CS 564: Sequence Learning: Hidden Markov Model

Observations leading to why probability is needed

- Many intelligence tasks are sequence labeling tasks
- Tasks carried out in layers
- Within a layer, there are limited windows of information
- This naturally calls for strategies for dealing with uncertainty
- Probability and Markov process give a way

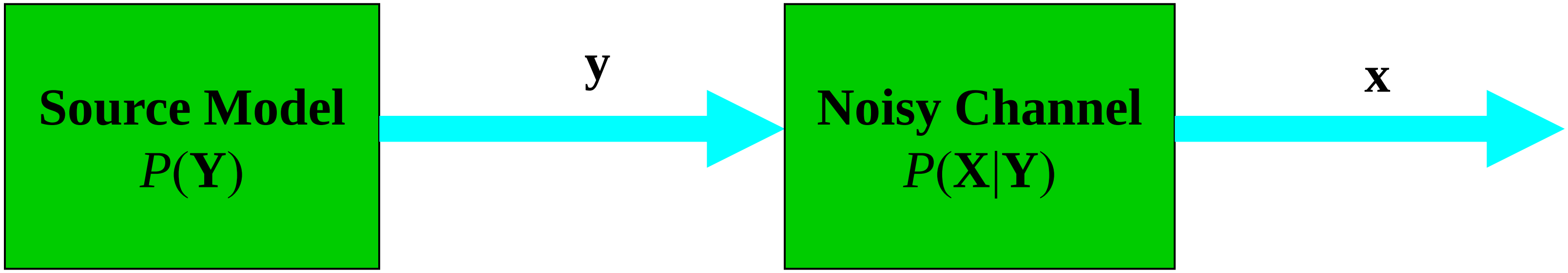
"I went with my friend to the bank to withdraw some money, but was disappointed to find it closed"

POS	Bank (N/V)	closed (V/ adj)
Sense	Bank (financial institution)	withdraw(take away)
Pronoun drop	But I/friend/money/bank	was disappointed
SCOPE	With	my friend
Co-referencing	It -> bank	

Different Models for Sequence Learning

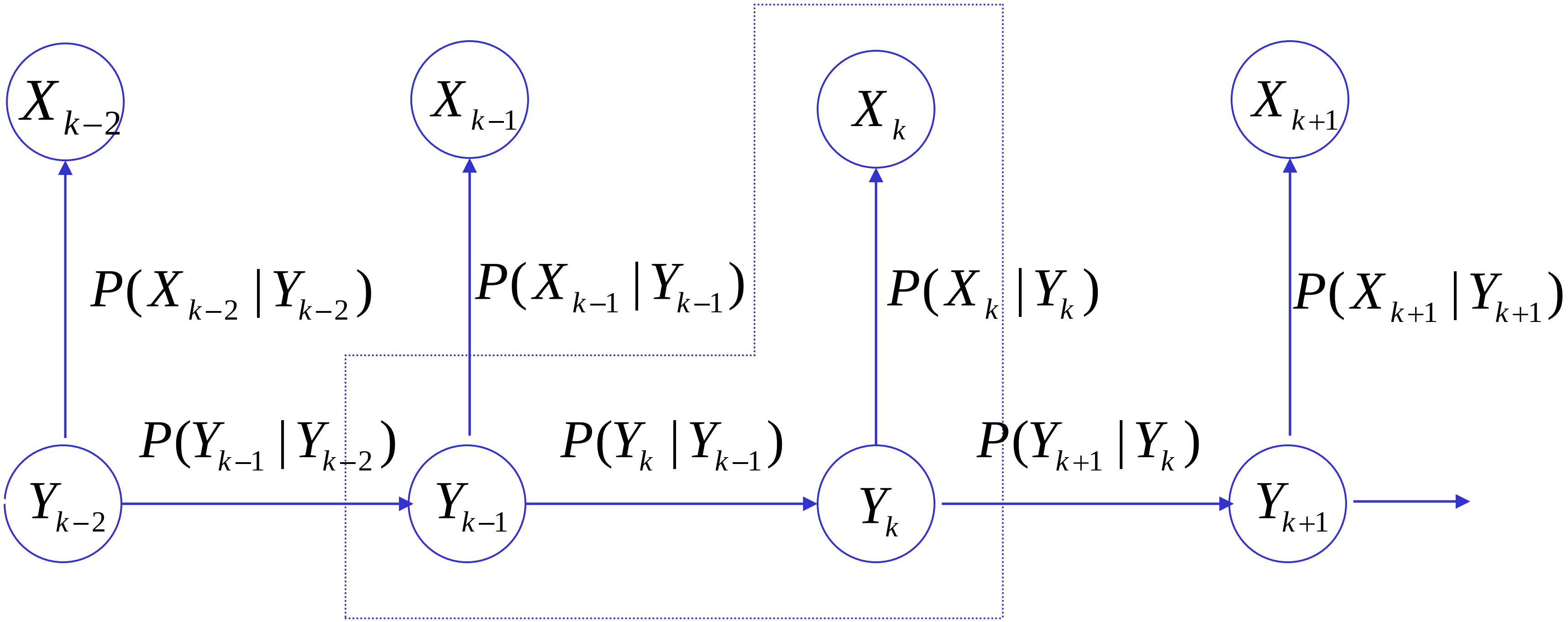
- HMM
- Maximum Entropy Markov Models
- Conditional Random Fields

