STAT 510 Mathematical Statistics, Fall 2022
Midterm Exam, 10/20/2022, Thursday, 9:30am – 10:50am

Name: ________________________ UIN: ________________________

Problem 1. [**24 points**] Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) continuous random variables with the probability density function

$$f(x \mid \beta, \theta) = \begin{cases} \frac{1}{\beta} e^{-\frac{x-\theta}{\beta}} & \text{if } x \geq \theta \\ 0 & \text{if } x < \theta \end{cases},$$

where $\beta > 0$ is a scale parameter and $\theta \in \mathbb{R}$ is a shift parameter. Both β and θ are unknown.

1. [**6 points**] Find a method of moments estimator $(\tilde{\beta}, \tilde{\theta})$ for (β, θ).
2. [**6 points**] Find the maximum likelihood estimator (MLE) $(\hat{\beta}, \hat{\theta})$ for (β, θ).
3. [**7 points**] Compute the bias of the MLEs $\hat{\beta}$ and $\hat{\theta}$. Are $\hat{\beta}$ and $\hat{\theta}$ biased?
4. [**5 points**] Find a sufficient statistic for (β, θ). [We are not interested in the trivial sufficient statistic $(X_1, \ldots, X_n)$.]

Problem 2. [**14 points**] Let $X_1, \ldots, X_n$ be the discrete i.i.d. random variables with the geometric distribution whose probability mass function is given by

$$P(X_1 = k) = (1-p)^{k-1} p, \qquad \text{for } k = 1, 2, 3, 4, \ldots,$$

where $p \in (0, 1)$ is the unknown parameter.

1. [**5 points**] Find the maximum likelihood estimator for p.
2. [**9 points**] Suppose the parameter p is a random variable taking values in the unit interval $(0, 1)$ with a prior distribution as the Beta distribution, whose probability density function is given by

   $$\pi(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} \qquad \text{for } 0 < p < 1.$$

   Here $\alpha, \beta > 0$ are hyperparameters and $\Gamma(\cdot)$ is the Gamma function satisfying recursion $\Gamma(x+1) = x\Gamma(x)$ for any $x > 0$. Find the posterior distribution $\pi(p \mid X_1, \ldots, X_n)$ and compute the posterior mean $\mathbb{E}[p \mid X_1, \ldots, X_n]$.

Problem 3. [**12 points**] Let $X_1, \ldots, X_n$ be an i.i.d. random variables drawn from $N(\theta, 1)$ with parameter $\theta \in \mathbb{R}$.

1. [**6 points**] Let $T_1 = X_1 + X_2$ and $T_2 = X_3 + X_4$. Is (T_1, T_2) a sufficient statistic for estimating θ? Why?
2. [**6 points**] Let $T_3 = X_1 + 2X_2$ and $T_3 = X_3 + X_4$. Is (T_3, T_4) a sufficient statistic for estimating θ? Why?