**STAT 510 — Mathematical Statistics**

**Assignment:** Problem Set 5 **Due Date:** October 24 2022, 11:59 PM

**Problem 1.**
The joint pdf of $\mathbf{X} = (X_1, \cdots, X_n)$ is

$$\begin{aligned} f(\mathbf{x}|\mu,\lambda) &= \prod_{i=1}^{n} \left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} exp\left[-\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right] \\ &= \left(\prod_{i=1}^{n} \frac{1}{x_i^{3/2}}\right)\left(\frac{\lambda}{2\pi}\right)^{n/2} exp\left[-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}\right] \end{aligned}$$

By Neyman's factorization theorem, $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \frac{1}{X_i})$ is a sufficient statistics for $(\mu, \lambda)$ ■

**Problem 2.**
From the fact that

$$Y := (n-1)s_n^2/\sigma^2 \sim \chi_{n-1}^2$$

we have $\mathbb{E}Y = n-1$, and its pdf is

$$f_Y(y) = \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} y^{\frac{n-3}{2}} e^{-\frac{y}{2}}, \quad y > 0$$

Since $s_n^2 = h(Y) := \frac{\sigma^2}{n-1}Y$, we have $\mathbb{E}s_n^2 = \sigma^2$. The pdf of $s_n^2$ can be computed by

$$\begin{aligned} f_{s_n^2}(s^2|\mu,\sigma^2) &= f_Y(h^{-1}(s^2))\left|\frac{dh^{-1}(s^2)}{ds^2}\right| \\ &= \frac{n-1}{\sigma^2} f_Y(\frac{n-1}{\sigma^2}s^2) \\ &= \frac{n-1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})\sigma^2}\left(\frac{n-1}{\sigma^2}s^2\right)^{\frac{n-3}{2}} e^{-\frac{(n-1)s^2}{2\sigma^2}} \end{aligned}$$

and consequently,

$$log f_{s_n^2}(s^2|\mu,\sigma^2) = C - \frac{n-1}{2} log\sigma^2 - \frac{(n-1)s^2}{2\sigma^2}$$

where C is a costant of $\mu$ and $\sigma^2$.
Therefore, we have

$$\begin{cases} \dfrac{\partial}{\partial \mu} log f_{s_n^2}(s^2|\mu,\sigma^2) = 0 \\ \\ \dfrac{\partial}{\partial \sigma^2} log f_{s_n^2}(s^2|\mu,\sigma^2) = \dfrac{(n-1)s^2}{2\sigma^4} - \dfrac{n-1}{2\sigma^2} \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{\partial^2}{\partial \mu^2} log f_{s_n^2}(s^2|\mu,\sigma^2) = \dfrac{\partial^2}{\partial \mu \partial \sigma^2} log f_{s_n^2}(s^2|\mu,\sigma^2) = \dfrac{\partial^2}{\partial \sigma^2 \partial \mu} log f_{s_n^2}(s^2|\mu,\sigma^2) = 0 \\ \\ \dfrac{\partial^2}{\partial (\sigma^2)^2} log f_{s_n^2}(s^2|\mu,\sigma^2) = \dfrac{n-1}{2\sigma^4} - \dfrac{(n-1)s^2}{\sigma^6} \end{cases}$$

$$\Longrightarrow \begin{cases} I_{11}(\mu,\sigma^2) = I_{12}(\mu,\sigma^2) = I_{21}(\mu,\sigma^2) = 0 \\ \\ I_{22}(\mu,\sigma^2) = -\mathbb{E}\left[\dfrac{\partial^2}{\partial (\sigma^2)^2} log f_{s_n^2}(s^2|\mu,\sigma^2)\right] = \dfrac{n-1}{2\sigma^4} \end{cases}$$

Thus the Fisher information matrix based on $s_n^2$ is

$$I_{s_n^2}(\theta) = \begin{pmatrix} I_{11}(\mu,\sigma^2) & I_{11}(\mu,\sigma^2) \\ I_{11}(\mu,\sigma^2) & I_{11}(\mu,\sigma^2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(n-1)\sigma^{-4} \end{pmatrix}$$

■

**Problem 3.**
(i) We have $log f_X(x|p) = plogx + (1-x)log(1-p)$ and thus

$$\frac{d}{dp} log f_X(x|p) = \frac{x}{p} - \frac{1-x}{1-p} \Longrightarrow \frac{d^2}{dp^2} log f_X(x|p) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$$

which implies

$$I_{X_1}(p) = -\mathbb{E}\left[\frac{d^2}{dp^2} log f_X(x|p)\right] = \frac{1}{p(1-p)}$$

and consequently, $I_{\mathbf{X}}(p) = 3I_{X_1}(p) = \dfrac{3}{p(1-p)}$

(ii) Since $Y := X_1 + X_2 + X_3 \sim Bin(3,p)$, then $f_Y(y|p) = \binom{3}{y} p^y (1-p)^{3-y}$.

