STAT 510 Mathematical Statistics Fall 2022

Problem set 2: Due on 11:59pm, Tuesday, 9/20/2022

1. Suppose that $X_1, \ldots, X_n$ are iid distributed as $Gamma(\alpha, \beta)$ random variables where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the rate parameter, i.e., the probability density function of $X \sim Gamma(\alpha, \beta)$ is given by

$$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad x > 0.$$

   Derive the method of moments estimator for α and β.

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the probability density function

$$f(x|\theta) = \theta x^{-2}, \qquad 0 < \theta \leq x < \infty.$$

   (a) Find the method of moments estimator of θ.

   (b) Find the maximum likelihood estimator (MLE) of θ.

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the probability density function $f(x|\theta)$, where if $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

   while if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Find the MLE of θ.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the cumulative distribution function

$$P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases},$$

   where the parameters α and β are positive. Find the MLE for (α, β).

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution $N(\theta, \sigma^2)$ and suppose the parameter θ is random with a prior distribution $N(\mu, \tau^2)$. Assume that σ^2, μ, τ^2 are all known.

   (a) Find the joint probability density function of $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ and θ.

   (b) Show that the marginal distribution $m(x)$ of $\overline{X}$ is $N(\mu, (\sigma^2/n) + \tau^2)$.

   (c) Derive the posterior distribution $\pi(\theta | X_1, \ldots, X_n)$.

6. In most situations, improper priors are not a problem as long as the resulting posterior is a well-defined probability distribution. Below is an example that illustrates this point. Let $X \sim N(\theta, \sigma^2)$, where σ^2 is known. Let the prior density $\pi(\theta) = 1, \theta \in \mathbb{R}$ to be the improper uniform density over the real line. Find the posterior distribution, $\pi(\theta|x)$ and posterior mean.

7. Let $X_1, \ldots, X_n$ be i.i.d. random variables with Poisson distribution

$$P(X_i = k|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

   Suppose the intensity parameter λ has a $Gamma(\alpha, \beta)$ distribution.

   (a) Find the posterior distribution of λ.

   (b) Calculate the posterior mean and variance.

   (c) Conclude whether or not the Gamma distributions form a conjugate family of Poisson distributions.