Automatica 122 (2020) 109282

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Finite-time stability of discrete autonomous systems[☆]

Wassim M. Haddad [*], Junsoo Lee

School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA

A R T I C L E I N F O

Article history:
Received 10 October 2019
Received in revised form 28 April 2020
Accepted 20 July 2020
Available online 28 September 2020

Keywords:
Stability
Finite time stability
Settling-time function
Lyapunov theorems
Converse Lyapunov theorems
Discrete-time systems

A B S T R A C T

Finite-time stability involves dynamical systems whose trajectories converge to a Lyapunov stable equilibrium state in finite time. In this paper, we address finite time stability of discrete-time dynamical systems. Specifically, we show that finite time stability leads to uniqueness of solutions in forward time. Furthermore, we provide Lyapunov and converse Lyapunov theorems for finite-time stability of discrete autonomous systems involving scalar difference fractional inequalities and minimum operators. In addition, lower semicontinuity of the settling-time function capturing the finite settling time behavior of the dynamical system is studied and illustrated through several examples. In particular, it is shown that the regularity properties of the Lyapunov function and those of the settling-time function are related. Consequently, converse Lyapunov theorems for finite time stability of discrete-time systems can only assure the existence of lower semicontinuous Lyapunov functions.

1. Introduction

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. For continuous-time systems, finite-time convergence to a Lyapunov stable equilibrium, that is, *finite-time stability*, was first addressed by Roxin (1966) and rigorously studied in Bhat and Bernstein (2000, 2005) for time-invariant deterministic systems using continuous Lyapunov functions. Extensions of finite-time stability to time-varying nonlinear dynamical systems are presented in Haddad and L'Afflitto (2015), Haddad, Nersesov, and Du (2009) and Moulay and Perruquetti (2008), whereas extensions of finite-time stability for stochastic dynamical systems are reported in Chen and Jiao (2010), Rajpurohit and Haddad (2017) and Yin, Khoo, Man, and Yu (2011). Unlike continuous-time systems, finite-time stability for discrete-time dynamical systems has received far less attention in the literature. Notable contributions include Polyakov, Efimov, and Brogliato (2018), wherein a discretization algorithm for a homogeneous system is presented that preserves a finite-time convergence property, and Hamrah, Sanyal, and Viswanathan (2019), which addresses finite time stable position tracking control of unmanned vehicles.

In this paper, we built on the results of Bhat and Bernstein (2000) to address finite time stability for discrete-time autonomous dynamical systems. Specifically, we define finite-time stability for equilibria of discrete autonomous dynamical systems possessing unique solutions in forward time. Continuity and uniqueness of solutions in forward time result in solutions being continuous functions of the system initial conditions, and hence, system solutions define a continuous semiflow on the state space. Using uniqueness of solutions, we define a settling-time function and establish lower semicontinuity of this function. Then we present Lyapunov and converse Lyapunov theorems for finite time stability. Specifically, the Lyapunov functions are assumed to be continuous with converse results shown to hold under the assumption that the settling-time function is lower semicontinuous.

Most of the available techniques for feedback stabilization of discrete-time systems lead to closed-loop systems that guarantee exponential convergence with an infinite settling time. Thus, a key application area of the proposed finite time stability theorems is to build on the results of the paper and extend the framework to address the problem of *optimal finite-time discrete stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite time stability of the closed-loop system. Such a framework can allow us to explore connections between finite time stability and time optimality, and relate the results of this paper to results on the time optimal and inverse optimal control problems for discrete-time systems.

☆ This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-20-1-0038. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Björn S. Rüffer under the direction of Editor Daniel Liberzon.

* Corresponding author.
E-mail addresses: wm.haddad@aerospace.gatech.edu (W.M. Haddad), j.s.lee@gatech.edu (J. Lee).

https://doi.org/10.1016/j.automatica.2020.109282
0005-1098/

2. Mathematical preliminaries

In this section, we establish notation, definitions, and present some key results needed for developing the main results of this paper. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of positive integers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denote transpose. We write $\mathscr{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm in $\mathbb{R}^n$, $\Delta V(x)$ for the difference operator of $V : \mathbb{R}^n \to \mathbb{R}$ at x, $\lceil\alpha\rceil$ for the ceiling function denoting the smallest integer greater than or equal to α, and $\lfloor\alpha\rfloor$ for the floor function denoting the greatest integer less than or equal to α. Furthermore, we write $\overline{\mathscr{S}}$ and $\partial\mathscr{S}$ to denote the closure and boundary of the set $\mathscr{S} \subset \mathbb{R}^n$, respectively.

Consider the discrete-time nonlinear dynamical system

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{1}$$

where $x(k) \in \mathscr{D} \subseteq \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the system state vector, $\mathscr{D}$ is an open set, $0 \in \mathscr{D}$, $f : \mathscr{D} \to \mathscr{D}$, and $f(0) = 0$. We denote the solution to (1) with initial condition $x(0) = x_0$ by $s(\cdot, x_0)$ so that the *map* of the dynamical system given by $s : \overline{\mathbb{Z}}_+ \times \mathscr{D} \to \mathscr{D}$ is continuous on $\mathscr{D}$ and satisfies the *consistency* property $s(0, x_0) = x_0$ and the *semigroup* property $s(\kappa, s(k, x_0)) = s(k+\kappa, x_0)$ for all $x_0 \in \mathscr{D}$ and $k, \kappa \in \overline{\mathbb{Z}}_+$. We use the notation $s(k, x_0)$, $k \in \overline{\mathbb{Z}}_+$, and $x(k)$, $k \in \overline{\mathbb{Z}}_+$, interchangeably as the solution of the nonlinear discrete-time dynamical system (1) with initial condition $x(0) = x_0$. By a *solution* to (1) with initial condition $x(0) = x_0$ we mean a function $x : \overline{\mathbb{Z}}_+ \to \mathscr{D}$ that satisfies (1). Given $k \in \overline{\mathbb{Z}}_+$ and $x \in \mathscr{D}$, we denote the map $s(k, \cdot) : \mathscr{D} \to \mathscr{D}$ by s_k and the map $s(\cdot, x) : \overline{\mathbb{Z}}_+ \to \mathscr{D}$ by s^x.

If $f(\cdot)$ is continuous, then it follows that $f(s(k-1, \cdot))$ is also continuous since it is constructed as a composition of continuous functions. Hence, $s(k, \cdot)$ is continuous on $\mathscr{D}$. If $f(\cdot)$ is such that $f : \mathbb{R}^n \to \mathbb{R}^n$, then we can construct the *solution sequence* or *discrete trajectory* $x(k) = s(k, x_0)$ to (1) iteratively by setting $x(0) = x_0$ and using $f(\cdot)$ to define $x(k)$ recursively by $x(k+1) = f(x(k))$. This iterative process can be continued indefinitely, and hence, a solution to (1) exists for all $k \geq 0$.

