MAT 1750

A Fast Multipole Algorithm for the Helmholtz Equation in 3D

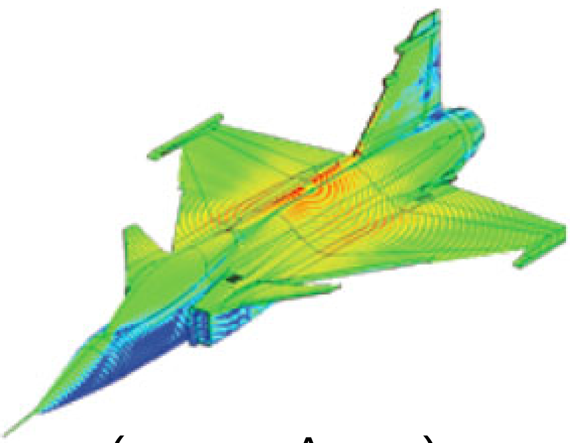
Shashwat Sharma

Dec 04, 2018

Background

- Boundary integral equation methods:
  - ✔ Dimensionality reduction
  - ✔ Linear system of equations
  - ✗ Large problems --> dense matrices
    - ✗ Storage
    - ✗ Factorization
- Acceleration methods needed

(source: FEKO)



(source: Ansys)

Formulation

- Helmholtz equation in 3D:

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = \vec{f}(\vec{r})$$

- In the context of electromagnetics:

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\vec{J}(\vec{r})\mu$$

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- We want to solve for the associated electric field:

$$\vec{E}\left(\vec{r}\right) = -j\omega\mu \int_{\mathcal{S}} dS' \, \overline{\overline{\mathbf{G}}}\left(\vec{r}, \vec{r}'\right) \cdot \vec{J}\left(\vec{r}'\right)$$

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- The dyadic Green's function can be simplified:

$$\vec{E}\left(\vec{r}\right) = -j\omega\mu \int_{\mathcal{S}} dS' \left(1 + \frac{\nabla\nabla\cdot}{k^2}\right) G\left(\vec{r}, \vec{r}'\right) \vec{J}\left(\vec{r}'\right)$$

Acceleration

- 3D scalar Green's function:

$$G\left(\vec{r}, \vec{r}'\right) = \frac{\mathrm{e}^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

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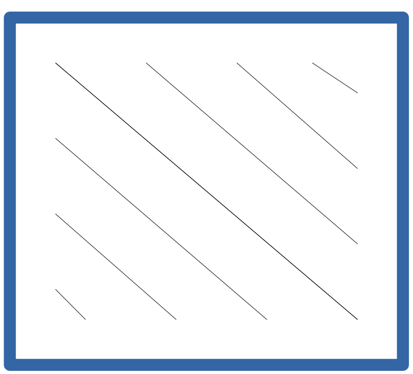

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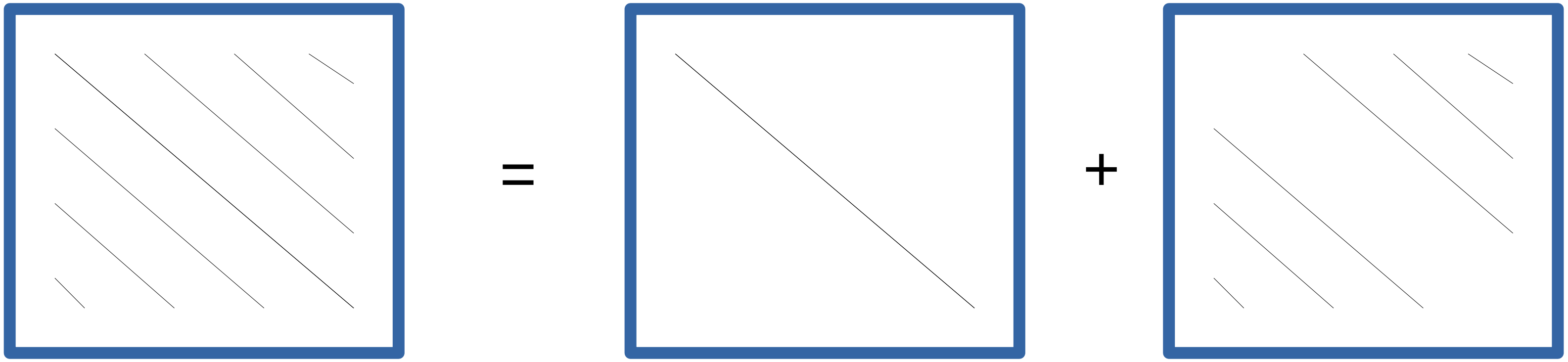

