STAT 410 - Section 1 - Fall 2021 Homework #10

Sharvi Tomar

TOTAL POINTS

10 / 10

QUESTION 1

7 5 pts

1.1 7kl **2 / 2**

✓ **- 0 pts** Correct

- 1 pts (k) wrong arrange of distribution

- 1 pts (k) wrong final confidence interval

- 0.5 pts (l) arithmetic miss

1.2 7m **0.5 / 0.5**

✓ **- 0 pts** Correct

- 0.5 pts wrong final sufficient statistic

1.3 7no **2.5 / 2.5**

✓ **- 0 pts** Correct

- 1 pts (n) wrong method for Fisher Information

- 0.5 pts (n) arithmetic miss

- 1 pts (o) wrong conclusion or wrong efficiency

QUESTION 2

8 4.5 pts

2.1 8hi **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts h) Wrong degree of freedom

- 0.5 pts h) Wrong interval

- 0.5 pts i) Wrong answer

2.2 8j **0.5 / 0.5**

✓ **- 0 pts** Correct

- 0.5 pts Not correct

2.3 8kl **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts Incorrect first/second order derivative(s) in 8k

- 0.5 pts Incorrect final ans in 8k

- 0.5 pts Calculation mistake in var (epsilon hat hat)

- 0.5 pts Wrong value of RCLB

- 0.5 pts Wrong value of efficiency

QUESTION 3

3 9d **0.5 / 0.5**

✓ **- 0 pts** Correct

- 0.5 pts Incorrect likelihood

- 0.5 pts Incorrect final ans

7) k) $-\ln\left(1-\frac{X}{2}\right) \sim \text{Gamma}\left(\alpha=1,\ \theta=\frac{1}{\psi}\right)$

$$Y = \sum_{i=1}^{n} -\ln\left(1-\frac{X_i}{2}\right) \sim \text{Gamma}(\alpha=n,\ \theta=1/\psi)$$

$$2\psi \sum_{i=1}^{n} -\ln\left(1-\frac{X_i}{2}\right) \sim \chi^2(2n)$$

$$P\left(\underline{\chi^2_{1-\alpha/2}(2n)} < 2\psi \sum_{i=1}^{n} -\ln\left(1-\frac{X_i}{2}\right) < \chi^2_{\alpha/2}(2n)\right) = 1-\alpha$$

$$= P\left(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} -\ln\left(1-\frac{X_i}{2}\right)} < \psi < \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} -\ln(1-X_i/2)}\right) = 1-\alpha$$

A $(1-\alpha)100\%$ confidence interval for ψ:

$$\left(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} -\ln(1-X_i/2)},\ \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} -\ln(1-X_i/2)}\right)$$

l) $\chi^2_{0.95}(6) = 1.635$ $\qquad$ $\chi^2_{0.05}(6) = 12.59$

$$2\sum_{i=1}^{n} -\ln(1-X_i/2) = 8.9999$$

$$\left(\frac{\chi^2_{0.95}(6)}{2\sum -\ln(1-X_i/2)},\ \frac{\chi^2_{0.05}(6)}{2\sum -\ln(1-X_i/2)}\right) \rightarrow$$ 90% confidence interval for ψ

$$(0.181666,\ 1.39888)$$

1.1 7kl 2 / 2

✓ **- 0 pts** Correct

- 1 pts (k) wrong arrange of distribution

- 1 pts (k) wrong final confidence interval

- 0.5 pts (l) arithmetic miss

7. m) $\prod_{i=1}^{n} f(x_i, \psi) = \prod_{i=1}^{n} \frac{\psi}{2^\psi} (2-x_i)^{\psi-1} \qquad 0<x<2$

$$= \left(\frac{\psi}{2^\psi}\right)^n \prod_{i=1}^{n} (2-x_i)^{\psi-1}$$

By Factorization Theorem, $\prod_{i=1}^{n} (2-x_i)$ is a sufficient statistic for ψ.

