STAT 510 Mathematical Statistics Fall 2022

Problem set 1: Due on 11:59pm, Wednesday, 9/7/2022

1. Suppose $\underline{X}$ is a 3×1 random vector with $\text{cov}(\underline{X}) = \sigma^2 \mathbf{I}_3$. Find the matrix A so that
$$A\underline{X} = \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ X_3 - \bar{X} \end{pmatrix},$$
the vector of deviations. Here $\bar{X} = (X_1 + X_2 + X_3)/3$. The covariance matrix of the deviations is
$$Cov(A\underline{X}) = \sigma^2 B$$
What is the matrix B?

2. The double exponential random variable X has space $\mathbb{R}$ and pdf $f(x) = (1/2)e^{-|x|}$. (a) Show that the moment generating function (mgf) of a double exponential is $M_X(t) = 1/(1-t^2)$. [Break the integral into two parts according to the sign of x.] For which t is it finite?

   (b) Suppose U and V are independent Exponential(1), and let $Y = U - V$. Find the mgf of Y. What is the distribution of Y?

   (c) The mean of a double exponential is 0 and the variance is 2. Suppose $X_1, \ldots, X_n$ are independent and identically distributed double exponentials, and let W_n be the standardized mean, $W_n = \frac{\bar{X}}{\sqrt{2/n}}$. Show that the mgf of W_n is $M_n(t) = \frac{1}{(1-t^2/(2n))^n}$.

   (d) What is the limit of $M_n(t)$ as $n \to \infty$? What distribution corresponds to the limit of $M_n(t)$?

3. Suppose $Z \sim N(0,1)$ and $U \sim Uniform(0,1)$, and Z and U are independent. Let $\underline{Y} = (Y_1, Y_2) = g(Z, U)$ be given by
$$Y_1 = \frac{Z}{U} \text{ and } Y_2 = U.$$
Y_1 is said to have the "slash" distribution.

   (a) What is the space of $\underline{Y}$?

   (b) Find $g^{-1}(\underline{y})$.

(c) Find the pdf of $\underline{Y}$?

(d) Are Y_1 and Y_2 independent? Why or why not?

(e) Show that the marginal pdf of Y_1 is

$$f_1(y_1) = c\frac{1 - e^{-y_1^2/2}}{y_1^2}$$

What is the constant c?

4. Suppose $X \sim Beta(\alpha, \beta)$. Find (a) $\mathbb{E}(X(1-X))$; (b) $\mathbb{E}(X^a(1-X)^b)$ for nonnegative integers a and b.