STAT 510 Mathematical Statistics Fall 2022

Problem set 7: Due on 11:59pm, Tuesday, 12/06/2022

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables following Poisson distribution with the rate parameter $\lambda > 0$. We have seen in class that the sample mean (also the maximum likelihood) estimator $\hat{\lambda} := \overline{X}_n$ achieves the C-R lower bound and thus it is the UMVUE. In this problem, we shall derive this fact and its variant with the Rao-Blackwell theorem.

   (a) Starting from the unbiased estimator X_1 for λ. Compute
   $$\phi(t) = \mathbb{E}[X_1 \mid \sum_{i=1}^{n} X_i = t]$$
   and show that it is the UMVUE for estimating λ. [Hint: use symmetry of $X_1, \ldots, X_n$.]

   Next we want to estimate $\theta = e^{-\lambda}$.

   (b) Show that $\tilde{\theta} = \mathbf{1}(X_1 = 0)$ is an unbiased estimator for θ.

   (c) Compute
   $$\hat{\theta} = \mathbb{E}[\mathbf{1}(X_1 = 0) \mid \sum_{i=1}^{n} X_i = t].$$
   Here, $\hat{\theta}$ should only be a function of t (and the sample size n). Use the Rao-Blackwell theorem to conclude $\mathrm{Var}(\hat{\theta}) \le \mathrm{Var}(\tilde{\theta})$.

2. Let $X \sim \text{Exponential}(\lambda)$ with the probability density function given by
   $$f(x \mid \lambda) = \lambda e^{-\lambda x}, \quad x \ge 0,\, \lambda > 0.$$
   For the hypothesis testing problem
   $$H_0 : \lambda = 1 \quad \text{and} \quad H_1 : \lambda \ne 1,$$
   consider the test that rejects H_0 if $|X - 1| > 3/4$.

   (a) Find the size α of this test.

   (b) Find the power function of this test as a function of λ.

   (c) Find the value of λ that minimizes the power. Find the power at this λ value.

3. Let X_1, X_2 be i.i.d. uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests: (i) $\phi_1(X_1)$ rejects H_0 if $X > 0.95$ and $\phi_2(X_1, X_2)$ reject H_0 if $X_1 + X_2 > C$.

   (a) Find the value of C such that ϕ_2 has the same size as ϕ_1.

   (b) Find the power function of each test. Find the regime in $\theta > 0$ where ϕ_2 is less powerful than ϕ_1.

   (c) Find a test that has the same size but is more powerful than ϕ_2.

4. Let $X_1, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$ distribution with known $\sigma^2 > 0$. A likelihood ratio test (LRT) of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is a test that rejects H_0 if $\sqrt{n}|\overline{X}_n - \theta_0|/\sigma > c$.

   (a) Find an expression in terms of standard Gaussian probabilities for the power function of this test.

   (b) One desires a type I error probability of 0.05 and a maximum type II error probability of 0.25 at $\theta = \theta_0 + \sigma$. Find values of (n, c) that will achieve this.

5. Let $f(x|\theta)$ be the Cauchy probability density function

$$f(x|\theta) = \frac{1}{\pi(1 + (x-\theta)^2)}, \qquad x \in \mathbb{R}, \theta \in \mathbb{R}.$$

   Show that the test function

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

   is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Find the type I and type II error probabilities.