STAT 410 - Section 1 - Fall 2021 Homework #01

Sharvi Tomar

TOTAL POINTS

10 / 10

QUESTION 1

1 7 pts

1.1 1ab 2 / 2

✓ **- 0 pts Correct**

- 0.5 pts Wrong calculation in 1b

- 1 pts Wrong limit of the integral in 1b, hence, wrong answer

1.2 1cd 2 / 2

✓ **- 0 pts Correct**

1.3 1ef 3 / 3

✓ **- 0 pts Correct**

- 0.5 pts (e) Missiing domain of p.d.f. (Or incorrect domain)

- 0.5 pts (f) Incorrect conclusion

- 0.5 pts (f) Incorrect calculation for E(Y)

QUESTION 2

2 2 3 / 3

✓ **- 0 pts Correct**

- 2 pts wrong initial setting ex) fail to arrange domain for y

- 1 pts wrong density for $1 < y < 4$

- 1 pts wrong density for $y > 4$

- 3 pts no valid answer

STAT 410 HW-01

1 $$f_x(x) = \begin{cases} \frac{x+2}{C} & 16 \le x \le 80 \\ 0 & \text{elsewhere} \end{cases}$$

a) Valid PDF implies $\int_{\mathbb{R}} f_x(x)\,dx = 1$

$$\int_{16}^{80} f_x(x)\,dx = 1$$

$$\int_{16}^{80} \frac{(x+2)}{C}\,dx = 1$$

$$\frac{1}{C}\left[\frac{x^2}{2} + 2x\right]_{16}^{80} = 1$$

$$\frac{(80^2 - 16^2)}{2} + 2(80-16) = C$$

$$\boxed{C = 3200} \quad \#\ C$$

b) $F_X(x) = P(X \leq x)$

$$F_X(x) = \begin{cases} 0 & x < 16 \\ \int_{16}^{x} f(u)\,du & 16 \leq x < 80 \\ 1 & x \geq 80 \end{cases}$$

Calculating $\int_{16}^{x} f(u)\,du$

$$= \frac{1}{3200}\left[\frac{u^2}{2} + 2u\right]_{16}^{x}$$

$$= \frac{1}{3200}\left(\frac{x^2 - 16^2 + 4x - 64}{2}\right)$$

$$= \frac{x^2 + 4x - 320}{6400}$$

Hence, $F_X(x) = \begin{cases} 0 & x < 16 \\ \frac{x^2 + 4x - 320}{6400} & 16 \leq x < 80 \\ 1 & x \geq 80 \end{cases}$

1.1 1ab 2 / 2

✓ **- 0 pts** **Correct**

- 0.5 pts Wrong calculation in 1b

- 1 pts Wrong limit of the integral in 1b, hence, wrong answer

c) $Y = g(X) = 8\sqrt{X+20}$, $16 \leq X \leq 80$

@ $X = 16$, $Y = 8\sqrt{16+20} = 48$

@ $X = 80$, $Y = 8\sqrt{80+20} = 80$

$\therefore S_Y = [48, 80]$

d) $F_Y(y) = P(Y \le y)$

$= P(g(X) \le y)$

$= P(8\sqrt{X+20} \le y)$

$= P\left(X \le \frac{y^2}{64} - 20\right)$

$= F_X\left(\frac{y^2}{64} - 20\right)$

Using $F_X(x)$ from part (b)

$$F_Y(y) = \frac{\left(\frac{y^2}{64} - 20\right)^2 + 4\left(\frac{y^2}{64} - 20\right) - 320}{6400}$$

$$= \frac{y^4 - 2304y^2}{262144000}$$

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$$F_Y(y) = \begin{cases} 0 & y < 48 \\ \frac{y^4 - 2304y^2}{262144000} & 48 \le y < 80 \\ 1 & y \ge 80 \end{cases}$$

1.2 1cd **2 / 2**

✓ **- 0 pts** **Correct**

e) $f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|$

STEP-1 $S_Y = [48, 80]$

STEP-2 $y = g(x)$

$= 8\sqrt{x+20}$

$g^{-1}(y) = \frac{y^2}{64} - 20$

STEP 3 $x = \frac{y^2}{64} - 20 = g^{-1}(y)$

$dx = \frac{y\,dy}{32} \quad \therefore \quad \frac{dx}{dy} = \frac{y}{32}$

$f_Y(y) = f_X\left(\frac{y^2}{64} - 20\right) \left|\frac{y}{32}\right| \quad , 48 \le y \le 80$

$= \frac{\left(\left(\frac{y^2}{64} - 20\right) + 2\right)}{3200} \left|\frac{y}{32}\right| \quad \because f_X(x) = \frac{x+2}{3200}$

$= \left(\frac{\frac{y^2}{64} - 18}{3200}\right)\left(\frac{y}{32}\right) \quad , 48 \le y \le 80$

$$f_Y(y) = \begin{cases} \frac{y^3 - 1152y}{6,553,600} & 48 \le y \le 80 \\ 0 & \text{otherwise} \end{cases}$$

f) $\mu_X = E(X)$

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$ $\quad \therefore S_X = [16, 80]$

