CS217 - Data Structures & Algorithm Analysis (DSAA)

Lecture #4

- **HeapSort**

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Reading: Chapter 6

Aims of this lecture

- To introduce the **HeapSort** algorithm. 1.
- To show how a **clever data structure**, a **heap**, can lead to a **fast** and **in place** sorting algorithm 2.
  - In place: *O(1)* additional space.
- To **practice the design and analysis of algorithms**. 3.

➤ Idea behind HeapSort

- Idea:
  - Find the largest element.
  - Move it to the end of the array (put another one in its place).
  - Repeat with remaining elements.
- Like SelectionSort but …
  - SelectionSort compares lots of elements to find the largest.
  - Can we store knowledge gained from these comparisons for the future?
  - Use this knowledge to make future iterations faster!

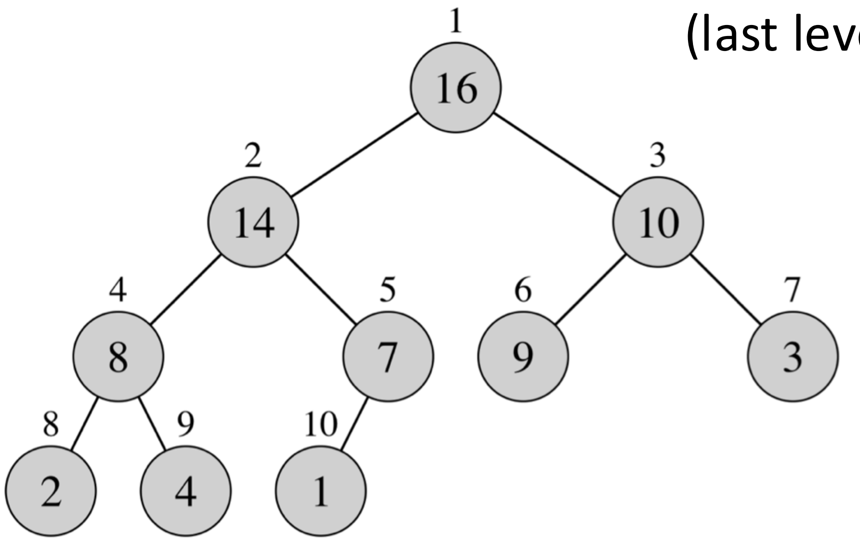
➢ Use your imagination…

Photo : Thomas Bresson

➢ A Heap

- Essentially an array **imagined** as being a **binary tree**!
- Elements are arranged row by row from left to right.

(last level may be incomplete)



(a)

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

(b)

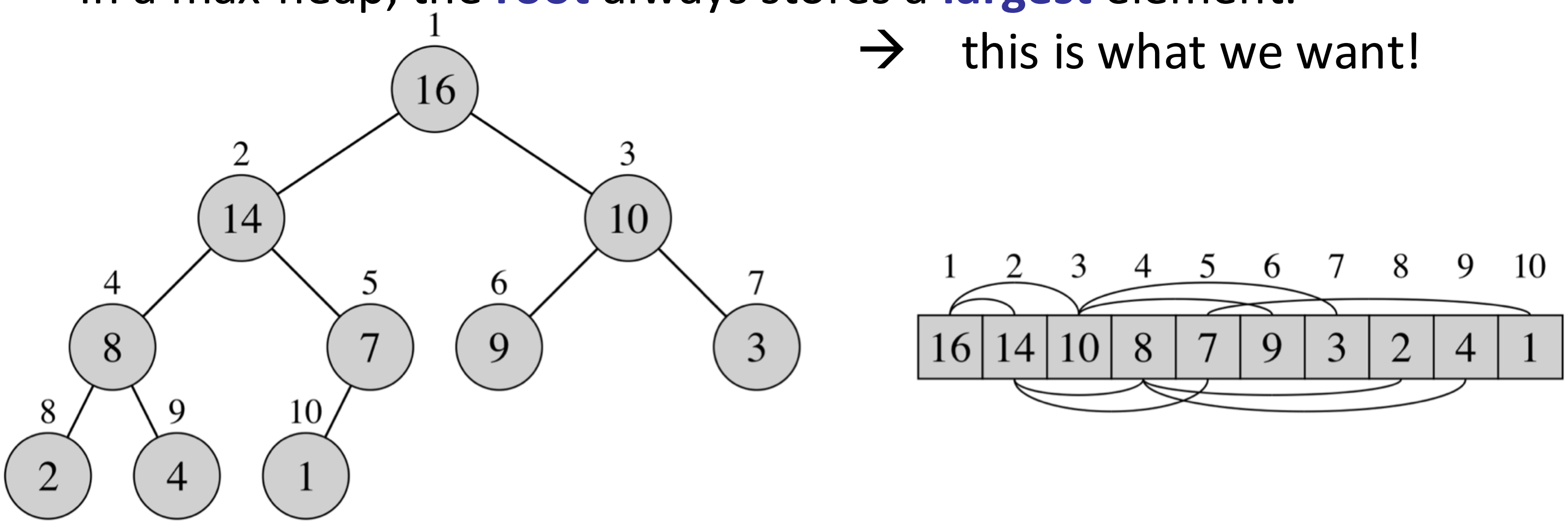
- Navigate through the array/imaginary tree using these operations:
- $\text{Parent}(i) = \left\lfloor \frac{i}{2} \right\rfloor$ (“floor of *i/2*”), $\text{Left}(i) = 2i$, $\text{Right}(i) = 2i + 1$

To Induct.

➢ Heap Properties

- **Max-heap property**: for every node other than the root, the parent is no smaller than the node, $A[Parent(i)] \geq A[i]$.
- In a max-heap, the **root** always stores a **largest** element.

→ this is what we want!



(a) (b)

- **Min-heap property**: for every node other than the root, the parent is no larger than the node, $A[Parent(i)] \leq A[i]$.