Fast Multipole Algorithm (FMA)

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$$G\left(\vec{r},\vec{r}'\right) = \frac{\mathrm{e}^{-jk\left|\vec{r}-\vec{r}'\right|}}{\left|\vec{r}-\vec{r}'\right|}$$

- Can be decomposed:

$$\frac{\mathrm{e}^{-jk\left|\vec{r}-\vec{r}'\right|}}{\left|\vec{r}-\vec{r}'\right|} = \int_1 T\left(\vec{r}',\hat{n}_r\right) T_L\left(k,\hat{n}_r,\vec{r}_{ab}\right) R\left(\vec{r},\hat{n}_r\right) dS$$

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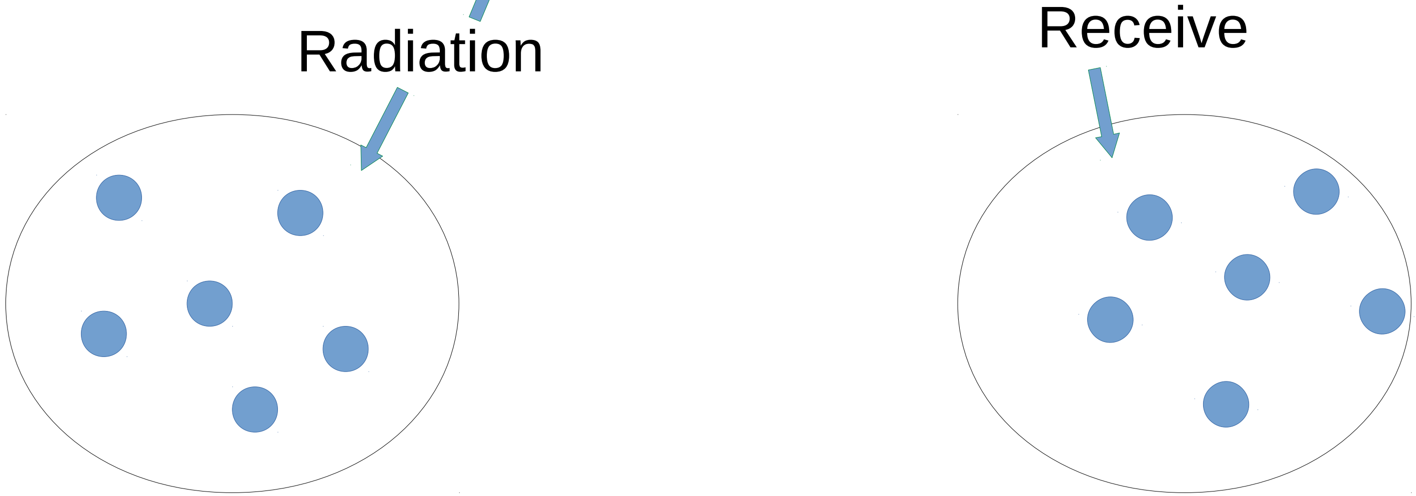

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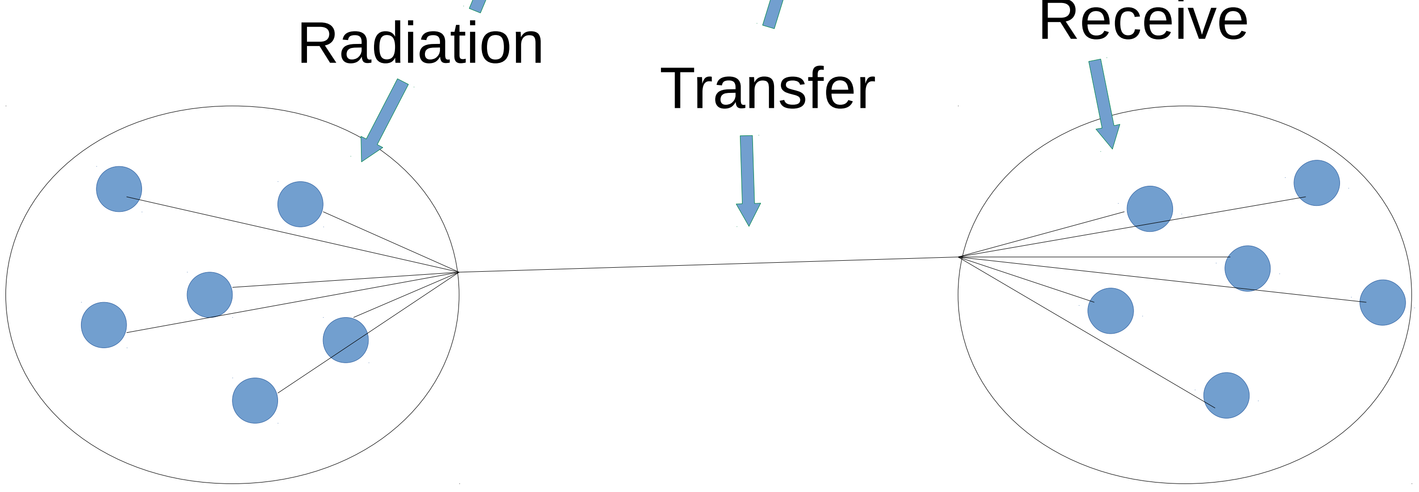

