STAT 410 - Section 1 - Fall 2021 Homework #07

Sharvi Tomar

TOTAL POINTS

10 / 10

QUESTION 1

6 3.5 pts

1.1 6ab **1 / 1**

✓ **- 0 pts** Correct

- 0.5 pts Wrong answer for (a)

- 0.5 pts Wrong answer for (b)

- 1 pts No valid answer

1.2 6cd **2.5 / 2.5**

✓ **- 0 pts** Correct

- 1 pts (c) Wrong setting(wrong parameters, wrong bounds..)

- 0.5 pts (c) Wrong final answer

- 0.5 pts (d) Wrong final quantiles

- 2.5 pts (c),(d) No valid answer

- 1 pts (d) No valid answer

QUESTION 2

7 4 pts

2.1 7a **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts Limits of integral are wrong

- 0.5 pts Final value of the integral is wrong

- 0.5 pts Calculation mistake part ii.

- 0.5 pts Final expression for estimator is not found out/ found out but wrong/ not consistent with the value of E(X) already found out.

2.2 7b **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts Calculation mistake in part ii / The ans isn't consistent with the formula for MLE, already found out.

2.3 7c **1 / 1**

✓ **- 0 pts** Correct

- 0.5 pts Range of W is wrong.

- 0.5 pts Wrong density of W.

- 0.5 pts Range of W needs to be found out

QUESTION 3

8 2.5 pts

3.1 8a **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts (i) Not correct

- 0.5 pts (ii) Not correct

3.2 8b **1 / 1**

✓ **- 0 pts** Correct

- 0.5 pts Not corrrect

- 1 pts Missing

6. $\lambda = 1.5$

a) $P(X_1 \le 2)$

$= P(\text{Poisson}(1.5) \le 2) = 0.809$

b) $P(T_7 = 5) = P(X_5 = 7)$

$= \dfrac{e^{-5(1.5)} [(1.5)(5)]^7}{\lfloor 7} = 0.1465$

c) $P(4 < T_7 < 5) = P(X_4 \le 6) - P(X_5 \le 6)$

$= P(\text{Poisson}(6) \le 6) - P(\text{Poisson}(7.5) \le 6)$

$= 0.606 - 0.378$

$= 0.228$

d) $P(a < T_7 < b) = 0.90$

$\alpha = 7, \quad \lambda = 1.5$

$P(2 \cdot 1.5a < 2\lambda T_\alpha < 2^*(1.5)b) = 0.90$

Since, $2\lambda T_\alpha \sim \chi^2(2\alpha)$

Thus, $2(1.5)T_7 \sim \chi^2(14)$ [T_7 has gamma distribution]

$P(3a < \chi^2(14) < 3b) = 0.9$ --- ⓐ

Using χ^2 table for 14 degrees of freedom,

$P(6.571 < \chi^2(14) < 23.68) = 0.90$ --- ⓑ

Comparing ⓐ & ⓑ

$3a = 6.571$

$\boxed{a = 2.19}$

$3b = 23.68$

$\boxed{b = 7.893}$

1.1 6ab 1 / 1

- ✓ - **0 pts** Correct
- - **0.5 pts** Wrong answer for (a)
- - **0.5 pts** Wrong answer for (b)
- - **1 pts** No valid answer

6. $\lambda = 1.5$

a) $P(X_1 \leq 2)$

$= P(\text{Poisson}(1.5) \leq 2) = 0.809$

b) $P(T_7 = 5) = P(X_5 = 7)$

$= \dfrac{e^{-5(1.5)}[(1.5)(5)]^7}{\lfloor 7}$ $= 0.1465$

c) $P(4 < T_7 < 5) = P(X_4 \leq 6) - P(X_5 \leq 6)$
$= P(\text{Poisson}(6) \leq 6) - P(\text{Poisson}(7.5) \leq 6)$
$= 0.606 - 0.378$
$= 0.228$

d) $P(a < T_7 < b) = 0.90$
$\alpha = 7, \lambda = 1.5$

$P(2 * 1.5a < 2\lambda T_\alpha < 2^*(1.5)b) = 0.90$

Since, $2\lambda T_\alpha \sim \chi^2(2\alpha)$
Thus, $2(1.5)T_7 \sim \chi^2(14)$ [T_7 has gamma distribution]

