Fall 2022 STAT 510 — Mathematical Statistics

Assignment: Problem Set 1 **Due Date:** September 7th 2022, 11:59pm

Problem 1.
Solution. Note that $X_1 - \bar{X} = \frac{2}{3}X_1 - \frac{1}{3}X_2 - \frac{1}{3}X_3$ and similarly for $X_2 - \bar{X}$ and $X_3 - \bar{X}$. Thus

$$A = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

Moreover, since $\sigma^2 B = Cov(A\underline{X}) = ACov(\underline{X})A^T = \sigma^2 AA^T = \sigma^2 A$, we could get $B = A$. ■

Problem 2.
Solution. (a) Note first

$$M_X(t) = E[e^{tX}] = \frac{1}{2}\int_{-\infty}^{+\infty} e^{tx-|x|}dx = \frac{1}{2}\int_{-\infty}^{0} e^{tx+x}dx + \frac{1}{2}\int_{0}^{+\infty} e^{tx-x}dx$$

The first integral is

$$\frac{1}{2}\int_{-\infty}^{0} e^{tx+x}dx = \frac{1}{2(1+t)}e^{x(1+t)}\Big|_{-\infty}^{0} = \frac{1}{2(1+t)},\ for\ 1+t>0$$

The second integral is

$$\frac{1}{2}\int_{0}^{+\infty} e^{tx-x}dx = \frac{1}{2(1-t)}e^{-x(1-t)}\Big|_{+\infty}^{0} = \frac{1}{2(1-t)},\ for\ 1-t>0$$

So the mgf is finite if both $1+t>0$ and $1-t>0$ that is $-1<t<1$. If so, then

$$M_X(t) = \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{1}{1-t^2}.$$

(b) The mgf of Exponential(1) is $\frac{1}{1-t}$ for $t<1$. Since U and V are independent Exponential(1), we have

$$M_Y(t) = E[e^{tY}] = E[e^{t(U-V)}] = E[e^{tU}]E[e^{-tV}] = \frac{1}{1-t}\cdot\frac{1}{1+t} = \frac{1}{1-t^2},$$

when $-1 < t < 1$. Thus Y is double exponential.
(c)

$$\begin{aligned}
M_n(t) &= E[e^{t\frac{\bar{X}}{\sqrt{2/n}}}] \\
&= E[e^{\frac{t}{\sqrt{2n}}(X_1+\cdots+X_n)}] \\
&= E[e^{\frac{t}{\sqrt{2n}}X_1}]\cdots E[e^{\frac{t}{\sqrt{2n}}X_n}] \\
&= M_{X_1}(\frac{t}{\sqrt{2n}})\cdots M_{X_n}(\frac{t}{\sqrt{2n}}) \\
&= \left(M_{X_1}(\frac{t}{\sqrt{2n}})\right)^n \\
&= \left(\frac{1}{1-t^2/(2n)}\right)^n \\
&= \frac{1}{(1-t^2/(2n))^n}.
\end{aligned}$$

(d) Since the limit of $(1+\frac{z}{n})^n$ is e^z as $n \to \infty$, then,

$$M_n(t) = \frac{1}{(1-t^2/(2n))^n} \to \frac{1}{e^{-t^2/2}} = e^{t^2/2}$$

which is the mgf of $N(0,1)$. ■

Problem 3.
Solution. (a) $\mathbb{R} \times [0,1]$.
(b) Since $z = uy_1 = y_1y_2$, then $g^{-1}(\underline{y}) = (y_1y_2, y_2)$.
(c)

$$J_{g^{-1}}(y) = \begin{vmatrix} \frac{\partial y_1y_2}{\partial y_1} & \frac{\partial y_1y_2}{\partial y_2} \\ \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

Since Z and U are independent, the pdf of (Z,U) is

$$f_{Z,U}(z,u) = \frac{1}{\sqrt{(2\pi)}}e^{-z^2/2},$$

for $z \in \mathbb{R}$ and $u \in [0,1]$. Then

$$f_{\underline{Y}}(y_1,y_2) = f_{Z,U}(g^{-1}(y_1,y_2))|J_{g^{-1}}(y)| = \frac{1}{\sqrt{2\pi}}e^{-\frac{y_1^2y_2^2}{2}}y_2$$

(d) Since we cannot factorize $f_{\underline{Y}}(y_1,y_2)$ (the joint pdf of Y_1 and Y_2) to $h_1(y_1)h_2(y_2)$, where $h_1(y_1)$ and $h_2(y_2)$ are just functions of y_1 and y_2 respectively. Thus they are not independent.

(e)

$$f_{Y_1}(y_1) = \int_0^1 f_{\underline{Y}}(y_1, y_2) dy_2 = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{y_1^2 y_2^2}{2}} y_2 dy_2 = \frac{-\frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2 y_2^2}{2}}}{y_1^2} \Bigg|_0^1 = \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-y_1^2/2}}{y_1^2}$$

which means $c = \frac{1}{\sqrt{2\pi}}$. ■

Problem 4.
Solution. (a)

$$\begin{aligned}
E[X(1-X)] &= \int_0^1 x(1-x) f(x) dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{(\beta+1)-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \\
&= \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}
\end{aligned}$$

(b)

$$\begin{aligned}
E[X^a(1-X)^b] &= \int_0^1 x^a(1-x)^b f(x) dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+a)-1}(1-x)^{(\beta+b)-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+a)\Gamma(\beta+b)}{\Gamma(\alpha+\beta+a+b)}
\end{aligned}$$

■