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- With radiation, receive and transfer functions:

$$T\left(\vec{f}_n, \hat{n}_r\right) = \left[1 - \hat{n}_r \hat{n}_r \cdot\right] \int_{\vec{f}_n} \vec{f}_n\left(\vec{r}'\right) \mathrm{e}^{jk\hat{n}_r \cdot \vec{r}'} d\vec{r}'$$

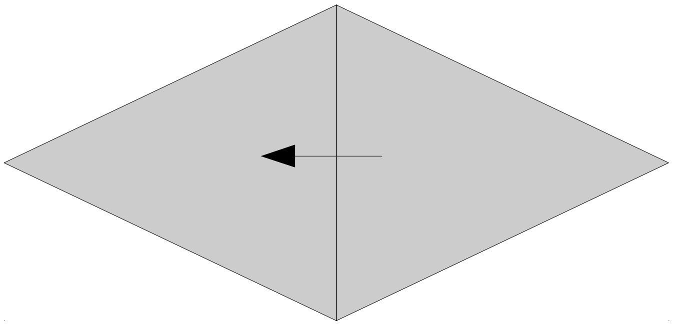
$$R\left(\vec{f}_m, \hat{n}_r\right) = j\omega\mu \int_{\vec{f}_m} \vec{f}_m\left(\vec{r}\right) \mathrm{e}^{-jk\hat{n}_r \cdot \vec{r}} d\vec{r}$$

$$T_L\left(k, \hat{n}_r, \vec{r}_{ab}\right) = \frac{k}{(4\pi)^2} \sum_{l=0}^{L} (-j)^{l+1} (2l+1)\, h_l^{(2)}\left(k\left|\vec{r}_{ab}\right|\right) P_l\left(\hat{n}_r \cdot \vec{r}_{ab}\right)$$

Fast Multipole Algorithm (FMA)

- Surface discretization:

Raviart-Thomas
basis functions



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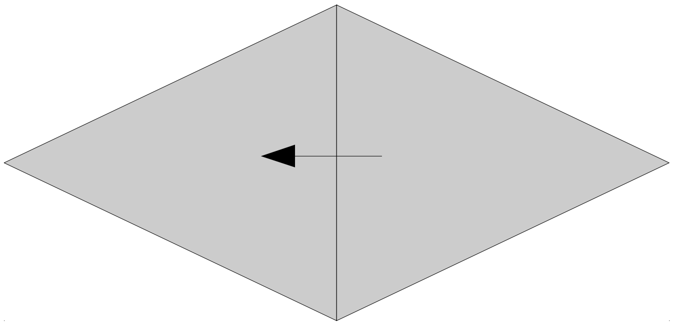
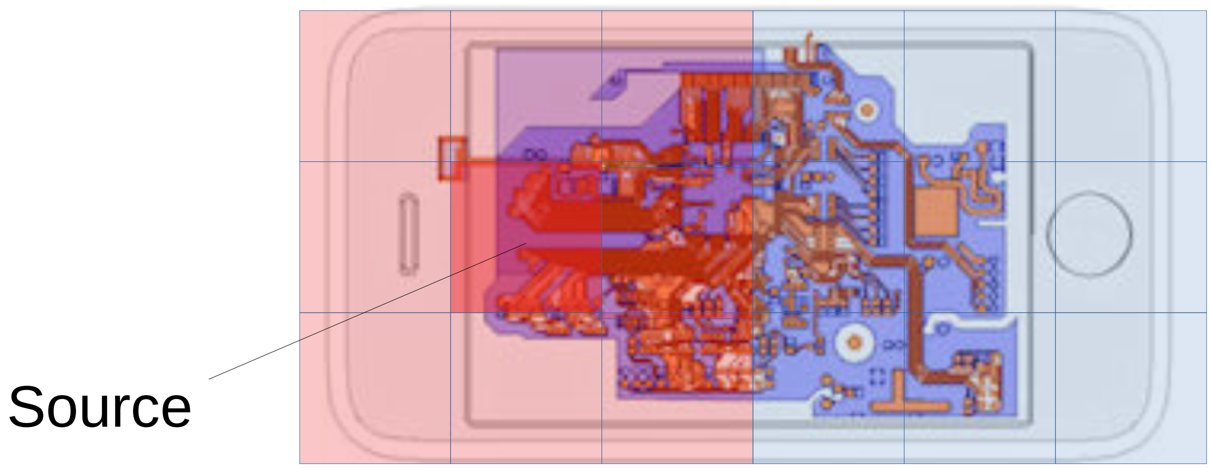


