HW-5

Thursday, November 10, 2022 10:57 PM

2. $Y_1, Y_2, \ldots Y_n \sim N(\underbrace{\mu, \sigma^2}_{\text{both unknown}})$

$\hat{\sigma}_{MLE} = \frac{1}{n} \sum_i (y_i - \bar{y})^2$

a) Likelihood:

$P(Y|\mu,\sigma^2) = \prod_{i=1}^{n} P(y_i|\mu,\sigma^2)$

$\propto \frac{1}{\sigma^n} \exp\left\{-\sum_i \frac{(y_i-\mu)^2}{2\sigma^2}\right\}$

• Case-1: 'Standard' Non-informative prior $\pi(\mu,\sigma^2) \propto \frac{1}{\sigma^2}$ $(\sigma^2>0)$

Joint Posterior (un-normalized)

$P(\mu,\sigma^2|y) \propto P(y|\mu,\sigma^2).\,\pi(\mu,\sigma^2)$

$\propto \frac{1}{\sigma^n} \exp\left\{-\sum_i \frac{(y_i-\mu)^2}{2\sigma^2}\right\} . \frac{1}{\sigma^2}$

$\propto \frac{1}{\sigma^{n+2}} \exp\left\{-\frac{(n-1)s^2 + n(\bar{y}-\mu)^2}{2\sigma^2}\right\}$

This is a special case of normal inverse chi-squared distribution

Marginal Posterior

$P(\mu|y) = \int_0^\infty P(\mu,\sigma^2|y)\, d\sigma^2$

$\propto \int_0^\infty \frac{1}{\sigma^{n+2}} \exp\left\{-\frac{(n-1)s^2+n(\bar{y}-\mu)^2}{2\sigma^2}\right\} d\sigma^2$

[Density of inverted chi-squared distribution is:

$P(\sigma^2) = \frac{(\nu_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)} . (\sigma_0^2)^{\nu_0/2} . (\sigma^2)^{-(\frac{\nu_0}{2}+1)} . \exp\left(\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right)$, $\sigma^2>0$

where $\nu_0 = n$,

$\sigma_0^2 = \left(\frac{n-1}{n}\right)s^2 + (\bar{y}-\mu)^2$]

$\propto (\sigma_0^2)^{-n/2} \int_0^\infty \frac{(n/2)^{-n/2}}{\Gamma(n/2)} (\sigma_0^2)^{n/2} . (\sigma^2)^{-(\frac{n}{2}+1)} . \exp\left\{-\frac{n\sigma_0^2}{2\sigma^2}\right\} d\sigma^2$

$\propto \left(\frac{(n-1)s^2}{n} + (\bar{y}-\mu)^2\right)^{-n/2}$

$\propto \left(1 + \frac{1}{(n-1)}\left(\frac{\mu-\bar{y}}{s/\sqrt{n}}\right)^2\right)^{-\frac{(n-1)+1}{2}}$

$\Rightarrow P(\mu|y)$ is the kernel of the non-std t-distribution with df = n-1

$\mu|Y = y \sim t_{n-1}\left(\bar{y}, \frac{s^2}{n}\right)$

$\Rightarrow$ Scaled & shifted μ follows std. t-distribution with df = n-1

$\frac{\mu-\bar{y}}{s/\sqrt{n}} \Big| Y = y \sim t_{n-1}(0,1)$

95% credible interval for μ

95% CI for μ = $\bar{y} \pm t_{0.025,n-1} \cdot \frac{s}{\sqrt{n}}$

• Case-2 Jeffreys' Prior $\pi(\mu,\sigma^2) \propto \frac{1}{(\sigma^2)^{3/2}}$, $\sigma^2>0$

JP is an improper prior. For the scenario when there are atleast 2 distinct observed values of y, we obtain a proper posterior.

Under those conditions, the posterior turns out to be proper & is actually normal-inverse gamma:

$\mu|\sigma^2, y \sim \text{Normal}\,(\bar{y}, \sigma^2/n)$

$\sigma^2|y \sim \text{InvGamma}\left(\frac{n}{2}, \frac{n-1}{2}s^2 = \frac{n\hat{\sigma}^2}{2}\right)$

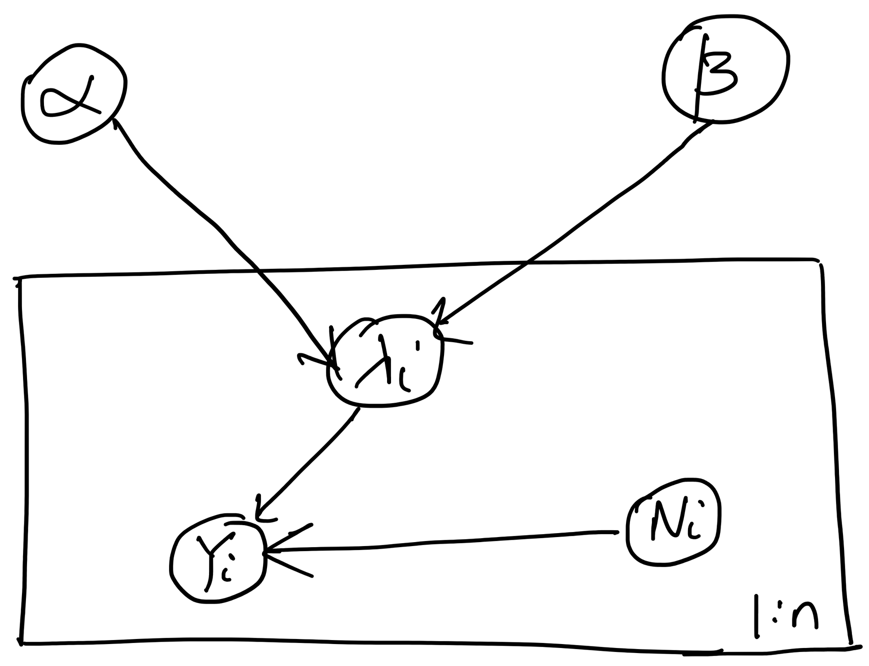
It follows that the marginal posterior for μ is

$\mu|y \sim t_n(\bar{y}, \hat{\sigma}^2/n)$

which gives the following 95% credible interval for μ:

$\bar{y} \pm t_{0.025,n} \cdot \frac{\hat{\sigma}}{\sqrt{n}}$

1) a)



Here,

n = length(y)