$P(3a < \chi^2(14) < 3b) = 0.9$ --- ⓐ
Using χ^2 table for 14 degrees of freedom,

$P(6.571 < \chi^2(14) < 23.68) = 0.90$ --- ⓑ
Comparing ⓐ & ⓑ
$3a = 6.571$
$\boxed{a = 2.19}$

$3b = 23.68$
$\boxed{b = 7.893}$

1.2 6cd **2.5 / 2.5**

- ✓ **- 0 pts** Correct
- **- 1 pts** (c) Wrong setting(wrong parameters, wrong bounds..)
- **- 0.5 pts** (c) Wrong final answer
- **- 0.5 pts** (d) Wrong final quantiles
- **- 2.5 pts** (c),(d) No valid answer
- **- 1 pts** (d) No valid answer

$\boxed{\theta = \Psi}$

7. $f(x;\theta) = \begin{cases} \frac{\theta}{2^\theta}(2-x)^{\theta-1} & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$

a) i) MOM

$E(x) = \int_0^\infty x f(x) dx$

$= \int_0^2 x \frac{\theta}{2^\theta}(2-x)^{\theta-1} dx$

$E(x) = \frac{2}{\theta+1}$

$\bar{x} = \frac{2}{\theta+1}$ Solving for θ

$\boxed{\tilde{\theta} = \frac{2}{\bar{x}} - 1}$

ii) $\bar{x} = \frac{\sum_{i=1}^{3} x_i}{3} = 1.34$

$\tilde{\theta} = \frac{2}{\bar{x}} - 1 = \frac{2}{1.34} - 1 = 0.49253$

b) i) MLE

$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2^\theta}(2-x_i)^{\theta-1}$

$= \frac{\theta^n}{2^{\theta n}} \prod_{i=1}^{n} (2-x_i)^{\theta-1}$

$\log_e L(\theta) = n \log_e \theta - \theta n \log_e 2 + (\theta-1) \log_e \prod_{i=1}^{n} (2-x_i)$

$\frac{d \log_e L(\theta)}{d\theta} = \frac{n}{\theta} - n \log_e 2 + \log_e \prod_{i=1}^{n} (2-x_i)$

2.1 7a **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts Limits of integral are wrong

- 0.5 pts Final value of the integral is wrong

- 0.5 pts Calculation mistake part ii.

- 0.5 pts Final expression for estimator is not found out/ found out but wrong/ not consistent with the value of E(X) already found out.

$\boxed{\theta = \Psi}$

7. $f(x;\theta) = \begin{cases} \dfrac{\theta}{2^\theta}(2-x)^{\theta-1} & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$

a) i) MOM

$E(x) = \int_0^\infty x f(x) dx$

$= \int_0^2 x \frac{\theta}{2^\theta} (2-x)^{\theta-1} dx$

$E(X) = \frac{2}{\theta+1}$

$\bar{X} = \frac{2}{\theta+1}$ Solving for θ

$\boxed{\tilde{\theta} = \frac{2}{\bar{X}} - 1}$

ii) $\bar{X} = \frac{\sum_{i=1}^{3} x_i}{3} = 1.34$

$\tilde{\theta} = \frac{2}{\bar{X}} - 1 = \frac{2}{1.34} - 1 = 0.49253$

b) i) MLE

$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2^\theta} (2-x_i)^{\theta-1}$

$= \frac{\theta^n}{2^{\theta n}} \prod_{i=1}^{n} (2-x_i)^{\theta-1}$

$\log_e L(\theta) = n \log_e \theta - \theta n \log_e 2 + (\theta-1) \log_e \prod_{i=1}^{n} (2-x_i)$

$\frac{d \log_e L(\theta)}{d\theta} = \frac{n}{\theta} - n \log_e 2 + \log_e \prod_{i=1}^{n} (2-x_i)$

