1. prove if G is a simple group with order 60, then $G \cong A_5$.

Step 1. Let $H < G$ consider $\Omega = \{H, Hg_2, \dots Hg_t\}$, $t = [G:H]$

Then $\rho: G \longrightarrow Sym(\Omega) \cong S_t$ is a group homomorphism.

$g \longmapsto \rho(g): \Omega \longrightarrow \Omega$

$Hg_i \longmapsto Hg_i g$

Since G is simple. $\ker\rho = 1$ so ρ is an embedding.

$|S_2| = 2$, $|S_3| = 6$ $|S_4| = 24$, $2, 6, 24 < 60$

So G can not embedding into S_2, S_3, S_4

which means that $t \neq 2, 3, 4$, i.e $\forall H < G$, $[G:H] \neq 2, 3, 4$

if $t = 5$ $|S_5| = 120 > 60$

Suppose G can embedding into S_5, then $\rho(G) \leq S_5$ and $|S_5 : \rho(G)| = 2$ it shows $\rho(G) = A_5$, i.e $G \cong A_5$.

We came to the conclusion that:

If exists $H < G$ s.t $|H| = \frac{60}{5} = 12$. then $G \cong A_5$.

Now consider Sylow 2-subgroup of G.

$n_2 \equiv 1 \mod 2$ and $n_2 | 15$, $n_2 = 1, 3, 5, 15$

$\Rightarrow$ ① $n_2 = 1$, then $P_2 \lhd G$ $\lightning$ G is simple.

② $n_2 = 3$, since $n_2 = [G : N_G(P_2)]$. $\lightning$

③ $n_2 = 5$, then $|N_G(P_2)| = \frac{60}{5} = 12$. done

④ $n_2 = 15$.

since $|P_2| = 4$. abelian. $|N_G(P_2)| = 4$

$\Rightarrow N_G(P_2) = C_G(P_2)$ By Burnside thm. G is 2-nilpotent

so $\exists N \triangleleft G$ and $[G:N] = 4$. ↯ G is simple

Or follow this proof:

(i). If for any different Sylow 2-subgroup of G

$P_2 . P_2'$, $P_2 \cap P_2' = 1$

Then $\exists$ $1 + 3 \times 15 = 46$ 2-elements

and $n_5 = 6$ shows 14 elements need to form 6 different

5-ordered cyclic group. which is impossible.

(ii) Hence $\exists$ 2 different Sylow 2-subgroups of G

P_2 and P_2' $A = P_2 \cap P_2'$ and $|A| = 2$

Consider $C_G(A)$, $P_2 \cup P_2' \subseteq C_G(A)$

so $|C_G(A)| > 4$ and $4 \mid |C_G(A)|$ $(P_2 \leq C_G(A).\ P_2' \leq C_G(A))$

also $|C_G(A)| \mid 60$ it forces $|C_G(A)| = 12$. done.

2 (1). no G $G' \simeq S_3$.

If $G' \simeq S_3$, since $S_3' = \mathbb{Z}_3$, $G > S_3$

$G' = S_3$ $G'' = \mathbb{Z}_3$ Let $G'' = \langle a \rangle$ since $G'' \triangleleft G$

By N-C lemma.

$$G/C_G(G'') \lesssim \mathrm{Aut}(G'') = C_2$$

$\Rightarrow$ $G' \le C_G(G'')$ since $G/C_G(G'')$ abelian.

$\Rightarrow$ All elements in G' commutes all elements in G''

$C_{G'}(G'') = G'$ ~~y~~ to $G' \simeq S_3$ and $G'' \simeq \mathbb{Z}_3$.

(2). no $G' \simeq S_4$.

If $G' \simeq S_4$ $G'' \simeq A_4$ $G''' \simeq V_4$ $G'''' \simeq 1$.

Let $\bar{G} = G/G''' \simeq S_3$

$\bar{G}''' = e$ $\bar{G}'' \simeq \mathbb{Z}_3$ $\bar{G}' \simeq S_3$ by (1), it is not true.

3. 3 cycle in A_n $n \ge 5$ can be repr. by a commutator.

Consider $(ijk) \in A_n$ since $n \ge 5$, exists s, t in Ω

Consider $[(isj), (itk)] = (ijk)$

It shows A_n' contains all 3-cycles in A_n

We know that A_n can be generated by all 3-cycles.

Thus $A_n = A_n'$

4. $F = \mathbb{Z}_p$, $G = GL_n(F)$ write a Sylow p subgroup of G

{ special upper triangular matrixes }

5. take $[h,k] = h^{-1}k^{-1}hk$

1. $h^{-1}k^{-1}h \in K \Rightarrow [h,k] \in K$

2. $k^{-1}hk \in H \Rightarrow [h,k] \in H$

$\Rightarrow [h,k] \in H \cap K = 1 \Rightarrow hk = kh.$

6. $N \trianglelefteq_{min} G$ then $N = T_1 \times \cdots T_k$, where $T_i \cong T_j$ are isomorphic simple groups.

Recall: Exercise class 2 Notes

Thm: Finite characteristic simple groups are product of iso. simple grups.

$N \trianglelefteq_{min} G \Rightarrow N$ char simple.

If not, $\exists$ K char $N \trianglelefteq G \Rightarrow K \trianglelefteq G$ $\lightning$

Thus N char simple $\Rightarrow N$ is product of iso simple grups.

