STAT 431/531 Homework 1 Solutions

2022-09-05

Problem 1

Let C be the event of a convention. Let $X = I(C)$ be the random variable which indicates whether there was an event or not (under event C, $X = 1$ but under event $\bar{C}$, $X = 0$).

In the problem statement, we are given the following information:

$$
\begin{aligned}
f(X=1) &= Prob(C) = \frac{1}{4} \\
f(X=0) &= Prob(\bar{C}) = \frac{3}{4} \\
f(Y|X=1) &= \begin{cases} \frac{1}{15} & 15 \le Y \le 30 \\ 0 & Y < 15 \text{ or } Y > 30 \end{cases} \\
f(Y|X=0) &= \begin{cases} \frac{1}{5} & 15 \le Y \le 20 \\ 0 & Y < 15 \text{ or } Y > 20 \end{cases}
\end{aligned}
$$

Part a

$$
\begin{aligned}
f(X=1|Y=18) &= \frac{f(Y=18|X=1)f(X=1)}{f(Y=18)} \\
&= \frac{f(Y=18|X=1)f(X=1)}{f(Y=18|X=1)f(X=1) + f(Y=18|X=0)f(X=0)} \\
&= \frac{\frac{1}{15} * \frac{1}{4}}{\frac{1}{15} * \frac{1}{4} + \frac{1}{5}\frac{3}{4}} \\
&= \frac{\frac{1}{60}}{\frac{1}{6}} = \frac{1}{10}
\end{aligned}
$$

Part b

Can also solve this using the method from Part a.

$$f(X=1|Y=28) = 1 - f(X=0|Y=28) = 1 - 0 = 1$$

Intuitively, a 28 minute commute is impossible on a day with no convention. So, we can be sure that there is a convention today.

Problem 2

Part a

- i: $Prob(X_1 = 1)$
- ii: $Prob(X_2 = 0|X_1 = 1)$
- iii: $Prob(X_2 = 0)$
- iv: $Prob(X_1 = 1|X_2 = 0)$

Part b

- i:

$Prob(X_1 = 1) = \theta$

- ii:

$X_2|X_1 = 1 \sim Binomial(n, \lambda)$, so $Prob(X_2 = 0|X_1 = 1) = (1 - \lambda)^n$

- iii:

$Prob(X_2 = 0) = Prob(X_2 = 0|X_1 = 1)Prob(X_1 = 1) + Prob(X_2 = 0|X_1 = 0)Prob(X_1 = 0)$.

$X_2|X_1 = 0 \sim Binomial(n, 0) = 0$ always, so $Prob(X_2 = 0|X_1 = 0) = 1$. Plug in to the above to get:

$Prob(X_2 = 0) = (1 - \lambda)^n\theta + 1(1 - \theta) = (1 - \lambda)^n\theta + (1 - \theta)$

- iv:

$Prob(X_1 = 1|X_2 = 0) = \frac{Prob(X_2=0|X_1=1)Prob(X_1=1)}{Prob(X_2=0)} = \frac{(1-\lambda)^n\theta}{(1-\lambda)^n\theta+(1-\theta)}$

Problem 3

Part a

$X \sim Exponential(1)$, so the marginal is $f(x) = e^{-x}$

$Y|X = x \sim Poisson(x)$, so the conditional is $f(y|x) = \frac{e^{-x}x^y}{y!}$

Part b

For $y = 0, 1, 2, ...,$

$$
\begin{aligned}
f(y) &= \int_0^\infty f(y|x)f(x)dx && (1)\\
&= \int_0^\infty \frac{e^{-x}x^y}{y!}e^{-x}dx && (2)\\
&= \frac{1}{y!}\int_0^\infty x^{(y+1)-1}e^{-2x}dx && (3)\\
&= \frac{\Gamma(y+1)}{(2^{y+1})(y!)}\int_0^\infty \frac{2^{y+1}}{\Gamma(y+1)}x^{(y+1)-1}e^{-2x}dx && (4)\\
&= \frac{y!}{(2^{y+1})(y!)} \cdot 1 && (5)\\
f(y) &= \left(\frac{1}{2}\right)^{y+1}, y = 0, 1, 2, ... && (6)
\end{aligned}
$$

Explanation:

Note the form of the gamma distribution: If $Z \sim Gamma(a, b)$, then $f(z) = \frac{b^a}{\Gamma(a)}z^{a-1}e^{-bz}$.

The terms of $f(z)$ that involve z are known as the kernel, and we state $f(z) \propto z^{a-1}e^{-bz}$. The rest ($\frac{b^a}{\Gamma(a)}$) is known as the normalizing constant. Note that if a is a positive integer, then $\Gamma(a) = (a-1)!$

Lines 1 & 2: We start by writing out the integral to calculate the marginal $f(y)$. **Line 3:** We isolate the kernel of the gamma distribution inside the integral. **Line 4:** By adding in the normalizing constant inside the interval and its inverse outside the integral, we can put the full $Gamma(y+1, 2)$ density function inside the integral. **Line 5:** The integral of any density over its support is 1.

Part c

Geometric and Negative binomial are both correct.

Explanation:

$$
\begin{aligned}
f(y) &= \left(\frac{1}{2}\right)^{y+1}\\
&= \left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right)^y
\end{aligned} \tag{7}
$$

So, $Y \sim Geometric(\frac{1}{2})$ or $Y \sim Negative - Binomial(\theta = \frac{1}{2}, m = 1)$