On the Functional Observer Design With Sampled Measurement

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***Abstract*—In this article, a functional observer for continuous systems with sampled measurement is designed for a class of nonlinear systems, operating under unknown inputs, and the existence conditions are derived. The nonlinear function, as a function of full states, is decomposed to Lipschitz term and an unknown one. The stability and boundedness of the estimation dynamics are analyzed and accordingly, the design procedure is formulated as a linear matrix inequality problem. Moreover, the design algorithm for some special cases is obtained. The numerical simulations are performed to evaluate the effectiveness of the proposed approach.**

***Index Terms*—Continuous nonlinear systems, functional observer, LMI, sampled measurements, unknown input.**

I. Introduction

The observer design plays a crucial role in most industrial systems for different purposes, such as fault detection [1] and control design [2]. Among all the available schemes, the functional observer, which was proposed in [3], has been widely studied in the last decade for different systems (see [4] and references therein). The functional observer has been originated in [5], [6], and [7]. Systems with unknown inputs have been emerged in [8] and [9] where generalized and reduced-order forms were developed, and further studied in some works, e.g., [10], [11], [12]. The functional observer has been also adopted with different applications, such as power systems [13] and multiagent systems [14]. The privilege of the functional observer over the others is the reduced size of the observer. On the other hand, the reduced number of estimated states removes the need for heavy computational burden and data acquisition framework. The necessary and sufficient conditions for the existence of a functional observer with order r and minimum order were derived in [3] and [15], respectively. The tradeoff between the use of functional or full-order observers is that the functional observer's design is less conservative since the observability or detectability condition is not needed.

Manuscript received 11 January 2022; revised 2 July 2022 and 4 November 2022; accepted 27 November 2022. Date of publication 1 December 2022; date of current version 30 August 2023. Recommended by Associate Editor S. K. Spurgeon. *(Corresponding author: Hamed Habibi.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2022.3226109.

Digital Object Identifier 10.1109/TAC.2022.3226109

The system representations are mainly categorized as continuous and discrete-time models. Consequently, the continuous or discrete observers are designed for corresponding systems. The continuous representation of the system takes the advantage of the instantaneous information of the output variation. This, in turn, enables almost real-time states estimation. However, this might require an expensive data acquisition scheme. In contrast, in the discrete approach, the data are sampled at a sequential sampling period [16]. Accordingly, the discrete-observer design has drawn a great deal of attention in the last few decades (see [17] and references therein).

In most discrete-time system representation and corresponding observer design, an implicit trivial assumption has been made, i.e., there is not any states variation during the sampling periods. Indeed, it is assumed that the system dynamics is varied exactly at the sampling time, at which the new output measurement is sampled. This, in turn, imposes a significant practical issue, i.e., the industrial systems operate in a continuous manner, and only the output measurement is discretely sampled. This is the case in, for example, bioreactors [18], chemical processes [16], biological denitrification process [19], and waste water treatment process [20]. Even though with the advent of digital sensors, the sampling time can be small compared to system response time, but it is worth investigating the observer design of the continuous system with sampled measurement, considering applicability to industrial systems.

To motivate the contributions of this article, the following points are worth mentioning. In [21], a high-gain observer is adapted to the continuous-discrete context. The robustness with respect to time discretization is investigated in [22]. In [23], an output predictor is used to design the observer. In [19], sufficient conditions of sampling time and system input to preserve the observability and stability of the observer are given. Furthermore, an algebraic Riccati equation is to be solved at each sampling time. In [24], an approximation of a reachable set is synthesized in real time. Similarly, in [25], the dynamics of the estimation error in each sampling period is governed by the state matrix. Therefore, in the case of unstable systems with a long sampling period, the estimation error can be large. In [18], the problem of high-gain observer design for a class of triangular nonlinear systems with sampled and delayed output measurements is addressed. In the mentioned works, when the updated measurement is not available, the state estimate is computed by integrating the model. Hence, the system dynamics is required to be stable. On the other hand, with the injection of new measurement, there can be a discontinuity in the dynamics of the observer, which in turn, imposes an obstacle on the convergence of estimation. Furthermore, when the unknown input is considered, the integration of the model is not feasible. Also, the full-order observer is sought for the considered systems, as it is required for the estimation of the nonlinear function, which is a function of the whole system states. The nonlinear function is usually considered as a Lipschitz function. On the other hand, the stability of the estimation dynamics is proven with some online constraints, to be solved at each sampling time.

Motivated by the abovementioned considerations, in this article, the problem of functional observer design of continuous systems with the discrete measurement with unknown input is studied. To the best of authors' knowledge, it is the first time that this problem is systematically tackled. The main contributions are as follows.

