STAT 410 | Fall 2021 | A. Stepanov

Homework #07

(due Friday, October 22, by 5:00 p.m. CDT)

No credit will be given without supporting work.

6. After a homework assignment is posted on Compass2g, the weak-minded lazy Chegg enthusiasts flock to Chegg according to a Poisson process with the average rate of 1.5 weak-minded lazy Chegg enthusiasts per minute. (The pudding-brain lazy CourseHero worshipers show similar behavior.)

a) Find the probability that at most two weak-minded lazy Chegg enthusiasts would show up on Chegg during the first minute after the homework assignment is posted on Compass2g.

b) Find the probability that exactly seven weak-minded lazy Chegg enthusiasts would show up on Chegg during the first five minutes after the homework assignment is posted on Compass2g.

c) Find the probability that the seventh weak-minded lazy Chegg enthusiast would show up on Chegg during the fifth minute after the homework assignment is posted on Compass2g.

d) Construct a 90% prediction interval for the time when the seventh weak-minded lazy Chegg enthusiast would show up on Chegg.

That is, find a and b such that

$$P(a < T_7 < b) = 0.90.$$

7. Let $\psi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\psi) = \frac{\psi}{2^{\psi}} \cdot (2-x)^{\psi-1}, \qquad 0 < x < 2, \qquad \text{zero otherwise.}$$

a) (i) Obtain a method of moments estimator of ψ, $\tilde{\psi}$.

(ii) Suppose $n = 3$, and $x_1 = 0.62$, $x_2 = 1.54$, $x_3 = 1.86$.

Find a method of moments estimate of ψ.

① Find $E(X)$. It will depend on ψ, so it will be a function of ψ, say, $E(X) = h(\psi)$.

② Replace $E(X)$ with $\overline{X}$, so $\overline{X} = h(\psi)$.

③ Solve $\overline{X} = h(\psi)$ for ψ. Add a tilde.

b) (i) Obtain the maximum likelihood estimator of ψ, $\hat{\psi}$.

(ii) Suppose $n = 3$, and $x_1 = 0.62$, $x_2 = 1.54$, $x_3 = 1.86$.

Find the maximum likelihood estimate of ψ.

That is, find $\hat{\psi} = \arg\max L(\psi) = \arg\max \ln L(\psi)$, where $L(\psi) = \prod_{i=1}^{n} f(x_i;\psi)$.

① Multiply: $L(\psi) = f(x_1;\psi) \cdot f(x_2;\psi) \cdot \ldots \cdot f(x_n;\psi)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to** ψ.

⑤ Set equal to zero. Solve for ψ. Add a hat.

c) Show that $W = -\ln\left(1 - \frac{X}{2}\right)$ follows a Gamma distribution.

What are the parameters α and θ for this Gamma distribution?

No credit will be given without proper justification.

8. Let $\xi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\xi) = \frac{1}{2}\,\xi^4 x^{11} e^{-\xi x^3}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

a) (i) Obtain the maximum likelihood estimator $\hat{\xi}$ of ξ.

(ii) Suppose $n = 5$, $x_1 = 0.3$, $x_2 = 0.6$, $x_3 = 1.2$, $x_4 = 1.3$, $x_5 = 1.8$.
Obtain the maximum likelihood estimate of ξ.

That is, find $\hat{\xi} = \arg\max \mathrm{L}(\xi) = \arg\max \ln \mathrm{L}(\xi)$, where $\mathrm{L}(\xi) = \prod_{i=1}^{n} f(x_i;\xi)$.

① Multiply: $\mathrm{L}(\xi) = f(x_1;\xi) \cdot f(x_2;\xi) \cdot \ldots \cdot f(x_n;\xi)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to** ξ.

⑤ Set equal to zero. Solve for ξ. Add a hat.

b) Show that $W = X^3$ follows a Gamma distribution.

What are the parameters α and θ for this Gamma distribution?

No credit will be given without proper justification.

"Hint": $f_W(w) = f_X(g^{-1}(w)) \left| \dfrac{dx}{dw} \right|$.