MA215 Probability Homework-4

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1 Question 1

Suppose that the probability mass function of X is given by:

X	-3	-1	0	2	3	5
p(m)	0.20	0.08	0.40	0.10	0.02	0.20

Table 1: Question1.1

Find the probability mass function of $Y = X^2$, that is, find $P(Y = m)$

Answer:

$P(Y = m)$ equals to $P(X = \sqrt{m}$ or $-\sqrt{m})$

Y	0	1	4	9	25
p(m)	0.40	0.08	0.10	0.22	0.20

Table 2: Question1.2

2 Question 2

Three fair dice (six-sided) are rolled. Let X denote the maximum of the three numbers on the dice and Y the minimum of the three numbers.

a) Find the probability mass function of X

b) Find the probability mass function of Y

Answer:

1. X is a discrete random variable, whose image = {1,2,3,4,5,6}, and sample space S has $6 \times 6 \times 6 = 216$ possible situations.

$$P(X = 1) = \frac{1}{216}$$

$$P(X = 2) = \frac{1 \times 2 \times 2 + 1 \times 1 \times 2 + 1 \times 1 \times 1}{216} = \frac{7}{216}$$

$$P(X = 3) = \frac{1 \times 3 \times 3 + 2 \times 1 \times 3 + 2 \times 2 \times 1}{216} = \frac{19}{216}$$

$$P(X = 4) = \frac{1 \times 4 \times 4 + 3 \times 1 \times 4 + 3 \times 3 \times 1}{216} = \frac{37}{216}$$

$$P(X = 5) = \frac{1 \times 5 \times 5 + 4 \times 1 \times 5 + 4 \times 4 \times 1}{216} = \frac{61}{216}$$

$$P(X = 6) = 1 - \frac{5 \times 5 \times 5}{216} = \frac{91}{216}$$

X	1	2	3	4	5	6
p(m)	$\frac{1}{216}$	$\frac{7}{216}$	$\frac{19}{216}$	$\frac{37}{216}$	$\frac{61}{216}$	$\frac{91}{216}$

Table 3: Question2.1

2. Y is a discrete random variable, whose image = {1,2,3,4,5,6}, and sample space S has $6 \times 6 \times 6 = 216$ possible situations.

$$P(Y = 1) = 1 - \frac{5 \times 5 \times 5}{216} = \frac{91}{216}$$

$$P(Y = 2) = 1 - P(Y = 1) - \frac{4 \times 4 \times 4}{216} = \frac{61}{216}$$

$$P(Y=3) = 1 - P(Y=1) - P(Y=2) - \frac{3\times 3\times 3}{216} = \frac{37}{216}$$

$$P(Y=4) = 1 - P(Y=1) - P(Y=2) - P(Y=3) - \frac{2\times 2\times 2}{216} = \frac{19}{216}$$

$$P(Y=5) = \frac{1\times 2\times 2 + 1\times 1\times 2 + 1\times 1\times 1}{216} = \frac{7}{216}$$

$$P(Y=6) = \frac{1}{216}$$

Y	1	2	3	4	5	6
p(m)	$\frac{91}{216}$	$\frac{61}{216}$	$\frac{37}{216}$	$\frac{19}{216}$	$\frac{7}{216}$	$\frac{1}{216}$

Table 4: Question2.1

3 Question 3

We choose a number from the set $\{10, 11, 12, \dots, 99\}$ uniformly at random.

a) Let X be the first digit and Y the second digit of the chosen number. Find the probability mass functions of X and Y. Show that for any $1 \le i \le 9$ and $0 \le j \le 9$,

$$P(X=i, Y=j) = P(X=i) \times P(Y=j)$$

b) Let X be the first digit of the chosen number and Z the sum of the two digits. Find the probability mass functions of X and Y. Show that there exist some $1 \le n \le 9$ and $0 \le m \le 18$,

$$P(X=n, Z=m) \ne P(X=n) \times P(Z=m)$$

Answer:

Let $S = \{10, 11, 12, \dots, 99\}$, be the sample space. Obviously, $|S| = 90$

a) Let us calculate LHS and RHS.

$$LHS = P(X=i, Y=j) = \frac{1}{90}$$

$$P(X=i) = \frac{1}{9} \text{ and } P(Y=j) = \frac{1}{10}$$

So:

$$RHS = P(X=i) \times P(Y=j) = \frac{1}{90}$$

So:

$$LHS = RHS$$

b) To prove this, we only need to give a negative example.
Let $n = 1$,$m = 18$

$$P(X=1, Z=18) = 0$$

$$P(X=1) \times P(Z=18) = \frac{1}{9} \times \frac{1}{90} = \frac{1}{810} \ne 0$$

So there exsit some n, m such that:

$$LHS \ne RHS$$

4 Question 4

Six distinct numbers are randomly distributed to players numbered 1 through 6. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P(X = i)$ for $i \in \{0, 1, 2, 3, 4, 5\}$

Answer:

Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be each player's number. Then for a fixed sequence, the i is fixed.
Moreover, $i =$ the number $(i+1)^{\text{th}}$ player which is bigger than player 1.
Let S be the sample space, then we easily have:

$$|S| = 6! = 720$$

Now we calculate $P(X = i)$:

$$P(X=0) = \frac{4!}{6!} + \frac{4! \times 2}{6!} + \frac{4! \times 3!}{6!} + \frac{4! \times 4}{6!} + \frac{5!}{6!} = \frac{1}{2}$$

$$P(X=1) = \frac{4!}{6!} + \frac{3! \times 2 \times 3}{6!} + \frac{3! \times 3 \times 2}{6!} + \frac{4!}{6!} = \frac{1}{6}$$

$$P(X=2) = \frac{4!}{6!} + \frac{3 \times 2 \times 2 \times 2!}{6!} + \frac{2! \times 3!}{6!} = \frac{1}{12}$$

$$P(X=3) = \frac{4!}{6!} + \frac{3! \times 2!}{6!} = \frac{1}{20}$$

$$P(X=4) = \frac{4!}{6!} = \frac{1}{30}$$

$$P(X=5) = \frac{5!}{6!} = \frac{1}{6}$$

X	0	1	2	3	4	5
p(i)	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{6}$

Table 5: Question4.1

5 Question 5

20 balls are to be distributed among 6 urns, with each ball going into urn i with probability p_i, $\Sigma_{i=1}^{6}(p_i) = 1$. Let X_i denote the number of balls that go into urn i. Assume that events corresponding to the locations of different balls are independent.

a) For each $1 \le i \le 6$, find the probability mass function of X_i

b) For each $1 \le i \le j \le 6$, find the probability mass function of $X_i + X_j$

c) Find $P(X_2 + X_3 + X_4 = 7)$

Answer:

a) For each X_i, it obeys Binomial Distribution, i.e. $X_i \sim Bin(20, p_i)$, so the p.m.f. of X_i is:

$$P(X_i = k) = \binom{20}{k} p_i^k (1 - p_i)^{20-k} \quad \forall k \in [0, 20]$$

b) From answer to question a, we know that:

$$P(X_i = k) = \binom{20}{k} p_i^k (1 - p_i)^{20-k} \quad \forall k \in [0, 20] \text{ and } i \in [1, 6]$$

$X_i + X_j \in [0, 20]$, consider i and j as a new whole group. $p_{ij} = p_i + p_j$, so it is easy to write that:

$$P(X_i + X_j = k)\binom{20}{k}(p_i + p_j)^k(1 - p_i - p_j)^{20-k} \quad \forall k \in [0, 20]$$

c) Similar to the thought in question b, we see three urns as a whole, then:

$$P(X_2 + X_3 + X_4 = 7) = \binom{20}{7}(p_2 + p_3 + p_4)^7(1 - p_2 - p_3 - p_4)^{13} = 77520(p_2 + p_3 + p_4)^7(1 - p_2 - p_3 - p_4)^{13}$$