Evolutionary Multi-Objective Optimization

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Motivation

Most problems in nature have several (possibly conflicting) objectives to be satisfied (e.g., design a bridge for which want to minimize its weight and cost while maximizing its safety). Many of these problems are frequently treated as single-objective optimization problems by transforming all but one objective into constraints.

Formal Definition

Find the vector $\vec{x}^* = \left[x_1^*, x_2^*, \ldots, x_n^*\right]^T$ which will satisfy the m inequality constraints:

$$g_i(\vec{x}) \leq 0 \quad i = 1, 2, \ldots, m \tag{1}$$

the p equality constraints

$$h_i(\vec{x}) = 0 \quad i = 1, 2, \ldots, p \tag{2}$$

and will optimize the vector function

$$\vec{f}(\vec{x}) = [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_k(\vec{x})]^T \tag{3}$$

Basic Concepts

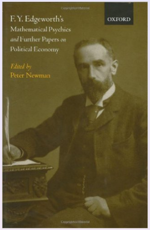
In multiobjective optimization problems, there are three possible situations:

- Minimize all the objective functions
- Maximize all the objective functions
- Minimize some and maximize others

For simplicity reasons, normally all the functions are converted to a maximization or minimization form. For example, the following identity may be used to convert all the functions which are to be maximized into a form which allows their minimization:

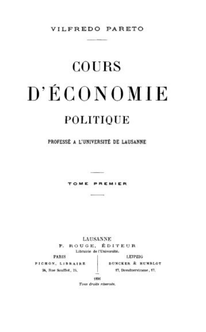
$$\max \ f_i(\vec{x}) = \min(-f_i(\vec{x})) \tag{4}$$

Basic Concepts



Having several objective functions, the notion of "optimum" changes, because in MOPs, the aim is to find good compromises (or "trade-offs") rather than a single solution as in global optimization.

The notion of "optimum" that is most commonly adopted is that originally proposed by Francis Ysidro Edgeworth (in 1881) in his book entitled **Mathematical Psychics.**



VILFREDO PARETO

COURS

D'ÉCONOMIE

POLITIQUE

PROFESSÉ A L'UNIVERSITÉ DE LAUSANNE

TOME PREMIER

LAUSANNE
F. ROUGE, ÉDITEUR
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24, Rue Soufflot, 24.

LEIPZIG
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1896
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This notion was generalized by the italian economist Vilfredo Pareto (in 1896) in his book **Cours d'Economie Politique.** Although some authors call *Edgeworth-Pareto optimum* to this notion (originally called **ophelimity**) it is normally preferred to use the most commonly accepted term: **Pareto optimum.**

Pareto Optimality

We say that a vector of decision variables $\vec{x}^* \in \mathcal{F}$ is **Pareto optimal** if there does not exist another $\vec{x} \in \mathcal{F}$ such that $f_i(\vec{x}) \leq f_i(\vec{x}^*)$ for all $i = 1, \ldots, k$ and $f_j(\vec{x}) < f_j(\vec{x}^*)$ for at least one j (assuming that all the objectives are being minimized).

Other important definitions

In words, this definition says that $\vec{x}^*$ is **Pareto optimal** if there exists no feasible vector of decision variables $\vec{x} \in \mathcal{F}$ which would decrease some criterion without causing a simultaneous increase in at least one other criterion. This concept normally produces a set of solutions called the **Pareto optimal set**. The vectors $\vec{x}^*$ corresponding to the solutions included in the Pareto optimal set are called **nondominated**. The image of the Pareto optimal set is called the **Pareto front**.

Basic Concepts

Pareto Dominance

A vector $\vec{u} = (u_1, \ldots, u_k)$ is said to **dominate** $\vec{v} = (v_1, \ldots, v_k)$ (denoted by $\vec{u} \preceq \vec{v}$) if and only if u is partially less than v, i.e., $\forall i \in \{1, \ldots, k\},\ u_i \leq v_i \wedge \exists i \in \{1, \ldots, k\} : u_i < v_i$.