Hidden Markov Model (HMM) : Generative Modeling



$$P(\mathbf{y}) = \prod_i P(y_i \mid y_{i-1}) \qquad P(\mathbf{x} \mid \mathbf{y}) = \prod_i P(x_i \mid y_i)$$

Dependency (1st order)



Markov Models

- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$

•Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

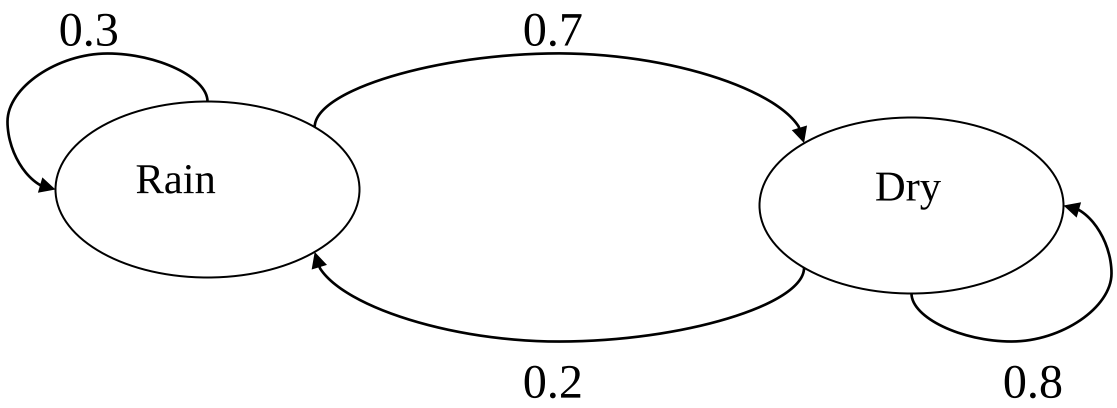
•To define Markov model, the following probabilities have to be specified: *transition probabilities*

and *initial probabilities*

$$a_{ij} = P(s_i \mid s_j)$$

$$\pi_i = P(s_i)$$

Example of Markov Model



- Two states : ‘Rain’ and ‘Dry’.
- Transition probabilities: P(‘Rain’ | ‘Rain’)=0.3 , P(‘Dry’ | ‘Rain’)=0.7 , P(‘Rain’ | ‘Dry’)=0.2, P(‘Dry’ | ‘Dry’)=0.8

- Initial probabilities: say P(‘Rain’)=0.4 , P(‘Dry’)=0.6 .

Calculation of sequence probability

- By Markov chain property, probability of state sequence can be found by the formula:

$$P(s_{i1}, s_{i2}, \ldots, s_{ik}) = P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1})$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1}) = \ldots$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{ik-1} \mid s_{ik-2}) \ldots P(s_{i2} \mid s_{i1}) P(s_{i1})$$

•Suppose we want to calculate a probability of a sequence of states in our example, {'Dry','Dry','Rain',Rain'}.

P({'Dry','Dry','Rain',Rain'}) =
P('Rain'|'Rain') P('Rain'|'Dry') P('Dry'|'Dry') P('Dry')=
= 0.3*0.2*0.8*0.6

Hidden Markov models

- Set of states: $\{s_1, s_2, \ldots, s_N\}$

- Process moves from one state to another generating a sequence of states :

$$s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$$

- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- States are not visible, but each state randomly generates one of M observations (or visible states)

