SQ HW-7

1.a Rao Blackwell Theorem,

If the unbiased estimator is a fun^n of a complete sufficient statistic (CS) then it is UMVUE

If $W(X)$ is any unbiased estimator with finite variance then,

$$Var(W(X)) = E_\theta[Var(W(X)|T(X))]$$

Sufficiency of $T(X)$ guarantees that W^* is not a fun^n of θ & then W^* is a statistic.

The conditional variance formula states that,

$$Var_\theta(W^*(X)) \le Var_\theta(W(X))$$

and by Lehman - Scheffe theorem $W^*(T(X))$ is UMVUE

Here we have $\sum X_i$ as CSS for λ. so for any $T(\theta)$,

$$h(\sum) = E(h(\sum)) = T(\theta)$$

UMVUE $= E[h(\sum)|T] = g(T)$

$$T = \sum X_i \Rightarrow E(X) = \lambda$$

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b) To show : $\theta = 1\{X_1 = 0\}$ is an unbiased estimator of $\theta = e^{-\lambda}$

Notice that $E(X_1 = 0) = e^{-\lambda}$

This satisfies the following estimator.

as $\theta = 1\{X_1 = 0\}$ so that

$\theta \sim Bernoulli(e^{-\lambda})$

$E(\theta) = e^{-\lambda}$

Hence $E(\theta) = \theta = e^{-\lambda}$

$\Rightarrow \theta$ is an unbiased estimator of θ.

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c) Given that

$1(X_1 = 0)$ is an unbiased estimator of $e^{-\lambda}$

$\sum X_i$ is a sufficient statistic for $e^{-\lambda}$

Using Rao Blackwell theorem,

$$\hat{\theta} = \mathbb{E}(1\{X_1 = 0\} \mid \sum X_i = t)$$

then, $Var_\lambda \hat{\theta} \le Var_\lambda \theta$

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2. Given that : $\lambda \sim$ exponential (λ)

$$f(x|\lambda) = \lambda e^{-\lambda x} \quad , x > 0 \; , \lambda > 0$$

$$H_0 : \lambda = 1 \qquad vs \qquad H_1 : \lambda \neq 1$$

$f(x|\lambda = 1) = e^{-x}$ [$f(\theta)$ is uniquely specified under the hypothesis $H_0 : \lambda = 1$ $\therefore$ simple hypothesis]

a) Size α of the test

$\alpha = P(\text{Type I error}) = P(|X-1| > 1/4 \mid H_0)$

$= P(X-1 > 1/4 \mid \lambda = 1) + P(X-1 < -1/4 \mid \lambda = 1)$

$= P(X < 1/4 \mid \lambda = 1) + P(X > 7/4 \mid \lambda = 1)$

$= 0.2212 + 1 - 0.8826$

$= 0.3386$

b) Power fun^n

$\beta(\lambda) = 1 - P(\text{Type I error})$

$= 1 - P(|X-1| \le 1/4 \mid H_1)$

$= 1 - P(3/4 < X \le 7/4 \mid \lambda)$

$= 1 + e^{-7/4\lambda} - e^{-3/4\lambda}$

c) To find the value of λ that minimizes it we can set $\frac{d\beta}{d\lambda} = 0$

$$0 = -7/4\, e^{-7/4\lambda} + e^{-3/4\lambda}$$

$$e^{3/4} = 7 e^{7/4\lambda} \Rightarrow \boxed{\lambda = \frac{1.386}{4}}$$

3.a) Size of ϕ is $\alpha_1 = P(X \in R \mid H_0)$

$= P(X_1 > 0.95 \mid \theta = 0)$

$= 0.05$

Size of ϕ_2 is $\alpha_2 = P(X_1 + X_2 > C \mid \theta = 0)$

$$= \iint_{1<C} dx_1 dx_2 \qquad 1 \le C \le 2$$

$$= \frac{(2-C)^2}{2}$$

equating α_1 & α_2 & solving for C

$$0.05 = \frac{(2-C)^2}{2}$$

$\Rightarrow C = 1.68$

b) Power fun^n $\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \text{if } \theta < -0.5 \\ \theta + 0.95 & \text{if } -0.5 < \theta \le 0.95 \\ 1 & \text{if } 0.95 < \theta \end{cases}$

