STAT 410 - Section 1 - Fall 2021 Homework #09

Sharvi Tomar

TOTAL POINTS

10 / 10

QUESTION 1

7 4.5 pts

1.1 7i **2 / 2**

✓ **- 0 pts** **Correct**

- 0.5 pts distribution for Y1

- 0.5 pts distribution for Un

- 0.5 pts limit distribution

1.2 7j **2.5 / 2.5**

✓ **- 0 pts** **Correct**

- 0.5 pts distribution for Yn

- 0.5 pts beta

- 0.5 pts distribution for Vn

- 0.5 pts limit distribution

QUESTION 2

2 8g **2 / 2**

✓ **- 0 pts** **Correct**

QUESTION 3

3 9abc **3.5 / 3.5**

✓ **- 0 pts** **Correct**

- 1 pts Wrong formula for MLE

- 0.5 pts Wrong value of MLE based on given data & the formula found out

- 0 pts P(lambda hat<= lambda - e)=0

- 1 pts Incorrect argument in 9b

- 0.5 pts Wrong final ans in 9b

- 0.5 pts Wrong/ invalid unbiased estimator in 9c

- 1 pts Incorrect argument in 9c

- 0.5 pts Calculation mistake in 9c

HW-09

i) $U_n = nY_1 = n \min X_i$

$$F_{Y_1}(x) = F_{\min X_i}(x) = 1 - [1 - F_X(x)]^n$$

$$= 1 - \left[\cancel{1} - \cancel{1} + \left(1 - \frac{x}{2}\right)^{\psi}\right]^{n}$$

$$F_{Y_1}(x) = 1 - \left(1 - \frac{x}{2}\right)^{\psi n}$$

$$F_{U_n}(v) = P(nY_n \leq v)$$

$$= P\left(Y_n \leq \frac{v}{n}\right) = F_{Y_n}\left(\frac{v}{n}\right)$$

$$F_{U_n}(u) = 1 - \left(1 - \frac{u}{2n}\right)^{\psi n}$$

$$F_{U_n}(u) \to 1 - e^{-\frac{u}{2}\psi} \quad \text{as } n \to \infty$$

$U_n \xrightarrow{D} X$, where X has Exponential Distribution with mean $= 2/\psi$

ii) $$F_{Y_n}(x) = F_{\max X_i}(x) = [F_X(x)]^n$$

$$= \left[1 - \left(1 - \frac{x}{2}\right)^{\psi}\right]^n$$

$$F_{V_n}(v) = P(V_n \leq v)$$

$$= P(n^{\beta}(2 - Y_n) \leq v)$$

$$= P\left(Y_n \geq 2 - \frac{v}{n^{\beta}}\right)$$

$$= 1 - F_{Y_n}\left(2 - \frac{v}{n^{\beta}}\right)$$

$$= 1 - \left[1 - \left(\cancel{1} - \frac{\cancel{2}}{2} + \frac{v}{2n^{\beta}}\right)^{\psi}\right]^n = 1 - \left[1 - \left(\frac{v}{2n^{\beta}}\right)^{\psi}\right]^n$$

1.1 7i **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts distribution for Y1

- 0.5 pts distribution for Un

- 0.5 pts limit distribution

HW-09

i) $U_n = nY_1 = n \min X_i$

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$$= 1 - \left[\cancel{1} - \cancel{1} + \left(1 - \frac{x}{2}\right)^{\psi}\right]^{n}$$

$$F_{Y_1}(x) = 1 - \left(1 - \frac{x}{2}\right)^{\psi n}$$

$$F_{U_n}(u) = P(nY_n \le u)$$
$$= P\left(Y_n \le \frac{u}{n}\right) = F_{Y_n}\left(\frac{u}{n}\right)$$

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$$F_{V_n}(v) = 1 - \left(1 - \frac{(v/2)^{\cancel{\beta}\psi}}{(n\psi)^{\beta}}\right)^{n} \qquad 0 \leq v < 2n^{\beta}$$

<u>Case-1</u> If $\beta = 1/\psi$

$$F_W(w) = \lim_{n \to \infty} F_{V_n}(v) = 1 - e^{-(v/2)^{\psi}} \qquad 0 < v < \infty$$

Thus, limiting distribution of V_n when $\beta = 1/\psi$ is a Weibull distribution.