Procedures (what do we need)

1. **Build-Max-Heap:** produces a Max-Heap from an unordered array
2. **Max-Heapify**: maintains the max-heap property once the maximum has been removed
3. **HeapSort:** sorts an array in place

- New variable A.heap-size indicates how many elements of A are stored in a heap: $0 \leq$ A.heap-size $\leq$ A.length.
  - Decreasing A.heap-size by 1 effectively removes the last element from the heap (we imagine a heap without it)
- There are analogous operations for min-heaps: Min-Heapify and Build-Min-Heap.

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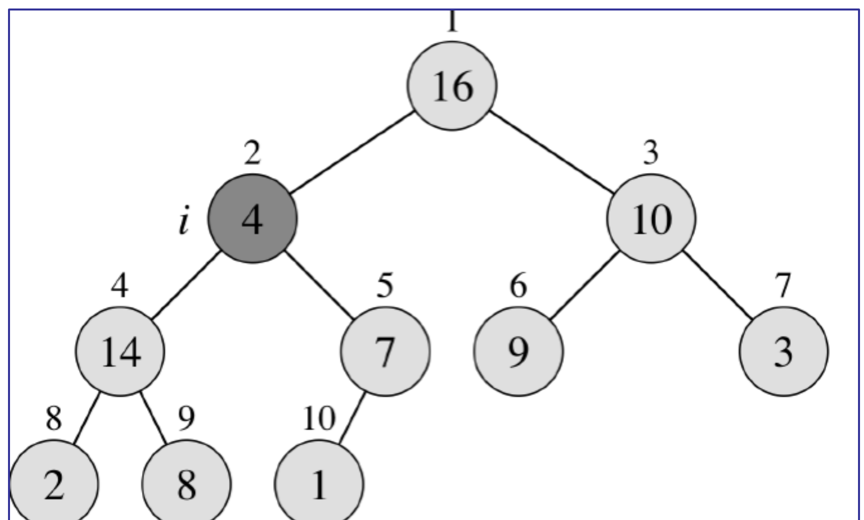
Max-Heapify(*A, i*)

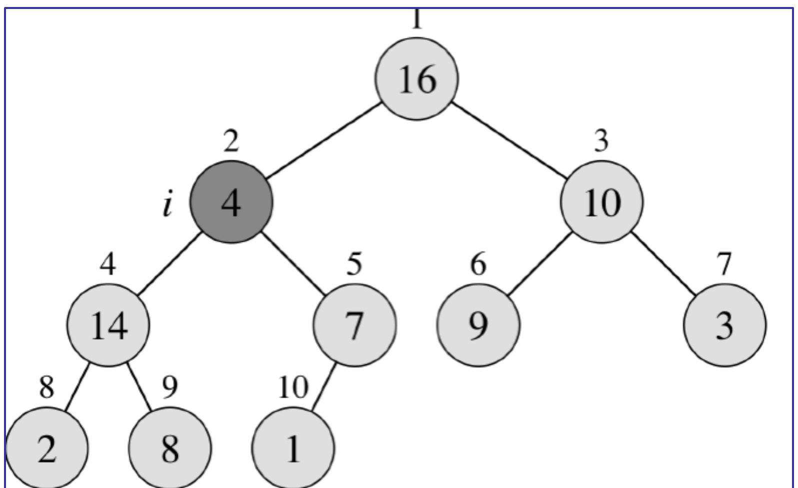
- **Assumes subtrees Left(*i*) and Right(*i*) are max-heaps**, but max-heap property might be violated in root of subtree at *i*.
  - "Subtree x": the part of the tree including x and everything below.
- Lets the value at *A[i]* "float down" if necessary, to restore max-heap property at *i*
- At the end of Max-Heapify the subtree at *i* is a max-heap.



(a)

➢ Max-Heapify: informal and in pseudocode

- Compare *A[i]* with all existing children
- If **largest child** is larger than *A[i]*, swap and recurse on child



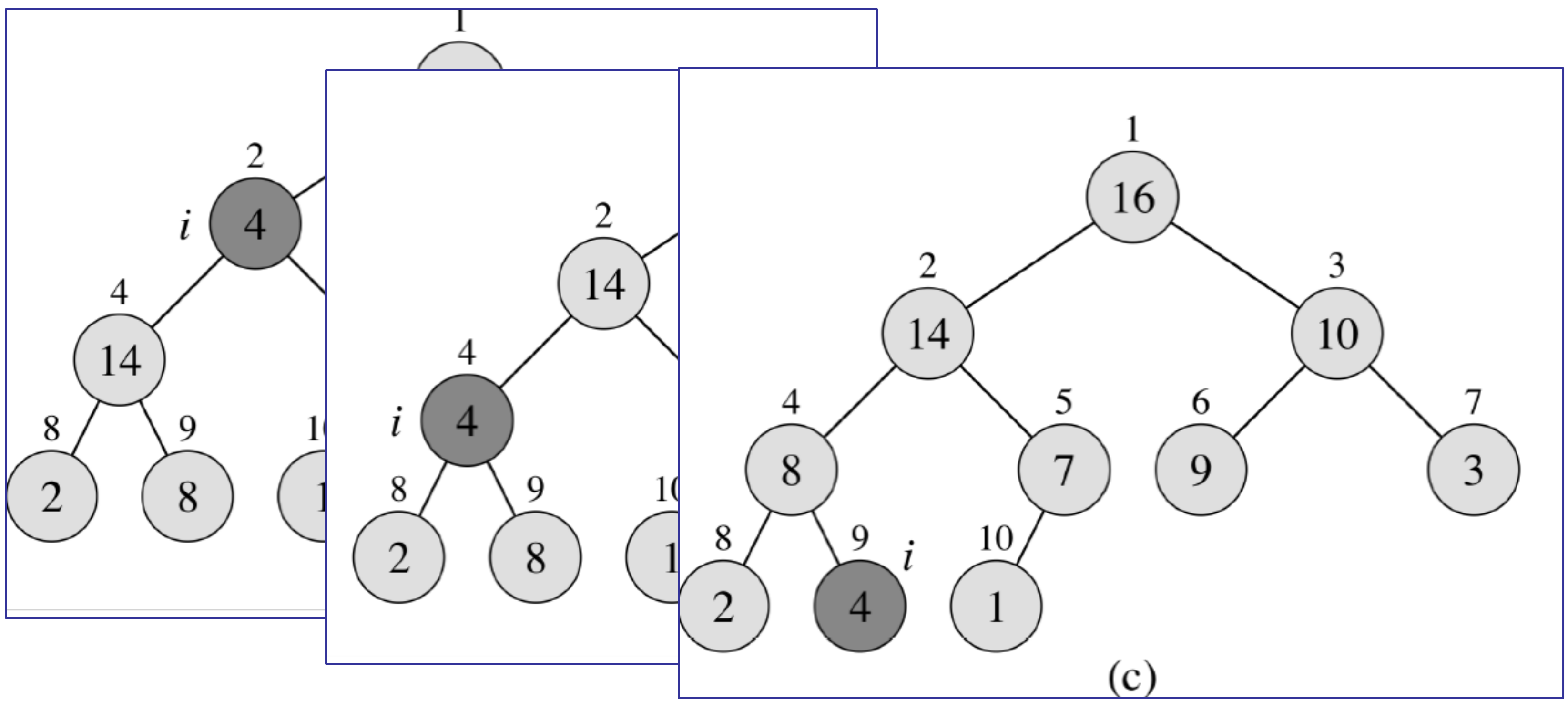
(a)

