where $A, B$ are open, nonempty, disjoint.
Fix $x \in A$ $y \in B$. Say WLOG that $x < y$ (use Dedekind completeness for reals).
Let $U = \{z \geq x | [x, z) \subseteq A\}$, nonempty bounded by $y$.
Let $s = sup(u)$

**Case 1:** If $s \in A$ (note: $s < y \leq b$) since $A$ is open, have $\epsilon > 0$ such that $[s, s + \epsilon) \subseteq A$.
Then, taking $z = s + \frac{\epsilon}{2}$ have $[x, z] \subseteq A$. So $z \in U$, a contradiction since $z > s$.

**Case 2:** If $s \in B$. Since $B$ is open (note $s > x \geq a$), have $\epsilon > 0$ such that $(s - \epsilon, s] \subseteq B$. Taking any $z > s - \epsilon$ have $[x, z) \nsubseteq A$.
So $z \notin U$ so $s - \epsilon$ is already a bound for $U$. So $\sup(U) \leq s - \epsilon$, a contradiction.

Next, **WTS** a lot of functions are continuous, including $x \mapsto e^x$
$\Rightarrow$ general operations preserve continuity
$\Rightarrow$ rephrase continuity in terms of limits of sequences, to use what we proved before about limit of sequences.

## 22.4 Limit of functions

In $X, Y$ metric spaces where $E \subseteq X$ $f : E \to Y$

**Definition:** Let $p$ be a limit point of $E$.
We write $\lim\limits_{x \to p, x \in E} f(x) = q$ or $f(x) \to q$ as $x \to p, x \in E$ if $(x \in Ex \neq p)$
(If $\forall \epsilon > 0, \exists N, n \geq N \to d(f(n), q) < \epsilon) \to$ redefine into

$\forall \epsilon > 0, \exists \delta > 0$ if $d(x, p) < \delta$ then $d(f(x), q) < \epsilon$

If there is $q$ as above, we say $\lim\limits_{x \to p} f(x)$ exists.
**Note:** $f$ is continuous at $p$ iff $\lim\limits_{x \to p} f(x) = f(p)$ (exactly the definition.

**Theorem:** $f$ is countinuous iff $\lim\limits_{x \to p} f(x) = f(p)$

**Theorem:** $f(x) \to q$ as $x \to p$ iff for every sequence $\{x_n\}_{n \geq 1}$ of points in $E \backslash \{p\}$ which converges to $p$, $\{f(x_n)\}_{n \geq 1}$ converges to $q$.

**Proof:**
$(\Rightarrow)$ Fix sequence $\{x_n\}$ converging to $p$. $(x_n \in E = -\{p\})$

Need to show $f(x_n) \to q$.
Fix $\epsilon > 0$. Need $N$ such that $n \geq N \to d(f(x_n), q) < \epsilon$.
By continuity, have $\delta > 0$ such that $d(x, p) < \delta \to d(f(x), q) < \epsilon$.
Since $x_n \to p$, can find tail end such that we have $N$ such that $n \geq N \to d(x_n, p) < \delta$.
Then $n \geq N \to d(f(x_n), q) < \epsilon$.

($\Leftarrow$) Fix $\epsilon > 0$ Need $\delta > 0$ such that $d(x, p) < \delta \to d(f(x), q) < \epsilon$.
Suppose not.
Then, for each $\delta > 0$ have some $x$ such that $d(x, p) < \delta$ and $d(f(x), q) \geq \epsilon$.
Using this with $\delta = \frac{1}{n}$, can find $x_n$ such that $d(x_n, p) < \frac{1}{n}$, $d(f(x_n), q) \geq \epsilon$.
Then $x_n \to p$ yet $f(x_n) \nrightarrow q$, a contradiction.

