

CS217: Data Structures & Algorithm Analysis (DSAA)

Lecture #1

Getting Started

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Reading (that means **homework!**): read Chapter 1 and Chapter 2, Sections 2.1-2.2, (skip problems, exercises, and pseudocode conventions)

➤ Aims of this lecture

- to set the scene for the analysis of algorithms
- to define **correctness** of algorithms and to demonstrate how to show that an algorithm is correct
- to show how the running time of an algorithm can be **analysed**
- to analyse **InsertionSort** as a simple sorting algorithm

➤ Algorithms

- An algorithm is a well-defined **computational procedure** that takes some **input** and produces some **output**.
 - It is a tool for solving a well-specified **computational problem**.
- Example: the **sorting** problem
 - **Input**: a sequence of n numbers $\langle a_1, a_2, \dots, a_n \rangle$.
 - **Output**: a permutation (reordering) $\langle a_1', a_2', \dots, a_n' \rangle$ of the input sequence such that $a_1' \leq a_2' \leq \dots \leq a_n'$.

A sequence like $\langle 31, 41, 59, 26, 41, 58 \rangle$ is called an **instance** of the sorting problem.

- We expect an algorithm to **solve** any instance of the problem

➤ How we describe algorithms

We use an abstract language, **pseudocode**, for two reasons:

1. See that algorithms exist independent from any particular programming language
2. Focus on **ideas** rather than syntax issues, error-handling, etc.

“If you wish to implement any of the algorithms, you should find the translation of our pseudocode into your favourite programming language to be a fairly straightforward task.

[...]

We attempt to present each algorithm simply and directly without allowing the idiosyncrasies of a particular programming language to obscure its essence.”

➤ **What's an ideal algorithm?**

➤ Correctness

- An algorithm is **correct** if for every input instance it halts with the correct output. A correct algorithm **solves** the problem.
- ? How do you know whether an algorithm is correct?
- ? Who would you rather be?
 - Person A: “I designed an algorithm and I think it is correct.”
 - Person B: “I tested my algorithm on 3 instances and it worked.”
 - Person C: “I can **prove** that my algorithm is **always** correct.”
- Ideally, all algorithms should be taught with a proof of correctness!

➤ How to measure time?

- Computers are different (clock rate, speed of memory...)
- Computer architecture can be complex (memory hierarchy, pipelining, multi-core...)
- Choice of programming language affects execution time
- We need a model that provides a **good level of abstraction:**
 - Gives a good idea about the time an algorithm needs
 - Allows us to compare different algorithms
 - Without us getting bogged down with details

➤ Random-access machine (RAM) model

- A generic random-access machine; instructions are executed one after another, with no concurrent operations
- Elementary operations:
 - **Arithmetic:** Add, subtract, multiply, divide, remainder
 - **Logical** operations, shifts, comparisons
 - **Data movement:** variable assignments (storing, retrieving)
 - **Control instructions:** loops, subroutine/method calls
- The RAM model assumes that each elementary operation takes the same amount of time (a *constant* independent of the problem size)
- The elementary operations are those commonly found in real computers

➤ Runtime

- Common cost model: count the number of elementary operations in the RAM model.
- Assumes all operations take the same time.

Runtime of Algorithm A on instance I:

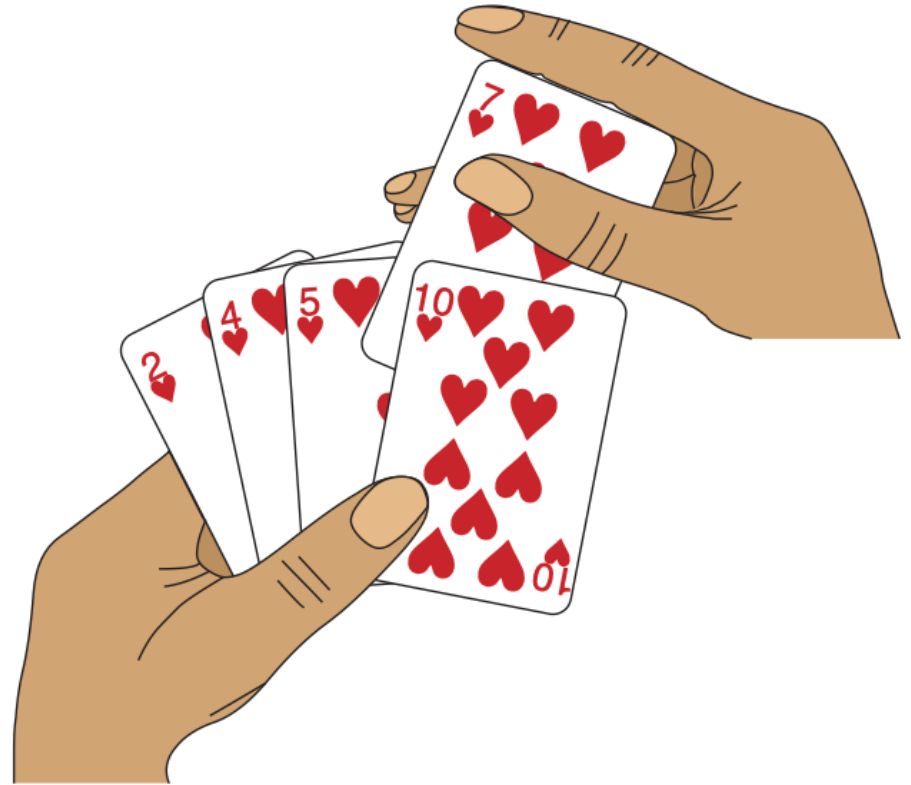
The number of elementary operations in the RAM model A takes on I.

- But... don't get obsessed counting operations in detail
- We'll often abstract from constants (you'll see how)
- Focus on **asymptotic growth** of runtime with problem size
- We'll meet some Greek friends to help us: $\Theta, O, \Omega, o, \omega$

➤ Example: InsertionSort

Idea: build up a sorted sequence by inserting the next element at the right position.

Like sorting a hand of cards!



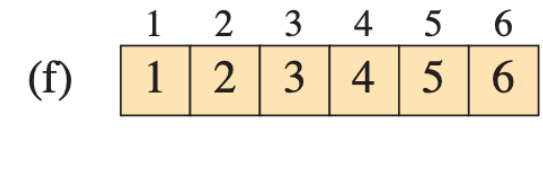
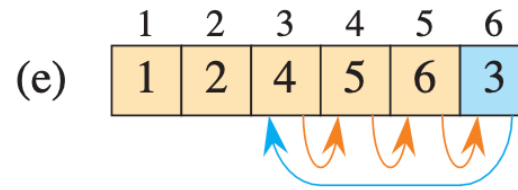
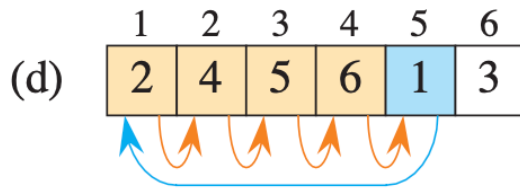
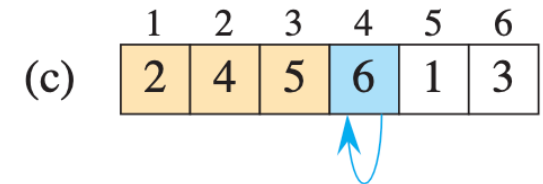
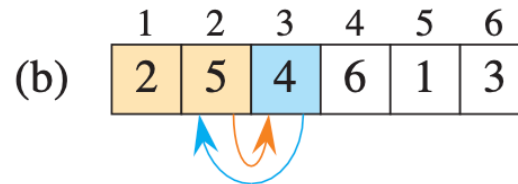
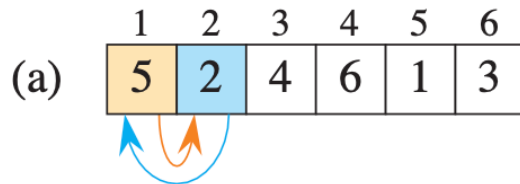
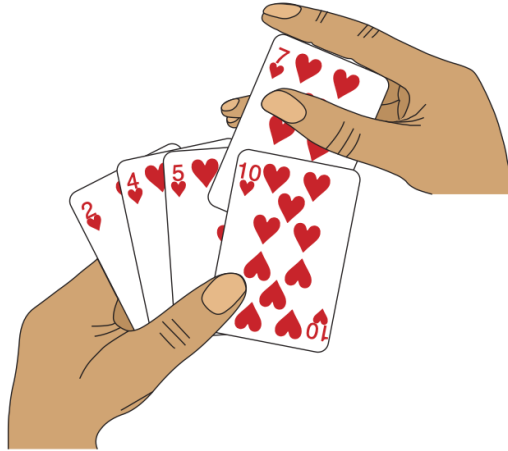
➤ InsertionSort