$\textsc{Max-Heapify}(A, i)$

$l = \text{Left}(i)$
$r = \text{Right}(i)$
if $l \leq A.\text{heap-size}$ and $A[l] > A[i]$ **then**
    largest $= l$
else
    largest $= i$
if $r \leq A.\text{heap-size}$ and $A[r] > A[\text{largest}]$ **then**
    largest $= r$
if largest $\neq i$ **then**
    exchange $A[i]$ with $A[\text{largest}]$
    $\textsc{Max-Heapify}(A, \text{largest})$

Max-Heapify: Example

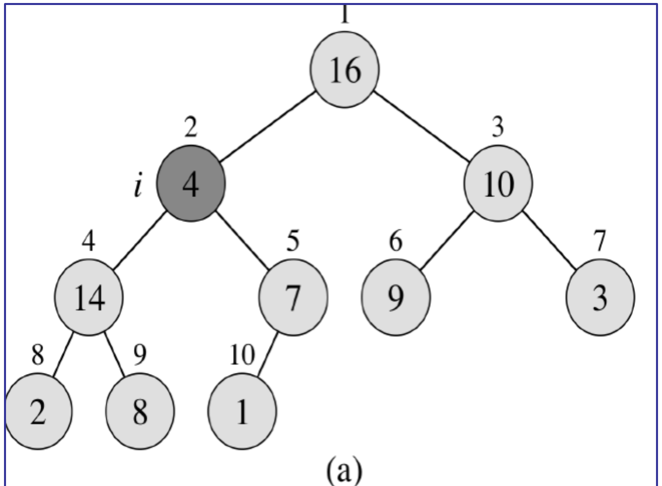
- Compare *A[i]* with all existing children
- If **largest child** is larger than *A[i]*, swap and recurse on child



Runtime of Max-Heapify

- Define the height of a node as the longest number of simple downward edges from the node to a **leaf**.
- **Leaf**: a node without children.
- Max-Heapify takes constant time, $\Theta(1)$, on each level.
- Running time of Max-Heapify on a node of height h is $O(h)$.
- It's not $\Omega(h)$ as Max-Heapify may stop early, e.g. if heap-property holds at i.
- For leaves $h = 0$ and the time is $O(1)$.

```
MAX-HEAPIFY(A, i)
l = Left(i)
r = Right(i)
if l ≤ A.heap-size and A[l] > A[i] then
    largest = l
else
    largest = i
if r ≤ A.heap-size and A[r] > A[largest] then
    largest = r
if largest ≠ i then
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```



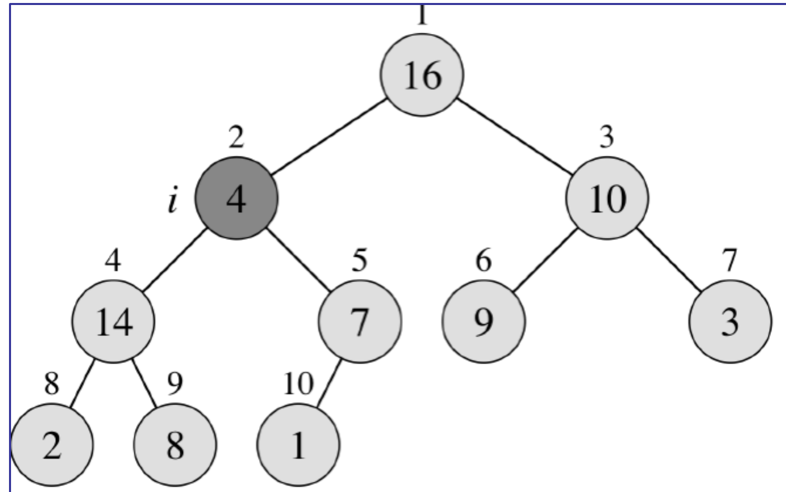
(a)

Bounding the height of a heap

- **Claim**: the **height of a heap** = height of the root is at most *log n*.
- **Proof**: the number *n* of elements in a heap of height *h* is
  - Doubling on each level
  - At least 1 node on the last level
  - Hence in total at least

$$1 + 2 + 4 + \cdots + 2^{h-1} + 1 = 2^h$$

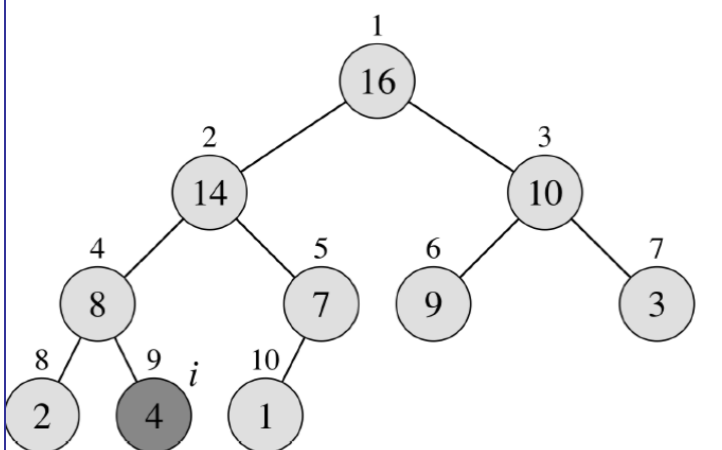
(we used $\sum_{i=0}^{k-1} 2^i = 2^k - 1$)



(a)

- So size and height are related as $n \geq 2^h \Leftrightarrow \log n \geq h$
- "the height of the root is at most $\log n$"
- So the runtime of Max-Heapify is O(log n)

Max-Heapify: Correctness

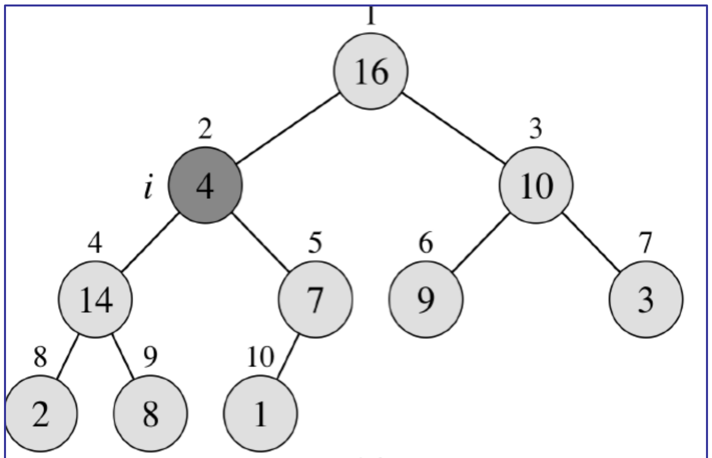


(c)