Pareto Optimal Set

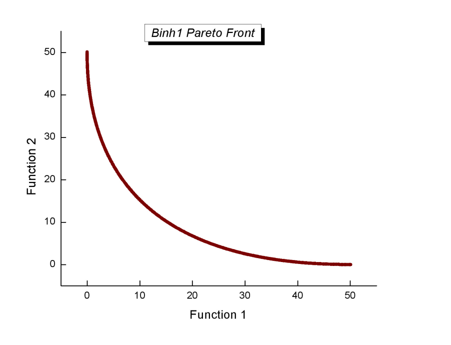
For a given MOP $\vec{f}(x)$, the Pareto optimal set ($\mathcal{P}^*$) is defined as:

$$\mathcal{P}^* := \{x \in \mathcal{F} \mid \neg\exists\ x' \in \mathcal{F}\ \ \vec{f}(x') \preceq \vec{f}(x)\}. \tag{5}$$

Pareto Front

For a given MOP $\vec{f}(x)$ and Pareto optimal set $\mathcal{P}^*$, the Pareto front ($\mathcal{PF}^*$) is defined as:

$$\mathcal{PF}^* := \{\vec{u} = \vec{f} = (f_1(x), \ldots, f_k(x)) \mid x \in \mathcal{P}^*\}. \tag{6}$$

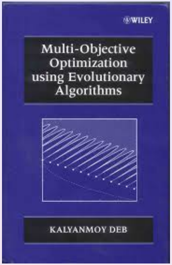


Pareto Front

For a given multi-objective optimization problem $\vec{f}(x)$ and a Pareto optimal set $\mathcal{P}^*$, the Pareto Front ($\mathcal{PF}^*$) is defined as:

$$\mathcal{PF}^* := \{\vec{u} = \vec{f} = (f_1(x), \ldots, f_k(x)) \mid x \in \mathcal{P}^*\}. \tag{7}$$

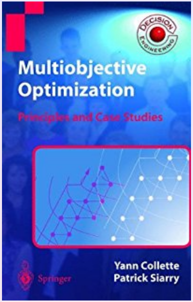
A Taxonomy of MOEAs



The Old Days

- Non-Elitist Non-Pareto-based Methods
  - Lexicographic Ordering
  - Linear Aggregating Functions
  - VEGA
  - ε-Constraint Method
  - Target Vector Approaches

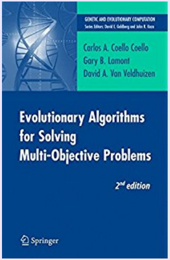
A Taxonomy of MOEAs



The Old Days

- Non-Elitist Pareto-based Methods
  - Pure Pareto ranking
  - MOGA
  - NSGA
  - NPGA and NPGA 2

A Taxonomy of MOEAs



Contemporary Approaches

- Elitist Pareto-based Methods
  - SPEA and SPEA2
  - NSGA-II
  - PAES, PESA and PESA II
  - Micro-genetic Algorithm for Multi-Objective Optimization and μGA^2
  - MOEAs that the world forgot

A Taxonomy of MOEAs

Recent Approaches

- MOEA/D
- Indicator-Based Approaches
  - SMS-EMOA
  - HyPE
  - Other Approaches
- NSGA-III

What is Elitism?

In single-objective optimization, **elitism** is an operator by which the best solution in the population passes intact to the next generation (i.e., it is not affected by crossover or mutation).

In the context of multi-objective optimization, elitism operates in a similar way, but in this case, we need to retain (all) the nondominated solutions generated by a MOEA. Since it is impractical to retain all of these solutions, it is normally the case, that some sort of bound is set on the maximum number of solutions that are retained. This is particularly important in MOEAs in which the elitist solutions play a role in the selection mechanism (e.g., SPEA).

Elitist Pareto-based Methods

Forms of Elitism

The two main forms in which elitism is normally implemented are:

1. Through the use of an **external archive** (also called **external population**), which is a data structure that resides in main memory and which stores the nondominated solutions generated during the evolutionary process
2. Using a **plus selection mechanism** in which the population of parents is merged with the population of offspring and only the best half is retained.

Elitist Pareto-based Methods

Why is elitism important?

Elitism is required to guarantee convergence of a MOEA to the true Pareto optimal set of a multi-objective optimization problem, as proved by Rudolph and Agapie [2001].

See:
Günter Rudolph and Alexandru Agapie, "**Convergence Properties of Some Multi-Objective Evolutionary Algorithms**", in *Proceedings of the 2000 IEEE Conference on Evolutionary Computation*, Vol. 2, pp. 1010–1016, IEEE Press, Piscataway, New Jersey, USA July 2000.