$$\{v_1, v_2, \ldots, v_M\}$$

Hidden Markov Models

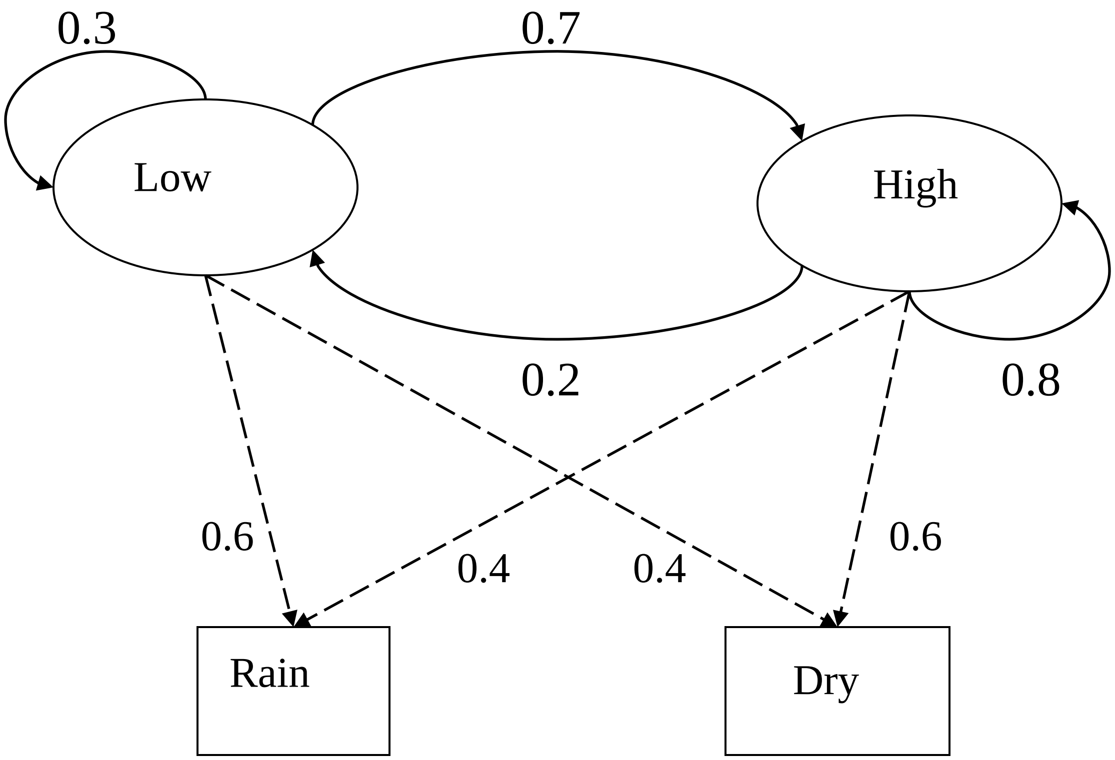
To define hidden Markov model, the following probabilities have to be specified:

- *matrix of transition probabilities* $A=(a_{ij})$, $a_{ij}= P(s_i \mid s_j)$

- *matrix of observation probabilities* $B=(b_i(v_m))$, $b_i(v_m) = P(v_m \mid s_i)$
- *vector of initial probabilities* $\pi=(\pi_i)$, $\pi_i = P(s_i)$

- Model is represented by $M=(A, B, \pi)$

Example of Hidden Markov Model



Example of Hidden Markov Model

- *Two states* : ‘Low’ and ‘High’ atmospheric pressure
- *Two observations* : ‘Rain’ and ‘Dry’
- *Transition probabilities*:
  P(‘Low’|‘Low’)=0.3, P(‘High’|‘Low’)=0.7,
  P(‘Low’|‘High’)=0.2, P(‘High’|‘High’)=0.8
- *Observation probabilities*:
  P(‘Rain’|‘Low’)=0.6   P(‘Dry’|‘Low’)=0.4
  P(‘Rain’|‘High’)=0.4   P(‘Dry’|‘High’)=0.36
- Initial probabilities: say P(‘Low’)=0.4 , P(‘High’)=0.6

Calculation of observation sequence probability

•Suppose we want to calculate a probability of a sequence of observations in our example, {'Dry','Rain'}

•Consider all possible hidden state sequences:
P({'Dry','Rain'})=P({'Dry','Rain'},{'Low','Low'}) + P({'Dry','Rain'}, {'Low','High'}) + P({'Dry','Rain'} , {'High','Low'}) + P({'Dry','Rain'} , {'High','High'})

where first term is :
P({'Dry','Rain'} , {'Low','Low'})=
P({'Dry','Rain'} | {'Low','Low'}) P({'Low','Low'}) =
P('Dry'|'Low')P('Rain'|'Low') P('Low')P('Low'|'Low)
= 0.4*0.4*0.6*0.4*0.3

