CMPS 203 – HM3

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Quetiion 1

Question. In the WHILE language, prove or disprove the equivalence of the two commands:

$$\mathrm{t := x; x := y; y := t;}$$

and

$$\mathrm{t := y; y := x; x := t;}$$

Proof. we define the first commands as C_1 and the second commands as C_2. Hence, the two commands are equivalence only if

$$\langle \mathrm{C}_1, s\rangle \rightarrow s' and \ \langle \mathrm{c}_2, s\rangle \rightarrow s'$$

However, we assume s is $s[t : t_0, x : x_0, y : y_0]$.
It easy to get the first commands $\langle \mathrm{C}_1, s\rangle \rightarrow s'[t : x_0, x : y_0, y : x_0]$.
But the second commands' state is $\langle \mathrm{C}_2, s\rangle \rightarrow s''[t : y_0, x : y_0, y : x_0]$.
Hence, $s' \neq s'' \Rightarrow \mathrm{C}_1 \neq \mathrm{C}_2$
As a result, we disprove the semanticlly equivalent of these two commands.

Quetiion 2

Question. In the WHILE , prove that if

$$\langle \mathrm{while\ b\ do\ y := y - x, s}\rangle \Downarrow \mathrm{s}'$$

then there exists an integer k such that

$$s(y) = s'(y) + k \ * \ s(x)$$

Please make it explicit if/when you reason by induction on derivations, stating your induction hypothesis.

Proof. We assume the command always terminate. we prove the command with induction.

- Base Case: $p(0)$ means this while run zero times. we have

$$\frac{\langle b, s\rangle \Downarrow \mathrm{false} \qquad \langle skip, s\rangle \Downarrow \ s'}{\langle \mathrm{while\ b\ do\ y\ :=\ y\ -\ x}, s\rangle \Downarrow s'}$$

It's easy to see $s(y) = s'(y) + 0 \times s(x)$

- We assume p(n) means while run n loops and we have

$$s(y) = s'(y) + n \times s(x)$$

We need to prove $p(n+1)$ which is $\langle while\ b\ do\ y\ :=\ y\ -\ x, s\rangle \Downarrow s'$ we have

$$s(y) = s''(y) + (n+1) \times s(x)$$

- We can build the derivation tree as blow

$$\frac{\langle b, s\rangle \Downarrow \text{true} \qquad \langle \text{while b do y } := \text{ y } - \text{ x}, s\rangle \Downarrow s' \qquad \langle \text{y} := \text{y } - \text{ x}, s'\rangle \Downarrow s''}{\langle \text{while b do y } := \text{ y } - \text{ x}, s'\rangle \Downarrow s''}$$

by induction hypothiesi, we know that

$$s(y) = s'(y) + n \times s(x)$$

we also know the derivation tree about $p(n+1)$ that

$$s'(y) = s''(y) + s(x)$$

Because of $\langle \text{y} := \text{y } - \text{ x}, s'\rangle \Downarrow s''$, Hence, we have

$$\begin{aligned} s(y) &= s'(y) + n \times s(x) \\ &= s''(y) + x + n \times s(x) \\ &= s''(y) + (n+1) \times s(x) \end{aligned}$$

Hence, the statement is proved.

Quetiion 3

$Question.$ In the WHILE , prove

$$\forall c_1, c_2, c_3\ .\ c_1; (c_2; c_3)\ \approx\ (c_1; c_2); c_3$$

$Prove.$ We assume

$$\langle c_1, s_0\rangle \Downarrow s_1$$

$$\langle c_2, s_1\rangle \Downarrow s_2$$

$$\langle c_3, s_2\rangle \Downarrow s_3$$

We have derivation tree of right part.

$$\frac{\dfrac{\langle c_1, s_0\rangle \Downarrow s_1 \qquad \langle c_2, s_1\rangle \Downarrow s_2}{\langle c_1; c_2, s_0\rangle \Downarrow s_2} \qquad \langle c_3, s_2\rangle \Downarrow s_3}{\langle (c_1; c_2); c_3, s_0\rangle \Downarrow s_3}$$

derivation tree of left part.

$$\frac{\langle c_1, s_0\rangle \Downarrow s_1 \qquad \dfrac{\langle c_2, s_1\rangle \Downarrow s_2 \qquad \langle c_3, s_2\rangle \Downarrow s_3}{\langle c_2; c_3, s_1\rangle \Downarrow s_3}}{\langle c_1; (c_2; c_3), s_0\rangle \Downarrow s_3}$$

Hence we have

$$\langle (c_1; c_2); c_3, s_0\rangle \Downarrow s_3$$

$$\langle c_1; (c_2; c_3), s_0\rangle \Downarrow s_3$$

Hence, we prove that

$$\forall c_1, c_2, c_3 \ . \ c_1; (c_2; c_3) \ \approx \ (c_1; c_2); c_3$$