Note that $\bar{X} = \frac{1}{3}Y$, then we have

$$f_{\bar{X}}(\bar{x}|p) = 3\binom{3}{3\bar{x}}p^{3\bar{x}}(1-p)^{3-3\bar{x}}$$

$$\implies logf_{\bar{X}}(\bar{x}|p) = log3 + log\binom{3}{3\bar{x}} + 3\bar{x}logp + (3-3\bar{x})log(1-p)$$

Consequently, we can compute the Fisher information by

$$I_{\bar{X}}(p) = -\mathbb{E}\left[\frac{d^2}{dp^2}logf_{\bar{X}}(\bar{x}|p)\right] = \mathbb{E}\left[\frac{3\bar{X}}{p^2} + \frac{3(1-\bar{X})}{(1-p)^2}\right] = \frac{3}{p(1-p)}$$

Note that $I_{\mathbf{X}}(p) = I_{\bar{X}}(p) = \dfrac{3}{p(1-p)}$, we could conculde that $\bar{X}$ is sufficient for p.

(iii) Since $T = X_1 + X_2 \sim Bin(2,p)$, then $f_T(t|p) = \binom{2}{t}p^t(1-p)^{2-t}$. We can compute $I_T(p)$ by

$$I_T(p) = -\mathbb{E}\left[\frac{d^2}{dp^2}logf_T(t|p)\right] = \mathbb{E}\left[\frac{T}{p^2} + \frac{2-T}{(1-p)^2}\right] = \frac{2}{p(1-p)}$$

Since $I_T(p) < I_{\mathbf{X}}(p)$, we could conclude that $T$ is not sufficient for p.

$\blacksquare$

**Problem 4.**

$$\begin{aligned}
\mathbb{E}[s_\theta(X)] &= \int \nabla_\theta logf(x|\theta)f(x|\theta)dx \\
&= \int \frac{1}{f(x|\theta)}\nabla_\theta f(x|\theta)f(x|\theta)dx \\
&= \int \nabla_\theta f(x|\theta)dx \\
&= \nabla_\theta \int f(x|\theta)dx \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[\nabla^2 log f(x|\theta)] &= \int [\nabla(\nabla log f(x|\theta))] f(x|\theta) dx \\
&= \int [\nabla(\frac{\nabla f(x|\theta)}{f(x|\theta)})] f(x|\theta) dx \\
&= \int \frac{(\nabla^2 f(x|\theta)) f(x|\theta) - (\nabla f(x|\theta))(\nabla f(x|\theta))^T}{f^2(x|\theta)} f(x|\theta) dx \\
&= \int \nabla^2 f(x|\theta) dx - \int \frac{\nabla f(x|\theta)}{f(x|\theta)} \frac{(\nabla f(x|\theta))^T}{f(x|\theta)} f(x|\theta) dx \\
&= \nabla^2 \int f(x|\theta) dx - \int (\nabla log f(x|\theta))(\nabla log f(x|\theta))^T f(x|\theta) dx \\
&= 0 - \mathbb{E}\left[s_\theta(X) s_\theta(X)^T\right] \\
&= -I_X(\theta)
\end{aligned}
$$

■

**Problem 5.**
(a)

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{(x>0)}$$

(i) $\alpha$ known,

$$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \mathbb{1}_{(x>0)}, \quad c(\beta) = \beta^\alpha, \quad w_1(\beta) = -\beta, t_1(x) = x$$

(ii) $\beta$ known,

$$h(x) = e^{-\beta x} \mathbb{1}_{(x>0)}, \quad c(\alpha) = \frac{\beta^\alpha}{\Gamma(\alpha)}, \quad w_1(\alpha) = \alpha - 1, t_1(x) = log x$$

(iii) $\alpha, \beta$ unknown,

$$h(x) = \mathbb{1}_{(x>0)}, c(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}, w_1(\alpha, \beta) = \alpha - 1, t_1(x) = log x, w_2(\alpha, \beta) = -\beta, t_2(x) = x$$

Thus, gamma distribution with either parameter $\alpha$ or $\beta$ known, or both unknown, belongs to the exponential family.