Using the distribution of $Y = X_1 + X_2$, given by

$$f_Y(y|\theta) = \begin{cases} y - 2\theta & \text{if } 2 \le y < 2\theta + 1 \\ 2\theta + 2 - y & \text{if } 2\theta + 1 \le y < 2\theta + 2 \\ 0 & \text{otherwise} \end{cases}$$

We obtain the power fun^n for the second test

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta + \frac{C}{2} > 1 \\ \frac{(2\theta + 2 - C)^2}{2} & \text{if } \frac{C}{2} - 1 < \theta < \frac{C-1}{2} \\ 1 - \frac{(C - 2\theta)^2}{2} & \text{if } \frac{C-1}{2} < \theta < \frac{C}{2} \\ 1 & \text{if } \frac{C}{2} < \theta \end{cases}$$

for θ slightly larger than 0, $\beta_1(\theta) > \beta_2(\theta)$, so that ϕ_1 is the one as that ϕ_1 is far powerful than ϕ_2, otherwise ϕ_2 is more powerful than ϕ_1.

c) Let $\phi_3 = \{X_1, X_2\}$ = reject H_0 if $X_1 + X_2 > C$ w.p. $X_1 > 1$ or $X_2 > 1$

$P(X_1 > 1 \mid H_0) = P(X_1 > 1 \mid X \sim Unif(0,1)) = 0$

$\alpha_3 = P(X_1 + X_2 > C$ or $X_1 > 1$ or $X_2 > 1 \mid H_0)$

$= P(X_1 + X_2 > C \mid H_0) \quad = \alpha_2$ (same as)

$\beta_3 = P(X_1 + X_2 > C$ or $X_1 > 1$ or $X_2 > 1 \mid H_1)$

$\ge P(X_1 + X_2 > C \mid H_1) = \beta_2$

$\Rightarrow \theta > 0$ i.e. slightly greater (not equal to zero)

$\therefore \beta_3 > \beta_2$

$\Rightarrow \phi_3$ is more powerful than ϕ_2

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4. $X_1, X_2, \ldots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

LRT $\quad H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$

Reject H_0 if $\sqrt{n}\,\frac{|\bar{X} - \theta_0|}{\sigma} > C$

a) Power fun^n:

$$\beta(\theta) = P_\theta\left(\frac{|\bar{X}_n - \theta_0|}{\sigma/\sqrt{n}} > C\right)$$

$$= P_\theta\left(\frac{\bar{X}_n - \theta_0}{\sigma/\sqrt{n}} > C\right) + P_\theta\left(\frac{\bar{X}_n - \theta_0}{\sigma/\sqrt{n}} < -C\right)$$

$$= 1 - P_\theta\left(\frac{\bar{X}_n - \theta_0}{\sigma/\sqrt{n}} \le C\right) + P_\theta\left(\frac{\bar{X}_n - \theta}{\sigma/\sqrt{n}} < -C - \frac{\theta - \theta_0}{\sigma/\sqrt{n}}\right)$$

$$= 1 - P_\theta\left(\frac{\bar{X}_n - \theta}{\sigma/\sqrt{n}} \le C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) + P_\theta\left(Z < -C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$= 1 - P_\theta\left(Z \le C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) + \Phi\left(-C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$= 1 - \Phi\left(C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) + \Phi\left(-C + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