SPEA

Eckart Zitzler [1998,1999] proposed in his PhD thesis the **Strength Pareto Evolutionary Algorithm** (SPEA) as a MOEA that integrates different mechanisms from previous approaches.

SPEA

See:
Eckart Zitzler and Lothar Thiele, "**Multiobjective Evolutionary Algorithms: A Comparative Case Study and the Strength Pareto Approach**", *IEEE Transactions on Evolutionary Computation*, **3**(4):257-271, November 1999.

SPEA adopts an external archive and somehow generalized this notion of elitism within MOEAs. At each generation, the nondominated solutions from the population are copied to this archive, and the archive participates in the selection process. For each individual in the external archive, a "strength" value is computed. This value is similar to the rank in MOGA, since it is proportional to the number of solutions that a certain individual dominates.

SPEA

The fitness of each individual in the population is computed based on the strengths of all the individuals in the external archive to which a certain individual dominates.

Zitzler realized that if the size of the archive was not bounded, the selection pressure would dilute as the number of nondominated solutions grew very quickly. Thus, he decided to prune the archive using a clustering technique called **average linking method** [Morse, 1989], once a certain (pre-defined) limit was reached.

SPEA

Some applications of this approach are the following:

- Exploration of software schedules for digital signal processors [Zitzler, 1999].
- Planning of medical treatments [Petrovski, 2001].
- Dose optimization problems in brachytherapy [Lahanas, 2001].
- Non-invasive atrial disease diagnosis [de Toro, 2003].
- Rehabilitation of a water distribution system [Cheung, 2003].

Elitist Pareto-based Methods

SPEA2

A revised (and improved) version of SPEA (called SPEA2) was proposed by Eckart Zitzler and his colleagues in 2001.

See:
Eckart Zitzler, Marco Laumanns and Lothar Thiele, "**SPEA2: Improving the Strength Pareto Evolutionary Algorithm**, in K. Giannakoglou, D. Tsahalis, J. Periaux, P. Papailou and T. Fogarty (eds.), *EUROGEN 2001, Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems*, pp. 95–100, Athens, Greece, 2002.

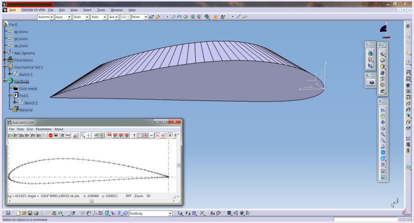
Elitist Pareto-based Methods

SPEA2

SPEA2 has three main differences with respect to the original SPEA:

- It incorporates a fine-grain fitness assignment strategy which takes into consideration both the number of individuals that a solution dominates and the number of solutions by which it is dominated.
- A more efficient density estimator (a better clustering algorithm).
- A mechanism to truncate the external archive, which guarantees that bound solutions are retained.

Elitist Pareto-based Methods



SPEA2

Some applications of this approach are the following:

- Reduction of bloat in genetic programming [Bleuler, 2001].
- Airfoil design [Willmes, 2003].
- Portfolio optimization [Garcia, 2011].
- Optimization of diesel engine emissions and fuel economy [Hiroyasu, 2005].

Elitist Pareto-based Methods

NSGA-II

The *Nondominated Sorting Genetic Algorithm II* (NSGA-II) was originally proposed by Kalyanmoy Deb and his students in 2000. However, most people only know the journal version of this paper, which appeared in 2002: Kalyanmoy Deb, Amrit Pratap, Sameer Agarwal and T. Meyarivan, “**A Fast and Elitist Multiobjective Genetic Algorithm: NSGA–II**, *IEEE Transactions on Evolutionary Computation*, Vol. 6, No. 2, pp. 182–197, April 2002.

Elitist Pareto-based Methods

NSGA-II

NSGA-II is actually quite different from the original NSGA. It still adopts nondominated sorting, but only in a single pass (as MOGA). Also, it adopts a plus selection mechanism by which the parents population is merged with the offspring population, such that only the best half survives (this is an implicitly elitist scheme).