```
MAX-HEAPIFY(A, i)
l = Left(i)
r = Right(i)
if l ≤ A.heap-size and A[l] > A[i] then
    largest = l
else
    largest = i
if r ≤ A.heap-size and A[r] > A[largest] then
    largest = r
if largest ≠ i then
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```

- By induction (on the height):
- **Base case:** height = 0 (*i* is a leaf)
- Then left(i) and right(i) are larger than A.heap-size and the algorithm returns a heap!

Max-Heapify: Correctness



(a)

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MAX-HEAPIFY(A, i)
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        largest = r
if largest ≠ i then
        exchange A[i] with A[largest]
        MAX-HEAPIFY(A, largest)
```

- By induction (on the height):
- **Inductive case:** assume it works for height $h = i - 1$ and show it works for $h = i$
- Then the algorithm swaps A[i] with the larger between Left(i) and Right(i) (if any) and one subtree was already a heap and the other will be by inductive hypothesis.

Procedures (what do we need)

1. **Build-Max-Heap:** produces a Max-Heap from an unordered array
2. **Max-Heapify**: maintains the max-heap property once the maximum has been removed
3. **HeapSort:** sorts an array in place

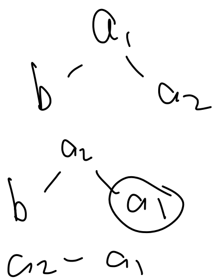
➢ Building a Heap

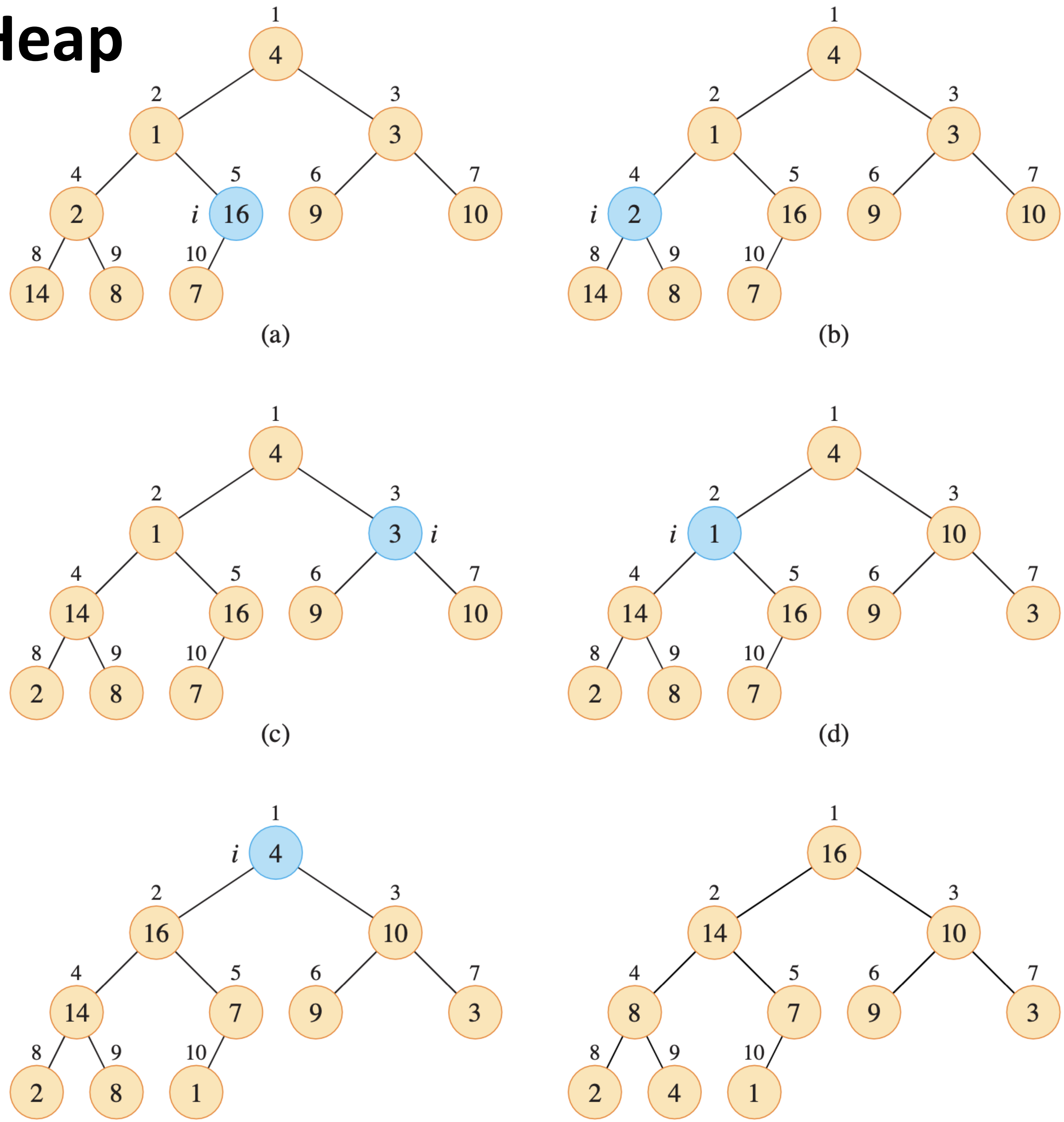
- Idea: use Max-Heapify repeatedly to create a heap.
- Which order of nodes: top-down or bottom-up?
- Answer: **bottom-up** – Max-Heapify assumes Left(*i*) and Right(*i*) are heaps. Top-down wouldn't work, bottom-up does.
- Note: nodes in $A\left[\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right), \ldots, n\right]$ are all leaves. Leaves are max-heaps, so no work required.

