STAT 510: Homework 02

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Due: Monday, February 7, 11:59 PM

Exercise 1 (Some Simple Expectations)

Let X denote the number of times Ron Weasley manages to irritate Professor Snape in one day. Then X has the following probability distribution:

x	$f(x)$
0	0.15
1	0.20
2	0.20
3	0.30
4	0.15

- Find the expected number of times Ron will get in trouble with Professor Snape, $\text{E}[X]$.
- Find the standard deviation of the number of times Ron will get in trouble with Professor Snape, $\text{SD}[X]$.
- Each day, Professor Snape takes 20 points from Gryffindor, simply because he can. Additionally, Professor Snape takes 10 points from Gryffindor each time Ron Weasley irritates him. If these are the only two sources of point deductions for Gryffindor, what is the expected point loss for Gryffindor each day?
- What is the standard deviation of points lost for Gryffindor each day?

Solution:

First, we add two useful columns to our table.

x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$
0	0.15	0	0
1	0.20	0.20	0.20
2	0.20	0.4	0.80
3	0.30	0.9	2.7
4	0.15	0.60	2.4

So, we have,

$$\sum_{x=0}^{4} f(x) = 1, \quad \sum_{x=0}^{4} x \cdot f(x) = 2.1, \quad \sum_{x=0}^{4} x^2 \cdot f(x) = 6.1$$

Thus,

$$\mathrm{E}[X] = \sum_{x=0}^{4} x \cdot f(x) = \boxed{2.1}$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \sum_{x=0}^{4} \big(x^2 \cdot f(x)\big) - (\mathrm{E}[X])^2 = 6.1 - 2.1^2 = 1.69$$

$$\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]} = \sqrt{1.69} = \boxed{1.3}$$

Define Y to be the number of points deducted from Gryffindor in a day.

Then, we have

$$Y = 10 \cdot X + 20$$

Thus,

$$\mathrm{E}[Y] = \mathrm{E}[10 \cdot X + 20] = 10 \cdot \mathrm{E}[X] + 20 = 10 \cdot 2.1 + 20 = \boxed{41}$$

Alternatively, we could recreate the table above for the distribution of our new random variable Y.

y	$f(y)$
20	0.15
30	0.20
40	0.20
50	0.30
60	0.15

Then we would simply repeat the process from above.

$$\mathrm{Var}[Y] = 10^2 \cdot \mathrm{Var}[X] = 100 \cdot 1.69 = 169$$

$$\mathrm{SD}[Y] = \sqrt{\mathrm{Var}[Y]} = \sqrt{169} = \boxed{13}$$

Exercise 2 (Expectation of a Maximum)

(**LW** 3.3) Let $X_1, \ldots, X_n \sim \mathrm{Uniform}(0, 1)$. Define $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[Y_n]$.

Solution

First, we find the CDF of Y_n.

$$F_{Yn}(y) = P(Y_n < y) = [P(X_i < y)]^n = y^n \quad 0 < y < 1.$$

Differentiating, we obtain the PDF of Y_n.

$$f_{Y_n}(y) = \frac{d}{dy} y^n = n(y)^{n-1} \quad 0 < y < 1.$$

Finally, we calculate the required expectation.

$$\mathbb{E}[Y_n] = \int_0^1 y \cdot n(y)^{n-1} dy = \boxed{\frac{n}{n+1}}.$$

Exercise 3 (A Random Walk)

(**LW** 3.4) A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1-p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n jumps. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Solution

First we define Y_i to be the jump at time i. We will use -1 to denote a movement left and 1 to denote a movement right.

$$Y_i = \begin{cases} -1 & \text{with probabilty} \;\; p \\ 1 & \text{with probabilty} \;\; (1-p) \end{cases}$$

We then calculate the mean and variance for each Y_i.

$$\mathbb{E}[Y_i] = 1 - 2p$$

$$\mathbb{E}[Y_i^2] = 1$$

$$\mathbb{V}[Y_i] = 4p(1-p)$$

Then, utilizing the independence of the Y_i, we find the requested expectations.

$$\mathbb{E}[X_n] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = \boxed{n(1-2p)}$$

$$\mathbb{V}[X_n] = \sum_{i=1}^{n} \mathbb{V}[Y_i] = \boxed{4np(1-p)}$$

Exercise 4 (A Lazy Statistican Does Algebra)

(**LW** 3.10) Let $X \sim \text{Normal}(0, 1)$. Define $Y = e^X$. Find $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$. Your answer should be a function of e, not a decimal representation.