A key element of NSGA-II is its density estimator, which is called **crowded comparison operator**. This approach requires that solutions are sorted with respect to one objective. Then, each individual uses its previous and further neighbors to build a rectangle. When comparing two solutions, if there is a tie (i.e., either both are nondominated or both are dominated), the one with the larger perimeter wins (i.e., preference is given to solutions in more isolated regions of objective function space). This density estimator requires no extra parameters and is quite efficient.

Elitist Pareto-based Methods

NSGA-II

The elegance, effectiveness and efficiency of NSGA-II made it a standard in evolutionary multi-objective optimization for more than 10 years.

The fact that its source code has been made available also contributed to its popularity (it is probably the most popular MOEA ever).

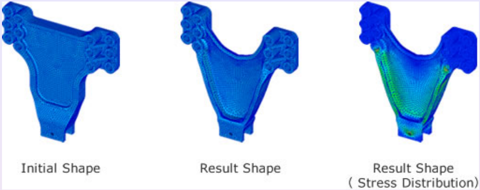
Elitist Pareto-based Methods

NSGA-II

However, NSGA-II does not work properly with more than 3 objectives, mainly because of its density estimator, which was conceived only for two objectives.

Additionally, there is experimental evidence that indicates that NSGA-II works better with real-numbers encoding than with binary encoding.

Elitist Pareto-based Methods



NSGA-II

Some applications of this approach are the following:

- Shape optimization [Deb, 2001].
- Safety systems optimum design [Greiner, 2003].
- Optimization of processing conditions for polymer twin-screw extrusion (Gaspar-Cunha, 2002).
- Watershed water quality management [Dorn, 2003].
- Intensity modulated beam radiation therapy dose optimization [Lahanas, 2003].

Elitist Pareto-based Methods

PAES

The *Pareto Archived Evolution Strategy* (PAES) was proposed in 1999, but its journal version appeared in 2000.

See:
Joshua D. Knowles and David W. Corne, "**Approximating the Nondominated Front Using the Pareto Archived Evolution Strategy**", *Evolutionary Computation*, Vol. 8, No. 2, pp. 149–172, 2000.

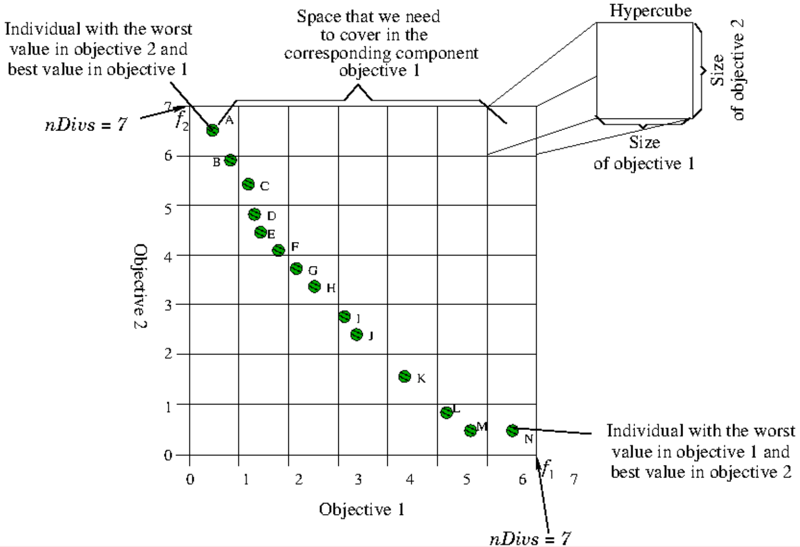
Elitist Pareto-based Methods

PAES

Conceptually speaking, PAES is perhaps the most simple MOEA that one can possibly design. It consists of a (1+1)-ES (i.e., a single parent which is mutated to produce an offspring). If the offspring dominates its parent, it is stored in an external archive and it becomes the parent in the next iteration.

The most interesting aspect of this approach is its external archive, which adopts a density estimator called *adaptive grid*. This density estimator only requires one parameter: the number of sub-divisions to be adopted in objective function space. Its main problem is that it was conceived only for two objectives and its generalization to any number of objectives doesn't seem possible.

Elitist Pareto-based Methods



Elitist Pareto-based Methods

PAES

Some applications of this approach are the following:

- Telecommunications problems [Knowles, 1999].
- The adaptive distributed database management problem [Knowles, 2000].
- Flexible job shop scheduling [Rabiee, 2012].