Equating $\frac{d \log_e L(\theta)}{d\theta} = 0$

$$\frac{n}{\theta} - n\log_e 2 + \sum_{i=1}^{n} \log_e(2-x_i) = 0$$

$$\frac{n}{\theta} = n\log_e 2 - \sum_{i=1}^{n} \log_e(2-x_i)$$

$$\frac{1}{\theta} = \log_e 2 - \frac{1}{n}\sum_{i=1}^{n} \log_e(2-x_i)$$

$$\boxed{\hat{\theta} = \frac{1}{\log_e 2 - \frac{1}{n}\sum_{i=1}^{n} \log_e(2-x_i)}}$$

ii) $$\hat{\theta} = \frac{1}{\log_e 2 - \frac{1}{3}\left[\log_e(2-0.62) + \log_e(2-1.54) + \log_e(2-1.86)\right]}$$

$$\hat{\theta} = \cancel{1.53505}\ 2/3 = 0.667$$

c) $W = -\ln(1 - X/2) \quad ; \quad 0 < x < 2$

Using change of variable method as the func^n satisfies requirements

$$x = 2 - 2e^{-w}$$

$$\frac{dx}{dw} = -2(-e^{-w}) = 2e^{-w}$$

$$f_W(w) = f_X(2 - 2e^{-w})\,|2e^{-w}|$$

$$= \frac{\theta}{2^\theta}(\cancel{2} - \cancel{2} + 2e^{-w})^{\theta-1}\,|2e^{-w}|$$

$$= \frac{\theta}{\cancel{2^\theta}}\,\cancel{2^{\theta-1+1}}\,e^{-w(\theta-1)-w} = \theta e^{-w\theta} \quad \underline{0 < w < \infty}$$

2.2 7b **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts Calculation mistake in part ii / The ans isn't consistent with the formula for MLE, already found out.

Equating $\frac{d \log_e L(\theta)}{d\theta} = 0$

$$\frac{n}{\theta} - n\log_e 2 + \sum_{i=1}^{n} \log_e(2-x_i) = 0$$

$$\frac{n}{\theta} = n\log_e 2 - \sum_{i=1}^{n} \log_e(2-x_i)$$

$$\frac{1}{\theta} = \log_e 2 - \frac{1}{n}\sum_{i=1}^{n} \log_e(2-x_i)$$

$$\boxed{\hat{\theta} = \frac{1}{\log_e 2 - \frac{1}{n}\sum_{i=1}^{n} \log_e(2-x_i)}}$$

ii) $$\hat{\theta} = \frac{1}{\log_e 2 - \frac{1}{3}\left[\log_e(2-0.62) + \log_e(2-1.54) + \log_e(2-1.86)\right]}$$

$$\hat{\theta} = \cancel{1.53505}\ 2/3 = 0.667$$

c) $W = -\ln(1 - X/2)$; $0 < x < 2$

Using change of variable method as the func^n satisfies requirements

$$x = 2 - 2e^{-w}$$

$$\frac{dx}{dw} = -2(-e^{-w}) = 2e^{-w}$$

$$f_W(w) = f_X(2 - 2e^{-w})\,|2e^{-w}|$$

$$= \frac{\theta}{2^\theta}(\cancel{2} - \cancel{2} + 2e^{-w})^{\theta-1}\,|2e^{-w}|$$

$$= \frac{\theta}{\cancel{2^\theta}}\,\cancel{2^{\theta-1+1}}\,e^{-w(\theta-1)-w} = \theta e^{-w\theta} \quad \underline{0 < w < \infty}$$

$w = \theta e^{-w\theta}$

Matching with Gamma distribution

$f(x; \alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \Rightarrow f(w, \alpha, \theta) = \frac{w^{\alpha-1} e^{-w/\theta}}{\theta^{\alpha} \Gamma(\alpha)}$

$w = \psi e^{-w\psi}$ [Replacing θ with ψ]

$\alpha = 1$, $\theta = \frac{1}{\psi}$

8. a) i) $L(\theta) = \prod_{i=1}^{n} \frac{1}{2} \theta^4 x_i^{11} e^{-\theta x_i^3}$

$\boxed{\theta = \xi}$

$= \frac{\theta^{4n}}{2^n} \prod_{i=1}^{n} x_i^{11} e^{-\theta x_i^3}$

$\log_e L(\theta) = 4n \log_e \theta - n \log_e 2 + \sum_{i=1} \log_e x_i^{11} e^{-\theta x_i^3}$

$= 4n \log_e \theta - n \log_e 2 + 11 \sum_{i=1}^{n} \log_e x_i - \theta \sum_{i=1}^{n} x_i^3$

