Solution to Quiz 1

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1. Let $C \subset \mathrm{M}_n(F)$ be the scalar matrices over F. Prove that C is the center of $\mathrm{GL}_n(F)$, denoted by $Z(\mathrm{GL}_n(F))$.

证明. Step 1: $\forall A \in C$, $A \in Z(\mathrm{GL}_n(F))$ which means that $C \subseteq Z(\mathrm{GL}_n(F))$. This is because scalar matrices commute with all matrices.

Step 2: $\forall A = (a_{ij}) \in Z(\mathrm{GL}_n(F))$, A commutes with $B = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ where $a_i \in F$, $a_i \neq 0$ for all i and $a_i \neq a_j$ for all $i \neq j$. That forces $a_{ij} = 0$ for all $i \neq j$. Further more, A also commutes with all cyclic permutation matrices. So A is a scalar matrix. That shows $Z(\mathrm{GL}_n(F)) \subseteq C$ and hence $Z(\mathrm{GL}_n(F)) = C$ (as a set).

Step 3: C is subgroup of $\mathrm{GL}_n(F)$ and C is a normal subgroup of $\mathrm{GL}_n(F)$, this is beacause C is closed under matrix multiplication and C is invariant under conjugation (since elements of C are scalar matrices, commute with all matrices). So C is a normal subgroup of $\mathrm{GL}_n(F)$. Therefore, C is the center of $\mathrm{GL}_n(F)$. $\square$

2. Let G be a finite group, $H, K \leq G$. Prove: $|HK| = \frac{|H||K|}{|H \cap K|}$. (Hint: HK may not be a group).

证明. Note that $HK = \cup_{k \in K} Hk$ and $|H||K| = \sum_{k \in K} |Hk|$. If $Hk_1 = Hk_2$, then $k_1 k_2^{-1} \in H$. For a fixed $k_0 \in K$, define $S_{k_0} = \{k \in K : kk_0^{-1} \in H\}$. Since $|S_{k_0}| = |H \cap K|$, then $|HK| = \frac{|H||K|}{|H \cap K|}$. $\square$

3. Let N, H be two different maximal normal subgroup of G, then $N \cap H$ is a maximal normal subgroup of H (also N).

证明. Note that $N \cap H$ is a normal subgroup of G. Since N, H are maximal normal subgroups, then G/N and G/H are simple. Since $NH = HN$, then NH is a normal subgroup of G and so $G = NH$. Note that $G/N = NH/N \cong N/H \cap N$ and $G/H = NH/H \cong N/N \cap H$ are simple, then $N \cap H$ is a maximal normal subgroup of H and N. $\square$

4. Let G be a finite group, $\varphi \in \mathrm{Aut}(G)$, Let

$$I = \{g \in G | \varphi(g) = g^{-1}\}$$

(1). Suppose $|I| > \frac{3}{4}|G|$, prove G is abelian.

(2). Suppose $|I| = \frac{3}{4}|G|$, prove: $\exists H \leqslant G$ s.t. H is abelian and $[G : H] = 2$.

证明. (1). If $\forall x, y \in I$, $xy = yx$, then I is a subgroup of G. Since $|I| > \frac{3}{4}|G|$, then $|I| = |G|$, then $I = G$ and G is abelian.

If $\exists x, y \in I$ s.t. $xy \neq yx$, then $xy \notin I$. Since $C_G(x) \leqslant G$, $|C_G(x)| \leqslant \frac{1}{2}|G|$. Let $S = \{g \in G | gx \neq xg\}$, then $|S| \geqslant \frac{1}{2}|G| > \frac{1}{4}|G|$, and $|xS| = |S| > \frac{1}{4}|G|$, contradict to $|I| > \frac{3}{4}|G|$.

(2). By (1), now G is non-abelian. And $\exists x, y \in I$ s.t. $xy \neq yx$, we have $|C_G(x)| \leqslant \frac{1}{2}|G|$. If $|C_G(x) \cap I| < \frac{1}{2}|G|$ then $|S \cap I| \geqslant \frac{1}{2}|G| > \frac{1}{4}|G|$ which contradict to $|I| = \frac{3}{4}|G|$. So $|C_G(x) \cap I| \geqslant \frac{1}{2}|G|$, but $|C_G(x)| \leqslant \frac{1}{2}|G|$ and $|C_G(x) \cap I| \leqslant |C_G(x)|$. It forces $|C_G(x) \cap I| = |C_G(x)| \Rightarrow C_G(x) \cap I = C_G(x)$ and $[G : C_G(x)] = 2$. And since $C_G(x) \subset I$ we have $\forall a, b \in C_G(x)$, $ab = ba$. Thus $C_G(x)$ is actually the H we want. □

5. Prove $\mathrm{Aut}\,(Z_2 \oplus Z_2) \simeq S_3$.

证明. Note that $\varphi(0) = 0$, then φ is a permutation on $\Omega = \{(1,0), (0,1), (1,1)\}$. Verify $\mathrm{Sym}(\Omega)$ are all automorphism. □

(Alternating proof: remark that $Z_2 \oplus Z_2 \cong \mathbb{F}_2^2$, then $\mathrm{Aut}(Z_2 \oplus Z_2) = \mathrm{GL}(2,2) \cong S_3$.)

6. Let $H, K \trianglelefteq G, G/H, G/K$ are all soluble groups. Prove: $G/H \cap K$ is also a soluble group.

证明. Since $G/H \cong (G/H \cap K)/(H/H \cap K)$ and $H/H \cap K \cong HK/K \leqslant G/K$ are solvable groups, then $G/H \cap K$ is also a solvable group. □