Recent Approaches

MOEA/D

The *Multi-Objective Evolutionary Algorithm based on Decomposition* (MOEA/D) proposed by Zhang and Li [2007] is one of the most competitive MOEAs in current use. This approach decomposes a multi-objective problem into several single-objective optimization problems, which are simultaneously solved. Each subproblem is optimized using information from its neighboring subproblems, in contrast with similar approaches (e.g., MOGLS [Ishibuchi & Murata, 1996]). This MOEA is inspired on a mathematical programming technique called *Normal Boundary Intersection* (NBI) [Das, 1998].

Qingfu Zhang and Hui Li, "**MOEA/D: A Multiobjective Evolutionary Algorithm Based on Decomposition**", *IEEE Transactions on Evolutionary Computation*, Vol. 11, No. 6, pp. 712–731, December 2007.

Indicator-based Selection

Perhaps the most important trend on the design of moderns MOEAs is the use of a performance measure in their selection mechanism.

ESP

The **Evolution Strategy with Probability Mutation** uses a measure based on the hypervolume, which is scale independent and doesn't require any parameters, in order to truncate the contents of the external archive [Huband et al., 2003].

Simon Huband, Phil Hingston, Lyndon White and Luigi Barone, "**An Evolution Strategy with Probabilistic Mutation for Multi-Objective Optimisation**", in *Proceedings of the 2003 Congress on Evolutionary Computation (CEC'2003)*, Vol. 3, pp. 2284–2291, IEEE Press, Canberra, Australia, December 2003.

Recent Approaches

IBEA

The **Indicator-Based Evolutionary Algorithm** is an algorithmic framework that allows the incorporation of any performance indicator in the selection mechanism of a MOEA [Zitzler et al., 2004]. It was originally tested using the hypervolume and the binary ϵ indicator.

Eckart Zitzler and Simon Künzli, "**Indicator-based Selection in Multiobjective Search**, in Xin Yao et al. (editors), *Parallel Problem Solving from Nature - PPSN VIII*, Springer-Verlag, Lecture Notes in Computer Science, Vol. 3242, pp. 832–842, Birmingham, UK, September 2004.

Recent Approaches

SMS-EMOA

Emmerich et al. [2005] proposed an approach based on NSGA-II and the archiving techniques proposed by Knowles, Corne and Fleischer. This approach was called *S Metric Selection Evolutionary Multiobjective Algorithm*.

SMS-EMOA creates an initial population and generates a single solution per iteration (i.e., it uses steady state selection) using the crossover and mutation operators from NSGA-II. Then, it applies Pareto ranking. When the last nondominated front has more than one solution, SMS-EMOA uses hypervolume to decide which solution should be removed.

Michael Emmerich, Nicola Beume and Boris Naujoks, "**An EMO Algorithm Using the Hypervolume Measure as Selection Criterion**, in Carlos A. Coello Coello et al. (editors), *Evolutionary Multi-Criterion Optimization. Third International Conference, EMO 2005*, pp. 62–76, Springer. Lecture Notes in Computer Science Vol. 3410, Guanajuato, México, March 2005.

SMS-EMOA

Beume et al. [2007] proposed a new version of SMS-EMOA in which the hypervolume contribution is not used when, in the Pareto ranking process, we obtain more than one front. In this case, they use the number of solutions that dominate to a certain individual (i.e., the solution that is dominated by the largest number of solutions is removed).

The authors of this approach indicate that their motivation to use the hypervolume is to improve the distribution of solutions along the Pareto front (in other words, hypervolume is used only as a density estimator).

Nicola Beume, Boris Naujoks and Michael Emmerich, “**SMS-EMOA: Multiobjective selection based on dominated hypervolume**”, *European Journal of Operational Research*, Vol. 181, No. 3, pp. 1653–1669, 16 September, 2007.

Recent Approaches

MO-CMA-ES

This is a multi-objective version of the **covariance matrix adaptation evolution strategy** (CMA-ES). It was proposed by Igel et al. [2007].

Its selection mechanism is based on a nondominated sorting that adopts as its second selection criterion either the crowding distance or the hypervolume contribution (two versions of the algorithm were tested, and the one based on the hypervolume has the best overall performance).

