HW-4

Friday, October 21, 2022 7:13 PM

Problem 1

```
n1 <- 24
ybar1 <- 7.8730
s1 <- 0.05353
n2 <- 123
ybar2 <- 7.9725
s2 <- 0.01409

### Independent draws from posterior
Nsim <- 100000

sigma1.2s <- 1 / rgamma(Nsim, (n1-1)/2, (n1-1)*s1^2/2)
sigma2.2s <- 1 / rgamma(Nsim, (n2-1)/2, (n2-1)*s2^2/2)
```

```
### Part(a)- approx. posterior mean
round(mean(sigma1.2s / sigma2.2s), 3)
```

```
## [1] 15.779
```

```
### Part(b)- Monte Carlo error of the approx. posterior mean
round(sd(sigma1.2s / sigma2.2s) / sqrt(Nsim), 3)
```

```
## [1] 0.017
```

```
### Part(c)- approx. standard deviation
round(sd(sigma1.2s / sigma2.2s), 3)
```

```
## [1] 5.506
```

```
### Part(d)- approx. 95% credible interval
round(quantile(sigma1.2s / sigma2.2s, c(0.025, 0.975)), 3)
```

```
##   2.5%  97.5%
##  8.121 29.288
```

```
### Part(e)- approx. 95% confidence interval
F1 = qf(p=.05, n1-1, n2-1, lower.tail=FALSE)
F2 = qf(p=.05, n2-1, n1-1, lower.tail=FALSE)
x = s1**2 / s2**2
round(c((1/F1)*x, F2*x), 3)
```

```
## [1]  8.920 26.152
```

2.a) Given: $D(y|\theta) = -2\ln(f(y|\theta))$

$$f(y|\theta) = \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \quad [\text{Binomial distribution}]$$

$$\Rightarrow D(y|\theta) = -2\ln\left[\binom{n}{y}\theta^{y}(1-\theta)^{n-y}\right]$$

$n = 70$

$$\therefore D(y=12|\theta) = -2\ln\left[\binom{70}{12}\theta^{12}(1-\theta)^{58}\right]$$

$$= -2\left[\ln\binom{70}{12} + 12\ln\theta + 58\ln(1-\theta)\right]$$

b) We have,

$$\text{Numerator} = \int_0^1 \theta\, D(y|\theta).\pi(\theta).d\theta$$

$$\text{Denominator} = \int_0^1 D(y|\theta).\pi(\theta).d\theta$$

$\pi(\theta) \sim \text{Beta}(0.5, 0.5)$

$$\Rightarrow \pi(\theta) = \frac{1}{B(0.5,0.5)}.\theta^{-0.5}(1-\theta)^{-0.5}$$

$$\text{Numerator} = \frac{1}{B(0.5,0.5)}\int_0^1 \theta\left[-2\times(12\ln\theta + 58\ln(1-\theta)\right].\left(\theta^{-0.5}(1-\theta)^{-0.5}\right)d\theta$$

$$= \frac{1}{B(0.5,0.5)}\int_0^1 \theta^{0.5}\left[-2\times(12\ln\theta + 58\ln(1-\theta)\right](1-\theta)^{-0.5}d\theta$$

$$\text{Denominator} = \frac{1}{B(0.5,0.5)}\int_0^1 -2\times\left[12\ln\theta + 58\ln(1-\theta)\right]\theta^{-0.5}(1-\theta)^{-0.5}d\theta$$

$$\text{Posterior Expected Deviance} = \frac{\text{Numerator}}{\text{Denominator}}$$

$$= \frac{\int_0^1 \theta^{0.5}\left[-2\times(12\ln\theta + 58\ln(1-\theta))\right].(1-\theta)^{-0.5}d\theta}{\int_0^1 \theta^{-0.5}\left[-2\times(12\ln\theta + 58\ln(1-\theta))\right](1-\theta)^{-0.5}d\theta}$$

