1. A_5: $\quad n_2 = 5$

check. $A_4 \hookrightarrow A_5$ $\quad V_4 \hookrightarrow A_4$. only sylow 2 subgroup

$C_5^4 = 5 \quad \Rightarrow 5$ for A_5.

in fact. possible n_2 : $\quad 1, \; 3, \; 5, \; 15$

$\quad\quad$ $\uparrow$ X, simple $\quad$ $\uparrow$ X $\quad$ $\uparrow$ at least

If has 15 sylow 2. $\quad$ sylow 3 $\quad 10 \quad$ sylow 5 $\quad 6 \quad\quad e$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 20 \quad\quad\quad\quad 24 \quad\quad\quad 1$

left 15 elements. ↯

$n_3 = 1, \; 4, \; 10$
$\quad\;\; X \;\; X \;\; \checkmark$

for C_3 $\quad (123)$ $\quad\quad C_5^3 = 10$. at least 10 group.

$n_5 = 1, \; 6$
$\quad\;\; X \;\; \checkmark$

S_5.

$n_2 = 1 \quad 3 \quad 5 \quad 15$
$\quad\quad X \quad X \quad \uparrow$ at least.

$n_3 = 10 \quad n_5 = 6.$

$n_2 = 15$. check.

$P \in Syl_2(S_5) \quad P \cong \underbrace{D_8 \hookrightarrow S_4}_{3} \xhookrightarrow{C_5^4} S_5$

$3 \times 5 = 15$.

$D_8(1) = \{ e, (12), (34), (12)(34), (1324), (14)(23), (4231) \}$

$D_8(2) = \{ e, (13), (24), (13)(24), (1234), (14)(32), (4321) \}$

$D_8(3) = \{ e, (14), (23), (14)(23), (1243), (13)(42), (3421) \}$

2. For $D_8 = \langle a, b \mid a^4 = b^2 = 1, \; bab = a^{-1} \rangle$

$\{1\}$

$\{a^2\}$

$\{a, a^3\}$

$\{ab, a^3b\}$

$\{b, a^2b\}$

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$\{1\}, \{-1\}, \{\pm i\}$ $\{\pm j\}$ $\{\pm k\}$

A_4.

$\{(1)\}$

$\{(12)(34), (13)(24), (14)(23)\}$

$\{(123), (243), (134), (142)\}$

$\{(132), (234), (143), (124)\}$

$\mathbb{Z}_2 \times S_3$.

$\{(1, (1))\}$ $\{(1, (12)), (1, (13)), (1, (23))\}$

$\{(1, (123)), (1, (132))\}$

$\{(-1, (1))\}$ $\{(-1, \ldots\ldots$ 把上面 $(1, -)$ 改 $(-1, -)$

~~3.~~ (a) $\{4, 9\}$ $\{9, 4\}$

(b). $\{2^2, 2\cdot 3^2\}$ $\{2^2\cdot 3^2, 2\}$

(c). 除了第三个

(d). 最后两个

~~4.~~ (a). $|G| = 16$ $|\langle v^4 \rangle| = 2$ $\Rightarrow |\bar{G}| = 8$

(b) $\bar{I}$ $\bar{v}$ $\overline{v^2}$ $\overline{v^3}$ $\bar{u}$ $\overline{uv}$ $\overline{uv^2}$ $\overline{uv^3}$

(c) 1 4 4 4 2 2 2 2

(d) $\overline{u v^3}$ $\overline{v^2}$ $\overline{1}$

(e) $\overline{u^{-1}v^{-1}uv} = \overline{1} \Rightarrow [\bar{u},\bar{v}] = \bar{1} \Rightarrow$ abelian.

$\varphi : \bar{G} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4$

$\bar{u}^a \bar{v}^b \longmapsto (x^a, y^b)$ where $\mathbb{Z}_2 = \langle x \rangle$ $\mathbb{Z}_4 = \langle y \rangle$

Show it's iso.

3. (a) $72 = 2^3 \times 3^2$ $\mathbb{Z}_8$ $\mathbb{Z}_{4\times 2}$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $\mathbb{Z}_9$ $\mathbb{Z}_3 \times \mathbb{Z}_3$

$\mathbb{Z}_8 \times \mathbb{Z}_9$
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9$ 6 in total
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

(b) Consider $Syl_r(G)$ $n_r \equiv 1 \mod r$ $n_r \mid pq$

Thus $n_r = 1$ or $n_r = pq$. if $n_r = 1$, $R \in Syl_r(G)$ unique

$R \lhd G$

Therefore. assume $n_r = pq$

Consider $Syl_q(G)$. $n_q \equiv 1 \mod q$, $n_q \mid pr$

Thus $n_q = 1$ or r or pr. to guarantee $Q \ntriangleleft G$

$|Syl_q(G)| \geq r$, assume $n_q = r$

Consider $Syl_p(G)$, $n_p \equiv 1 \mod p$ $n_p \mid qr$

Thus $n_p = 1$ or q or r or qr .... $|Syl_p(G)| \geq q \Rightarrow$ assume $n_p = q$

Thus in G, elements of

order r: $pq(r-1)$

order q: $r(q-1)$

order p: $q(p-1)$

$$pq(r-1)+r(q-1)+q(p-1)$$
$$=pqr-pq+rq-r+pq-q$$
$$=pqr+r(q-1)-q$$
$$>pqr+r-q>pqr$$ contradiction!