Alternatively, if $f(\cdot)$ is such that $f : \mathscr{D} \to \mathbb{R}^n$, then the solution may cease to exist at some point if $f(\cdot)$ maps $x(k)$ into some point $x(k+1)$ outside the domain of $f(\cdot)$. In this case, the solution sequence $x(k) = s(k, x_0)$ will be defined on the maximal interval of existence $x(k)$, $k \in \mathscr{I}^+_{x_0} \subset \overline{\mathbb{Z}}_+$. Furthermore, note that the solution sequence $x(k)$, $k \in \mathscr{I}^+_{x_0}$, is uniquely defined for every initial condition $x_0 \in \mathscr{D}$ irrespective of whether or not $f(\cdot)$ is a continuous function. That is, any other solution sequence $y(k)$ starting from x_0 at $k = 0$ will take exactly the same values as $x(k)$ and can be continued to the same interval as $x(k)$. It is important to note that if $k \in \overline{\mathbb{Z}}_+$, then uniqueness of solutions backward in time need not necessarily hold. This is due to the fact that $(k, x_0) = f^{-1}(s(k+1, x_0))$, $k \in \overline{\mathbb{Z}}_+$, and there is no guarantee that $f(\cdot)$ is invertible for all $k \in \overline{\mathbb{Z}}_+$. However, if $f : \mathscr{D} \to \mathscr{D}$ is a homeomorphism for all $k \in \overline{\mathbb{Z}}_+$, then the solution sequence is unique for all $k \in \mathbb{Z}$.

In light of the above discussion the following theorem is immediate.

Theorem 2.1 (Haddad & Chellaboina, 2008). *Consider the nonlinear dynamical system (1). Then, for every $x_0 \in \mathscr{D}$, there exists $\mathscr{I}^+_{x_0} \subseteq \overline{\mathbb{Z}}_+$ such that (1) has a unique solution $x : \mathscr{I}^+_{x_0} \to \mathbb{R}^n$. Moreover, if $f(\cdot)$ is continuous, then the solution $s(k, \cdot)$ is continuous for each $k \in \mathscr{I}^+_{x_0}$. If, in addition, $f(\cdot)$ is a homeomorphism of $\mathscr{D}$ onto $\mathbb{R}^n$, then the solution $x : \mathscr{I}^+_{x_0} \to \mathbb{R}^n$ is unique in all $\mathscr{I}_{x_0} \subseteq \mathbb{Z}$ and $s(k, \cdot)$ is continuous for all $k \in \mathscr{I}_{x_0}$. Finally, if $\mathscr{D} = \mathbb{R}^n$, then $\mathscr{I}_{x_0} = \mathbb{Z}$.*

The following definition introduces the notion of class $\mathscr{W}_{\mathrm{d}}$ functions involving *nondecreasing* functions.

Definition 2.1. A function $w : \mathbb{R} \to \mathbb{R}$ is of *class* $\mathscr{W}_{\mathrm{d}}$ if $w(z') \leq w(z'')$ for all $z', z'' \in \mathbb{R}$ such that $z' \leq z''$.

To develop the theory for finite time stability of discrete autonomous systems, we will require several key results on difference inequalities and the discrete-time comparison principle. Consider the scalar discrete-time nonlinear dynamical system given by

$$z(k+1) = w(z(k)), \quad z(k_0) = z_0, \quad k \in \mathscr{I}_{z_0}, \tag{2}$$

where $z(k) \in \mathscr{Q} \subseteq \mathbb{R}$, $k \in \mathscr{I}_{z_0}$, is the system state vector, $\mathscr{I}_{z_0} \subseteq \mathbb{Z}$ is the maximal interval of existence of a solution $z(k)$ to (2), $\mathscr{Q}$ is an open set, $0 \in \mathscr{Q}$, and $w : \mathscr{Q} \to \mathbb{R}$ is a continuous function on $\mathscr{Q}$.

Theorem 2.2. *Consider the discrete-time nonlinear dynamical system (2). Assume that the function $w : \mathscr{Q} \subseteq \mathbb{R} \to \mathbb{R}$ is continuous and $w(\cdot)$ is of class $\mathscr{W}_{\mathrm{d}}$. If there exists a continuous function $V : \mathscr{I}_{z_0} \to \mathscr{Q}$ such that*

$$V(k+1) \leq w(V(k)), \quad k \in \mathscr{I}_{z_0}, \tag{3}$$

then

$$V(k_0) \leq z_0, \quad z_0 \in \mathscr{Q}, \tag{4}$$

implies

$$V(k) \leq z(k), \quad k \in \mathscr{I}_{z_0}, \tag{5}$$

where $z(k)$, $k \in \mathscr{I}_{z_0}$, is the solution to (2).

Proof. Suppose, *ad absurdum*, that inequality (5) does not hold on the entire interval $\mathscr{I}_{z_0}$. Then there exists $\hat{k} \in \mathscr{I}_{z_0}$ such that $V(k) \leq z(k)$, $k_0 \leq k < \hat{k}$, and

$$V(\hat{k}) > z(\hat{k}). \tag{6}$$

Since $w(\cdot) \in \mathscr{W}_{\mathrm{d}}$, it follows from (2), (3), and (6) that

$$w(z(\hat{k}-1)) = z(\hat{k}) < V(\hat{k}) \leq w(V(\hat{k}-1)) \leq w(z(\hat{k}-1)), \tag{7}$$

which is a contradiction. □

The following result is a direct corollary of Theorem 2.2

Corollary 2.1. *Consider the discrete-time nonlinear dynamical system (1). Assume there exists a continuous function $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathscr{Q}$ such that*

$$V(f(x)) \leq w(V(x)), \quad x \in \mathscr{D}, \tag{8}$$

where $w : \mathscr{Q} \subseteq \mathbb{R} \to \mathbb{R}$ is continuous, $w(\cdot) \in \mathscr{W}_{\mathrm{d}}$, and

$$z(k+1) = w(z(k)), \quad z(k_0) = z_0, \quad k \in \mathscr{I}_{z_0}. \tag{9}$$

If $\{k_0, \ldots, k_0+\tau\} \subseteq \mathscr{I}_{x_0} \cap \mathscr{I}_{z_0}$, then

$$V(x_0) \leq z_0, \quad z_0 \in \mathscr{Q}, \tag{10}$$

implies

$$V(x(k)) \leq z(k), \quad k \in \{k_0, \ldots, k_0+\tau\}. \tag{11}$$

Proof. For every given $x_0 \in \mathscr{D}$, the solution $x(k)$, $k \in \mathscr{I}_{x_0}$, to (1) is well defined. With $\eta(k) \triangleq V(x(k))$, $k \in \mathscr{I}_{x_0}$, it follows from (8) that

$$\eta(k+1) \leq w(\eta(k)), \quad k \in \mathscr{I}_{x_0}. \tag{12}$$

Moreover, if $\{k_0, \ldots, k_0+\tau\} \subseteq \mathscr{I}_{x_0} \cap \mathscr{I}_{z_0}$, then it follows from Theorem 2.2 that $V(x_0) = \eta(k_0) \leq z_0$ implies

$$V(x(k)) = \eta(k) \leq z(k), \quad k \in \{k_0, \ldots, k_0+\tau\}, \tag{13}$$

which establishes the result. □

Note that if the solutions to (1) and (9) are globally defined for all $x_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}$, then Corollary 2.1 holds for all $k \geq k_0$. For the remainder of the paper we assume, without loss of generality, that $k_0 = 0$.

