STAT 510 — Mathematical Statistics

Assignment: Problem Set 2 **Due Date:** September 20 2022, 11:59 PM

Problem 1.

Solution. If $X \sim Gamma(\alpha, \beta)$, then $E[X] = \frac{\alpha}{\beta}$ and $Var(X) = \frac{\alpha}{\beta^2}$. Then we can get the method of moments estimators by

$$\begin{cases} \frac{\alpha}{\beta} = \bar{X}_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i \\ \frac{\alpha}{\beta^2} = S_n^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \bar{X}_n)^2 \end{cases} \Longrightarrow \begin{cases} \hat{\alpha} = \dfrac{\bar{X}_n^2}{S_n^2} \\ \\ \hat{\beta} = \dfrac{\bar{X}_n}{S_n^2} \end{cases}$$

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Problem 2.

Solution. (a) Note that for any $n \in \mathbb{Z}^+$, we have

$$E[X^n] = \int_\theta^\infty \theta x^{n-2} dx = \begin{cases} \theta log x|_\theta^\infty = \infty, & n = 1 \\ \frac{\theta x^{n-1}}{n-1}|_\theta^\infty = \infty, & n > 1 \end{cases}$$

which implies that the method of moments estimator of θ does not exist.

(b) From $f(x|\theta) = \theta x^{-2}, 0 < \theta \leq x < \infty$, we have

$$f(x|\theta) = \theta x^{-2}\mathbb{I}_{\{x \geq \theta\}}, \quad x \in \mathbb{R}, \theta > 0$$

Let $x_{(1)} = min\{x_1, \cdots, x_n\}$, then we have

$$L(\theta|x_1, \cdots, x_n) = \prod_{i=1}^{n} f(x_i|\theta) = \frac{\theta^n}{\prod\limits_{i=1}^{n} x_i^2}\mathbb{I}_{\{x_{(1)} \geq \theta\}} = \begin{cases} \dfrac{\theta^n}{\prod\limits_{i=1}^{n} x_i^2}, & if\ \theta \leq x_{(1)} \\ \\ 0, & if\ \theta > x_{(1)} \end{cases}$$

Therefore,

$$\hat{\theta}_{MLE} = argmax\ L(\theta|X_1, \cdots, X_n) = X_{(1)}$$

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Problem 3.

Solution. Let $x_{(1)} = min\{x_1, \cdots, x_n\}, x_{(n)} = max\{x_1, \cdots, x_n\}$, then we have

$$L(\theta|x_1, \cdots, x_n) = \prod_{i=1}^{n} f(x_i|\theta) = \begin{cases} \mathbb{I}_{\{0<x_{(1)}\leq x_{(n)}<1\}}, & if\ \theta = 0 \\ \dfrac{1}{2^n \left(\prod\limits_{i=1}^{n} \sqrt{x_i}\right)} \mathbb{I}_{\{0<x_{(1)}\leq x_{(n)}<1\}}, & if\ \theta = 1 \end{cases}$$

Note that

$$L(1) > L(0) \Longleftrightarrow \prod_{i=1}^{n} x_i < \frac{1}{4^n}$$

then we have

$$\hat{\theta}_{MLE} = argmax\ L(\theta|X_1, \cdots, X_n) = \begin{cases} 0, & if\ \prod\limits_{i=1}^{n} X_i \geq 4^{-n} \\ 1, & if\ \prod\limits_{i=1}^{n} X_i < 4^{-n} \end{cases}$$

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Problem 4.

Solution. From $P(X_i \leq x|\alpha, \beta)$ we can get the pdf of $X_1, \cdots, X_n$,

$$f(x|\alpha, \beta) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \mathbb{I}_{\{0\leq x\leq \beta\}}$$

and the likelihood function is

$$L(\alpha, \beta|x_1, \cdots, x_n) = \prod_{i=1}^{n} f(x_i|\alpha, \beta) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \mathbb{I}_{\{0\leq x_{(1)}\leq x_{(n)}\leq \beta\}}$$

For any fixed α, $L(\alpha, \beta|x_1, \cdots, x_n)$ is decreasing w.r.t. β when $\beta \geq x_{(n)}$. Thus $\hat{\beta}_{MLE} = X_{(n)}$. The profile log-likelihood function is

$$l(\alpha, \hat{\beta}_{MLE}|x_1, \cdots, x_n) = nlog\alpha - n\alpha log\hat{\beta}_{MLE} + (\alpha - 1) \sum_{i=1}^{n} logx_i$$

Then from

$$0 = \frac{\partial l(\alpha, \hat{\beta}_{MLE}|x_1, \cdots, x_n)}{\partial \alpha} = \frac{n}{\alpha} - nlog\hat{\beta}_{MLE} + \sum_{i=1}^{n} logx_i$$

we could get $\hat{\alpha}_{MLE} = \dfrac{n}{nlogX_{(n)} - \sum\limits_{i=1}^{n} logX_i}$ ■

Problem 5.

