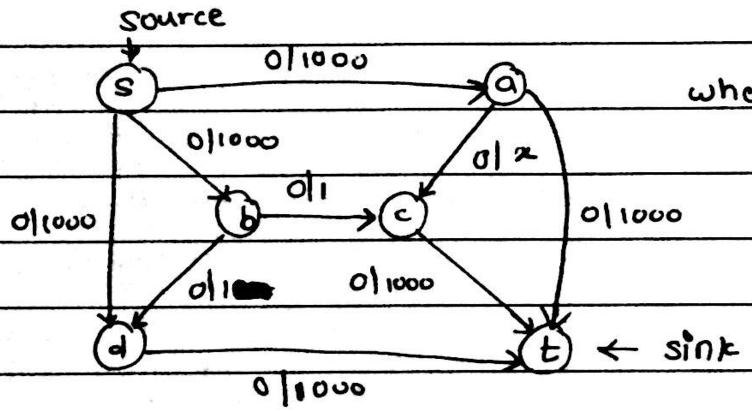
1. ⓐ To get the integers out of rational numbers, we can multiply all the rational capacities of the graph by the divisors least common multiple (LCM).

- At least one capacity for each augmenting path needs to be augmented according to Ford-Fulkerson Algorithm.
- Therefore, now in order to find Max-Flow we need to divide the output by LCM.
- In case of irrational capacities, the augmenting path can be chosen arbitarily. and Ford-Fulkerson may not terminate because of the number of loops that can lead to infinity
- Consider below diagram / network;



where,

$\cancel{x < \sqrt{5}-1}$

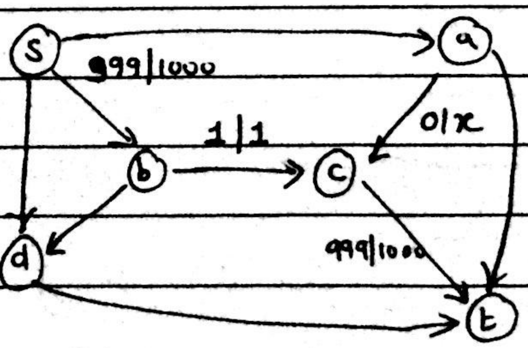
$$x = \frac{\sqrt{5}-1}{2} = \underline{\underline{0.6180}}$$

$$1 - x = 1 - 0.6180$$
$$= \underline{\underline{0.382}}$$

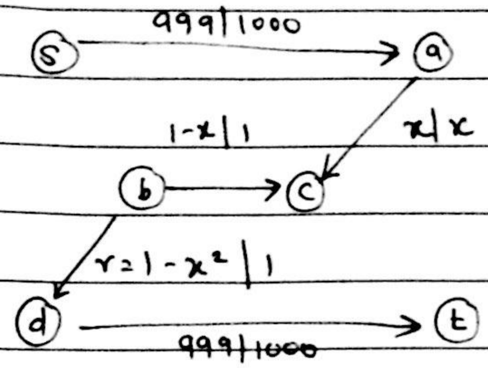
$$\therefore x^2 = \underline{\underline{0.382}}$$

$$\therefore \boxed{x^2 = 1 - x}$$

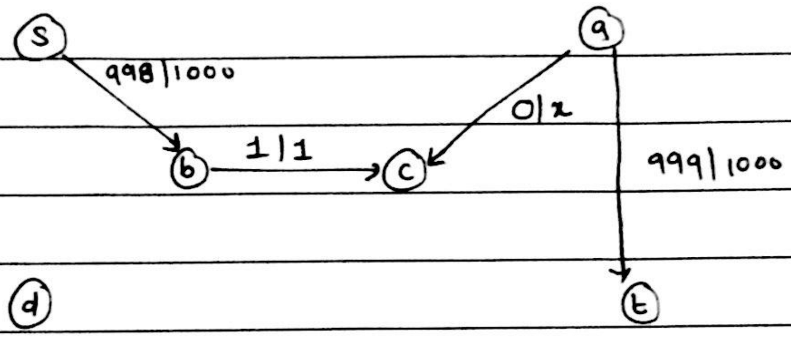
- Augmenting path 1 – < s, b, c, t > { min capacity = 1 }



