STAT 510 — Mathematical Statistics

Assignment: Problem Set 4 **Due Date:** October 12 2022, 11:59 PM

Problem 1.

(i) We choose $g(x^*|x) = N(x, 1)$ as the proposal. Then the acceptance probability is

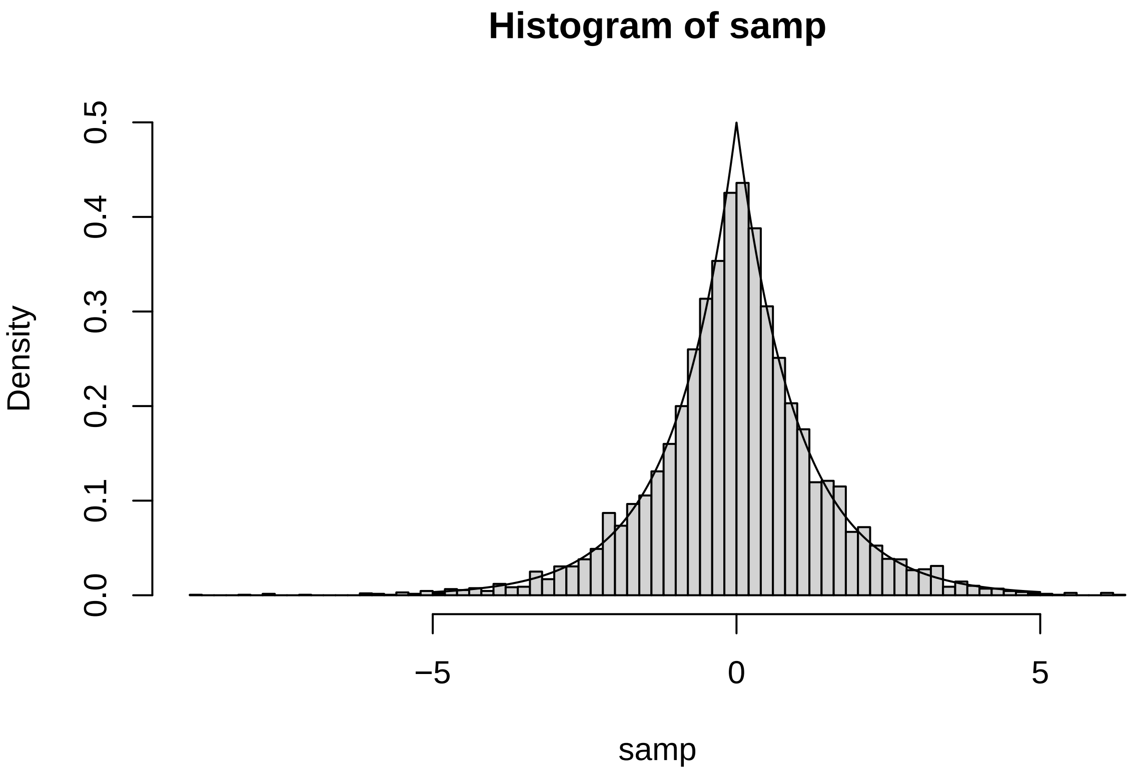
$$r = min(1, \frac{\pi(x^*)g(x|x^*)}{\pi(x)g(x^*|x)}) = min(1, e^{|x|-|x^*|})$$

```
set.seed(510)
n <- 1e4
samp <- rep(NA, n)
samp[1] <- rnorm(1)

r_accep <- function(x_star, x){
  exp(abs(x)-abs(x_star))
}

for (i in 2:n){
  proposal <- rnorm(1, samp[i-1], 1)
  r <- r_accep(proposal, samp[i-1])
  u <- runif(1)
  if (u < r){
    samp[i] <- proposal
  }
  else{
    samp[i] <- samp[i-1]
  }
}
```

```
hist(samp, breaks = 100, freq = FALSE, ylim=c(0, 0.5))
x <- seq(-5,5, length.out = 1e4)
lines(x, exp(-abs(x))/2)
```



(ii) Denote the cdf of X as F_X, since

$$F_X(x) = P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$$

then $X = F^{-1}(U)$ has the cdf as F.

