1 $Y|\theta \sim NegBinomial(\theta, m=5)$

a) θ : probability of success

i.e probability that the call made by the salesperson results in a sale

b) $\theta = 0.2$

From Geometric Distribution we know how many trials needed to get the first success

$$P(Y=y) = q^{y-1}p \quad , y=1,2,3\ldots$$

$$\therefore \mathbb{E}(Y) = p\sum_{y=1}^{\infty} y q^{y-1}$$
$$= \frac{1}{p}$$

$\therefore$ Expected total no. of calls to have 5 successes

$$\mathbb{E}\left[\sum_i Y_i\right] = 5 * \frac{1}{p}$$

Given $\theta = 0.2$ $\quad (\theta = p)$

$$\therefore \mathbb{E}\left[\sum_i Y_i\right] = \frac{5}{0.2} = 25$$

A : 25 calls

c) Using Bayes Rule, we know that

$$L(\theta|y) \propto f(y|\theta) * \pi(\theta)$$

Given that,

$Y|\theta \sim Neg\,Binomial(\theta, m=5)$

$$f(y|\theta) = \binom{y-m-1}{m-1}\theta^m (1-\theta)^y$$

and $y = 20$

$$\Rightarrow L(\theta|y) \propto \binom{y+m-1}{m-1}\theta^m (1-\theta)^y * \pi(\theta)$$

Substituting $y=20$, $m=5$

$$L(\theta|20) \propto \binom{24}{4}\theta^5 (1-\theta)^{20} * \pi(\theta)$$

$$\boxed{L(\theta|20) \propto \theta^5 (1-\theta)^{20} * \pi(\theta)}$$

d) Given,

$$\pi(\theta) = \frac{1}{1-0} = 1 \quad , 0 \le \theta \le 1$$

From Bayes rule,

$$f(\theta|y) = \frac{f(y|\theta) * \pi(\theta)}{m(y)}$$

Posterior Density $(\theta|y)$ = $\frac{\text{Likelihood } (\theta|y)}{\text{Normalizing constant}}$

$\therefore$ Posterior Density $(\theta|y) \propto$ Likelihood $(\theta|y)$

From above, we know that

Likelihood $(\theta|y) \propto \theta^5 (1-\theta)^{20} * \pi(\theta)$

Substituting $\pi(\theta) = 1$

$$\therefore f(\theta|y) \propto L(\theta|y)$$
$$f(\theta|y) \propto \theta^5 (1-\theta)^{20} * 1$$
$$\boxed{f(\theta|y) \propto \theta^5 (1-\theta)^{20}}$$

e) $f(\theta|y) \propto \theta^5 (1-\theta)^{20}$

Binomial distribution : ${}^nC_x\, p^x (1-p)^{n-x}$

$\Rightarrow f(\theta|y)$ resembles Binomial distribution

where $\theta = p$, $x = 5$, $n = 25$

$f(\theta|y) \sim Binomial(\theta, n=25, x=5)$

2.a) Beta prior is the conjugate prior to the family of binomial distributions which includes negative binomial distribution

i.e If $X|\theta \sim NB$ & $\theta \sim Beta(\alpha, \beta)$

then $\theta|X \sim Beta$

$$\pi(\theta|Y) \propto f(Y|\theta) * \pi(\theta)$$
$$f(Y|\theta) = \binom{y+m-1}{m-1}\theta^m (1-\theta)^y$$
$$f(y|\theta) \propto \theta^m (1-\theta)^y \quad ; m=5, y=20$$

Take $\pi(\theta) \sim Beta(\alpha', \beta')$

$$\pi(\theta) \propto \theta^{\alpha'-1}(1-\theta)^{\beta'-1}$$
$$\therefore \pi(\theta|y) \propto \theta^5 (1-\theta)^{20} \cdot \theta^{\alpha'-1}(1-\theta)^{\beta'-1}$$
$$\theta^{5-1}(1-\theta)^{20-1} \propto \theta^{5+\alpha'-1}(1-\theta)^{20+\beta'-1}$$

