Abstract Algebra
: Lecture 10

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Today we talk about group actions.

Example 1. *Recall: Let $G = \mathrm{GL}(V)$, this is a linear group. And we have a set V, also a vector space.*

1. Each $g \in G$ is a bijection from V to V;

2. $\forall v \in V$ and $1 \in G$ we have $v^1 = v$;

3. $\forall v \in V$ and $g, h \in G$ we have $v^{gh} = (v^g)^h$.

We called G acts on V.

Definition 2. *Let G be a group and Ω be a set. We say that G acts on Ω if*

(1). each element of G is a bijection from Ω to Ω;

(2). $\forall \omega \in \Omega$, $\omega^1 = \omega$;

(3). $\forall \omega \in \Omega$ and $g, h \in G$ we have $\omega^{gh} = (\omega^g)^h$.

We denote this action by $G \curvearrowright \Omega$.

Example 3. *$Sym(\Omega)$ and $Alt(\Omega)$ acts on Ω naturally.*

Example 4. *Let G be a group, for each element $g \in G$, we define action $g : x \mapsto xg$ forall $x \in G$. We denote this map by $\hat{g}$, called right multiplication. This is a group action. We denote the group of this action by $\hat{G}$.*

Example 5. *Let G be a group, for each element $g \in G$, we define action $g : x \mapsto g^{-1}xg$ forall $x \in G$. We denote this map by $\tilde{g}$, called conjugation. This is a group action. We denote the group of this action by $\tilde{G}$.*

Example 6. *Let G be a group, for each element $g \in G$, we define action $g : x \mapsto g^{-1}x$ forall $x \in G$. We denote this map by $\check{g}$, called left multiplication. This is a group action. We denote the group of this action by $\check{G}$.*

Exercise 7. *If we right group action in another way, i.e. $(gh)\omega = g(h\omega)$, check example 4,5,6.*

Proposition 8. *1. $\hat{g}, \check{g}, \tilde{g} \in \mathrm{Sym}(G)$.*

2. $\hat{G}, \check{G}, \tilde{G} \leqslant \mathrm{Sym}(G)$.

3. $\forall g, h \in G$, $\hat{g}\check{h} = \check{h}\hat{g}$.
4. $\forall g \in G$, $\hat{g}\check{g} = \tilde{g}$;
5. $\hat{G} \cap \check{G} = Z(\hat{G}) = Z(\check{G})$, $\hat{G} \cap \tilde{G} = \check{G} \cap \tilde{G} = \{1\}$;
6. $\left\langle \hat{G}, \check{G} \right\rangle = \hat{G}\check{G} = \hat{G} \circ \check{G} = \hat{G} : \check{G} = \check{G} : \hat{G}$;

Definition 9. *For two groups G and H, assume there exist $C \lesssim Z(G)$ and $C \lesssim Z(H)$. s.t. $C \neq \{1\}$. Let $Z_1 \leqslant Z(G)$ and $Z_2 \leqslant Z(H)$. s.t. $Z_1 \simeq Z_2 \simeq C$. Let ϕ be an isomorphism from Z_1 to Z_2. Let $X = (G \times H)/\left\langle (x, x^{\phi}) | x \in Z_1 \right\rangle \simeq (G \times H)/C$. This group X is called a central product of G and H, denoted by $G \circ H$.*

Definition 10. *Let $H, K < G$ s.t. $H \lhd G$ and $H \cap K = \{1\}$. Then $\langle H, K \rangle = HK = H \rtimes K = H : K$ called a semi-direct product of H and K.*

G acts on Ω also can be said as an action of G on Ω or group action of G on Ω.