3. Finite-time stability of discrete-time nonlinear dynamical systems

In this section, we develop the notion of finite time stability for discrete-time nonlinear dynamical systems. The notion of finite time stability involves finite time convergence along with Lyapunov stability as detailed in the next definition.

Definition 3.1. Consider the nonlinear dynamical system (1). The zero solution $x(k) \equiv 0$ to (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $K : \mathcal{N}\backslash\{0\} \to \mathbb{Z}_+$, called the *settling time function*, such that the following statements hold:

(i) *Finite-time convergence.* For every $x \in \mathcal{N} \setminus \{0\}$ and $k \geq K(x)$, $s^x(k)$ is contained in $\mathcal{N} \cap \{0\}$.

(ii) *Lyapunov stability.* For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s(k, x) \in \mathcal{B}_\varepsilon(0)$ for all $k \in \{0, \ldots, K(x_0) - 1\}$.

The zero solution $x(k) \equiv 0$ to (1) is *globally finite time stable* if it is finite time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Note that if the zero solution $x(k) \equiv 0$ to (1) is finite time stable, then it is asymptotically stable; however, the converse is not true.

Remark 3.1. The notion of finite-time stability introduced here is different from the same term discussed in Amato, Carbone, Ariola, and Cosentino (2004), Kang, Zhong, Shi, and Cheng (2016) and Mastellone, Dorato, and Abdallah (2004). Specifically, the term finite-time stability discussed in Amato et al. (2004), Kang et al. (2016) and Mastellone et al. (2004) deals with systems whose operation is constrained to a fixed finite interval of time and requires bounds on the system state variables.

For continuous-time dynamical systems with a finite-time stable equilibrium, the vector field $f(\cdot)$ is necessarily non-Lipschitzian at the system equilibrium because of backward nonuniqueness at the system equilibrium. This leads to standard existence and uniqueness results not applying to solutions reaching the system equilibrium. Consequently, finite time stability is defined over the time interval that the solution takes to reach the system equilibrium with solutions after the settling time function being given as a separate result (see Proposition 2.3 of Bhat & Bernstein, 2000). In contrast, for finite time discrete autonomous systems forward uniqueness is always guaranteed, and hence, such a result is not necessary.

It is easy to see from Definition 3.1 that

$$K(x) = \min\{k \in \mathbb{Z}_+ : s(k, x) = 0\}, \quad x \in \mathcal{N}, \tag{14}$$

and $K(0) = 0$ for the equilibrium point $x_e = 0$.

The following two scalar examples illustrate finite time stability and are used later in the paper.

Example 3.1. Consider the scalar discrete-time nonlinear dynamical system given by

$$x(k+1) = x(k) - \operatorname{sign}(x(k)) \min\{|x(k)|, c\}, \quad x(0) = x_0, \quad k \geq 0, \tag{15}$$

where $x(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, $\operatorname{sign}(x) \triangleq x/|x|$, $x \neq 0$, $\operatorname{sign}(0) \triangleq 0$, and $c > 0$. Note that the right-hand side of (15) is continuous everywhere and, for every initial condition in $\mathbb{R}\backslash\{0\}$, (15) has a unique solution in forward time. Furthermore, note that if $|x(k)| \leq c$, $k \in \overline{\mathbb{Z}}_+$, then $x(k+1) = x(k) - \operatorname{sign}(x(k))\,|x(k)| = 0$, and hence, ($i$) of Definition 3.1 is satisfied with $\mathcal{N} = \mathcal{D} = \mathbb{R}$ and with the settling time function given by

$$K(x_0) = \left\lceil \frac{|x_0|}{c} \right\rceil. \tag{16}$$

Lyapunov stability follows by considering the Lyapunov function $V(x) = x^2$. Thus, the zero solution $x(k) \equiv 0$ is globally finite-time stable. $\triangle$

Example 3.2. Consider the scalar discrete-time nonlinear dynamical system given by

$$x(k+1) = x(k) - c \operatorname{sign}(x(k)) \min\left\{\frac{|x(k)|}{c}, |x(k)|^\alpha\right\}, \quad x(0) = x_0, \quad k \geq 0, \tag{17}$$

where $x(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, $\alpha \in (0, 1)$, and $c > 0$. Note that the right-hand side of (17) is continuous everywhere and, for every initial condition in $\mathbb{R}\backslash\{0\}$, (17) has a unique solution in forward time. Furthermore, note that if $|x(k)| \leq c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then $x(k+1) = 0$, and if $|x(k)| > c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then

$$\begin{aligned} |x(k)| &= \left|x(k-1)\left(1 - c|x(k-1)|^{\alpha-1}\right)\right| \\ &< |x(k-1)|, \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{18}$$

Since $\alpha \in (0, 1)$, $|x(k)|^{\alpha-1} > |x(k-1)|^{\alpha-1}$, $k \in \overline{\mathbb{Z}}_+$, and

$$1 - c\,|x(k)|^{\alpha-1} < 1 - c\,|x(k-1)|^{\alpha-1}, \quad k \in \overline{\mathbb{Z}}_+. \tag{19}$$

Next, it follows from (18), (19), and $|x(k)| > c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, that

$$\begin{aligned} |x(k)| &= |x(k-1)|\left(1 - c\,|x(k-1)|^{\alpha-1}\right) \\ &= |x(k-2)|\left(1 - c\,|x(k-2)|^{\alpha-1}\right) \\ &\qquad \left(1 - c\,|x(k-1)|^{\alpha-1}\right) \\ &\ \ \vdots \\ &= |x_0|\left(1 - c\,|x_0|^{\alpha-1}\right)\cdots\left(1 - c\,|x(k-1)|^{\alpha-1}\right) \\ &< |x_0|\left(1 - c\,|x_0|^{\alpha-1}\right)^k, \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{20}$$

Now, if $\left|x_0\left(1 - c\,|x_0|^{\alpha-1}\right)^k\right| \leq c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then $|x(k)| < c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, which implies $x(k+1) = 0$ for

$$k \geq \log_{[1-c|x_0|^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{|x_0|}, \quad |x_0| > c^{\frac{1}{1-\alpha}},$$

and hence, (i) of Definition 3.1 is satisfied with $\mathcal{N} = \mathcal{D} = \mathbb{R}$ and with the settling time function $K(x_0)$ given by either

$$K(x_0) \leq \left\lceil \log_{[1-c|x_0|^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{|x_0|} \right\rceil + 1, \quad |x_0| > c^{\frac{1}{1-\alpha}}, \tag{21}$$

or

$$K(x_0) = 1, \quad |x_0| \leq c^{\frac{1}{1-\alpha}}, \quad x_0 \neq 0. \tag{22}$$

Lyapunov stability follows by considering the Lyapunov function $V(x) = x^2$. Thus, the zero solution $x(k) \equiv 0$ to (1) is globally finite-time stable. $\triangle$

The following definition is needed for the statement of the next result.

