STAT 431 — Applied Bayesian Analysis — Course Notes

Improper Priors

Fall 2022

Improper Densities

beyond distributions
do not correspond to

An **improper density** $f(\boldsymbol{x})$ satisfies

- $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$ (non-negative)
- $\sum_{\boldsymbol{x}} f(\boldsymbol{x}) \;=\; \infty$ (discrete) or divergent
  $\int f(\boldsymbol{x})\, d\boldsymbol{x} \;=\; \infty$ (continuous)

  (i.e. it cannot be *normalized* to a **proper** density)
  true PDF

flat prior also

When an improper density is used as a (“noninformative”) prior density, it is called an **improper prior**.

Eg: Population proportion θ (binomial model)

$$\pi(\theta) \;=\; \frac{1}{\theta(1-\theta)} \qquad 0<\theta<1$$

$\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$

(like a beta density with "$\alpha = 0$" and "$\beta = 0$")

$\alpha \to$ # successes
$\beta \to$ # failures
i.e. no info at all

motivation

Improper because

$$\int_0^1 \pi(\theta)\, d\theta \;=\; \infty$$

→ How is it improper?
→ Area under curve

[Draw density ...]

Infinite area b'coz of behaviour near 0 & 1

Warning: Improper priors may lead to improper posteriors!

problematic

Eg: (continued)

If $Y \sim \text{Binomial}(n, \theta)$ has prior

$$\pi(\theta) = \frac{1}{\theta(1-\theta)} \qquad 0 < \theta < 1$$

and either $y = 0$ or $y = n$, then can show that $p(\theta \mid y)$ is improper!

Alternative: Use “vague” priors — proper, but close to improper.

Eg: Use $\mathrm{Beta}(\alpha, \beta)$ with α and β “near” zero.

Example: Mean-Only Normal Sample

$$Y_1, \ldots, Y_n \mid \mu \;\sim\; iid\ \text{Normal}\big(\mu, \sigma^2\big) \qquad \sigma^2 \text{ known}$$

Recall likelihood

$$\propto \quad e^{-\frac{n}{2\sigma^2}\,(\mu-\bar{y})^2}$$

Try improper "flat" prior

$$\pi(\mu) \;\propto\; 1 \qquad -\infty < \mu < \infty$$

$\alpha = \beta \sim 1$

Flat prior

Need atleast 1 observation

We must check that the posterior is proper ...

$$p(\mu \mid \boldsymbol{y}) \quad \propto \quad f(\boldsymbol{y} \mid \mu)\, \underbrace{\pi(\mu)}_{\propto 1} \quad \propto \quad e^{-\frac{n}{2\sigma^2}(\mu - \bar{y})^2}$$

Recognize as the kernel of $\mathrm{Normal}(\bar{y}, \sigma^2/n)$ (why?), so the posterior is indeed proper:

$$\mu \mid \boldsymbol{y} \;\sim\; \mathrm{Normal}\big(\bar{y}, \sigma^2/n\big)$$

Gives us posterior mean

Note: The posterior mean is $\bar{y}$ and the posterior standard deviation is the (frequentist) standard error of $\bar{y}$.

Can show that, under this improper prior,

- credible intervals for μ are the same as confidence intervals

- the posterior probability of a *one-sided* H_0 is the same as a p-value

  $\leq$, $>$

  (Not true for the two-sided case.)

- this posterior is the limit as $\tau_0^2 \to 0$ (equiv. $m \to 0$) of the posterior when the $\mathrm{Normal}(\mu_0,\, 1/\tau_0^2)$ prior is used

Notation: For a flat prior, write, e.g.

$$\boxed{\mu \sim 1\, d\mu}$$

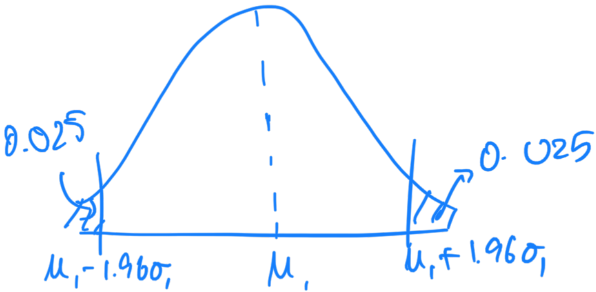
Eg: Jevons's Coin Data

Posterior under flat prior:

$$\mu \mid \boldsymbol{y} \quad \sim \quad \text{Normal}(\mu_1 = \bar{y} = 7.8730,\ \sigma_1^2 = \sigma^2/n \approx 0.0001194)$$

(so $\sigma_1 \approx 0.01093$)

[Draw density w/ probability limits ...]



So an approx. 95% credible interval is

$$\mu_1 \pm 1.96\,\sigma_1 \quad \approx \quad (7.8516,\ 7.8944)$$

It excludes all values meeting the min. legal weight (7.9379).

1-sided

Indeed, for $\quad H_0 :\ \mu \geq 7.9379 \quad$ we find

$$\mathrm{Prob}(H_0 \mid \boldsymbol{y}) \;=\; 1 - \Phi\left(\frac{7.9379 - \mu_1}{\sigma_1}\right) \;\approx\; 10^{-9}$$

(same as a p-value, in this case)

Atleast on avg, the coins are not meeting the min legal weight

Now consider randomly selecting another coin of the same kind (minted before 1830). Its (random) weight will be Y^*.

Using the flat prior and previous formulas for the posterior predictive distribution,

$$Y^* \mid \boldsymbol{y} \quad \sim \quad \text{Normal}\big(7.8730,\ 0.0001194 + (0.05353)^2\big)$$

The posterior predictive standard deviation works out to be about 0.05463.

The posterior predictive prob. that *this coin* is of legal weight:

$$\begin{aligned} \text{Prob}(Y^* \geq 7.9379 \mid \boldsymbol{y}) \quad &\approx \quad 1 \, - \, \Phi\left(\frac{7.9379 - 7.8730}{0.05463}\right) \\ &\approx \quad 0.1174 \end{aligned}$$

Remarks

- An improper prior may be hard to interpret subjectively, since it can't produce probabilities for $\boldsymbol{\theta}$.

  Perhaps interpret as equivalent to a series of proper priors whose posteriors converge to the posterior produced by the improper prior.

  (e.g. as $\sigma_0^2 \to \infty$, $\mathrm{Normal}(0, \sigma_0^2)$ priors produce posteriors converging to the flat prior's posterior)

- Representing the condition of "no prior information" often requires considering improper priors ("noninformative" priors — next).