- Grouping of basis functions:



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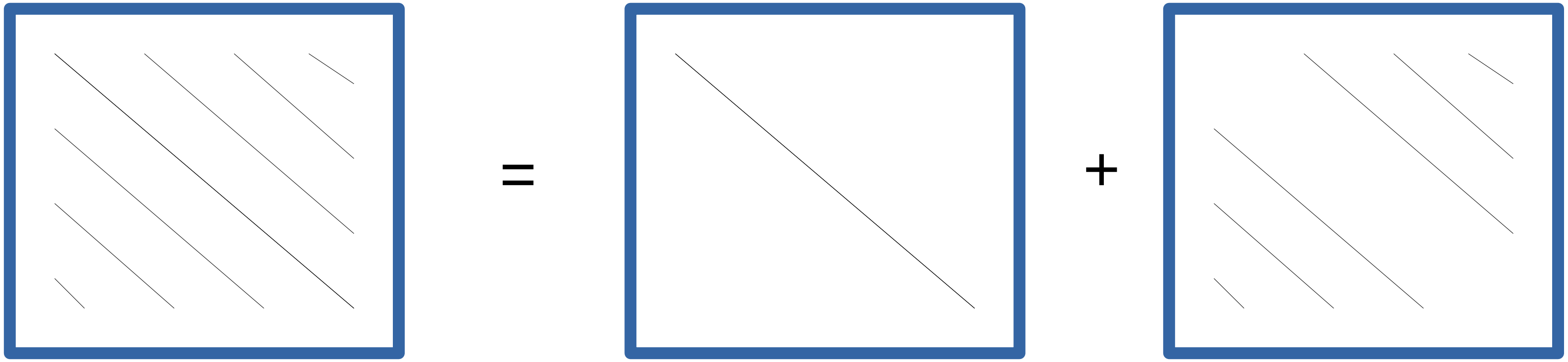


Already have:

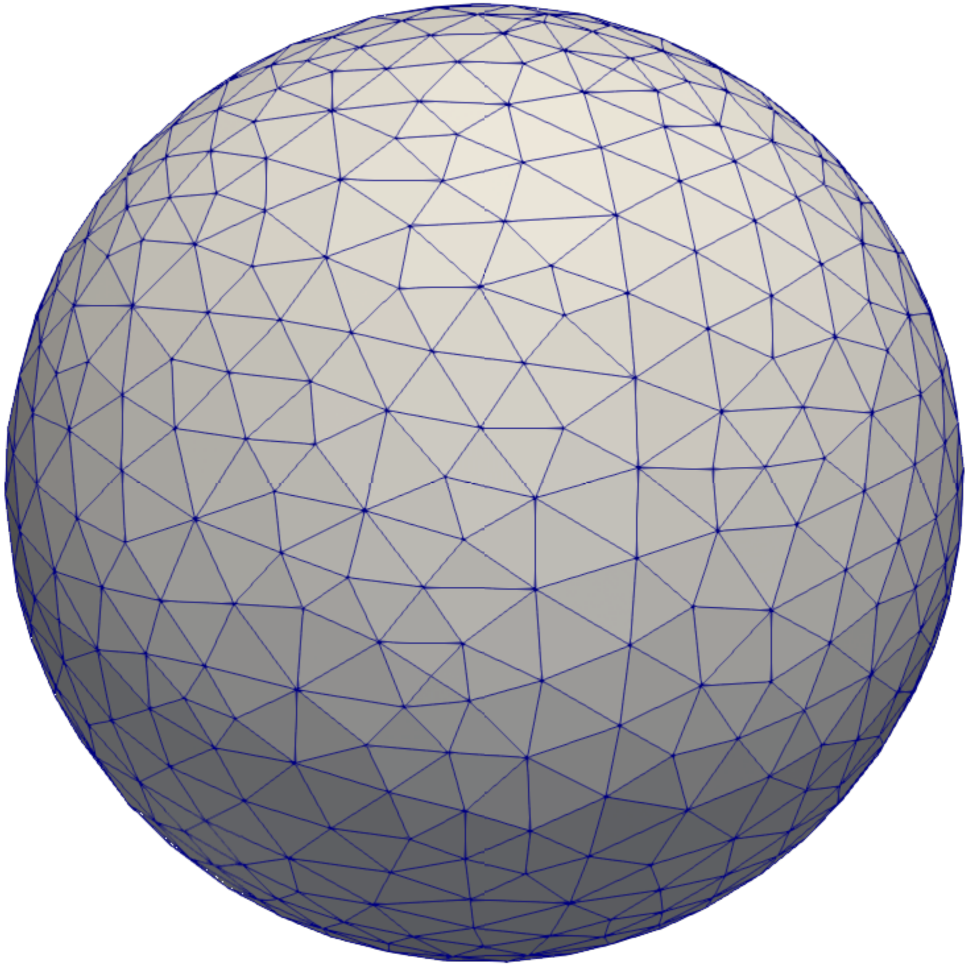
- Meshing tools
- Integration over basis functions
- Direct solver for dense matrices

This project:

- FMA cube subdivisions
- Transfer, radiation, receive functions
- Iterative solver