This MOEA is rotation invariant, as the original single-objective optimizer on which it is based.

Christian Igel, Nikolaus Hansen and Stefan Roth, "**Covariance Matrix Adaptation for Multi-objective Optimization**", *Evolutionary Computation*, Vol. 15, No. 1, pp. 1–28, Spring 2007.

Recent Approaches

SPAM

The *Set Preference Algorithm for Multiobjective optimization* is a generalization of IBEA which allows to adopt any set preference relation in its selection mechanism [Zitzler et al., 2008].

Eckart Zitzler, Lothar Thiele and Johannes Bader, "**SPAM: Set Preference Algorithm for Multiobjective Optimization**", in Günter Rudolph et al. (editors), *Parallel Problem Solving from Nature–PPSN X*, pp. 847–858, Springer, Lecture Notes in Computer Science Vol. 5199, Dortmund, Germany, September 2008.

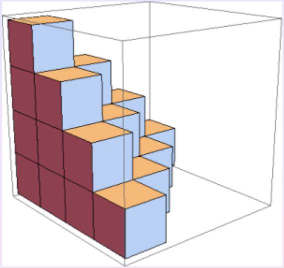
Recent Approaches

HyPE

The *hypervolume estimation algorithm for multi-objective optimization*, was proposed by Bader and Zitzler [2011]. In this case, the author proposes a quick search algorithm that uses Monte Carlo simulations to approximate the hypervolume contributions.

The core idea is that the actual hypervolume contribution value is not that important, but only the actual ranking that is produced with it. Although this proposal is quite interesting, in practice its performance is rather poor with respect that of MOEAs that use the exact hypervolume contributions.

Johannes Bader and Eckart Zitzler, "**HypE: An Algorithm for Fast Hypervolume-Based Many-Objective Optimization**", *Evolutionary Computation*, Vol. 19, No. 1, pp. 45–76, Spring, 2011.



The Hypervolume

The **hypervolume** (also known as the *S* metric or the Lebesgue measure) of a set of solutions, measures the size of the portion of objective space that is dominated by such solutions, collectively.

The hypervolume is the only performance indicator that is known to be monotonic with respect to Pareto dominance. This guarantees that the true Pareto front achieves the maximum possible hypervolume value, and any other set will produce a lower value for this indicator.

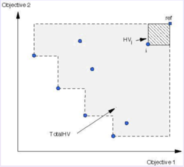
The Hypervolume

Fleischer [2003] proved that, given a finite search space and a reference point, maximizing the hypervolume is equivalent to obtaining the Pareto optimal set. Therefore, a bounded set that contains the maximum possible hypervolume value for a certain population size, will only consist of Pareto optimal solutions.

This has been experimentally validated [Knowles, 2003; Emmerich, 2005], and it has been observed that such solutions also have a good distribution along the Pareto front.

M. Fleischer, "**The Measure of Pareto Optima. Applications to Multi-objective Metaheuristics**", in Carlos M. Fonseca et al. (editors), *Evolutionary Multi-Criterion Optimization. Second International Conference, EMO 2003*, pp. 519–533, Springer. Lecture Notes in Computer Science. Volume 2632, Faro, Portugal, April 2003.

Recent Approaches

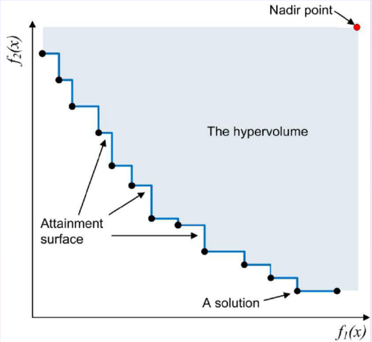


The Hypervolume

The computation of the hypervolume depends on the reference point that we adopt, and this point can have a significant influence on the results. Some researchers have proposed to use the worst objective function values available in the current population, but this requires a scaling of the objectives.

However, the main drawback of using the hypervolume is its high computational cost. The best known algorithms currently available to compute the hypervolume have a complexity that is polynominal on the number of points, but such a complexity grows exponentially with the number of objectives.

Recent Approaches



The Hypervolume

The fact that no algorithm of polynomial complexity exists for computing the hypervolume in an exact manner, gave rise to the hypothesis that such an algorithm may not be at all possible. This is remarkable if we consider that the tight lower bound for the complexity of the hypervolume computation is $O(N \log N)$ [Beume, 2007].