**Corollary:** If $f$ has a limit at $p$, this limit is unique.

**Proof:** Immediate from last theorem, and the uniqueness of limits for sequences.

**Corollary:** Let $f, g$ be functions. Then (immediate from theorem and limits of sequences)

- $\lim\limits_{x \to p} f(x) + g(x) = \lim\limits_{x \to p} f(x) + \lim\limits_{x \to p} g(x)$ (assuming limits on the RHS exists)
- Similarly for $f(x)g(x)$
  for $f(x)\backslash g(x)$ if $\lim\limits_{x \to p} g(x) \neq 0$
  for $f(x)^{g(x)}$ if $\lim\limits_{x \to p} f(x) \geq 0$ (with restrictions)

**Corollary:** If $f, g$ are continuous at $p$ into $\mathbb{R}$, then the following are also continuous at $p$

- $f + g$ $(f + g)(x) = f(x) + g(x)$
- $fg$
- $f/g$ if $g(p) \neq 0$
- $f^g$ if $f(p) > 0$

**Proof:** By the last theorem and characterization of continuity and using limits.

**Note:** The functions $x \mapsto x$ and for any $c \in \mathbb{R}$ $x \mapsto c$ are continuous.

**Corollary:** The following functions on $\mathbb{R}$ are continuous:

- $x, y \mapsto x + y$
- $x, y \mapsto xy$

- $x, y \mapsto cx$
- $x, y \mapsto cx + dy$
- $x, y \mapsto \frac{x}{y}$ for $y \neq 0$
- $x, y \mapsto x^y$ for $x > 0$

**Theorem:** If $f_i : X_i \to Y$ are continuous and $g : Y_1 \times ... \times Y_n \to Z$ is continuous, then $x_1...x_n \mapsto g(f_1(x_1), ..., f_n(x_n))$ is continuous (using product space).
Metric on $Y_1 \times ... \times Y_n$ is $d(\{a_1...a_n\}, \{b_1, ...b_n\}) = \sqrt{\sum_{i=1}^{n} d(a_i, b_i)^2}$

**Theorem:** For each $i$ the function $x_1, ...x_n \mapsto x_i$ from $Y_i \times ... \times Y_n$ to $Y_i$ is continuous. (projections)

**Example:** $x_1, x_2, x_3 \mapsto \sqrt{3x_1^2 + 5^{x_2} + x_3^8}$ is continuous.
Functions mapping to each term is continuous and whole thing is continuous by composition.
**Another example:** $x \mapsto e^x$ is continuous

**Proposition:** for any $y > 0$ there is $x$ such that $e^x = y$.
**Proof:** (using IVT)
Have some $N$ such that $e^N > y$
(*"By early claims a long time ago, using some strange induction"* - Neeman)
Have some $M$ such that $e^M > \frac{1}{y}$.
Then, $e^{-M} < y < e^{-N}$, and by intermediate value theorem, there is $x$ between $-M, N$ such that $e^x = y$.

## 22.5 Continuity and Compactness

**Claim:** Suppose $f : X \to Y$ is continuous and $f$ is compact. Then $f(X)$ is compact. $(f(X) = \{f(u|u \in X)\})$

**Proof:** Show every open cover has finite subcover.
Let $\{v_i\}i \in I$ be an open cover for $f(X)$.
Then $f^{-1}(v_i)$ is open by continuity and $\bigcup_{i \in I} f^{-1}(v_i)$ covers $X$.
By compactness of $X$, there is a finite $F \subseteq I$ such that $\bigcup_{i \in F} f^{-1}(v_i)$ covers $X$.
Then $\bigcup_{i \in F} v_i$ covers $f(X)$.

**Theorem:** Let $f : X \to \mathbb{R}$ be continuous. Suppose $X$ is compact, then $f$ achieves maximum and minimum values. i.e. $\exists p, q \in X$ such that $f(p) \leq f(x) \leq f(q)$ $\forall x \in X$.

**Another way:** let $M = \sup_{x \in X} f(x)$ $m = \inf_{x \in X}$.
Then $\exists p, q \in X$ such that $f(p) = m$ $f(q) = M$.

**Proof 1:** By last theorem, $f(X)$, (compact in $\mathbb{R}$) is closed and bounded.
By boundedness, $\inf f(X)$, $\sup f(X)$ exist. By closedness $\inf f(X)$, $\sup f(X) \in f(X)$.

