STAT 410 Fall 2021

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Homework #08

(due Friday, October 29, by 5:00 p.m. CDT)

No credit will be given without supporting work.

7. Let $\psi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\psi) = \frac{\psi}{2^{\psi}} \cdot (2-x)^{\psi-1}, \qquad 0 < x < 2, \qquad \text{zero otherwise.}$$

Recall: $E(X) = \dfrac{2}{\psi+1}$. A method of moments estimator of ψ is $\tilde{\psi} = \dfrac{2}{\overline{X}} - 1$.

The maximum likelihood estimator of ψ is

$$\hat{\psi} = \frac{n}{n \cdot \ln 2 - \sum_{i=1}^{n} \ln(2 - X_i)} = \frac{n}{\sum_{i=1}^{n} \left(-\ln\left(1 - \frac{X_i}{2}\right)\right)}.$$

$W = -\ln\left(1 - \dfrac{X}{2}\right)$ has an Exponential distribution with mean $\theta = \dfrac{1}{\psi}$.

d) Is the method of moments estimator of ψ, $\tilde{\psi}$, an unbiased estimator of ψ? If $\tilde{\psi}$ is not an unbiased estimator of ψ, does $\tilde{\psi}$ underestimate or overestimate ψ (on average)?

Hint: $\tilde{\psi} = g(\overline{X})$. Is $g(x)$ a linear function?

If it is not a linear function, does it curve up or down?

e) Is $\tilde{\psi}$ a consistent estimator of ψ? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

"Hint": Start with the WLLN and $\overline{X}$.

f) Is the maximum likelihood estimator $\hat{\psi}$ an unbiased estimator of ψ?

If $\hat{\psi}$ is not an unbiased estimator of ψ, construct an unbiased estimator of ψ based on $\hat{\psi}$.

"Hint" 0: If U has a Gamma(α_1, θ) distribution, V has a Gamma(α_2, θ) distribution, U and V are independent, then U + V has a Gamma$(\alpha_1 + \alpha_2, \theta)$ distribution.

"Hint" 1: $\mathrm{E}(a ☺) = a\,\mathrm{E}(☺)$. "Hint" 2: $\dfrac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

g) Find $\mathrm{MSE}(\hat{\psi}) = (\mathrm{bias}(\hat{\psi}))^2 + \mathrm{Var}(\hat{\psi})$.

"Hint" 1: $\mathrm{bias}(\hat{\psi}) = \mathrm{E}(\hat{\psi}) - \psi$. You have $\mathrm{E}(\hat{\psi})$ from part (f).

"Hint" 2: $\mathrm{Var}(a ☺) = a^2\,\mathrm{Var}(☺)$. $\mathrm{Var}(☺) = \mathrm{E}(☺^2) - [\mathrm{E}(☺)]^2$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

h) Is $\hat{\psi}$ a consistent estimator of ψ? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

"Hint": Start with the WLLN and $\overline{\mathrm{W}}$.

8. Let $\xi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\xi) = \frac{1}{2}\xi^4 x^{11} e^{-\xi x^3}, \quad x > 0, \quad \text{zero elsewhere.}$$

Recall: The maximum likelihood estimator of ξ is $\hat{\xi} = \dfrac{4n}{\sum_{i=1}^{n} X_i^3}$.

$W = X^3$ has a Gamma($\alpha = 4$, $\theta = \frac{1}{\xi}$) distribution.

c) Is the maximum likelihood estimator $\hat{\xi}$ an unbiased estimator of ξ?

If $\hat{\xi}$ is not an unbiased estimator of ξ, construct an unbiased estimator of ξ based on $\hat{\xi}$.

d) Find $\mathrm{MSE}(\hat{\xi}) = (\mathrm{bias}(\hat{\xi}))^2 + \mathrm{Var}(\hat{\xi})$.

e) Is $\hat{\xi}$ a consistent estimator of ξ? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

f) Suppose $n = 5$ and $\xi = 1.5$. Find the probability $P\left(\sum_{i=1}^{5} X_i^3 > 10\right)$.