Question 01 (Exponential Likelihood)

Consider $X_1, \dots, X_n \sim \text{Exp}(\lambda)$. That is,

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0.$$

(a) [2 points] Show that the MLE for λ is $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$.

(b) [2 points] Find $I_n(\lambda)$ and $\text{se}\left(\hat{\lambda}\right)$.

(c) [1 point] Give an approximate 95% confidence interval for λ.

(d) [3 points] Find the MLE and an approximate 95% confidence interval for $2 \log \lambda$.

(e) [3 points] Suppose that in addition to the exponential likelihood above, we use a Gamma(α, β) prior. Derive the posterior distribution for λ. (Note that Gamma is a conjugate prior in this case. You simply need to show which Gamma distribution.) Use the following density for a Gamma(α, β) distribution:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

Note that with this parameterization, the mean is α/β. (This fact might be useful later.)

(a) $f(x) = f(x_1) \cdots f(x_n) = \lambda^n e^{-\lambda \sum x_i}$. $\ell(\lambda) = n \log \lambda - \lambda \sum x_i$.

$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i$. $\frac{\partial \ell}{\partial \lambda}(\hat{\lambda}) = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i}$.

(b) $\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2}$. $\Rightarrow I(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda^2}\right] = \frac{n}{\lambda^2}$

$\text{se}(\hat{\lambda}) = \sqrt{\frac{1}{I(\hat{\lambda})}} = \frac{\hat{\lambda}}{\sqrt{n}}$

(c) $\hat{\lambda} \pm 1.96\, \text{se}(\hat{\lambda}) = \hat{\lambda} \pm 1.96 \frac{\hat{\lambda}}{\sqrt{n}} = \left(1 \pm \frac{1.96}{\sqrt{n}}\right) \cdot \frac{n}{\sum x_i}$

(d) Consistency $\Rightarrow (2 \log \lambda)_{MLE} = 2 \log \hat{\lambda}_{MLE} = 2 \log \frac{n}{\sum x_i}$.

$g(\lambda) := 2 \log \lambda$. $g'(\lambda) = \frac{2}{\lambda}$.

$\frac{\sqrt{n}(\lambda - \hat{\lambda})}{\hat{\lambda}} \xrightarrow{d} N(0,1)$. $\frac{\sqrt{n}(g(\lambda) - g(\hat{\lambda}))}{\hat{\lambda}} \xrightarrow{d} N\left(0, \frac{4}{\lambda^2}\right)$ (δ-method).

i.e. $\sqrt{n}(g(\lambda) - g(\hat{\lambda})) \xrightarrow{d} N(0, 4) \Rightarrow 2 \log \frac{n}{\sum x_i} \pm \frac{4}{n} \cdot 1.96$

(e) $$p(\lambda \mid \underset{\sim}{x}) = \frac{p(\underset{\sim}{x} \mid \lambda) \cdot m(\lambda)}{p(\underset{\sim}{x})}$$

$$\propto \lambda^{n} e^{-\lambda \sum x_i} \cdot \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \lambda^{n+\alpha-1} e^{-\lambda(\beta + \sum x_i)}$$

So $\lambda \mid \underset{\sim}{x} \sim \underline{\text{Gamma}(n+\alpha,\ \beta + \sum x_i)}$

Question 02 (Geometric Likelihood)

Consider $X_1, \ldots, X_n \sim \text{Geometric}(p)$. That is,

$$f(x) = (1-p)^{x-1} \cdot p, \quad x = 1, 2, 3, \ldots, \quad 0 < p < 1.$$

(a) [2 points] Show that the MLE for p is $\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i}$.

(b) [2 points] Find $I_n(p)$ and $\text{se}(\hat{p})$.

(c) [2 points] Find the test statistic for the Wald test of

$$H_0 : p = 0.5 \quad \text{versus} \quad H_1 : p \neq 0.5.$$

(d) [3 points] Perform a simulation study to estimate the power function, $\beta(p)$ for $0 < p < 1$ when $\alpha = 0.05$. Use $n = 50$. Use a "reasonably" large number of simulations for each value of p. (That is, use enough values of p and enough simulations for each such that the resulting curve is reasonably smooth.) Plot the resulting curve.