Definition 3.2. Let $\mathcal{D} \subseteq \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}$, and $x \in \mathcal{D}$. g is *lower semicontinuous at* $x \in \mathcal{D}$ if for every sequence $\{x_n\}_{n=0}^\infty \subset \mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$, $g(x) \leq \liminf_{n\to\infty} g(x_n)$. g is *lower semicontinuous on* $\mathcal{D}$ if g is lower semicontinuous at every point $x \in \mathcal{D}$.

The next proposition shows that if the settling-time function of a finite-time stable system is lower semicontinuous at the origin, then it is lower semicontinuous on $\mathcal{N}$.

Proposition 3.1. *Consider the nonlinear dynamical system (1). Assume that the zero solution $x(k) \equiv 0$ to (1) is finite time stable, let $\mathcal{N} \subseteq \mathcal{D}$ be as in Definition 3.1, and let $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ be the settling-time function. Then $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$.*

Proof. Let $y \in \mathcal{N}$, consider the sequence $\{y_n\}_{n=0}^{\infty}$ in $\mathcal{N}$ converging to y, and let $\tau^- = \liminf_{n\to\infty} K(y_n)$. Let $\{y_m^-\}_{m=0}^{\infty}$ be a subsequence of $\{y_n\}_{n=0}^{\infty}$ such that $K(y_m^-) \to \tau^-$ as $m \to \infty$. Since K only takes integer values, it follows that there exists $M > 0$ such that $K(y_m^-) = \tau^-$ for all $m > M$. Since $s(k, \cdot)$ is continuous for each k, and since $K(y_m^-) = \tau^-$ for all $m > M$, it follows that $s(K(y_m^-), y_m^-) \to s(\tau^-, y)$ as $m \to \infty$. Now, it follows from (14) that $s(K(y_m^-), y_m^-) = 0$ for each m. Since the set $\{0\}$ is closed, we conclude that $s(\tau^-, y) = 0$. Eq. (14) now implies that

$$K(y) \le \tau^- = \liminf_{n\to\infty} K(y_n), \tag{23}$$

which implies that $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. □

Remark 3.2. In the case of continuous-time systems, it is known that the settling-time function $K(\cdot)$ of a finite-time stable equilibrium is continuous in the domain of convergence if and only if it is continuous at the equilibrium (see Proposition 2.4 of Bhat & Bernstein, 2000). In the case of discrete-time systems, the integer-valued function $K(\cdot)$ is continuous at a point only if it is locally constant. Thus, if $K(\cdot)$ is continuous at an equilibrium point x_e, then x_e necessarily has to satisfy $f(x_e) = x_e$. On the other hand, the set of equilibrium points is closed. Hence, $K(\cdot)$ can be continuous at any equilibrium point only in the uninteresting case where the set of equilibria is either empty or $f(x) = x$ for all $x \in \mathcal{D}$.

4. Lyapunov theorems for finite time stability

In this section, we present sufficient conditions for finite time stability using Lyapunov theorems involving a scalar difference fractional inequality and a minimum operator. For the results in this section, define $\Delta V(x) \triangleq V(f(x)) - V(x)$ for a given continuous function $V : \mathcal{D} \to \mathbb{R}$.

Theorem 4.1. *Consider the nonlinear dynamical system (1). Assume that there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $\alpha \in (0, 1)$ and $c > 0$, and a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that*

$$V(0) = 0, \tag{24}$$

$$V(x) > 0, \quad x \in \mathcal{M}\backslash\{0\}, \tag{25}$$

$$\Delta V(x) \le -c \min\left\{\frac{V(x)}{c}, V(x)^{\alpha}\right\}, \quad x \in \mathcal{M}\backslash\{0\}. \tag{26}$$

Then the zero solution $x(k) \equiv 0$ to (1) is finite time stable. Moreover, there exist an open neighborhood $\mathcal{N}$ of the origin and a settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ such that either

$$K(x_0) \le \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathcal{N}, \quad V(x_0) > c^{\frac{1}{1-\alpha}}, \tag{27}$$

or

$$K(x_0) = 1, \quad x_0 \in \mathcal{N}\backslash\{0\}, \quad V(x_0) \le c^{\frac{1}{1-\alpha}}, \tag{28}$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (26) holds on $\mathbb{R}^n$, then the zero solution $x(k) \equiv 0$ to (1) is globally finite time stable.

Proof. Since $V(\cdot)$ is positive definite and $\Delta V(\cdot)$ takes negative values on $\mathcal{M}\backslash\{0\}$, it follows from Lyapunov stability that $x(k) \equiv 0$ is the unique solution of (1) for $k \in \overline{\mathbb{Z}}_+$ satisfying $x(0) = 0$. Thus, for every initial condition in $\mathcal{D}$, (1) has a unique solution in forward time. Moreover, $\Delta V(0) = 0$, and hence, (26) holds on $\mathcal{M}$.

Let $\mathcal{V} \subseteq \mathcal{M}$ be a bounded open set such that $0 \in \mathcal{V}$ and $\overline{\mathcal{V}} \subset \mathcal{D}$. Then $\partial\mathcal{V}$ is compact and $0 \notin \partial\mathcal{V}$. Now, it follows from Weierstrass' theorem [Haddad & Chellaboina, 2008, p. 44] that the continuous function $V(\cdot)$ attains a minimum on $\partial\mathcal{V}$ and since $V(\cdot)$ is positive definite, $\min_{x\in\partial\mathcal{V}} V(x) > 0$. Let $0 < \beta < \min_{x\in\partial\mathcal{V}} V(x)$ and define $\mathcal{D}_\beta$ to be the arcwise connected component of $\{x \in \mathcal{V} : V(x) \le \beta\}$. Note that $\mathcal{D}_\beta$ is nonempty since $0 \in \mathcal{D}_\beta$, closed since $V(\cdot)$ is continuous, and bounded since $\mathcal{V}$ is bounded. It follows from (26) that $\mathcal{D}_\beta \subset \mathcal{M}$ is positively invariant with respect to (1). Furthermore, it follows from (26), the positive definiteness of $V(\cdot)$, and standard Lyapunov arguments that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_\beta \subset \mathcal{M}$ and

$$\|x(k)\| \le \varepsilon, \quad \|x_0\| < \delta, \quad k \in \mathcal{I}_{x_0}. \tag{29}$$

Moreover, since the solution $x(k)$ to (1) is bounded for all $k \in \mathcal{I}_{x_0}$, it can be extended on $\overline{\mathbb{Z}}_+$, and hence, $x(k)$ is defined for all $k \ge k_0$. Furthermore, it follows from Corollary 2.1 with $w\,(V(x)) = V(x) - c \min\left\{\frac{V(x)}{c}, V(x)^{\alpha}\right\}$ and $z(k) = s(k, V(x_0))$, where $\alpha \in (0, 1)$ and $s(\cdot, \cdot)$ is the solution to (17), that

$$V(x(k)) \le s(k, V(x_0)), \quad x_0 \in \mathcal{B}_\delta(0), \quad k \in \overline{\mathbb{Z}}_+. \tag{30}$$

