MA215 Probability Homework7

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Contents

1 Question 1

The joint density of X and Y is:

$$f(x,y) = c(x^2 - y^2)e^{-x},\ 0 < x < \infty, -x \le y \le x.$$

Find the conditional distribution of Y, given $X = x$

Answer:

By the definition of the conditional probability density function, we know that:

$$P_{Y|X}(y|x) = \frac{f(x,y)}{P_X(x)}$$

So we only need to calculate the marginal probability density function of X

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\ &= \int_{-x}^{x} c(x^2 - y^2)e^{-x}\,dy \\ &= cx^2e^{-x}(x-(-x)) - \frac{1}{3}c(x^3 - (-x)^3)e^{-x} \\ &= \frac{4}{3}cx^3e^{-x} \end{aligned}$$

To calculate the specific value of c, we need to use the formula:

$$\iint_{-\infty}^{\infty} f(x,y)\,dxdy = 1$$

Do a simple computation, we get:

$$\begin{aligned} \int_0^{\infty} \frac{4}{3}cx^3e^{-x}\,dx &= 1 \\ \frac{4}{3}c \times \Gamma(4) &= 1 \\ 8c &= 1 \\ c &= \frac{1}{8} \end{aligned}$$

So:

$$f_X(x) = \frac{1}{6}x^3e^{-x}$$

$$P_{Y|X}(y|x) = \frac{f(x,y)}{P_X(x)} = \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{1}{6}x^3e^{-x}} = \frac{3(x^2 - y^2)}{4x^3}$$

2 Question 2

A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 plasma set on that day?

Answer:

For a fixed customer Let A_1 be the event that the number of customers who brought an ordinary television.
A_2 be the event that the number of customers who brought a plasma television.
A_3 be the event that the number of customers who was just browsing.

It is obvious that in the given day:

$$A_1 + A_2 + A_3 = 5$$
$$A_i \geq 0 \,\forall\, i \in \{1, 2, 3\}$$

The probability we need to calculate is $P(A_1 = 2 \cap A_2 = 1 \cap A_3 = 2) = P_{Ans}$

$$\begin{aligned}
P_{Ans} &= \binom{5}{2}(0.45)^2\binom{3}{1}(0.15)^1\binom{2}{2}(0.4)^2 \\
&= \frac{5 \times 4 \times 9^2}{2 \times 1 \times 20^2} \times \frac{3}{20} \times \frac{2^2}{5^2} \\
&= \frac{81 \times 3}{20 \times 10 \times 25} \\
&= \frac{273}{5000} \\
&= 0.0546
\end{aligned}$$

3 Question 3

Suppose that X and Y are independent geometric random variables with the same parameter p

a) Without any computation, what do you think is the value of:

$$P(X = i | X + Y = n)$$

b) Verify your conjecture in part (a).

Answer:

1. We can consider all the situations of $X + Y = n$ form a newly deduced sample space. Under this assumption, we know that the sequence look like:

1	...	X-1	X	X+1	X+2	...	n-2	n-1	n
head	...	head	tail	head	head	...	head	head	tail

Table 1: Question3-1

So the position of where the X^{th} element "tail" at is uniformly distributed. So:

$$P = \frac{1}{n-1}$$

2. Since both random variables are geometrically distributed, so:

$$P(X = k) = (1-p)^{k-1}p \ \forall k \geq 1$$
$$P(Y = k) = (1-p)^{k-1}p \ \forall k \geq 1$$

By simple computation:

$$P(X = i|X + Y = n) = \frac{P(X = i \cap X + Y = n)}{P(X + Y = n)}$$

Since X and Y are independent, so:

$$
\begin{aligned}
&= \frac{P(X = i)P(Y = n - i)}{\sum_{k=1}^{n-1} P(X = k)P(Y = n - k)} \\
&= \frac{(1 - p)^{i-1}p \times (1 - p)^{n-i}p}{\sum_{k=1}^{n-1}((1 - p)^{k-1}p \times (1 - p)^{n-k}p)} \\
&= \frac{(1 - p)^{n-1}p^2}{\sum_{k=1}^{n-1}((1 - p)^{n-1}p^2)} \\
&= \frac{1}{n - 1}
\end{aligned}
$$

