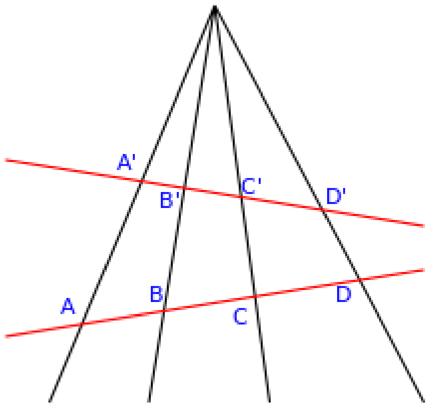
Cross-ratio

In geometry, the **cross-ratio**, also called the **double ratio** and **anharmonic ratio**, is a number associated with a list of four collinear points, particularly points on a projective line. Given four points *A*, *B*, *C* and *D* on a line, their cross ratio is defined as

$$(A,B;C,D) = \frac{AC \cdot BD}{BC \cdot AD}$$

where an orientation of the line determines the sign of each distance and the distance is measured as projected into Euclidean space. (If one of the four points is the line's point at infinity, then the two distances involving that point are dropped from the formula.) The point *D* is the harmonic conjugate of *C* with respect to *A* and *B* precisely if the cross-ratio of the quadruple is −1, called the *harmonic ratio*. The cross-ratio can therefore be regarded as measuring the quadruple's deviation from this ratio; hence the name *anharmonic ratio*.



Points *A*, *B*, *C*, *D* and *A*′, *B*′, *C*′, *D*′ are related by a projective transformation so their cross ratios, (*A*, *B*; *C*, *D*) and (*A*′, *B*′; *C*′, *D*′) are equal.

The cross-ratio is preserved by linear fractional transformations. It is essentially the only projective invariant of a quadruple of collinear points; this underlies its importance for projective geometry.

The cross-ratio had been defined in deep antiquity, possibly already by Euclid, and was considered by Pappus, who noted its key invariance property. It was extensively studied in the 19th century.[1]

Variants of this concept exist for a quadruple of concurrent lines on the projective plane and a quadruple of points on the Riemann sphere. In the Cayley–Klein model of hyperbolic geometry, the distance between points is expressed in terms of a certain cross-ratio.

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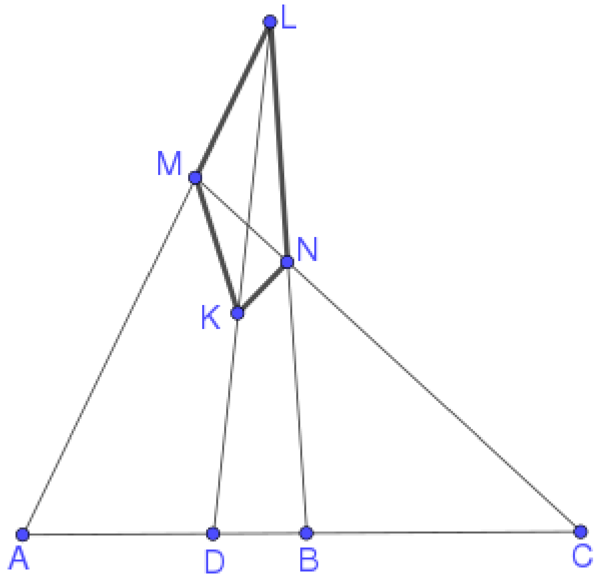
Terminology and history

Pappus of Alexandria made implicit use of concepts equivalent to the cross-ratio in his *Collection: Book VII*. Early users of Pappus included Isaac Newton, Michel Chasles, and Robert Simson. In 1986 Alexander Jones made a translation of the original by Pappus, then wrote a commentary on how the lemmas of Pappus relate to modern terminology.[2]



D is the harmonic conjugate of *C* with respect to *A* and *B*, so that the cross-ratio (*A*, *B*; *C*, *D*) equals −1.

Modern use of the cross ratio in projective geometry began with Lazare Carnot in 1803 with his book *Géométrie de Position*. The term used was *le rapport anharmonique* (Fr: anharmonic ratio). German geometers call it *das Doppelverhältnis* (Ger: double ratio).

