HW-6

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1) $X_1, X_2 \dots X_n \overset{iid}{\sim} \text{Gamma}(\alpha, 1/\beta)$

$$f(x|\beta) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \cdot \exp\left(-\frac{x}{\beta}\right) & x > 0 \\ 0 & x \le 0 \end{cases}$$

where α is known & β is unknown

ii) Likelihood : $L\, f(\beta|\underset{\sim}{x}) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left(\frac{-x_i}{\beta}\right)$

$$= \left(\frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta^{\alpha}}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \cdot \exp\left(-\frac{\sum_{i=1}^{n} x_i}{\beta}\right)$$

$$\log L = -n \log\Gamma(\alpha) - n\alpha \log\beta + (\alpha-1)\log\sum_i x_i - \sum_i \frac{x_i}{\beta}$$

I) $\frac{\partial \log L}{\partial \beta} = -n\frac{\alpha}{\beta} + \sum_i \frac{x_i}{\beta}$

II) $\frac{\partial \log L}{\partial \beta} = \underbrace{\frac{n}{\beta^2}}_{a(\beta)} \Big(\underbrace{\sum_i \frac{x_i}{n}}_{W(\tilde{X})} - \underbrace{\alpha\beta}_{\tau(\beta)} \Big)$

III) $E_\beta\left(\sum_i \frac{x_i}{n}\right) = \alpha\beta$ (To be verified)

$$E_\beta\left(\sum_i \frac{x_i}{n}\right) = \frac{1}{n}\mathbb{E}\left(\sum_i x_i\right) \qquad [\because x_i \text{ are iid}]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(x_i)$$

$$= \frac{1}{\cancel{n}}\cancel{n}\,\alpha\beta$$

Since $\mathbb{E}(W(\tilde{X})) = \alpha\beta = \tau(\beta)$

$\Rightarrow W(\tilde{X}) = \sum_{i=1}^{n}\frac{x_i}{n}$ is an unbiased estimator of β.

Since $W(\tilde{X})$ is an unbiased estimator of β, $\therefore$ $W(\tilde{X})$ is UMVUE.

i) From ii) we know that,

$\sum_{i=1}^{n}\frac{x_i}{n}$ is an unbiased estimate of β.

CRLB : $W(\tilde{X})$: Unbiased estimator of β

$\sum_i \frac{x_i}{n}$

$\mathbb{E}(W(\underset{\sim}{X})) = \alpha\beta = \tau(\beta)$

$$Var_\beta(W(\underset{\sim}{X})) \geq \frac{1}{n I_{x_1}(\beta)}$$

$$I_{x_1}(\beta) = \mathbb{E}_{x_1 \sim f}\left[\frac{\partial}{\partial\beta}\log f(x_1|\beta)\right]^2$$

$$= \mathbb{E}_{x_1 \sim f}\left[-\frac{\partial^2}{\partial\beta^2}\log f(x_1|\beta)\right]$$

$$f(x_1|\beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}} \cdot \exp\left(-\frac{x}{\beta}\right), \quad x>0$$

$$\log f(x_1|\beta) = (\alpha-1)\log x - \log\Gamma(\alpha) - \alpha\log\beta - \frac{x_1}{\beta}$$

$$\frac{\partial}{\partial\beta}\log f(x_1|\beta) = -\frac{\alpha}{\beta} + \frac{x_1}{\beta^2}$$

$$\frac{\partial^2}{\partial\beta^2}\log f(x_1|\beta) = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}$$

$$I_{x_1}(\beta) = \mathbb{E}\left[-\frac{\partial^2}{\partial\beta^2}\log f(x_1|\beta)\right]$$

$$= \mathbb{E}\left[-\frac{\alpha}{\beta^2} + \frac{2x}{\beta^3}\right]$$

$$= \int\left[-\frac{\alpha}{\beta^2} + \frac{2x}{\beta^3}\right] \cdot f(x_1|\beta)\,dx$$

$$= \int_0^\infty \left(\frac{2x}{\beta} - \alpha\right)\cdot\frac{1}{\beta^2}\cdot\frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot\exp\left(-\frac{x}{\beta}\right)dx$$

$$= 2\int_0^\infty \frac{x^{\alpha}\cdot\alpha}{\beta^3\cdot\alpha\cdot\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot\exp\left(\frac{-x}{\beta}\right)dx - \frac{\alpha}{\beta^2}\int_0^\infty\frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot\exp\left(\frac{-x}{\beta}\right)dx$$

$$= \frac{2\alpha}{\beta^2}\int_0^\infty\frac{x^{\alpha}}{\Gamma(\alpha+1)\cdot\beta^{\alpha+1}}\cdot\exp\left(\frac{-x}{\beta}\right)dx - \frac{\alpha}{\beta^2}\int_0^\infty\frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot\exp\left(-\frac{x}{\beta}\right)dx$$

$$= \frac{2\alpha}{\beta^2} - \frac{\alpha}{\beta^2}$$

$$= \frac{\alpha}{\beta^2}$$

$$Var_\beta(W(\hat{X})) \geq \frac{1}{n I_{x_1}(\beta)}$$

$$\geq \frac{\beta^2}{n\cdot\alpha}$$

$\therefore$ The Cramer-Rao lower bound for the variance of the unbiased estimator is $\frac{\beta^2}{n\cdot\alpha}$.