4 Question 4

A model proposed for NBA basketball supposes that when two teams with roughly the same record play each other, the number of points scored in a quarter by the home team minus the number scored by the visiting team is approximately a normal random variable with mean 1.5 and variance 0.6. In addition, the model supposes that the point differentials for the four quarters are independent. Assume that this model is correct. You may express the following answers in terms of some integrals regarding $\phi(x)$ with $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

a) What is the probability that the home team win?

b) What is the conditional probability that the home team wins, given that it is behind 5 points at halftime?

c) What is the conditional probability that the home team wins, given that it is ahead by 5 points at the end of the first quarter?

Answer:
Let D_i be the points of the home team minus the visiting team in the i^{th} quarter. So we know:

$$D_i \sim Normal(1.5, 0.6)$$

Let D be the points of the home team minus the visiting team in the whole game. Then we have:

$$D = D_1 + D_2 + D_3 + D_4$$

So:

$$D \sim Normal(6, 2.4)$$

a) The probability of the home team win is $P(D > 0)$.

$$
\begin{aligned}
P(D > 0) &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \text{ where } \sigma^2 = 2.4 \text{ and } \mu = 6 \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,d(\frac{x - \mu}{\sigma})
\end{aligned}
$$

Let $t = \dfrac{x - \mu}{\sigma}$:

$$
\begin{aligned}
&= \int_{-\frac{\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt \\
&= \frac{1}{2} + \int_{-\frac{\mu}{\sigma}}^0 \phi(x)\,dx \\
&= \frac{1}{2} + \int_{-\sqrt{15}}^0 \phi(x)\,dx
\end{aligned}
$$

b) Let P_b be the probability we want to know.

$$P_b = \frac{P((D \geq 0) \cap (D_1 + D_2 = -5))}{P(D_1 + D_2 = -5)}$$

Let $A_1 = D_1 + D_2$ be a new normal distributed random variable which satisfies: $A_1 \sim Normal(3, 1.2)$.
$A_2 = D_3 + D_4$ be a new normal distributed random variable which also satisfies: $A_2 \sim Normal(3, 1.2)$. So P_b can be simplified into:

$$P_b = \frac{P((A_2 \geq 5) \cap (A_1 = -5))}{P(A_1 = -5)}$$

A_1 and A_2 are obviously independent random variables, so:

$$= P(A_2 \geq 5)$$

$$= \int_5^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Let $t = \dfrac{x-\mu}{\sigma}$:

$$= \int_{\frac{5-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

$$= \int_{\frac{\sqrt{30}}{3}}^{\infty} \phi(t)\, dt$$

$$= \frac{1}{2} - \int_0^{\frac{\sqrt{30}}{3}} \phi(t)\, dt$$

c) Let P_c be the probability we want to know.

$$P_c = \frac{P((D \geq 0) \cap (D_1 = 5))}{P(D_1 = 5)}$$

Let $A_1 = D_1$ be a new normal distributed random variable which satisfies: $A_1 \sim Normal(1.5, 0.6)$.
$A_2 = D_2 + D_3 + D_4$ be a new normal distributed random variable which also satisfies: $A_2 \sim Normal(4.5, 1.8)$. So P_b can be simplified into:

$$P_c = \frac{P((A_2 \geq 5) \cap (A_1 = 5))}{P(A_1 = 5)}$$

A_1 and A_2 are obviously independent random variables, so:

$$= P(A_2 \geq -5)$$

$$= \int_{-5}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Let $t = \dfrac{x-\mu}{\sigma}$:

$$= \int_{\frac{-5-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

$$= \int_{-\frac{19\sqrt{5}}{6}}^{\infty} \phi(t)\, dt$$

$$= \frac{1}{2} + \int_{-\frac{19\sqrt{5}}{6}}^{0} \phi(t)\, dt$$