MA215 Homework 13

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1 Question 1

Nate is a competitive eater specializing in eating hot dogs. From past experience we know that it takes him on average 15 seconds to consume one hot dog, with a standard deviation of 4 seconds. In this year's hot dog eating contest he hopes to consume 64 hot dogs in just 15 minutes. Use the CLT to approximate the probability that he achieves this feat of skill.

Answer:

Let N be the random variable that represents the time Nate need to consume one hot dog.

$$\mu_N = 15 \text{ and } \sigma_N = 4$$

Now we define a sequence of i.i.d. random variable: $X_1, X_2, , \ldots, X_{64}$ and all of them is equivalent with N. Let P_a represents the probability we are solving.

$$\begin{aligned}
P_a &= P(\sum_{i=1}^{64} X_i \leq 15 \times 60) \\
&= P(\sum_{i=1}^{64} X_i \leq 900) \\
&= P(\frac{\sum_{i=1}^{64} X_i - 64\mu_N}{\sqrt{64\sigma_N^2}} \leq -\frac{15}{8})
\end{aligned}$$

We use Central Limit Theorem to estimate this:

$$P_a = P(Z \leq -1.875) = 0.0304$$

2 Question 2

A four year old is going to spin around with his arms stretched out 100 times. From past experience, his father knows it takes approximately 1/2 second to perform one full spin, with a standard deviation of 1/3 second. Consider the probability that it will take this child over 55 seconds to complete spinning. Give an upper bound with Chebyshev's inequality and an approximation with the CLT.

Answer:

Let T be the random variable that represents the time Nate need to consume one hot dog.

$$\mu_T = 0.5 \text{ and } \sigma_N = \frac{1}{3}$$

Now we define a sequence of i.i.d. random variable: $X_1, X_2, , \ldots, X_{100}$ and all of them is equivalent with T. Let P_a represents the probability we are solving.

$$\begin{aligned}
P_a &= P(\sum_{i=1}^{100} X_i \geq 55) \\
&= 1 - P(\sum_{i=1}^{100} X_i < 55) \\
&= 1 - P(\frac{\sum_{i=1}^{100} X_i - 100\mu_T}{\sqrt{100\sigma_T^2}} < \frac{3}{2})
\end{aligned}$$

We use Central Limit Theorem to estimate this:

$$P_a = 1 - P(Z < 1.5) = 0.0668$$

Let $X = \sum_{i=1}^{100} X_i$.

$$\mu_X = 50 \text{ and } \sigma_X = \frac{10}{3}$$

We use Chebyshev's inequality to estimate this:

$$\begin{aligned} P_a &= P(X \geq 55) \\ &\leq P(\mid X - \mu_X \mid \geq 5) \\ &\leq \frac{\sigma_X^2}{5^2} \\ &= \frac{4}{9} \end{aligned}$$

3 Question 3

Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

(a) Approximate the probability that the average test score in the class of size 25 exceeds 80.

(b) Repeat part (a) for the class of size 64.

(c) Approximate the probability that the average test score in the larger class exceeds that of the other class by over 2.2 points.

(d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by over 2.2 points.

Answer:

Let S be the random variable of the score the student will get in the exam.

$$\mu_S = 74 \text{ and } \sigma_T = 14$$

Let $X_1, X_2, \ldots, X_{25}$ be the sequence of i.i.d. random variables. All are same to S
Let $Y_1, Y_2, \ldots, Y_{64}$ be the sequence of i.i.d. random variables. All are same to S
Let $C_1 = \sum_{i=1}^{25} X_i$ and $C_2 = \sum_{i=1}^{64} Y_i$.

a) We use Central Limit Theorem to estimate this probability.

$$\begin{aligned} P_a &= P(C_1 \geq 2000) \\ &= 1 - P(\frac{C_1 - 2000}{\sqrt{25\sigma_T^2}} < \frac{15}{7}) \\ &= 1 - P(X < \frac{15}{7}) \\ &= 0.0162 \end{aligned}$$

b) We also use Central Limit Theorem to estimate this probability.

$$\begin{aligned} P_b &= P(C_2 \geq 64 \times 80) \\ &= 1 - P(\frac{C_2 - 5120}{\sqrt{64\sigma_T^2}} < \frac{15}{7}) \\ &= 1 - P(X < \frac{24}{7}) \\ &= 0.0003 \end{aligned}$$

c) We also use Central Limit Theorem to estimate this probability. Here we assume 25 and 64 is large enough to assume it be a normal distribution random variable.

$$C_1 \sim N(\mu_S, \frac{\sigma_S^2}{25}) \text{ and } C_2 \sim N(\mu_S, \frac{\sigma_S^2}{64})$$

So:

$$C_1 - C_2 \sim N(\mu_S, \frac{89}{1600}\sigma_S^2)$$

So:

$$\begin{aligned} P_c &= P(C_2 - C_1 \geq 2.2) \\ &= 1 - P(Z < 0.67) \\ &= 0.2512 \end{aligned}$$

d) By the symmetric of normal distribution:

$$\begin{aligned} P_d &= P(C_1 - C_2 \geq 2.2) \\ &= 1 - P(Z < 0.67) \\ &= 0.2512 \end{aligned}$$

4 Question 4

Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1?

Answer:

Let W be a random variable that represent the amount of weight (in units of 1000 pounds) that a certain span of a bridge cancwithstand .

$$\mu_W = 400 \text{ and } \sigma_W = 40$$

Let C be the random variable that the weight of a car.

$$\mu_C = 3 \text{ and } \sigma_C = 0.3$$

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables that each is same as C.

$$\begin{aligned} P_4 &= P(\sum_{i=1}^{n} X_i \geq W) \\ &= P(\sum_{i=1}^{n} X_i - W \geq 0) \\ \text{Let } X = \sum_{i=1}^{n} X_i - W \text{ be a new R.V. } X &\sim N(3n - 400, 0.09n + 1600) \\ &= P(X \geq 0) \end{aligned}$$

Since we need:

$$P(X \geq 0) > 0.1$$

So:

$$\frac{400 - 3n}{\sqrt{1600 + 0.09n}} < 1.285 \Rightarrow n \geqslant 117$$

So we need about 117 cars.