Solution. (a) Note that $\bar{X} \sim N(\theta, \frac{\sigma^2}{n})$, then the joint pdf of $\bar{X}$ and θ is

$$f(\bar{x}, \theta) = f(\bar{x}|\theta)f(\theta) = \frac{1}{2\pi\sqrt{\sigma^2\tau^2/n}} exp\left(-\frac{(\bar{x}-\theta)^2}{2\sigma^2/n} - \frac{(\theta-\mu)^2}{2\tau^2}\right)$$

(b) Since

$$\frac{(\bar{x}-\theta)^2}{2\sigma^2/n} + \frac{(\theta-\mu)^2}{2\tau^2} = \left(\frac{1}{2\sigma^2/n} + \frac{1}{2\tau^2}\right)\left(\theta - \frac{\frac{\bar{x}}{\sigma^2/n} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}}\right)^2 + \frac{(\bar{x}-\mu)^2}{2(\sigma^2/n + \tau^2)}$$

then the marginal pdf of $\bar{X}$ is

$$\begin{aligned} m(\bar{x}) &= \int f(\bar{x}, \theta) d\theta \\ &= \frac{1}{2\pi\sqrt{\sigma^2\tau^2/n}} exp\left(-\frac{(\bar{x}-\mu)^2}{2(\sigma^2/n+\tau^2)}\right) \int exp\left(-\frac{1}{2\left(\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}\right)^{-1}}\left(\theta - \frac{\frac{\bar{x}}{\sigma^2/n} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}}\right)^2\right) d\theta \\ &= \frac{1}{\sqrt{2\pi(\sigma^2/n+\tau^2)}} exp\left(-\frac{(\bar{x}-\mu)^2}{2(\sigma^2/n+\tau^2)}\right) \end{aligned}$$

which is $N(\mu, (\sigma^2/n) + \tau^2)$

(c) From (a) and (b), we could get that

$$\begin{aligned} \pi(\theta|x_1, \cdots, x_n) &= \frac{f(\bar{x}, \theta)}{m(\bar{x})} \\ &= \frac{1}{\sqrt{2\pi\left(\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}\right)^{-1}}} exp\left(-\frac{1}{2\left(\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}\right)^{-1}}\left(\theta - \frac{\frac{\bar{x}}{\sigma^2/n} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}}\right)^2\right) \end{aligned}$$

which implies that

$$\theta|\bar{x} \sim N\left(\frac{\frac{\bar{x}}{\sigma^2/n} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}}, \left(\frac{1}{\sigma^2/n} + \frac{1}{\tau^2}\right)^{-1}\right)$$

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Problem 6.
Solution. Note that

$$m(x) = \int_{\mathbb{R}} f(x|\theta)\pi(\theta)d\theta = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(\theta - x)^2}{2\sigma^2}\right) d\theta = 1$$

then we have

$$\pi(\theta|x) = \frac{f(x,\theta)}{m(x)} = f(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(\theta - x)^2}{2\sigma^2}\right)$$

which means that $\theta|x \sim N(x, \sigma^2)$ and the posterior mean $E[\theta|x] = x$. ■

Problem 7.
Solution. (a) Since

$$\begin{aligned}
\pi(\lambda|x_1, \cdots, x_n) &\propto f(x_1, \cdots, x_n|\lambda)\pi(\lambda) \\
&\propto \prod_{i=1}^{n} \left(e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \\
&\propto \lambda^{\alpha + \sum\limits_{i=1}^{n} x_i - 1} e^{-(\beta+n)\lambda}
\end{aligned}$$

which implies that $\lambda|x_1, \cdots, x_n \sim Gamma(\alpha + \sum\limits_{i=1}^{n} x_i, \beta + n)$
(b) It follows (a) that

$$E[\lambda|x_1, \cdots, x_n] = \frac{\alpha + \sum\limits_{i=1}^{n} x_i}{\beta + n}, \quad Var(\lambda|x_1, \cdots, x_n) = \frac{\alpha + \sum\limits_{i=1}^{n} x_i}{(\beta + n)^2}$$

(c) From (a), (b) we could know that the when the prior distribution is Gamma distribution, the posterior distribution is still Gamma distribution. Thus Gamma distributions form a conjugate family of Poisson distributions.
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