Abstract Algebra
: Lecture 9

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Theorem 1. *A_n is a simple group when $n \geqslant 5$.*

Remark 2. *Let $G = \mathrm{Sym}(\Omega) = S_n$, if $|\Omega| = n$.*

- *each element of G is a product of transpositions.*
- *each even permutation of Ω is a product of some 3-cycles.*

证明. Let $N \lhd A_n$, s.t. $N \neq \{e\}$. We aim to prove $N = A_n$.

Claim 1: N cotains a 3-cycle.

Let $e \neq g \in N$ s.t. $|\operatorname{Fix}(g)| \geqslant |\operatorname{Fix}(x)|$ for all $x \in A_n$. In which $\operatorname{Fix}(x) = \{\omega \in \Omega | \omega^x = \omega,\ x \in S_n\}$ called the set of fixed points of x.

Remark 3. *Since g is a even permutation, $|\operatorname{Fix}(g)| \leqslant n-3$.*

If $|\operatorname{Fix}(g)| = n-3$ then g is a 3-cycle as claimed.

To complete the proof of the claim, we assume that $|\operatorname{Fix}(g)| < n-3$. Then relabling if necessary, we may write $g = \begin{cases} (12345\ldots)\ldots \\ (1234)(5\ldots)\ldots \\ (123)(45\ldots)\ldots \\ (12)(34)(5\ldots)\ldots \end{cases}$ or $g = (12)(34)$ which means that we write g into a product of disjoint cycles and put the longest cylce in the first place, and relabling if necessary.

If $\operatorname{Fix}(g) \cap \{1,2,3,4,5\} = \varnothing$ and so $\operatorname{Fix}(g) \subseteq \{6,7,\ldots,n\}$. In particular, $1^g = 2$.

Let $\sigma = (345)$ and $h = [\sigma, g]$. Then $h \in N$ and $1^\sigma = 2^{g^{-1}} = 1$. So $1 \in \operatorname{Fix}(h)$ and h fixes each point in $\operatorname{Fix}(g)$. Thus $|\operatorname{Fix}(h)| \geqslant |\operatorname{Fix}(g)| + 1$, and $h = 1$ by our assumption. Then $[\sigma, g] = 1 \Rightarrow \sigma g = g\sigma$ it is obviously wrong, just need to consider the result of conjugacy action of σ.

This is a contradiction, so g is not a product of two disjoint cycles. It is just a 3-cycle as we claimed. □

Remark 4. *If $g = (12)(34)$ then $h = (354)$ also has more fixed points.*

Lemma 5. *Let $g = (ijk\ldots)$, $\sigma \in S_n$, $g^\sigma = (i^\sigma j^\sigma k^\sigma \ldots)$.*

Remark 6. *1.* $|\operatorname{Fix}(g)| = n - 3 \Leftrightarrow g$ *is a* 3*-cycle.*

2. As g *is even, if* g *is not a* 3*-cycle, then either* g *moves at least* 5 *points or* g *is a product of two disjoint transpositions.*

Claim 2: N contains all 3-cycles.

Remark 7. *each* 3*-cycle is conjugate to* (123) *by even permutations. Let* (ijk) *be an arbitrary* 3*-cycle. Then there exists* $x \in S_n$ *such that* $(ijk)^x = (123)$*.*

1. If $\{i, j, k\} \cap \{1, 2, 3\} = \oslash$*, then let* $x = (1i)(2k)(3j)(23)$*. Then* $(ijk)^x = (123)$*.*

2. Assume $|\{i, j, k\} \cap \{1, 2, 3\}| = 1$*. WLOG,* $i = 1$ *and* $\{j, k\} \cap \{2, 3\} = \oslash$*. Let* $x = (2j)(3k)$*. Then* $(ijk)^x = (123)$*.*

3. Assume $|\{i, j, k\} \cap \{1, 2, 3\}| = 2$*. WLOG,* $i = 1$ *,* $j = 2$ *and* $k = 4$*. Let* $x = (k35)$*.*

That means all 3-cycles are in N since $N \lhd A_n$, and each element fo A_n is a product of 3-cycles so $N = A_n$.

Remark 8. *If* G *is simple then for all* $e \neq g \in G$*, the normal closure* $g^G = G$ *where* $g^G = \langle g^x | x \in G \rangle$*.*

Permutation Group

Definition 9. *For a set* Ω*, each subgroup of* $\operatorname{Sym}(\Omega)$ *is called a permutation group on* Ω*.*

Theorem 10. *(Cayley) Every finite group is isomorphic to a permutation group.*

证明. Let G be a group, $G = \{g_1, g_2, \dots g_n\} = \Omega$. For each element $x \in G$ define a permutation on Ω by $\hat{x} : g_i \mapsto g_i x$. Then $\hat{x} \in \operatorname{Sym}(\Omega)$ (the proof of $\hat{x}$ is a bijection is easy to check).

Let $\hat{G} = \{\hat{x} | x \in G\}$ then $\hat{G}$ is a group which isomorphic to G since $\hat{x}\hat{y} = \widehat{xy}$. Also $\hat{G} \leqslant \operatorname{Sym}(\Omega)$ □

For any $g_i, g_j \in G$ there exists $\hat{x} \in \hat{G}$ s.t. $g_i^{\hat{x}} = g_j$. So $\hat{G}$ is transitive on Ω. And such $\hat{x}$ is unique. So $\hat{G}$ is regular on Ω which means only the identity permutation fixes some point of Ω.