MA215 Probability Homework-6

HONGLI YE 12311501

November 7^{th} 2024

Contents

1 Question 1

If the joint probability mass function of X, Y is given by:

$X \backslash Y$	-1	0	1
-1	a	0	0.2
0	0.1	b	0.1
1	0	0.2	c

Table 1: Question 1

and $P(X \times Y \neq 0) = 0.4, P(X \leq 0|Y \leq 0) = \frac{2}{3}$.

a) Find the values of a, b, c.

b) Compute the marginal probability mass function of X and Y .

c) Find the probability mass function of $X + Y$.

Answer:

a) Obviously, X and Y are two discrete random variables. By the axiom of probability, we have:

$$\Sigma_{(x,y)\in S}(P(X = x, Y = y)) = 1$$

Observing the table, we could get the following equation by counting one by one:

$$0.1 + 0.1 + 0.2 + 0.2 + a + b + c = 1 \tag{1}$$

ANd for another two conditions: $P(X \times Y \neq 0) = 0.4, P(X \leq 0|Y \leq 0) = \frac{2}{3}$.

$$a + 0.2 + c + 0 = 0.4 \tag{2}$$
$$\frac{a + b + 0.1}{a + b + 0.1 + 0.2} = \frac{2}{3} \tag{3}$$

So together the 3 equations, we could get: $\begin{cases} a = 0.1 \\ b = 0.2 \\ c = 0.1 \end{cases}$

b) By definition, we could simply calculate:

$$p_X(x) = \begin{cases} 0.3 & x = -1 \\ 0.4 & x = 0 \\ 0.3 & x = 1 \end{cases}$$

$$p_Y(y) = \begin{cases} 0.2 & y = -1 \\ 0.4 & y = 0 \\ 0.4 & x = 1 \end{cases}$$

c) $X + Y \in \{-2, -1, 0, 1, 2\}$

$$P_{X+Y}(X + Y = m) = \begin{cases} 0.1 & m = -2 \\ 0.1 & m = -1 \\ 0.4 & m = 0 \\ 0.2 & m = 1 \\ 0.2 & m = 2 \end{cases}$$

2 Question 2

The joint probability density function of (X, Y) is given by:

$$f(x,y) = \begin{cases} cx^4y & x^4 < y < 1 \\ 0 & otherwise \end{cases}$$

where $c > 0$ is some constant.

a) Find the marginal probability density functions f_X and f_Y.

b) Calculate EX and EY .

Answer:
First, by the property of probability dense function, we have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$$

By simplifying the equation, we could get:

$$\begin{aligned} \int_{-1}^{1} dx \int_{x^4}^{1} cx^4y\,dy &= 1 \\ \int_{-1}^{1} \frac{1}{2}c(x^4 - x^{12})\,dx &= 1 \\ \frac{c}{5} - \frac{c}{13} &= 1 \\ c &= \frac{65}{8} \end{aligned}$$

a) By definition:

$$f_X(m) = \int_{-\infty}^{\infty} f(m,y)\,dy$$

$$f_X(m) = \begin{cases} 0 & \text{otherwise} \\ \frac{65}{16}m^4 - \frac{65}{16}m^{12} & -1 < m < 1 \end{cases}$$

Similarly, we could get $f_Y(n)$

$$f_Y(n) = \int_{-\infty}^{\infty} f(x,n)\,dx$$

$$f_X(n) = \begin{cases} 0 & \text{otherwise} \\ \frac{13}{4}y^{\frac{9}{4}} & 0 < n < 1 \end{cases}$$

b) By the definition of expectation of continuous random variable:

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} xf_X(x)\,dx = \int_{-1}^{1} \frac{65}{16}x(x^4 - x^{12})\,dx \\ EY &= \int_{-\infty}^{\infty} yf_Y(y)\,dy = \int_{0}^{1} y\frac{13}{4}y^{\frac{9}{4}}\,dy \end{aligned}$$

Since $\frac{65}{16}x(x^4 - x^{12})$ is an odd function, then we know:

$$EX = 0$$

And do some simple calculation on EY, we could get:

$$\begin{aligned} EY &= \int_0^1 y\frac{13}{4}y^{\frac{9}{4}}\,dy \\ &= \frac{13}{4}\int_0^1 y^{\frac{13}{4}}\,dy \\ &= \frac{13}{4}\times\frac{4}{17}[y^{\frac{17}{4}}]|_0^1 \\ &= \frac{13}{17} \end{aligned}$$

3 Question 3

You spend the night in a teepee shaped as a right circular cone whose base is a disk of radius r centered at the origin and the height at the apex is h. A fly is buzzing around the teepee at night. At some time point the fly dies in mid-flight and falls directly on the floor of the teepee at a random location (X, Y). Assume that the position of the fly at the moment of its death was uniformly random in the volume of the teepee.

a) Derive the joint probability density function $f_{XY}(x, y)$ of the point (X, Y) where you find the dead fly in the morning.

b) Let Z be the height from which the dead fly fell to the floor. Find the probability density function $f_Z(z)$ of Z.