Comparing powers of θ, we get $\alpha' = 1$, $\beta' = 5$

$\therefore$ Prior $\pi(\theta) \sim Beta(\alpha = 1, \beta = 5)$

b) Posterior mean $= \frac{\alpha}{\alpha+\beta} = \frac{6}{6+28} = \frac{6}{31}$

Posterior SD $= \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$

$$= \sqrt{\frac{6 * 25}{(31)^2 * 32}}$$

$$= \frac{5\sqrt{3}}{124}$$

c) 95% Credible Interval

$$MAP = \frac{6-1}{(6-1)+(28-1)} = \frac{5}{5+27} = \frac{5}{27}$$

We want interval $[a, b]$ s.t.

$P(a \le \theta \le b) = 95$ (centered ; 0.025)

$F_{Beta} \rightarrow$ CDF

$$a = F^{-1}_{Beta}(0.025) \quad \& \quad b = F^{-1}_{Beta}(0.975)$$

$a \sim 0.771$, $b \sim 0.347$

d) $P(\theta \le 0.2 \mid Y \ldots) = \int_0^{0.2} Beta_\theta(\alpha=6, \beta=28)\, d\theta$

e) Posterior predictive probability

$$P(x_{new}|x) = \int_\theta P(x_{new}|\theta, x)\, P(\theta|x)\, d\theta$$
$$= \int P(x_{new}|\theta)\, P(\theta|x)\, d\theta$$

3.a) $\theta \sim Beta(\alpha, \beta)$

$y|\theta \sim Binomial(n, \theta)$, $P(y=k)$

$\Rightarrow \theta|Y=y \sim Beta[\alpha+y, \beta+n-y]$

SD of Beta Distribution $B(\alpha,\beta) = \sqrt{\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}}$

SD of Beta $(\alpha+y, \beta+n-y) = \frac{1}{(\alpha+\beta+n)}\sqrt{\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n+1)}}$

$$= \left(\frac{1}{\alpha+\beta+n}\right)\sqrt{\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n+1)}}$$

b) $SE(\hat{\theta}) = \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$, $\hat{\theta} = \frac{Y}{n}$ ②

$$\sqrt{Var(\theta|Y)} = \frac{1}{(\alpha+\beta+n)}\sqrt{\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n+1)}} \quad ①$$

$$\frac{①}{②} = \frac{1}{(\alpha+\beta+n)} \cdot \sqrt{\frac{(\alpha+y)(\beta+n-y)}{(\alpha+\beta+n+1)}} \cdot \sqrt{\frac{Y}{n^2}\cdot\left(1-\frac{Y}{n}\right)}$$

$$= \frac{1}{1+\left(\frac{\alpha+\beta}{n}\right)} \cdot \sqrt{\frac{(\alpha+y)(\beta-y+n)}{(\alpha+\beta+n+1)\, Y.(1-Y)}}$$

$$= \frac{1}{1+\left(\frac{\alpha+\beta}{n}\right)} \cdot \sqrt{\frac{\left(\frac{\alpha+y}{n}\right)\left(1+\left(\frac{\beta-y}{n}\right)\right)}{\left(1+\left(\frac{\alpha+\beta+1}{n}\right)\right)\cdot Y\left(1-\frac{Y}{n}\right)}}$$

As $n \to \infty$, $\frac{k}{n} \to 0$ where k is any constant.

$$\therefore \frac{\sqrt{Var(\theta|Y)}}{SE(\hat{\theta})} \xrightarrow[n\to\infty]{} \frac{1}{(1+0)} \cdot \sqrt{\frac{0.(1+0)}{(1+0)Y(1-0)}}$$

$$\frac{\sqrt{Var(\theta|Y)}}{SE(\hat{\theta})} \xrightarrow[n\to\infty]{} 1$$