Here $Z = \left(\frac{\bar{X}_n - \theta}{\sigma/\sqrt{n}}\right) \sim N(0,1)$ & $\Phi \Rightarrow$ std. normal CDF

b) Type-I error probability

$P(\text{Type-I error}) = P(X \in R \mid H_0)$

$= \beta(\theta_0)$

$= 1 - \Phi(C) + \Phi(-C)$

$= 1 - \Phi(C) + 1 - \Phi(C)$

$0.05 = 2 - 2\Phi(C)$

$0.975 = \Phi(C)$

$\Rightarrow \boxed{1.96 = C}$

Power $= 1 -$ Type II error probability

$P(\theta_1 + \sigma) = 1 -$ Type II error prob

Type II error prob $= 1 - P(\theta_1 + \sigma) \le 0.25$

$P(\theta_1 + \sigma) \ge 0.75$

$1 - \Phi(C - \sqrt{n}) + \Phi(-C - \sqrt{n}) \ge 0.75$

letting $C = 1.96$

$1 - \Phi(1.96 - \sqrt{n}) + \Phi(-1.96 - \sqrt{n}) \ge 0.75$

$-\Phi(-1.96 - \sqrt{n}) + \Phi(1.96 - \sqrt{n}) \le 0.25$

≈ 0

$\Rightarrow \quad \Phi(1.96 - \sqrt{n}) \le 0.25$

$1.96 - \sqrt{n} \le -0.67$

$\Rightarrow \quad \sqrt{n} \ge 2.63 \qquad \boxed{\therefore \; n \ge 6.91 \quad n = 7}$

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5) $f(x|\theta) = \frac{1}{\pi\left(1 + (x-\theta)^2\right)}$, $x \in R, \theta \in R$

$H_0 : \theta = 0 \quad vs \quad H_1 : \theta = 1$ {simple Hypothesis}

a) Power fun^n

$$\beta(\theta) = P(1 < X < 3 \mid \theta) = \int_1^3 \frac{1}{\pi\left(1 + (x-\theta)^2\right)} dx$$

Let $x - \theta = t$ then $x = t + \theta$, $dx = dt$, $t = x - \theta = \begin{matrix} 3-\theta \\ 1-\theta \end{matrix}$

$$\Rightarrow \beta(\theta) = \frac{1}{\pi} \int_{1-\theta}^{3-\theta} \frac{1}{1 + t^2} dt$$

$$= \frac{1}{\pi}\left(\arctan(3-\theta) - \arctan(1-\theta)\right)$$

We know that $\alpha = \beta(\theta) =$ Type I error

$$\Rightarrow \alpha = \frac{1}{\pi}\left(\arctan(3) - \arctan(1)\right)$$

b) Type II error $= 1 - \beta(\theta) = 1 - \beta(1)$

$$\beta(1) = \frac{1}{\pi}\left(\arctan(3-1) - \arctan(1-1)\right)$$

$$= \frac{1}{\pi}\left(\arctan(2) - \arctan(0)\right)$$

Type II error $= 1 - \frac{1}{\pi}\left(\arctan(2) - \arctan(0)\right)$

c) Using Neyman Pearson lemma, for a test to be UMP, we reject H_0 if $\frac{f(x|\theta=1)}{f(x|\theta=0)} > K$

Let $\phi'(x)$ is UMP then,

$$\phi'(x) = \begin{cases} 1 & \text{if } \frac{f(x|\theta=1)}{f(x|\theta=0)} > K \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\frac{f(x|\theta=1)}{f(x|\theta=0)} = \frac{\frac{1}{\pi\left(1 + (x-1)^2\right)}}{\frac{1}{\pi\left(1 + (x-0)^2\right)}}$

$$= \frac{1 + x^2}{1 + (x-1)^2} > K$$

Let us take $K = 2$, the $\phi(x)$ will be equivalent to $\phi'(x)$

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{f(x|\theta=1)}{f(x|\theta=0)} > 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & , \; 1 < x < 3 \\ 0 & , \; \text{otherwise} \end{cases}$$

$\therefore \phi(x)$ is the most powerful test among tests of size α.