Answer:

a) Let $R = \sqrt{x^2 + y^2}$, and $S = \{(x, y)|x^2 + y^2 \le r^2\}$by the symmetry properties. We know that for those points with the same R, we have the same probability.
Since the position of the fly at the moment of its death was uniformly random in the volume of the teepee. We know that:

$$P_{XYZ}(x,y,z) = \begin{cases} \frac{3}{\pi r^2 h} & 0 < \frac{zr}{r-\sqrt{x^2+y^2}} \le h \\ 0 & \text{otherwise} \end{cases}$$

Then we have:

$$\begin{aligned} f_{XY}(x,y) &= \int_{-\infty}^{\infty} P_{XYZ}(x,y,z)\,dz \\ &= \int_0^{\frac{h(r-\sqrt{x^2+y^2})}{r}} \frac{3}{\pi r^2 h}\,dz \\ &= \frac{h(r-\sqrt{x^2+y^2})}{r} \times \frac{3}{\pi r^2 h} \\ &= \frac{3(r-\sqrt{x^2+y^2})}{\pi r^3} \end{aligned}$$

b) By observing the structure of the problem, we could directly give the formula:

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1 - (\frac{h-z}{h})^3 & 0 < z < h \\ 1 & h \le z \end{cases}$$

Differentiate to obtain:

$$f_Z(z) = \begin{cases} -\frac{3}{h}(\frac{h-z}{h})^2 & 0 < z < h \\ 0 & \text{otherwise} \end{cases}$$

4 Question 4

If X is exponential with rate λ, find:

$$P([X] = n, X - [X] \leq x)$$

for $n \in \mathbb{Z}$ with $n \geq 0$ and $x \in \mathbb{R}$ with $x > 0$. Here $[x]$ is defined as the largest integer less than or equal to x.

Answer:

By the definition of $[x]$, we know that:

$$0 \leq x - [x] < 1$$

Since $X \sim Exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Let $p(x, n) = P([X] = n, X - [X] \leq x)$ Then we have:

$$\begin{aligned} p(x, n) &= P(n \leq X < n + 1, n \leq X < n + x) \\ &= P(n \leq X < min(n + 1, n + x)) \\ &= \begin{cases} 0 & x \leq 0 \\ P(n \leq X < n + x) & 0 < x \leq 1 \\ P(n \leq X < n + 1) & 1 < x \end{cases} \end{aligned}$$

Then we do some simple calculations:

$$p(x, n) = \begin{cases} 0 & x \leq 0 \\ e^{-\lambda n}(1 - e^{-\lambda x}) & 0 < x \leq 1 \\ e^{-\lambda n}(1 - e^{-\lambda}) & 1 < x \end{cases}$$

5 Question 5

If X and Y are independent exponential random variables with parameters λ_1, λ_2, express the density function of:

1. $Z = X \backslash Y$
2. $Z = XY$

Answer: By definition, we know:

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & x \geq 0 \\ 0 & x < 0 \end{cases} \text{ and } f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

1. Since X, Y, Z are all continuous random variables, we shall calculate $F_Z(z)$ first.

$$\begin{aligned}
F_Z(z) &= P(X \leq Yz) \\
&= \int_0^{\infty} dy \int_0^{zy} f_X(x) f_Y(y)\, dx \\
&= \int_0^{\infty} dy \int_0^{zy} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}\, dx \\
&= \lambda_2 \int_0^{\infty} dy \int_0^{zy} e^{-(\lambda_1 x + \lambda_2 y)}\, d(\lambda_1 x + \lambda_2 y) \\
&= \lambda_2 \int_0^{\infty} e^{-\lambda_2 y} - e^{-(\lambda_1 zy + \lambda_2 y)}\, dy \\
&= \frac{\lambda_1 z}{\lambda_1 z + \lambda_2}
\end{aligned}$$

Differentiate to obtain:

$$f_Z(z) = \begin{cases} 0 & z \leq 0 \\ \frac{\lambda_1 \lambda_2}{(\lambda_1 z + \lambda_2)^2} & z > 0 \end{cases}$$

