Part 1 ...

1) $X(\theta, \sigma^2)$; σ is known

a) $\log p(\theta|x)$ can be expressed as better than $\log p(x|\theta)$ for the exponential family

$f(x|\theta) = A(x) \cdot h(\theta) \exp\{\sum_{i=1}^{k} w_i(\theta) t_i(x)\}$ —①

So, $\pi(\theta|x)$

$$= \frac{f(x,\theta)\pi(\theta)}{\int f(x,\theta)\pi(\theta)d\theta}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\theta)^2}{2\sigma^2}\}}{\int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\theta)^2}{2\sigma^2}\}}$$

$$= \frac{\exp\{-\frac{1}{2\sigma^2}(x^2 - 2x\theta + \theta^2)\}}{\int \exp\{-\frac{1}{2\sigma^2}x^2 + \frac{2x\theta}{2\sigma^2} - \frac{\theta^2}{2\sigma^2}\}}$$

$$= \frac{\exp\{-\frac{1}{2\sigma^2}(-2x\theta + \theta^2)\}}{\int \exp\{-\frac{1}{2\sigma^2}(x^2)\} \exp\{\frac{1}{2\sigma^2}(-2x\theta + \theta^2)\}}$$

Comparing ① with ②

$A(x) = \frac{1}{\int \exp\{-\frac{1}{2\sigma^2}x^2\}}$

$c(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{\theta^2}{2\sigma^2}\}$

$w_1(\theta) = \frac{1}{2\sigma^2}$, $t_1(x) = -x^2$

$w_2(\theta) = \frac{\theta}{\sigma^2}$, $t_2(x) = x$

Since ② can be expressed as ①, the given distribution $X(\theta, \sigma^2)$ belongs to the exponential family.

b) Parameter $= \{\theta, \sigma^2\}$

where $-\infty < \theta < \infty$ & $\sigma > 0$

As $\theta \propto \sigma^2$

$\therefore$ Space not an a parabola

(ii)

2) $f(x|\theta) = c \exp\{-(x-\theta)^2\}$

i) $= c \exp\{-(x^2 - 2x\theta + \theta^2)\} \cdot \exp\{-(x^2)\}$

$= c \exp\{-x^2\} \cdot \exp\{2x\theta\} \exp\{-(x^2 - 2x\theta + \theta^2)\}$

Comparing ① with ②

$A(x) = \frac{1}{\int \exp\{-x^2\}}$

$c(\theta) = c \exp\{-\theta^2\}$

$w_1(\theta) = 2\theta$, $t_1(x) = x$

Since ② can be expressed as ①, the given distribution $f(x|\theta) = c \exp\{-(x-\theta)^2\}$ belongs to the exponential distribution family.

ii) The curve is a spiral in 3-space.

3) Cannot graph 3D plot on 2D

4) $f(x, \lambda) = \frac{e^{-\lambda\lambda}}{x!} x^{x} I_{\{x\}}$

i) Assume λ is known.

$f(x|\lambda) = \frac{1}{\Gamma(x)} I_{\{x\}}$ $\exp\{x \log \lambda - \lambda \log(x)\}$

Comparing ① with ②

$A(x) = \frac{1}{\Gamma(x)} I_{\{x\}}$

$c(\lambda) = \exp\{-\lambda \log \lambda - \log(\Gamma(\lambda))\}$

$w_1(\lambda) = \lambda$, $t_1(x) = x$

Hence $f(x|\lambda)$ is a distribution from the exponential family

ii) Assume λ is known

$f(x|\lambda) = \exp\{-\frac{1}{2}\} I_{\{x\}}$ $\exp\{x \log \lambda - \lambda \log(x)\}$

$= \exp\{-\frac{1}{2}\} I_{\{x\}}$ $\exp\{x \log \lambda - \log(\Gamma(x))\}$ $\exp(\log x)$ ②

Comparing ① with ②

$A(x) = \exp\{-\frac{1}{2}\} I_{\{x\}}$

$c(\lambda) = \exp\{-\lambda \log \lambda - \log(\Gamma(\lambda))\}$

$w_1(\lambda) = \lambda$, $t_1(x) = \log x$

Hence $f(x|\lambda)$ is a distribution from the exponential family

iii) Assume λ is both unknown

$$f(x|\lambda, \lambda) = \frac{1}{\lambda^{\lambda}\Gamma(\lambda)} \exp\{-\frac{x}{\lambda} + (\lambda-1)\log x\}$$ ①

Comparing ① with ②

$A(x) = \frac{1}{\Gamma(\lambda)}$

$c(\lambda) = \exp\{-\lambda \log \lambda - \log(\Gamma(\lambda))\}$

$w_1(\lambda) = -\frac{1}{\lambda}$, $t_1(x) = x$

$w_2(\lambda, \lambda) = (\lambda - 1)$, $t_2(x) = \log x$

Hence, $f(x|\lambda, \lambda)$ distribution belongs to the exponential family.

5) $f(x = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$

$= \frac{1}{k!} \exp\{-\lambda\} \exp\{k \log \lambda\}$ ①

Comparing ① with ②

$A(x = k) = \frac{1}{k!}$

$c(\lambda) = \exp\{-\lambda\}$, $\lambda \in (0, \infty)$

$w_1(\lambda) = \log \lambda$, $t_1(x = k) = k$

Hence, $f(x = k|\lambda)$ distribution belongs to the exponential family

6) $f(x|\theta, \sigma, \lambda) = \frac{\sigma}{\sqrt{2\pi}} \exp\{-\frac{1}{2}\sigma^2 x^2 + \lambda x\}$

$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}\sigma^2 x^2 + \lambda \frac{\sigma}{\sigma} x\}$

$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(\sigma x)^2 + \lambda \frac{\sigma}{\sigma} x\}$

$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(\sigma x - \frac{\lambda}{\sigma})^2 + \frac{\lambda^2}{2\sigma^2}\}$

$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(\sigma x - \frac{\lambda}{\sigma})^2\} \exp\{\