Main issues using HMMs

•**Evaluation problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O

• **Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence O

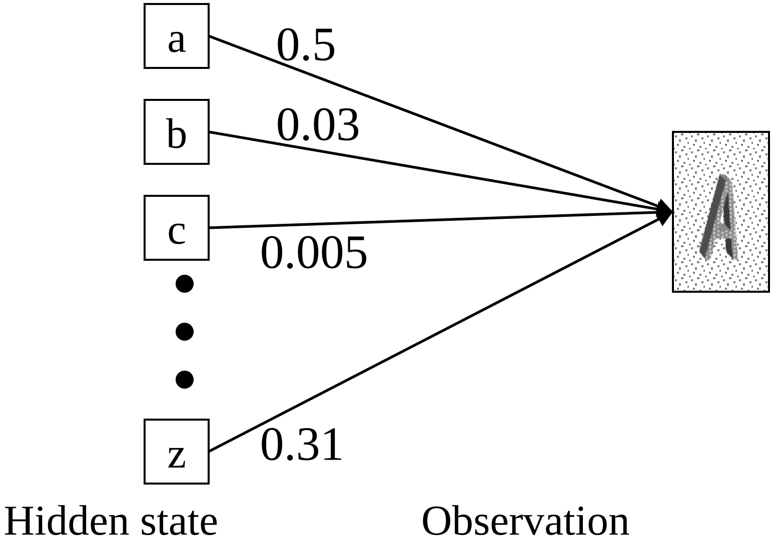
•**Learning problem.** Given some training observation sequences $O=o_1 o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data

$O=o_1 \ldots o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

Word recognition example(1)

- Typed word recognition, assume all characters are separated

- Character recognizer outputs probability of the image being particular character, P(image|character)



Word recognition example(2)

- Hidden states of HMM = characters

- Observations = typed images of characters segmented from the image v_α
- Note that there is an infinite number of observations

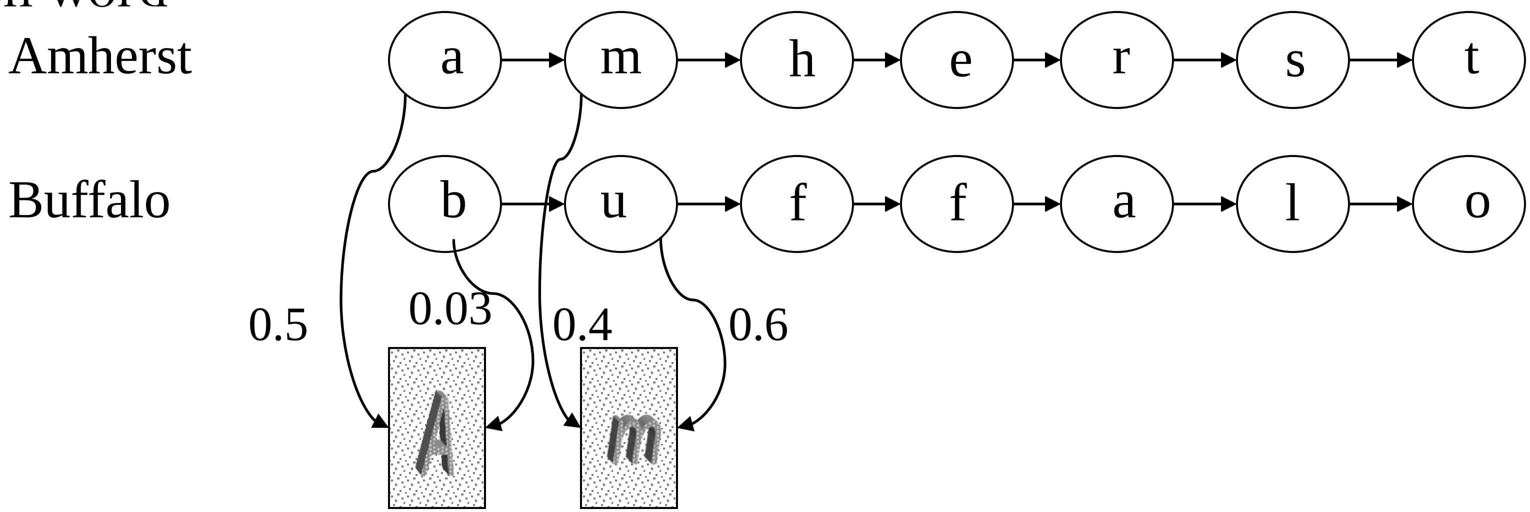
- Observation probabilities = character recognizer scores

$$B = \left(b_i(v_\alpha)\right) = \left(P(v_\alpha \mid s_i)\right)$$

- Transition probabilities will be defined differently in two subsequent models

Word recognition example(3)