INSERTIONSORT(A)

```
1: for  $j = 2$  to  $A.length$  do  
2:    $key = A[j]$   
3:   // Insert  $A[j]$  into the sorted sequence  $A[1 \dots j - 1]$ .  
4:    $i = j - 1$   
5:   while  $i > 0$  and  $A[i] > key$  do  
6:      $A[i + 1] = A[i]$   
7:      $i = i - 1$   
8:    $A[i + 1] = key$ 
```



➤ Example for InsertionSort



Coming up

1. How do we know whether InsertionSort is always correct?
 - Proof by loop invariant
2. How long does InsertionSort take to run?
 - Naïve and messy approach for now to motivate a cleaner and easier way (next week).

➤ Loop invariants

- A popular way of proving correctness of algorithms with loops.
- A **loop invariant** is a statement that is always true and that reflects the progress of the algorithm towards producing a correct output.
 - Example: “After i iterations of the loop, at least i things are nice.”
 - The hard bit is finding out what is “nice” for your algorithm!
 - **Initialisation**: the loop invariant is true at initialisation.
 - Often trivial: “After 0 iterations of the loop, at least 0 things are nice.”
 - **Maintenance**: if the loop invariant is true after i iterations, it is also true after $i+1$ iterations.
 - Need to prove that the loop turns i nice things into $i+1$ nice things.
 - **Termination**: when the algorithm terminates, the loop invariant tells that the algorithm is correct.
 - “When terminating, all is nice and that means the output is correct!”

➤ Loop invariant: Example

INSERT-ALL-FIVES(A, n)

1: **for** $i = 1$ to n **do**

2: $A[i] = 5$

- **Loop invariant:** *“At the start of each iteration of the for loop, each element of the subarray $A[1..i-1]$ is a 5”*
- **Initialisation:** For $i=1$ the empty subarray has no elements (trivial).
- **Maintenance:** Loop invariant says that at step i of the *for* loop the subarray $A[1..i-1]$ contains 5s. During the i_{th} iteration we insert a 5 in $A[i]$, so by the end of the iteration the loop invariant still holds for step $i+1$.
- **Termination:** The algorithm terminates when $i=n+1$. Then the loop invariant for $i=n+1$ says that all the elements of the subarray $A[1..n]$ contain 5s, so the algorithm returns the correct output!

➤ Correctness of InsertionSort

- **Loop invariant:** *“At the start of each iteration of the for loop of lines 1-8, the subarray $A[1..j-1]$ consists of the elements originally in $A[1..j-1]$, but in sorted order.”*
- **Initialisation:** For $j=2$ the subarray $A[1]$ is the original $A[1]$ and it is sorted (trivially).
- **Maintenance:** The while loop moves $A[j-1]$, $A[j-2]$, ... one position to the right and inserts $A[j]$ at the correct position $i+1$. Then $A[1..j]$ contains the original $A[1..j]$, but in sorted order:

$$\underbrace{A[1] \leq A[2] \leq \dots \leq A[i-1]}_{\text{sorted before}} \leq \underbrace{A[i]}_{\text{from while loop}} \leq \underbrace{A[i+1] \leq A[i+2] \leq \dots \leq A[j]}_{\text{sorted before}}$$

- **Termination:** The for loop ends when $j=n+1$. Then the loop invariant for $j=n+1$ says that the array contains the original $A[1..n]$ in sorted order!

➤ Runtime of InsertionSort

INSERTIONSORT(A)	Cost	Times
1: for $j = 2$ to $A.length$ do		
2: $key = A[j]$		
3: // Insert $A[j]$ into ...		
4: $i = j - 1$		
5: while $i > 0$ and $A[i] > key$ do		
6: $A[i + 1] = A[i]$		
7: $i = i - 1$		
8: $A[i + 1] = key$		

Define t_j as the number of times the while loop is executed for that j .

