STAT 431 HW3 Solution

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Problem 1

Part a

$$
\begin{aligned}
f(y_1,...,y_n|\mu,\sigma^2) &= \prod_{i=1}^{n}\sqrt{\frac{1}{2\pi\sigma^2}}e^{-\frac{(y_i-\mu)^2}{2\sigma^2}} \\
f(y_1,...,y_n|0,1/\tau^2) &= \prod_{i=1}^{n}\sqrt{\frac{\tau^2}{2\pi}}e^{-\frac{\tau^2}{2}(y_i-0)^2} \\
&\propto (\sqrt{\tau^2})^n e^{-\frac{\tau^2}{2}\sum_{i=1}^n y_i^2} \\
&= (\tau^2)^{n/2}e^{-\tau^2\frac{s_n}{2}}
\end{aligned}
$$

Part b

Let the prior be $\tau^2 \sim Gamma(\alpha,\beta)$. Then,

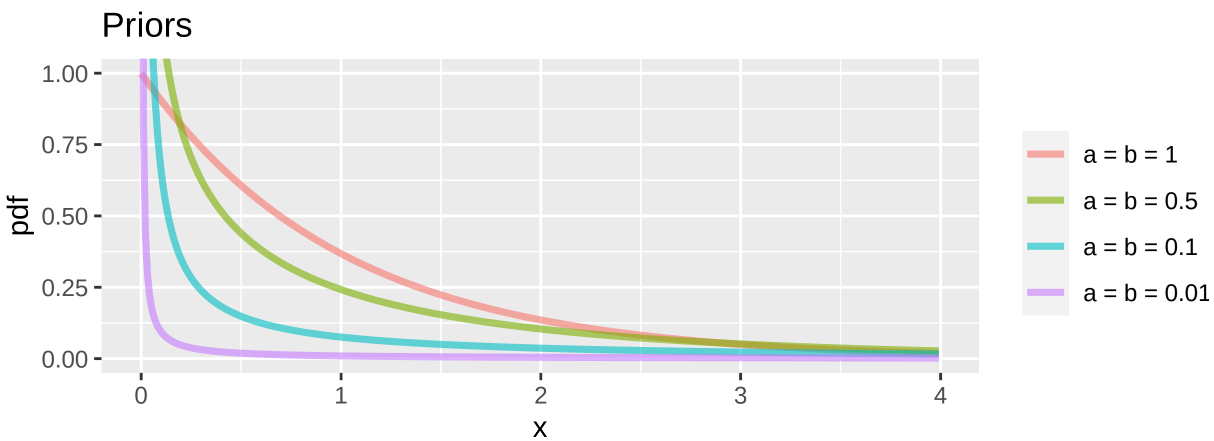
$$
\begin{aligned}
p(\tau^2|y_1,...,y_n) &\propto f(y_1,...,y_n|0,1/\tau^2)\pi(\tau^2) \\
&\propto (\tau^2)^{n/2}e^{-\frac{s_n}{2}\tau^2}(\tau^2)^{\alpha-1}e^{-\beta\tau^2} \\
&= (\tau^2)^{\alpha+n/2-1}e^{-(\beta+\frac{s_n}{2})\tau^2}
\end{aligned}
$$

So, the $Gamma(\alpha,\beta)$ family is conjugate for τ^2, resulting in the following posterior:

$$\tau^2|Y_1,...,Y_n \sim Gamma(\alpha+n/2,\beta+\frac{s_n}{2})$$

Part c

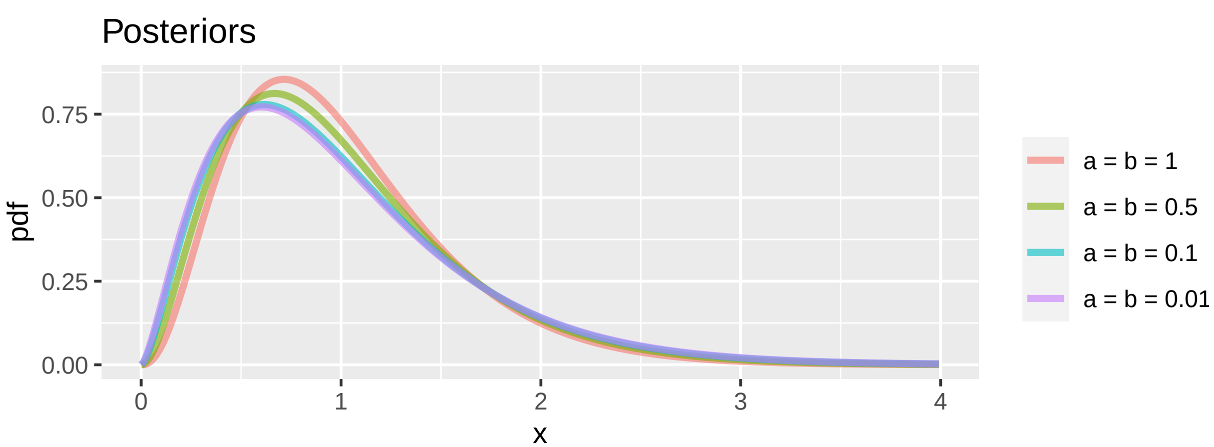
```r
x = seq(0.001,4,0.01)
ab = c(1,0.5,0.1,0.01)
pdf = lapply(ab,function(ab){dgamma(x,ab,ab)})
plot_data = data.frame(
  x = rep(x,4),
  pdf = unlist(pdf),
  ab = factor(rep(paste("a = b =",ab),each=length(x)),levels = paste("a = b =",ab))
)
ggplot(plot_data)+
  geom_line(aes(x,pdf,color=ab),size=1.25,alpha=0.6)+
  labs(color=NULL,title='Priors')+
  coord_cartesian(ylim=c(0,1))
```



These PDF curves do not appear to be converging to a valid PDF: as $\alpha = \beta$ goes to 0, the distribution is approaching a degenerate distribution with infinite density at $x = 0$.

