MA215 Probability Homework 9

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1 Question 1

The county hospital is located at the center of a square whose sides are 520 meters wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are (0, 0), to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

Answer:

By describing the problem, $X, Y \sim Uniform(-260, 260)$. Since X and Y are independent. So:

$$f_{XY}(x, y) = \frac{1}{520^2}\,\forall x, y \in [-260, 260]$$

So:

$$\begin{aligned}
E[|X| + |Y|] &= \iint_{-\infty}^{\infty} (|X| + |Y|) f_{XY}(x, y)\,dxdy \\
&= \frac{1}{520^2}\int_{-260}^{260} dy \int_{-260}^{260} (|X| + |Y|)\,dx \\
&= \frac{1}{520^2}\int_{-260}^{260} 260^2 + 520|Y|\,dy \\
&= \frac{1}{520^2}(520 \times 260^2 + 520 \times 260^2) \\
&= 260
\end{aligned}$$

2 Question 2

Suppose that A and B each randomly and independently choose 4 of 11 objects. Find the expected number of objects:

1. Chosen by both A and B
2. Not chosen by either A or B
3. Chosen by exactly one of A and B

Answer:

1. Let X_i be the Characteristic function of the i^{th} object.

$$X_i = \begin{cases} 1 & \text{If the object is chosen by both } A \text{ and } B \\ 0 & \text{otherwise} \end{cases}$$

So:

$$\begin{aligned}
EX_{\text{total}} &= \sum_{i=1}^{11} EX_i \\
\text{By symmetric: } &= 11EX_i
\end{aligned}$$

Since $EX_i = \frac{4}{11} \times \frac{4}{11} = \frac{16}{121}$.
So $EX_{\text{total}} = \frac{16}{11}$

2. Similar to the idea in 2.1. Let Y_i be the Characteristic function of the i^{th} object.

$$Y_i = \begin{cases} 1 & \text{If the object not chosen by neither } A \text{ nor } B \\ 0 & \text{otherwise} \end{cases}$$

So:

$$\begin{aligned} EY_{\text{total}} &= \sum_{i=1}^{11} EY_i \\ \text{By symmetric: } &= 11EY_i \end{aligned}$$

Since $EY_i = \frac{7}{11} \times \frac{7}{11} = \frac{49}{121}$.
So $EX_{\text{total}} = \frac{49}{11}$

3. Let Z_i be the Characteristic function of the i^{th} object.

$$Z_i = \begin{cases} 1 & \text{If the object not chosen by neither } A \text{ nor } B \\ 0 & \text{otherwise} \end{cases}$$

So:

$$\begin{aligned} EZ_{\text{total}} &= \sum_{i=1}^{11} EZ_i \\ \text{By symmetric: } &= 11EZ_i \end{aligned}$$

Since for each ball, it only has X, Y, Z three situations, so we have:

$$X_i + Y_i + Z_i = 1 \,\forall\, i \in [1, 11]$$

So:

$$\begin{aligned} EZ_i &= \frac{56}{121} \\ EZ_{\text{total}} &= \frac{56}{11} \end{aligned}$$

3 Question 3

A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find

1. the expected number of urns that are empty;

2. the probability that none of the urns is empty.

Answer:

1. Let X_i be the characteristic function of the i^{th} urn:

$$X_i = \begin{cases} 1 & \text{if the urn is empty} \\ 0 & \text{if the urn not empty} \end{cases}$$

$$EX_i = \prod_{k=i}^{n} (1 - \frac{1}{k})$$

Then:

$$\begin{aligned}
EX_{total} &= \sum_{i=1}^{n} EX_i \\
&= \sum_{i=1}^{n} (\prod_{k=i}^{n} (1 - \frac{1}{k})) \\
&= n - \sum_{i=1}^{n} \frac{i-1}{n} \\
&= \frac{n-1}{2}
\end{aligned}$$

2. We let the probability we want be P_2, then we have:

$$P_2 = P(X_i = 1 \,\forall\, i \in [1, n])$$

It is easy to induce that the i^{th} ball must be in i^{th} urns. So:

$$P_2 = \frac{1}{\prod_{k=1}^{n} k} = \frac{1}{n!}$$

4 Question 4

Suppose that X is a continuous random variable with density function f. Show that $E(|X - a|)$ is minimized when $a \in \mathbb{R}$ satisfies $F(a) = \frac{1}{2}$.

