IE531: Algorithms for Data Analytics
Spring, 2023
Homework 3: SVD and Related Topics
Due Date: 3 March 2023

Instructions

1. You will submit a PDF-version of your answers on Canvas on-or-before midnight of the due date.

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1. (*40 points*) Suppose a matrix $n \times d$ matrix $\mathbf{A}$ has an SVD decomposition that can be written as

$$\underbrace{\mathbf{A}}_{n \times d} = \left(\underbrace{\mathbf{U}_1}_{n \times r_1} \times \overbrace{\mathbf{\Sigma}_1}^{r_1 \times r_1} \times \underbrace{\mathbf{V}_1^T}_{r_1 \times d} \right) + \left(\underbrace{\mathbf{U}_2}_{n \times r_2} \times \overbrace{\mathbf{\Sigma}_2}^{r_2 \times r_2} \times \underbrace{\mathbf{V}_2^T}_{r_2 \times d} \right)$$

where the singular-values in $\mathbf{\Sigma_1}$ (resp. $\mathbf{\Sigma_2}$) are greater than (resp. lesser than) some $\gamma \in \mathcal{R}$. Show that

(a) (*20 points*)
$$\mathbf{U}_1^T \times \mathbf{U}_2 \times \mathbf{\Sigma_2} \times \mathbf{V}_2^T = \underbrace{\mathbf{0}}_{r_1 \times d}, \text{ and}$$

(b) (*20 points*)
$$\mathbf{U}_2 \times \mathbf{\Sigma_2} \times \mathbf{V}_2^T \times \mathbf{V}_1 = \underbrace{\mathbf{0}}_{d \times r_1}$$

2. (*40 points*) Suppose a matrix $n \times d$ matrix $\mathbf{A}$ with rank r, and has an SVD

$$\underbrace{\mathbf{A}}_{n \times d} = \underbrace{\mathbf{U}}_{n \times r} \times \overbrace{\mathbf{\Sigma}}^{r \times r} \times \underbrace{\mathbf{V}^T}_{r \times d}.$$

Let us suppose $\mathbf{A}$ gets "corrupted" by a $n \times d$, *noise-matrix* $\mathbf{E}$, and $\mathbf{A}_1 = \mathbf{A} + \mathbf{E}$. Suppose the corrupted-matrix $\mathbf{A}_1$ has an SVD

$$\underbrace{\mathbf{A}_1}_{n \times d} = \underbrace{\mathbf{U}_1}_{n \times r_1} \times \overbrace{\mathbf{\Sigma}_1}^{r_1 \times r_1} \times \underbrace{\mathbf{V}_1^T}_{r_1 \times d}.$$

Suppose $\widehat{\mathbf{\Sigma}}_1$ is obtained from $\mathbf{\Sigma}_1$ by keeping only the top r-many entries (i.e. we zero-out all diagonal-values that not in the list of top r-many SVs). Let $\widehat{\mathbf{A}} = \mathbf{U}_1 \times \widehat{\mathbf{\Sigma}}_1 \times \mathbf{V}_1^T$. Show that

$$||\widehat{\mathbf{A}} - \mathbf{A}||_F \leq \sqrt{8r} \times ||\mathbf{E}||_2.$$

3. (*20 points*) Let $\mathbf{A}$ be an $n \times d$ matrix (of real numbers) that can be partitioned as

$$\underbrace{\mathbf{A}}_{n\times d} = \begin{pmatrix} \underbrace{\mathbf{A}_1}_{n_1\times d_1} & \underbrace{\mathbf{A}_2}_{n_1\times d_2} \\ \underbrace{\mathbf{A}_3}_{n_2\times d_1} & \underbrace{\mathbf{A}_4}_{n_2\times d_2} \end{pmatrix},$$

where $n = n_1 + n_2$ and $d = d_1 + d_2$ (obviously).

(a) (*10 points*) Show that

$$rank(\mathbf{A}) \leq rank(\mathbf{A}_1) + rank(\mathbf{A}_2) + rank(\mathbf{A}_3) + rank(\mathbf{A}_4)$$

(b) (*10 points*) Suppose for each $\{\mathbf{A}_i\}_{i=1}^4$ there is a corresponding set of matrices $\{\mathbf{B}_i\}_{i=1}^4$ such that $\forall i, \|\mathbf{A}_i - \mathbf{B}_i\|_F \leq \epsilon$ show that

$$\left\| \mathbf{A} - \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \right\|_F \leq 4\epsilon$$