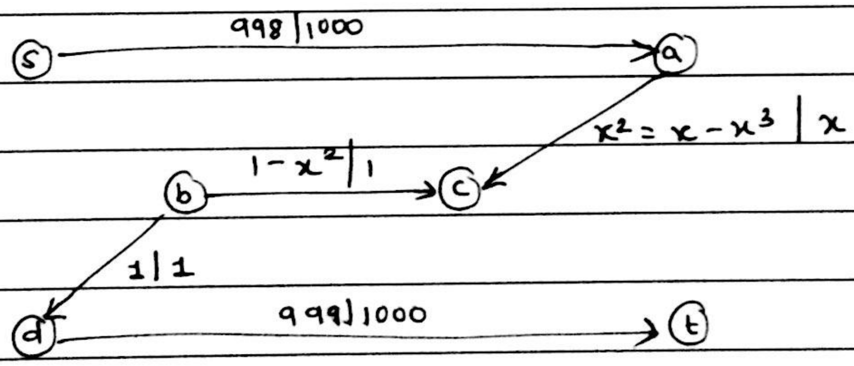
1. ⓐ - Augmenting path 2 → < s, a, c, b, d, t > { min capacity = x }



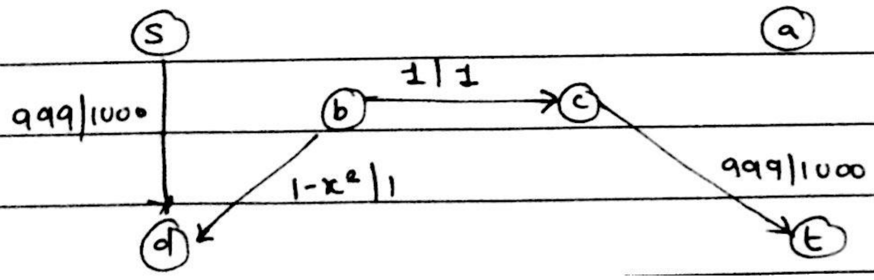
- Augmenting path 3 → < s, b, c, a, t > { min capacity = x }



- Augmenting path 4 → < s, a, c, b, d, t > { min capacity = x^2 }



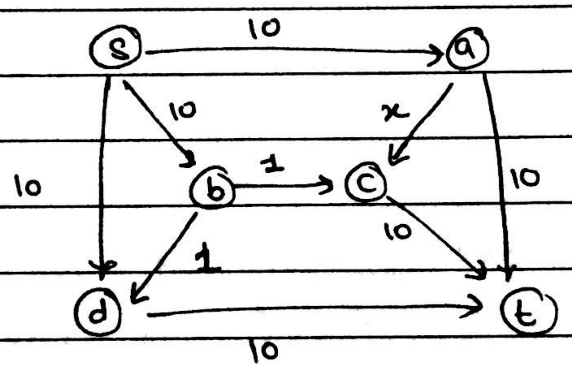
- Augmenting path 5 → < s, d, b, c, t > { min capacity = x^2 }



1. (a) The flows of the above network augmenting paths are :- $1, x, x, x^2, x^2, \ldots$ which never terminates

- Hence, the Ford-Fulkerson Algorithm tends to fail to return max-flow of this network which produces the above augmented paths.
- Hence proved that Ford-Fulkerson algorithm/method fail to terminate only if edge capacities are irrational numbers.

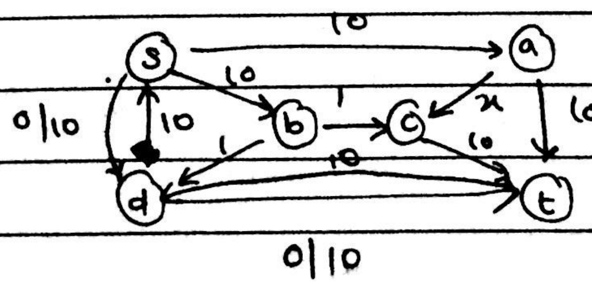
(b) Consider same network with irrational capacities as above; (in 1.a)



where;

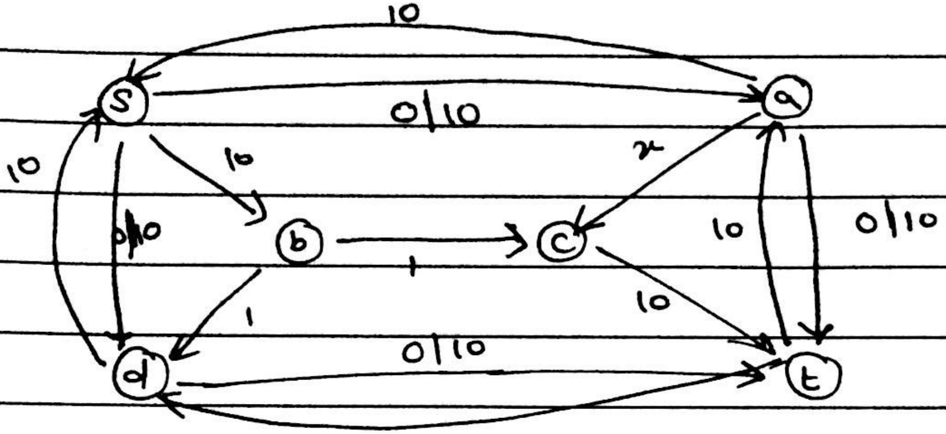
$$x = \frac{\sqrt{5} - 1}{2} = \underline{0.6180}$$

