HW-4

Tuesday, October 11, 2022 5:27 PM

1. Target distribution :

$$\pi(x) = \frac{1}{2} e^{-|x|} \quad , \ x \in \mathbb{R}$$

i) MH algorithm :

- Target distribution : $\pi(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$
- Consider Normal distribution as the proposal distribution centered at x_i & Sd $= 1$

$$q(x_{i+1} | x_i) \sim \mathcal{N}(x_i, 1)$$

- Initial $x_0 = 0$ (any value) & maintain all sampled values in a list values = [ ]
- Iterate for i in range$(1, n)$: ($n \to$ large say 50K)
  - proposed_x $\sim \mathcal{N}(x_{i-1}, 1)$
  - compute ratio A:

$$A = \frac{\text{target\_dist}^n(\text{proposed\_x})}{\text{target\_dist}^n(x_{i-1})}$$

    Here, target_dist^n := standard double exponential func^n

  - if $u < A$: [$u \sim$ Uniform$(0,1)$]

    values.append(proposed_x) ["accept" the move with prob. $\min(1, A)$]
  - else :

    values.append(x_{i-1}) ["reject" move, & stay where we are]

- Acceptance probability is $\min(1, A)$ where

$$A = \frac{\text{target\_distr}^n(\text{proposed\_x})}{\text{target\_dist}^n(x_{i-1})}$$

2. $\pi(x) = 1/6 \quad \forall \ x \in [1, 2, 3, 4, 5, 6]$

Transition probabilities :

$q(\hat{x} = 2 \mid x = 1) = 1$, $q(\hat{x} = 5 \mid x = 6) = 1$

$q(\hat{x} = 1 \mid x = 2) = 0.5$ $\quad$ $q(\hat{x} = 3 \mid x = 2) = 0.5$

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$q(\hat{x} = 4 \mid x = 5) = 0.5$ $\quad$ $q(\hat{x} = 6 \mid x = 5) = 0.5$

MH algo:

1) Initialize x_0 (say $x_0 = 2$)
2) Iterate for i in range $(1, n)$
   - sample

$$x_{new} \sim q(\cdot \mid x_{current})$$

   - compute ratio:

$$A = \frac{\cancel{\pi(x_{new})}^* \ q(x_{current} \mid x_{new})}{\cancel{\pi(x_{current})}^* \ q(x_{new} \mid x_{current})}$$

$\because \pi(x)$ for $x = [1, 2 \ldots 6] = 1/6$

if $u < A$: $\quad$ $u \sim$ Uniform$(0,1)$

accept x_{new} with acceptance probability $\min\{1, A\}$

else :

reject x_{new}

3.
$$f_i(x|\theta) = \begin{cases} e^{i\theta - x} & x \geq i\theta \\ 0 & x < \theta \end{cases}$$

$x_i = 1 \ldots n$

$$L = \prod_{i=1}^{n} f_i(x|\theta)$$

$$= \prod_{i=1}^{n} e^{(i\theta - x_i)} \cdot I_{(i\theta, \infty)}(x_i) \cdot \mathbb{I}(x_i \in N^+)$$

$$= \prod_{i=1}^{n} e^{(i\theta - x_i)} \cdot \mathbb{I}(i\theta \leq x < \infty) \cdot \mathbb{I}(x_i \in N^+)$$

$$= \underbrace{e^{\sum_i (i\theta - x_i)} \cdot \prod_{i=1}^{n} \mathbb{I}(i\theta \leq x_i < \infty)}_{g(T(x) | \theta)} \cdot \underbrace{\mathbb{I}(x_i \in N^+)}_{h(x)}$$

Here, $I(i\theta \leq x < \infty) = \begin{cases} 1 & , x_i \geq i\theta \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ By Neymann Factorization, we can see that $T = \min_{1 \leq i \leq n} (X_i | i)$ is a sufficient statistic for θ

4.
$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha - 1} e^{-\beta x}$$

$$L(f(x|\alpha, \beta)) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x_i^{\alpha - 1} \cdot e^{-\beta x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \prod_{i=1}^{n} x_i^{\alpha - 1} \cdot e^{-\beta x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^{n} x_i^{\alpha - 1}\right) \cdot e^{-\beta \sum_i x_i}$$

Let $h(x) = 1$, $\theta = \{\alpha, \beta\}$

Then, using the Neymann Factorization Theorem, joint sufficient statistic for (α, β) is

$$= \left(\prod_{i=1}^{n} x_i^{\alpha - 1}\right) \cdot e^{-\beta \sum_i x_i}$$

5. $X_1, X_2 \overset{iid}{\sim} \mathcal{N}(\theta, 1)$

$$P\{X_1, X_2\} = \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}(x_1 - \theta)^2\right] \cdot \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x_2 - \theta)^2\right]$$

To check if $T = X_1 + 2X_2$

$$P\{X_1 = 0, X_2 = 1 \mid X_1 + 2X_2 = 2\} = \frac{P\{X_1 = 0, X_2 = 1\}}{P\{X_1 + 2X_2 = 2\}}$$

$$= \frac{P\{X_1 = 0, X_2 = 1\}}{P\{X_1 = 0, X_2 = 1\} + P\{X_1 = 2, X_2 = 0\}}$$

$$= \frac{\cancel{\frac{1}{2\pi}} \exp\left\{\frac{-\theta^2}{2} - \frac{(1 - \theta^2 - 2\theta)}{2}\right\}}{\cancel{\frac{1}{2\pi}} \exp\left\{-\frac{\theta^2}{2} - \frac{(1 - \theta^2 - 2\theta)}{2}\right\} + \cancel{\frac{1}{2\pi}} \exp\left\{-\frac{(4 + \theta^2 - 4\theta)}{2} - \frac{\theta^2}{2}\right\}}$$

$$= \exp\{\theta^2 + 2\theta + 4\}$$

The final expression is not independent of θ.

$\therefore T = X_1 + 2X_2$ is not sufficient for θ.

Also,

$$\prod_{i=1}^{2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i - \theta)^2\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left[\sum_{i=1}^{2} -\frac{1}{2}(x_i - \theta)^2\right]$$

$$= \frac{1}{2\pi} \exp\left(-\frac{(X_1 - \theta)^2}{2} - \frac{(X_2 - \theta)^2}{2}\right)$$

$$= \frac{1}{2\pi} \exp\left[\frac{-(X_1^2 + X_2^2) - 2\theta^2 + 2\theta(X_1 + X_2)}{2}\right]$$

$$= \underbrace{\frac{1}{2\pi} \exp\left(\frac{-x_1^2 - x_2^2}{2}\right)}_{h(x)} \cdot \underbrace{\exp(-\theta^2) \cdot \exp(\theta(X_1 + X_2))}_{g(T(x)|\theta)}$$

Using Neymann Factorization theorem, we can see that $T = X_1 + X_2$ is a sufficient statistic for θ.

1(ii)
$$f_X(x) = \frac{1}{2} e^{-|x|}$$

$$\Rightarrow F_X(x) = \begin{cases} \frac{e^x}{2} & , x < 0 \\ 1 - \frac{e^{-x}}{2} & , x \geq 0 \end{cases}$$

Let $\frac{e^x}{2} = y \Rightarrow x = \log 2y$

$1 - \frac{e^{-x}}{2} = y \Rightarrow 2(1 - y) = e^{-x}$

$\Rightarrow x = -\log(2(1 - y))$

$$\Rightarrow F_X^{-1}(U) = \begin{cases} \log(2u) & U < 1/2 \\ -\log(2(1 - U)) & U \geq 1/2 \end{cases}$$