STAT 510 — Mathematical Statistics

Assignment: Problem Set 6 **Due Date:** November 11 2022, 11:59 PM

Problem 1.
Solution.
(i)

$$
\begin{aligned}
& log f(x|\beta) = (\alpha-1)logx - log\Gamma(\alpha) - \alpha log\beta - x/\beta, \quad x>0 \\
\implies \quad & \frac{d}{d\beta} log f(x|\beta) = -\frac{\alpha}{\beta} + \frac{x}{\beta^2} \\
\implies \quad & \frac{d^2}{d\beta^2} log f(x|\beta) = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3} \\
\implies \quad & I_X(\beta) = -\mathbb{E}\left[\frac{d^2}{d\beta^2} log f(x|\beta)\right] = \frac{2\mathbb{E}[X]}{\beta^3} - \frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2}
\end{aligned}
$$

Then the C-R lower bound for the variance of unbiased estimators of $\tau(\beta) = \beta$ is $CR = \frac{[\tau'(\beta)]^2}{nI_X(\beta)} = \frac{\beta^2}{n\alpha}$

(ii) Since $X_1, X_2, \cdots, X_n \overset{\text{iid}}{\sim} f(x|\beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} exp(-x/\beta)$, then $T = \sum_{i=1}^n X_i \sim \Gamma(n\alpha, \beta)$ is the CSS for β. And we also have $\mathbb{E}[T] = n\alpha\beta$.
Then let

$$\phi(T) = \frac{T}{n\alpha} = \frac{\sum_{i=1}^n X_i}{n\alpha}$$

we have $\mathbb{E}[\phi(T)] = \beta$, and thus $\phi(T) = \frac{\sum_{i=1}^n X_i}{n\alpha}$ is the UMVUE of β. ∎

Problem 2.
Solution. Let $Y = X^\beta$, then we can obtain the pdf of Y, i.e.,

$$f_Y(y|\alpha) = f_X(y^{1/\beta}|\alpha)\frac{1}{\beta}y^{1/\beta-1} = \frac{1}{\alpha}exp(-\frac{y}{\alpha}) \sim Exp(\alpha)$$

and $\mathbb{E}[Y] = \mathbb{E}[X^\beta] = \alpha$.Then we can evaluate $I_X(\alpha)$.

$$
\begin{aligned}
& log f_X(x|\alpha) = -log\alpha + log\beta + (\beta - 1)log x - x^\beta/\alpha \\
\Longrightarrow \quad & \frac{d}{d\alpha} log f_X(x|\alpha) = -\frac{1}{\alpha} + \frac{x^\beta}{\alpha^2} \\
\Longrightarrow \quad & \frac{d^2}{d\alpha^2} log f_X(x|\alpha) = \frac{1}{\alpha^2} - \frac{2x^\beta}{\alpha^3} \\
\Longrightarrow \quad & I_X(\alpha) = -\mathbb{E}\left[\frac{d^2}{d\alpha^2} log f_X(x|\alpha)\right] = \frac{2}{\alpha^3}\mathbb{E}[X^\beta] - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}
\end{aligned}
$$

(i) The C-R lower bound for the variance of unbiased estimators of $\tau_1(\alpha) = \alpha$ is

$$CR_1 = \frac{[\tau_1'(\alpha)]^2}{nI_X(\alpha)} = \frac{\alpha^2}{n}$$

(ii) The C-R lower bound for the variance of unbiased estimators of $\tau_2(\alpha) = \alpha^2$ is

$$CR_2 = \frac{[\tau_2'(\alpha)]^2}{nI_X(\alpha)} = \frac{4\alpha^4}{n}$$

(iii) The C-R lower bound for the variance of unbiased estimators of $\tau_3(\alpha) = \alpha^{-1}$ is

$$CR_3 = \frac{[\tau_3'(\alpha)]^2}{nI_X(\alpha)} = \frac{1}{n\alpha^2}$$

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Problem 3.
Solution.
(i) Note that

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n (\prod_{i=1}^{n} x_i)^{\theta-1}, \quad 0 < x_i < 1, \theta > 0$$

then we have

$$log L(\theta|\underline{x}) = n log(\theta) + (\theta - 1)\sum_{i=1}^{n} log x_i$$

and consequently,

$$\frac{d}{d\theta} log L(\theta|\underline{x}) = n\left(\frac{1}{n}\sum_{i=1}^{n} log x_i - (-\frac{1}{\theta})\right) = a(\theta)[W(\underline{x}) - g(\theta)]$$

where $a(\theta) = n, W(\underline{X}) = \frac{1}{n}\sum_{i=1}^{n} log X_i$ and $g(\theta) = -\frac{1}{\theta}$

Now we verify that $W(\underline{X})$ is an unbiased estimator of $g(\theta)$

$$\begin{aligned}
\mathbb{E}[W(\underline{X})] &= \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} logX_i] \\
&= \mathbb{E}[logX_1] \\
&= \int_0^1 logx\ \theta x^{\theta-1}dx \\
&= x^{\theta}logx|_0^1 - \int_0^1 x^{\theta-1}dx \\
&= -\frac{1}{\theta}
\end{aligned}$$

Then we conclude that the variance of $W(\underline{X}) = \frac{1}{n}\sum_{i=1}^{n} logX_i$ attains the CR lower bound of $g(\theta) = -\frac{1}{\theta}$.

(ii) Note that

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \left(\frac{log\theta}{\theta-1}\right)^n \theta^{\sum_i x_i}, \quad 0 < x_i < 1, \theta > 1$$

then we have

$$logL(\theta|\underline{x}) = nloglog\theta - nlog(\theta-1) + (log\theta)\sum_{i=1}^{n} x_i$$

and consequently,

$$\frac{d}{d\theta}logL(\theta|\underline{x}) = \frac{n}{\theta}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{\theta}{\theta-1} + \frac{1}{log\theta}\right) = a(\theta)[W(\underline{x}) - g(\theta)]$$

where $a(\theta) = \frac{n}{\theta}, W(\underline{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $g(\theta) = \frac{\theta}{\theta-1} - \frac{1}{log\theta}$

Now we verify that $W(\underline{X})$ is an unbiased estimator of $g(\theta)$

$$
\begin{aligned}
\mathbb{E}[W(\underline{X})] &= \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} x_i] \\
&= \mathbb{E}[X_1] \\
&= \int_0^1 \frac{log\theta}{\theta - 1}\theta^x x dx \\
&= \frac{log\theta}{\theta - 1}\int_0^1 \theta^x x dx \\
&= \frac{log\theta}{\theta - 1}\int_1^\theta t\frac{logt}{log\theta} d\frac{logt}{log\theta} \qquad\qquad (here\ we\ let\ t = \theta^x) \\
&= \frac{1}{(log\theta)(\theta - 1)}\int_1^\theta logtdt \\
&= \frac{\theta}{\theta - 1} - \frac{1}{log\theta}
\end{aligned}
$$

Then we conclude that the variance of $W(\underline{X}) = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ attains the CR lower bound of $g(\theta) = \dfrac{\theta}{\theta - 1} - \dfrac{1}{log\theta}$.

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