STAT 410 # Homework #09 Fall 2021
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(due Friday, November 5, by 5:00 p.m. CDT)

No credit will be given without supporting work.

7. Let $\psi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\psi) = \frac{\psi}{2^{\psi}} \cdot (2-x)^{\psi-1}, \qquad 0 < x < 2, \qquad \text{zero otherwise.}$$

Recall: $\quad F_X(x) = 1 - \frac{1}{2^{\psi}} \cdot (2-x)^{\psi} = 1 - \left(1 - \frac{x}{2}\right)^{\psi}, \qquad 0 \le x < 2.$

Let $Y_1 < Y_2 < \ldots < Y_n$ denote the corresponding order statistics.

i) $\left[\right.$ Proving that $Y_1 = \min X_i \xrightarrow{P} 0$ is super easy, barely an inconvenience. $\left.\right]$

Let $U_n = n\,Y_1 = n \min X_i$. Find the limiting distribution of U_n.

"Hint": ① Find the c.d.f. of Y_1, $F_{Y_1}(x) = F_{\min X_i}(x)$.

② Use $F_{Y_1}(x)$ to find the c.d.f. of U_n, $F_{U_n}(u) = P(U_n \le u)$.

③ $F_{\infty}(u) = \lim_{n \to \infty} F_{U_n}(u)$. If the limit exists, and if $F_{\infty}(u)$ is a c.d.f. of a probability distribution, then that is the limiting distribution of U_n.

j) $\left[\right.$ Proving that $Y_n = \max X_i \xrightarrow{P} 2$ is super easy, barely an inconvenience. $\left.\right]$

Find β so that $V_n = n^{\beta}(2 - Y_n) = n^{\beta}(2 - \max X_i)$ converges in distribution. Find the limiting distribution of V_n.

"Hint": ① Use $F_X(x)$ to find the c.d.f. of Y_n, $F_{Y_n}(x) = F_{\max X_i}(x)$.

② Use $F_{Y_n}(x)$ to find the c.d.f. of V_n, $F_{V_n}(v) = P(V_n \le v)$.

③ $F_\infty(v) = \lim_{n\to\infty} F_{V_n}(v)$. IF the limit exists and IF $F_\infty(v)$ is a c.d.f. of a probability distribution, then that is the limiting distribution of V_n.

④ $\lim_{n\to\infty}\left(1+\frac{a}{n^{\mathbf{1}}}\right)^n = e^a$. Only "interesting" case is interesting.

8. Let $\xi > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\xi) = \frac{1}{2}\,\xi^4\,x^{11}\,e^{-\xi x^3}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Recall: The maximum likelihood estimator of ξ is $\hat{\xi} = \dfrac{4n}{\sum_{i=1}^{n} X_i^3}$.

$W = X^3$ has a Gamma$(\alpha = 4, \theta = \frac{1}{\xi})$ distribution.

g) Show that $\hat{\xi}$ is asymptotically normally distributed (as $n \to \infty$). Find the parameters.

"Hint": ① By CLT, $\sqrt{n}\left(\overline{W} - \mu_W\right) \xrightarrow{D} N\left(0, \sigma_W^2\right)$.

② If $g(x)$ is differentiable at μ and $g'(\mu) \ne 0$, then

$$\sqrt{n}\left(g\left(\overline{X}\right) - g\left(\mu_W\right)\right) \xrightarrow{D} N\left(0, \left[g'(\mu_W)\right]^2 \sigma_W^2\right).$$

That is, for large n,

$$g(\overline{X}) \text{ is approximately } N\left(g(\mu_W), [g'(\mu_W)]^2 \frac{\sigma_W^2}{n}\right).$$

9. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\lambda) = \frac{\lambda}{x^2}, \qquad x \ge \lambda, \qquad \text{zero otherwise.}$$

a) (i) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

(ii) Suppose $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 3$, $x_4 = 20$.
Find the maximum likelihood estimate of λ.

b) Is $\hat{\lambda}$ a consistent estimator of λ? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

c) Is $\hat{\lambda}$ an unbiased estimator of λ? If $\hat{\lambda}$ is not an unbiased estimator of λ, construct an unbiased estimator of λ based on $\hat{\lambda}$. (Assume $n \ge 2$.)