Solution

$$\begin{aligned} \mathbb{E}[Y] &= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2-1}{2}} dx \\ &= e^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2-4}{2}} dx \\
&= e^2
\end{aligned}
$$

$$\mathbb{V}[Y] = \mathbb{E}\left[Y^2\right] - (\mathbb{E}[Y])^2 = \boxed{e^2 - e}$$

Note that as defined here, Y follows a log-normal distribution.

Exercise 5 (A Simple Hierarchical Model)

(**LW** 3.13) Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a Uniform$(0, 1)$ distribution. If the coin is tails, take X to have a Uniform$(3, 4)$ distribution. Find the mean and standard deviation of X.

Solution

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \mid \text{coin is heads}] \cdot \mathbb{P}(\text{coin is heads}) + \mathbb{E}[X \mid \text{coin is tails}] \cdot \mathbb{P}(\text{coin is tails}) \\
&= \frac{1}{2} \cdot \frac{1}{2} + \frac{7}{2} \cdot \frac{1}{2} \\
&= \boxed{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[X^2 \mid \text{coin is heads}\right] \mathbb{P}(\text{coin is heads}) + \mathbb{E}\left[X^2 \mid \text{coin is tails}\right] \mathbb{P}(\text{coin is tails}) \\
&= \frac{1}{3} \cdot \frac{1}{2} + \frac{37}{3} \cdot \frac{1}{2} \\
&= \frac{19}{3}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{SD}[X] &= \sqrt{\mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2} \\
&= \boxed{\sqrt{\frac{7}{3}}}
\end{aligned}
$$

Exercise 6 (Variance and Covariance)

(**LW** 3.15) Let

$$f_{X,Y}(x, y) = \frac{1}{3}(x + y), \quad 0 < x < 1,\ 0 < y < 2$$

Find $\mathbb{V}[2X - 3Y + 8]$.

Solution

$$\mathbb{E}[2X - 3Y + 8] = \int_0^1 \int_0^2 (2x - 3y + 8) \frac{1}{3}(x + y) dy dx = \frac{49}{9}$$

$$\mathbb{E}\left[(2X - 3Y + 8)^2\right] = \int_0^1 \int_0^2 (2x - 3y + 8)^2 \frac{1}{3}(x + y) dy dx = \frac{98}{3}$$

$$\mathbb{V}[2X - 3Y + 8] = \mathbb{E}\left[(2X - 3Y + 8)^2\right] - (\mathbb{E}[2X - 3Y + 8])^2 = \boxed{\frac{245}{81}}$$

Alternatively, you could find $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{V}[X]$, $\mathbb{V}[Y]$, and $\text{Cov}[X, Y]$ and apply the appropriate rules.

Exercise 7 (Checking Independence)

(**LW** 3.22) Let $X \sim \text{Uniform}(0, 1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

and let

$$Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are Y and Z independent? (Why or why not?) Find $\mathbb{E}[Y \mid Z]$.

Solution

We first show that Y and Z are **not** independent.

$$\mathbb{P}(Y = 1 \mid Z = 1) = \frac{\mathbb{P}(Y = 1, Z = 1)}{\mathbb{P}(Z = 1)} = \frac{\mathbb{P}(a < X < b)}{\mathbb{P}(a < X < 1)} = \frac{b - a}{1 - a}$$

$$\mathbb{P}(Y = 1) = \mathbb{P}(0 < X < b) = b$$

Thus we have

$$\mathbb{P}(Y = 1 \mid Z = 1) \neq \mathbb{P}(Y = 1).$$

We then move to finding $\mathbb{E}[Y \mid Z]$. We first note that Z takes only two possible values, 0 and 1.

$$\mathbb{E}[Y \mid Z = 1] = 1 \cdot \mathbb{P}(Y = 1 \mid Z = 1) + 0 \cdot \mathbb{P}(Y = 0 \mid Z = 1) = \frac{b - a}{1 - a}$$

$$\mathbb{E}[Y \mid Z = 0] = 1 \cdot \mathbb{P}(Y = 1 \mid Z = 0) + 0 \cdot \mathbb{P}(Y = 0 \mid Z = 0) = 1$$

Exercise 8 (Using Moment Generating Functions)

(**LW** 3.24) Let $X_1, \ldots, X_n \sim \text{Exp}(\beta)$. Find the moment generating function of X_i and use this to show that

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta).$$

Solution

We first note, using the parameterization in the textbook, the MGF of an $\text{Exp}(\beta)$ random variable is given by

$$\psi(t) = \frac{1}{1-\beta t}, \quad t < 1/\beta.$$

We also note that the MGF of a $\text{Gamma}(\alpha, \beta)$ random variable is given by

$$\psi(t) = \frac{1}{(1-\beta t)^{\alpha}}, \quad t < 1/\beta.$$

We then proceed by using the fact that the MGF of the sum of IID random variables is the product of their MGFs. So we have

$$\psi_{\sum X_i}(t) = \prod_{i=1}^{n} \psi_{X_i}(t) = \frac{1}{(1-\beta t)^{n}}, \quad t < 1/\beta.$$

We then conclude by inspection that this is the MGF of a $\text{Gamma}(n, \beta)$ random variable, and thus

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta).$$

Exercise 9 (The Classic Setup)

(**LW** 3.8) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$.