1.2 7m **0.5 / 0.5**

✓ **- 0 pts** Correct

- 0.5 pts wrong final sufficient statistic

7n) $f(x;\psi) = \frac{\psi}{2^{\psi}}(2-x)^{\psi-1} \qquad 0<x<2$

$\ln f = \ln\psi - \psi\ln 2 + (\psi-1)\ln(2-x)$

$\frac{\partial \ln f}{\partial \psi} = \frac{1}{\psi} - \ln 2 + \ln(2-x)$

$\frac{\partial^2 \ln f}{\partial \psi^2} = -\frac{1}{\psi^2}$

$I(\psi) = -E\left(\frac{\partial^2 \ln f}{\partial \psi^2}\right)$

$= -E\left(-\frac{1}{\psi^2}\right) = \boxed{\frac{1}{\psi^2}} \rightarrow I(\psi)$

o) $\hat{\psi} = \frac{n-1}{\sum_{i=1} -\ln(1-Xi/2)}$

$Var(\hat{\psi}) = Var\left(\frac{n-1}{\sum_{i=1}^{n} \ln\left(1-\frac{X_i}{2}\right)}\right)$

$= \frac{\cancel{(n-1)^2}\,\psi^2}{\cancel{(n-1)^2}(n-2)} = \frac{\psi^2}{(n-2)}$

$RCLB = \frac{1}{nI(\psi)}$

$= \frac{\psi^2}{n}$

$\Rightarrow Var(\hat{\psi}) > RCLB$, $\therefore \hat{\psi}$ is not efficient.

$Efficiency = \frac{RCLB}{Var(\psi)} = \frac{n-2}{n}$

$Efficiency = \frac{n-2}{n} \xrightarrow[n\to\infty]{} 1$

1.3 7no **2.5 / 2.5**

- ✓ **- 0 pts** Correct
- **- 1 pts** (n) wrong method for Fisher Information
- **- 0.5 pts** (n) arithmetic miss
- **- 1 pts** (o) wrong conclusion or wrong efficiency

8. a) $X^3 \sim \text{Gamma}(\alpha = 4, \theta = 1/\xi)$

$Y = \sum_{i=1}^{n} X_i^3 \sim \text{Gamma}(\alpha = 4n, \theta = 1/\xi)$

$2\xi \sum_{i=1}^{n} X_i^3 \sim \chi^2(8n)$

$$P\left(\chi^2_{1-\alpha/2}(8n) < 2\xi \sum_{i=1}^{n} X_i^3 < \chi^2_{\alpha/2}(8n)\right) = 1-\alpha$$

A $(1-\alpha)\,100\%$ confidence interval for ξ is:

$$P\left(\frac{\chi^2_{1-\alpha/2}(8n)}{2\sum_{i=1}^{n} X_i^3} < \xi < \frac{\chi^2_{\alpha/2}(8n)}{2\sum_{i=1}^{n} X_i^3}\right) = 1-\alpha$$

Confidence Interval

$$\boxed{\left(\frac{\chi^2_{1-\alpha/2}(8n)}{2\sum_{i=1}^{n} X_i^3}, \frac{\chi^2_{\alpha/2}(8n)}{2\sum_{i=1}^{n} X_i^3}\right)}$$

i) $\chi^2_{0.95}(40) = 26.51$ $\qquad$ $\chi^2_{0.05}(40) = 55.76$

90% confidence interval for ξ:

$$\left(\frac{\chi^2_{0.95}(40)}{2\sum_{i=1}^{5} X_i^3}, \frac{\chi^2_{0.05}(40)}{2\sum_{i=1}^{5} X_i^3}\right)$$

$= (1.3255, 2.788)$

2.1 8hi **2 / 2**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** h) Wrong degree of freedom
- **- 0.5 pts** h) Wrong interval
- **- 0.5 pts** i) Wrong answer

j) $f(x; \xi) = \frac{1}{2} \xi^4 x^{11} e^{-\xi x^3}$

$$\prod_{i=1}^{n} f(x_i, \xi) = \prod_{i=1}^{n} \frac{1}{2} \xi^4 x_i^{11} e^{-\xi x_i^3}$$

$$= \underbrace{\frac{\xi^{4n}}{2^n} e^{-\xi \sum_{i=1}^{n} x_i^3}}_{\phi(u, \xi)} \underbrace{\prod_{i=1}^{n} x_i^{11}}_{h(x)}$$

By Factorization Theorem, $\boxed{\sum_{i=1}^{n} x_i^3}$ is a sufficient statistic for ξ.