•If lexicon is given, we can construct separate HMM models for each lexicon word

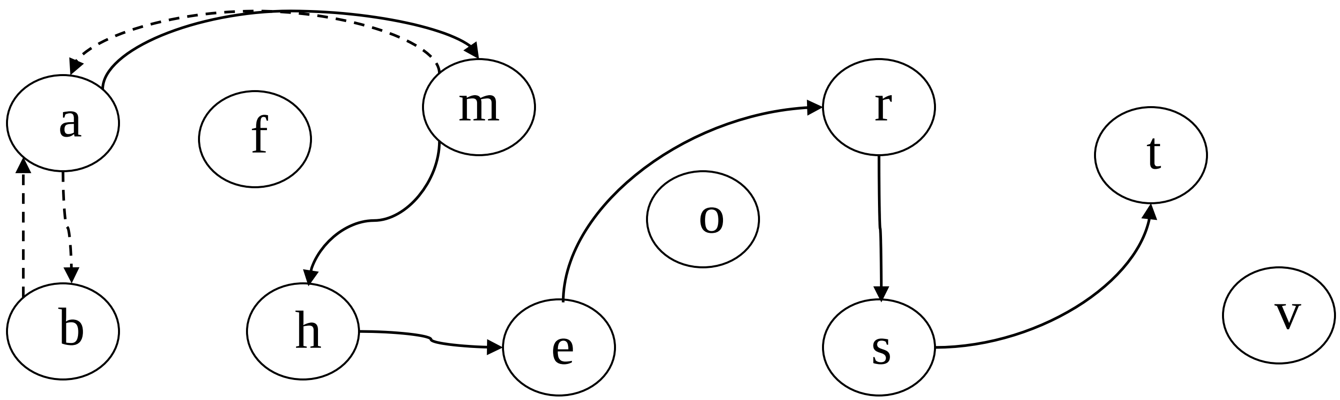


• Here recognition of word image is equivalent to the problem of evaluating a few HMM models

•This is an application of **Evaluation problem**

Word recognition example(4)

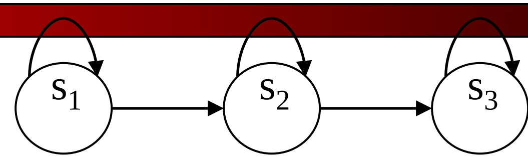
- We can construct a single HMM for all words
- Hidden states = all characters in the alphabet
- Transition probabilities and initial probabilities: calculated from language model
- Observations and observation probabilities (as before)



- ***Here we have to determine the best sequence of hidden states, the one that most likely produced word image***
- This is an application of ***Decoding problem***

Exercise: Character recognition with HMM(1)

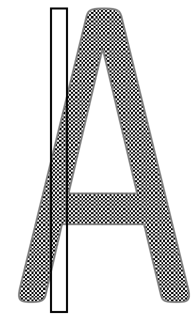
The structure of hidden states:



- Observation = number of islands in the vertical slice
- HMM for character 'A' :

Transition probabilities: $\{a_{ij}\}= \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

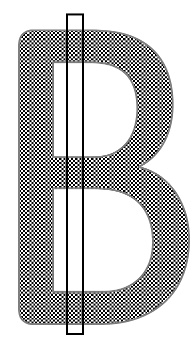
Observation probabilities: $\{b_{jk}\}= \begin{pmatrix} .9 & .1 & 0 \\ .1 & .8 & .1 \\ .9 & .1 & 0 \end{pmatrix}$



- HMM for character 'B' :

Transition probabilities: $\{a_{ij}\}= \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

Observation probabilities: $\{b_{jk}\}= \begin{pmatrix} .9 & .1 & 0 \\ 0 & .2 & .8 \\ .6 & .4 & 0 \end{pmatrix}$



Exercise: character recognition with HMM(2)

- Suppose that after character image segmentation the following sequence of island numbers in 4 slices were observed:

  { 1, 3, 2, 1}

- Which HMM is more likely to generate this observation sequence, HMM for ‘A’ or HMM for ‘B’ ?

Exercise: character recognition with HMM(3)

Consider likelihood of generating given observation for each possible sequence of hidden states:

- HMM for character 'A':

Hidden state sequence	Transition probabilities		Observation probabilities	
$s_1 \to s_1 \to s_2 \to s_3$	$.8 * .2 * .2$	$*$	$.9 * 0 * .8 * .9$	$= 0$
$s_1 \to s_2 \to s_2 \to s_3$	$.2 * .8 * .2$	$*$	$.9 * .1 * .8 * .9$	$= 0.0020736$
$s_1 \to s_2 \to s_3 \to s_3$	$.2 * .2 * 1$	$*$	$.9 * .1 * .1 * .9$	$= 0.000324$
			Total	$= 0.0023976$

- HMM for character 'B':

Hidden state sequence	Transition probabilities		Observation probabilities	
$s_1 \to s_1 \to s_2 \to s_3$	$.8 * .2 * .2$	$*$	$.9 * 0 * .2 * .6$	$= 0$
$s_1 \to s_2 \to s_2 \to s_3$	$.2 * .8 * .2$	$*$	$.9 * .8 * .2 * .6$	$= 0.0027648$
$s_1 \to s_2 \to s_3 \to s_3$	$.2 * .2 * 1$	$*$	$.9 * .8 * .4 * .6$	$= 0.006912$
			Total	$= 0.0096768$

