HW-3. ...

3) $f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}}$ $x>0$

MOM for α & β

$E[X] = \frac{\beta}{\alpha-1}$

$E[X^2] = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$

Solving for α & β

$\frac{1}{\beta} = m_1$ & $\frac{1}{\alpha} = \frac{m_2}{m_1}$

$\frac{1}{\beta} = m_1$

Substituting for $\frac{1}{\beta}$ in $\frac{1}{\alpha}$

$\left(\frac{m_2 m_1^2}{m_2 - m_1^2}\right) m_1^2 = m_2$

$\left(\frac{m_2 m_1^2}{m_2 - m_1^2}\right) m_1^2 = m_2$

$\left(\frac{1}{\alpha}\right) m_1^2 = \frac{m_2}{m_1} - m_1$

$\frac{1}{\alpha} = \frac{m_2}{m_1^2} - 1$

So

$$\boxed{\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum_i (X_i - \bar{X})^2}}$$

Substituting $\hat{\alpha}$ in $\beta = \frac{1}{m_1}$

$$\boxed{\hat{\beta} = \frac{1}{\bar{X}} = \frac{\frac{1}{n}\sum_i (X_i - \bar{X})^2}{\bar{X}}}$$

4) Likelihood

$L(\theta|x) = 1$, $0 < x < 1$

$L(\theta|x) = \frac{1}{\theta x}$, $0 < x < 1$

Thus, MLE is 0 if $\frac{1}{\theta x} \leq 1$ &

MLE is 1 if $\frac{1}{\theta x} >$

5. $f(x|\theta) = \theta x^{\theta}$ $0<x<1$

$0<\theta<\infty$

a) MLE of θ

$f(x|\theta) = \theta x^{\theta}$ $0<x<1$

$L(x|\theta) = \prod_i \theta x_i^{\theta}$ $0<x<1$

$= \theta^n \prod_i x_i^{\theta}$ $0<\theta<\infty$

$l(\theta|x) = n\log\theta + \theta\sum_i \log x_i$

To compute the likelihood, it depends on minimum. Only if $\theta \leq \min(x_i)$

$\therefore$ MLE $\hat{\theta} = \infty$

b) MLE of θ

$E[X] = \int_0^1 x f(x)\,dx$

$= \int_0^1 \theta x^{\theta}\,dx$

$= \int_0^1 \theta x^{\theta}\,dx$

$= \theta \frac{x^{\theta+1}}{\theta+1} \rightarrow$ (lowerpt)

$E[X] = \int_0^1 x f(x)\,dx$

$= \int_0^1 \theta x^{\theta}\,dx$

$= \theta \frac{x^{\theta+1}}{\theta+1} \rightarrow \infty$

$E[X^2] = \int_0^1 x^2 f(x)\,dx$

$= \int_0^1 \theta x^{\theta+2}\,dx$

$= \theta \frac{x^{\theta+3}}{\theta+3} \rightarrow \infty$

$E[X^2] = \int_0^1 x^2 f(x)\,dx$

$= \int_0^1 \theta x^{\theta+2}\,dx$

$= \theta \frac{x^{\theta+3}}{\theta+3} \rightarrow \infty$

Integral integral, $\therefore$ MOM does not exist

c) $\bar{X} = \frac{1}{n}\sum_i x_i$

$X_1 \ldots X_n$ iid $N(\theta, \sigma^2)$

θ $\sim N(\mu, \tau^2)$

Linear transformation of a normally distributed variable. Consider

$X_i \sim N(\theta, \sigma^2)$

$\frac{X_i}{n} \sim N\left(\frac{\theta}{n}, \frac{\sigma^2}{n^2}\right)$

Sum of iid normally distributed r.v.

$\sum_i \frac{X_i}{n} \sim \frac{\sigma^2}{n} \sim N\left(\frac{\theta}{n}, \frac{\sigma^2}{n^2}\right)$

Multiplying $\sum_i \frac{X_i}{n}$ by n $\sim N\left(\theta, \frac{\sigma^2}{n}\right)$

$\Rightarrow$ $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$

$f(\bar{X}, \theta) \sim N\left(\theta, \frac{\sigma^2}{n}\right)$

Normal distribution with mean θ & variance $\frac{\sigma^2}{n}$

d) Marginal distribution of $\bar{X}$

$f(\bar{X}) = \int f(\bar{X}, \theta) \cdot f(\theta)\,d\theta$

$f(\bar{X}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\frac{(\bar{X}-\theta)^2}{\frac{\sigma^2}{n}}} \cdot \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2}\frac{(\theta-\mu)^2}{\tau^2}}\,d\theta$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\frac{2\pi\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left[\frac{(\bar{X}^2-2\bar{X}\theta+\theta^2)}{\frac{\sigma^2}{n}} + \frac{(\theta^2-2\theta\mu+\mu^2)}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left[\frac{\bar{X}^2}{\frac{\sigma^2}{n}} - \frac{2\bar{X}\theta}{\frac{\sigma^2}{n}} + \frac{\theta^2}{\frac{\sigma^2}{n}} + \frac{\theta^2}{\tau^2} - \frac{2\theta\mu}{\tau^2} + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{\bar{X}^2}{\frac{\sigma^2}{n}} - \frac{2\bar{X}\theta}{\frac{\sigma^2}{n}} + \frac{\theta^2}{\frac{\sigma^2}{n}} + \frac{\theta^2}{\tau^2} - \frac{2\theta\mu}{\tau^2} + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} - 2\bar{X}\theta\frac{n}{\sigma^2} + \theta^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\frac{\mu}{\tau^2} + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu}{\tau^2} + \frac{\bar{X}n}{\sigma^2}\right) + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right) + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu + \bar{X}n\tau^2/\sigma^2}{\tau^2}\right) + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu + \bar{X}n\tau^2/\sigma^2}{\tau^2}\right) + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right) + \frac{\mu^2}{\tau^2}\right]}\,d\theta$

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$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \frac{\mu^2}{\tau^2}\right]} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right)\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \frac{\mu^2}{\tau^2}\right]} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right) + \left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right)^2 - \left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right)^2\right]}\,d\theta$

$= \frac{1}{2\pi\sqrt{\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left[\bar{X}^2\frac{n}{\sigma^2} + \frac{\mu^2}{\tau^2} - \left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right)^2\right]} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\theta^2\left(\frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\right) - 2\theta\left(\frac{\mu\sigma^2 + \bar{X}n\tau^2}{\sigma^2\tau^2}\right) +