```
BUILD-MAX-HEAP(A, n)
    A.heap-size = n
    for i = ⌊n/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

A	4	1	3	2	16	9	10	14	8	7

Build-Max-Heap



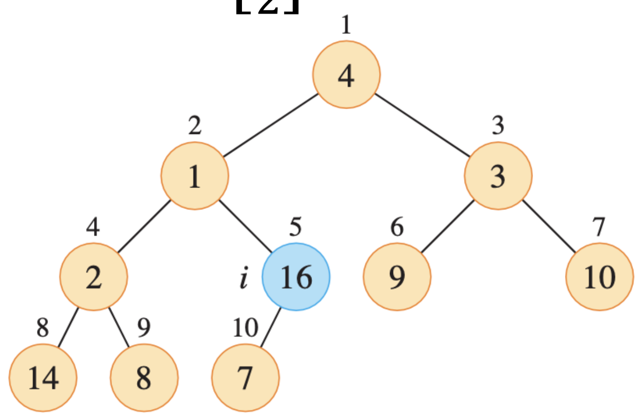


(a) (b) (c) (d) (e) (f)

➢ Correctness of Build-Max-Heap

```
BUILD-MAX-HEAP(A, n)
    A.heap-size = n
    for i = ⌊n/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

- **Loop invariant:** At the start of each iteration of the for loop, each node $i+1, i+2, \ldots, n$ is the root of a max-heap.
- **Initialisation:** true for leaves $\left\lfloor\frac{n}{2}\right\rfloor + 1, \ldots, n$.



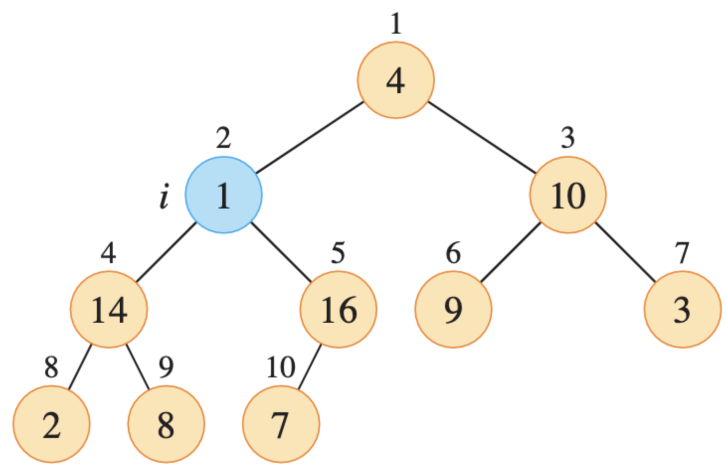
(a)

Correctness of Build-Max-Heap

```
BUILD-MAX-HEAP(A, n)
    A.heap-size = n
    for i = ⌊n/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

- **Loop invariant:** At the start of each iteration of the for loop, each node $i+1, i+2, \ldots, n$ is the root of a max-heap.

- **Maintenance:** by loop invariant, all children of i are roots of max-heaps (as their numbers are larger than i).
  Then Max-Heapify(A, i) turns the subtree at i into a max-heap.

