Abstract Algebra
: Lecture 8

Leo

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Lemma 1. *Let G be a abelian group with $|G| = p^e$. Let $g \in G$ which has the largest order. Then $G =< g > \times H$ for some subgroup $H < G$.*

证明. Assume G is not cyclic.

Claim: $\exists h \in G$ s.t. $< h > \cap < g >= 1$ and $o(h) = p$.

Since $G \neq< g >$, then $\exists y \in G- < g >$ s.t. $y^p \in< g >$. Then $\exists k \in \mathbb{Z}_{\geqslant 1}$ s.t. $y^p = g^{kp}$. Let $h = y^{-1}g^k$, then $h \neq 1$ and $h^p = 1$. Thus $o(h) = p$. And if $h \in< g >$ then $y^{-1}g^k \in< g >\Rightarrow y \in< g >$, contradiction.

Let $\bar{G} = G/ < h >$. Then $|\bar{G}| < |G|$, and $o(g) = o(\bar{g})$ is the largest order of element in $\bar{G}$. By indction, Let $\bar{G} =< \bar{g} > \times\bar{H}$. Let H be the full preimage of $\bar{H}$ in G.

Claim: $< g > \cap H = 1$.

Suppose there exists $x \in< g > \cap H$. Then $\bar{x} = \bar{1}$ as $< \bar{g} > \times\bar{H} = \bar{1}$, so $x \in< h >$ and $x \in< g >\Rightarrow< h >\subseteq< g >$, contradiction.

Therefore $G =< g > \times H$.

□

Theorem 2. *(Fundemental Theorem of Finite Abelian Groups) Let G be a finite abelian group of order n. Let $n = p_1^{e_1} \dots p_r^{e_r}$. Then:*

(1). $G = G_1 \times \cdots \times G_r$ where $|G_i| = p_i^{e_i}$.

(2). G is a direct product of cyclic groups.

证明. (1). Let $n = p^e m$ s.t. p is a prime and $(p, m) = 1$. Let $H = \{g^m | g \in G\}$. Then H is a subgroup and every element of H has order p-power. Moreover $|H| = p^e$, and $G = H \times K$ where K has order m. By induction on K we can prove (1).

(2). By lemma 1. □

Solvable Groups:

Definition 3. *Let G be a finite group. For $x, y \in G$, let $[x, y] = x^{-1}y^{-1}xy$, called the commutator of x and y.*

Definition 4. *Let $H = \langle [x,y] | x, y \in G \rangle$. Then H is called the commutator subgroup of G, denoted by G'.*

Proposition 5. $H \lhd G$.

证明. $\forall g \in G$, $[x,y]^g = g^{-1}[x,y]g = [x^g, y^g]$. □

Example 6. *1. Let $G = \langle a, b \rangle = D_{2n}$, where $o(a) = n$, $o(b) = 2$ and $a^b = a^{-1}$. Then $G' = \begin{cases} \langle a \rangle, if\ o(a)\ odd, \\ \langle a^2 \rangle, if\ o(a)\ even. \end{cases}$ and $G/G' = \begin{cases} \langle \bar{a} \rangle = C_2, if\ o(a)\ odd, \\ \langle \bar{a}, \bar{b} \rangle = C_2 \times C_2, if\ o(a)\ even. \end{cases}$*

Example 7. *Let G be abelian. Then $G' = 1$.*

Lemma 8. *G/G' is abelian.*

证明. Let $\bar{x}, \bar{y} \in \bar{G}$, let x, y be preimages of $\bar{x}, \bar{y}$ under $\pi : G \to \bar{G}$ respectively. Then by definition $[x.y] = x^{-1}y^{-1}xy \in G'$, hence $[\bar{x}, \bar{y}] = \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y} = \overline{x^{-1}y^{-1}xy} = \bar{1} \Rightarrow \bar{G}$ is abelian. □

Lemma 9. *For any $H \lhd G$, G/H is abelian $\Leftrightarrow G' \leqslant H$.*

证明. ($\Leftarrow$) If $G' \leqslant H$, then $G/H \simeq \frac{G}{G'}/\frac{H}{G'}$ is abelian.

($\Rightarrow$) Assume G/H is abelian, then foe any $\bar{x}, \bar{y} \in \bar{G} = G/H$, $\bar{x}\bar{y} = \bar{y}\bar{x}$ i.e. $[\bar{x}, \bar{y}] = \bar{1}$, it shows $\forall x, y \in G$, $[x, y] \in H$. Thus $G' \leqslant H$. □

We can come to the conclusion that G' is the smallest subgroup of G s.t. G/G' is abelian.

Definition 10. *$G \rhd G' \rhd G'' \rhd \cdots \rhd G^{(n)} \rhd \ldots$ where $G^{(n)} = (G^{(n-1)})'$. Since G is finite, there exists n s.t. $G^{(n)} = G^{(n+1)} = \ldots$.*

Example 11. *If G is nonabelian simple, then $G = G'$.*

Definition 12. *If $G = G'$, then G is called a perfect group.*

Definition 13. *A finite group G is called a solvable group if $G = G^{(n)}$ for some n. Otherwise, G is called a nonsolvable group.*

Proposition 14. *A group G is solvable iff there exists a subgroup chain:*

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_s = 1$$

s.t. G_{i+1} is normal in G_i and G_i/G_{i+1} is abelian for all i.

证明. ($\Rightarrow$) obviously.

($\Leftarrow$) Assume $G = G_0 \rhd G_1 \rhd \cdots \rhd G_s = 1$ s.t. G_i/G_{i+1} is abelian.

Claim: $G^{(i)} \leqslant G_i$ for all i.

First, $G' \leqslant G_1$ as G/G_1 is abelian. Suppose $G^{(i)} \leqslant G_i$ for some $i \geqslant 1$. Since G_i/G_{i+1} is abelian, $(G_i)' \leqslant G_{i+1}$. Thus $G^{(i+1)} = (G^{(i)})' \leqslant (G_i)' \leqslant G_{i+1}$. □

Definition 15. *Let $x, g \in G$, consider the conjugation action $x^g = g^{-1}xg$. It induces an automorphism of G s.t. $x \mapsto g^{-1}xg$, called the inner automorphism induced by g.*