Definition 11. *Let G act on Ω. Then G partitions Ω into orbits, where an orbit is $\Delta = \omega^G = \{\omega^g | g \in G\}$, where $\omega \in \Omega$. So $\Omega = \bigsqcup_{\omega \in \Omega} \omega^G$.*

Example 12. *$\tilde{G}$ acts on G naturally, i.e. $x^{\tilde{g}} = g^{-1}xg$. $G = x_0^G \sqcup x_1^G \sqcup \cdots \sqcup x_t^G$, where $x_0 = e$, $x_i^G = \{g^{-1}x_i g | g \in G\}$ called a conjugacy class of x_i, denoted by $C(x_i)$. $C(x_i) = \{g^{-1}x_i g | g \in G\}$ is a orbit of the action of $\tilde{G}$ on G.*

Definition 13. *For G acting on Ω, $G_\omega = \{g \in G | \omega^g = \omega\}$ called the stabilizer of ω in G. It's easy to check that G_ω is a subgroup of G.*

Theorem 14. *(Orbit-Stabilizer Theorem) For G acting on Ω, For $\omega \in \Omega$, $|G| = |\omega^G| \cdot |G_\omega|$.*

证明. Let $\Delta = \omega^G = \{\delta_1, \delta_2, \ldots, \delta_m\}$ write $\delta = \delta_1$. Let g_i s.t. $\delta^{g_i} = \delta_i$ for $1 \leqslant i \leqslant m$.

Claim: For any element $x \in G$, $\delta^x = \delta_i \Leftrightarrow x \in G_\delta g_i$. Which is due to $\delta^x = \delta_i = \delta^{g_i} \Leftrightarrow \delta^{xg_i^{-1}} = \delta \Leftrightarrow xg_i^{-1} \in G_\delta \Leftrightarrow x \in G_\delta g_i$.

Recall $G = G_\delta \sqcup G_\delta g_2 \sqcup \cdots \sqcup G_\delta g_m$, we have $|G| = |G_\delta| + |G_\delta g_2| + \cdots + |G_\delta g_m|$, i.e. $|G| = |\Delta||G_\delta|$. □

Oberserve: consider conjugate action of G on itself, $|x^G| = 1$ iff $x \in Z(G)$. So by Orbit-Stabilizer Theorem, $|G| = |Z(G)| + |C(g_1)| + \cdots + |C(g_r)|$ and $|g_i^G| \mid |G|$ due to $|G| = |C(g_i)| \cdot |C_G(g_i)|$.

Theorem 15. *(Sylow's 1st Theorem) Let G be a finite group, $|G| = p^e m$ s.t. p is a prime and $(p, m) = 1$, a subgroup H of G s.t. $|H| = p^e$ exists. And H is called a Sylow p-subgroup of G, denoted by $H \in Syl_p(G)$.*

Recall: If G is abelian, then let $H = \{g \in G || g| \mid p^e\}$. Then H is a subgroup of G and $|H| = p^e$. In other words H is a Sylow p-subgroup of G.

证明. Write $|G| = |Z(G)| + |C(g_1)| + \cdots + |C(g_r)|$. If $p||Z(G)|$ then $Z(G)$ has a Sylow p-subgroup N and $N \lhd G$. Then $\bar{G} = G/N$ has order $|\bar{G}| < |G|$. If $|N| = p^e$ then N is a Sylow p-subgroup of G. If $|N| < p^e$ then $|\bar{G}| < p^e m$. By induction, $\bar{G}$ has a Sylow p-subgroup $\bar{N}$, the preimage of $\bar{N}$ is a Sylow p-subgroup of G.

Now suppose $p \nmid |Z(G)|$. Then $p \nmid |C(g_i)|$ for some i. By Orbit-Stabilizer Theorem, $|G| = |C(g_i)| \cdot |C_G(g_i)|$. Then $p^e \mid |C_G(g_i)|$ and by induction $C_G(g_i)$ has a Sylow p-subgroup N, which is also a Sylow p-subgroup of G. $\square$

Theorem 16. *(Cauthy) If $p \mid |G|$ then G has a subgroup of order p.*

Lemma 17. *If G is a p-group i.e. $|G| = p^n$ then G has a non-trivial center.*