(b)

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}}$$

$$h(x) = \frac{1}{x!}\mathbb{1}_{x \in \mathbb{N}}, \quad c(\lambda) = e^{-\lambda}, \quad w_1(\lambda) = log\lambda, \quad t_1(x) = x$$

implying that Poisson distribution belongs to the exponential family.
(c)

$$f(x|n) = \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,2,\ldots,n\}}$$

which cannot be written in the form $h(x)c(\theta)exp\{\sum_{i=1}^{k} w_k(\theta)t_k(x)\}$. Thus it does not belong to the exponential family.

**Problem 6:**

**(a)** $X \sim N(\theta, a\theta^2)$

(i)

$$f(x|\theta) = (2\pi a\theta^2)^{-1/2} exp\left(-\frac{(x-\theta)^2}{2a\theta^2}\right)$$
$$= (2\pi a\theta^2)^{-1/2} exp\left(-\frac{1}{2a}\right) exp\left(\frac{1}{a\theta}x - \frac{1}{2a\theta^2}x^2\right)$$

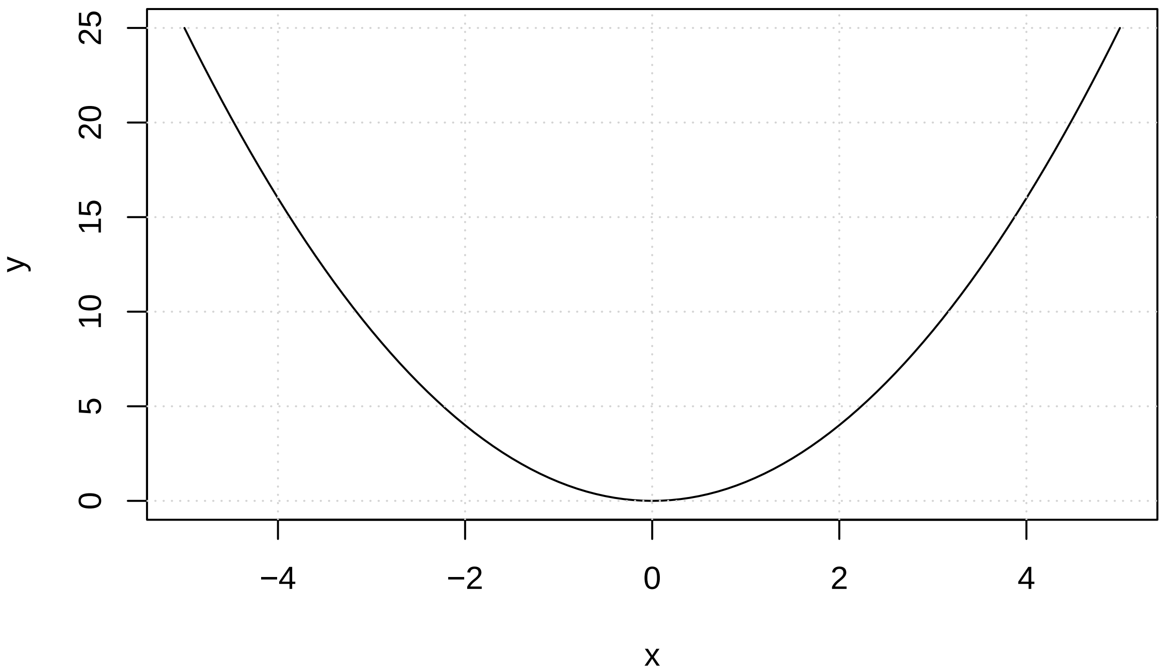
In this case, $h(x) = exp\left(-\frac{1}{2a}\right), c(\theta) = (2\pi a\theta^2)^{-1/2}, w_1(\theta) = \frac{1}{\theta}, t_1(x) = \frac{x}{a}, w_2(x) = \frac{1}{\theta^2}, t_2(x) = -\frac{x^2}{2a}$ implying that this family is an exponential family.

(ii) from(i),

$$\boldsymbol{\theta} = (\theta_1, \theta_2) = (w_1(\theta), w_2(\theta)) = (\frac{1}{\theta}, \frac{1}{\theta^2})$$

which implies that the $\boldsymbol{\theta}$ parameter vector lies on the parabola.