Prove that,

$$\mathbb{E}[\bar{X}_n] = \mu, \quad \mathbb{V}[\bar{X}_n] = \frac{\sigma^2}{n}, \quad \text{and} \quad \mathbb{E}[S_n^2] = \sigma^2.$$

Solution

$$\begin{aligned}
\mathbb{E}\left[\bar{X}_n\right] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] \\
&= \mu
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}\left[\bar{X}_n\right] &= \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}[X_i] \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[S_n^2\right] &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] \\
&= \frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n} X_i^2 - n(\bar{X}_n)^2\right] \quad \text{(via some algebra)} \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\right] - n\mathbb{E}\left[\bar{X}_n^2\right]\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(\mathbb{V}\left[X_i\right] + \left(\mathbb{E}\left[X_i\right]\right)^2\right) - n\left(\mathbb{V}[\bar{X}_n] + \left(\mathbb{E}[\bar{X}_n]\right)^2\right)\right) \\
&= \frac{1}{n-1}\left(n\left(\sigma^2 + \mu^2\right) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right) \\
&= \sigma^2
\end{aligned}
$$

Exercise 10 (Unintuitive Intuitions)

(Based on **LW** 3.9) Let $X_1, \ldots, X_n \sim \text{Normal}(0, 1)$ and $Y_1, \ldots, Y_n \sim \text{Cauchy}$. (In particular a Cauchy distribution with a location parameter of 0 and a scale parameter of 1.) Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Generate a (single) random sample of size $n = 10,000$ for both distributions and plot $\bar{x}_n$ and $\bar{y}_n$ versus n for $n = 1, \ldots, 10,000$ on a single plot. (Use a different color for each distribution.) Repeat this process three times. Display the plots in a 1×3 grid. Explain why there is such a difference between the two distributions.

Solution

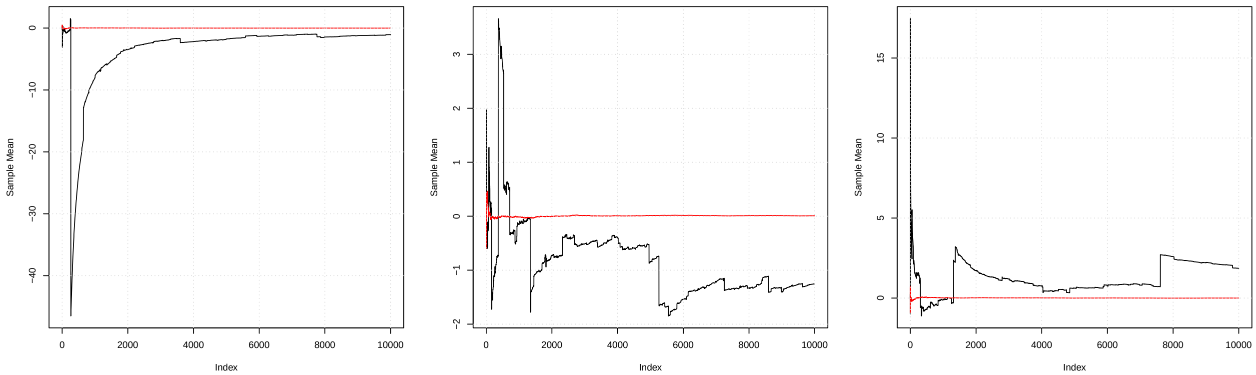
```r
# define parameters of simulation
sample_size = 10000

# set seed
set.seed(42)
```

```r
# helper function
sim_and_plot = function(sample_size) {
  plot(cumsum(rcauchy(n = sample_size)) / 1:sample_size,
       type = "l",
       ylab = "Sample Mean")
  lines(cumsum(rnorm(n = sample_size)) / 1:sample_size,
        type = "l",
        col = "red")
  grid()
}
```

```r
# setup plot structure
par(mfrow = c(1, 3))

# plot results
sim_and_plot(sample_size)
sim_and_plot(sample_size)
sim_and_plot(sample_size)
```



Our intuition (and the WLLN which we will see later) suggests that the sample mean should converge to the true mean. We see this happening with the normal distribution. However, the mean of a Cauchy distribution does not exist, so these plots never "settle down" as there are sporadic observations that are very far from 0.

