Groups.

Def: A set G with a multiplication $*$ is a group if

("multiplication" is a bimap (binary operation))

$$* : G \times G \longrightarrow G$$
$$(a, b) \longmapsto a*b$$

(1). $a*(b*c) = (a*b)*c$ association law.

(2). $\exists\, e \in G$ s.t $a*e = e*a = a$ identity.

(3). For any $a \in G$. $\exists\, b \in G$ s.t $a*b = b*a = e$ inverse

denote by $(G, *)$, If $\underbrace{a*b = b*a}_{\text{commutation law}}$, G is called abelian.

Fields.

Def: A set F with "$+$" and "$\times$" is called a field if

(1). $(F, +)$ is abelian group with addition identity "0"

(2) $(F \setminus \{0\}, \times)$ is abelian group with multiplication identity "1"

(3) "$\times$" and "$+$" are compitable, i.e.

$$a \times (b+c) = a \times b + a \times c$$
$$(a+b) \times c = a \times c + b \times c$$

distribution law.

(I think since "$\times$" is commutative $a \times (b+c) = a \times b + a \times c$ is enough)

Rings

Def: A set R with "$+$" and "$\times$" is called a ring if

(1). $(R, +)$ form a abelian group with addition identity "0"

(2). $(R, \times)$ form a "semigroup", that is

$$a \times (b \times c) = (a \times b) \times c$$

(3). "$+$" and "$\times$" are compitable. that is:

$$a \times (b+c) = a \times b + a \times c$$
$$(a+b) \times c = a \times c + b \times c$$

(need two equation since "$\times$" may not commutative)

Obesenation :

① A field is a ring ② A ring is not necessary a field.

Moreover (i) if $ab=ba$, $\forall a,b\in R$. R is called commutative ring

(ii). if $e\in R$ s.t $ae=ea=a$, $\forall a\in R$. the e is called the identity of R.

Q: ① How to construct gp. G ?

② What do we ask about a gp. G ?

Let $(G,\times)$, $(H,*)$ be groups

Def: $X=G\times H=\{(g,h)\mid g\in G, h\in H\}$ mulei. " $\cdot$ "

$$(g_1,h_1)\cdot(g_2,h_2)=(g_1\times g_2,\ h_1 * h_2)$$

claim $(G\times H,\cdot)$ is a group. Check it !

Direct Product of $(G,\times)$ and $(H,*)$