Part d

```
n = 5
sn = 5
pdf = lapply(ab,function(ab){dgamma(x,ab+n/2,ab+sn/2)})
plot_data = data.frame(
  x = rep(x,4),
  pdf = unlist(pdf),
  ab = factor(rep(paste("a = b =",ab),each=length(x)),levels = paste("a = b =",ab))
)
ggplot(plot_data)+
  geom_line(aes(x,pdf,color=ab),size=1.25,alpha=0.6)+
  labs(color=NULL,title='Posteriors')
```



This time, we are approaching a valid PDF. Since the prior hyperparameters are shrinking to 0, we converge to $Gamma(2.5, 2.5)$.

Problem 2

Part a

$$
\begin{aligned}
\pi(\theta) &\propto \sqrt{I(\theta)} && (1)\\
&= \sqrt{-E\left(\frac{\partial^2}{\partial\theta^2} ln\left(f(Y|\theta)\right)|\theta\right)} && (2)\\
&= \sqrt{-E\left(\frac{\partial^2}{\partial\theta^2} ln\left(\frac{e^\theta}{(e^\theta+1)^{Y+1}}\right)|\theta\right)} && (3)\\
&= \sqrt{-E\left(\frac{\partial^2}{\partial\theta^2}\theta-(Y+1)ln(e^\theta+1)|\theta\right)} && (4)\\
&= \sqrt{-E\left(\frac{\partial}{\partial\theta}1-(Y+1)\frac{e^\theta}{e^\theta+1}|\theta\right)} && (5)\\
&= \sqrt{-E\left(-(Y+1)\frac{e^\theta}{(e^\theta+1)^2}|\theta\right)} && (6)\\
&= \sqrt{E((Y+1)|\theta)\frac{e^\theta}{(e^\theta+1)^2}} && (7)\\
&= \sqrt{(e^{-\theta}+1)\frac{e^\theta}{(e^\theta+1)^2}} && (8)\\
&= \sqrt{\frac{e^\theta+1}{(e^\theta+1)^2}} && (9)\\
&= \sqrt{\frac{1}{e^\theta+1}} && (10)\\
& && (11)
\end{aligned}
$$

So, the Jeffreys prior is $\pi(\theta) \propto \sqrt{\frac{1}{e^\theta+1}}$, $-\infty < \theta < \infty$

Part b

$e^\theta > 0$ for all $-\infty < \theta < \infty$.

So, $\frac{\partial}{\partial\theta}\sqrt{\frac{1}{e^\theta+1}} = -\frac{1}{2}e^\theta\left(\frac{1}{e^\theta+1}\right)^{\frac{3}{2}}$ is always negative.

Therefore, $\pi(\theta)$ is monotone decreasing.

$\pi(0) = \sqrt{\frac{1}{e^0+1}} = \sqrt{\frac{1}{2}}$.

Because $\pi(\theta)$ is monotone decreasing, this shows for all $\theta < 0$, $\pi(\theta) > \sqrt{\frac{1}{2}}$.

So, the Jeffreys prior has infinite area under the density on the interval $(-\infty, 0]$, and is therefore improper.

Problem 3

Part a

$$\pi(\tau^2|\phi_0) \propto \prod_{j=1}^{m} f(y_j^0|\tau^2) \tag{12}$$

$$= \prod_{j=1}^{m} \left(\frac{\tau^2}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{\tau^2}{2}(y_j^0-0)^2} \tag{13}$$

$$= \left(\frac{\tau^2}{2\pi}\right)^{\frac{m}{2}} e^{-\frac{\tau^2}{2}\sum_{j=1}^{m}(y_j^0)^2} \tag{14}$$

$$= \left(\frac{\tau^2}{2\pi}\right)^{\frac{m}{2}} e^{-\frac{\tau^2}{2}s_m^0} \tag{15}$$

$$\propto (\tau^2)^{m/2} e^{-\frac{s_m^0}{2}\tau^2} \tag{16}$$

Part b

The expression in Part a looks like the kernel of the $Gamma(m/2+1, s_m^0/2)$ density. The Gamma PDF is defined for $\alpha > 0$ and $\beta > 0$. So, for $s_m^0 > 0$, the previous expression is proportional to a proper density. For $s_m^0 = 0$, the normalizing constant of the Gamma PDF will be 0, resulting in an invalid PDF.

In conclusion, the expression is proportional to a valid density for all values of s_m^0 on the interval $(0, \infty)$. This means $y_1^0, ..., y_m^0$ must not all be equal to 0.

Part c

All members of this natural conjugate family belong to the Gamma distribution. The hyperparameters are $\alpha = m/2 + 1$ and $\beta = s_m^0/2, \quad s_m^0 > 0$.