# Homework-1

September 21, 2024

1. Prove: $GL_n(p)/SL_n(p) \simeq Z_{p-1}$.

2. Let $H, K \leq G$, prove: $HK \leq G$ if and only if $HK = KH$. In particular, $HK = G$ if and only if $KH = G$.

3. Let $G$ be a finite group, $H, K \leq G$. Prove: $|HK| = \frac{|H||K|}{|H \cap K|}$. (Hint: $HK$ may not be a group).

4. Let $H$ be a subgroup of $G$ with index 2, i.e. $[G : H] = 2$, show that $H \lhd G$.

5. Let $S$ be a non-empty subset of $G$. Let

$$C_G(S) = \{x \in G | xa = ax, \forall a \in S\}$$

$$N_G(S) = \{x \in G | xSx^{-1} = S\}$$

$C_G(S)$ and $N_G(S)$ are called the *centralizer* and *normalizer* of $S$ respectively. Prove that:
(1). $C_G(S)$ and $N_G(S)$ are all subgroups of $G$;
(2). $C_G(S) \trianglelefteq N_G(S)$

6. Let $G$ be a group. Let $Z(G) = \cap_{g \in G} C_G(g)$, $Z(G)$ is called the *center* of $G$. Prove that $Z(G) \trianglelefteq G$.

7. Let $H \trianglelefteq G$ where $|H| = 2$, prove $H \leq Z(G)$.

8. Let $Z(G)$ be the center of $G$ and $G/Z(G)$ is a cyclic group, prove that $G$ is abelian.

9. Prove: $N$ is the maximal normal subgroup of $G$ if and only if $G/N$ is a simple group.

10. Let $N, H$ be two different maximal normal subgroup of $G$, then $N \cap H$ is a maximal normal subgroup of $H$ (also $N$).

11. Let $G$ ba a finite group, $\varphi \in \mathrm{Aut}(G)$, let

$$I = \{g \in G | \varphi(g) = g^{-1}\}$$

(1). Suppose $|I| > \frac{3}{4}|G|$, prove: $G$ is abelian;
(2). Suppose $|I| = \frac{3}{4}|G|$, prove: $\exists H \leq G$ s.t. $H$ is abelian and $[G : H] = 2$.