Exercise 11 (Simulating a Stock Market)

(Based on **LW** 2.11) Let $Y_1, Y_2, \ldots$ be independent random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = 0.5.$$

Define

$$X_n = \sum_{i=1}^{n} Y_i.$$

Think of $Y_i = 1$ as "the stock price increased by one dollar," $Y_i = -1$ as "the stock price decreased by one dollar," and X_n as the value of the stock on day n. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Simulate X_n and plot X_n versus n for $n = 10,000$. Repeat this process three times. (So, simulate the price of three stocks for $10,000$ days.) Use a single plot with a different color for each stock. Notice two things. First, it is easy to "see" patterns in the sequence even though it is random. Second, you will find that the three runs look very different even though they were generated the same way. How do the expectations you found explain this observation? (Also, consider repeating this process more times than needed to simulate three stocks for more intuition.)

Note: This is an incredibly simplistic model of a market. Please do not make any decisions in the real world based on this model.

Solution

First, we plug $p = 0.5$ into the results from Exercise 3.

$$\mathbb{E}[X_n] = 0$$

$$\mathbb{V}[X_n] = n$$

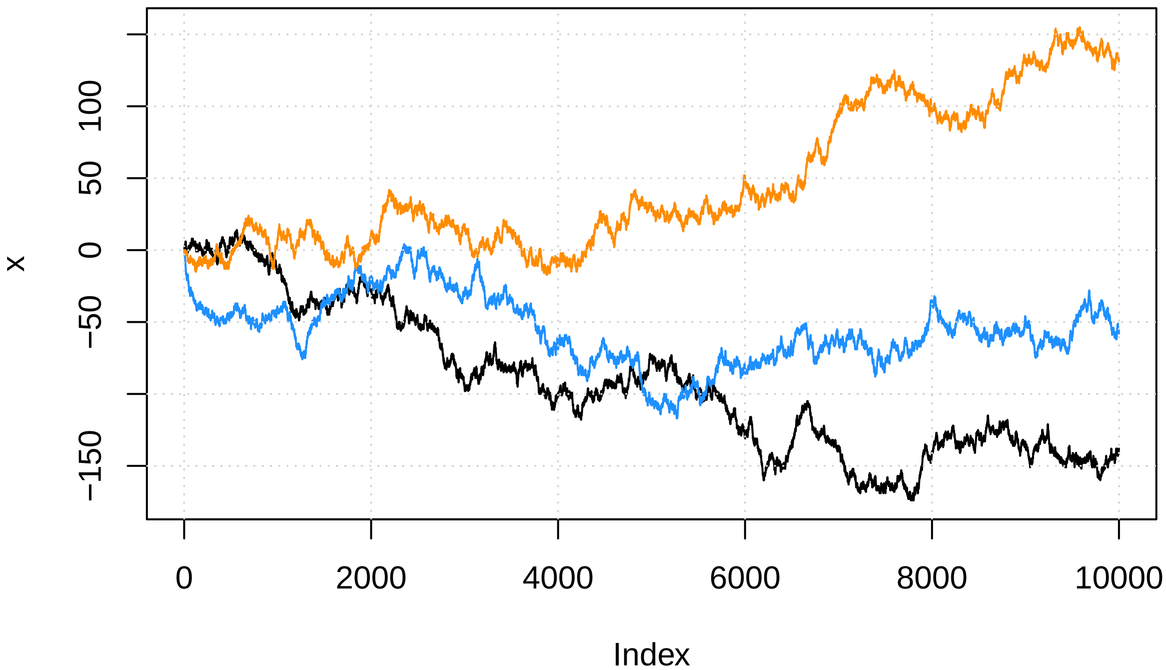
```
# define parameters of simulation
sample_size = 10000

# set seed
set.seed(11)
```

```r
# simulate stocks
x = cumsum(sample(c(-1, 1), replace = TRUE, size = sample_size))
y = cumsum(sample(c(-1, 1), replace = TRUE, size = sample_size))
z = cumsum(sample(c(-1, 1), replace = TRUE, size = sample_size))
```

```r
# setup plot structure
par(mfrow = c(1, 1))

# plot results
plot(x, type = "l", ylim = range(c(x, y, z)))
lines(y, type = "l", col = "dodgerblue")
lines(z, type = "l", col = "darkorange")
grid()
```



Here we've done a bit of seed hunting to find a representative seed, but to truly grasp this result, more "stocks" should be simulated. First we see that while one stock is a "winner" and two are "losers" at the 10,000th step the average gains here are closer to zero. We also see that as time goes on, it's more likely to see a individual stock deviate from zero, which matches the variance result which is dependent on n.