$\frac{d}{d\theta} \log_e L(\theta) = \frac{4n}{\theta} - \sum_{i=1}^{n} x_i^3$

Equating $\frac{d \log L(\theta)}{d\theta} = 0$

$\frac{4n}{\theta} - \sum_{i=1}^{n} x_i^3 = 0$

$\hat{\theta} = \frac{4n}{\sum_{i=1}^{n} x_i^3}$

2.3 7c **1 / 1**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Range of W is wrong.
- **- 0.5 pts** Wrong density of W.
- **- 0.5 pts** Range of W needs to be found out

$$w = \theta e^{-w\theta}$$

Matching with Gamma distribution

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \Rightarrow f(w,\alpha,\theta) = \frac{w^{\alpha-1} e^{-w/\theta}}{\theta^{\alpha}\Gamma(\alpha)}$$

$$w = \psi e^{-w\psi}$$ [Replacing θ with ψ]

$$\alpha = 1 \quad , \quad \theta = \frac{1}{\psi}$$

8.a)i) $$L(\theta) = \prod_{i=1}^{n} \frac{1}{2}\theta^4 x_i^{11} e^{-\theta x_i^3}$$

$\boxed{\theta = \varepsilon}$

$$= \frac{\theta^{4n}}{2^n} \prod_{i=1}^{n} x_i^{11} e^{-\theta x_i^3}$$

$$\log_e L(\theta) = 4n\log_e\theta - n\log_e 2 + \sum_{i=1}^{n} \log_e x_i^{11} e^{-\theta x_i^3}$$

$$= 4n\log_e\theta - n\log_e 2 + 11\sum_{i=1}^{n}\log_e x_i - \theta\sum_{i=1}^{n} x_i^3$$

$$\frac{d}{d\theta}\log_e L(\theta) = \frac{4n}{\theta} - \sum_{i=1}^{n} x_i^3$$

Equating $\frac{d\log L(\theta)}{d\theta} = 0$

$$\frac{4n}{\theta} - \sum_{i=1}^{n} x_i^3 = 0$$

$$\hat{\theta} = \frac{4n}{\sum_{i=1}^{n} x_i^3}$$

7i) $\hat{\Theta} = \frac{4 * 5}{(0.3^3 + 0.6^3 + 1.2^3 + 1.3^3 + 1.8^3)} = 2$

b) $W = X^3$

$X = W^{1/3}$

$\frac{dx}{dW} = \frac{1}{3 w^{2/3}}$

$f_W(w) = f_X(w^{1/3}) \cdot \left| \frac{1}{3 w^{2/3}} \right|$

$= \frac{1}{2} \theta^4 w^{1/3} e^{-\theta w} \frac{1}{3 w^{2/3}}$

$= \frac{\theta^4 w^3 e^{-\theta w}}{6}$

$f_W(w) = \frac{1}{6} \Sigma^4 w^3 e^{-\Sigma w}$ [Replacing back θ with Σ

Comparing with gamma func^n

$f(w, \alpha, \theta) = \frac{w^{\alpha-1} e^{-w/\theta}}{\theta^\alpha \Gamma(\alpha)}$

$\boxed{\alpha = 4, \quad \theta = \frac{1}{\Sigma}}$

3.1 8a **1.5 / 1.5**

✓ **- 0 pts** Correct

- 0.5 pts (i) Not correct

- 0.5 pts (ii) Not correct

7i) $\hat{\Theta} = \dfrac{4*5}{(0.3^3 + 0.6^3 + 1.2^3 + 1.3^3 + 1.8^3)} = 2$

b) $W = X^3$

$X = W^{1/3}$

$\dfrac{dx}{dW} = \dfrac{1}{3W^{2/3}}$

$f_W(w) = f_X(w^{1/3}) \cdot \left| \dfrac{1}{3w^{2/3}} \right|$

$= \dfrac{1}{2} \theta^4 w^{1/3} e^{-\theta w} \dfrac{1}{3w^{2/3}}$

$= \dfrac{\theta^4 w^3 e^{-\theta w}}{6}$

$f_W(w) = \dfrac{1}{6} \Sigma^4 w^3 e^{-\Sigma w}$ [Replacing back θ with Σ]

Comparing with gamma func^n

$f(w, \alpha, \theta) = \dfrac{w^{\alpha-1} e^{-w/\theta}}{\theta^{\alpha} \Gamma(\alpha)}$

$\boxed{\alpha = 4, \ \theta = \dfrac{1}{\Sigma}}$

3.2 8b **1 / 1**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Not corrrect
- **- 1 pts** Missing