Given three points on a line, a fourth point that makes the cross ratio equal to minus one is called the projective harmonic conjugate. In 1847 Carl von Staudt called the construction of the fourth point a **Throw** (**Wurf**), and used the construction to exhibit arithmetic implicit in geometry. His Algebra of Throws provides an approach to numerical propositions, usually taken as axioms, but proven in projective geometry.[3]

The English term "cross-ratio" was introduced in 1878 by William Kingdon Clifford.[4]

Definition

The cross-ratio of a quadruple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

It can also be written as a "double ratio" of two division ratios of triples of points:

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

The cross-ratio is normally extended to the case when one of z_1, z_2, z_3, z_4 is infinity (∞); this is done by removing the corresponding two differences from the formula.

For example: if $z_1 = \infty$ the cross ratio becomes:

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - \infty)(z_4 - z_2)}{(z_3 - z_2)(z_4 - \infty)} = \frac{(z_4 - z_2)}{(z_3 - z_2)}.$$

In geometry, if *A*, *B*, *C* and *D* are collinear points, then the cross ratio is defined similarly as

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD},$$

where each of the distances is signed according to a consistent orientation of the line.

The same formulas can be applied to four different complex numbers or, more generally, to elements of any field and can also be extended to the case when one of them is the symbol ∞, by removing the corresponding two differences from the formula. The formula shows that cross-ratio is a function of four points, generally four numbers z_1, z_2, z_3, z_4 taken from a field.

Properties

The cross ratio of the four collinear points *A*, *B*, *C*, *D* can be written as

(A,B;C,D)={\frac {AC}{CB}}:{\frac {AD}

where { describes the ratio with which the point *C* divides the line segment *AB*, and { describes the ratio with which the point *D* divides that same line segment. The cross ratio then appears as a ratio of ratios, describing how the two points *C*, *D* are situated with respect to the line segment *AB*. As long as the points *A*, *B*, *C* and *D* are distinct, the cross ratio (*A*, *B*; *C*, *D*) will be a non-zero real number. We can easily deduce that

- (*A*, *B*; *C*, *D*) < 0 if and only if one of the points *C*, *D* lies between the points *A*, *B* and the other does not
- (*A*, *B*; *C*, *D*) = 1 / (*A*, *B*; *D*, *C*)
- (*A*, *B*; *C*, *D*) = (*C*, *D*; *A*, *B*)
- (*A*, *B*; *C*, *D*) ≠ (*A*, *B*; *C*, *E*) ↔ *D* ≠ *E*

Six cross-ratios

Four points can be ordered in 4! = 4 x 3 x 2 x 1 = 24 ways, but there are only six ways for partitioning them into two non-ordered pairs. Thus, four points can have only six different cross-ratios, which are related as:

{\displaystyle (A,B;C,D)=(B,A;D,C)=(C,D;A,B)=

{\displaystyle (A,B;D,C)=(B,A;C,D)=(C,D;B,A)=(D,C;A,B)={\fra {1}{\lambda }}}

{\displaystyle (A,C;B,D)=(B,D;A,C)=(C,A;D,B)=(D,B;C,A)=1-\lambd

{\displaystyle (A,C;D,B)=(B,D;C,A)=(C,A;B,D)=(D,B;A,C)={\frac {1} {1-\lambda }}}

{\displaystyle (A,D;B,C)=(B,C;A,D)=(C,B;D,A)=(D,A;C,B)={\fra {\lambda -1}{\lambda }}}

{\displaystyle (A,D;C,B)=(B,C;D,A)=(C,B;A,D)=(D,A;B,C)={\fra {\lambda }{\lambda -1}}}

Projective geometry

Cross-ratio is a **projective invariant** in the sense that it is preserved by the projective transformations of a projective line.

In particular, if four points lie on a straight line *L* in $\mathbf{R}^2$ then their cross-ratio is a well-defined quantity, because any choice of the origin and even of the scale on the line will yield the same value of the cross-ratio.

