Analytical derivation of zero rates

Shasa Foster

December 2019

1 Deriving periodically compounded zero rates

1.1 Wikipedia example

Given: 0.5-year spot rate, $Z_1 = 4\%$, and 1-year spot rate, $Z_2 = 4.3\%$ (we can get these rates from T-Bills which are zero-coupon); and the par rate on a 1.5-year semi-annual coupon bond, $C = 4.5\%$. We then use these rates to calculate the 1.5 year spot rate. We solve the 1.5 year spot rate, Z_3, for semi-annual compounding by the formula below, resulting in a Z_3 of 5.32%. [1]

$$100 = \frac{\frac{4.5}{2}}{(1+\frac{4\%}{2})^1} + \frac{\frac{4.5}{2}}{(1+\frac{4.3\%}{2})^2} + \frac{100+\frac{4.5}{2}}{(1+\frac{Z_3\%}{2})^3} \tag{1}$$

1.2 Generalising the equation

We can generalize the previous example by defining the following parameters:
n is the compounding frequency per year.
p is the coupon period length in years.
C is the nominal coupon rate.
t is the time in the future in years.
T is the time til maturity in years.

	n	p
Monthly	1	0.0833
Quarterly	3	0.25
Semi Annual	6	0.5
Annual	12	1

Table 1: Numeric values of n and p for various frequencies

Replacing the coupon rate of 4.5% with C and the semi-annual payment frequency (of value 0.5) with p we get:

$$100 = \frac{Cp}{(1+\frac{4\%}{n})^n} + \frac{Cp}{(1+\frac{4.3\%}{n})^{2n}} + \frac{100+Cp}{(1+\frac{Z_3\%}{n})^{3n}} \tag{2}$$

We can then write in the summation notation. We can sum over all payment's except for the final payment, as per formula (1), the notional amount is paid back at maturity.

$$100 = \frac{100 + Cp}{(1 + \frac{Z_T}{n})^{Tn}} + \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}} \tag{3}$$

1.3 Rearrange and solve for Z_T

$$\frac{100 + Cp}{(1 + \frac{Z_T}{n})^{Tn}} = 100 - \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}} \tag{4}$$

$$\frac{1}{(1 + \frac{Z_T}{n})^{Tn}} = \frac{100 - \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}}}{100 + Cp} \tag{5}$$

$$(1 + \frac{Z_T}{n})^{Tn} = \frac{100 + Cp}{100 - \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}}} \tag{6}$$

$$1 + \frac{Z_T}{n} = \sqrt[Tn]{\frac{100 + Cp}{100 - \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}}}} \tag{7}$$

$$Z_T = n \sqrt[Tn]{\frac{100 + Cp}{100 - \sum_{t=p}^{T-p} \frac{Cp}{(1 + \frac{Z_t}{n})^{tn}}}} - n \tag{8}$$

2 Deriving continuously compounded zero rates

We observe the theorem:

$$\lim_{x \to \infty} \frac{1}{(1 + \frac{r}{x})^x} = \frac{1}{e^{rx}}$$

We observe that $\frac{1}{(1+\frac{r}{x})^x}$ looks look the familiar discounting term in our equations above. We can hence substitute $\frac{1}{e^{rx}}$ for occurrences of the form $\frac{1}{(1+\frac{r}{x})^x}$. We will start by applying continuous compounding to our example in equation (1).

$$100 = \frac{\frac{4.5}{2}}{e^{4\%}} + \frac{\frac{4.5}{2}}{e^{8.6\%}} + \frac{100 + \frac{4.5}{2}}{e^{3Z_3}} = \frac{4.5}{2}(e^{-4} + e^{-8.6}) + \frac{100 + \frac{4.5}{2}}{e^{3Z_3}} \tag{9}$$

2.1 Generalising the equation

We can generalize the previous example.

$$100 = Cp(e^{-Z_1 t1} + e^{-Z_2 t2} + (100 + Cp)e^{-Z_3 t3} \tag{10}$$

We can then write in the summation notation.

$$100 = (100 + Cp)e^{-r_T T} + Cp \sum_{t=p}^{T-p} e^{-r_t t} \tag{11}$$

2.2 Rearrange and solve for Z_T

$$(100 + Cp)e^{-r_T T} = Cp \sum_{t=p}^{T-p} e^{-r_t t} - 100 \tag{12}$$

$$(100 + Cp)e^{-r_T T} = Cp \sum_{t=p}^{T-p} e^{-r_t t} - 100 \tag{13}$$

$$e^{-r_T T} = \frac{Cp \sum_{t=p}^{T-p} e^{-r_t t} - 100}{100 + Cp} \tag{14}$$

$$-r_T T = \ln(\frac{Cp \sum_{t=p}^{T-p} e^{-r_t t} - 100}{100 + Cp}) \tag{15}$$

$$r_T = -\frac{\ln(\frac{Cp \sum_{t=p}^{T-p} e^{-r_t t} - 100}{100 + Cp})}{T} \tag{16}$$

References

[1] *Bootstrapping (finance)*. https://en.wikipedia.org/wiki/Bootstrapping_(finance)/ [Accessed: 2019-12-8].