Week 5 - 910

1) Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Given that $\underline{x} \sim \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$

Considering $\underline{y}$

$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - 2x_1 + 2x_2 \end{pmatrix}$

$\underline{y} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\underline{y} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$

$\underline{y} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

Transpose

$\underline{y} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$

$\mathrm{Cov}(\underline{y}) = \underline{A}\ \mathrm{Cov}(\underline{x})\ \underline{A}^T$

Since $\underline{y}$ is a function of $\underline{x}$ and $\underline{A}^T$ is symmetric

$(\underline{A}\,\underline{x})$ is idempotent $(\underline{A}\,\underline{A})^T$

$\mathrm{Cov}(\underline{y}) = \sigma^2 \underline{A}$

Given that

$\mathrm{Cov}(\underline{A}\,\underline{x}) = \sigma^2 \underline{B}$

$\sigma^2 \underline{A} = \sigma^2 \underline{B}$

so $\underline{B} = \underline{A} = \frac{1}{3} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

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2) $f(x) = \frac{1}{2} e^{-|x|}$

$$f(x) = \begin{cases} \frac{1}{2} e^{x} & , x < 0 \\ \frac{1}{2} e^{-x} & , x \geq 0 \end{cases}$$

a) $M_X(t) = E[e^{tX}]$

$= \int_{-\infty}^{0} e^{tx} \frac{1}{2} e^{x}\, dx + \int_{0}^{\infty} e^{tx} \frac{1}{2} e^{-x}\, dx$

$= \frac{1}{2} \int_{-\infty}^{0} e^{(t+1)x}\, dx + \frac{1}{2} \int_{0}^{\infty} e^{(-1-t)x}\, dx$

$= \frac{1}{2} \left[\frac{e^{(t+1)x}}{(t+1)} \right]_{-\infty}^{0} + \frac{1}{2} \left[\frac{e^{(-1-t)x}}{(-1-t)} \right]_{0}^{\infty}$

$= \frac{1}{2} \left[\frac{1}{t+1} - 0 + 0 - \frac{1}{-1-t} \right]$, $|t| < 1$

$M_X(t) = \frac{1}{1-t^2}$, $|t| < 1$

$M_X(t)$ is finite for $|t| < 1$ i.e. $t \in (-1, 1)$

b) $U \sim$ Exponential (1) & $V \sim$ Exponential (1)

$f_U(u) = \begin{cases} e^{-u} & u > 0 \\ 0 & \text{otherwise} \end{cases}$ | $f_V(v) = \begin{cases} e^{-v} & v > 0 \\ 0 & \text{otherwise} \end{cases}$

Given that U & V are independent R.V.

$X = U + (-V)$

$M_Y(t) = M_U(t) * M_{-V}(t)$ [U & V independent]

$M_U(t) = \int_0^\infty e^{tu} f_U(u)\, du$

$= \int_0^\infty e^{tu} \cdot e^{-u}\, du$

$= \frac{1}{t-1} \cdot e^{(t-1)u} \Big]_0^\infty$

$= \frac{1}{1-t}$

$M_{-V}(t) = M_V(-t)$ [MGF of linear transformation]

$= \int_0^\infty e^{-tv} \cdot f_V(v)\, dv$

$= \int_0^\infty e^{-tv} \cdot e^{-v}\, dv$

$= \frac{1}{-(t+1)} e^{-(t+1)v} \Big]_0^\infty$

$= \frac{1}{(t+1)}$

$\therefore M_Y(t) = M_U(t) * M_{-V}(t)$

$= \frac{1}{(1-t)} \cdot \frac{1}{(1+t)}$

$M_Y(t) = \frac{1}{(1-t^2)}$

From Q2 a), we know MGF of a double exponential fn is $\frac{1}{1-t^2}$, $\therefore Y$ has a double exponential distribution

$f(y) = \frac{1}{2} e^{-|y|}$

c) $W_n = \frac{\bar{X}}{\sigma/\sqrt{n}}$

$= \frac{\bar{X} \cdot \sqrt{n}}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + \dots X_n}{\sigma/\sqrt{n}}$

$M_X(t) = M_{X_1}(t) * M_{X_2}(t) * \dots M_{X_n}(t)$

$M_{\frac{\bar{X}}{\sigma/\sqrt{n}}} = M_{X_1}\left(\frac{t}{\sigma/\sqrt{n}}\right)$ [MGF of linear transfor]

From Q2 a), we know that $M_X(t) = \frac{1}{1-t^2}$ where $X \sim$ Double exponential $(\mu=0, \beta=1)$

$\therefore M_X\left(\frac{t}{\sigma/\sqrt{n}}\right) = \frac{1}{1 - \left(\frac{t}{\sigma/\sqrt{n}}\right)^2}$

$M_{W_n}(t) = M_{X_1}\left(\frac{t}{\sigma/\sqrt{n}}\right) * M_{X_2}\left(\frac{t}{\sigma/\sqrt{n}}\right) * \dots M_{X_n}\left(\frac{t}{\sigma/\sqrt{n}}\right)$ [X_1, X_2 independent]

$= \left[\frac{1}{1 - \left(\frac{t}{\sigma/\sqrt{n}}\right)^2} \right]^n$

$= \frac{1}{\left(1 - \frac{t^2}{n\sigma^2}\right)^n}$

d) Lim $M_{W_n}(t)$ $= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{(-t^2/\sigma^2)}{n}\right)^n}$