(1) Path $P_0 \rightarrow \langle s, d, t \rangle$ {Minimum capacity = $\underline{10}$}

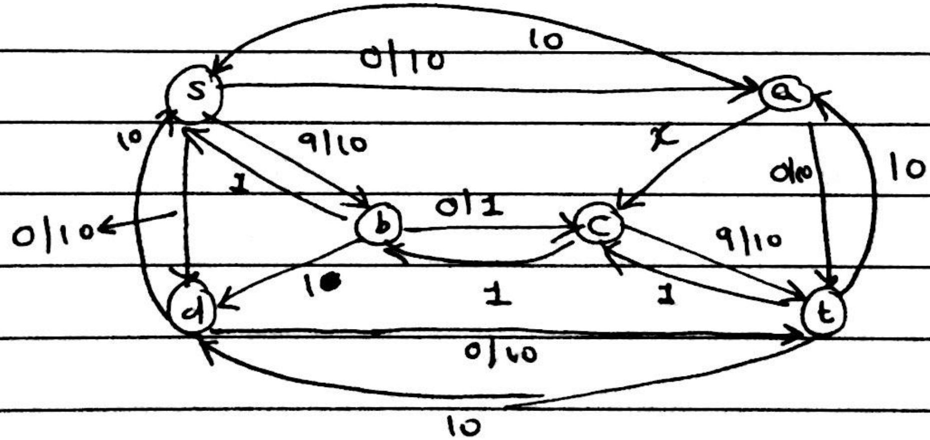


1. (b)

(2) Path $P_1 \rightarrow \langle s, a, t \rangle$ {minimum capacity (bottleneck) = 10}



(3) Path $P_2 \rightarrow \langle s, b, c, t \rangle$ {minimum capacity = 1}



⇒ We cannot proceed further for new path as source (s) = 0

1. (b) $\therefore$ Max Flow = capacities of paths $(P_0 + P_1 + P_2)$
{minimum capacities}

$= 10 + 10 + 1$ ... from (1), (2), (3)

$= \underline{\underline{21}}$

However, the algorithm Ford-Fulkerson returns

$1 + x + x + x^2 + x^2 \ldots$ (from Ans. 1. (a))

$\therefore$ After $(1+4k)^{th}$ augmented path $= 1 + 2 \sum_{i=1}^{k} x^i$

$\leq 1 + 2 \sum_{i=1}^{\infty} x^i$ where $x = \frac{\sqrt{5}-1}{2}$

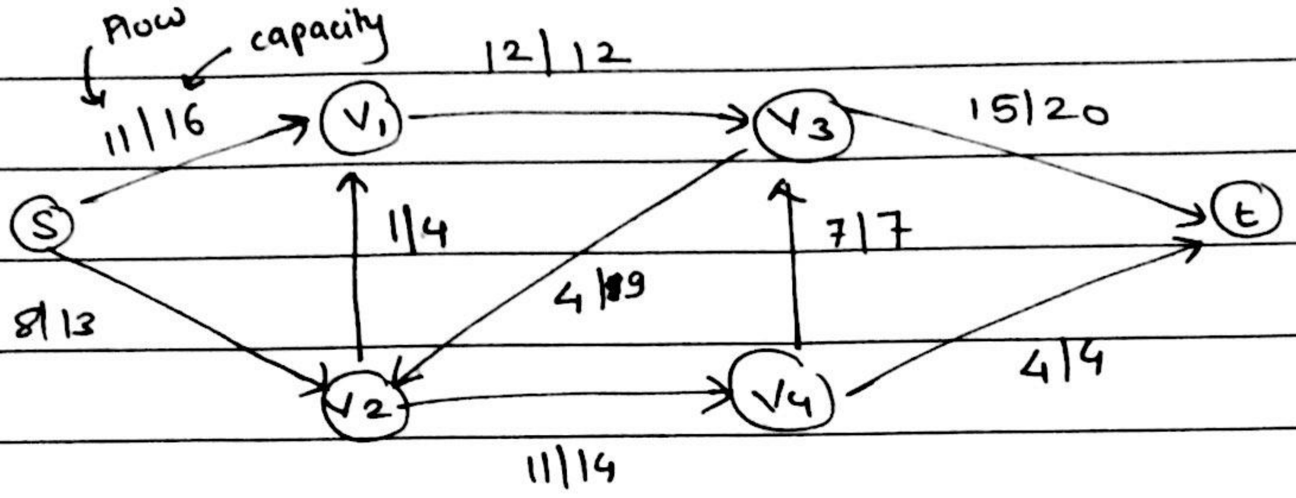
$= 3 + 2x$

< 5 and $\neq$ max-flow of 21

Thus, Ford-Fulkerson algorithm fails when the capacities are irrational and it does not ~~cover~~ converge to the maximum flow.