Evaluation Problem

•**Evaluation problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\ o_2\ \dots\ o_K$, calculate the probability that model M has generated sequence O

•Trying to find probability of observations $O=o_1\ o_2\ \dots\ o_K$ by means of considering all hidden state sequences (*as was done in example*) is impractical:

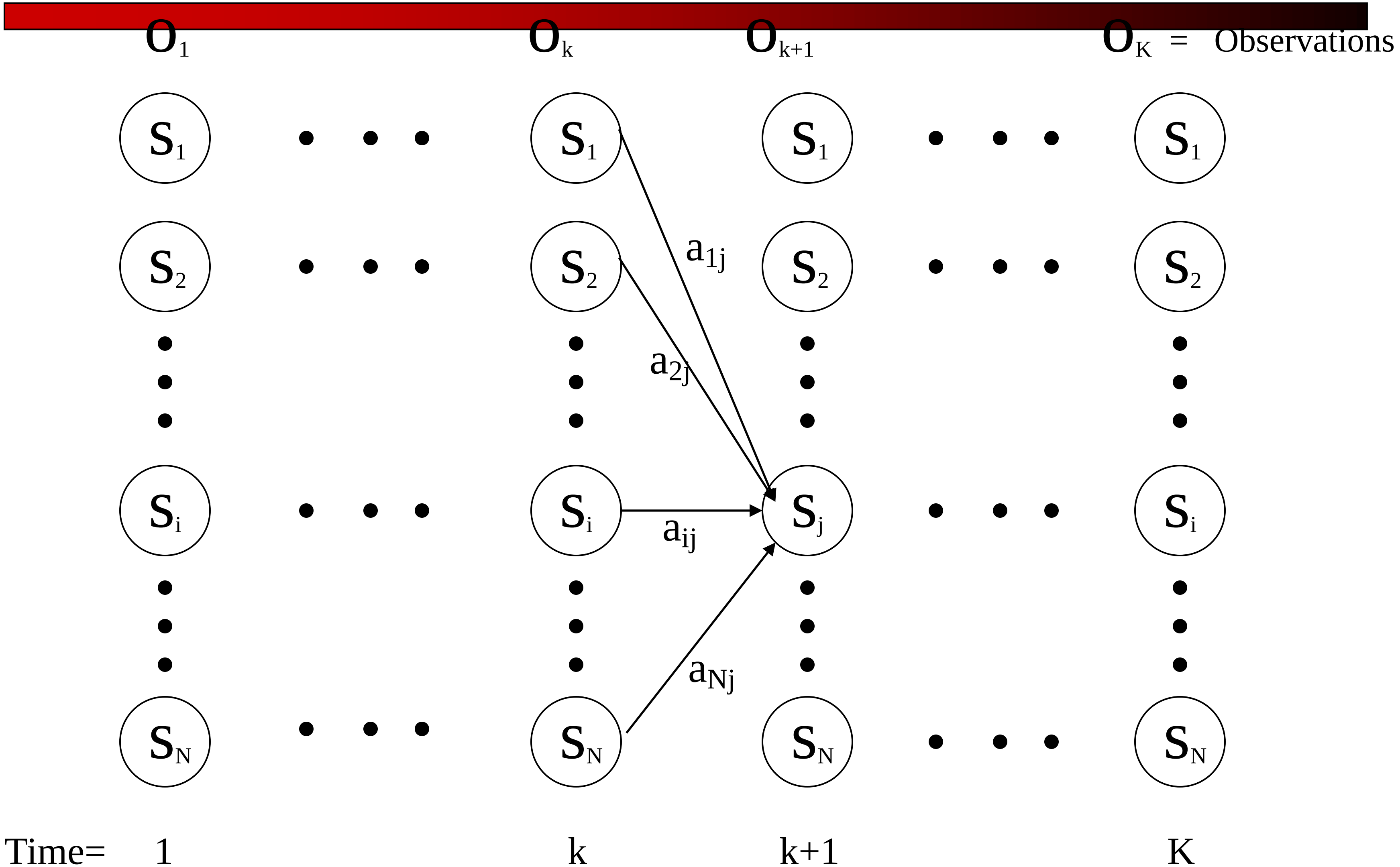
N^K hidden state sequences-exponential complexity