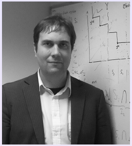
Recent Approaches

The Hypervolume

Recent theoretical results strenghten this hypothesis: Bringmann and Friedrich [2009] proved that computing the hypervolume is #P-Complete. This means that no polynomial complexity algorithm exists, because otherwise, this would imply that $NP = P$.

Nevertheless, this indicator has triggered a significant amount of research. See for example:

- `http://ls11-www.cs.uni-dortmund.de/rudolph/hypervolume/start`
- `http://people.mpi-inf.mpg.de/~tfried/HYP/`
- `http://iridia.ulb.ac.be/~manuel/hypervolume`



The Hypervolume

It is worth noting that the use of the hypervolume to select solutions is not straightforward. This indicator operates on a set of solutions, and the selection operator considers only one solution at a time. Therefore, when using the hypervolume to select solutions, a fitness assignment strategy is required.

The strategy that has been most commonly adopted in the specialized literature consists of performing first a nondominated sorting procedure and then ranking the solutions within each front based on the hypervolume loss that results from removing a particular solution [Knowles and Corne, 2003; Emmerich et al., 2005; Igel et al., 2007; Bader et al., 2010].

Recent Approaches

The Hypervolume

The main motivation for using indicators in the selection mechanism is scalability (in objective function space). However, the high computational cost of the hypervolume has motivated the exploration of alternative performance indicators, such as Δ_p.

Oliver Schütze, Xavier Esquivel, Adriana Lara and Carlos A. Coello Coello, **Using the Averaged Hausdorff Distance as a Performance Measure in Evolutionary Multi-Objective Optimization**, *IEEE Transactions on Evolutionary Computation*, Vol. 16, No. 4, pp. 504–522, August 2012.

Δ_p Indicator

The Δ_p indicator can be seen as an “averaged Hausdorff distance” between our approximation and the true Pareto front. Δ_p combines some slight variations of two well-known performance indicators: generational distance [Van Veldhuizen, 1999] and inverted generational distance [Coello & Cruz, 2005].

Δ_p is a pseudo-metric that simultaneously evaluates proximity to the true Pareto front and the distribution of solutions along it. Although it is not a Pareto compliant indicator, in practice, it seems to work reasonably well, being able to deal with outliers. This makes it attractive as a performance indicator. Additionally, its computational cost is very low.

Δ_p Indicator

Nevertheless, it is worth mentioning that in order to incorporate Δ_p in the selection mechanism of a MOEA, it is necessary to have an approximation of the true Pareto front at all times. This has motivated the development of techniques that can produce such an approximation in an efficient and effective manner.

For example, Gerst et al. [2011] lineralized the nondominated front produced by the current population and used that information in the so-called Δ_p-EMOA, which was used to solve bi-objective problems. This algorithm is inspired on the SMS-EMOA and adopts an external archive.

K. Gerstl, G. Rudolph, O. Schütze and H. Trautmann, "**Finding Evenly Spaced Fronts for Multiobjective Control via Averaging Hausdorff-Measure**", in *The 2011 8th International Conference on Electrical Engineering, Computer Science and Automatic Control (CCE'2011)*, pp. 975–980, IEEE Press, Mérida, Yucatán, México, October 2011.

Recent Approaches

Δ_p Indicator

There was a further extension of this MOEA for dealing with problems having three objectives [Trautmann, 2012]. In this case, the algorithm requires some prior steps, which include reducing the dimensionality of the nondominated solutions and computing their convex hull.

This version of the Δ_p-EMOA generates solutions with a better distribution, but requires more parameters and has a high computational cost when is used for solving many-objective optimization problems.

Heike Trautmann, Günter Rudolph, Christian Dominguez-Medina and Oliver Schütze, "**Finding Evenly Spaced Pareto Fronts for Three-Objective Optimization Problems**, in Oliver Schütze et al. (editors), *EVOLVE - A Bridge between Probability, Set Oriented Numerics, and Evolutionary Computation II*, pp. 89–105, Springer, Advances in Intelligent Systems and Computing Vol. 175, Berlin, Germany, 2012, ISBN 978-3-642-31519-0.

