431 : HW-3

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1) $Y_1, Y_2 \dots Y_n \sim N(0, 1/\tau^2)$

$$f(y|0, 1/\tau^2) = \frac{1}{(2\pi 1/\tau^2)^{1/2}} \exp\left(-\frac{1}{2}(y^2\tau^2)\right)$$

$$\text{Likelihood} = \prod_{i=1}^{n} f(y_i | 0, 1/\tau^2)$$

$$= \prod_{i=1}^{n} \frac{1}{(2\pi 1/\tau^2)^{1/2}} \exp\left(-\frac{1}{2} y^2\tau^2\right)$$

$$\text{Likelihood} \propto (\tau^2)^{n/2} \cdot \exp\left(\frac{-SSE}{2 \cdot 1/\tau^2}\right)$$

in terms of sufficient statistic

$SSE = S_n = \sum_i y_i^2 \Rightarrow$ Likelihood depends on S_n.

b) Show: Gamma is the conjugate prior for τ^2

Suppose $\tau^2 \sim \text{Gamma}(\alpha, \beta)$ $\left[f(\tau^2) = \frac{(\tau^2)^{\alpha-1} e^{-\beta\tau^2} \cdot \beta^\alpha}{\Gamma(\alpha)}\right]$

Then,

$$p(\tau^2|y) \propto (\tau^2)^{n/2} e^{\frac{-SSE}{2(1/\tau^2)}} \cdot (\tau^2)^{\alpha-1} \cdot e^{-\beta\tau^2} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$\propto (\tau^2)^{\alpha+\frac{n}{2}-1} \cdot e^{-\left(\frac{SSE}{2}+\beta\right)\tau^2}$$

which is the kernel of $\text{Gamma}\left(\alpha+\frac{n}{2}, \beta+\frac{\sum y_i^2}{2}\right)$

$$\therefore \tau^2 | y \propto \text{Gamma}\left(\alpha+\frac{n}{2}, \beta+\frac{\sum_i y_i^2}{2}\right)$$

As the prior Gamma gave a result of Posterior Gamma, thus Gamma is a conjugate prior of τ^2.

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2. a) $f(y|\theta)$, $-\infty < \theta < \infty$

Mean$(y) = e^{-\theta}$

Var$(y) = e^{-\theta} \cdot (e^{-\theta}+1)$

$$f(y|\theta) \propto \frac{e^\theta}{(e^\theta+1)^{y+1}} \quad -\infty < \theta < \infty$$

JF prior: $\pi(\theta) \propto \sqrt{I(\theta)}$

$$f(y|\theta) \propto \frac{e^\theta}{(e^\theta+1)^{y+1}}, \quad -\infty < \theta < \infty$$

$$\ln f(y|\theta) \propto \ln e^\theta - (y+1)\ln(e^\theta+1)$$

$$\propto \theta - (y+1)\ln(e^\theta+1)$$

$$\frac{\partial}{\partial\theta} \ln f(y|\theta) \propto 1 - \frac{(y+1)}{(e^\theta+1)} \cdot e^\theta$$

$$\frac{\partial^2}{\partial\theta^2} \ln f(y|\theta) \propto -(y+1)\frac{e^\theta}{(e^\theta+1)^2}$$

$$E\left(\frac{\partial^2}{\partial\theta^2} \ln f(y|\theta)\right) \propto -E\left[(y+1)\frac{e^\theta}{(1+e^\theta)^2}\right]$$

$$-I(\theta) \propto -E\left[(y+1)\frac{e^\theta}{(1+e^\theta)^2}\right]$$

$$I(\theta) \propto [E(Y)+1]\frac{e^\theta}{(1+e^\theta)^2}$$

$$I(\theta) \propto \frac{(e^{-\theta}+1)e^\theta}{(1+e^\theta)^2}$$

$$I(\theta) \propto \frac{1}{(1+e^\theta)}$$

JP prior:

$$\boxed{\pi(\theta) \propto \sqrt{I(\theta)} \propto \left(\frac{1}{e^\theta+1}\right)^{1/2}}$$

b) $J(P)$:

$$\pi(\theta) \propto \sqrt{\frac{1}{(e^\theta+1)}}, \quad -\infty < \theta < \infty$$

This JP is improper as the integral is divergent since $\theta \in (-\infty, \infty)$ in $\mathbb{R}$.

$$\int_{-\infty}^{\infty} \pi(\theta)\, d\theta = \int_{-\infty}^{\infty} \frac{1}{(e^\theta+1)}\, d\theta = \infty$$

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3) a) Given:

$$\pi(\tau^2 | \phi_0) \propto \prod_{j=1}^{n} f(y_j|\tau^2)$$

for every $\phi_0 = (y_1^0, \dots, y_m^0, m)$

From Problem-1, we have

$$\pi(1/\tau^2 | \theta) \propto \prod_{i=1}^{n} f(y_i|\theta)$$

$$\propto \left(\frac{\tau}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\tau^2 \sum_{i=1}^{n} y_i^2\right)$$

We see that,

$$P(\theta|Y) \propto \pi(\theta | y_1^0, y_2^0, \dots, y_m^0, m) \cdot \prod_{i=1}^{n} f(Y_i|\theta)$$

$$\propto \prod_{j=1}^{M} f(y_j^*|\theta)$$

where $M = m+n$

$y_j^* = y_j$ for $j = 1, \dots m$

$y_j^* = Y_{j-m}$ for $j = (m+1), \dots (m+n$

Thus, the posterior distribution belongs to some class of distribution as for the one in Problem-1.

$$\text{Therefore, } \pi(\tau^2 | \phi_0) = \prod_{j=1}^{m} f(y_j^0 | \phi_0)$$

$$= \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi\tau^2}} \cdot \exp\left(-\frac{1}{2}\frac{(y_j^0)^2}{1/\tau^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi\tau^2}}\right)^m \exp\left(-\frac{1}{2} \cdot \tau^2 \cdot \sum_{j=1}^{m} (y_j^0)^2\right)$$

$$\pi(\tau^2 | \phi_0) \propto \left(\frac{1}{\tau}\right)^m \cdot \exp\left(-\frac{1}{2}\tau^2 \sum_{j=1}^{m} (y_j^0)^2\right)$$

$$\propto \left(\frac{1}{\tau}\right)^m \exp\left(-\frac{\tau^2}{2} S_m^0\right)$$

b) Proper density exists if there exists m s.t.

$$\int_0^\infty \prod_{j=1}^{m} f(y_j^0 | \phi)\, d\theta = 1$$

$$\therefore \int_0^\infty \frac{1}{\sqrt{2\pi\tau^2}} \cdot \exp\left(-\frac{1}{2}\tau^2 S_m^0\right) d\tau^2$$

So, S_m^0 can take any real value ~~to~~ keep this integral definite.

c) Let $f(y|\theta) \sim \text{Normal}(0, \sigma^2)$

We have $f(y|\sigma^2) \propto \exp\left(-\frac{(y-0)^2}{2\sigma^2}\right)$

$$\pi(1/\tau^2 | y_1^0, \dots, y_m^0) \propto \exp\left(-\frac{1}{2}\sum_{j=1}^{m}(y_j^0-0)^2 / 1/\tau^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\sum_{j=1}^{m}(y_j^0)^2 \tau^2\right)$$

$$\propto \exp\left(-\frac{1}{2} S_m^0 \tau^2\right) \quad \because \sum_{j=1}^{m}(y_j^0)^2 = S_m^0$$

$$f(y|1/\tau^2) \sim \text{Normal}\left(0, \frac{m}{\tau^2}\right)$$