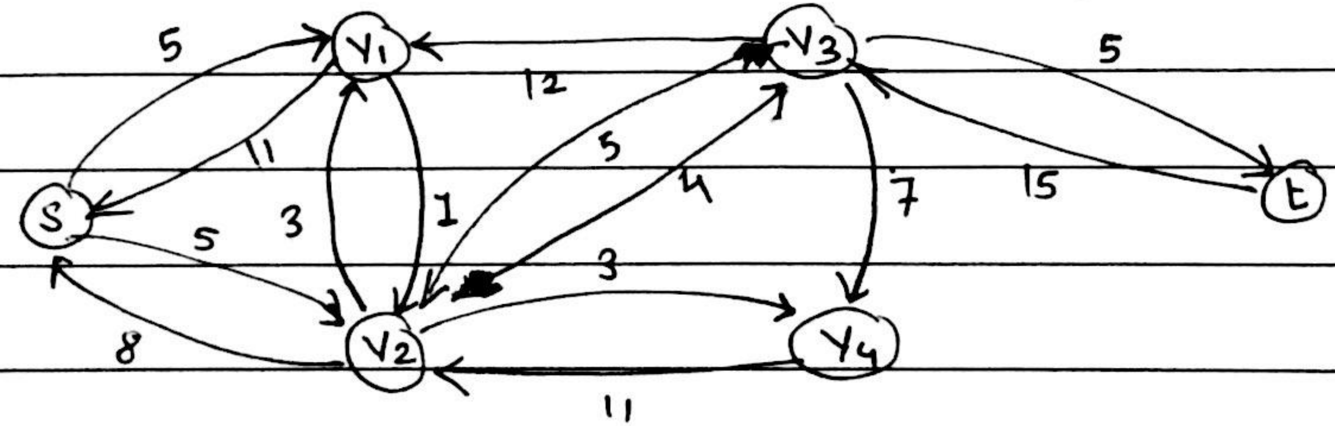
- References Ans. 1 (a), (b) $\rightarrow$ mit.edu, cse.unt.edu

2. Consider below network graph ;



① Find an edge (u,v) where flow(u,v) is (minimum) → 0 (zero)

② Now, find a path from source to sink (t) containing (u,v) & reduce the flow on the path by f(u,v)



2. The edge between v_1 and v_3 is removed since its flow is zero (0).

- Similarly, edge v_4 to t is removed.
- These edges will not be selected again as ~~its flow~~ ~~is~~ their flow are zero now

(3) Since one edge is removed each time, we do this at most $|E|$ times.

(4) Each path on which we reduce flow could be augmented path.

(5) From the above observations and mechanism we could get to our max-flow with at most $|E|$ augmentations/augmenting paths.

3. Here we need to prove two things. Firstly, any overflow vertex say 'x' in initial step i.e. when network is initialized has a simple path back to source 's'. Secondly, simple path from $x \rightarrow s$ exist and after RELABLE or PUSH operation the new overflowing path at vertex 'u' still have a simple path to 's'. $\{u \rightarrow s\}$

① Initialization:

- Initially s is at height $|V|$ and all the adjacent / neighbouring nodes from source 's' are at height zero (0)
- source sends the flow to these group of vertices (G_s)
- The vertices in G_s are the only overflowing vertices currently
- In residual network, their are exactly same number of edges from neighbouring vertices (G_s) to s. So, considering this as simple paths from overflowing vertices then their exist simple path ~~to between~~ from $G_s \xrightarrow{to} s$.

[redacted]:

② Now suppose their exist simple path ~~betwe~~ from 'a' to 's'.

- If RELABELED is done on this node then only height of 'a' is changed, but their exist a simple path ~~between~~ from $a \rightarrow s$.
- If PUSH operation takes place, the 'a' is connected to s by initial path (previous operation path)

3. - If PUSH affects path $a \to u \to v \ldots \to z$, then in residual network there must be path $z \to \ldots v \to u \to a$ in G_f.

- Hence, in G_f any new overflowing vertex in the affected path $u \to \ldots \to v$ has a simple path to $a \to s$.