AI1103 : Assignment 2

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Download all python codes from

```
https://github.com/shashank−anirudh−rachapalle/
    probability−and−random−variables/tree/main/
    Assignment2/codes
```

and latex codes from

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https://github.com/shashank−anirudh−rachapalle/
    probability−and−random−variables/tree/main/
    Assignment2/Assignment2.tex
```

Problem statement(GATE 68)

Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

The covariance between the random variables X and Y is

Solution(GATE 68)

Covariance between X and Y is $E(XY) - E(X)E(Y)$

Lemma 0.1.

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx = \sqrt{2\pi} y^{\frac{3}{2}} \tag{0.0.2}$$

Proof.

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{0.0.3}$$

$$= \int_{-\infty}^{\infty} (x-y)\, e^{-\frac{1}{2y}(x-y)^2}\,dx + y \int_{-\infty}^{\infty} e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{0.0.4}$$

$$= 0 + \sqrt{2\pi} y^{\frac{3}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{\frac{-1}{2(\sqrt{y})^2}(x-y)^2}\,dx \tag{0.0.5}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}} \lim_{x_0 \to -\infty} Q\left(\frac{x_0 - y}{\sqrt{y}}\right) \tag{0.0.6}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}} \tag{0.0.7}$$

□

Lemma 0.2.

$$E(XY) = \frac{1}{3} \tag{0.0.8}$$

Proof.

$$E(XY) = \int_0^1 \int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dx\,dy \tag{0.0.9}$$

$$= \int_0^1 \int_{-\infty}^{\infty} xy \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{0.0.10}$$

$$= \int_0^1 \frac{y}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx\right) dy \tag{0.0.11}$$

$$\text{from } (0.0.7) \tag{0.0.12}$$

$$= \int_0^1 y^2\,dy \tag{0.0.13}$$

$$E(XY) = \frac{1}{3} \tag{0.0.14}$$

□

Lemma 0.3.

$$E(Y) = \frac{1}{2} \tag{0.0.15}$$

Proof. Y has marginal probability ($y \in (0,1)$)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = 1 \tag{0.0.16}$$

$$\implies E(Y) = \int_0^1 y f_Y(y)\,dy \tag{0.0.17}$$

$$E(Y) = \frac{1}{2} \tag{0.0.18}$$

□

Lemma 0.4.

$$E(X) = \frac{1}{2} \tag{0.0.19}$$

Proof.

$$E(X) = \int_0^1 \int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy \tag{0.0.20}$$

$$= \int_0^1 \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{0.0.21}$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx \right) dy \tag{0.0.22}$$

$$\text{from } (0.0.7) \tag{0.0.23}$$

$$= \int_0^1 y\,dy \tag{0.0.24}$$

$$E(X) = \frac{1}{2} \tag{0.0.25}$$

□

From (0.0.14),(0.0.18),(0.0.25)

$$Cov(X,Y) = E(XY) - E(X)E(Y) \tag{0.0.26}$$

$$= \frac{1}{3} - \frac{1}{2} \times \frac{1}{2} \tag{0.0.27}$$

$$Cov(X,Y) = \frac{1}{12} \tag{0.0.28}$$