7. if $|G| = p^3$, non-abelian. Then G has exactly 2 types.

Discuss on the existence of p^2 order element

First. Suppose G has a element of order p^2, which is $a \in G$

Then $\langle a \rangle$ is a normal subgroup of G since index p.

This is because. consider:

Let $\langle a\rangle = H < G$.

$\rho: G \longrightarrow Sym(H)$ is the right multiplication permutation.

i.e. $G/Core_G(H) \lesssim S_p$ $\quad |S_p| = p!$

$\Rightarrow Core_G(H) = H \Rightarrow H \trianglelefteq G$.

then pick $b \in G \setminus \langle a\rangle$, we have $b^p \in \langle a\rangle$

If $b^p = 1$. $G = \mathbb{Z}_{p^2} = \mathbb{Z}_p$ or D_8.

If $o(b) = p^2$. $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c\rangle$ or $G = Q_8$.

(Ref. Dummit P183 − P184)

8. $|G| = p^3 q$

Recall our midterm, what about $|G| = p^a q$, $a \geq 1$?

1° If $p = q$. G is a p-group with order $\geq p^2$. G is not simple.

2° If $p \neq q$

$n_p(G) \mid q \Rightarrow n_p(G) = 1$ or q

If $n_p(G) = 1$. $P \trianglelefteq G$. G is not simple.

If $n_p(G) = q$

①. If for any $S, T \in Syl_p(G)$ where $S \neq T$, we have $S \cap T = 1$.

Then $\left|\bigcup_{S \in Syl_p(G)} S\right| = q(p^a - 1) + 1$ where $|G| - \left|\bigcup_{S \in Syl_p(G)} S\right| = q - 1$

It show we can only form one Sylow q-subgrp in G.

So G is not simple.

②. If $\exists$ S, T s.t $S \neq T$ and $|S \cap T| \neq 1$.

Choose S, T s.t $S \cap T$ as large as possible.

Let $N = N_G(S \cap T)$

Then $N_S(S \cap T) = S \cap N$

Lemma: Let G be a finite p-grp $H < G$ then $H < N_G(H)$

$|G| > 1 \Rightarrow |Z(G)| > 1 \Rightarrow Z(G) \subseteq N_G(H)$

if $Z(G) \not\subseteq H$, $H < N_G(H)$

if $Z(G) \subseteq H$, $\pi : G \to G/Z(G) = \bar{G}$

$\pi(N_G(H)) = \overline{N_G(H)} = N_{\bar{G}}(\bar{H})$

Induction on $|G|$ we have $\bar{H} < N_{\bar{G}}(\bar{H})$

$\Rightarrow H < N_G(H)$

Since S is a finite p group. $S \cap T < S$

$\Rightarrow S \cap T < N_S(S \cap T) = S \cap N$

$\Rightarrow S \cap T < N_T(S \cap T) = T \cap N$

If N is a p-grp. then $N \subseteq P$ for some $P \in Syl_p(G)$

$S \cap T < S \cap N \subseteq S \cap P$ since $S \cap T$ is the largest. $\lightning$.

Thus N is not a p-group. we have $q \mid |N|$

take $Q \in Syl_q(N)$

and Q is a cyclic group. $Q = \{e, a, a^2, \dots a^{q-1}\}$

show $G = \{ S, Sa, \dots, Sa^{q-1} \}$.

if $Sa^i = Sa^j$ then $Sa^{i-j} = S \Rightarrow$ $i-j=0$ or $a \in S$ ↯

$\Rightarrow \forall g \in G$, $g = sx$ for some $s \in S$ and $x \in Q$

$$S \cap T = \bigcap_{x \in Q} x^{-1}(S \cap T)x \subseteq \bigcap_{x \in Q} x^{-1} S x$$

on the other hand

$$\bigcap_{g \in G} g^{-1} S g = \bigcap_{g \in G} x^{-1} s^{-1} S s x \quad \text{for } g = sx \text{ for some } s, x.$$

$$= \bigcap_{x \in Q} x^{-1} S x$$

it shows $1 < |S \cap T| \leq \left| \bigcap_{x \in Q} s^{-1} S x \right| = \left| \bigcap_{g \in G} g^{-1} S g \right|$

and $\bigcap_{g \in G} g^{-1} S g$ is p grup, so $\bigcap_{g \in G} g^{-1} S g \trianglelefteq G$

$\Rightarrow G$ is not simple.

9. $A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\varphi C_3$

$V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$
$= \{ e, (12)(34), (13)(24), (14)(23) \}$

$$\varphi: C_3 \longrightarrow \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$$

$e \longmapsto \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$
$\sigma \longmapsto \sigma$

$a \longmapsto \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$
$\sigma \longmapsto a^{-1} \sigma a$

$$a^2 \longmapsto \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\sigma \longmapsto a^{-2} \sigma a^{2}$$

i.e. write $a = (123)$ $\sigma = (12)(34)$

$$a \longmapsto \rho_a :$$

$$(12)(34) \longmapsto (132)(12)(34)(123) = (23)(14)$$

10. $S_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\psi} S_3$

$$\psi : S_3 \to \mathrm{Aut}(V_4) = S_3 \simeq \widehat{S_3}$$

$$x \longmapsto \widehat{x}$$