Homework-5

October 18, 2024

1. (1). Let $H < G$. Describe the subgroup of G generated by the complement set of H.
(2). Let G be a finite group with order n, $S \subseteq G$ and $|S| > n/2$. Prove that for all $g \in G$, there exist $a, b \in S$ s.t. $g = ab$.

2. If G is a finite non-trivial group with only one maximal subgroup, show that G is cyclic and $|G| = p^k$ where p is a prime number and k is a positive integer. Then we can deduce that if G is a finite group, and suppose that for any two subgroups H and K either $H \subseteq K$ or $K \subseteq H$. Then G is cyclic of order p^k.

3. Let A be a finite abelian group and let p be a prime number. Let

$$A^p = \{a^p \mid a \in A\} \quad \text{and} \quad A_p = \{x \mid x^p = 1\}$$

(so A^p and A_p are the image and kernel of the p^{th} -power map, respectively).
(a) Prove that $A/A^p \simeq A_p$ [Show that they are both elementary abelian and they have the same order, an elementary abelian group is an abelian group in which all elements other than the identity have the same order. i.e. If G is elementary abelian then $G \simeq (\mathbb{Z}_p)^k$ where p is a prime number and k is a positive integer.].
(b) Prove that the number of subgroups of A of order p equals the number of subgroups of A of index p [Reduce to the case where A is an elementary abelian p-group.]
(c) Let $A = Z_{60} \times Z_{45} \times Z_{12} \times Z_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.

4. Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either (i) $K \leq H$ or (ii) $G = HK$ and $|K : K \cap H| = p$.

5. If G is a group of odd order, prove for any nonidentity element $x \in G$ that x and x^{-1} are not conjugate in G.

6. Let G be a finite group with order n, $a_1, a_2, \ldots a_n$ are arbitary n elements of G (not necessary different). Prove that there exist integers p, q where $1 \leqslant p \leqslant q \leqslant n$, s.t. $a_p a_{p+1} \ldots a_q = 1$.

7. For $\sigma \in \mathrm{Aut}(G)$, if $\forall g \neq 1$, $\sigma(g) \neq g$, then σ is called an automorphism with no fixed point. If for a finite group G there exists an automorphism σ with no fixed point and $\sigma^2 = 1$. Prove that G must be an abelian group with odd order.

8. Let a, b be two elements of group G. If $aba = ba^2b$, $a^3 = 1$ and $b^{2n-1} = 1$ for some integer n, prove $b = 1$.

9. Let $A \leqslant G$, prove $C_G C_G C_G(A) = C_G(A)$.

10. Let G be a finite group with odd order, $\alpha \in \mathrm{Aut}(G)$ and $\alpha^2 = 1$. Let

$$G_1 = \{g \in G | \alpha(g) = g\}, \ G_{-1} = \{g \in G | \alpha(g) = g^{-1}\}$$

Prove: $G = G_1 G_{-1}$ and $G_1 \cap G_{-1} = 1$.