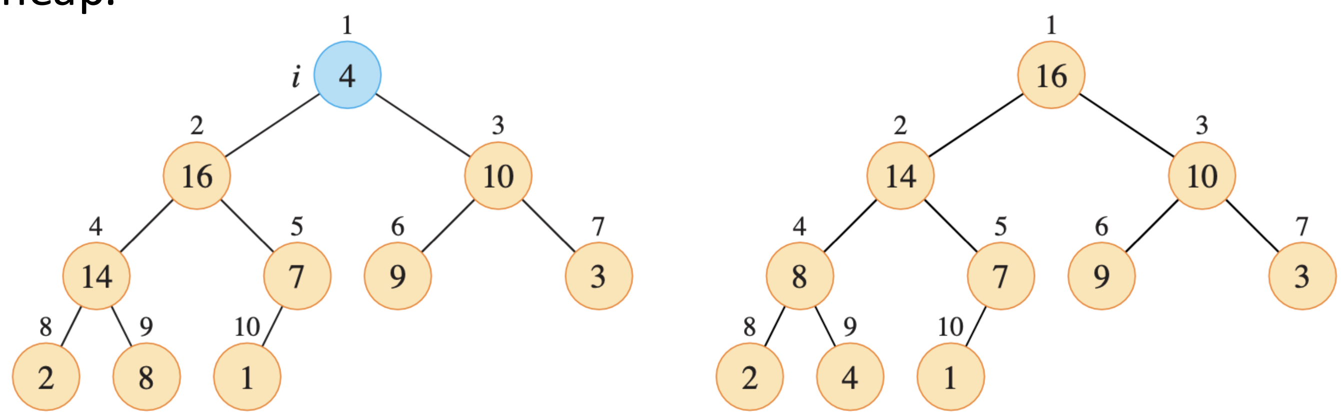


(d)

