Homework 12 (Due December 19)

Grade Distribution (Total=6+6+6+6+6+6+6=42).

1. If X_n converges in probability to X and X_n converges in probability to Y both hold, then $P(X = Y) = 1$.

2. If X_n converges in probability to X and Y_n converges in probability to Y both hold, then $X_n + Y_n$ converges in probability to $X + Y$.

3. Show that X_n converges in probability to some constant $c \in \mathbb{R}$ if and only if

$$\lim_{n\to\infty} E\frac{(X_n - c)^2}{1 + (X_n - c)^2} = 0.$$

   Hint: Consider the function $f(x) = \frac{(x-c)^2}{1+(x-c)^2}$ and bound $f(X_n)$ for $|X_n - c| > \varepsilon$ and $|X_n - c| < \varepsilon$.

4. If X_n converges in probability to X, then X_n^2 converges in probability to X^2.

5. If X_n converges in probability to 1, then X_n^{-1} converges in probability to 1.

6. If X_n converges in probability to X, then $g(X_n)$ converges in probability to $g(X)$ whenever g is a continuous function on $\mathbb{R}$.

7. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma_i^2$. Show that if $n^{-2}\sum_{i=1}^n \sigma_i^2 \to 0$, then $\frac{X_1+\cdots+X_n}{n}$ converges in probability to μ. [Hint: Use Chebyshev's inequality.]