**STAT 510 Mathematical Statistics Fall 2022**

Problem set 6: Due on 11:59pm, Friday, 11/11/2022

1. Let $X_1, \ldots, X_n$ follow i.i.d. Gamma distribution with the probability density function

$$f(x \mid \beta) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp(-x/\beta) & x > 0 \\ 0 & x \leq 0 \end{cases},$$

where $\alpha > 0$ is known and $\beta > 0$ is the scale parameter to be estimated.

(i) Find the C-R lower bound for the variance of unbiased estimators of $\beta$.

(ii) Find an UMVUE of $\beta$.

2. Suppose that $X_1, \ldots, X_n$ follow i.i.d. Weibull distribution parametrized by the probability density function

$$f(x \mid \alpha) = \begin{cases} \alpha^{-1}\beta x^{\beta-1} \exp(-x^\beta/\alpha) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

where $\beta > 0$ is assumed to be known and $\alpha > 0$ is the unknown parameter. Find the C-R lower bound for the variance of unbiased estimators of (i) $\alpha$; (ii) $\alpha^2$, and (iii) $\alpha^{-1}$.

3. For each of the following distributions, let $X_1, \ldots, X_n$ be a random sample. Is there a function of $\theta$, say $\tau(\theta)$, for which there exists an unbiased estimator whose variance attains the C-R lower bound? If so, find it. If not, show why not.

(i) $f(x|\theta) = \theta x^{\theta-1}$ for $x \in (0, 1)$ and $\theta > 0$.

(ii) $f(x|\theta) = \frac{\log(\theta)}{\theta-1}\theta^x$ for $x \in (0, 1)$ and $\theta > 1$.