Problem 2

```
### Part(c):  Population Proportion: Numerical Integration

n <- 70
y <- 12

### Define likelihood and prior (up to proportionality)
like <- function(theta) (-2 * (y * log(theta) + (n-y) * log(1-theta)))
prior <- function(theta) dbeta(theta, 0.5, 0.5)

numerator <- integrate(function(theta) theta * like(theta) * prior(theta), 0, 1)
numerator
```

```
## 143.0403 with absolute error < 0.00089
```

```
denominator <- integrate(function(theta) like(theta) * prior(theta), 0, 1)
denominator
```

```
## 194.0812 with absolute error < 0.0021
```

```
sprintf("posterior expected deviance is %f", numerator$value / denominator$value)
```

```
## [1] "posterior expected deviance is 0.737013"
```

3.a) Full conditional of θ by definition is:

$$p(\theta|y,n)$$

Given that:

- $Y|N=n,\theta \sim \text{Binomial}(n,\theta)$
- $\pi(n) \propto (0.9)^{n}$, $n=0,1,2\ldots$
- $\pi(\theta) \propto 1$, $0<\theta<1$
  $\pi(\theta) \propto \text{Beta}(\alpha=1, \beta=1)$

Likelihood : $P(Y|n,\theta) \sim \text{Binomial}(n,\theta)$
Prior : $\pi(\theta) \sim \text{Beta}(\alpha=1, \beta=1)$

$\therefore$ By Binomial - Beta conjugacy, we know that the posterior distribution of θ conditional on $Y=y$ will be Beta distribution i.e

Posterior : $\pi(\theta|Y=y,n) \sim \text{Beta}(\alpha+y, \beta+n-y)$
$\pi(\theta|Y=y,n) \sim \text{Beta}(1+y, 1+n-y)$

$\therefore$ (Posterior) full conditional distribution of θ

$$p(\theta|Y=y, N=n) \sim \text{Beta}(1+y, 1+n-y)$$

Beta distribution with params: $1+y$, $1+n-y$

b) Given: $Y|N=n,\theta \sim \text{Binomial}(n,\theta)$

$R = N - Y$

so, $Y = N - R$

$\therefore \quad N-R|N=n,\theta \sim \text{Binomial}(n,\theta)$

$$f(N-R|N=n,\theta) \sim \text{Binomial}(n,\theta)$$

$$\propto \theta^{N-R}(1-\theta)^{N-(N-R)}$$

$$\propto \underbrace{\theta^{n-R}(1-\theta)^{R}}_{\text{Kernel of Negative Binomial Dist.}}$$

$\therefore$ R follows NegBinomial $(\hat{\theta}, m)$ density where

$\hat{\theta} = R$
$m = n - R = Y$

c) $N = R+Y$ & $y = 20$
$n = r+20$

We can sample r from its distribution calculated in b)

From a), $\theta|y,n \sim \text{Beta}(1+y, 1+n-y)$
$\sim \text{Beta}(y+1, r+1)$

& $r|\theta,y \sim$ Negative Binomial $(p = 0.1+0.9\theta,\ k = y+1)$; $y=20$

We can simulate r, Nsim = 10000 times by choosing $\theta_0 = 0.5$ & iterating

$N = r+20$

$$P(N/2 \le 20) = P(r+20 \le 40)$$
$$= \text{mean}(r\_s + 20 \le 40)$$
$$= 0.94$$

Q3.(c)

```
# Gibbs Sampling:

Nsim_gibbs = 10000
r_s <- numeric(Nsim_gibbs)
theta_s <- numeric(Nsim_gibbs)

# initialize

theta_s[1] <- 0.5  # 0 to 1 (unif)

r_s[1] <- rnbinom(1, size = 21, prob = 0.1 + 0.9*theta_s[1])

# iterate

for(k in 2:Nsim){
  theta_s[k] = rbeta(1, 21, r_s[k-1]+1)

  r_s[k] = rnbinom(1, size = 21, prob = 0.1 + 0.9*theta_s[k])
}

N = r_s + 20  # Sample for N

# Probability for N to be less than 2*Y
mean(N <= 40)
```

```
## [1] 0.938
```