Moreover, the cdf of double exponential distribution is

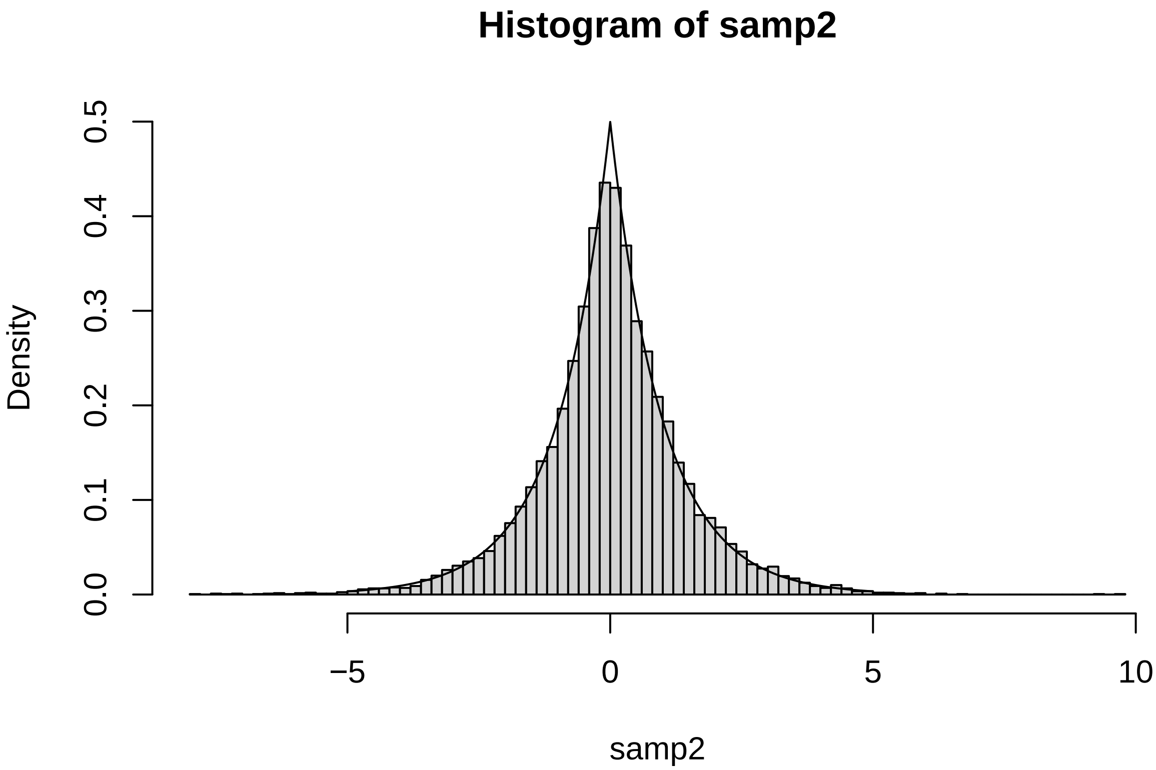
$$F(x) = \begin{cases} \frac{1}{2}e^{x} & x < 0 \\ 1 - \frac{1}{2}e^{-x} & x \geq 0 \end{cases}$$

Then

$$F^{-1}(u) = \begin{cases} log(2u) & u < \frac{1}{2} \\ -log(2-2u) & u \geq \frac{1}{2} \end{cases}$$

```
U = runif(n)
F_inv <- function(u){
  if(u<1/2){
    log(2*u)
  }
  else{
    -log(2-2*u)
  }
}
samp2 <- sapply(U, F_inv)
```

```r
hist(samp2, breaks = 100, freq = FALSE, ylim=c(0, 0.5))
x <- seq(-5,5, length.out = 1e4)
lines(x, exp(-abs(x))/2)
```



Problem 2.

The proposal is

$$g(x^*|x) = \begin{cases} 1, & if\ x = 1\ or\ 6 \\ 0.5, & if\ 2 \leq x \leq 5 \\ 0, & o.w. \end{cases}$$

Then the acceptance ratio is

$$r = min(1, \frac{p(x^*)g(x|x^*)}{p(x)g(x^*|x)}) = \begin{cases} 0.5, & if\ x = 1\ or\ 6 \\ 1, & o.w. \end{cases}$$