2. Since X, Y, Z are all continuous random variables, we shall calculate $F_Z(z)$ first.

$$\begin{aligned}
F_Z(z) &= P(X \leq z/Y) \\
&= \int_0^{\infty} dy \int_0^{z/y} f_X(x) f_Y(y)\, dx \\
&= \int_0^{\infty} dy \int_0^{z/y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}\, dx \\
&= \lambda_2 \int_0^{\infty} dy \int_0^{z/y} e^{-(\lambda_1 x + \lambda_2 y)}\, d(\lambda_1 x + \lambda_2 y) \\
&= \lambda_2 \int_0^{\infty} e^{-\lambda_2 y} - e^{-(\lambda_1 \frac{z}{y} + \lambda_2 y)}\, dy \\
&= 1 - \lambda_2 \int_0^{\infty} e^{-(\lambda_1 \frac{z}{y} + \lambda_2 y)}\, dy
\end{aligned}$$

Differentiate to obtain:

$$f_Z(z) = \begin{cases} 0 & z \leq 0 \\ function & z > 0 \end{cases}$$

6 Question 6

Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over (0, L/2) and Y is uniformly distributed over (L/2, L).]

1. Let $Z = |X - Y|$ be the distance between the two points. Find the probability that Z is greater than $L/3$.
2. Compute EZ.

Answer:

1. First, we need to calculate the $f_X(x)$ and $f_Y(y)$, by definition, it is obvious to know that $X \sim Uniform(0, L/2)$

and $Y \sim Uniform(L/2, L)$, so:

$$f_X(x) = \begin{cases} \frac{2}{L} & 0 < x < L/2 \\ 0 & \text{otherwise} \end{cases} \text{ and } f_Y(y) = \begin{cases} \frac{2}{L} & L/2 < x < L \\ 0 & \text{otherwise} \end{cases}$$

Now we calculate the cdf of Z

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(Y - X \leq z) \\ &= P(Y \leq X + z) \\ &= \int_0^{\infty} dx \int_{L/2}^{x+z} f_X(x) f_Y(y) dy \\ &= \int_0^{L/2} \frac{2}{L} dx \int_{L/2}^{x+z} f_Y(y) dy \\ &= \frac{2}{L} \int_0^{L/2} g(x) dx \end{aligned}$$

Then we need to do a classified discussion on whether $x + z$ is bigger than L or not.

$$g(x) = \int_{L/2}^{x+z} f_Y(y) dy = \begin{cases} 0 & x \leq L/2 - z \\ (x + z - L/2)(\frac{2}{L}) & L/2 - z < x \leq L - z \\ 1 & L - z < x \end{cases}$$

So:

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ \frac{2z^2}{L^2} & 0 < z \leq L/2 \\ \frac{-2z^2+4zL-L^2}{L^2} & L/2 < z \leq L \\ 1 & L < z \end{cases}$$

Differentiate to obtain it:

$$f_Z(z) = \begin{cases} \frac{4z}{L^2} & 0 < z \leq L/2 \\ \frac{-4z+4L}{L^2} & L/2 < z \leq L \\ 0 & \text{otherwise} \end{cases}$$

So:

$$P(Z \geq \frac{L}{3}) = 1 - P(Z \leq \frac{L}{3}) = 1 - F_Z(\frac{L}{3}) = \frac{7}{9}$$

2. By definition, we can get the formula of EZ.

$$\begin{aligned} EZ &= \int_{-\infty}^{\infty} z f_Z(z) \, dz \\ &= \int_0^{L/2} z \frac{4z}{L^2} \, dz + \int_{L/2}^{L} z \frac{4L - 4z}{L^2} \, dz \\ &= \frac{L}{6} + \frac{3L}{2} - \frac{7L}{6} \\ &= \frac{L}{2} \end{aligned}$$

7 Question 7

Let X, Y have a bivariate normal distribution $N(\mu_x, \mu_y; \sigma_x^2, \sigma_y^2; \rho)$

1. Show that:

$$E[(X-\mu_x)(Y-\mu_y)] = \rho\sigma_x\sigma_y$$

2. Let:

$$G(\lambda) := \{(x,y) \in \mathbb{R}^2 \,:\, (\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \le \lambda^2\}$$

   Calculate $P((X,Y) \in G(\lambda))$

Answer:

1. By definition:

$$\begin{aligned}
E[(X-\mu_x)(Y-\mu_y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)f(x,y)\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)(e^{\{-\frac{1}{2(1-p^2)}[(\frac{x-\mu_x}{\sigma_x})^2+(\frac{y-\mu_y}{\sigma_y})^2-2p\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}]\}})\,dxdy \\
\text{We do some substitutions: } u &= \frac{x-\mu_x}{\sigma_x} \text{ and } u = \frac{y-\mu_y}{\sigma_y} \\
&= \iint_{R^2}\frac{\sigma_x\cdot\sigma_y\cdot u\cdot v}{2\pi\sqrt{1-p^2}}\cdot e^{[-\frac{1}{2(1-p^2)}(u^2+v^2-2p\cdot uv)]}\,dudv \\
&= \int_{-\infty}^{+\infty}dv\cdot\int_{-\infty}^{+\infty}\frac{\sigma_x v_y uv}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\cdot[(u-\rho v)^2+(1-\rho^2)v^2]}\,du \\
&= \frac{\sigma_x\sigma_y}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}v\cdot e^{-\frac{v^2}{3}}\,dv\cdot\int_{-\infty}^{+\infty}[(u-\rho v)+\rho v]\cdot e^{[-(\frac{u-\rho v}{\sqrt{2(1-\rho^2)}})^2]}\,du \\
\text{Let: } t &= \frac{u-pv}{2\sqrt{1-\rho^2}} \\
&= \frac{\sigma_x\sigma_y}{2\pi\sqrt{1-\rho^2}}\cdot\int_{-\infty}^{+\infty}v\cdot e^{-\frac{v^2}{2}}\,dv\cdot\int_{-\infty}^{+\infty}(\sqrt{2(1-\rho^2)}\cdot t+\rho v)\cdot e^{-t^2}\cdot\sqrt{2(1-\rho^2)}dt
\end{aligned}$$

   Since te^{-t^2} is a odd function, we get:

$$\begin{aligned}
&= \frac{\sigma_x\cdot\sigma_y\cdot\rho}{\sqrt{2\pi}}\cdot\int_{-\infty}^{+\infty}v^2\cdot e^{-\frac{v^2}{2}}\,dv \\
\text{Let } k &:= \frac{v^2}{2} \\
&= \sigma_x\cdot\sigma_y\cdot\rho\cdot\frac{2}{\sqrt{\pi}}\cdot\Gamma(\frac{3}{2}) \\
&= \sigma_x\cdot\sigma_y\cdot\rho
\end{aligned}$$

2. By definition:

$$P((X,Y)\in G(\lambda)) = \iint_{G(\lambda)} f(x,y)\,dxdy$$

$u := \dfrac{x-\mu_x}{\sigma_x} : v := \dfrac{y-\mu_y}{\sigma_y}$

$\Rightarrow G(\lambda) = \{(\sigma_x \cdot u + \mu_x, \sigma_y \cdot u + \mu_y) \in \mathbb{R}^2 : u^2 + v^2 - 2\rho \cdot u \cdot v \leq \lambda^2\}.$

i.e. $\therefore (x,y) \in G(\lambda) \Leftrightarrow (u - \rho \cdot v)^2 + (1-\rho^2)v^2 \leq \lambda^2$

$\Rightarrow P((X,Y) \in G(\lambda)) = \displaystyle\int\int 1_{(X,Y)\in G(\lambda)} f(x,y)dxdy$

Let $u - \rho \cdot v = r \cdot \cos\theta, v \cdot \sqrt{1-\rho^2} = r \cdot \sin\theta$

$\Rightarrow P((X,Y) \in G(\lambda)) = \dfrac{1}{2\pi} \cdot \dfrac{1}{\sqrt{1-\rho^2}} \cdot \displaystyle\int_0^\lambda \int_0^{2\pi} \exp[-\dfrac{r^2}{2(1-\rho^2)}] \cdot \dfrac{r}{\sqrt{1-\rho^2}} d\theta \cdot dr$

$= \displaystyle\int_0^\lambda \exp(-\dfrac{r^2}{2(1-\rho^2)}) \cdot \dfrac{r}{\sqrt{1-\rho^2}} \cdot d(\dfrac{r}{\sqrt{1-\rho^2}})$

$= \displaystyle\int_0^{\lambda/\sqrt{1-\rho^2}} e^{-\frac{t^2}{2}} \cdot t \cdot dt = 1 - e^{-\frac{1}{2}\cdot\frac{\lambda^2}{1-\rho^2}}$

$hence : P((X,Y) \in G(\lambda)) = 1 - exp(-\dfrac{1}{2} \cdot \dfrac{\lambda^2}{1-\rho^2})$