MA215 Probability Homework-5

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1 Question 1

Suppose that the cumulative distribution function of the random variable X is given by:

$$F(x) = \begin{cases} 1 - e^{-2x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Calculate: (a)EX (b)$Var(X)$ (c)$E(e^{X^2})$

Answer:

By definition: we know that:

$$F(X \leq b) = 1 - e^{-2b^2}$$

Take the derivative of it, we could get a probability distribution function of X.

$$f(b) = F'_X(b) = \begin{cases} 4be^{-2b^2} & b \geq 0 \\ 0 & b < 0 \end{cases}$$

a)

$$EX = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} xf(x)\,dx + \int_{0}^{\infty} 4x^2e^{-2x^2}\,dx = \int_{0}^{\infty} 4x^2e^{-2x^2}\,dx$$

Let $u = 2x^2$, then:

$$EX = \frac{1}{\sqrt{2}}\int_{0}^{\infty} u^{\frac{1}{2}}e^{-u}\,du = \frac{1}{\sqrt{2}}\Gamma(\frac{3}{2}) = \frac{\sqrt{2\pi}}{4}$$

b)

$$Var(X) = E(X^2) - (EX)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} 4x^3e^{-2x^2}\,dx = \int_{0}^{\infty} 2ue^{-2u}\,du \text{ where } u = x^2$$

$$E(X^2) = \frac{1}{2}\int_{0}^{\infty} ue^{-u}\,du = \frac{1}{2}\Gamma(2) = \frac{1}{2}$$

So:

$$Var(X) = \frac{1}{2} - \frac{\pi}{8} = \frac{4-\pi}{8}$$

c)

$$E(e^{X^2}) = \int_{-\infty}^{\infty} e^{x^2}f(x)\,dx = \int_{0}^{\infty} 4xe^{-x^2}\,dx = \int_{0}^{\infty} 2e^{-x^2}\,d(x^2) = -2(0 - e^0) = 2$$

2 Question 2

To be a winner in a certain game, you must be successful in three successive rounds. The game depends on the value of U, a uniform random variable on $(0, 1)$. If $U > 0.15$, then you are successful in round 1; if $U > 0.24$, then you are successful in round 2; and if $U > 0.45$, then you are successful in round 3.

1. Find the probability that you are successful in round 1
2. Find the conditional probability that you are successful in round 2 given that you were successful in round 1.
3. Find the conditional probability that you are successful in round 3 given that you were successful in rounds 1 and 2.
4. Find the probability that you are a winner.

Answer:

From the discription we know that $U \sim Uniform(0, 1)$ Let A_1 be the event that successful in round 1, A_2 be the

event that successful in round 2, A_3 be the event that successful in round 3.
By observing the structure, it is obvious that $A_3 \subset A_2 \subset A_1$.

1. $$P(A_1) = P(U > 0.15) = P(0.15 \leq U \leq 1) = \int_{0.15}^{1} 1\,dx = 1 - 0.15 = 0.85$$

2. By the definition of conditional probability and independence, we know that:
$$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{P(A_2)}{P(A_1)} = \frac{76}{85}$$

3. By the definition of conditional probability, we know that:
$$P(A_3|(A_2 \cap A_1)) = \frac{P(A_3)}{P(A_2)} = \frac{55}{76}$$

4. Being a winner means being successful in total 3 games, so the probability equals to:
$$P(A_1 \cap A_2 \cap A_3) = P(A_3) = 0.55$$

3 Question 3

A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him, independently of each other, with probability 0.7, whereas if he has an off day, this probability will be reduced to 0.4. Suppose that the student will pass the examination if a majority of the examiners pass him. Assume that the student is twice as likely to have an off day as he is to have an on day (that is, off day with probability $\frac{2}{3}$, and on day $\frac{1}{3}$). Suppose that the students requests an examination with 7 examiners.

1. What is the mean and variance for the number of the 7 examiners that pass the student?
2. What is the probability for the student to finally pass the examination?

Answer:
From the description of the problem, let X_i be the number of the examiners who passes him, then $X_i = Bin(7, p_i)$, where i represents whether the day is "on" or "off".

1. From the properties of binomial random variable. $EX = np$ and $Var(X) = np(1-p)$
$$EX = \frac{1}{3}EX_1 + \frac{2}{3}EX_2 = \frac{7 \times 0.7 + 2 \times 7 \times 0.4}{3} = 3.5$$
$$Var(X) = Var(\frac{1}{3}X_1 + \frac{2}{3}X_2) = \frac{1}{9}Var(X_1) + \frac{4}{9}Var(X_2) = 0.91$$
So we need to find $E(X^2)$, i.e. find the distribution of X_i^2

2. The probability for passing the examination is:
$$P(X \geq 4) = \frac{1}{3}P(X_1 \geq 4) + \frac{2}{3}P(X_2 \geq 4)$$
$$P(X_i \geq 4) = P(X_i = 4) - P(X_i = 5) - P(X_i = 6) - P(X_i = 7)$$
Meanwhile:
$$P(X_i \geq 4) = 1 - P(X_i = 0) - P(X_i = 1) - P(X_i = 2) - P(X_i = 3)$$
We have:
$$P(X_i = k) = P(X_i = 7 - k)$$

So:

$$P(X_i \geq 4) = 1 - P(X_i \geq 4)$$
$$P(X_i \geq 4) = \frac{1}{2}$$

So:

$$P(X \geq 4) = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{1}{2}$$

4 Question 4

Suppose there is a room with three windows, only one of which is open. Two birds in the room try to escape through the windows.

