1. Prove $GL_n(\mathbb{F}_p)/SL_n(\mathbb{F}_p) \simeq \mathbb{Z}_{p-1}$

Let $\det: GL_n(\mathbb{F}_p) \longrightarrow \mathbb{Z}_{p-1}$

$x \longmapsto \det x$

$\forall x, y \in GL_n(\mathbb{F}_p)$, $\det(xy) = \det x \det(y)$ $\Rightarrow$ det is group homo.

$\forall z \in GL_n(\mathbb{F}_p)$ $z \in \ker \det \Leftrightarrow \det z = 1 \Leftrightarrow z \in SL_n(\mathbb{F}_p)$

$\Rightarrow \ker \det = SL_n(\mathbb{F}_p) \Rightarrow GL_n(\mathbb{F}_p)/_{SL_n(\mathbb{F}_p)} \simeq \mathbb{Z}_{p-1}$

2. Let $H, K \leq G$, prove $HK \leq G \Leftrightarrow HK = KH$.

($\Rightarrow$) If $HK \leq G$ then $\underset{HK}{\underbrace{(HK)^{-1}}} = K^{-1}H^{-1} = KH$

($\Leftarrow$) If $HK = KH$, then $HK \cdot (HK)^{-1} = HKH = HHK = HK \Rightarrow HK \leq G$.

3. $H \leq G$, $K \leq G$, $|G| < \infty$, then $|HK| = \dfrac{|H| \cdot |K|}{|H \cap K|}$

define a equivalence relation on the set $H \times K$ s.t.

$(h_1, k_1) \sim (h_2, k_2)$

iff $\exists\ a \in H \cap K$ s.t. $h_2 = h_1 a$ $k_2 = a^{-1} k_1$

Then $[(u, v)] \longrightarrow uv$ is one-to-one.

$\left| H \times K/_\sim \right| = \dfrac{|H| \cdot |K|}{|H \cap K|} = |HK|$

4. $[G:H] = 2$, show $H \trianglelefteq G$.

$G = \{H, gH\}_{g \notin H} = \{H, Hg\} \Rightarrow gH = Hg$, $\forall g \in G \setminus H$

$\Rightarrow H \lhd G$

5. (1). $C_G(S), N_G(S) \le G$ (trivial: $e \in C_G(S)$, $e \in N_G(S)$)

$\forall x, y \in C_G(S)$, $\forall s \in S$

$xys = xsy = sxy \Rightarrow xy \in C_G(S)$

$x^{-1}s = (s^{-1}x)^{-1} = (xs^{-1})^{-1} = sx^{-1}$, $x^{-1} \in C_G(S)$

$xys(xy)^{-1} = x\, ysy^{-1} x^{-1}$

$\exists s_1 = ysy^{-1} \in S$ $\quad ||$

$xs_1x^{-1} \in S \Rightarrow xy \in N_G(S)$

$(x^{-1})s(x^{-1})^{-1} = x^{-1}sx = (x^{-1}s^{-1}x)^{-1} = s^{-1} = s$

$\Rightarrow x^{-1} \in N_G(S)$

$\Rightarrow C_G(S), N_G(S) \le G$

(2). $C_G(S) \trianglelefteq N_G(S)$.

①. $C_G(S) \le N_G(S)$. obviously.

②. $\forall a \in C_G(S)$, $\forall b \in N_G(S)$ $\forall s \in S$

($\exists s' = b^{-1}sb$.

Consider.

$bab^{-1}s$

$= bas'b^{-1}$

$= bs'ab^{-1}$

$= sbab^{-1} \Rightarrow bab^{-1} \in C_G(S)$

$\Rightarrow C_G(S) \trianglelefteq N_G(S)$.

6. $Z(G)=\bigcap_{g\in G} C_G(g)$. prove $Z(G)\trianglelefteq G$.

①. $e\in Z(G)$

②. $\forall\ x,y\in Z(G)$ $\forall\ g\in G$.

$xyg=xgy=gxy \Rightarrow xy\in\bigcap_{g\in G} C_G(g)$.

③. $x^{-1}g=(g^{-1}x)^{-1}=(xg^{-1})^{-1}=gx^{-1} \Rightarrow x^{-1}\in\bigcap_{g\in G} C_G(g)$

$\Rightarrow Z(G)\leq G$.

