IE531: Algorithms for Data Analytics
Spring, 2023
Homework 2: Semidefinite Programming and Data Analytics
Due Date: February 17, 2023

Instructions

1. You will submit a PDF-version of your answers on Canvas on-or-before midnight of the due date.

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In Section 12 of Lesson 1, you were introduced to the canonical *Semidefinite Program* (SDP)

$$\left(\begin{array}{c} min : \mathbf{C} \circ \mathbf{X} \\ \mathbf{A}_i \circ \mathbf{X} = b_i, i \in \{1, 2, \ldots, m\} \\ \mathbf{X} \geq \mathbf{0} \end{array}\right) \equiv \left(\begin{array}{c} min : trace(\mathbf{C}^T\mathbf{X}) \\ trace(\mathbf{A}_i^T\mathbf{X}) = b_i, i \in \{1, 2, \ldots, m\} \\ \mathbf{X} \geq \mathbf{0} \end{array}\right) \equiv \left(\begin{array}{c} min : trace(\mathbf{C}\mathbf{X}) \\ trace(\mathbf{A}_i\mathbf{X}) = b_i, i \in \{1, 2, \ldots, m\} \\ \mathbf{X} \geq \mathbf{0} \end{array}\right)$$

where $\mathbf{C}$ and $\{\mathbf{A}_i\}_{i=1}^m$ are symmetric matrices that are known, and $\mathbf{X}$ is a symmetric matrix of unknown variables. I showed you how to use the package `cvxpy` to solve instance of an SDP, as well. In this homework, you are going to show that a some of the archetypal problems in Data Analytics can be cast as an instance of an SDP (and effectively solved using an SDP-solver).

We will pay attention to a particular problem called the *Norm Minimization* problem in this HW. We have a r-many, d-dimensional data vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r \in \mathcal{R}^d$. We wish to find a d-dimensional vector that "represents" this set of vectors, by solving

$$\min_{\mathbf{y}} \left\{ \max_{i=1,2,\ldots,r} \|\mathbf{y} - \mathbf{b}_i\|_2 \right\}.$$

That is, $\mathbf{y} \in \mathcal{R}^d$ is a vector that minimizes the maximum 2-norm/Euclidean-distance between itself and each of the r-many data vectors. You will show that this problem is essentially an instance of an SDP, which can be solved by using an SDP-solver.

1. (*20 points*) **Gershgorin's Theorem**

   (a) (*10 points*) Let $\mathbf{A}$ be an $n \times n$ matrix of real numbers. You can view the matrix as a collection of rows, or as a collection of columns. Let us start with a row-view of the matrix. For $i \in \{1, 2, \ldots, n\}$, let

   $$R_i = \sum_{j=1; j \neq i}^{n} \mid a_{i,j} \mid$$

   denote the sum of the absolute-values of the entries of $\mathbf{A}$ in the i-th row, not including the diagonal-element (i.e. $a_{i,i}$). Using this, define the interval $[a_{i,i} - R_i, a_{i,i} + R_i]$. These intervals are called the set of *Gershgorin Row Intervals*. Show that the eigenvalues of $\mathbf{A}$ lie in some Gershogorin Row Interval.

(b) (*10 points*) As a follow on to problem 1a, define the Gershgorin Column Intervals

$$\{[a_{i,i} - C_i, a_{i,i} + C_i]\}_{i=1}^{n}, \text{ where } C_i = \sum_{j=1; j \neq i}^{n} | a_{j,i} |,$$

and show that the eigenvalues of $\mathbf{A}$ lie in some Gershgorin Column Interval.

2. *(30 points)* **Second-Order Cone Programming**: The *Second-Order Cone* (also called *Lorenz Cone*, *Ice-Cream Cone*, etc) is the set of n-dimensional vectors, Q_n, where

$$Q_n := \left\{ \mathbf{x} \in \mathcal{R}^n \mid \mathbf{x} = \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix}, y_0 \geq \|\mathbf{y}\|_2, y_0 \in \mathcal{R}, \mathbf{y} \in \mathcal{R}^{n-1} \right\}.$$

Show that

(a) Q_n is convex, and

(b)

$$(\mathbf{x} \in Q_n) \Leftrightarrow \left(\begin{pmatrix} y_0 & \mathbf{y}^T \\ \mathbf{y} & y_0 \times \mathbf{I} \end{pmatrix} \geq \mathbf{0} \right)$$

3. (*30 points*) **Norm Minimization**: We have a r-many, d-dimensional data vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r \in \mathcal{R}^d$. We wish to find a $\mathbf{y} \in \mathcal{R}^d$ that solves

$$\min_{\mathbf{y}} \left\{ \max_{i=1,2,\ldots,r} \|\mathbf{y} - \mathbf{b}_i\|_2 \right\}.$$

Show that

$$\left(\begin{array}{c} \min t \\ \|\mathbf{y} - \mathbf{b}_i\| \leq t \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \max -t \\ \begin{pmatrix} 0 \\ \mathbf{b}_i \end{pmatrix} - \begin{pmatrix} -t \\ \mathbf{y} \end{pmatrix} \in Q_{d+1} \text{ for } i \in \{1, 2, \ldots, r\} \end{array} \right)$$

4. (*20 points*) **SDP-formulation of the Norm-Minimization Problem** Using the above observations present an SDP-formulation for finding the vector $\mathbf{y} \in \mathcal{R}^d$ that solves the Norm-minimization problem.

Hint: Use your Google-skills to look for a formal treatment that Second-Order Cone Programming (SOCP) Problems are a special instance of Semidefinite Programming (SDP) Problems.