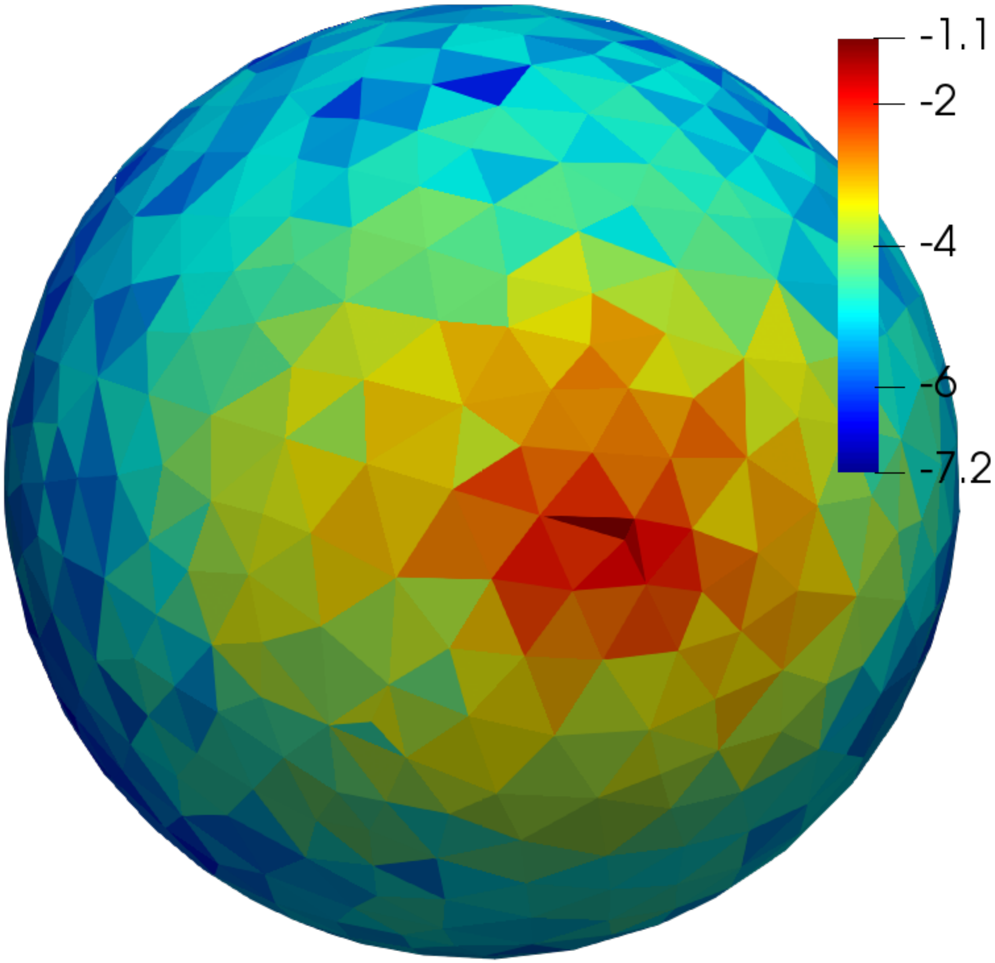
Test Case

3D sphere with 4120 basis functions



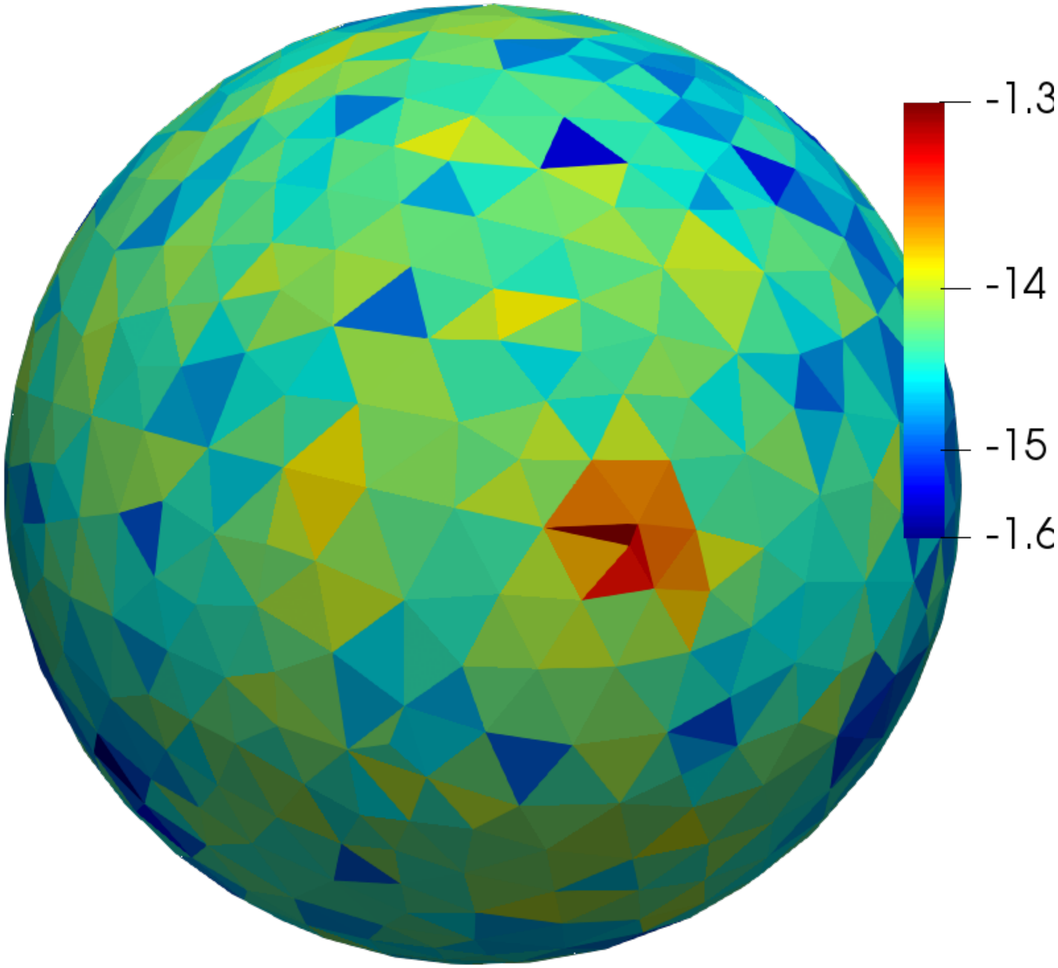
Test Case

Solution (current density) from direct solver



Test Case

Solution (current density) with FMA



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Iterative solver with FMA – Issues:

- Convergence

  ➔ Solution only converged to 30% residual norm error

  - Tried diagonal and ILU preconditioners

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  - ➔ Formulation itself needs improvement
    - cond(A) ~ 10^14