STAT 510 — Mathematical Statistics

Assignment: Problem Set 7 **Due Date:** December 6 2022, 11:59 PM

Problem 1.

Solution.

(a) $\phi(t) = \mathbb{E}[X_1 | \sum_{i=1}^n X_i = t] = \mathbb{E}[\sum_{i=1}^n X_i | \sum_{i=1}^n X_i = t]/n = \dfrac{t}{n}$

Moreover, since $\sum_{i=1}^n X_i$ is CSS for λ and $\mathbb{E}X_1 = \lambda$, by Rao-Blackwell theroem, $\mathbb{E}[X_1 | \sum_{i=1}^n X_i] = \bar{X}_n$ is the UMVUE for λ.

(b) $\mathbb{E}\tilde{\theta} = \mathbb{E}[\mathbf{1}(X_1 = 0)] = P(X_1 = 0) = e^{-\lambda}$ which is an unbiased estimator for θ

(c)

$$\begin{aligned}
\hat{\theta} &= \mathbb{E}[\mathbf{1}(X_1 = 0) | \sum_{i=1}^n X_i = t] \\
&= P(X_1 = 0 | \sum_{i=1}^n X_i = t) \\
&= \frac{P(X_1 = 0, \sum_{i=2}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} \\
&= \frac{e^{-\lambda} \dfrac{((n-1)\lambda)^t e^{-(n-1)\lambda}}{t!}}{\dfrac{(n\lambda)^t e^{-n\lambda}}{t!}} \\
&= (\frac{n-1}{n})^t
\end{aligned}$$

Since $\sum_{i=1}^n X_i$ is a sufficient statistic for λ, and by (b) we get $\tilde{\theta}$ is an unbiased estimator for θ. Thus we could conclude that $Var(\hat{\theta}) \leq Var(\tilde{\theta})$ by using the Rao-Blackwell theorem.

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Problem 2.

Solution.

(a)$\alpha = P(|X-1| > 3/4 | H_0) = P(X > 7/4 | H_0) + P(X < 1/4 | H_0) = \int_{7/4}^{\infty} e^{-x} dx + \int_0^{1/4} e^{-x} dx = e^{-7/4} - e^{-1/4} + 1$

(b)$\beta(\lambda) = P_\lambda(|X-1| > 3/4) = \int_{7/4}^{\infty} \lambda e^{-\lambda x} dx + \int_0^{1/4} \lambda e^{-\lambda x} dx = e^{-\frac{7}{4}\lambda} - e^{-\frac{1}{4}\lambda} + 1$

(c)let $g(\lambda) = e^{-\frac{7}{4}\lambda} - e^{-\frac{1}{4}\lambda} + 1$, then $0 = f'(\lambda) = -\frac{7}{4}e^{-\frac{7}{4}\lambda} + \frac{1}{4}e^{-\frac{1}{4}\lambda} \Longrightarrow \lambda = \frac{2}{3}log7$
Therefore $\beta(\frac{2}{3}log7) = 7^{-7/6} - 7^{-1/6} + 1$

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Problem 3.
Solution.
(a) The size of test ϕ_1 is

$$\alpha_1 = P(X_1 > 0.95|H_0) = P(X_1 > 0.95|X_1 \sim uniform(0,1)) = 0.05$$

and similarly, the size of test ϕ_2 is

$$\begin{aligned}
\alpha_2 &= P(X_1 + X_2 > C|H_0) \\
&= P(X_1 + X_2 > C|X_1, X_2 \overset{\text{iid}}{\sim} uniform(0,1)) \\
&= \begin{cases} \int_{C-1}^{1}\int_{C-\nu}^{1} dud\nu = \frac{(2-C)^2}{2} & C > 1 \\ \\ 1 - \int_0^C \int_0^{C-\nu} dud\nu = 1 - \frac{1}{2}C^2 > \frac{1}{2} & C \le 1 \end{cases}
\end{aligned}$$

To make $\alpha_1 = \alpha_2$, it suffices to consider the case $C > 1$, and C can be solved by $\frac{(2-C)^2}{2} = 0.05$, i.e., $C = 2 - \frac{1}{\sqrt{10}}$
(b) The power of test ϕ_1 is

$$\begin{aligned}
\beta_1 &= P(X_1 > 0.95|X_1 \sim uniform(\theta, \theta+1)) \\
&= \begin{cases} 0 & \theta \le 0.05 \\ \theta + 0.05 & -0.05 < \theta \le 0.95 \\ 1 & \theta > 0.95 \end{cases}
\end{aligned}$$

Let $W = X_1 + X_2$, then

$$\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} \mathbb{1}_{x\in(\theta,\theta+1)}\mathbb{1}_{w-x\in(\theta,\theta+1)}dx \\
&= \begin{cases} \int_{\theta}^{w-\theta} dx & 2\theta \le w < 2\theta + 1 \\ \int_{w-\theta-1}^{\theta+1} dx & 2\theta + 1 \le w < 2\theta + 2 \\ 0 & o.w. \end{cases} \\
&= \begin{cases} w - 2\theta & 2\theta \le w < 2\theta + 1 \\ 2\theta + 2 - w & 2\theta + 1 \le w < 2\theta + 2 \\ 0 & o.w. \end{cases}
\end{aligned}$$

Then (Also from (a), $C = 2 - \frac{1}{\sqrt{10}}$)