Correctness of Build-Max-Heap

```
BUILD-MAX-HEAP(A, n)
A.heap-size = n
for i = ⌊n/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

- **Loop invariant:** At the start of each iteration of the for loop, each node $i+1, i+2, \ldots, n$ is the root of a max-heap.
- **Termination:** the loop terminates at $i = 0$, hence node 1 is the root of a max-heap.

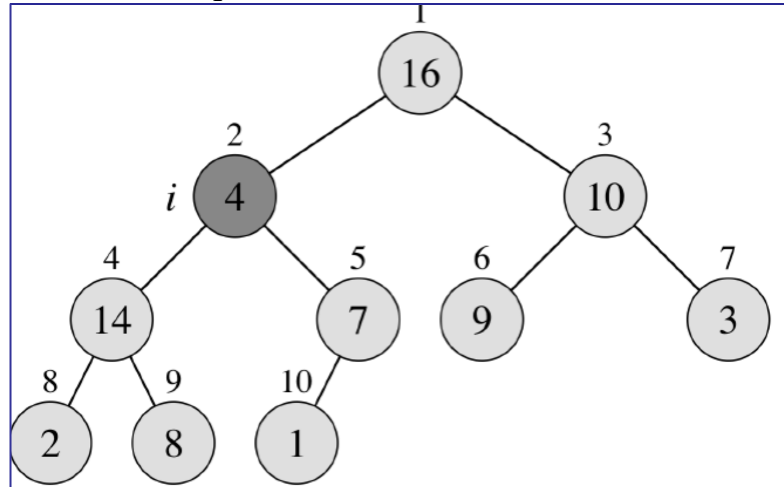


(e) (f)

Runtime of Build-Max-Heap

- The **height of a heap** = height of the root is at most *log n*.
- So all nodes have height at most *log n*.
- Every call to Max-Heapify takes time $O(\log n)$.
- Build-Max-Heap calls Max-Heapify $O(n)$ times.
- Total time is at most $O(n) \cdot O(\log n) = O(n \log n)$.
  - The time can be improved to *O(n)* since most nodes have small height.
  - $O(n \log n)$ is sufficient for us, though.

Refined Analysis of Build-Max-Heap



(a)

- **Observation: most nodes have small height!**
- One can show: there are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of height h. (by induction)
- $O(\log n)$ time bound is correct, but crude for most nodes.
- A better bound:

$$\sum_{h=1}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=1}^{\lfloor \log n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=1}^{\infty} \frac{h}{2^h} \right) = O(n)$$

as the infinite series of $\frac{h}{2^h}$ is 2.

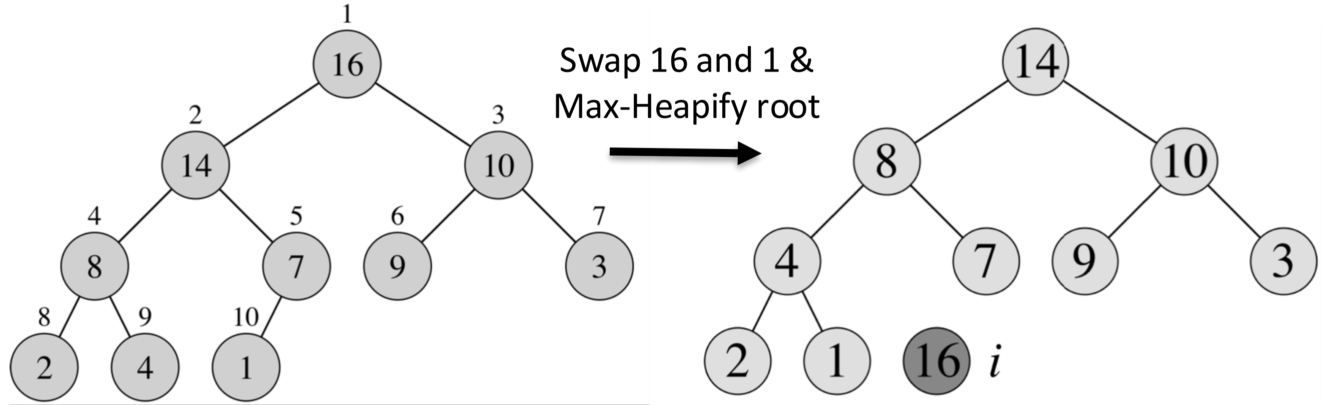
- 1st equality, we used that: $\lceil x \rceil \leq 2x$ for $x \geq 1/2$

$\Longrightarrow$ for $h \leq \log n$, $\frac{n}{2^{h+1}} \geq 1/2$ for all $n \geq 2^h$ which is necessary to have a tree of height h

- 2nd equality, we used that $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$ for $|x| < 1$

➢ Procedures (what do we need)

1. **Build-Max-Heap:** produces a Max-Heap from an unordered array ✓
2. **Max-Heapify**: maintains the max-heap property once the maximum has been removed ✓
3. **HeapSort:** sorts an array in place

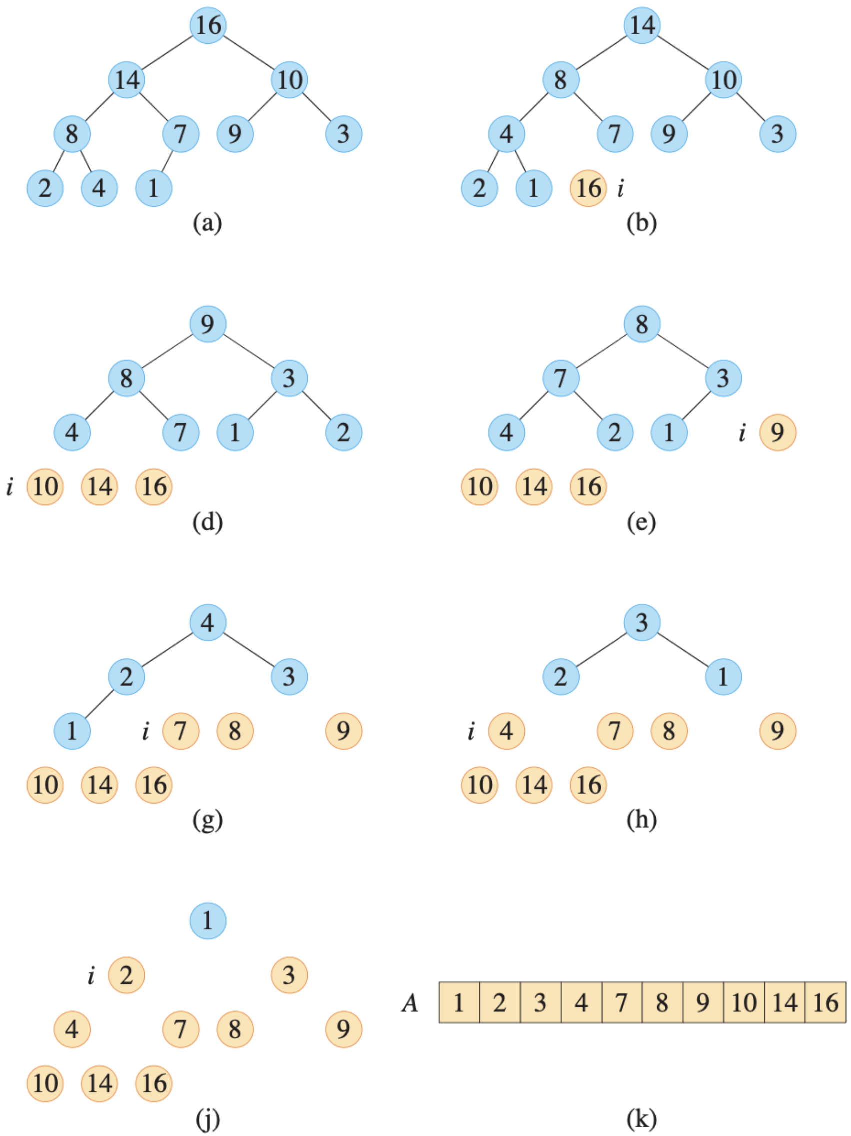


➢ HeapSort

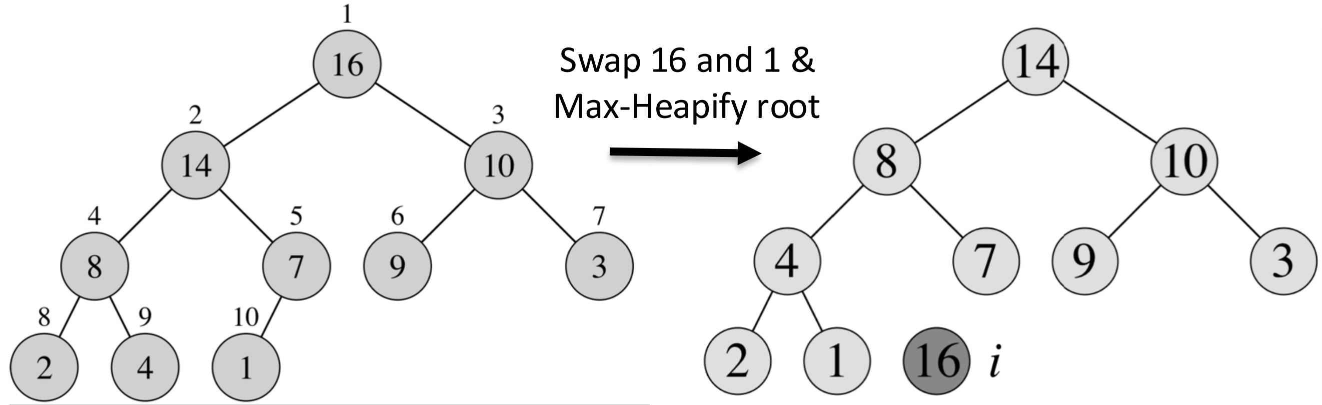
- Ideas:
  1. Build a max-heap, such that the root contains largest element.
  2. Swap the root with the last element of the heap/array.
  3. Discard the last element from the heap by reducing heap.size. (We simply imagine a smaller heap.)
  4. Call Max-Heapify(*A, 1*) to restore heap property at the root.

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i = A.length downto 2 do
    exchange A[1] with A[i]
    A.heap-size = A.heap-size − 1
    MAX-HEAPIFY(A, 1)
```

HeapSort: Example



HeapSort



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  1. Build a max-heap, such that the root contains largest element.
  2. Swap the root with the last element of the heap/array.
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  4. Call Max-Heapify(*A, 1*) to restore heap property at the root.