Δ_p Indicator

Another possible way of incorporating Δ_p into a MOEA is to use the currently available nondominated solutions in a stepped way, in order to build an approximation of the true Pareto front.

This was the approach adopted by the Δ_p-DDE [Rodríguez & Coello, 2012], which uses differential evolution as its search engine. This MOEA provides results of similar quality to those generated by SMS-EMOA, but at a much lower computational cost (in high dimensionality). Its main limitation is that its solutions are normally not well-distributed in many-objective problems. Additionally, it has difficulties to deal with disconnected Pareto fronts.

Cynthia A. Rodríguez Villalobos and Carlos A. Coello Coello, "**A New Multi-Objective Evolutionary Algorithm Based on a Performance Assessment Indicator**", in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*, pp. 505–512, ACM Press, Philadelphia, USA, July 2012, ISBN: 978-1-4503-1177-9.

R2

Recently, some researchers have recommended the use of the *R*2 indicator, which was originally proposed by Hansen (1998) for comparing sets of solutions using utility functions [Brockhoff, 2012]. A utility function is a model of the decision maker preferences that maps each point from the objective function space to a utility value.

Dimo Brockhoff, Tobias Wagner and Heike Trautmann, "**On the Properties of the *R*2 Indicator**, in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*, pp. 465–472, ACM Press, Philadelphia, USA, July 2012, ISBN: 978-1-4503-1177-9.

R2

It is worth indicating that *R2* is weakly monotonic and that it's correlated to the hypervolume, but has a much lower computational cost. Due to these properties, its use is recommended for dealing with many-objective problems. Nevertheless, the utility functions that are required to compute this indicator have to be properly scaled.

According to Brockhoff [2012], the unary version of the *R2* indicator for a constant reference set can be expressed as follows:

$$R2(A, U) = -\frac{1}{|U|} \sum_{u \in U} \max_{\mathbf{a} \in A} \{u(\mathbf{a})\}, \tag{8}$$

where A is the Pareto set approximation and U is a set of utility functions.

With respect to the choice of the utility functions $u : \mathbb{R}^m \to \mathbb{R}$, there are several posibilities: weighted linear, weighted Tchebycheff or augmented Tchebycheff functions.

Recent Approaches

R2

Currently, there are already several MOEAs based on *R*2.

Raquel Hernández Gómez and Carlos A. Coello Coello, **MOMBI: A New Metaheuristic for Many-Objective Optimization Based on the R2 Indicator**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 2488–2495, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

Dimo Brockhoff, Tobias Wagner and Heike Trautmann, *R*2 **Indicator-Based Multiobjective Search**, *Evolutionary Computation*, Vol. 23, No. 3, pp. 369–395, Fall 2015.

Alan Díaz-Manríquez, Gregorio Toscano-Pulido, Carlos A. Coello Coello and Ricardo Landa-Becerra, **A Ranking Method Based on the R2 Indicator for Many-Objective Optimization**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 1523–1530, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

Recent Approaches

R2

Dúng H. Phan and Junichi Suzuki, **R2-IBEA: R2 Indicator Based Evolutionary Algorithm for Multiobjective Optimization**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 1836–1845, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

Raquel Hernández Gómez and Carlos A. Coello Coello, "**Improved Metaheuristic Based on the** $R2$ **Indicator for Many-Objective Optimization**", in *2015 Genetic and Evolutionary Computation Conference (GECCO 2015)*, pp. 679–686, ACM Press, Madrid, Spain, July 11-15, 2015, ISBN 978-1-4503-3472-3.

Recent Approaches

NSGA-III

The *Nondominated Sorting Genetic Algorithm III* (NSGA-III) was proposed by Deb and Jain [2014] as an extension of NSGA-II specifically designed to deal with many-objective problems (i.e., multi-objective optimization problems having 4 or more objectives). NSGA-III still uses nondominated sorting (producing different levels), but in this case, the density estimation is done through adaptively updating a number of well-spread reference points.

Kalyanmoy Deb and Himanshu Jain, "**An Evolutionary Many-Objective Optimization Algorithm Using Reference-Point-Based Nondominated Sorting Approach, Part I: Solving Problems With Box Constraints**", *IEEE Transactions on Evolutionary Computation*, Vol. 18, No. 4, pp. 577–601, August 2014.