STAT 410 - Section 1 - Fall 2021 Homework #08

Sharvi Tomar

TOTAL POINTS

9.5 / 10

QUESTION 1

7 5.5 pts

1.1 7de **1.5 / 2**

- **- 0 pts** Correct
- ✓ **- 0.5 pts** **Need to argue that as X_bar isn't a constant r.v. in order to get strict inequality.**
- **- 0.5 pts** Need to mention about continuity of g(x)=2/x -1 in 7e
- **- 0.5 pts** Need to show why 1/X_bar goes to 1/E(X) in probability.
- **- 0.5 pts** Need to show the application of Jensen Inequality in 7d
- **- 0.5 pts** Inequality in Jensen should be in the opposite direction

1.2 7fg **2.5 / 2.5**

- ✓ **- 0 pts** **Correct**
- **- 0.5 pts** f) Expectation wrong
- **- 1 pts** f) Missing unbiased estimator
- **- 0.5 pts** g)Not Correct

1.3 7h **1 / 1**

- ✓ **- 0 pts** **Correct**
- **- 0.5 pts** Not correct, need to use continuous mapping theorem
- **- 0.5 pts** Not answer the question
- **- 0.5 pts** Not correct

QUESTION 2

8 4.5 pts

2.1 8cd **2.5 / 2.5**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** (c) Wrong bias or wrong conclusion for first estimator
- **- 0.5 pts** (c) Wrong final estimator
- **- 0.5 pts** (d) Wrong final answer
- **- 0.5 pts** (c), (d) arithmetic miss
- **- 1.5 pts** No valid answer for (c)
- **- 1 pts** No valid answer for (d)

2.2 8ef **2 / 2**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** (e) Wrong method or wrong conclusion
- **- 1 pts** (f) Wrong method or wrong distribution used
- **- 0.5 pts** (e),(f) arithmetic miss
- **- 2 pts** No valid work
- **- 1 pts** (f) No valid work
- **- 1 pts** (e) No valid work

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7. d) $\hat{\psi} = \frac{2}{\bar{X}} - 1$

Let $g(\bar{X}) = \hat{\psi}$

$g(\bar{X}) = \frac{2}{\bar{X}} - 1$

$g(x) = \frac{2}{x} - 1$, $0 < x < 2$ [$g(x)$ is not linear]

$g'(x) = \frac{-2}{x^2}$, $0 < x < 2$

$g''(x) = \frac{4}{x^3}$, $0 < x < 2$ [$g''(x) > 0$ $\therefore$ convex]

Using Jenson's inequality,

$E(\hat{\psi}) = E[g(\bar{x})] > g(E(\bar{x})) = g\left(\frac{2}{\psi+1}\right)$

$E(\hat{\psi}) > g\left(\frac{2}{\psi+1}\right)$

$E(\hat{\psi}) > \psi$

On average, $\hat{\psi}$ overestimates ψ

e) By WLLN, $\bar{X} \xrightarrow{P} \mu = E(X) = \frac{2}{\psi+1}$

$g(x) = \frac{2}{x} - 1$ is continuous at $\frac{2}{\psi+1}$

$g(\bar{X}) = \hat{\psi}$, $g\left(\frac{2}{\psi+1}\right) = \psi$

$\Rightarrow \hat{\psi} \xrightarrow{P} \psi$

$\therefore$ $\hat{\psi}$ is a consistent estimator of ψ.

1.1 7de **1.5 / 2**

- **- 0 pts** Correct
- ✓ **- 0.5 pts** **Need to argue that as X_bar isn't a constant r.v. in order to get strict inequality.**
- **- 0.5 pts** Need to mention about continuity of g(x)=2/x -1 in 7e
- **- 0.5 pts** Need to show why 1/X_bar goes to 1/E(X) in probability.
- **- 0.5 pts** Need to show the application of Jensen Inequality in 7d
- **- 0.5 pts** Inequality in Jensen should be in the opposite direction

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f) MLE of $\psi = \hat{\psi} = \dfrac{n}{\sum_{i=1}^{n} \left(-\ln\left(1 - \frac{X_i}{2}\right)\right)}$

From HW4, $-\ln\left(1 - \frac{X}{2}\right) \sim \text{Gamma}(\alpha = 1, \lambda = \psi)$

$\therefore \sum_{i=1}^{n} \left(-\ln\left(1 - \frac{X_i}{2}\right)\right) \sim \text{Gamma}(\alpha = 1 \cdot n, \lambda = \psi)$

Take $Y = \sum_{i=1}^{n} \left(-\ln\left(1 - \frac{X_i}{2}\right)\right) \sim \text{Gamma}(\alpha = n, \lambda = \psi)$

finding $E(\hat{\psi})$

$= E\left(\dfrac{n}{\sum_{i=1}^{n}\left(-\ln\left(1 - \frac{X_i}{2}\right)\right)}\right) = n\,E\left(\dfrac{1}{Y}\right)$