Furthermore, let $\{L_i \mid 1 \le i \le 4\}$ be four distinct lines in the plane passing through the same point *Q*. Then any line *L* not passing through *Q* intersects these lines in four distinct points P_i (if *L* is parallel to L_i then the corresponding intersection point is "at infinity"). It turns out that the cross-ratio of these points (taken in a fixed order) does not depend on the choice of a line *L*, and hence it is an invariant of the 4-tuple of lines $\{L_i\}$.

This can be understood as follows: if L and L' are two lines not passing through Q then the perspective transformation from L to L' with the center Q is a projective transformation that takes the quadruple $\{P_i\}$ of points on L into the quadruple $\{P_i'\}$ of points on L'.

Therefore, the invariance of the cross-ratio under projective automorphisms of the line implies (in fact, is equivalent to) the independence of the cross-ratio of the four collinear points $\{P_i\}$ on the lines $\{L_i\}$ from the choice of the line that contains them.

Use of cross-ratios in projective geometry to measure real-world dimensions of features depicted in a perspective projection. A, B, C, D and V are points on the image, their separation given in pixels; A', B', C' and D' are in the real world, their separation in metres.

- In (1), the width of the side street, W is computed from the known widths of the adjacent shops.
- In (2), the width of only one shop is needed because a vanishing point, V is visible.

Definition in homogeneous coordinates

If four collinear points are represented in homogeneous coordinates by vectors a, b, c, d such that $c = a + b$ and $d = ka + b$, then their cross-ratio is k.[5]

Role in non-Euclidean geometry

Arthur Cayley and Felix Klein found an application of the cross-ratio to non-Euclidean geometry. Given a nonsingular conic C in the real projective plane, its stabilizer G_C in the projective group $G = \mathrm{PGL}(3, \mathbf{R})$ acts transitively on the points in the interior of C. However, there is an invariant for the action of G_C on *pairs* of points. In fact, every such invariant is expressible as a function of the appropriate cross ratio.

Hyperbolic geometry

Explicitly, let the conic be the unit circle. For any two points P, Q, inside the unit circle . If the line connecting them intersects the circle in two points, X and Y and the points are, in order, X, P, Q, Y. Then the hyperbolic distance between P and Q in the Cayley–Klein model of the hyperbolic plane can be expressed as

{\displaystyle d_{h}(P,Q)={\frac {1}{2}}\left|\log \left({\frac {|XQ||PY|}

(the factor one half is needed to make the curvature −1). Since the cross-ratio is invariant under projective transformations, it follows that the hyperbolic distance is invariant under the projective transformations that preserve the conic C.

Conversely, the group G acts transitively on the set of pairs of points (p, q) in the unit disk at a fixed hyperbolic distance.

Later, partly through the influence of Henri Poincaré, the cross ratio of four complex numbers on a circle was used for hyperbolic metrics. Being on a circle means the four points are the image of four real points under a Möbius transformation, and hence the cross ratio is a real number. The Poincaré half-plane model and Poincaré disk model are two models of hyperbolic geometry in the complex projective line.

These models are instances of Cayley–Klein metrics.

The anharmonic group

The cross-ratio may be defined by any of these four expressions:

$$(A,B;C,D)=(B,A;D,C)=(C,D;A,B)=(D,C;B,A).\,$$

These differ by the following permutations of the variables:

{\displaystyl {\begin{alig (A,B) (C,D)\\[6pt] (A,C)

These three and the identity permutation leave the cross ratio unaltered. They make up a realization of the Klein four-group, a group of order 4 in which the order of every non-identity element is 2.

Other permutations of the four variables alter the cross-ratio so that it may take any of the following six values.

$$\begin{aligned}(A,B;C,D)&=\lambda &(A,B;D,C)&={\frac {1}{\lambda }}\\[6pt](A,C;D,B)&={\frac {1}{1-\lambda }}&(A,C;B,D)&=1-\lambda \\[6pt](A,D;C,B)&={\frac {\lambda }{\lambda -1}}&(A,D;B,C)&={\frac {\lambda -1}{\lambda }}\end{aligned}$$

As functions of λ, these form a non-abelian group of order 6 with the operation of composition of functions. This is the **anharmonic group**. It is a subgroup of the group of all Möbius transformations. The six cross-ratios listed above represent torsion elements (geometrically, elliptic transforms) of PGL(2, **Z**). Namely, , 1, and \fra {\1a are of order 2 in PGL(2, **Z**), with fixed points, respectively, −1, 1/2, and 2 (namely, the orbit of the harmonic cross-ratio). Meanwhile, elements \fra {1 and \fra 1} are of order 3 in PGL(2, **Z**) – in PSL(2, **Z**) (this corresponds to the subgroup A_3 of even elements). Each of them fixes both values e^{ of the "most symmetric" cross-ratio.