Furthermore. Since. $Z(G)=\bigcap_{g\in G} C_G(g)$ it's obvious that

$\forall\ g\in G$, $gZ(G)=Z(G)g$, which means that $Z(G)\trianglelefteq G$.

7. $H\trianglelefteq G$ $|H|=2$. prove $H\leq Z(G)$.

Let $H=\{e,h\}$ then $\forall\ g\in N_G(H)$, $ghg^{-1}=h$

$\forall\ k\in C_G(H)$, $khk^{-1}=h$

it shows $N_G(H)=C_G(H)$ Since $H\trianglelefteq G \Rightarrow N_G(H)=G \Rightarrow C_G(H)=G$

$\Rightarrow H\leq Z(G)$

8. $G/Z(G)$ cyc. prove G abelian.

Let $G/Z(G)=\langle\bar{a}\rangle$

$\forall\ b,c\in G$, let $\pi(b)=\bar{a}^m$, $\pi(c)=\bar{a}^n$

then $\exists\ b_1, c_1\in Z(G)$ s.t $b=a^m b_1$, $c=a^n c_1$

$bc=a^m b_1 a^n c_1=a^n c_1 a^m b_1=cb \Rightarrow G$ is abelian.

9. N maximal normal $\Leftrightarrow$ G/N simple.

N maximal normal $\Leftrightarrow$ $\nexists M$ s.t. $N \triangleleft M \triangleleft G$

$\Leftrightarrow$ G/N has no non-trivial normal subgrp

(3rd iso. thm)

$\Leftrightarrow$ G/N simple.

10. $N, H \underset{max}{\triangleleft} G$, $N \neq H$ $\Rightarrow$ $N \cap H \underset{max}{\triangleleft} H$

$$H/H\cap N \simeq HN/N$$

Since $NH \triangleleft G$, $N \leq NH$ $\Rightarrow$ $NH = G$

N max $\Rightarrow$ G/N simple $\Rightarrow$ $H/H\cap N$ simple $\Rightarrow$ $H \cap N \underset{max}{\triangleleft} H$

11. $\varphi \in Aut(G)$ $|G| < \infty$ $I = \{g \in G \mid \varphi(g) = g^{-1}\}$

(a). $|I| > \frac{3}{4}|G|$, G abelian

<1>. if $\forall x, y \in I$, $xy = yx$ then $I \leq G$ and $|I| > \frac{3}{4}|G|$

$\Rightarrow$ $I = G$ $\Rightarrow$ G abelian

<2> if $\exists x, y \in I$, $xy \neq yx$, then $xy \notin I$

Since $C_G(x) \lneq G$ $\Rightarrow$ $|C_G(x)| \leq \frac{1}{2}|G|$

$\Rightarrow$ let $S = \{g \in G \mid gx \neq xg\}$.

$\Rightarrow$ $|S| > \frac{1}{4}|G|$

$\Rightarrow$ $|xS| = |S| = |\{xg \mid gx \neq xg, g \in G\}| > \frac{1}{4}|G|$

$xy \in I$

$\Leftrightarrow$ $\varphi(xy) = (xy)^{-1}$

$\Leftrightarrow$ $\varphi(x)\varphi(y) = y^{-1}x^{-1}$

$\Leftrightarrow$ $x^{-1}y^{-1} = y^{-1}x^{-1}$

$\Leftrightarrow$ $xy = yx$.

$\lightning$ to $I > \frac{3}{4}|G|$

(b). $|I| = \frac{3}{4}|G|$. prove $\exists$ $H < G$ s.t H abelian, $[G:H] = 2$

by (a). G is non-abelian.

$\exists$ $x, y \in I$ s.t $xy \neq yx$. $\Rightarrow$ $|C_G(x)| \leq \frac{1}{2}|G|$

If $|C_G(x) \cap I| < \frac{1}{2}|G|$ then $|S \cap I| > \frac{1}{4}|G|$ $\lightning$.

$\Rightarrow$ $|C_G(x) \cap I| \geq \frac{1}{2}|G|$ but $\leq |C_G(x)|$

$\Rightarrow$ $|C_G(x) \cap I| = \frac{1}{2}|G|$

$\Rightarrow$ $[G : C_G(x)] = 2$.

$\Rightarrow$ $C_G(x) \subset I$ thus. $\forall$ $g_1, g_2 \in C_G(x)$, then $g_1 g_2 \in I$

i.e. $g_1 g_2 = g_2 g_1$ $\Rightarrow$ $C_G(x)$ is abelian.