(a) $$\mathcal{L}(\underset{\sim}{x} \mid p) = \prod_{i=1}^{n} f(x_i) = (1-p)^{\sum x_i - n} \cdot p^{n}$$

$$\ell(p) = \log \mathcal{L}(\underset{\sim}{x}) = \left(\sum x_i - n\right) \log(1-p) + n \log p$$

$$\frac{\partial \ell}{\partial p} = \frac{n - \sum x_i}{1-p} + \frac{n}{p} = \frac{\cancel{pn} - p\sum x_i + n - \cancel{np}}{p(1-p)} = \frac{n - p\sum x_i}{p(1-p)}$$

$$\frac{\partial \ell}{\partial p} = 0 \Rightarrow \hat{p} = \frac{n}{\sum x_i}.$$

(b) $$\frac{\partial^2 l}{\partial p^2} = \frac{\partial}{\partial p}\left(\frac{n-\sum x_i}{1-p} + \frac{n}{p}\right)$$

$$= \frac{n-\sum x_i}{(1-p)^2} - \frac{n}{p^2}.$$

$$I(p) = -E\left[\frac{\partial^2 l}{\partial p^2}\right] = -\left(\frac{n-nEX_i}{(1-p)^2} - \frac{n}{p^2}\right).$$

$$EX = \sum_{k=1}^{\infty}(1-p)^{k-1}p\cdot k = p\sum_{k=1}^{\infty}(1-p)^{k-1}k$$

$$= p\cdot\left(-\frac{d}{dp}\sum_{k=1}^{\infty}(1-p)^k\right)$$

$$= p\cdot\left(-\frac{d}{dp}\left(\frac{1-p}{p}\right)\right)$$

$$= \cancel{p}\cdot\left(-\frac{-p-(1-p)}{p^{\cancel{2}}}\right) = \frac{1}{p}.$$

$$I(p) = +\left(\frac{-n+\frac{n}{p}}{(1-p)^2} + \frac{n}{p^2}\right)$$

$$= \frac{np}{p^2(1-p)} + \frac{n(1-p)}{p^2(1-p)}$$

$$= \frac{n}{p^2(1-p)}$$

$$s.e.(\hat{p}) = \sqrt{\frac{1}{I(\hat{p})}} = \frac{p\sqrt{1-p}}{\sqrt{n}}.$$

(c) $W = \frac{p_0 - \hat{p}}{s.e.(\hat{p})} = \frac{\frac{1}{2} - \frac{n}{\sum x_i}}{\hat{p}\sqrt{1-\hat{p}}/\sqrt{n}} = \frac{\sqrt{n}\left(1 - \frac{2n}{\sum x_i}\right)}{2\hat{p}\sqrt{1-\hat{p}}} = \boxed{\frac{\sqrt{n}\left(1 - \frac{2n}{\sum x_i}\right)}{\frac{2n}{\sum x_i}\sqrt{1-\frac{n}{\sum x_i}}}}$.

(d)

Question 03 (Poisson Likelihood)

Consider $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. That is,

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 1, 2, 3, \ldots, \qquad \lambda > 0.$$

(a) [2 points] Show that the MLE for λ is $\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$.

(b) [2 points] Find $I_n(\lambda)$ and $\text{se}\left(\hat{\lambda}\right)$.

(c) [3 points] Recall that if we add a Gamma(α, β) prior for λ, the posterior distribution is given by

$$\lambda \mid X_1, \ldots X_n \sim \text{Gamma}\left(\alpha + \sum_{i=1}^{n} x_i,\ \beta + n\right).$$

Note that we are assuming the same parameterization as Question 01. Also note that the second parameter here is considered the "rate" parameter.