➤ Runtime of InsertionSort

INSERTIONSORT(A)

1: **for** $j = 2$ to $A.length$ **do**

2: $key = A[j]$

3: // Insert $A[j]$ into ...

4: $i = j - 1$

5: **while** $i > 0$ and $A[i] > key$ **do**

6: $A[i + 1] = A[i]$

7: $i = i - 1$

8: $A[i + 1] = key$

Cost

Times

c_1

n

c_2

$n - 1$

c_4

$n - 1$

c_5

$t_2 + t_3 + \dots = \sum_{j=2}^n t_j$

c_6

$(t_2 - 1) + (t_3 - 1) + \dots = \sum_{j=2}^n (t_j - 1)$

c_7

$(t_2 - 1) + (t_3 - 1) + \dots = \sum_{j=2}^n (t_j - 1)$

c_8

$n - 1$

Define t_j as the number of times the while loop is executed for that j .

➤ Runtime of InsertionSort

- How to analyse the runtime of InsertionSort (in a naïve way):
 1. Assume that line i is run in time (cost) c_i .
 2. Count the number of times that line is executed.
 - Use t_j for the number of times the while loop was executed
 3. Sum up products of costs and times.
- Result (it's messy; our Greek friends will help keep things tidy):

$$T(n) = c_1n + c_2(n - 1) + c_4(n - 1) + c_5 \sum_{j=2}^n t_j + c_6 \sum_{j=2}^n (t_j - 1) + c_7 \sum_{j=2}^n (t_j - 1) + c_8(n - 1)$$

➤ Runtime of InsertionSort: Best case

$$T(n) = c_1n + c_2(n - 1) + c_4(n - 1) + c_5 \sum_{j=2}^n t_j + c_6 \sum_{j=2}^n (t_j - 1) + c_7 \sum_{j=2}^n (t_j - 1) + c_8(n - 1)$$

Best case: the array is sorted, $t_j = 1$ (1x head of while loop)

$$\begin{aligned} T(n) &= c_1n + c_2(n - 1) + c_4(n - 1) + c_5(n - 1) + c_8(n - 1) \\ &= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \\ &= an + b \end{aligned}$$

for constants a, b composed of c_1, c_2 , etc.

Note: $an + b$ is a **linear function** in n .

➤ Runtime of InsertionSort: Worst case

$$T(n) = c_1n + c_2(n - 1) + c_4(n - 1) + c_5 \sum_{j=2}^n t_j + c_6 \sum_{j=2}^n (t_j - 1) + c_7 \sum_{j=2}^n (t_j - 1) + c_8(n - 1)$$

Worst case: the array is reverse sorted, $t_j = j$

The following formula is very helpful:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

So

$$\sum_{j=2}^n j = \frac{n(n+1)}{2} - 1 \quad \text{and} \quad \sum_{j=2}^n (j-1) = \sum_{j=1}^{n-1} j = \frac{(n-1)n}{2}$$

➤ Runtime of InsertionSort: Worst case (2)

Worst case: the array is reverse sorted, $t_j = j$

Using these formulas gives

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5 \left(\frac{n(n+1)}{2} - 1 \right) \\ &\quad + c_6 \left(\frac{n(n-1)}{2} \right) + c_7 \left(\frac{n(n-1)}{2} \right) + c_8(n-1) \\ &= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2} \right) n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8 \right) n \\ &\quad - (c_2 + c_4 + c_5 + c_8) \\ &= an^2 + bn + c \end{aligned}$$

For constants a, b, c composed of c_1, c_2 , etc.

Note: a **quadratic function** in n

Summary

- **Correctness** means that an algorithm always produces the intended output for any input.
- Runtime describes the number of **elementary operations in a RAM machine**.
- Seen **InsertionSort** as a first example of an algorithm
 - Idea: build up sorted sequence by slotting in the next element.
 - Used a **loop invariant** to prove that the algorithm is **correct**.
 - A loop invariant is a statement that is always true.
 - Captures the progress towards producing a correct output at termination.
 - Analysed the **runtime** of InsertionSort.