• Use **Forward-Backward HMM algorithms** for efficient calculations

• Define the forward variable $\alpha_k(i)$ as the joint probability of the partial observation sequence $o_1\ o_2\ \dots\ o_k$ and the hidden state s_i at time k:

$$\alpha_k(i)= P(o_1\ o_2\ \dots\ o_k\ , q_k= s_i\ )$$

Trellis representation of an HMM



Forward recursion for HMM

Initialization:

$$\alpha_1(i) = P(o_1, q_1 = s_i) = \pi_i \, b_i(o_1), \; 1 <= i <= N$$

- Forward recursion:

$$\alpha_{k+1}(i) = P(o_1 \, o_2 \dots o_{k+1}, q_{k+1} = s_j) =$$
$$\Sigma_i P(o_1 \, o_2 \dots o_{k+1}, q_k = s_i, q_{k+1} = s_j) =$$
$$\Sigma_i P(o_1 \, o_2 \dots o_k, q_k = s_i) \, a_{ij} \, b_j(o_{k+1}) =$$
$$[\Sigma_i \, \alpha_k(i) \, a_{ij}] \, b_j(o_{k+1}), \quad 1 <= j <= N, \; 1 <= k <= K-1$$

- Termination:

$$P(o_1 \, o_2 \dots o_K) = \Sigma_i P(o_1 \, o_2 \dots o_K, q_K = s_i) = \Sigma_i \, \alpha_K(i)$$

- Complexity :

N^2K operations

Backward recursion for HMM

• Define the backward variable $\beta_k(i)$ as the joint probability of the partial observation sequence $o_{k+1}\, o_{k+2} \ldots o_K$ given that the hidden state at time k is s_i : $\beta_k(i)= P(o_{k+1}\, o_{k+2} \ldots o_K \mid q_k= s_i)$

• <u>Initialization:</u>

$\beta_K(i)= 1$, $1<=i<=N$.

• <u>Backward recursion:</u>

$$\beta_k(j)= P(o_{k+1}\, o_{k+2} \ldots o_K \mid q_k= s_j) =$$
$$\Sigma_i\, P(o_{k+1}\, o_{k+2} \ldots o_K\, , q_{k+1}= s_i \mid q_k= s_j) =$$
$$\Sigma_i\, P(o_{k+2}\, o_{k+3} \ldots o_K \mid q_{k+1}= s_i)\, a_{ji}\, b_i\, (o_{k+1}) =$$
$$\Sigma_i\, \beta_{k+1}(i)\, a_{ji}\, b_i\, (o_{k+1}) , \quad 1<=j<=N,\ 1<=k<=K-1 .$$

• <u>Termination:</u>

$$P(o_1\, o_2 \ldots o_K) = \Sigma_i\, P(o_1\, o_2 \ldots o_K\, , q_1= s_i) =$$
$$\Sigma_i\, P(o_1\, o_2 \ldots o_K \mid q_1= s_i)\, P(q_1= s_i) = \Sigma_i\, \beta_1(i)\, b_i\, (o_1)\, \pi_i$$

Decoding problem

•**Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence.

•We want to find the state sequence $Q= q_1 \ldots q_K$ which maximizes $P(Q \mid o_1 o_2 \ldots o_K)$, or equivalently $P(Q, o_1 o_2 \ldots o_K)$.

• Brute force consideration of all paths takes exponential time. Use efficient **Viterbi algorithm** instead.

• Define variable $\delta_k(i)$ as the maximum probability of producing observation sequence $o_1 o_2 \ldots o_k$ when moving along any hidden state sequence $q_1 \ldots q_{k-1}$ and getting into $q_k = s_i$.

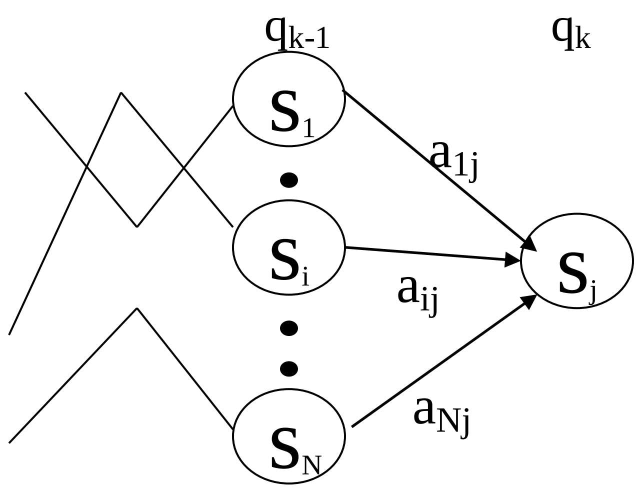
$$\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k = s_i, o_1 o_2 \ldots o_k)$$

where max is taken over all possible paths $q_1 \ldots q_{k-1}$.

Viterbi algorithm (1)

•General idea:

if best path ending in $q_k = s_j$ goes through $q_{k-1} = s_i$ then it should coincide with best path ending in $q_{k-1} = s_i$



• $\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k = s_j, o_1 o_2 \ldots o_k) =$
$\max_i [\, a_{ij}\, b_j(o_k) \max P(q_1 \ldots q_{k-1} = s_i, o_1 o_2 \ldots o_{k-1}) \,]$

• To backtrack best path keep info that predecessor of s_j was s_i.