2) i) CRLB (α)

$$f(x_1|\alpha) = \frac{\beta x_1^{\beta-1}}{\alpha}\cdot\exp\left(\frac{-x_1^{\beta}}{\alpha}\right)$$

$$\log L = \log\left[\left(\frac{\beta}{\alpha}\right)\right] + \log\left(x_1^{\beta-1}\right) - \frac{x_1^{\beta}}{\alpha}$$

$$\frac{\partial}{\partial\alpha}\log L = -\frac{1}{\alpha} + \frac{x_1^{\beta}}{\alpha^2}$$

$$\frac{\partial^2 \log L}{\partial\alpha^2} = \frac{1}{\alpha^2} - 2\frac{x_1^{\beta}}{\alpha^3}$$

$$I_{x_1}(\alpha) = -\mathbb{E}\left[\frac{\partial^2}{\partial\alpha^2}\log L\right]$$

$$-\mathbb{E}\left[\frac{\partial^2}{\partial\alpha^2}\log L\right] = -\frac{1}{\alpha} + 2\frac{\mathbb{E}(x_1^{\beta})}{\alpha^3} \qquad -①$$

Let $z = x^{\beta}$

then $\frac{\partial z}{\partial x} = \beta x^{\beta-1}$, $\Rightarrow \left|\frac{dx}{dz}\right| = \frac{1}{\beta x^{\beta-1}}$

Using change of variables,

$$f(z) = \frac{\beta x^{\beta-1}}{\alpha}\exp\left(\frac{-x^{\beta}}{\alpha}\right)\cdot\frac{1}{\beta x^{\beta-1}}$$

$$f(z) = \frac{1}{\alpha}\cdot\exp\left(-\frac{z}{\alpha}\right)$$

$\Rightarrow x^{\beta} \sim$ exponential $(1/\alpha)$

$$\mathbb{E}(X^{\beta}) = \alpha$$

$$I_{x_1}(\alpha) = \frac{-1}{\alpha} + \frac{2\alpha}{\alpha^3} = \frac{1}{\alpha^2}$$

$$\text{CRLB:} = \frac{1}{n I_{x_1}(\alpha)} = \frac{\alpha^2}{n}$$

ii) CRLB (α^2)

$$\log L = \log\beta - \log(\alpha^2)^{1/2} + \log\left(x_1^{\beta-1}\right) - \frac{x_1^{\beta}}{(\alpha^2)^{1/2}}$$

$$\frac{\partial}{\partial(\alpha^2)}\log L = -\frac{1}{2\alpha^2} + \frac{1}{2}\frac{x_1^{\beta}}{(\alpha^2)^{3/2}}$$

$$\frac{\partial^2}{\partial(\alpha^2)^2}\log L = \frac{1}{2\alpha^4} - \frac{3}{4}\frac{x_1^{\beta}}{(\alpha^2)^{5/2}} \qquad -②$$

We know that, $\mathbb{E}[X_1^{\beta}] = \alpha$

$$\text{Eq ②}: = \frac{1}{2\alpha^4} - \frac{3}{4\alpha^4} = -\frac{1}{4\alpha^4}$$

$$I_{x_1}(\alpha^2) = -\mathbb{E}\left[-\frac{1}{4\alpha^4}\right] = \frac{1}{4\alpha^4}$$

$$\text{CRLB}: \frac{1}{n I_{x_1}(\alpha^2)} = \frac{4\alpha^4}{n}$$

iii) CRLB (α^{-1})

$$f(x|\alpha) = (\alpha^{-1})\beta x^{\beta-1}\exp\left(-x^{\beta}\alpha^{-1}\right)$$

$$\log L = \log\alpha^{-1} + \log\beta + \log x_1^{\beta-1} - x_1^{\beta}\cdot\alpha^{-1}$$

$$\frac{\partial}{\partial(\alpha^{-1})}L = -\frac{1}{\alpha^{-1}} - (\alpha^{-1})$$

