Abstract Algebra
: Lecture 12

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Definition 1. *Let G be a group, we have following subnormal chain/ subnormal series:*

$$G = G_0 \rhd G_1 \rhd \cdots \rhd \{1\}$$

Where $G_{i+1} \lhd G_i$ for all i. All G_i are subnormal subgroups of G, denoted by $G_i \lhd \lhd G$.
*If G_i/G_{i+1} is simple for all i, then this chain is called a **composition series/chain** of G, and each factor group G_i/G_{i+1} is called a **composition factor** of G.*

Example 2. *Let $G = S_4$, $G = G_0 \rhd A_4 \rhd V_4 \rhd Z_2 \rhd 1$, we have $G_0/G_1 \simeq Z_2$, $G_1/G_2 \simeq Z_3$, $G_2/G_3 \simeq Z_2$, $G_3/G_4 \simeq Z_2$.*

Example 3. *Let $G = S_5$, $G = G_0 \rhd A_5 \rhd 1$. $G_0/G_1 \simeq Z_2$, $G_1/G_2 \simeq A_5$.*

Theorem 4. *The number and the set of composition factors of a finite group G is uniquely determined by G.*

Example 5. *Let $G = S_4 \times S_5$.*

$$G = G_0 = S_4 \times S_5 \rhd A_4 \times S_5 \rhd A_4 \times A_5 \rhd A_4 \rhd Z_2 \times Z_2 \rhd Z_2 \rhd 1$$

$$G_0/G_1 \simeq Z_2, G_1/G_2 \simeq Z_2, G_2/G_3 \simeq A_5, G_3/G_4 \simeq Z_3, G_4/G_5 \simeq Z_2, G_5/G_6 \simeq Z_2$$

Composition factors set $\{Z_2, Z_2, Z_2, Z_2, Z_3, A_5\}$.
Choose another composition chain:

$$G = G_0 = S_4 \times S_5 \rhd S_4 \times A_5 \rhd S_4 \rhd A_4 \rhd Z_2 \times Z_2 \rhd Z_2 \rhd 1$$

$$G_0/G_1 \simeq Z_2, G_1/G_2 \simeq A_5, G_2/G_3 \simeq Z_2, G_3/G_4 \simeq Z_3, G_4/G_5 \simeq Z_2, G_5/G_6 \simeq Z_2$$

Composition factors set is the same.

Theorem 6. *Any two subnormal series of a finite group G have subnormal refinements that are equivalent.*

Example 7. *Let* $G = A_4 \rtimes Z_2$.
Exerise*: Find* (1234) *by this semidirect product.*

Definition 8. *Let* G *be a group, and* $A = \mathrm{Aut}(G)$*, Let* $X = \{(g,\sigma)|g \in G, \sigma \in A\}$, $|X| = |G||\mathrm{Aut}(G)|$, $(g_1,\sigma_1)(g_2,\sigma_2) = (g_1 g_2^{\sigma_1^{-1}}, \sigma_1\sigma_2)$*,* X *is a group, called the holomorph of* G*, denoted by* $\mathrm{Hol}(G)$.

- $G \simeq G \times \{1\} = \{(g,1)|g \in G\} \lhd \mathrm{Hol}(G)$;
- $A \simeq \{1\} \times A = \{(1,\sigma)|\sigma \in A\} < \mathrm{Hol}(G)$;
- $\mathrm{Hol}(G) = G \rtimes A$.

Example 9. *Let* $G = Z_3 \times Z_5 = \langle a\rangle \times \langle b\rangle$*. Let* $\sigma \in \mathrm{Aut}(G)$ *s.t.* $a^\sigma = a^{-1}, b^\sigma = b^{-1}$, $G \rtimes \langle\sigma\rangle = D_{30}$

Example 10. *Let* $G = Z_3 \times Z_5 = \langle a\rangle \times \langle b\rangle$*, Let* $\tau \in \mathrm{Aut}(G)$ *s.t.* $a^\tau = a^{-1}, b^\tau = b$*, then* $G \rtimes \langle\tau\rangle = D_6 \times Z_5$.

Example 11. *Let* $\rho \in \mathrm{Aut}(G)$ *s.t.* $a^\rho = a, b^\rho = b^{-1}$, $G \rtimes \langle\rho\rangle = Z_3 \times D_{10}$.

Example 12. *Let* $G = Z_5 = \langle a\rangle$*, Let* $\sigma \in \mathrm{Aut}(G) = Z_4$*, be such that* $a^\sigma = a^{-1}$*, then* $G \rtimes \langle\sigma\rangle = D_{10}$.
Let $\tau \in \mathrm{Aut}(G)$ *s.t.* $a^\tau = a^2$*, then* $G \rtimes \langle\tau\rangle = Z_5 \rtimes Z_4$.
How many different types of $Z_5 \rtimes Z_4$*?*
(1). $x^y = x$, Z_{20};
(2). $x^y = x^{-1}$, $Z_2 \times D_{10}$;
(3). $x^y = x^2$, $Z_5 \rtimes Z_4 = \langle x, y | x^5 = 1, y^4 = 1, y^{-1}xy = x^2\rangle$.

Exercise 13. *Identify* $Z_{17} \rtimes Z_{16}$.

Exercise 14. *Construct* $Z^3 \rtimes Q_8$ *s.t.* $Z(G) = 1$.

Definition 15. *Let* $X = \{x_1, \ldots, x_r\}$, $X' = \{x_1', \ldots, x_r'\}$, $Y = X \cup X'$*. Let* $\{\textit{word on } Y\} = \mathrm{Word}(Y)$.
Let $u, v \in \mathrm{Word}(Y)$*. If* $u = v x_i x_i'$ *then* u, v *are equivalent. Denoted by* $u \sim v$.
If $u = v_1 x_i x_i' x_j x_j' v_2$ *then* u *is equivalent to* $v_1 v_2$*. Denoted by* $u \sim v_1 v_2$.
Thus we have an equivalent relation on $\mathrm{Word}(Y)$.
Let $\bar{W}$ *be a equivalent class of* $w \in \mathrm{Word}(Y)$*. Define* $G = \{\bar{w}|w \in \mathrm{Word}(Y)\}$*. For* $\bar{w}_1, \bar{w}_2 \in G$*, let* $\bar{w_1}\bar{w_2} = \overline{w_1 w_2}$*. Then* G *is a group. Called a finite generated free group of rank* r.

Example 16. *Let* $G = \langle a, \sigma\rangle = D_{20}$ *There is a free group* F *of rank* 2 *s.t.* G *is a homomorphism image of* F *s.t.* $\phi : x_1 \mapsto a, x_2 \mapsto \sigma, x_1' \mapsto a^{-1}, x_2' \mapsto \sigma^{-1}, \varnothing \mapsto 1$, $\ker(\phi) = \langle x_1^5, x_2^2, x_1x_2x_1x_2\rangle$*. i.e.* $F/\ker(\phi) \simeq G$.