**Proof 2:** Show $f$ attains maximum values.
Suppose not, $\forall q \in X$ have $x_q$ such that $f(x_q) > f(q)$.
By continuity, have $\delta_q > 0$ such that $x \in B_{\delta_q}(q) \to f(x) < f(x_q)$.
The sets $B_{\delta_q}(q)$ for an open cover of $X$.
By compactness, have $q_1, ...q_n$ such that $x \subseteq B_{\delta_{q_1}}(q_1) \cup ... \cup B_{\delta_{q_n}}(q_n)$.
Then, $\forall x \in X$ $f(x) < \max\{f(x_{q_1})...f(x_{q_n})\}$. Contradiction since $\max\{f(x_{q_1})...f(x_{q_n})\} = f(x)$ for some $x$ (in fact $x_{q_i}$)

**Theorem:** Let $f : X \to Y$ be continuous. Suppose $X$ is compact. Then, $f$ is uniformly continuous.

**Proof:** Fix $\epsilon > 0$. Need $\delta > 0$ such that $\forall x, y \in X$ if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
By continuity, for each $p \in X$, have $\delta_p > 0$ such that If $x \in B_{\delta_p}(p)$ then $d(f(p), f(x)) < \frac{\epsilon}{2}$.
The sets $B_{\frac{\delta_p}{2}}(p)$ form an open cover of $X$.
By compactness, have $p_1, ...p_n$ such that $X \subseteq B_{\frac{\delta_{p1}}{2}}(p_1) \cup ... \cup B_{\frac{\delta_{pn}}{2}}(p_n)$.
Let $\delta = \min\{\frac{\delta_{p1}}{2}, ..., \frac{\delta_{p_n}}{2}\}$.
We prove $d(x, y) < \delta \to d(f(x), f(y)) < \epsilon$.
Suppose $d(x, y) < \delta$, fix $i$ such that $x \in B_{\frac{\delta_{p_i}}{2}}(p_i)$.
Then $y \in B_{d_{p_i}}(p_i)$ (since $d(p_i, y) \leq d(p_i, x) + d(x, y) \leq \frac{\delta_{p_i}}{2} + \frac{\delta_{p_i}}{2} = \delta_{p_i}$)
So $d(f(p), f(x))$, $d(f(p_i), f(y)) < \frac{\epsilon}{2}$, so $d(f(x), f(y)) < \epsilon$.

**Example:** $x \mapsto \frac{1}{x}$ from $(0, +\infty)$ to $(0, +\infty)$ continuously.
Does not attain max or min domain not compact, not uniformly continuous, i.e. can't find $\delta$ that works throughout intervals.

## 22.6 Application of Uniform Continuity

**Theorem:** Suppose $E \subseteq X$, $E$ is dense, $f : E \to Y$. $Y$ is complete, and $f$ is uniformly continuous. Then there is a unique $f* : X \to Y$ continuous, extending $f$.

**Proof sketch:**
**Lemma:** Suppose $f : X \to Y$ is uniformly continuous. If $\{x_n\}$ is cauchy in $X$ then $\{f(x_n)\}$ is cauchy in $Y$.

**Proof:** fix $\epsilon > 0$. By uniform continuity of $f$, have $\delta > 0$ such that $d(x, y) < \delta \to d(f(x), f(y)) < \epsilon$.
Since $\{x_n\}$ is cauchy, have $N$ such that $n, m \geq N \to d(x_n, x_m) < \delta \to d(f(x_n), f(x_m)) < \epsilon$.

To prove theorem, use lemma to show:
If $a_n \in E$, $p \in X$ $a_n \to p$, then $\{f(a_n)\}$ is cauchy has a limit $q$ in $Y$ (by completeness).
Prove $f(x) \to q$ as $x \to p \in X$ and $x \in E$.
Define $f*(p) = \lim_{x \to p, x \in E} f(x)$. Prove $f*$ extends $f$, prove $f*$ is continuous and prove uniqueness.