Homework-4

October 10, 2024

1. Prove the following 3 isomorphic theorems of rings:
(1). For a ring homomorphism $\varphi : R_1 \to R_2$, $R_1/\ker\varphi \simeq \operatorname{Im}\varphi \leqslant R_2$.
(2). Let $I \lhd R$ s.t. $\pi : R \to R/I : r \mapsto r + I$, natural homomorphism. Then:

1. The ideal(subring) of R containing I and ideal(subring) of R/I are in one-to-one correspondence;
2. If $I \lhd J \lhd R$ then $J/I \lhd R/I$ and $R/J \simeq \frac{R}{I}/\frac{J}{I}$.

(3). Let $I \lhd R$, $S \leqslant R$. Then $I + S$ is a subring of R, and:

1. $S \cap I \lhd S$ and $I \lhd I + S$;
2. $(I + S)/I \simeq S/S \cap I$

2. Let $F = \mathbb{F}_3 = \{0, 1, 2\}$ be a field. Let $p(x) = x^2 + 1$, check that $p(x)$ is irreducible over F. This means that $F[x]/(x^2 + 1)$ is a field. Find a basis for $F[x]/(x^2 + 1)$ as a vector space over F.

3. Let $G = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, show $\operatorname{End}(G)$.

4. Let R be a finite ring with identity, $a, b \in R$ and $ab = 1$. Prove that $ba = 1$.

5. Let R be a ring, $a \in R$. If there exist a positive number n s.t. $a^n = 0$, then a is called a nilpotent element. Prove that if a is a nilpotent element of a ring R with identity, then $1 - a$ is invertible.

6. Prove: in a commutative ring R all nilpotent element form an ideal. (Called the nilpotent ideal of R)

7. Prove that the addition and multiplication of ideals in a ring R with identity are compatible. i.e. for all ideals I, J, K in a ring R with identity, they satisfy:
(1). $(I + J)K = IK + JK$;
(2). $K(I + J) = KI + KJ$.

8. Prove that the matrix ring $\mathrm{M}_n(K)$ over field K has no non-trivial ideal. (Those kind of rings are called simple rings).

9. Let R be a non-zero commutative ring with identity. Prove that R is a simple ring if and only if R is a field.

10. Let K be a field, R be a ring, $\varphi : K \to R$ be a ring homomorphism. Prove that $\varphi(K) = 0$ or φ is injective.

11. Prove that all finite integral domains are fields.

12. Prove that the subset $\{\pm 1, \pm i, \pm j, \pm k\}$ of quaternion ring $\mathbb{H}$ is a group under the multiplication which is called quaternion group, denoted by Q_8. Prove that every subgroup of Q_8 is normal subgroup, but Q_8 is non-abelian.

13. Let L be a skew field, $a, b \in L$ and $ab \neq 0, 1$. Prove Hua's identity:

$$a - (a^{-1} + (b^{-1} - a)^{-1})^{-1} = aba$$

14. Let F be a field with $\operatorname{Char} F = p > 0$. Prove:

$$(a + b)^p = a^p + b^p, \ \forall a, b \in F$$