Now, it follows from (21), (30), and the positive definiteness of $V(\cdot)$ that

$$x(k) = 0, \quad k \ge \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathcal{B}_\delta(0), \quad V(x_0) > c^{\frac{1}{1-\alpha}}, \tag{31}$$

which implies finite-time convergence of the trajectories of (1) for all $x_0 \in \mathcal{B}_\delta(0)$ such that $V(x_0) > c^{\frac{1}{1-\alpha}}$. Alternatively, if $V(x_0) \le c^{\frac{1}{1-\alpha}}$, then it follows from (22) that the zero solution $x(k) \equiv 0$ is finite-time stable with the settling-time function $K(x_0) = 1$. This along with (29) implies finite-time stability of the zero solution $x(k) \equiv 0$ to (1) with $\mathcal{N} \triangleq \mathcal{B}_\delta(0)$.

Next, since $s(0, x) = x$, $s(k, \cdot)$ is continuous, and $\Delta V(s(k, x)) = 0$ is equivalent to $f(s(k, x)) = 0$ on $\mathcal{N}$, it follows that $\min\{k \in \overline{\mathbb{Z}}_+ : s(k, x) = 0\} > 0$, $x \in \mathcal{N}\backslash\{0\}$. Furthermore, it follows from (31) that $\min\{k \in \overline{\mathbb{Z}}_+ : s(k, x) = 0\} < \infty$, $x \in \mathcal{N}$. Now, define $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ by using (14). Then it follows that the zero solution $x(k) \equiv 0$ is finite time stable and, by Proposition 3.1, K is a lower-semicontinuous settling-time function on $\mathcal{N}$.

Finally, if $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then global finite time stability follows using identical arguments. □

Theorem 4.2. *Consider the nonlinear dynamical system (1). Assume that there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, a real number $c > 0$, and a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that*

$$V(0) = 0, \tag{32}$$

$$V(x) > 0, \quad x \in \mathcal{M}\backslash\{0\}, \tag{33}$$

$$\Delta V(x) \le -\min\{V(x), c\}, \quad x \in \mathcal{M}\backslash\{0\}. \tag{34}$$

Then the zero solution $x(k) \equiv 0$ to (1) is finite time stable. Moreover, there exist an open neighborhood $\mathcal{N}$ of the origin and a settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ such that

$$K(x_0) \le \left\lceil \frac{V(x_0)}{c} \right\rceil, \quad x_0 \in \mathcal{N}, \tag{35}$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (34) holds on $\mathbb{R}^n$, then the zero solution $x(k) \equiv 0$ to (1) is globally finite time stable.

Proof. The proof follows using identical arguments as in the proof of Theorem 4.1 with $w(V(x)) = V(x) - \min\{V(x), c\}$ and $z(k) = s(k, V(x_0))$, where $s(\cdot,\cdot)$ satisfies (15). □

Example 4.1. Consider the nonlinear discrete-time dynamical system $\mathcal{G}$ given by (15). For this system, we show that the zero solution $x(k) \equiv 0$ to (15) is globally finite-time stable using Theorem 4.2. To see this, consider $V(x) = x^2$ and let $|x_0| > c$. (Note that if $|x_0| \leq c$, then the zero solution $x(k) \equiv 0$ to (15) is finite-time stable with $K(x_0) = 1$.) Then,

$$\begin{aligned}\Delta V(x) &= V(f(x)) - V(x)\\ &= [x - \mathrm{sign}(x)\min\{|x|, c\}]^2 - x^2\\ &= -2|x|\min\{|x|, c\} + [\min\{|x|, c\}]^2\\ &= \min\{|x|, c\}[\min\{|x|, c\} - 2|x|]\\ &\leq -\min\{|x|, c\}|x|\\ &\leq -[\min\{|x|, c\}]^2\\ &= -\min\left\{|x|^2, c^2\right\}\\ &= -\min\left\{V(x), c^2\right\}, \quad x \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{36}$$

Hence, it follows from Theorem 4.2 that the zero solution $x(k) \equiv 0$ to (15) is globally finite-time stable with settling time function

$$K(x_0) = \left\lceil \frac{|x_0|}{c} \right\rceil \leq \left\lceil \frac{x_0^2}{c^2} \right\rceil = \left\lceil \frac{V(x_0)}{c^2} \right\rceil. \quad \triangle \tag{37}$$

Example 4.2. Consider the nonlinear dynamical system $\mathcal{G}$ given by (17). For this system, we show that the zero solution $x(k) \equiv 0$ to (17) is globally finite-time stable using Theorem 4.1. To see this, consider $V(x) = x^2$ and let $\alpha \in (0, 1)$ and $|x_0| > c^{\frac{1}{1-\alpha}}$. (Note that if $|x_0| \leq c^{\frac{1}{1-\alpha}}$, $x_0 \neq 0$, then the zero solution $x(k) \equiv 0$ to (15) is finite-time stable with $K(x_0) = 1$.) Then,

$$\begin{aligned}\Delta V(x) &= \left[x - c\,\mathrm{sign}(x)\min\left\{\frac{|x|}{c}, |x|^\alpha\right\}\right]^2 - x^2\\ &= c\,\min\left\{\frac{|x|}{c}, |x|^\alpha\right\}\left[c\,\min\left\{\frac{|x|}{c}, |x|^\alpha\right\} - 2|x|\right]\\ &\leq -c\,\min\left\{\frac{|x|}{c}, |x|^\alpha\right\}|x|\\ &\leq -c^2\left[\min\left\{\frac{|x|}{c}, |x|^\alpha\right\}\right]^2\\ &= -c^2\,\min\left\{\frac{|x|^2}{c^2}, |x|^{2\alpha}\right\}\\ &= -c^2\,\min\left\{\frac{V(x)}{c^2}, V(x)^\alpha\right\}, \quad x \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{38}$$

Hence, it follows from Theorem 4.1 that the zero solution $x(k) \equiv 0$ to (17) is globally finite-time stable with the settling-time function

$$\begin{aligned}K(x_0) &\leq \left\lceil \log_{[1-c^2V(x_0)^{\alpha-1}]} \frac{c^{\frac{2}{1-\alpha}}}{V(x_0)} \right\rceil + 1\\ &\leq \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1. \quad \triangle \end{aligned} \tag{39}$$

Example 4.3. Consider the discrete-time linear dynamical system given by

$$x(k+1) = Ax(k), \quad x(0) = x_0, \quad k \geq 0, \tag{40}$$

where $x(k) \in \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, $A \in \mathbb{R}^{n\times n}$, and assume that (32)–(34) hold with $c > 0$ and $V(x) = x^T Px$, where $P \in \mathbb{R}^{n\times n}$ is positive definite, that is, $P > 0$. In this case, it follows from Theorem 4.2 that the zero solution $x(k) \equiv 0$ to (40) is finite time stable. Furthermore, (34) implies

$$\begin{aligned}\Delta V(x) &= V(Ax) - V(x)\\ &= x^T\left(A^T PA - P\right)x\\ &\leq -\min\left\{x^T Px, c\right\}, \quad x \in \mathbb{R}^n\backslash\{0\}. \end{aligned} \tag{41}$$