The anharmonic group is generated by $\lambda \mapsto 1/\lambda$ and $\lambda \mapsto 1-\lambda$. Its action on $\{0, 1, \infty\}$ gives an isomorphism with S_3. It may also be realised as the six Möbius transformations mentioned,[6] which yields a projective representation of S_3 over any field (since it is defined with integer entries), and is always faithful/injective (since no two terms differ only by 1/−1). Over the field with two elements, the projective line only has three points, so this representation is an isomorphism, and is the exceptional isomorphism {\mathrm. In characteristic 3, this stabilizes the point -1=[-1:1], which corresponds to the orbit of the harmonic cross-ratio being only a single point, since 2=1/2=-1. Over the field with 3 elements, the projective line has only 4 points and {\mathrm, and thus the representation is exactly the stabilizer of the harmonic cross-ratio, yielding an embedding {\matl equals the stabilizer of the point .

Role of Klein four-group

In the language of group theory, the symmetric group S_4 acts on the cross-ratio by permuting coordinates. The kernel of this action is isomorphic to the Klein four-group K. This group consists of 2-cycle permutations of type {\display (in addition to the identity), which preserve the cross-ratio. The effective symmetry group is then the quotient group {\l, which is isomorphic to S_3.

Exceptional orbits

For certain values of λ there will be greater symmetry and therefore fewer than six possible values for the cross-ratio. These values of λ correspond to fixed points of the action of S_3 on the Riemann sphere (given by the above six functions); or, equivalently, those points with a non-trivial stabilizer in this permutation group.

The first set of fixed points is $\{0, 1, \infty\}$. However, the cross-ratio can never take on these values if the points *A*, *B*, *C* and *D* are all distinct. These values are limit values as one pair of coordinates approach each other:

$$(Z,B;Z,D)=$$

$$(Z,Z;C,D)=$$

$$(Z,B;C,Z)=$$

The second set of fixed points is $\{-1, 1/2, 2\}$. This situation is what is classically called the **harmonic cross-ratio**, and arises in projective harmonic conjugates. In the real case, there are no other exceptional orbits.

In the complex case, the most symmetric cross-ratio occurs when \lambda. These are then the only two values of the cross-ratio, and these are acted on according to the sign of the permutation.

Transformational approach

The cross-ratio is invariant under the projective transformations of the line. In the case of a complex projective line, or the Riemann sphere, these transformations are known as Möbius transformations. A general Möbius transformation has the form

$$f(z)={\frac {az+b}{cz+d}}\;,\quad {\mbox{where }}a,b,c,d\in {\mathbb {C}}{\mbox{ and }}ad-bc\neq 0.$$

These transformations form a group acting on the Riemann sphere, the Möbius group.

The projective invariance of the cross-ratio means that

$$(f(z_{1}),f(z_{2});f(z_{3}),f(z_{4}))=$$

The cross-ratio is real if and only if the four points are either collinear or concyclic, reflecting the fact that every Möbius transformation maps generalized circles to generalized circles.