$= \int_{16}^{80} x \cdot \left(\frac{x+2}{3200}\right) dx$

$= \frac{1}{3200} \int_{16}^{80} (x^2 + 2x)\,dx$

$= \frac{1}{3200} \left[\frac{x^3}{3} + \frac{2x^2}{2}\right]_{16}^{80}$

$= \frac{1}{3200} \left[\frac{80^3 - 16^3}{3} + 80^2 - 16^2\right] = 54,82667$

$g(x) = 8\sqrt{x+20}$

$g(\mu_X) = 8\sqrt{54.82667 + 20} = 69,20192$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$

$= \int_{48}^{80} y \left(\frac{y^3 - 1152y}{6553600}\right) dy$ $\quad \therefore S_Y = [48, 80]$

$= \frac{1}{6553600} \int_{48}^{80} (y^4 - 1152y^2)\,dy$

$= \frac{1}{6553600} \left[\frac{y^5}{5} - \frac{1152y^3}{3}\right]_{48}^{80}$ (384)

$= \frac{1}{6553600} \left[\frac{80^5 - 48^5}{5} - 384(80^3 - 48^3)\right] = 68.704$

$\therefore \mu_Y \neq g(\mu_X)$

1.3 1ef 3 / 3

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2. $f_X(x) = \begin{cases} \frac{x+2}{18} & -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

$$F_X(x) = \begin{cases} 0 & x < -2 \\ \int_{-2}^{x} f_X(u)\,du & -2 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

Calculating $\int_{-2}^{x} f_X(u)\,du$

$$= \int_{-2}^{x} \frac{(u+2)}{18}\,du$$

$$= \frac{1}{18}\left(\frac{u^2}{2} + 2u\right)\Big]_{-2}^{x}$$

$$= \frac{x^2 + 4x + 4}{36}$$

Hence, $F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{x^2+4x+4}{36} & -2 \le x < 4 \\ 1 & x \ge 4 \end{cases}$

$Y = g(X) = \frac{16}{X^2}$, $-2 \le X \le 4$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ \text{(A)} & 1 \le y \le 4 \\ \text{(B)} & y > 4 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P\left(\frac{16}{X^2} \le y\right)$$

$$= P\left(X \le \frac{-4}{\sqrt{y}}\right) + P\left(X \ge \frac{4}{\sqrt{y}}\right)$$

$$= F_X\left(\frac{-4}{\sqrt{y}}\right) + 1 - F_X\left(\frac{4}{\sqrt{y}}\right)$$

Calculating (A) when $1 \le y \le 4$

(A) $= F_X\left(\frac{-4}{\sqrt{y}}\right) + 1 - F_X\left(\frac{4}{\sqrt{y}}\right)$, $1 \le y \le 4$

$-4 \le -\frac{4}{\sqrt{y}} \le -2$ hence $F_X\left(\frac{-4}{\sqrt{y}}\right) = 0$

$2 \le \frac{4}{\sqrt{y}} \le 4$ hence $F_X\left(\frac{4}{\sqrt{y}}\right) = \frac{(4/\sqrt{y})^2 + 4(4/\sqrt{y}) + 4}{36}$

(A) $= 0 + 1 - \left(\frac{16/y^2 + 16/\sqrt{y} + 4}{36}\right)$

(A) $= 1 - \left(\frac{4/y + 4/\sqrt{y} + 1}{9}\right) = \frac{8}{9} - \frac{4}{9y} - \frac{4}{9\sqrt{y}}$

Calculating (B) when $4 < y < \infty$

(B) $= F_X\left(\frac{-4}{\sqrt{y}}\right) + 1 - F_X\left(\frac{4}{\sqrt{y}}\right)$, $y > 4$

$-2 < -\frac{4}{\sqrt{y}} < 0$ hence $F_X\left(\frac{-4}{\sqrt{y}}\right) = \frac{(4/\sqrt{y})^2 + 4\left(\frac{-4}{\sqrt{y}}\right) + 4}{36}$

$0 < \frac{4}{\sqrt{y}} < 2$ hence $F_X\left(\frac{4}{\sqrt{y}}\right) = \frac{(4/\sqrt{y})^2 + 4(4/\sqrt{y}) + 4}{36}$

(B) $= \frac{16/y - 16/\sqrt{y} + 4}{36} + 1 - \left(\frac{16/y + 16/\sqrt{y} + 4}{36}\right)$

(B) $= \frac{36 - 32/\sqrt{y}}{36}$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{8}{9} - \frac{4}{9y} - \frac{4}{9\sqrt{y}} & 1 \le y \le 4 \\ 1 - \frac{8}{9\sqrt{y}} & y > 4 \end{cases}$$

2 2 3 / 3

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