STAT 510 Mathematical Statistics Fall 2022

Problem set 4: Due on 11:59pm, Wednesday, 10/12/2022

1. Derive a Metropolis-Hastings (MH) algorithm for sampling from the (standard) double exponential distribution with the probability density function given by

$$\pi(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

   (i) Write down your pseudocode of the MH algorithm (namely, what is your proposal and what is the acceptance probability). Implement the algorithm and access the convergence of your algorithm.

   (ii) *Inverse sampling method.* Denote F as the cumulative distribution function (cdf) of the double exponential density π. Let U be a uniform random variable in $[0, 1]$. Show that $X = F^{-1}(U)$ has the cdf as F. Use this fact to design an inverse sampler for generating double exponential random variables from the uniform random variables. Compare your inverse sample with that generated from MH algorithm.

2. Suppose we want to sample from rolling a fair 6-sided dice, i.e., $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = 1/6$. But we neither have a dice to roll or a computer to simulate the rolls. We only can flip fair coins. The rule we flip the coin is: go up if it is a head and go down if it is a tail. For example, if $X = 1$, we always propose $X = 2$ for the next state of the dice, while if $X = 2$, we have 50% chance to propose $X = 1$ and 50% chance to propose $X = 3$, and if $X = 6$, we always propose $X = 5$. This random experiment is essentially a Metropolis-Hastings algorithm to modify the simple random walk as flipping a fair coin to have the right probabilities for a 6-sided dice. Write down your pseudocode of the MH algorithm. Implement the algorithm and access the convergence of your algorithm.

3. Let $X_i, i = 1, \dots, n$ be independent variables with probability density functions

$$f_i(x|\theta) = \begin{cases} e^{i\theta - x} & \text{if } x \geq i\theta \\ 0 & \text{if } x < i\theta \end{cases}.$$

   Show that $T = \min_{1 \leq i \leq n}(X_i/i)$ is a sufficient statistic for θ.

4. Let $X_1, \ldots, X_n$ be a random sample from a Gamma(α, β) distribution, i.e., the probability density function is given by

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

   Find a sufficient statistic for (α, β).

5. Suppose that X_1, X_2 are iid $N(\theta, 1)$, where θ is unknown. Let $T = X_1 + 2X_2$ be a statistic. Is T is a sufficient statistic for θ? Why?