<u>Case-2</u> If $\beta < 1/\psi$

$$\lim_{n \to \infty} F_{V_n}(v) = 1 \qquad 0 < v < \infty$$

Then, $V_n \xrightarrow{D} 0$ & thus, $V_n \xrightarrow{P} 0$.

<u>Case-3</u> If $\beta > 1/\psi$

$$\lim_{n \to \infty} F_{V_n}(v) = 0 \qquad 0 < v < \infty$$

then, V_n does not have a limiting distribution.

1.2 7j **2.5 / 2.5**

- ✓ **- 0 pts** **Correct**
- **- 0.5 pts** distribution for Yn
- **- 0.5 pts** beta
- **- 0.5 pts** distribution for Vn
- **- 0.5 pts** limit distribution

8. g) $\hat{\varepsilon} = \dfrac{4n}{\sum_{i=1}^{n} X_i^3}$, $W = X^3 \sim \text{Gamma}\left(\alpha=4, \theta=\dfrac{1}{\varepsilon}\right)$

By Central Limit Theorem (CLT),

$$\sqrt{n}\left(\overline{W} - \mu_W\right) \xrightarrow{D} N\left(0, \sigma_W^2\right)$$

- $W = X^3 \sim \text{Gamma}(\alpha=4, \theta=1/\varepsilon)$

$E(W) = \alpha\theta = \dfrac{4}{\varepsilon}$, $Var(W) = \alpha\theta^2 = \dfrac{4}{\varepsilon^2}$

- Let $g(x) = \dfrac{4}{x}$, $g(\overline{W}) = \dfrac{4}{\overline{W}} = \hat{\varepsilon}$ $g\left(\dfrac{4}{\varepsilon}\right) = \varepsilon$

$g'(x) = -\dfrac{4}{x^2}$, $g'(\mu_W) = g'\left(\dfrac{4}{\varepsilon}\right) = -\dfrac{\varepsilon^2}{4} \neq 0$

$\Rightarrow$ $g(x)$ is differentiable at $\mu_W = \dfrac{4}{\varepsilon}$ & $g'(\mu_W) \neq 0$

$$\sqrt{n}\left(g(\overline{W}) - g(\mu_W)\right) \xrightarrow{D} N\left(0, [g'(\mu_W)]^2 \cdot \sigma_W^2\right)$$

$$\sqrt{n}\left(\hat{\varepsilon} - \varepsilon\right) \xrightarrow{D} N\left(0, \left(\frac{-\varepsilon^2}{4}\right)^2 \cdot \left(\frac{4}{\varepsilon^2}\right)\right)$$

For large n,

$$\hat{\varepsilon} \sim N\left(\varepsilon, \frac{\varepsilon^2}{4n}\right)$$

2 8g 2 / 2

✓ - 0 pts Correct

9. a) i) $f(x, \lambda) = \begin{cases} \frac{\lambda}{x^2} & x \geq \lambda \\ 0 & \text{otherwise} \end{cases}$

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda}{x_i^2} = \lambda^n \prod_{i=1}^{n} \frac{1}{x_i^2}$$