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set.seed(42)
some_data = rpois(n = 25, lambda = 4)
```

Using the above data and a Gamma$(\alpha = 2, \beta = 5)$ prior, calculate three point estimates for λ.

- Using only the prior.
- Using only the data.
- Using both the prior and the data, that is, using the posterior.

(d) [3 points] Using the same data, prior, and scenarios as the previous part, calculate three 95% interval estimates.

(a) $\mathcal{L}(\tilde{x} \mid \lambda) \propto \lambda^{\sum x_i} e^{-n\lambda}$. $\quad \ell(\lambda) = \sum x_i \log \lambda - n\lambda + \text{const}.$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\sum x_i}{\lambda} - n = 0 \Rightarrow \boxed{\hat{\lambda} = \frac{\sum x_i}{n}}.$$

(b) $$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{\sum x_i}{\lambda^2}. \qquad I(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda^2}\right] = \frac{\sum \mathbb{E} X_i}{\lambda^2} = \boxed{\frac{n}{\lambda}}.$$

$$se(\hat{\lambda}) = \sqrt{\text{Var}\left(\frac{\sum x_i}{n}\right)} = \boxed{\sqrt{\frac{\lambda}{n}}}.$$

(c) L^2-loss $\Rightarrow$ Bayesian estimator is posterior mean

prior $\Rightarrow \frac{2}{5}$.

data $\Rightarrow \bar{X}_n$

posterior $\Rightarrow \frac{2 + \sum x_i}{5 + n}$.

Question 04 (Likelihood Ratio Test)

Assume

$$X_1, \ldots, X_{n_x} \sim \text{Exp}(\lambda_x) \quad \text{and} \quad Y_1, \ldots, Y_{n_y} \sim \text{Poisson}(\lambda_y).$$

Then, consider testing

$$H_0 : \lambda_x = \lambda_y \quad \text{versus} \quad \lambda_x \neq \lambda_y.$$

(a) [3 points] Find the MLE under the null. That is, find the MLE of λ assuming that $\lambda = \lambda_x = \lambda_y$.

(b) [2 points] Derive the LRT statistic for the above test. You do not need to do any algebraic simplifications.

(c) [1 point] State the asymptotic distribution of your statistic in the previous part.

(d) [3 points] Perform a simulation study to verify that this is a level α test for any α. To do so, repeatedly do the following:

- Generate data according to the data generating process under the null. For this question use $\lambda_x = \lambda_y = 2$.
- Calculate the test statistic for the generated sample.
- Calculate and store the (large sample) p-value.

Plot a histogram of these p-values. If done correctly, this histogram should indicate that the p-values are roughly uniform.

(a) $L(\underline{X}, \underline{Y} \mid \lambda_x, \lambda_y) = \lambda_x^{n_x} e^{-\lambda_x \sum X_i} \cdot \lambda_y^{n_y} e^{-\lambda_y \sum Y_i}$

$\ell(\lambda_x, \lambda_y) = n_x \log \lambda_x - \left(\sum X_i\right)\lambda_x + n_y \log \lambda_y - \left(\sum Y_i\right)\lambda_y.$

when $\lambda = \lambda_x = \lambda_y$

$\ell(\lambda) = (n_x + n_y)\log \lambda - \left(\sum X_i + \sum Y_i\right)\lambda.$

$\frac{\partial \ell}{\partial \lambda} = \frac{n_x + n_y}{\lambda} - \left(\sum X_i + \sum Y_i\right) = 0 \Rightarrow \boxed{\hat{\lambda} = \frac{n_x + n_y}{\sum X_i + \sum Y_i}}.$

(b) $\hat{\lambda}_x = \frac{n_x}{\sum X_i}, \quad \hat{\lambda}_y = \frac{n_y}{\sum Y_i}$

$$LR = \frac{\sup_{\lambda_x, \lambda_y} L(\underline{X}, \underline{Y})}{\sup_{\lambda_x = \lambda_y} L(\underline{X}, \underline{Y})} = \boxed{\frac{\left(\frac{n_x}{\sum X_i}\right)^{n_x} e^{-n_x} \left(\frac{n_y}{\sum Y_i}\right)^{n_y} e^{-n_y}}{\left(\frac{n_x + n_y}{\sum X_i + \sum Y_i}\right)^{n_x + n_y} \cdot e^{-n_x - n_y}}}$$

(c) $\underline{2 \log LR \mathrel{\dot\sim} \chi^2_1}.$