# Homework-6

October 24, 2024

1. Let $G$ be a finite group, $H < G$ and $[G : H] = n > 1$. Prove $G$ contains a non-trivial normal subgroup $K$ where $[G : K] \mid n!$ or $G$ is isomorphic to a subgroup of $S_n$. (Hint: consider the group action $G \curvearrowright X$ where $X$ is the set of all $H$-cosets in $G$).

2. Let $G$ be a finite group and $p$ be the least prime divisor of $|G|$. Prove that if $H < G$ and $[G : H] = p$ then $H \lhd G$. (Hint: use the same trick as 1).

3. Prove that if $|G| = p^2$ where $p$ is a prime. Then $G$ is abelian and $G$ has only two different types up to isomorphic. (Hint: the last lemma we learned from last class is useful, which is "the center of $p$-group is non-trivial". You should prove this lemma first. As the proof you just need to consider the conjugacy action of $G$ on itself, and consider the class equation).

4. Prove that every $p$-group $G$ is solvable. $p$-group means the order of $G$ is a $p^{th}$-power. (Hint: induction on the order of $G$).

5. Let $p, q, r$ be different primes. (Hint: learn Sylow 3rd theorem first and then do these exercises).
(1). Prove that if $|G| = pq$ then $G$ is not a simple group.
(2). Prove that if $|G| = p^2q$ then $G$ is not a simple group.
(3). Prove that if $|G| = pqr$ then $G$ is not a simple group.

6. Prove that if $|G| = 56$ then $G$ is not a simple group.

7. Let $p$ be a prime, $|G| = p^3$. If $G$ is non-abelian. Prove $G' = Z(G)$. (Hint: consider the existance of $x \in G$ where $o(x) = p^2$).

8. Let $G$ be a $p$-group, $N \trianglelefteq G$, $|N| = p$. Prove $N \leqslant Z(G)$. (Hint: consider $G$ acts on $N$ by conjugate and the class equation of this action).

9. Let $G$ be a finite group, $N \trianglelefteq G$, $P$ is a Sylow $p$-subgroup of $N$, prove that $G = NN_G(P)$. (Hint: consider $G$ acts on $Syl_p(G) = \{All\ Sylow\ p-subgroup\ of\ G\}$ by conjugate).

10. Let $G$ be a finite group, and $G$ has a non-trivial cyclic Sylow 2-subgroup. Prove that $G$ has a subgroup which index is 2. (Hint: consider Cayley Theorem, how many odd and even permutation in the isomorphic image of $G$ ?)

11. Lecture note 10, exercise 7

12. Determine $\mathrm{Aut}(\mathbb{Z}_p^d)$. (Hint: regard $\mathbb{Z}_p^d$ as d-dimensional vector space over $\mathbb{F}_p$ ).

13. Determine $\mathrm{Aut}(D_{2n})$. (Hint: this is a hard exercise, one reference is https://ysharifi.wordpress.com/2022/09/14/automorphisms-of-dihedral-groups/)