IE531: Algorithms for Data Analytics
Fall, 2022
Homework 5: Markov Chain Monte Carlo Methods
Due Date: 7 April 2023

Consider the 9-state Markov Chain shown in figure 1, with the desired stationary probability distribution of

$$\begin{array}{lll} \pi(x_{1,1}) = \frac{8}{37} & \pi(x_{1,2}) = \frac{6}{37} & \pi(x_{1,3}) = \frac{4}{37} \\ \pi(x_{2,1}) = \frac{3}{37} & \pi(x_{2,2}) = \frac{4}{37} & \pi(x_{2,3}) = \frac{2}{37} \\ \pi(x_{3,1}) = \frac{4}{37} & \pi(x_{3,2}) = \frac{4}{37} & \pi(x_{3,3}) = \frac{2}{37} \end{array}$$

Using the *Gibbs Sampling Procedure* find the 9×9 probability matrix $\mathbf{P}$ such that if $\mathbf{A} = \lim_{k\to\infty} P^k$, then

$$\mathbf{A} = \begin{pmatrix} \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \\ \pi(x_{1,1}) & \pi(x_{1,2}) & \pi(x_{1,3}) & \pi(x_{2,1}) & \pi(x_{2,2}) & \pi(x_{2,3}) & \pi(x_{3,1}) & \pi(x_{3,2}) & \pi(x_{3,3}) \end{pmatrix}$$

This problem is a corrected-version of the problem in page 85 of the text (cf. figure 2). There are two errors in the solution provided in the book. First, the book lists the desired stationary probability distribution as

$$\begin{array}{lll} \pi(x_{1,1}) = \frac{1}{3} & \pi(x_{1,2}) = \frac{1}{4} & \pi(x_{1,3}) = \frac{1}{6} \\ \pi(x_{2,1}) = \frac{1}{8} & \pi(x_{2,2}) = \frac{1}{6} & \pi(x_{2,3}) = \frac{1}{12} \\ \pi(x_{3,1}) = \frac{1}{6} & \pi(x_{3,2}) = \frac{1}{6} & \pi(x_{3,3}) = \frac{1}{12} \end{array}$$

It is not hard to see that the above stationary probability distribution does not sum to unity. In fact,

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{37}{24} > 1$$

While the method described in the book is correct (even though the above stationary distribution is incorrect), the authors have also computed the value of $P_{(1,1),(1,1)}$ incorrectly. This is the second error.

To make the stationary probability distribution correct, we have to normalize the incorrect distribution by multiplying each term by $\frac{24}{37}$. This results in

$$\begin{array}{lll} \pi(x_{1,1}) = \frac{1}{3} \times \frac{24}{37} = \frac{8}{37}; & \pi(x_{1,2}) = \frac{1}{4} \times \frac{24}{37} = \frac{6}{37}; & \pi(x_{1,3}) = \frac{1}{6} \times \frac{24}{37} = \frac{4}{37}; \\ \pi(x_{2,1}) = \frac{1}{8} \times \frac{24}{37} = \frac{3}{37}; & \pi(x_{2,2}) = \frac{1}{6} \times \frac{24}{37} = \frac{4}{37}; & \pi(x_{2,3}) = \frac{1}{12} \times \frac{24}{37} = \frac{2}{37}; \\ \pi(x_{3,1}) = \frac{1}{6} \times \frac{24}{37} = \frac{4}{37}; & \pi(x_{3,2}) = \frac{1}{6} \times \frac{24}{37} = \frac{4}{37}; & \pi(x_{3,3}) = \frac{1}{12} \times \frac{24}{37} = \frac{2}{37}; \end{array}$$

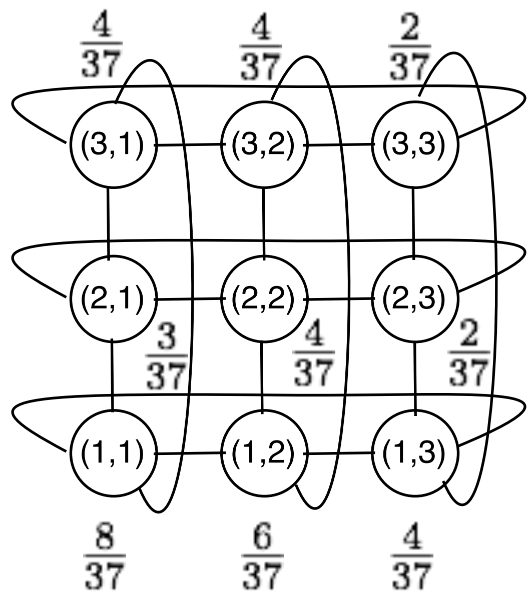

Figure 1: The Markov Chain for problem 1.

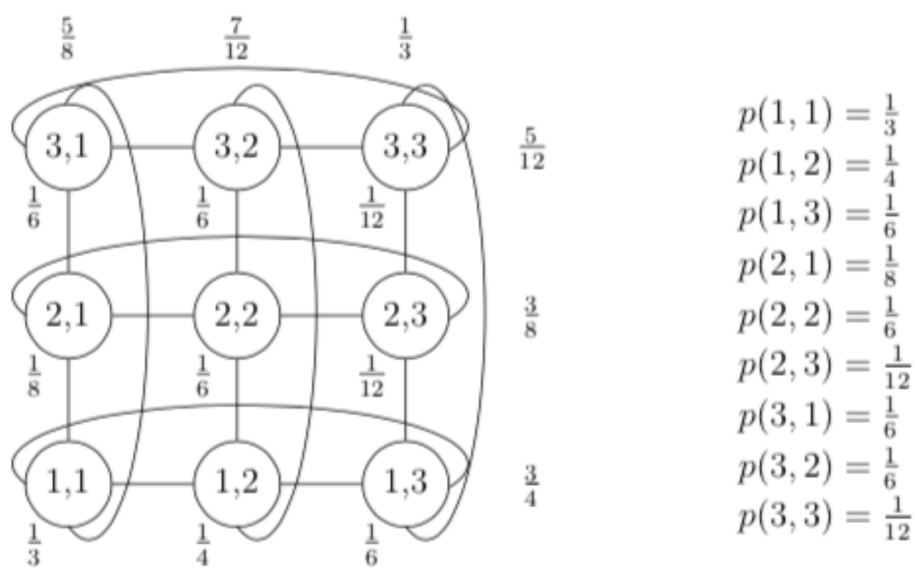

Figure 2: The Markov Chain in Figure 4.3 of the text (which has the same structure as the one shown in figure 1) with an incorrect stationary distribution (i.e. $p(i, j), i, j \in \{1, 2, 3\}$ do not add to unity).

which is what I used for this problem.

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1. (*50 points*) Present the (9×9)-stochastic matrix $\mathbf{P}$ that has the desired stationary distribution.

2. (*50 points*) Show (using Python Code, if you like) that the solution to problem 1 has the desired stationary distribution (i.e. "verify" your solution to problem 1).