MA215 Probability Homework8

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1 Question 1

If X and Y are independently and identically distributed uniform random variables on (0, 1), compute the joint density of:

1. $U = X + Y, V = X/Y$
2. $U = X, V = X/Y$
3. $U = X + Y, V = X/(X + Y)$

Answer:
By the formula of multiple substitution:

$$f_{UV}(u, v) = |J(u, v)| \times f_{XY}(q(u, v), r(u, v))$$

And also, we know that the probability density function:

$$f_{XY}(x, y) = \begin{cases} 1 & x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

We can calculate:

1. $U = X + Y, V = X/Y$, so:

$$\begin{cases} X & = q(u, v) = \frac{uv}{v+1} \\ Y & = r(u, v) = \frac{u}{v+1} \end{cases}$$

$$|J(u, v)| = \begin{vmatrix} \frac{v}{v+1} & \frac{u}{(v+1)^2} \\ \frac{1}{v+1} & -\frac{u}{(v+1)^2} \end{vmatrix} = -\frac{u}{(v+1)^2}$$

$$f_{UV}(u, v) = \begin{cases} \frac{u}{(v+1)^2} & u \in [0, 2], v \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

2. $U = X, V = X/Y$, so:

$$\begin{cases} X & = q(u, v) = u \\ Y & = r(u, v) = \frac{u}{v} \end{cases}$$

$$|J(u, v)| = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v^2}$$

$$f_{UV}(u, v) = \begin{cases} \frac{u}{v^2} & u \in [0, 1], v \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

3. $U = X + Y, V = X/(X + Y)$:

$$\begin{cases} X & = q(u, v) = uv \\ Y & = r(u, v) = u - uv \end{cases}$$

$$|J(u, v)| = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u$$

$$f_{UV}(u,v) = \begin{cases} u & u \in [0,2], v \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

2 Question 2

If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}, 0 < x < \infty$, derive the joint distribution of $U = X + Y, V = X + Z, W = Y + Z$.

Answer:

$U = X + Y, V = Z + X, W = Y + Z$:

$$\begin{cases} X & = q(u,v,w) = \frac{u+v-w}{2} \\ Y & = m(u,v,w) = \frac{u+w-v}{2} \\ Z & = n(u,v,w) = \frac{v+w-u}{2} \end{cases}$$

$$|J(u,v,w)| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4}$$

$$f_{UVW}(u,v,w) = \begin{cases} \frac{1}{4}e^{-\frac{u+v+w}{2}} & u,v,w \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

3 Question 3

Let $X_1, \ldots, X_n$ be a set of independent and identically distributed continuous random variables having cumulative distribution function F(x), and let $X(i), i = 1, \ldots, n$ denote their ordered values. If X, independent of the $X_i, i = 1, \ldots, n$, also has cumulative distribution function F, determine:

1. $P(X > X_{(n)})$
2. $P(X > X_{(1)})$
3. $P(X_{(i)} < X < X_{(j)}), 1 \leq i < j \leq n.$

[Hint: For any $1 \leq i \leq n, P(X_i = X_{(n)}) = P(X_i \text{ is the max}) = \frac{1}{n}$ by symmetry.]

Answer:

By the definition of ordered statistic, we know that the ordered joint p.d.f:

$$f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n) = n! \times f(x_1)f(x_2)\ldots f(x_n) \times 1_{x_1 < x_2 < \cdots < x_n}$$

1. Construct a new ordered statistic Y_j with $n+1$ random variables including: X and X_i where $i \in [1,n]$

$$P(X > X_{(1)}) = P(Y_j = Y_{(n+1)}) = \frac{1}{n+1}$$

2. Construct a new ordered statistic Y_j with $n+1$ random variables including: X and X_i where $i \in [1,n]$

$$P(X > X_{(1)}) = \sum_{i=1}^{n+1} P(Y_j = Y_{(i)}) = \sum_{i=1}^{n+1} \frac{1}{n+1} = \frac{n}{n+1}$$

3. Construct a new ordered statistic Y_j with $n+1$ random variables including: X and X_i where $i \in [1,n]$

$$P(X > X_{(1)}) = \sum_{k=i+1}^{j-1} P(X = Y_{(k)}) + \frac{1}{2}(P(X = Y_{(i)}) + P(X = Y_{(j)})) = \sum_{k=i+1}^{j-1} \frac{1}{n+1} + \frac{1}{2}(\frac{1}{n+1} + \frac{1}{n+1}) = \frac{j-i}{n+1}$$

4 Question 4

Let $X_{(1)}, X_{(2)}, X_{(3)}$ be the ordered values of 3 independent uniform (0, 1) random variables. Prove that for $1 \leq k \leq 4$,

$$P(X_{(k)} - X_{(k-1)} > t) = (1-t)^3, \forall t \in (0,1)$$

where $X_{(0)} = 0$ and $X_{(4)} = 1$.

Answer:

To simplify, we denote $P := P(X_{(k)} - X_{(k-1)} > t)$

$$\begin{aligned} P &= 3!P(X_1 < X_2 < X_3 \cap X_{(k)} - X_{(k-1)} > t) \\ &= 6 \times P(X_1 < X_2 < X_3 \cap X_k - X_{k-1} > t) \end{aligned}$$

Do a classification discussion:

1. $k = 1:$

$$\begin{aligned} P &= 6 \times P(X_1 < X_2 < X_3 \cap X_1 > t) \\ &= 6 \times P(t < X_1 < X_2 < X_3 < 1) \\ &= 6 \int_t^1 dx_3 \int_t^{X_3} dx_2 \int_t^{X_2} 1 \, dx_1 \\ &= 6 \int_t^1 dx_3 \int_t^{X_3} (x_2 - t) \, dx_2 \\ &= 6 \int_t^1 \frac{1}{2}(x_3 - t)^2 \, dx_3 \\ &= 6 \times \frac{1}{6}(1-t)^3 \\ &= (1-t)^3 \end{aligned}$$

2. $k = 2:$

$$\begin{aligned} P &= 6 \times P(X_1 < X_2 < X_3 \cap X_2 > X_1 + t) \\ &= 6 \times P(t < X_1 + t < X_2 < X_3 < 1) \\ &= 6 \int_t^1 dx_3 \int_t^{1-t} dx_1 \int_{X_1+t}^{X_3} 1 \, dx_2 \\ &= (1-t)^3 \end{aligned}$$

3. $k = 3:$

$$\begin{aligned} P &= 6 \times P(X_1 < X_2 < X_3 \cap X_3 > X_2 + t) \\ &= 6 \times P(t < X_1 + t < X_2 + t < X_3 < 1) \\ &= 6 \int_0^{1-t} dx_1 \int_0^{1-t} dx_2 \int_{X_2+t}^{1} 1 \, dx_3 \\ &= (1-t)^3 \end{aligned}$$

4. $k = 4:$

$$\begin{aligned}
P &= 6 \times P(X_1 < X_2 < X_3 \cap X_3 < 1-t) \\
&= 6 \times P(t < X_1 < X_2 < X_3 < 1-t) \\
&= 6 \int_0^{1-t} dx_1 \int_0^{1-t} dx_2 \int_{X_2}^{1-t} 1\, dx_3 \\
&= 6 \int_t^1 dx_1 \int_t^{X_3} (x_2 - t)\, dx_2 \\
&= (1-t)^3
\end{aligned}$$

So the statement:

$$P(X_{(k)} - X_{(k-1)} > t) = (1-t)^3, \forall t \in (0,1)$$

is correct for all $k \in [1, 4]$