$$\frac{\partial^2}{\partial^2(\alpha^{-1})}L = \frac{-1}{(\alpha^{-1})^2} = -\alpha^2, \quad I_{x_1}(\alpha^{-1}) = \alpha^2$$

$$\text{CRLB:} = \frac{1}{n I_{x_1}(\alpha^{-1})} = \frac{1}{n\alpha^2}$$

3) $f(x|\theta) = \theta x^{\theta-1}$ for $x \in (0,1)$ & $\theta > 0$

I) Define likelihood & calculate $\frac{\partial L}{\partial\theta}$

$$L(\theta|x) = \prod_{i=1}^{n} f(x|\theta) = \prod_{i=1}^{n}\theta x_i^{\theta-1} = \theta^n\prod_{i=1}^{n}x_i^{\theta-1}$$

$$\log L = n\log\theta + (\theta-1)\log\left(\prod_i x_i\right) = n\log\theta + \theta\log\left(\prod_i x_i\right) - \log\left(\prod_i x_i\right)$$

$$\frac{\partial}{\partial\theta}\log L = \frac{n}{\theta} + \log\left(\prod_i x_i\right) = \frac{n}{\theta} + \sum_i\log x_i \qquad —①$$

II) Factorize $\frac{\partial}{\partial\theta}\log L = a(\theta)[W(\underset{\sim}{X}) - \tau(\theta)]$

$$\frac{\partial}{\partial\theta}\log L = \left[\sum_i\log x_i + \frac{n}{\theta}\right]$$

$$= -n\left[-\sum_i\frac{\log x_i}{n} - \frac{1}{\theta}\right]$$

Here, $a(\theta) = -n$

$W(\underset{\sim}{X}) = -\sum_i\frac{\log x_i}{n}$; $\tau(\theta) = 1/\theta$

III) Check for unbiasedness of the estimator, i.e. $\mathbb{E}_\theta W(\underset{\sim}{X}) = \tau(\theta)$

$$\mathbb{E}_\theta W(\underset{\sim}{X}) = \mathbb{E}\left[-\frac{1}{n}\sum_i\log x_i\right] = \frac{-1}{n}\sum_i\mathbb{E}_\theta[\log x_i]$$

$$= \frac{-1}{n}\sum_i\left(-\frac{1}{\theta}\right) = \frac{-1}{\cancel{n}}\left(\frac{-\cancel{n}}{\theta}\right) = \frac{1}{\theta} = \tau(\theta)$$

Hence, it is proved that $-\sum_{i=1}^{n}\frac{\log x_i}{n}$ is the UMVUE of $1/\theta$ & attained the CRLB for $1/\theta$.

ii) $f(x|\theta) = \frac{\log\theta}{\theta-1}\theta^{x}$ for $x \in (0,1)$ & $\theta > 1$

I) Define likelihood & calculate $\frac{\partial}{\partial\theta}\log L$

$$L(\theta|x) = \prod_i f(x_i|\theta)$$

$$= \prod_i\frac{\log\theta}{(\theta-1)}\cdot\theta^{x_i} = \left(\frac{\log\theta}{\theta-1}\right)^n\prod_i\theta^{x_i}$$

$$\log L = n\log\left(\frac{\log\theta}{\theta-1}\right) + \left(\sum_i x_i\right)\log\theta$$

$$= n\log(\log\theta) - n\log(\theta-1) + \log\theta\left(\sum x_i\right)$$

$$\frac{\partial}{\partial\theta}\log L = \frac{n}{\log\theta}\cdot\frac{1}{\theta} - \frac{n}{\theta-1} + \frac{1}{\theta}\sum_i x_i = \frac{n}{\theta\log\theta} - \frac{n}{\theta-1} + \frac{n\bar{x}}{\theta}$$

$$= \frac{n}{\theta}\left[\frac{1}{\log\theta} - \frac{\theta}{(\theta-1)} + \bar{x}\right] \qquad \rightarrow ①$$

II) Factorize $\frac{\partial}{\partial\theta}\log L = a(\theta)[W(\tilde{X}) - \tau(\theta)]$

$$\frac{\partial}{\partial\theta}\log L = \underbrace{\frac{n}{\theta}}_{a(\theta)}\Big[\underbrace{\bar{x}}_{W(\tilde{X})} - \underbrace{\left(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right)}_{\tau(\theta)}\Big]$$

Thus, $\bar{X}$ is the UMVUE of $\tau(\theta) = \left(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right)$ & attains CRLB.

Since both i) & ii) probability densities belong to exponential family, the $\mathbb{E}\left[\frac{\partial}{\partial\theta}\log L(\theta|x)\right] = 0$ holds true for all exponential families,

thus, the estimator calculated are UMVUE