$\boxed{\text{Theorem}}$ $H, K \triangleleft G$, consider G/H, G/K

If $H < K$, then G/H is homomorphic to G/K $\quad G/H \to G/K$

with the kernel K/H, i.e. $\quad {G/H}\big/{K/H} \simeq G/K$ by the

1st isomorphism theorem.

And this is the 3rd isomorphism theorem.

$\boxed{\text{Definition}}$ Let G, H be groups. Let $X = G \times H$

i.e. $X = \{(g,h) \mid g \in G,\ h \in H\}$

s.t. $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$

Then $G \times H$ is a group called a "direct product" of G and H, or "direct sum" (Just consider finitely many copies).

$\boxed{\text{Definition}}$ cyclic group. $G = \langle g \rangle = \{ g^{\pm i} \mid i \in \mathbb{N} \}$

g is called a generator

E.g. ① $(\mathbb{Z}, +) = \{ (\pm i) \cdot 1 \mid i \in \mathbb{N} \} = \{ \pm i \mid i \in \mathbb{N} \}$

② $(\mathbb{Z}_p, \oplus) = \{ (\pm i) \cdot 1 \mid i \in \mathbb{N} \}$

$= \{ \bar{0}, \bar{1}, \ldots, \overline{p-1} \}$

every element (except of $\bar{0}$) is a generator of

$\boxed{\text{Definition}}$ Dihedral group

$$G = \{ 1, a, \ldots, a^{n-1}, \\ b, ab, \ldots a^{n-1}b \mid |a| = n,\ |b| = 2 \}$$

where $bab = a^{-1}$

Claim. G is a group of order $2n$.

Denoted by D_{2n}. It has Geometric Background.

To prove it is a group you need check.

① $\forall x \in D_{2n}$. x can be written into the form

$a^i b^j$ where $i = 0, \dots, n-1$, $j = 0, 1$

② $\forall\ i_1, i_2, j_1, j_2$

$(a^i b^j)^{-1} \in D_{2n}$ and $(a^{i_1} b^{j_1})(a^{i_2} b^{j_2}) \in D_{2n}$.

$\boxed{\text{Symmetric Group}}$

Let $\Omega = \{1, \dots, n\}$ A 1-1 map from Ω to Ω is called a permutation on Ω.

Let $Sym(\Omega) = \{$ all permutations on $\Omega\}$ $|Sym(\Omega)| = n!$

Define multiplication "$\cdot$" by composition. Then

$(Sym(\Omega), \cdot)$ is a group, called the symmetric group on Ω.

denoted by S_n

Eg: ϕ: $1 \to 2$ denote. $(124)(3)$

$2 \to 4$

$3 \to 3$ or (124)

$4 \to 1$

$\boxed{\text{HW. Write all elements of } S_4 = Sym(\Omega),\ \Omega = \{1, 2, 3, 4\}}$

$\boxed{\text{Lemma}}$ Each permutation of $\Omega = \{1, 2, \cdots n\}$ can be written as a product of disjoint cycles.

$\boxed{\text{Lemma 2}}$ Each permutation can be written into a product of transpositions. (not uniquely)

The number of transpositions is unique (mod 2)

$\boxed{\text{Def}}$ If the number of transpositions appear in a permutation is even, then the permutation is called even permutation.

$\boxed{\text{Lemma}}$ All even permutations in $Sym(\Omega)$ (S_n) form a subgroup of $Sym(\Omega)$ (S_n) called alternating group. denoted by $Alt(\Omega)$ or A_n.

$\boxed{\text{Claim}}$ $A_n \lhd S_n$ and $|S_n| = 2|A_n| \Rightarrow |A_n| = \frac{n!}{2}$