STAT 410 **Homework #10** Fall 2021
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(due Friday, November 12, by 5:00 p.m. CST)

No credit will be given without supporting work.

7. Let $\psi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\psi) = \frac{\psi}{2^{\psi}} \cdot (2-x)^{\psi-1}, \qquad 0 < x < 2, \qquad \text{zero otherwise.}$$

Recall: $W = -\ln\left(1 - \frac{X}{2}\right)$ has an Exponential distribution with mean $\theta = \frac{1}{\psi}$.

$$\hat{\hat{\psi}} = \frac{n-1}{\sum_{i=1}^{n}\left(-\ln\left(1-\frac{X_i}{2}\right)\right)} \text{ is an unbiased estimator of } \psi.$$

k) Suggest a confidence interval for ψ with $(1-\alpha)\,100\,\%$ confidence level.

① Use $Y = \sum_{i=1}^{n}\left(-\ln\left(1-\frac{X_i}{2}\right)\right)$.

② If T has a Gamma($\alpha, \theta = 1/\lambda$) distribution, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution.

l) Suppose $n = 3$, and $x_1 = 0.62$, $x_2 = 1.54$, $x_3 = 1.86$.
Use part (k) to construct a 90% confidence interval for ψ.

m) Find a sufficient statistic $u(X_1, X_2, \ldots, X_n)$ for ψ.

n) Find the Fisher information $I(\psi)$.

o) Is $\hat{\hat{\psi}}$ an efficient estimator of ψ?

If $\hat{\hat{\psi}}$ is not efficient, find its efficiency.

① Find $\mathrm{Var}(\hat{\hat{\psi}})$. ("Hint": Recall Homework #08 problem 7 part (g).)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\hat{\psi}}$ an efficient estimator of ψ? Does $\mathrm{Var}(\hat{\hat{\psi}})$ attain the R.C.L.B.?

If $\hat{\hat{\psi}}$ is not efficient, find its efficiency.

8. Let $\xi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\xi) = \frac{1}{2}\,\xi^4 x^{11} e^{-\xi x^3}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Recall: $W = X^3$ has a $\text{Gamma}(\alpha = 4,\ \theta = \frac{1}{\xi})$ distribution.

$$\hat{\hat{\xi}} = \frac{4n-1}{\sum_{i=1}^{n} X_i^3}$$ is an unbiased estimator of ξ.

h) Suggest a confidence interval for ξ with $(1-\alpha)\,100\,\%$ confidence level.

① Use $Y = \sum_{i=1}^{n} X_i^3$.

② If T has a $\text{Gamma}(\alpha,\ \theta = {}^{1}/_{\lambda})$ distribution, then

${}^{2T}/_{\theta} = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution.

i) Suppose $n = 5$, $x_1 = 0.3$, $x_2 = 0.6$, $x_3 = 1.2$, $x_4 = 1.3$, $x_5 = 1.8$.
Use part (h) to construct a 90% confidence interval for ξ.

j) Find a sufficient statistic $u(X_1, X_2, \ldots, X_n)$ for ξ.

k) Find the Fisher information $I(\xi)$.

(After you are done with part (k), glance back at Homework #09 problem 8 part (g).)

l) Is $\hat{\hat{\xi}}$ an efficient estimator of ξ?
If $\hat{\hat{\xi}}$ is not efficient, find its efficiency.

① Find $\operatorname{Var}(\hat{\hat{\xi}})$. ("Hint": Recall Homework #08 problem 8 part (d).)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\hat{\xi}}$ an efficient estimator of ξ? Does $\operatorname{Var}(\hat{\hat{\xi}})$ attain the R.C.L.B.?
If $\hat{\hat{\xi}}$ is not efficient, find its efficiency.

9. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\lambda) = \frac{\lambda}{x^2}, \qquad x \ge \lambda, \qquad \text{zero otherwise.}$$

d) Find a sufficient statistic $u(X_1, X_2, \ldots, X_n)$ for λ.