$= n\,E[Y^{-1}]$

$= \dfrac{n\psi\,\Gamma(n-1)}{\Gamma(n)}$

$= \dfrac{n\psi}{n-1}$

$\left[\begin{array}{l} Y \sim \text{Gamma}(\alpha = n, \lambda = \psi) \\ \therefore E[Y^{-1}] = \dfrac{1}{\psi^{-1}} \dfrac{\Gamma(n-1)}{\Gamma(n)} \end{array}\right]$

$\Rightarrow E(\hat{\psi}) = \dfrac{n\psi}{n-1} \neq \psi$

Hence, $\hat{\psi}$ is not unbiased estimator

$E(\overset{\square}{\psi}) = \psi \Rightarrow \overset{\square}{\psi}$ is unbiased estimator

$\therefore E(\overset{\square}{\psi}) = E\left(\dfrac{n-1}{n}\hat{\psi}\right) = \dfrac{n-1}{n} E(\hat{\psi}) = \hat{\psi}$

$$\boxed{\overset{\square}{\psi} = \frac{n-1}{n}\hat{\psi}}$$

unbiased estimator

MLE of ψ

g) $MSE(\hat{\psi}) = [Bias(\hat{\psi})]^2 + Var(\hat{\psi})$

$$[Bias(\hat{\psi})]^2 = (E(\hat{\psi}) - \psi)^2 = \left(\frac{n\psi}{n-1} - \psi\right)^2 = \frac{\psi^2}{(n-1)^2}$$

$$Var(\hat{\psi}) = Var\left(\frac{n}{\sum_{i=1} \left(-\ln\left(1 - \frac{x_i}{2}\right)\right)}\right)$$

$$= n^2 Var\left(\frac{1}{Y}\right) \qquad Y \sim Gamma(\alpha = n, \lambda = \psi)$$

$$Var\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - \left[E\left(\frac{1}{Y}\right)\right]^2$$

$$= \frac{1}{\psi^{-2}} \frac{\Gamma(n-2)}{\Gamma(n)} - \left[\frac{1}{\psi^{-1}} \frac{\Gamma(n-1)}{\Gamma(n)}\right]^2$$

$$= \frac{\psi^2}{(n-1)(n-2)} - \frac{\psi^2}{(n-1)^2}$$

$$= \frac{\psi^2}{(n-1)^2(n-2)}$$

$$\therefore Var(\hat{\psi}) = n^2 Var\left(\frac{1}{Y}\right)$$

$$= \frac{n^2\psi^2}{(n-1)^2(n-2)}$$

$$MSE(\hat{\psi}) = \frac{\psi^2}{(n-1)^2} + \frac{n^2\psi^2}{(n-1)^2(n-2)}$$

$$= \frac{\psi^2}{(n-1)^2}\left[1 + \frac{n^2}{n-2}\right]$$

$$= \frac{\psi^2(n+2)}{(n-1)(n-2)}$$

1.2 7fg **2.5 / 2.5**

✓ **- 0 pts** Correct

- 0.5 pts f) Expectation wrong

- 1 pts f) Missing unbiased estimator

- 0.5 pts g)Not Correct

h) $\hat{\psi} = \dfrac{n}{\sum_{i=1}^{n}\left(-\ln\left(1-\frac{x_i}{2}\right)\right)}$

$W_i = -\ln\left(1-\frac{x_i}{2}\right) \quad , i=1,2\ldots n$

$\therefore \hat{\psi} = \dfrac{1}{\overline{W}} \Rightarrow \overline{W} = \dfrac{1}{\hat{\psi}}$

By WLLN,

$\overline{W} \xrightarrow{P} E(W) = \dfrac{1}{\psi} \qquad \left[E\left(-\ln\left(1-\frac{x}{2}\right)\right) = \frac{1}{\psi}\right]$

$\dfrac{1}{\hat{\psi}} = \overline{W} \xrightarrow{P} E(W) = \dfrac{1}{\psi}$

$\Rightarrow \dfrac{1}{\hat{\psi}} \xrightarrow{P} \dfrac{1}{\psi}$

$\Rightarrow \hat{\psi} \xrightarrow{P} \psi \qquad \therefore \hat{\psi}$ is a consistent estimator of ψ.

1.3 7h **1 / 1**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Not correct, need to use continuous mapping theorem
- **- 0.5 pts** Not answer the question
- **- 0.5 pts** Not correct

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8. c) $E[\hat{\xi}] = E\left[\dfrac{4n}{\sum_{i=1}^{n} X_i^3}\right]$

$$= 4n\, E\left[\frac{1}{\sum_{i=1}^{n} X_i^3}\right]$$

Take $Y = \sum_{i=1}^{n} X_i^3 \sim \text{Gamma}(\alpha = 4n, \lambda = \xi)$

$$E[\hat{\xi}] = 4n\, E\left[\frac{1}{Y}\right]$$

$$= 4n\, E[Y^{-1}]$$

$$= \frac{4n}{\xi^{-1}} \frac{\Gamma(4n-1)}{\Gamma(4n)}$$

$$= \frac{4n\xi}{4n-1}$$

$$\Rightarrow E[\hat{\xi}] = \frac{4n\xi}{4n-1} \neq \xi$$

$\therefore\ \hat{\xi}$ is not an unbiased estimator.