Now, if $x_0^T Px_0 \leq c$, $x_0 \in \mathbb{R}^n\setminus\{0\}$, then $x_0^T A^T PAx_0 = 0$. Alternatively, if $x_0^T Px_0 > c$, $x_0 \in \mathbb{R}^n\backslash\{0\}$, then

$$\begin{aligned}V(x(k)) &\leq V(x(k-1)) - \min\{V(x(k-1)), c\}\\ &= V(x(k-1)) - c\\ &\leq V(x(k-2)) - 2c\\ &\;\;\vdots\\ &\leq V(x_0) - kc. \end{aligned} \tag{42}$$

Now, for $k \geq \left\lfloor \frac{V(x_0)}{c} \right\rfloor$, $V(x(k)) \leq c$, and hence, $V(x(k+1)) = 0$. Thus,

$$\begin{aligned}\Delta V(x(k)) &\leq -\min\{V(x(k)), c\}\\ &= -V(x(k)), \quad k \geq \left\lfloor \frac{V(x_0)}{c} \right\rfloor, \end{aligned} \tag{43}$$

and hence,

$$V(x(k+1)) = x^T(k)A^T PAx(k) \leq 0. \tag{44}$$

Next, since $P > 0$ and $x(k) = A^k x_0$, it follows that

$$Ax(k) = A^{k+1}x_0 = 0, \quad k \geq \left\lfloor \frac{V(x_0)}{c} \right\rfloor, \tag{45}$$

which implies that A is a nilpotent matrix with maximum index $q = \left\lceil \frac{V(x_0)}{c} \right\rceil$, that is, $A^q = 0$.

Conversely, if A is nilpotent, then finite time stability of the zero solution $x(k) \equiv 0$ to (40) is immediate. △

Finally, the next result establishes a relationship between finite time convergence and finite time stability.

Theorem 4.3. *Consider the nonlinear dynamical system (1). Assume that there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$ and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that*

$$V(0) = 0, \tag{46}$$

$$V(x) > 0, \quad x \in \mathcal{M}\backslash\{0\}, \tag{47}$$

$$V(f(x)) \leq w(V(x)), \quad x \in \mathcal{M}\backslash\{0\}, \tag{48}$$

where $w : \mathbb{R} \to \mathbb{R}$ is continuous, $w(0) = 0$, and

$$z(k+1) = w(z(k)), \quad z(0) = z_0, \quad k \geq 0. \tag{49}$$

If (49) is finite time convergent to the origin and the origin is a Lyapunov stable equilibrium point of (1), then the zero solution $x(k) \equiv 0$ to (1) is finite time stable. Moreover, the settling-time function of (1) is lower semicontinuous on an open neighborhood of the origin.

Proof. It follows from Lyapunov stability that $x(k) \equiv 0$ is the unique solution of (1) for $k \in \overline{\mathbb{Z}}_+$ satisfying $x(0) = 0$. Thus, for every initial condition in $\mathcal{D}$, (1) has a unique solution in forward time. Moreover, $V(f(0)) = 0$, and hence, (48) holds on $\mathcal{M}$.

Since $x(k) \equiv 0$ to (1) is Lyapunov stable, it follows that there exists an open positively invariant set $\mathcal{V} \subseteq \mathcal{M}$ such that $0 \in \mathcal{V}$. Moreover, since the solution $x(k)$ to (1) is bounded for all $k \in \mathcal{I}_{x_0}$, it can be extended on $\overline{\mathbb{Z}}_+$, and hence, $x(k)$ is defined for all $k \geq k_0$. Next, it follows from (48) that

$$V(s(k+1,x)) \leq w(V(s(k,x))), \quad x \in \mathcal{V}, \quad k \in \overline{\mathbb{Z}}_+. \tag{50}$$

Now, it follows from Corollary 2.1 that

$$V(s(k,x)) \leq \psi(k, V(x_0)), \quad x \in \mathcal{V}, \quad k \in \overline{\mathbb{Z}}_+, \tag{51}$$

where $\psi : \overline{\mathbb{Z}}_+ \times \mathbb{R} \to \mathbb{R}$ is the global semiflow of (49). Since (49) is finite time convergent to the origin for $\overline{\mathbb{R}}_+$, it follows from (47) and (51) that

$$V(s(k,x)) = 0, \quad k \geq \hat{K}(V(x)), \quad x \in \mathcal{V}, \tag{52}$$

where $\hat{K}(\cdot)$ denotes the settling-time function of (49).

Next, since $s(0,x) = x$, $s(\cdot,x)$ is lower semicontinuous, $s(k,\cdot)$ is continuous, and $V(s(k,x)) = 0$ is equivalent to $s(k,x) = 0$ on $\mathcal{V}$, it follows that $\min\{k \in \overline{\mathbb{Z}}_+ : s(k,x) = 0\} > 0$, $x \in \mathcal{V}\backslash\{0\}$. Furthermore, it follows from (52) that $\min\{k \in \overline{\mathbb{Z}}_+ : s(k,x) = 0\} < \infty$, $x \in \mathcal{V}$. Now, define $K : \mathcal{V} \to \mathbb{Z}_+$ by using (14). Then it follows that the origin is finite-time stable and, by Proposition 3.1, K is a lower-semicontinuous settling time function on $\mathcal{V}$. Furthermore, it follows from (52) that $K(x) \leq \hat{K}(V(x))$, $x \in \mathcal{V}$. □

5. Converse Lyapunov theorems for finite time stability

In this section, we present two converse theorems to Theorems 4.1 and 4.2 for finite-time stability of discrete-time nonlinear dynamical systems.

