STAT 431 — Applied Bayesian Analysis — Fall 2022

Homework 3

Due: October 7, 11:59 PM (US Central)

In your submission, please include any computer code and output that you used to answer the following problems. Any numerical values should be given to at least three significant digits, unless otherwise stated.

1. Suppose
$$Y_1, \ldots, Y_n \mid \tau^2 \;\sim\; \text{Normal}\big(0, 1/\tau^2\big) \qquad\qquad \tau^2 > 0$$
where the sampling precision τ^2 is the only unknown.

   (a) [2 pts] **Derive** a simplified expression for the likelihood function (up to proportionality) in terms of observations $y_1, \ldots, y_n$ (of $Y_1, \ldots, Y_n$), showing that it depends on these only through
$$s_n \;=\; \sum_{i=1}^{n} y_i^2$$

   (b) [4 pts] **Directly** show that the family of all gamma distributions is a conjugate prior family for τ^2. Also, express the posterior's hyperparameters in terms of the gamma prior's hyperparameters (α and β).

   (c) [1 pt] On the same plot, display the PDF curves of the Gamma(α, β) prior distributions with $\alpha = \beta = 1$, $\alpha = \beta = 0.5$, $\alpha = \beta = 0.1$, and $\alpha = \beta = 0.01$. Distinguish the four curves with different line types and/or colors. Let the horizontal axis of the plot go from 0 to 4. (Do these PDFs appear to be converging to some valid PDF?)

   (d) [1 pt] Suppose $n = 5$ and $s_n = s_5 = 5$. On the same plot (but separate from the plot you made for the previous part), display the PDF curves of the posterior distributions that result from each of the priors of the previous part. Distinguish the four curves with different line types and/or colors. Let the horizontal axis of the plot go from 0 to 4. (Do these PDFs appear to be converging to some valid PDF?)

2. Random Y is a non-negative integer from a distribution defined by unknown real-valued parameter θ, $-\infty < \theta < \infty$. Under this model, Y has mean $e^{-\theta}$ and variance $e^{-\theta}\big(e^{-\theta} + 1\big)$. For any observed data value y of Y, the likelihood is proportional to
$$\frac{e^{\theta}}{\big(e^{\theta} + 1\big)^{y+1}} \qquad\qquad -\infty < \theta < \infty$$

   (a) [5 pts] **Derive** Jeffreys' prior on θ. (Express its density up to proportionality.) Show your work.

   (b) [2 pts] Is this Jeffreys' prior proper or improper? Justify your answer.

3. GRADUATE SECTION ONLY

   Consider the situation of Problem 1. Let m be a constant positive integer, $y_1^0, \ldots, y_m^0$ be constant real values, and f be the PDF of $\text{Normal}\big(0, 1/\tau^2\big)$.

Recall that the **natural conjugate** prior family has densities

$$\pi\big(\tau^2 \mid \boldsymbol{\phi}_0\big) \;\; \propto \;\; \prod_{j=1}^{m} f\big(y_j^0 \mid \tau^2\big)$$

for every $\boldsymbol{\phi}_0 = \big(y_1^0, \ldots, y_m^0, m\big)$ for which this represents a proper density.

(a) [2 pts] For the situation of Problem 1, find a simplified expression proportional to $\pi\big(\tau^2 \mid \boldsymbol{\phi}_0\big)$, showing that it depends on $y_1^0, \ldots, y_m^0$ only through

$$s_m^0 \;\; = \;\; \sum_{j=1}^{m} \big(y_j^0\big)^2$$

(b) [1 pt] For what values of s_m^0 is your expression from the previous part proportional to a **proper** density (PDF) for τ^2 on the interval $(0, \infty)$?

(c) [2 pts] **Name** the type of distribution to which all members of this natural conjugate family belong, and express its usual arguments (hyperparameters) in terms of m and s_m^0.