Homework-1

September 12, 2024

1. Calculate the order of following groups:
(a). $GL_n(p)$;
(b). $SL_n(p)$.

2. Let $C \subset M_n(F)$ be the scalar matrices over F. Prove that C is the center of $GL_n(F)$, denoted by $Z(GL_n(F))$.

3. Prove Eular's Theorem: Let n be a positive integer. Let x be an integer which is coprime to n. If $1 \leq x \leq n$, then

$$x^{\varphi(n)} \equiv 1 \mod n,$$

in which φ is the Euler's Totient function.

4. Let G be a group, $a, b \in G$. If $aba^{-1} = b^r$, prove that $a^i ba^{-i} = b^{r^i}$.

5. Let G be a group. If $\forall a, b \in G$, $(ab)^2 = a^2b^2$. Prove that G is abelian. Show that if $exp(G) = 2$, then G is abelian. $exp(G)$ is the smallest integer n s.t. $\forall g \in G$, $g^n = e$.

6. Assume $|G|$ is even. Prove: $\exists a \neq e \in G$ s.t. $a^2 = e$.

7. Suppose $n > 2$. Prove that for a finite group G, there exists even number of x where $|x| = n$.

8. Consider $a, b \in SL_2(\mathbb{Q})$, where

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Prove that:
(a). $|a| = 4$;
(b). $|b| = 3$;
(c). $|ab| = \infty$.

9. If a group G has only finitely many subgroups, prove G is a finite group.