$E[\tilde{\xi}] = \xi \quad \Rightarrow \tilde{\xi}$ is an unbiased estimator

$$E[\tilde{\xi}] = E\left[\left(\frac{4n-1}{4n}\right)\hat{\xi}\right] = \frac{4n-1}{4n} E[\hat{\xi}] = \xi$$

$$\boxed{\tilde{\xi} = \frac{4n-1}{4n}\hat{\xi}}$$

$\tilde{\xi}$: Unbiased estimator

$\hat{\xi}$: MLE

d) $\left(\text{Bias}(\hat{\xi})\right)^2 = \left[E(\hat{\xi}) - \xi\right]^2$

$$= \left[\frac{4n}{4n-1}\xi - \xi\right]^2 = \left(\frac{\xi}{4n-1}\right)^2$$

$$\text{Var}(\hat{\xi}) = \text{Var}\left(\frac{4n}{\sum_{i=1} X_i^3}\right)$$

$$\left[Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}(\alpha = 4n, \lambda = \xi)\right]$$

$$= 16n^2 \, \text{Var}\left(\frac{1}{Y}\right)$$

$$\text{Var}\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - \left[E\left(\frac{1}{Y}\right)\right]^2$$

$$= E\left[Y^{-2}\right] - \left[E(Y^{-1})\right]^2$$

$$= \frac{1}{\xi^{-2}} \frac{\Gamma(4n-2)}{\Gamma(4n)} - \left[\frac{1}{\xi^{-1}} \frac{\Gamma(4n-1)}{\Gamma(4n)}\right]^2$$

$$= \frac{\xi^2}{(4n-1)(4n-2)} - \frac{\xi^2}{(4n-1)^2}$$

$$= \frac{\xi^2}{(4n-1)}\left[\frac{1}{(4n-2)} - \frac{1}{(4n-1)}\right]$$

$$= \frac{\xi^2}{(4n-1)^2(4n-2)}$$

$$\text{Var}(\hat{\xi}) = 16n^2 \, \text{Var}\left(\frac{1}{Y}\right) = \frac{16n^2\xi^2}{(4n-1)^2(4n-2)}$$

$$\text{MSE}(\hat{\xi}) = \frac{\xi^2}{(4n-1)^2} + \frac{16n^2\xi^2}{(4n-1)^2(4n-2)}$$

$$= \frac{(8n-2)(2n+1)\,\xi^2}{(4n-1)^2(4n-2)} = \frac{(2n+1)\,\xi^2}{(4n-1)(2n-1)}$$

2.1 8cd **2.5 / 2.5**

- ✓ **- 0 pts** Correct
- **- 1 pts** (c) Wrong bias or wrong conclusion for first estimator
- **- 0.5 pts** (c) Wrong final estimator
- **- 0.5 pts** (d) Wrong final answer
- **- 0.5 pts** (c), (d) arithmetic miss
- **- 1.5 pts** No valid answer for (c)
- **- 1 pts** No valid answer for (d)

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f) $\sum_{i=1}^{n} X_i^3 \sim \text{Gamma}(\alpha = 4n, \lambda = \lambda)$

$$P(T_{4n} > 10) = \int_{10}^{\infty} \frac{\lambda^{4n} x^{4n-1} e^{-\lambda x}}{\Gamma(4n)} dx$$

Put $n = 5$, $\lambda = 1.5$

$$P(T_{4n} > 10) = 0.875218$$

e) $\hat{\lambda} = \dfrac{4n}{\sum_{i=1}^{n} X_i^3}$

$W_i = X_i^3$, $i = 1, 2 \ldots n$

$\hat{\lambda} = \dfrac{4}{\bar{W}} \Rightarrow \bar{W} = \dfrac{4}{\hat{\lambda}}$

By WLLN,

$$\bar{W} \xrightarrow{P} E(W) = 4 \cdot \frac{1}{\lambda}$$

$\left[\begin{array}{l} W \sim \text{Gamma}(\alpha = 4, \theta = \frac{1}{\lambda}) \\ E(W) = \alpha\theta = 4/\lambda \end{array} \right]$

$$\frac{4}{\hat{\lambda}} = \bar{W} \xrightarrow{P} E(W) = \frac{4}{\lambda}$$

$$\Rightarrow \frac{4}{\hat{\lambda}} \xrightarrow{P} \frac{4}{\lambda}$$

$\Rightarrow \hat{\lambda} \xrightarrow{P} \lambda$ $\therefore$ $\hat{\lambda}$ is a consistent estimator of λ

2.2 8ef 2 / 2

✓ **- 0 pts** Correct

- 1 pts (e) Wrong method or wrong conclusion

- 1 pts (f) Wrong method or wrong distribution used

- 0.5 pts (e),(f) arithmetic miss

- 2 pts No valid work

- 1 pts (f) No valid work

- 1 pts (e) No valid work