(iii)




**(b)**

(i)

$$f(x|\theta) = Cexp\left(-(x-\theta)^4\right) = Cexp(-x^4 + 4x^3\theta - 6x^2\theta^2 + 4x\theta^3 - \theta^4)$$

In this case, $h(x) = Cexp(-x^4), c(\theta) = exp(-\theta^4), w_1(\theta) = \theta, t_1(x) = 4x^3, w_2(\theta) = \theta^2, t_2(x) = -6x^3, w_3(\theta) = \theta^3, t_3(x) = 4x$ which implies that this family is an exponential family.

(ii) from(i)

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (w_1(\theta), w_2(\theta), w_3(\theta)) = (\theta, \theta^2, \theta^3)$$

and consequently, the $\boldsymbol{\theta}$ parameters vector lies on a line in $\mathbb{R}^3$

(iii)

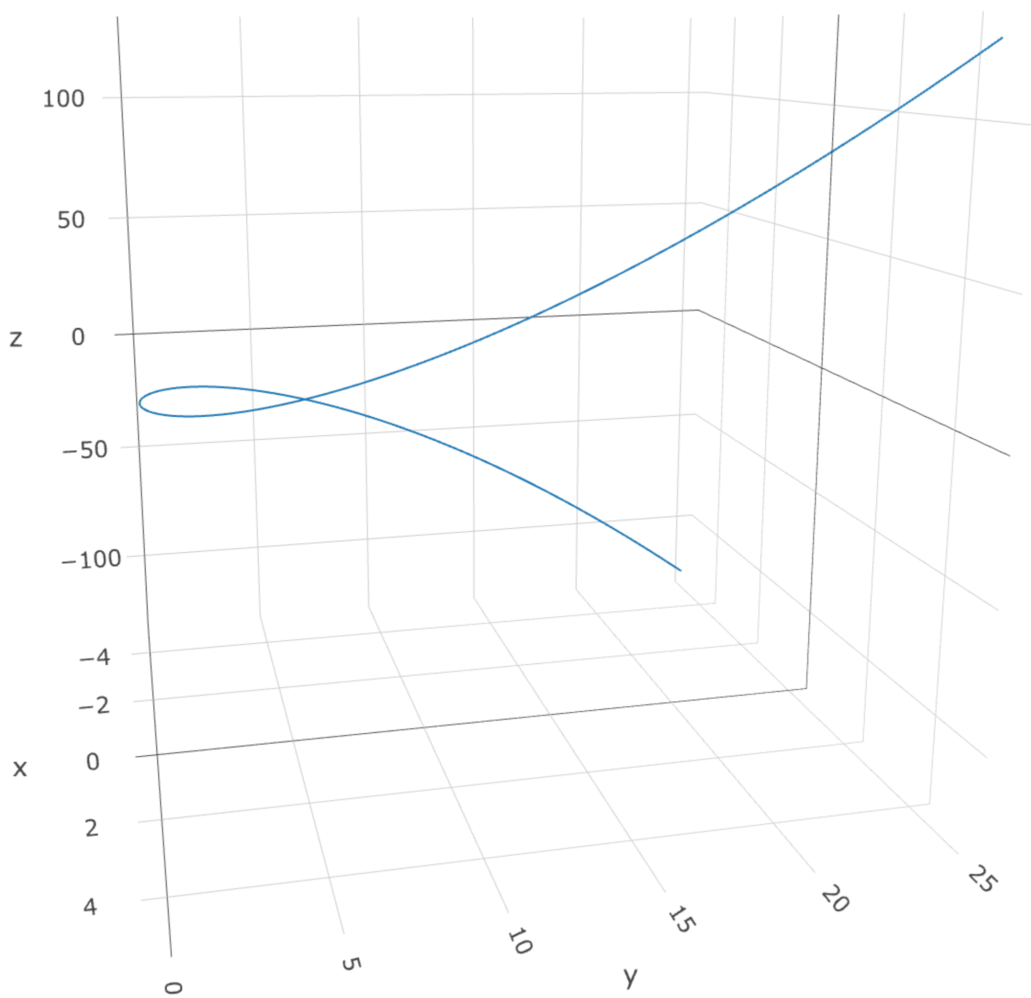



Figure 1: b(iii).