9. d) Define

$$I_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

$$f(x; \lambda) = \frac{\lambda}{x^2} \cdot I_{\{x \geq \lambda\}}$$

$$\prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \frac{\lambda}{x_i^2} I_{\{\min x_i \geq \lambda\}}$$

$$= \lambda^n I_{\{\min x_i \geq \lambda\}} \prod_{i=1}^{n} \frac{1}{x_i^2}$$

By Factorization Theorem, $\boxed{\min x_i}$ is a sufficient statistic for λ.

2.2 8j **0.5 / 0.5**

✓ **- 0 pts** Correct

- 0.5 pts Not correct

8. k) $f(x;\xi) = \frac{1}{2}\xi^4 x^{11} e^{-\xi x^3}$

$\ln f = -\ln 2 + 4\ln\xi + 11\ln x - \xi x^3$

$\frac{\partial \ln f}{\partial \xi} = \frac{4}{\xi} - x^3$

$\frac{\partial^2 \ln f}{\partial \xi^2} = -\frac{4}{\xi^2}$

$I(\xi) = -E\left(\frac{\partial^2 \ln f}{\partial \xi^2}\right)$

$= -E\left(-\frac{4}{\xi^2}\right) = \frac{4}{\xi^2}$

l) $\hat{\xi} = \frac{4n-1}{\sum_{i=1}^{n} x_i^3}$

$Var(\hat{\xi}) = Var\left(\frac{4n-1}{\sum_i x_i^3}\right)$

$= \frac{\cancel{(4n-1)^2}\,\xi^2}{\cancel{(4n-1)^2}(4n-2)} = \frac{\xi^2}{4n-2}$

$RCLB = \frac{1}{nI(\xi)}$

$= \frac{\xi^2}{4n}$

$\Rightarrow Var(\hat{\xi}) > RCLB$, $\therefore \hat{\xi}$ is not efficient

Efficiency $= \frac{RCLB}{Var(\hat{\xi})} = \frac{4n-2}{4n}$

Efficiency $= \frac{4n-2}{4n} \xrightarrow[n\to\infty]{} 1$

2.3 8kl 2 / 2

✓ **- 0 pts** **Correct**

- 0.5 pts Incorrect first/second order derivative(s) in 8k

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- 0.5 pts Calculation mistake in var (epsilon hat hat)

- 0.5 pts Wrong value of RCLB

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j) $f(x;\xi) = \frac{1}{2}\xi^4 x^{11} e^{-\xi x^3}$

$$\prod_{i=1}^{n} f(x_i,\xi) = \prod_{i=1}^{n} \frac{1}{2}\xi^4 x_i^{11} e^{-\xi x_i^3}$$

$$= \underbrace{\frac{\xi^{4n}}{2^n} e^{-\xi \sum_{i=1}^{n} x_i^3}}_{\phi(u,\xi)} \underbrace{\prod_{i=1}^{n} x_i^{11}}_{h(x)}$$

By factorization Theorem, $\boxed{\sum_{i=1}^{n} x_i^3}$ is a sufficient statistic for ξ.

9. d) Define

$$I_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

$$f(x;\lambda) = \frac{\lambda}{x^2} \cdot I_{\{x \geq \lambda\}}$$

$$\prod_{i=1}^{n} f(x_i;\lambda) = \prod_{i=1}^{n} \frac{\lambda}{x_i^2} I_{\{\min x_i \geq \lambda\}}$$

$$= \lambda^n I_{\{\min x_i \geq \lambda\}} \prod_{i=1}^{n} \frac{1}{x_i^2}$$

By factorization Theorem, $\boxed{\min x_i}$ is a sufficient statistic for λ.

3 9d 0.5 / 0.5

✓ **- 0 pts** Correct

- 0.5 pts Incorrect likelihood

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