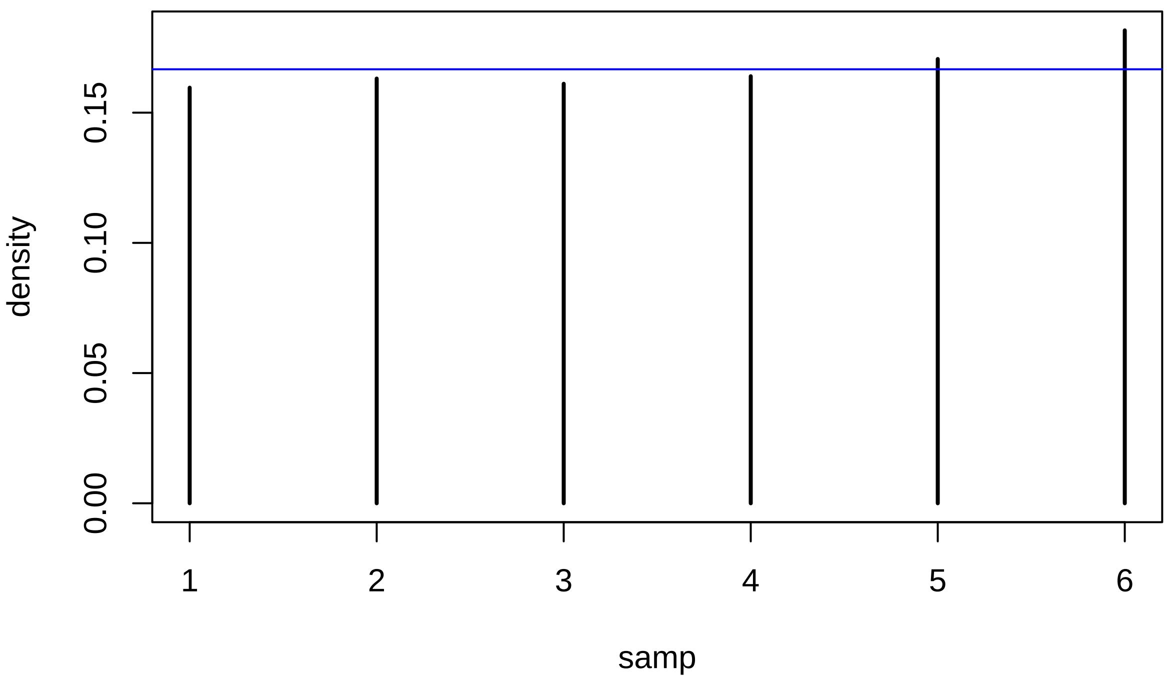
```r
n <- 1e4
samp <- rep(NA, n)
samp[1] <- 3

dice <- function(x){
  if(x==1){
    return(2)
  }else if(x==6){
    return(5)
  }else{
    u <- runif(1)
    if(u<=0.5){
      return(x-1)
    }else{
```

```
      return(x+1)
    }
  }
}

for (i in 2:n){
  proposal <- dice(samp[i-1])
  if (samp[i-1] %in% c(1,6)){
    u <- runif(1)
    if (u<=0.5){
      samp[i] <- samp[i-1]
    }else{
      samp[i] <- proposal
    }
  }else{
    samp[i] <- proposal
  }
}
```

```
plot(table(samp)/n, ylab="density")
abline(h=1/6, col="blue")
```



From the plot, we could notice that the density of the sample we get from MH algorithm is close to the density of rolling a fair 6-sided dice.

Problem 3.

Solution. Note that the pdf of X_i can be written as

$$f_{X_i}(x|\theta) = e^{i\theta - x}\mathbb{1}_{\{x \geq i\theta\}} = e^{i\theta - x}\mathbb{1}_{\{(x/i)\geq\theta\}}$$

then the joint pdf of $X_1, \cdots, X_n$ is

$$\begin{aligned} f(x_1, \cdots, x_n|\theta) &= \prod_{i=1}^{n} f_{X_i}(x_i|\theta) \\ &= e^{\theta \sum_i i - \sum_i x_i}\mathbb{1}_{\{x_1 \geq \theta, (x_2/2)\geq\theta, \cdots, (x_n/n)\geq\theta\}} \\ &= e^{-\sum_i x_i} e^{\theta \sum_i i}\mathbb{1}_{\{T \geq \theta\}} \end{aligned}$$

Thus $T = min_{1\leq i\leq n}(X_i/i)$ is a sufficient statistic for θ by Neyman's factorization theorem. ■

Problem 4.

Solution.

$$\begin{aligned} f(x_1, \cdots, x_n|\alpha, \beta) &= \prod_{i=1}^{n} f(x_i|\alpha, \beta) \\ &= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\beta \sum\limits_{i=1}^{n} x_i} \end{aligned}$$

Then $T = (\prod\limits_{i=1}^{n} x_i, \sum\limits_{i=1}^{n} x_i)$ is a sufficient statistic for (α, β) by Neyman's factorization theorem. ■

Problem 5.

Solution. From X_1, X_2 are iid $N(\theta, 1)$, we have $T = X_1 + 2X_2 \sim N(3\theta, 5)$. Let $p(\tilde{x}|\theta) = p(x_1, x_2|\theta)$ denote the joint pdf of $\tilde{X} = (X_1, X_2)$ and $q(t|\theta)$ denote the pdf of T, then we have

$$p(\tilde{x}|\theta) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1-\theta)^2 - \frac{1}{2}(x_2-\theta)^2}$$

$$q(t|\theta) = \frac{1}{\sqrt{10\pi}} e^{-\frac{1}{10}(x_1+2x_2-3\theta)^2}$$

Then we further have

$$\frac{p(\tilde{x}|\theta)}{q(t|\theta)} = \left(\frac{2}{5}\pi\right)^{-1/2} e^{-\frac{1}{10}(2x_1-x_2-\theta)^2}$$

which depends on θ. Thus T is not a sufficient statistics for θ. ■