The action of the Möbius group is simply transitive on the set of triples of distinct points of the Riemann sphere: given any ordered triple of distinct points, (z_2, z_3, z_4), there is a unique Möbius transformation $f(z)$ that maps it to the triple $(1, 0, \infty)$. This transformation can be conveniently described using the cross-ratio: since (z, z_2, z_3, z_4) must equal $(f(z), 1; 0, \infty)$, which in turn equals $f(z)$, we obtain

$$f(z)=$$

An alternative explanation for the invariance of the cross-ratio is based on the fact that the group of projective transformations of a line is generated by the translations, the homotheties, and the multiplicative inversion. The differences $z_j - z_k$ are invariant under the translations

z\mapsto

where a is a constant in the ground field F. Furthermore, the division ratios are invariant under a homothety

z\ma

for a non-zero constant b in F. Therefore, the cross-ratio is invariant under the affine transformations

In order to obtain a well-defined inversion mapping

T:z\mapsto

the affine line needs to be augmented by the point at infinity, denoted ∞, forming the projective line $P^1(F)$. Each affine mapping $f : F \to F$ can be uniquely extended to a mapping of $P^1(F)$ into itself that fixes the point at infinity. The map T swaps 0 and ∞. The projective group is generated by T and the affine mappings extended to $P^1(F)$. In the case $F = \mathbf{C}$, the complex plane, this results in the Möbius group. Since the cross-ratio is also invariant under T, it is invariant under any projective mapping of $P^1(F)$ into itself.

Co-ordinate description

If we write the complex points as vectors {\displaystyle and define {\displaystyle. Let (a be the dot product of with then the real part of the cross ratio is given by:

{\displaystyle C_{1}={\frac {(x_{12},x_{14})(x_{23},x_{34})-(x_{12},x_{34})(x_{14},x_{23})+(x_{12},x_{23})(x_{14},x_{34})}{|x_{23}|^{2}|x_{14}|^{2}}}}

This is an invariant of the 2D special conformal transformation such as inversion {\display x^{\mu.

The imaginary part must make use of the 2-dimensional cross product {\displaystyle a\times b=

{\displaystyle C_{2}={\frac {(x_{12},x_{14})[x_{34},x_{23}]-(x_{43},x_{23})[x_{12},x_{34}]}

Ring homography

The concept of cross ratio only depends on the ring operations of addition, multiplication, and inversion (though inversion of a given element is not certain in a ring). One approach to cross ratio interprets it as a homography that takes three designated points to 0, 1, and infinity. Under restrictions having to do with inverses, it is possible to generate such a mapping with ring operations in the projective line over a ring. The cross ratio of four points is the evaluation of this homography at the fourth point.

Differential-geometric point of view

The theory takes on a differential calculus aspect as the four points are brought into proximity. This leads to the theory of the Schwarzian derivative, and more generally of projective connections.

Higher-dimensional generalizations

The cross-ratio does not generalize in a simple manner to higher dimensions, due to other geometric properties of configurations of points, notably collinearity – configuration spaces are more complicated, and distinct k-tuples of points are not in general position.

While the projective linear group of the plane is 3-transitive (any three distinct points can be mapped to any other three points), and indeed simply 3-transitive (there is a *unique* projective map taking any triple to another triple), with the cross ratio thus being the unique projective invariant of a set of four points, there are basic geometric invariants in higher dimension. The projective linear group of n-space {\mathbf has $(n + 1)^2 - 1$ dimensions (because it is {\mathrm {PGL}}(n,K)= projectivization removing one dimension), but in other dimensions the projective linear group is only 2-transitive – because three collinear points must be mapped to three collinear points (which is not a restriction in the projective line) – and thus there is not a "generalized cross ratio" providing the unique invariant of n^2 points.

Collinearity is not the only geometric property of configurations of points that must be maintained – for example, five points determine a conic, but six general points do not lie on a conic, so whether any 6-tuple of points lies on a conic is also a projective invariant. One can study orbits of points in general position – in the line "general position" is equivalent to being distinct, while in higher dimensions it requires geometric considerations, as discussed – but, as the above indicates, this is more complicated and less informative.

However, a generalization to Riemann surfaces of positive genus exists, using the Abel–Jacobi map and theta functions.

See also

- Hilbert metric

Notes and references

1. A theorem on the anharmonic ratio of lines appeared in the work of Pappus, but Michel Chasles, who devoted considerable efforts to reconstructing lost works of Euclid, asserted that it had earlier appeared in his book *Porisms*.
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External links

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- Cross-Ratio at cut-the-knot
- Weisstein, Eric W. "Cross-ratio". *MathWorld*.
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