Homework 6 (Due November 7)

Grade Distribution (Total=10+8+8+4+8+8+10=56).

Please simply answer as much as possible.

1. If the joint probability mass function of X, Y is given by

X\Y	-1	0	1
-1	a	0	0.2
0	0.1	b	0.1
1	0	0.2	c

and $P(X \cdot Y \neq 0) = 0.4$, $P(X \leq 0 | Y \leq 0) = \frac{2}{3}$.

(a) (3 points) Find the values of a, b, c.

(b) (4 points) Compute the marginal probability mass function of X and Y.

(c) (3 points) Find the probability mass function of $X + Y$.

2. The joint probability density function of (X, Y) is given by

$$f(x,y) = \begin{cases} cx^4 y, & \text{if } x^4 < y < 1; \\ 0, & \text{otherwise,} \end{cases} \tag{0.1}$$

where $c > 0$ is some constant.

(a) (4 points) Find the marginal probability density functions f_X and f_Y.

(b) (4 points) Calculate EX and EY.

3. You spend the night in a teepee shaped as a right circular cone whose base is a disk of radius r centered at the origin and the height at the apex is h. A fly is buzzing around the teepee at night. At some time point the fly dies in mid-flight and falls directly on the floor of the teepee at a random location (X, Y). Assume that the position of the fly at the moment of its death was uniformly random in the volume of the teepee.

(a)(4 points) Derive the joint probability density function $f_{XY}(x, y)$ of the point (X, Y) where you find the dead fly in the morning.

(b)(4 points) Let Z be the height from which the dead fly fell to the floor. Find the probability density function $f_Z(z)$ of Z.

4. (4 points) If X is exponential with rate λ, find

$$P([X] = n, X - [X] \leq x)$$

for $n \in \mathbb{Z}$ with $n \geq 0$ and $x \in \mathbb{R}$ with $x > 0$. Here $[x]$ is defined as the largest integer less than or equal to x.

5. If X and Y are independent exponential random variables with parameters λ_1, λ_2, express the density function of

   (a) (4 points) $Z = X/Y$.

   (b) (4 points) $Z = XY$.

6. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.]

   (a) (4 points) Let $Z = |X - Y|$ be the distance between the two points. Find the probability that Z is greater than $L/3$.

   (b) (4 points) Compute EZ.

7. Let X, Y have a bivariate normal distribution $\mathcal{N}(\mu_x, \mu_y; \sigma_x^2, \sigma_y^2; \rho)$.

   (a) (3 points) Show that

$$E[(X - \mu_x)(Y - \mu_y)] = \rho\sigma_x\sigma_y.$$

   (b) (7 points) Let

$$G(\lambda) = \left\{(x, y) \in \mathbb{R}^2 : \left(\frac{x - \mu_x}{\sigma_x}\right)^2 + \left(\frac{y - \mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} \leq \lambda^2\right\}.$$

   Calculate $P((X, Y) \in G(\lambda))$. [Hint: Use Substitutions in Multiple Integrals.]