Theorem 5.1. *Let $\alpha \in (0,1)$ and let $\mathcal{N}$ be as in Definition 3.1. If the zero solution $x(k) \equiv 0$ to (1) is finite time stable, then there exist a lower-semicontinuous function $V : \mathcal{N} \to \mathbb{R}$ and real numbers $\alpha \in (0,1)$ and $c > 0$ such that $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{N}$, $x \neq 0$, and $\Delta V(x) \leq -c \min\left\{\frac{V(x)}{c}, V(x)^\alpha\right\}$, $x \in \mathcal{N}$.*

Proof. First, it follows from Proposition 3.1 that the settling-time function $K : \mathcal{N} \to \mathbb{Z}_+$ is lower semicontinuous on $\mathcal{N}$. Next, define $V : \mathcal{N} \to \mathbb{R}_+$ by

$$V(x) \triangleq \sup_{k \geq 0} \frac{1+bk}{1+ak} [K(s(k,x))]^\beta, \tag{53}$$

where $\beta > 2$, $\beta \in \mathbb{Z}_+$, and $b > a > 0$. Note that $V(\cdot)$ is lower semicontinuous and positive definite, and, by $s(K(x)+k, x) = 0$ for all $x \in \mathcal{N}$ and $k \in \mathbb{Z}_+$, $\Delta V(0) = 0$. Now, note that the supremum in the definition of $V(s(1,x))$ is reached at some time $\hat{k}$ such that $0 \leq \hat{k} \leq K(x)$. If $\hat{k} < K(x)$, then

$$\begin{aligned} V(s(1,x)) &= \frac{1+b\hat{k}}{1+a\hat{k}} \left[K(s(\hat{k}+1,x))\right]^\beta \\ &= \left[1 - \frac{b-a}{(1+b\hat{k}+b)(1+a\hat{k})}\right] \\ &\quad \frac{1+b\hat{k}+b}{1+a\hat{k}+a} \left[K(s(\hat{k}+1,x))\right]^\beta \\ &\leq \left[1 - \frac{a(b-a)}{b\,[1+aK(x)]^2}\right] V(x), \quad K(x) > \hat{k}. \end{aligned} \tag{54}$$

Alternatively, if $\hat{k} = K(x)$, then $V(s(1,x)) = 0$, which implies $x = 0$.

Next, if $x \neq 0$, then

$$V(x) = \sup_{k \geq 0} \frac{1+bk}{1+ak} [K(s(k,x))]^\beta \geq [K(x)]^\beta \geq 1, \tag{55}$$

and hence, $[1+aK(x)]^\beta \leq (1+a)^\beta V(x)$. Now, (54) yields

$$\begin{aligned} &V(f(x)) - V(x) \\ &\quad \leq -\frac{a(b-a)}{b} V(x)\,(1+K(x))^{-2} \\ &\quad \leq -\frac{a(b-a)}{b} V(x) \left[(1+a)^\beta V(x)\right]^{-\frac{2}{\beta}} \\ &\quad = -\frac{a(b-a)}{b(1+a)^2} V(x)^{\frac{\beta-2}{\beta}} \\ &\quad \leq -\frac{a(b-a)}{b(1+a)^2} \min\left\{\frac{b(1+a)^2}{a(b-a)} V(x), V(x)^{\frac{\beta-2}{\beta}}\right\}, \end{aligned} \tag{56}$$

which proves the result with $\alpha = \frac{\beta-2}{\beta} \in (0,1)$ and $c = \frac{a(b-a)}{b(1+a)^2} > 0$. □

Theorem 5.2. *Let $\mathcal{N}$ be as in Definition 3.1. If the zero solution $x(k) \equiv 0$ to (1) is finite time stable, then there exist a lower-semicontinuous function $V : \mathcal{N} \to \mathbb{R}$ and a real number $c > 0$ such that $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{N}$, $x \neq 0$, and $\Delta V(x) \leq -\min\{V(x), c\}$, $x \in \mathcal{N}$.*

Proof. First, note that it follows from Proposition 3.1 that the settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ is lower semicontinuous on $\mathcal{N}$. Next, define $V : \mathcal{N} \to \overline{\mathbb{R}}_+$ by $V(x) \triangleq cK(x)$, where $c > 0$. Note that $V(\cdot)$ is lower semicontinuous and nonnegative, and, by $s(K(x)+k, x) \in \mathcal{N} \cap \Delta f^{-1}(0)$ for all $x \in \mathcal{N}$ and $k \in \mathbb{Z}_+$, $\Delta V(x) = 0$, $x \in \mathcal{N} \cap \Delta f^{-1}(0)$. Now, since the zero solution $x(k) \equiv 0$ of (1) is finite time stable and $K(s(1,x)) = K(x) - 1$, it follows that

$$\begin{aligned} V(f(x)) &= cK(f(x)) \\ &= cK(s(1,x)) \\ &= c(K(x)-1) \\ &= V(x) - c, \quad x \neq 0, \end{aligned} \tag{57}$$

and hence,

$$V(f(x)) - V(x) = -c \leq -\min\{V(x), c\}. \quad \square \tag{58}$$

6. Illustrative numerical example

In this section, we provide an illustrative numerical example to demonstrate the interplay between α and c for finite time convergence in Theorem 4.1. Consider the scalar discrete-time nonlinear dynamical system (17). To examine the conservatism of the *guaranteed* settling-time function $K(x_0)$ with the *achieved* finite time convergence, we vary the parameters α and c in (17). Fig. 1 shows the state $x(k)$ versus the discrete time k, whereas Table 1 shows that the gap between the guaranteed settling-time function $K(x_0)$ versus the achieved settling time for different values of α and c.

Table 1
Initial condition and parameters for finite time convergence of (17).

	Case I	Case II	Case III	Case IV
x_0	1000	1000	1000	1000
α	0.5	0.5	0.4	0.6
c	2	4	4	4
$K(x_0)$	86	32	72	13
Actual steps	30	14	25	8

7. Conclusion

This paper addresses the notion of finite time stability for discrete-time autonomous systems. Specifically, Lyapunov and