1. Assume that the first bird is memoryless, that is, the bird randomly pick one of the three windows and attempts to escape. If it fails, it will again randomly pick one window out of the three. Let X be the number of attempts for the first bird. Find EX and $Var(X)$.

2. Assume that the second bird does have memory, that is, the bird randomly pick one of the three windows, and if it fails, it will randomly pick one of the remaining windows. Let Y be the number of attempts for the second bird. Find EY and $Var(Y)$.

Answer:

1. From the description of the bird, let X be the number of attempts the bird need to escape, then $X \sim Geo(\frac{1}{3})$. Use the property of geometry random variable.

$$EX = \frac{1-\frac{1}{3}}{\frac{1}{3}} = 2.Var(X) = \frac{1-\frac{1}{3}}{(\frac{1}{3})^2} = 6$$

2. Without loss of generosity, we assume 1 and 2 windows are closed while the 3^{rd} window is open. Then $Y \in \{1, 2, 3\}$

$$P(Y = 1) = \frac{1}{3}$$
$$P(Y = 2) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$
$$P(Y = 3) = \frac{2}{3} \times \frac{1}{2} \times 1 = \frac{1}{3}$$

So:

$$EY = \Sigma_{i=1}^{3} P(Y = i) \times i = 2$$
$$E(Y^2) = \Sigma_{i=1}^{3} P(Y = i) \times i^2 = \frac{1+4+9}{3} = \frac{14}{3}$$
$$Var(Y) = E(Y^2) - (EY)^2 = \frac{2}{3}$$

5 Question 5

Evidence concerning the guilt or innocence of a defendant in a criminal investigation can be summarized by the value of an exponential random variable X whose mean μ depends on whether the defendant is guilty. If innocent, $\mu = 1$; if guilty, $\mu = 2$. The deciding judge will rule the defendant guilty if $X > c$ for some suitably chosen value of c.

1. If the judge wants to be 98 percent certain that an innocent man will not be convicted, what should be the value of c?

2. Using the value of c found in part (1), what is the probability that a guilty defendant will be convicted?

Answer:

1. Since the man is innocent, $\mu = 1$, by the property of exponential random variable, $\lambda = \frac{1}{\mu} = 1$. So: $X \sim Exp(1)$.

$$P(x > c) = P(x \geq c) = \int_c^{\infty} e^{-x}\,dx = e^{-c}$$

Since we want 98 percent or more, then we need:

$$e^{-c} \geq 1 - 0.98$$

$$c \leq -ln(0.02) = ln(50) \approx 3.912$$

2. If the man is guilty, $\mu = 2$, $\lambda = \frac{1}{\mu} = \frac{1}{2}$

$$P(X > c) = P(x \geq c) = \int_c^{\infty} \frac{1}{2} e^{-\frac{1}{2}x}\,dx = e^{-\frac{1}{2}c} = \frac{\sqrt{2}}{10}$$

6 Question 6

We say X is a Weibull random variable with parameters $\nu \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$ if its cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & x \leq \nu \\ 1 - e^{-(\frac{x-\nu}{\alpha})^{\beta}} & x > \nu \end{cases}$$

1. Show that if $Y = e^{(\frac{x-\nu}{\alpha})^{\beta}}$ then Y is an exponential random variable with parameter $\lambda = 1$.
2. Show that if Y is an exponential random variable with parameter $\lambda = 1$, then $\xi = \alpha Y^{1/\beta} + \nu$(Notice that $Y = (\frac{\xi-\nu}{\alpha})^{\beta}$)is also a Weibull random variable.

Answer:

1. Since $F(x)$ is a cumulative distribution function, then:

$$lim_{x\to\infty}(e^{(\frac{x-\nu}{\alpha})^{\beta}}) = 0 \text{ and is not increasing}$$

Let $F_Y(t)$ be the cumulative distribution function of $Y = (\frac{x-\nu}{\alpha})^{\beta}$ So:

$$\begin{aligned} F_Y(t) &= P(Y \leq t) \\ &= P((\frac{x-\nu}{\alpha})^{\beta} \leq t) \\ &= P(x \leq x_0) \text{ where } (\frac{x_0-\nu}{\alpha})^{\beta} = t \\ &= F_X(x_0) \end{aligned}$$

So:

$$F_Y(t) = \begin{cases} 0 & t \leq 0 \\ 1 - e^{-t} & t > 0 \end{cases}$$

After taking the derivative of it.

$$f(t) = \begin{cases} 0 & t \leq 0 \\ e^{-t} & t > 0 \end{cases}$$

So,

$$Y \sim Exp(1)$$

2. Since $Y \sim Exp(1)$, so:

$$F_Y(t) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-t} & x > 0 \end{cases}$$

Now we calculate $F_\xi(x)$:

$$\begin{aligned} F_\xi(x) &= P(\xi \le x) \\ &= P(\alpha Y^{\frac{1}{\beta}} + \nu \le x) \\ &= P(Y \le \frac{x - \nu}{\alpha})^\beta) \\ &= F_Y((\frac{x - \nu}{\alpha})^\beta) \end{aligned}$$

So:

$$F_\xi(x) = \begin{cases} 0 & (\frac{x-\nu}{\alpha})^\beta \le 0 \\ 1 - e^{-(\frac{x-\nu}{\alpha})^\beta} & (\frac{x-\nu}{\alpha})^\beta > 0 \end{cases}$$

Simplify it, we will get:

$$F_\xi(x) = \begin{cases} 0 & x \le \nu \\ 1 - e^{-(\frac{x-\nu}{\alpha})^\beta} & x > \nu \end{cases}$$

By the definition, it is easy to prove ξ is a Weibull random variable.