**STAT 510 Mathematical Statistics Fall 2022**

Problem set 5: Due on 11:59pm, Monday, 10/24/2022

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the inverse Gaussian distribution whose pdf is given by

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], \quad 0 < x < \infty.$$

   Find a sufficient statistic for $(\mu, \lambda)$.

2. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$, where both $\mu$ and $\sigma^2$ are unknown parameters. Let $s_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$ be the sample variance. Show that the Fisher information matrix based on $s_n^2$ is

$$I_{s_n^2}(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(n-1)\sigma^{-4} \end{pmatrix}$$

   where $\theta = (\mu, \sigma^2)$.

3. Let $X_1, X_2, X_3$ be iid Bernoulli(p) where $0 < p < 1$ is the unknown parameter. Evaluate Fisher information $I_{\mathbf{X}}(p)$, $I_{\overline{X}}(p)$ and $I_T(p)$, where $\mathbf{X} = (X_1, X_2, X_3)$, $\bar{X} = (X_1 + X_2 + X_3)/3$ and $T = X_1 + X_2$. Is $\overline{X}$ sufficient for $p$? Is $T$ sufficient for $p$?

4. Let $X$ be a random variable with probability density function $f(x|\theta)$ where $\theta \in \mathbb{R}^K$ is the parameter vector. Let $s_\theta(x) = \nabla_\theta \log f(x|\theta)$ be the score function. Show that $\mathbb{E}[s_\theta(X)] = 0$ and the Fisher information matrix $I_X(\theta) := \mathbb{E}[s_\theta(X)s_\theta(X)^T]$ can be expressed as

$$I_X(\theta) = -\mathbb{E}[\nabla^2 \log f(X|\theta)].$$

5. (a) Show that gamma distribution with either parameter $\alpha$ or $\beta$ known, or both unknown, belongs to the exponential family.

   (b) Show that Poisson distribution belongs to the exponential family.

   (c) Does binomial distribution $b(n,p)$ with unknown $n$ belong to the exponential family? Why?

6. For each of the following families (a) and (b),

   (a) $N(\theta, a\theta^2)$, where $a$ is known,

   (b) $f(x|\theta) = C\exp(-(x-\theta)^4)$, where $C$ is a normalization constant,

   do the following:

   (i) verify that it is an exponential family.

   (ii) describe the curve on which the parameter $\theta$ lies.

   (iii) sketch a graph of the curved parameter space.