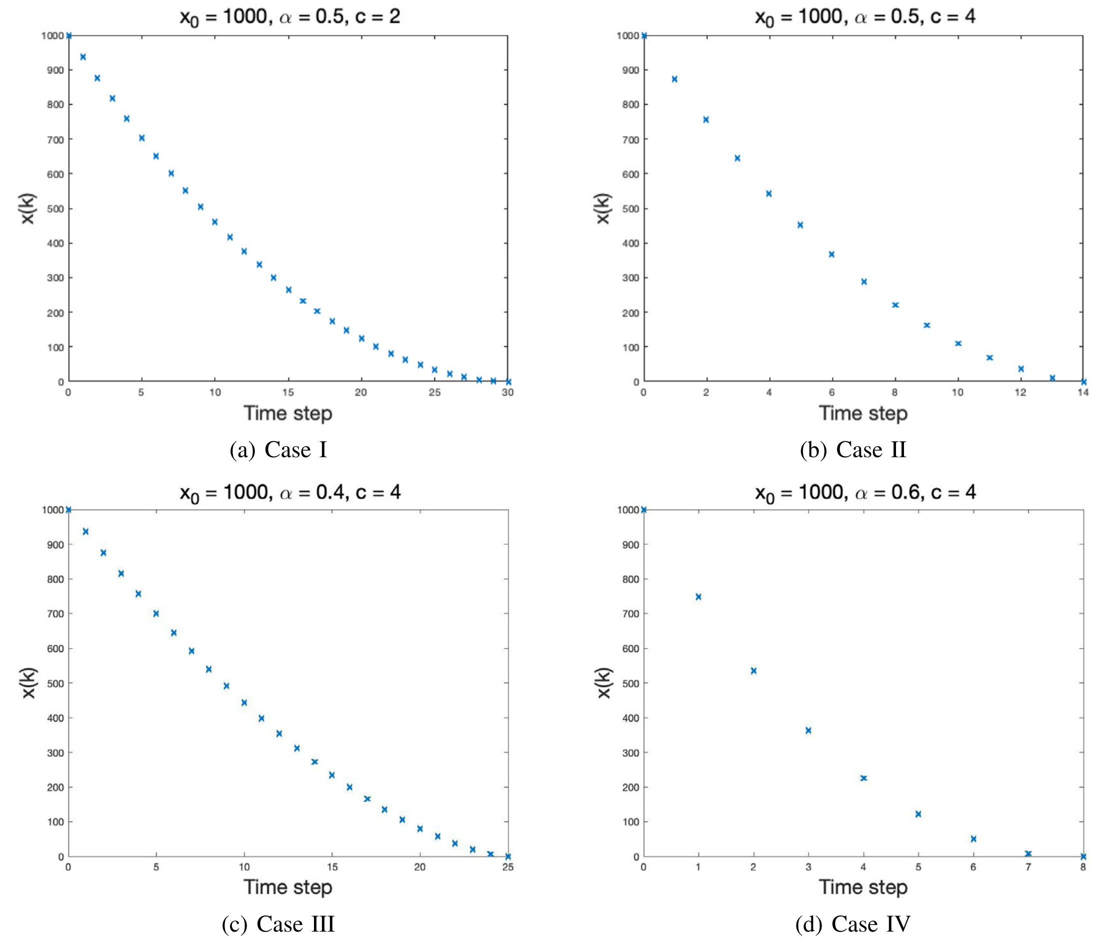

Fig. 1. Interplay between α and c for finite time convergence. Note that increasing both α and c results in faster finite-time convergence.

converse Lyapunov theorems for finite-time stability involving scalar difference fractional inequalities and minimum operators are established. The regularity properties of the Lyapunov functions satisfying the sufficient Lyapunov difference inequalities were shown to be strongly dependent on the regularity properties of the settling-time function. In addition, sufficient conditions for lower semicontinuity of the settling-time function are also presented.

Future extensions will focus on merging the results of this paper with Bhat and Bernstein (2000) to develop stronger Lyapunov finite time stability theorems for hybrid dynamical systems than the ones reported in Nersesov and Haddad (2008). In addition, the results in this paper can be extended to develop the notion of finite time semistability for developing consensus protocols for addressing finite time agreement in discrete-time networks. Finally, the results of this paper can be used to develop optimal finite time feedback stabilizing controllers for nonlinear discrete-time systems (Haddad & Lee, 2020).

Acknowledgment

The authors thank Dr. Sanjay P. Bhat for several helpful discussions.

References

Amato, F., Carbone, M., Ariola, M., & Cosentino, C. (2004). Finite-time stability of discrete-time systems. In *Proc. Amer. control conf.* (pp. 1440–1444).

Bhat, S. P., & Bernstein, D. S. (2000). Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization*, *38*(3), 751–766.

Bhat, S. P., & Bernstein, D. S. (2005). Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals, and Systems*, *17*(2), 101–127.

Chen, W., & Jiao, L. C. (2010). Finite-time stability theorem of stochastic nonlinear systems. *Automatica*, *46*(12), 2105–2108.

Haddad, W. M., & Chellaboina, V. (2008). *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton, NJ: Princeton University Press.

Haddad, W. M., & L'Afflitto, A. (2015). Finite-time partial stability, stabilization, and optimal feedback control. *Journal of The Franklin Institute*, *352*, 2329–2357.

Haddad, W., & Lee, J. (2020). Finite-time stabilization and optimal feedback control for nonlinear discrete-time systems. *IEEE Transactions on Automatic Control* (submitted for publication).

Haddad, W. M., Nersesov, S. G., & Du, L. (2009). Finite-time stability for time-varying nonlinear dynamical systems. In *Advances in nonlinear analysis: theory methods and application* (pp. 139–150). S. Sivasundaram, J. V. Devi, Z. Drici, and F. Mcrae, Eds.: Cambridge Scientific Publishers.

Hamrah, R., Sanyal, A. K., & Viswanathan, S. P. (2019). Discrete finite-time stable position tracking control of unmanned vehicles. In *Proc. IEEE conf. decision and control* (pp. 7025–7030).

Kang, W., Zhong, S., Shi, K., & Cheng, J. (2016). Finite-time stability for discrete-time system with time-varying delay and nonlinear perturbations. *ISA Tansactions*, *60*, 67–73.

Mastellone, S., Dorato, P., & Abdallah, C. T. (2004). Finite-time stability of discrete-time nonlinear systems: analysis and design. In *Proc. IEEE conf. decision and control* (pp. 2572–2577).

Moulay, E., & Perruquetti, W. (2008). Finite time stability conditions for non-autonomous continuous systems. *International Journal of Control*, *81*(5), 797–803.

Nersesov, S. G., & Haddad, W. M. (2008). Finite-time stabilization for nonlinear impulsive dynamical systems. *Nonlinear Analysis. Hybrid Systems*, *2*, 832–845.

Polyakov, A., Efimov, D., & Brogliato, B. (2018). Consistent discretization of finite-time stable homogeneous systems. In *VSS 2018 - 15th international workshop on variable structure systems and sliding mode control* (pp. 360–365).

Rajpurohit, T., & Haddad, W. M. (2017). Stochastic finite-time partial stability, partial-state stabilization, and finite-time optimal feedback control. *Mathematics of Control, Signals, and Systems*, *29*(2), 1–39.
Roxin, E. (1966). On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, *15*(3), 273–282.
Yin, J., Khoo, S., Man, Z., & Yu, X. (2011). Finite-time stability and instability of stochastic nonlinear systems. *Automatica*, *47*(12), 2671–2677.

Wassim M. Haddad received the B.S., M.S., and Ph.D. degrees in mechanical engineering from Florida Tech in 1983, 1984, and 1987. Since 1994 he has been with the School of Aerospace Engineering at Georgia Tech, where he holds the rank of Professor, the David Lewis Chair in Dynamical Systems and Control, and Chair of the Flight Mechanics and Control Discipline. He also holds a joint Professor appointment with the School of Electrical and Computer Engineering at Georgia Tech. He is the Co-Founder, Chairman of the Board, and Chief Scientific Advisor of Autonomous Healthcare, Inc. Dr. Haddad has made numerous contributions to the development of nonlinear control theory and its application to aerospace, electrical, and biomedical engineering. His transdisciplinary research in systems and control is documented in over 650 archival journal and conference publications, and 8 books in the areas of science, mathematics, medicine, and engineering. Dr. Haddad is an NSF Presidential Faculty Fellow; a member of the Academy of Nonlinear Sciences; an IEEE Fellow; and the recipient of the 2014 AIAA Pendray Aerospace Literature Award.

Junsoo Lee received the B.S. degree in mechanical and aerospace engineering and the M.S. degree in aerospace engineering from the Seoul National University, Republic of Korea, in 2016 and 2018, respectively. In 2018 he joined the Georgia Institute of Technology, where he is currently pursuing a doctoral degree in aerospace engineering. His research interests include stability theory, optimal control, multiagent systems, network control, and stochastic systems and control.