Viterbi algorithm (2)

- Initialization:

$$\delta_1(i) = \max P(q_1 = s_i\ ,\ o_1) = \pi_i\, b_i\,(o_1)\ ,\ 1 <= i <= N$$

- Forward recursion:

$$\delta_k(j) = \max P(q_1 \ldots\ q_{k-1}\ , q_k = s_j\ ,\ o_1\, o_2 \ldots\ o_k) =$$
$$\max_i [\ a_{ij}\ b_j\,(o_k\,)\ \max P(q_1 \ldots\ q_{k-1} = s_i\ ,\ o_1\, o_2 \ldots\ o_{k-1})\ ] =$$
$$\max_i [\ a_{ij}\ b_j\,(o_k\,)\ \delta_{k-1}(i)\ ]\ , \quad 1 <= j <= N,\ 2 <= k <= K.$$

- Termination: choose best path ending at time K

$$\max_i [\ \delta_K(i)\ ]$$

- Backtrack best path

This algorithm is similar to the forward recursion of evaluation problem, with Σ replaced by max and additional backtracking

Learning problem (1)

•**Learning problem.** Given some training observation sequences $O=o_1\ o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data, i.e. maximizes $P(O \mid M)$

• Finding optimal parameter values- *very difficult* (no existing algorithm)

• Use *iterative expectation-maximization algorithm* to find local maximum of $P(O \mid M)$-**Baum-Welch algorithm**

Learning problem (2)

- If training data has information about sequence of hidden states (as in word recognition example), then use maximum likelihood estimation of parameters:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Number of transitions from state } s_j \text{ to state } s_i}{\text{Number of transitions out of state } s_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Number of times observation } v_m \text{ occurs in state } s_i}{\text{Number of observation occurrences in state } s_i}$$

Baum-Welch algorithm

General idea:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Expected number of transitions from state } s_j \text{ to state } s_i}{\text{Expected number of transitions out of state } s_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Expected number of times observation } v_m \text{ occurs in state } s_i}{\text{Expected number of total observations in state } s_i}$$

$\pi_i = P(s_i) =$ Expected frequency in state s_i at time $k=1$.

Disadvantage of HMMs (1)

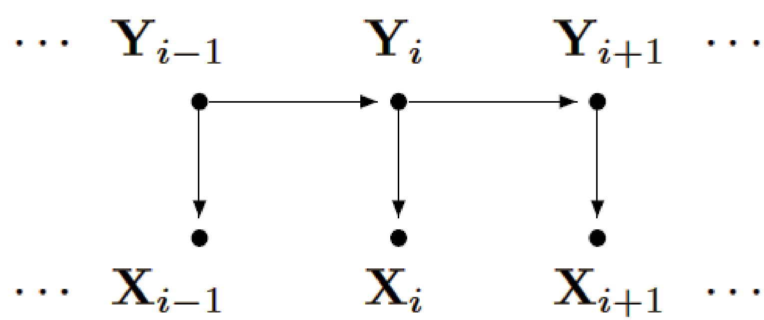
- No Rich Feature Information
  - Rich information are required
    - When x_k is complex
    - When data of x_k is sparse
- Example: Part-of-Speech (PoS) Tagging
  - How to evaluate $P(w_k \mid t_k)$ for unknown words w_k ?
  - Useful features
    - Suffix, e.g., -ed, -tion, -ing, etc.
    - Capitalization
- Generative Model
  - Parameter estimation: maximize the joint likelihood of training examples

$$\sum_{(\mathbf{x},\mathbf{y})\in T} \log_2 P(\mathbf{X}=\mathbf{x}, \mathbf{Y}=\mathbf{y})$$

Generative Models

- Hidden Markov models (HMMs) and stochastic grammars
  - Assign a joint probability to paired observation and label sequences
  - The parameters typically trained to maximize the joint likelihood of train examples

Standard tool is the hidden Markov Model (HMM).



$$P(\mathbf{X},\mathbf{Y}) \;=\; \textstyle\prod_i P(\mathbf{X}_i \,|\, \mathbf{Y}_i)\, P(\mathbf{Y}_i \,|\, \mathbf{Y}_{i-1})$$

Generative Models (cont'd)

- Difficulties and disadvantages
  - Need to enumerate all possible observation sequences
  - Not practical to represent multiple interacting features or long-range dependencies of the observations
  - Very strict independence assumptions on the observations

Making use of rich domain features

- A learning algorithm is as good as its features.
- There are *many* useful features to include in a model
- Most of them aren't independent of each other

- Identity of word
- Ends in "-shire"
- Is capitalized
- Is head of noun phrase
- Is in a list of city names
- Is under node X in WordNet
- Word to left is verb
- Word to left is lowercase
- Is in bold font
- Is in hyperlink anchor
- Other occurrences in doc
- …