Homework 5 (Due October 24)

Grade Distribution (Total=6+8+8+8+6+8=44).

Please simply answer as much as possible.

1. Suppose that the cumulative distribution function of the random variable X is given by

$$F(x) = \begin{cases} 1 - e^{-2x^2}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

   Evaluate (a) EX; (b) $Var(X)$; (c) $E(e^X)$.

2. To be a winner in a certain game, you must be successful in three successive rounds. The game depends on the value of U, a uniform random variable on $(0, 1)$. If $U > 0.15$, then you are successful in round 1; if $U > 0.24$, then you are successful in round 2; and if $U > 0.45$, then you are successful in round 3.

   (a) Find the probability that you are successful in round 1.

   (b) Find the conditional probability that you are successful in round 2 given that you were successful in round 1.

   (c) Find the conditional probability that you are successful in round 3 given that you were successful in rounds 1 and 2.

   (d) Find the probability that you are a winner.

3. A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him, independently of each other, with probability 0.7, whereas if he has an off day, this probability will be reduced to 0.4. Suppose that the student will pass the examination if a majority of the examiners pass him. Assume that the student is twice as likely to have an off day as he is to have an on day (that is, off day with probability 2/3, and on day 1/3). Suppose that the student requests an examination with 7 examiners.

   (a) What is the mean and variance for the number of the 7 examiners that pass the student?

   (b) What is the probability for the student to finally pass the examination?

4. Suppose there is a room with three windows, only one of which is open. Two birds in the room try to escape through the windows.

   (a) Assume that the first bird is memoryless, that is, the bird randomly pick one of the three windows and attempts to escape. If it fails, it will again randomly pick one window out of the three. Let X be the number of attempts for the first bird. Find EX and $Var(X)$.

(b) Assume that the second bird does have memory, that is, the bird randomly pick one of the three windows, and if it fails, it will randomly pick one of the remaining windows. Let Y be the number of attempts for the second bird. Find EY and $Var(Y)$.

5. Evidence concerning the guilt or innocence of a defendant in a criminal investigation can be summarized by the value of an exponential random variable X whose mean μ depends on whether the defendant is guilty. If innocent, $\mu = 1$; if guilty, $\mu = 2$. The deciding judge will rule the defendant guilty if $X > c$ for some suitably chosen value of c.

   (a) If the judge wants to be 98 percent certain that an innocent man will not be convicted, what should be the value of c?

   (b) Using the value of c found in part (a), what is the probability that a guilty defendant will be convicted?

6. We say X is a Weibull random variable with parameters $\nu \in \mathbb{R}, \alpha > 0, \beta > 0$ if its cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & \text{if } x \leq \nu; \\ 1 - e^{-(\frac{x-\nu}{\alpha})^\beta}, & \text{if } x > \nu. \end{cases} \tag{0.1}$$

   (a) Show that if $Y = (\frac{X-\nu}{\alpha})^\beta$, then Y is an exponential random variable with parameter $\lambda = 1$.

   (b) Show that if Y is an exponential random variable with parameter $\lambda = 1$, then $\xi = \alpha Y^{1/\beta} + \nu$ (notice that $Y = (\frac{\xi-\nu}{\alpha})^\beta$) is also a Weibull random variable.