Answer:

By definition, we know that:

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

Let $g(a) := E(|X - a|)$. Then we have:

$$\begin{aligned}
g(a) &= \int_{-\infty}^{\infty} |x - a| f(x)\,dx \\
&= \int_{-\infty}^{a} (a - x) f(x)\,dx + \int_{a}^{\infty} (x - a) f(x)\,dx \\
&= aF(a)\,dx - \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx - a \int_{a}^{\infty} f(x)\,dx \\
&= aF(a)\,dx - \int_{-\infty}^{a} x f(x)\,dx + (EX - \int_{-\infty}^{a} x f(x)\,dx) - a(1 - \int_{-\infty}^{a} f(x)\,dx) \\
&= 2aF(a) - 2\int_{-\infty}^{a} x f(x)\,dx + EX - a \\
&= 2aF(a) - 2((xF(x))|_{-\infty}^{a} - \int_{-\infty}^{a} F(x)\,dx) + EX - a \\
&= EX + 2\int_{-\infty}^{a} F(x)\,dx - a
\end{aligned}$$

Now we take derivative of $g(a)$:

$$g'(a) = 2F(a) - 1$$

Since $F(a)$ is an increasing funciton:

$$g(a) = \begin{cases} Decrease & F(a) < \frac{1}{2} \\ Increase & F(a) > \frac{1}{2} \end{cases}$$

So $E(|X-a|) = g(x)$ reaches its minimum when $F(a) = \frac{1}{2}$

5 Question 5

Twenty individuals consisting of 10 married couples are to be seated at 8 different tables, 2 of which have 4 seats and 6 of which have 2 seats. If the seating is done "at random," what is the expected number of married couples that are seated at the same table?

Answer:

Let X_i be the characteristic function of the i^{th} couple. Then:

$$X_i = \begin{cases} 1 & \text{the couple sits together} \\ 0 & \text{the couple not sits together} \end{cases}$$

Do a little classification discussion on the first person's table's type.

$$\begin{aligned} EX_i &= \frac{2}{5} \times \frac{3}{19} + \frac{3}{5} \times \frac{1}{19} \\ &= \frac{9}{95} \end{aligned}$$

So:

$$EX_{\text{total}} = \sum_{k=1}^{10} EX_k = \frac{18}{19}$$

6 Question 6

Individuals 1 through n, $n > 1$, are to be recruited into a firm in the following manner: Individual 1 starts the firm and recruits individual 2. Individuals 1 and 2 will then compete to recruit individual 3. Once individual 3 is recruited, individuals 1, 2, and 3 will compete to recruit individual 4, and so on. Suppose that when individuals 1, 2, . . . , i compete to recruit individual i + 1, each of them is equally likely to be the successful recruiter.

1. Find the expected number of the individuals 1, . . . , n who did not recruit anyone else.
2. Derive an expression for the variance of the number of individuals who did not recruit anyone else and evaluate it for $n = 5$.

Answer:

Let the X_i be the characteristic function of i^{th} person.

$$X_i = \begin{cases} 0 & \text{If the person has successfully recruit someone} \\ 1 & \text{otherwise} \end{cases}$$

1. We know for i^{th} person, he can only recruit k^{th} where $i+1 \leq k \leq n$. So:

$$\begin{aligned} EX_i &= 1 - \prod_{k=i}^{n-1}(1 - \frac{1}{k}) \\ &= 1 - \frac{i-1}{n-1} \end{aligned}$$

So:

$$\begin{aligned} EX_{\text{total}} &= \sum_{k=1}^{n} EX_k \\ &= \sum_{k=1}^{n} (1 - \frac{k-1}{n-1}) \\ &= n - \frac{n}{2} \\ &= \frac{n}{2} \end{aligned}$$

2. From the previous discussion, we know that:

$$X_i \sim Bernoulli(\frac{n-i}{n-1})$$

So:

$$Var(X_i) = \frac{(i-1)(n-i)}{(n-1)^2}$$

Obviously, X_i and X_j are not independent.

$$\begin{aligned} Var(X_{\text{total}}) &= E((X_{\text{total}})^2) - (E(X_{\text{total}}))^2 \\ &= E((X_{\text{total}})^2) - \frac{n^2}{4} \end{aligned}$$

Now we calculate $E[(X_{\text{total}})^2]$

$$\begin{aligned} E((X_{\text{total}})^2) &= \sum_{k=1}^{n} X_k^2 + 2 \sum_{1 \le i < j \le n} E[X_i X_j] \\ &= \frac{n}{2} + 2 \sum_{1 \le i < j \le n} (\prod_{k=j}^{n-1} (1 - \frac{2}{k}) \times \prod_{m=i}^{j-1} (1 - \frac{1}{m})) \\ &= \frac{n}{2} + 2 \sum_{i<j} \frac{i-1}{n-1} \times \frac{j-2}{n-2} \\ &= \frac{n}{2} + 2 \sum_{j=2}^{n} \sum_{i=1}^{j-1} \frac{i-1}{n-1} \times \frac{j-2}{n-2} \\ &= \frac{n}{2} + \sum_{j=2}^{n} \frac{(j-2)^2(j-1)}{(n-1)(n-2)} \\ &= \frac{n}{2} + \frac{n^2}{4} - \frac{n}{6} \\ &= \frac{n}{12} + \frac{n^2}{4} \end{aligned}$$

So:

$$\begin{aligned} Var(X_{\text{total}}) &= E((X_{\text{total}})^2) - \frac{n^2}{4} \\ &= \frac{n}{12} \end{aligned}$$

When $n = 5$:

$$Var(X_{\text{total}}) = \frac{5}{12}.$$