We know that,

$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

$\therefore \lim_{n \to \infty} M_{W_n}(t) = \frac{1}{e^{-t^2/\sigma^2}}$

$= e^{t^2/\sigma^2}$

$\therefore$ $M_{W_n}(t)$ corresponds to Exponential distribution

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2) Given,

$Z \sim N(0,1)$

$U \sim$ Uniform $(0,1)$

Z, U are independent

$\underline{Y} = (Y_1, Y_2) = g(Z, U)$

$Y_1 = \frac{Z}{U}$, $Y_2 = U$

a) Space of Z is $z \in \mathbb{R}$

Space of U is $x \in [0,1]$

From given statements,

Space of Y_1 is $x \in \mathbb{R}$

Space of Y_2 is $x \in [0,1]$

Space of Y

Space of $Y = \mathbb{R} \times [0,1]$

b) Given that,

$Y_1 = \frac{Z}{U}$ & $Y_2 = U$

So we have $Z = UY_1$ & $Y_1 = Y_2$

$U = Y_2$

$g^{-1}(y) = (z, u) = g^{-1}(Y_1 Y_2, Y_2)$

$J\, g^{-1}(y) = \begin{vmatrix} \frac{\partial z}{\partial Y_1} & \frac{\partial z}{\partial Y_2} \\ \frac{\partial U}{\partial Y_1} & \frac{\partial U}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial (Y_1 Y_2)}{\partial Y_1} & \frac{\partial (Y_1 Y_2)}{\partial Y_2} \\ \frac{\partial Y_2}{\partial Y_1} & \frac{\partial Y_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix}$

$\boxed{J\, g^{-1}(y) = -Y_2}$

c) $\underline{Y} = (Z, U)$ where $Z, U \to$ independent

$f_{Z,U}(z, u) = f_Z(z) \cdot f_U(u)$ — ①

PDF of Y

$f_Y(y) = f_{Z,U}\left(g^{-1}(Y_1, Y_2)\right) \cdot |J\, g^{-1}(y)|$ — ②

$Z \sim N(0,1)$

$\therefore f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$U \sim$ Uniform $(0,1)$

$f_U(u) = \begin{cases} 0 & \text{otherwise} \\ 1 & 0 \leq u \leq 1 \end{cases}$

From ①,

$f_{Z,U}(z, u) = f_Z(z) \cdot f_U(u)$

$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \frac{1}{\sqrt{2\pi}} e^{(y_1 y_2)^2/2}$

Substituting $f_{Z,U}(z, u)$ in ②

$f_Y(y_1, y_2) = \frac{1}{\sqrt{2\pi}} e^{(y_1 y_2)^2/2} \cdot y_2$

d) Y_1 & Y_2 are not independent

Since we cannot factorize pdf of Y in terms of Y_1 & Y_2 (separately) $\therefore Y_1$ & Y_2 are not independent

e) $f_Y(y_1, y_2) = \frac{1}{\sqrt{2\pi}} e^{(y_1 y_2)^2/2} \cdot y_2$ & $y_1 \in \mathbb{R}$, $y_2 \in [0,1]$

$f_1(y_1) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y_1^2 y_2^2/2}\, y_2\, dy_2$

Let $t = \frac{y_1^2 y_2^2}{2}$ $\Rightarrow \frac{dt}{dy_2} = \frac{2 y_1^2 y_2}{2}$ $= y_1^2 y_2$ (OR $y_2 = \frac{\sqrt{2t}}{y_1}$)

$\Rightarrow \frac{dt}{y_1^2 y_2} = dy_2$ (OR $y_2 = \frac{\sqrt{2t}}{y_1}$)

$f_1(y_1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-t}}{y_1^2}\, dt$

$= \frac{1}{\sqrt{2\pi}\, y_1^2} \left[e^{-t} \right]_0^\infty$

$= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{e^{-y_1^2/2}}{y_1^2} \right)$

This gives $\boxed{\text{CDF} = \frac{1}{\sqrt{2\pi}}}$

3a) $E(X(1-X)) = \int_0^1 x(1-x) f_X(x)\, dx$

$= \int_0^1 x\, (1-x) \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\, dx$

$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha} (1-x)^{\beta}\, dx$

$= \frac{B(\alpha+1, \beta+1)}{B(\alpha,\beta)} \underbrace{\int \frac{x^{\alpha} (1-x)^{\beta}}{B(\alpha+1, \beta+1)}\, dx}_{\alpha' = \alpha+1,\ \beta' = \beta+1}$

$= \frac{B(\alpha+1, \beta+1)}{B(\alpha,\beta)} \underbrace{\int_0^1 \frac{x^{\alpha'-1} (1-x)^{\beta'-1}}{B(\alpha', \beta')}\, dx}_{=1}$

$E(X(1-X)) = \frac{B(\alpha+1, \beta+1)}{B(\alpha,\beta)}$

b) $E\left(X^2 (1-X)^2\right)$

$= \int_0^1 x^2 (1-x)^2 \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\, dx$

$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+2-1} (1-x)^{\beta+2-1}\, dx$

$= \frac{B(\alpha+2, \beta+2)}{B(\alpha,\beta)} \underbrace{\int_0^1 \frac{x^{\alpha+2-1} (1-x)^{\beta+2-1}}{B(\alpha+2, \beta+2)}\, dx}_{=1}$

$E\left(X^2 (1-X)^2\right) = \frac{B(\alpha+2, \beta+2)}{B(\alpha,\beta)}$