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\begin{aligned}
\beta_2 &= P(W > C | X_1, X_2 \overset{\text{iid}}{\sim} uniform(\theta, \theta+1)) \\
&= \int_C^{\infty} f_W(w)dw \\
&= \begin{cases} 0 & C > 2\theta + 2 \\ \frac{(2\theta+2-C)^2}{2} & 2\theta + 1 < C \leq 2\theta + 2 \\ 1 - \frac{(C-2\theta)^2}{2} & 2\theta < C \leq 2\theta + 1 \\ 1 & C \leq 2\theta \end{cases} \\
&= \begin{cases} 0 & \theta < \frac{C-2}{2} \\ \frac{(2\theta+2-C)^2}{2} & \frac{C-2}{2} \leq \theta < \frac{C-1}{2} \\ 1 - \frac{(C-2\theta)^2}{2} & \frac{C-1}{2} \leq \theta < \frac{C}{2} \\ 1 & \theta \geq \frac{C}{2} \end{cases}
\end{aligned}
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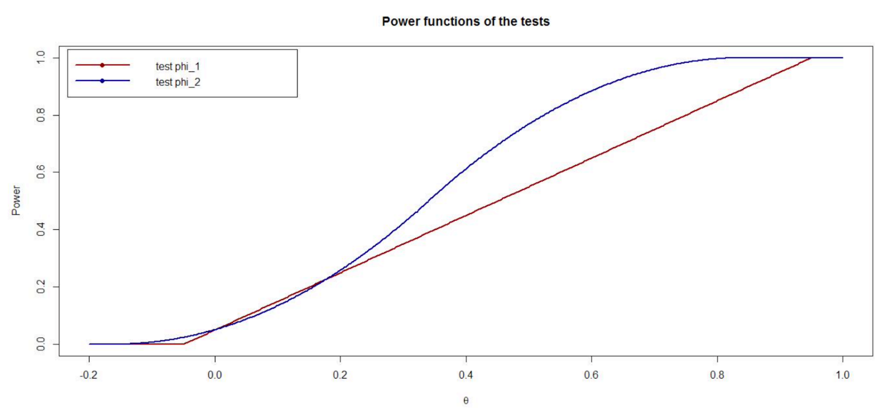

Figure 1.1

From the plot, we could easily see the regime where ϕ_2 is less power than ϕ_1.

(c) Consider the test

$$\phi_3(X_1, X_2) : Reject\ H_0\ if\ X_1 + X_2 > C,\ or\ X_1 > 1,\ or\ X_2 > 1$$

Note that $P(X_1 > 1|H_0) = P(X_1 > 1|X_1 \sim uniform(0,1)) = 0$, then the size of ϕ_3 is

$$\alpha_3 = P(X_1 + X_2 > C,\ or\ X_1 > 1,\ or\ X_2 > 1|H_0) = P(X_1 + X_2 > C|H_0) = \alpha_2$$

i.e., ϕ_3 has the same size as ϕ_2. Also note that

$$\beta_3 = P(X_1+X_2 > C, \textit{ or } X_1 > 1, \textit{ or } X_2 > 1|H_1) \geq P(X_1+X_2 > C|H_1) = \beta_2$$

and there exists $\theta > 0$ such that the inequality is strict, then we could conclude that ϕ_3 is more powerful than ϕ_2.

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Problem 4.
Solution.
(a) Let Z denote a standard normal r.v. and Φ denote the cdf of the standard normal distribution, then the power function is

$$\begin{aligned}\beta(\theta) &= P(\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}} > c|X_i \overset{\text{iid}}{\sim} N(\theta,\sigma^2))\\ &= P(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} > c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}}|\bar{X}\sim N(\theta,\sigma^2/n)+P(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} < -c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}}|\bar{X}\sim N(\theta,\sigma^2/n)\\ &= P(Z > c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}})+P(Z < -c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}})\\ &= 1-\Phi(c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}})+\Phi(-c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}})\end{aligned}$$

(b) It suffices to solve the following equations:

$$\begin{cases}\beta(\theta_0) = 0.05\\ \beta(\theta_0+\sigma) = 1-0.25\end{cases}$$

$$\Longrightarrow \begin{cases}1-\Phi(c)+\Phi(-c) = 0.05\\ 1-\Phi(c-\sqrt{n})+\Phi(-c-\sqrt{n}) = 0.75\end{cases}$$

$$\Longrightarrow \begin{cases}\Phi(-c) = 0.025\\ 1-\Phi(c-\sqrt{n})+\Phi(-c-\sqrt{n}) = 0.75\end{cases}$$

Thus we could get $c \approx 1.96$ and $n \approx 7$.

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Problem 5.
Solution. (a) Since $\beta(\theta) = P(1 < X < 3|\theta) = \int_1^3 \frac{1}{\pi(1+(x-\theta)^2)}dx = \frac{1}{\pi}\int_{1-\theta}^{3-\theta}\frac{1}{1+t^2}dt = \frac{1}{\pi}(arctan(3-\theta)-arctan(1-\theta))$ Then the type I error probability is

$$\beta(0) = \frac{1}{\pi}(arctan(3)-arctan(1))$$

(b) The type II error probability is

$$1-\beta(1) = 1-\frac{1}{\pi}(arctan(2)-arctan(0)) = 1-\frac{1}{\pi}arctan(2)$$

(c)By Neyman-Pearson Lemma, the UMP test rejects H_0 if $\dfrac{f(x|\theta=1)}{f(x|\theta=0)} > k$ for some constant $k > 0$.
Note that

$$\frac{f(x|\theta=1)}{f(x|\theta=0)} = \frac{1+x^2}{1+(x-1)^2}$$

let k=2, then the rejection region is

$$\frac{1+x^2}{1+(x-1)^2} > 2 \Longleftrightarrow 1 < x < 3$$

Then $\phi(x)$ is the UMP of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$

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