4. (a). Consider. $G=\{g_1,\dots,g_{2n}\}$

$$\varphi: G \hookrightarrow S_{2n}$$
$$g \mapsto \sigma_g: \begin{array}{l} G \to G \\ g_i \mapsto g_i g = g_{i^{\sigma_g}} \end{array}$$

Since $|G|=2n$. n is odd. By Sylow 1st Thm

$\exists\ P_2 \leq G$ s.t. $|P_2|=2$. Obviously P_2 is cyclic.

Let $P_2=\{e,\tau\}$

Consider σ_τ, ① it's of order 2. so it a production of 2-cycles

② σ_τ has no fixed point. so it's a production of n 2 cycles.

i.e. σ_τ is a odd permutation.

Let H be the preimage of $\varphi(G)\cap A_{2n}$ under φ

Then $H \trianglelefteq G$ and $G=H \sqcup \tau H$

$\Rightarrow H$ has index 2.

Furthermore. index 2 subgrp is normal since

left coset = right coset.

and $|H| = n$ is odd. By Feit Thompson Thm H is solvable.

G/H is solvable.

$\Rightarrow G$ is solvable.

(b). Let $|G| = 2^m \cdot n$, n is odd.

Then $P \in Syl_2(G)$ is sylow 2 subgrp of G and

P is cyclic with $|P| = 2^m$.

$\exists\ \tau \in P$ s.t $|\tau| = 2^m$.

Now consider $G \overset{\varphi}{\hookrightarrow} Sym(2^m \cdot n) \xrightarrow{sgn} \{\pm 1\}$

it shows $\varphi(\tau)$ is odd permu. i.e $sgn(\varphi(\tau)) = -1$

$\Rightarrow$ $sgn \circ \varphi$ surjective with $\ker \trianglelefteq G$ denoted by K.

$|K| = 2^{m-1} n$. $K \trianglelefteq G$ since each P is cyclic in G

$\Rightarrow K$ also has cyclic Syl_2 subgrp

$\Rightarrow \exists\ \tau_1 \in K$ s.t $|\tau_1| = 2^{m-1}$

repeat this process finally there $\exists\ M$ s.t $|M| = n$ is odd

and $M \trianglelefteq G$.

5. Let $h_1 \in Z(H)$, $h_1 = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}$

$\forall\ h \in H$, $h = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$

$\Rightarrow hh_1 = h_1 h \Rightarrow \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+a & b+xc+y \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow az = xc$, $\forall\ a.c \in \mathbb{R}$ $\Rightarrow x = z = 0$

$\Rightarrow Z(H) = \left\{ \begin{pmatrix} 1 & & y \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$.

Consider. $\varphi: H \longrightarrow \mathbb{R} \times \mathbb{R}$ s.t

$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \longmapsto (a.c)$

Then $H/Z(H) \cong \underbrace{(\mathbb{R} \times \mathbb{R}, +)}_{\text{abelian}}$

so $H/Z(H)$ abelian.

6. $K_1 \lhd K \Rightarrow H\cap K_1 \lhd H\cap K. \Rightarrow H_1(H\cap K_1) \lhd H_1(H\cap K)$

$\Rightarrow \quad H_1(H\cap K_1)/H_1(H\cap K)$

$= H_1(H\cap K_1)/(H_1(H\cap K))\cap(H_1(H\cap K_1))$

$= H_1(H\cap K_1)(H\cap K)/H_1(H\cap K)$

$= H\cap K/(H_1(H\cap K_1))\cap(H\cap K)$

$$(H_1(H\cap K_1))\cap(H\cap K) = (H\cap K_1)\cap(H_1\cap(H\cap K))$$
$$= (H\cap K_1)\cap(H_1\cap K)$$

$\Rightarrow$ left $\cong H\cap K/(H\cap K_1)\cap(H_1\cap K)$

Similarly. right $\cong H\cap K/(H\cap K_1)(H_1\cap K)$

Accordingly we can draw the lattice:

$$\begin{tikzcd}
H \arrow[dr, dash] & & & & K \arrow[dl, dash] \\
& H_1(H\cap K) \arrow[dr, dash] \arrow[dd, dash] & & K_1(K\cap H) \arrow[dl, dash] \arrow[dd, dash] & \\
& & H\cap K \arrow[dd, dash] & & \\
& H_1(H\cap K_1) \arrow[dl, dash] \arrow[dr, dash] & & K_1(K\cap H_1) \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & & (H_1\cap K)(K_1\cap H) \arrow[dl, dash] \arrow[dr, dash] & & K_1 \arrow[dl, dash] \\
& H_1\cap K & & K_1\cap H &
\end{tikzcd}$$