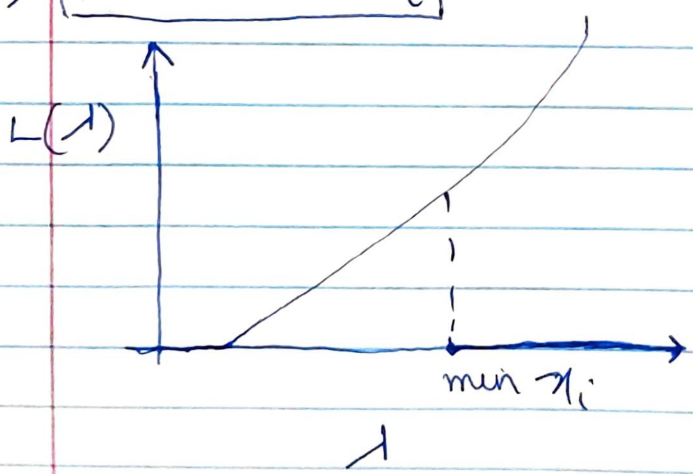
$$\log L(\lambda) = n \log \lambda - 2\sum_{i=1}^{n} \log x_i$$

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{\lambda} = 0$$

$$\lambda = ??$$

$x_1 > \lambda_1, \; x_2 > \lambda_2 \; \ldots \; x_n > \lambda$

$\Rightarrow \boxed{\lambda < \min X_i}$



$\Rightarrow \boxed{\hat{\lambda} = \min X_i} =$ Maximum

ii) $\hat{\lambda} = \min X_i$

$= \min(5, 10, 3, 20)$

$\hat{\lambda} = 3$

$\downarrow$

MLE of λ

b) Let $\varepsilon > 0$

$$P(|\hat{\lambda} - \lambda| \geq \varepsilon) = P(|\min X_i - \lambda| \geq \varepsilon)$$

$$= P(\min X_i \leq \lambda - \varepsilon) + P(\min X_i \geq \lambda + \varepsilon)$$

$$= 0 + P(\min X_i \geq \lambda + \varepsilon)$$

$$= [1 - F(\lambda + \varepsilon)]^n$$

$$F_X(x) = \int_\lambda^x \frac{\lambda}{y^2}\,dy = -\frac{\lambda}{x}\Big]_\lambda^x$$

$$F_X(x) = 1 - \frac{\lambda}{x} \qquad x \geq \lambda$$

$$P(|\hat{\lambda} - \lambda| \geq \varepsilon) = [1 - F_X(\lambda + \varepsilon)]^n$$

$$= \left[1 - \left(1 - \frac{\lambda}{\lambda + \varepsilon}\right)\right]^n$$

$$= \left(\frac{\lambda}{\lambda + \varepsilon}\right)^n$$

As $n \to \infty$, $\left(\frac{\lambda}{\lambda + \varepsilon}\right)^n \longrightarrow 0$

$$\Rightarrow P(|\hat{\lambda} - \lambda| \geq \varepsilon) \longrightarrow 0$$

Thus, $\hat{\lambda}$ is a consistent estimator of λ

c) From previous part we have,

$$F_X(x) = 1 - \frac{\lambda}{x} \qquad x \geq \lambda$$

$$F_{\min X_i}(x) = 1 - (1 - F_X(x))^n$$

$$= 1 - \left(\frac{\lambda}{x}\right)^n, \quad x \geq \lambda$$

$$f_{\min X_i}(x) = F'_{\min X_i}(x)$$

$$= \frac{n\lambda^n}{x^{n+1}}$$

$$E(\hat{\lambda}) = E(\min X_i) = \int_{\lambda}^{\infty} x \, f_{\min X_i} \, dx$$

$$= \int_{\lambda}^{\infty} x \frac{n\lambda^n}{x^{n+1}} \, dx$$

$$= \left(\frac{n}{n-1}\right)\lambda \neq \lambda \qquad (\text{as } n \geq 2)$$

Thus, $\hat{\lambda}$ is not an unbiased estimator of λ.

$$E(\hat{\lambda}) = \left(\frac{n}{n-1}\right)\lambda$$

$$\left(\frac{n-1}{n}\right) E(\hat{\lambda}) = \lambda$$

$$E\left(\left(\frac{n-1}{n}\right)\hat{\lambda}\right) = \lambda$$

Thus, $\left(\frac{n-1}{n}\right)\underset{\substack{\downarrow \\ \text{MLE of } \lambda}}{\hat{\lambda}}$ is an unbiased estimator of λ

3 9abc **3.5 / 3.5**

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