$\textsc{HeapSort}(A)$

$\textsc{Build-Max-Heap}(A)$
for $i = A.\text{length}$ downto 2 **do**
  exchange $A[1]$ with $A[i]$
  A.heap-size = A.heap-size -1
  $\textsc{Max-Heapify}(A, 1)$

Runtime:

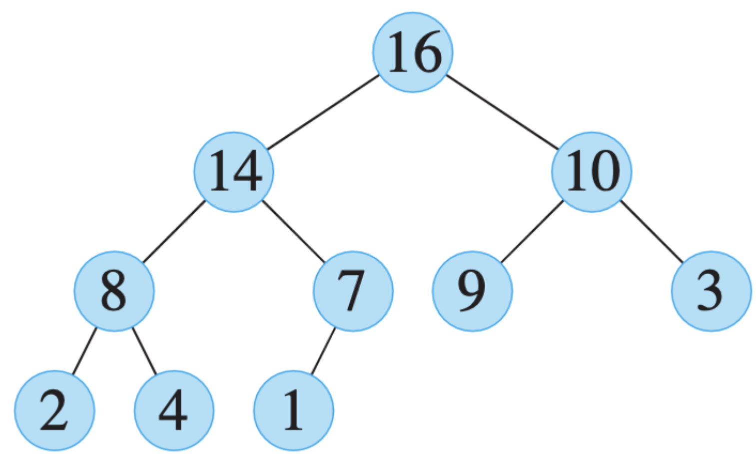
$$O(n \log n) + (n-1) \cdot O(\log n) = O(n \log n)$$

Correctness of HeapSort

Loop Invariant: "At the start of each iteration of the for loop of lines 2-5, the subarray $A[1..i]$ is a max-heap containing the i smallest elements of $A[1..n]$, and the subarray $A[i+1..n]$ contains the $n-i$ largest elements of $A[1..n]$, sorted."

- **Initialization:** The subarray $A[i+1..n]$ is empty, thus the invariant holds.

```
HEAPSORT(A)
1: BUILD-MAX-HEAP(A)
2: for i = A.length downto 2 do
3:     exchange A[1] with A[i]
4:     A.heap-size = A.heap-size −1
5:     MAX-HEAPIFY(A, 1)
```



(a)

Correctness of HeapSort

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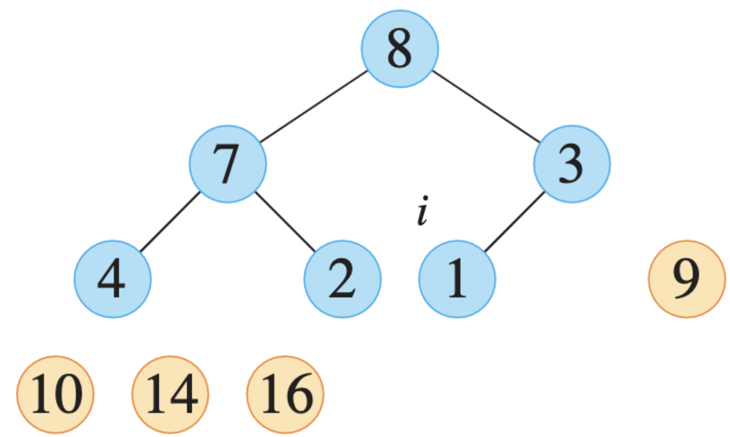
Maintenance: $A[1]$ is the largest element in $A[1..i]$ and it is smaller than the elements in $A[i+1..n]$. When we put it in the ith position, then $A[i..n]$ contains the largest elements, sorted. Decreasing the heap size and calling Max-Heapify turns $A[1..i-1]$ into a max-heap. Decrementing i sets up the invariant for the next iteration.

HEAPSORT(A)

1: BUILD-MAX-HEAP(A)
2: **for** i = A.length downto 2 **do**
3: exchange $A[1]$ with $A[i]$
4: A.heap-size = A.heap-size -1
5: MAX-HEAPIFY($A, 1$)



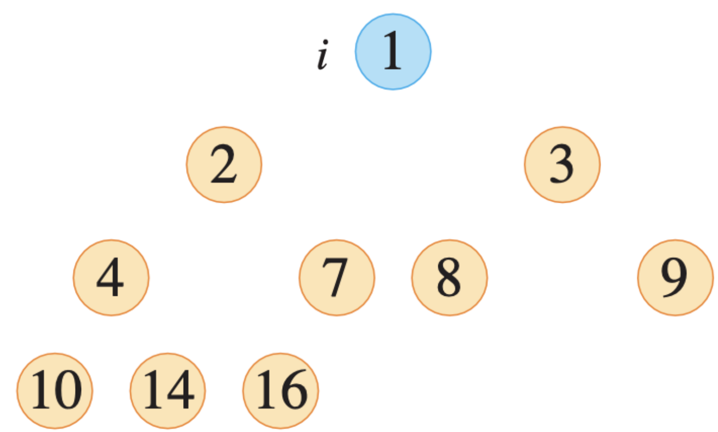
(e)

➢ Correctness of HeapSort

Loop Invariant: "At the start of each iteration of the for loop of lines 2-5, the subarray *A*[1..*i*] is a max-heap containing the *i* smallest elements of *A*[1..*n*], and the subarray *A*[*i+1*..*n*] contains the *n*−*i* largest elements of *A*[1..*n*], sorted."

- **Termination:** After the loop *i*=1. This means that *A*[2..*n*] is sorted and *A*[1] is the smallest element in the array, which makes the array sorted.

```
HEAPSORT(A)
1: BUILD-MAX-HEAP(A)
2: for i = A.length downto 2 do
3:     exchange A[1] with A[i]
4:     A.heap-size = A.heap-size −1
5:     MAX-HEAPIFY(A, 1)
```



(j)

Summary

- HeapSort sorts in place in time $O(n \log n)$.
  - Building a Heap in time $O(n)$.
  - Extracting the largest element and restoring the heap-property in total time $O(n \log n)$.
- The use of appropriate **data structures** can speed up computation (in contrast to SelectionSort).
  - The heap "memorises" information about comparisons of elements.
  - The heap is imaginary, no objects/pointers required!
- Heaps also play a role in **Priority Queues**.