Homework-3

September 27, 2024

1. Write down all elements of S_4. (E.g. all elements of S_2 is a set $\{(1)(2), (12)\}$)

2. Write down all normal subgroups of D_{20}.

3. Let $H, K \trianglelefteq G$, prove:
(1). $HK = KH$;
(2). $HK \trianglelefteq G$;
(3). If $H \cap K = \{e\}$, then G is isomorphic to a subgroup of $G/H \oplus G/K$.

4. Let $m, n \in \mathbb{Z}$. Prove that $\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ if and only if m and n are coprime.

5. Let H, K be subgroups of G with finite indexes, prove that $H \cap K$ is also a subgroup of G with finite index.

6. Classify all groups with order 4 up to isomorphic.

7. Let G be a group, $g \in G$. Prove that if $o(g) = n$, then $o(g^m) = n/(m, n)$. ((m, n) here means the largest common divisor of the integers m and n).

8. Prove that there is no subgroup of A_4 with order 6.

9. Prove: a finite group G is a dihedral group if and only if G can be generated by two "order 2 generators".

10. Let G be a group and $g \in G$ such that $|g| = rs$ where r and s are coprime. Prove that there exist $a, b \in G$ s.t. $g = ab$, $|a| = r$, $|b| = s$, and a, b are power of g.

11. Let G be a group, suppose there exist $g \in G$ s.t. $(g, k) = 1$ where k is a positive integer. Prove the equation $x^k = g$ has exact one solution in $< g >$.

12. Prove that any finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

13. Let G be a finitely generated abelian group. If any generator of G has finite order, prove that G is a finite group.

14. Prove the subgroup of finite index of a finitely generated group is finitely generated.

15. Let G be a group. For any positive integer k, let $G^k = \{g^k | g \in G\}$. Prove G is cyclic if and only if any subgroup of G is of G^k form as a set.

16. Let p be a prime and n positive integer, $G = Z_{p^n}$. Identify $\mathrm{Aut}(G)$.

17. Classify all abelian groups with order 36 up to isomorphic.

18. Let $G = Z_3 \oplus Z_9 \oplus Z_9 \oplus Z_{243}$. Find the number of:
(1). Cyclic subgroups of order 9;
(2). Non-cyclic subgroups of order 9.

19. Prove S_n can be generated by $n-1$ transpositions $(12), (13), \dots, (1n)$, or be generated by another $n-1$ transpositions $(12), (23), \dots, (n-1n)$.

20. Prove: if $n > 2$ and n is an even number, then A_n can be generated by $(123), (124), \dots, (12n)$, or be generated by $(123), (234), \dots, (n-2n-1n)$.

21. Prove: if $n > 2$ and n is an even number, then A_n can be generated by (123) and $(23 \dots n)$; if $n > 2$ and n is an odd number, then A_n can be generated by (123) and $(12 \dots n)$.

22. Let $\sigma = (12 \dots n)$, prove that $C_{S_n}(\sigma) =< \sigma >$, and prove the conjugacy class of σ in S_n has $(n-1)!$ elements.

23. Let $n > 2$, prove $Z(S_n) = \{(1)\}$.

24. Let $n \geqslant 5$, prove S_n has a unique non-trivial proper normal subgroup, which is A_n.

25. Let G be a group, $N \trianglelefteq G$, $N \cap G' = \{e\}$. Prove $N \leq Z(G)$. (Hint: Let G be a group, $a, b \in G$. $[a, b] = aba^{-1}b^{-1}$ is called the commutator of a, b. The subgroup generated by all commutators of G is called the commutator subgroup or derived subgroup of G, denoted by G').

26. Prove that the subgroups and quotient groups of a soluble group G are all soluble.

27. Let $H, K \trianglelefteq G$, $G/H, G/K$ are all soluble groups. Prove: $G/H \cap K$ is also a soluble group.

28. Let G be a group. If $|\mathrm{Aut}(G)| = 2$ prove that G is abelian.

29. Prove that: if G is a finite group and $|G| > 2$, then G has at least 2 automorphisms.

30. Prove $\mathrm{Aut}(Z_2 \oplus Z_2) \simeq S_3$.

31. Prove that S_n can be generated by (12) and $(12 \dots n)$.