1) We present the existence conditions of functional observer for a class of nonlinear systems affected by an unknown input. This unknown input can represent the system uncertainty, disturbance, fault, and noise. This is in contrast to the most of H_∞ approaches [4], [26], where the unknown input is assumed to be of finite energy.
2) The nonlinearity, which is a function of full states, is decomposed into two terms, including a Lipschitz term as a function of linear combination of states, and an unknown term, in contrast to some works, e.g., [2], [24].
3) We incorporate a correction term, introduced in [23], at each sampling time to resolve the problem of discontinuity due to the injection of the new measurement. Accordingly, the stability and boundedness of the estimation dynamics are proven. The observer design is obtained in the linear matrix inequality (LMI) optimization, to minimize the estimation error bound.

The rest of this article is organized as follows. In Section II, the system dynamics is described. Also, some technical preliminaries are given. In Section III, the observer structure is given. Also, the stability and boundedness of the estimation dynamics are studied. Then, the observer design algorithm is obtained for some special cases in Section IV. Numerical simulation results are investigated in Section V. Finally, the concluding remarks are given in Section VI. The following notations are used. $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{C}$ represent real, integer, and complex number sets, respectively. $||\bullet||$ denotes the Euclidean norm of a vector and induced norm of a matrix. I_n and $0_{n\times p}$ represent the identity matrix of size n, and zero matrix of size $n\times p$, respectively. For any matrix H, H^- denotes the generalized inverse of H satisfying $HH^-H=H$. Finally, ∇_x represents the gradient operator with respect to x.

II. System Description and Preliminaries

Consider a class of a nonlinear system, modeled as

$$\dot{x}(t) = Ax(t) + Bu(t) + Fd(t) + \phi(x,u) \tag{1a}$$

$$y(t_k) = Cx(t_k) \tag{1b}$$

$$z(t) = Lx(t) \tag{1c}$$

where $x(t)\in\mathbb{R}^n$ is the states vector, $u(t)\in\mathbb{R}^m$ is the control inputs vector, $y(t_k)\in\mathbb{R}^p$ is the outputs vector at time step t_k with $k\in\mathbb{Z}_{\geq 0}$, $z(t)\in\mathbb{R}^r$ is the vector to be estimated where $r\leq n$, $d(t)\in\mathbb{R}^q$ is the unknown input, and $\phi(x,u)\in\mathbb{R}^n$ is the nonlinear function vector. It should be noted that the output measurement $y(t)$ is only available at time instant t_k. Matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, $L\in\mathbb{R}^{r\times n}$, and $F\in\mathbb{R}^{n\times q}$ are known and constant. The sampling period $\delta = t_k - t_{k-1}$ is assumed to be constant.

Assumption 1: It is assumed that $q\leq p$, i.e., the number of unknown inputs to be decoupled are less than the number of measurements [27]. Also, without loss of generality, we assume $rank(F)=q$, $rank(C)=p$, and $rank(L)=r$.

Assumption 2: It is assumed that the variation of states of system (1) in each sampling period is finite, i.e., for the bounded inputs $u(t)$ and $d(t)$ the variation of the states are bounded by an unknown upper bound in every sampling period $t\in[t_k,t_{k+1})$. Moreover, (1a) does not exhibit the finite escape time. Also, the variation of $d(t)$ in each sampling period is assumed to be bounded.

According to Assumption 2, for $t\in[t_k,t_{k+1})$ we have

$$\begin{aligned} x(t)-x(t_k) &= \Delta x(t,t_k)\\ u(t)-u(t_k) &= \Delta u(t,t_k)\\ d(t)-d(t_k) &= \Delta d(t,t_k) \end{aligned} \tag{2}$$

where $\Delta x(t,t_k)$, $\Delta u(t,t_k)$, and $\Delta d(t,t_k)$ are bounded as $||\Delta x(t,t_k)||\leq\bar{\Delta}_x$, $||\Delta u(t,t_k)||\leq\bar{\Delta}_u$, and $||\Delta d(t,t_k)||\leq\bar{\Delta}_d$, where $\bar{\Delta}_x$, $\bar{\Delta}_u$, and $\bar{\Delta}_d$ are unknown positive constants. Accordingly, it is obtained that the nonlinear function vector $\phi(x,u)$ is differentiable function in the region of the operation. Thus, one can obtain $\int_{t_0}^{\infty}\phi(x(t),u(t))\,dt = \sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}}\phi(x(t),u(t))\,dt$. The term $\int_{t_k}^{t_{k+1}}\phi(x(t),u(t))\,dt$ can be considered as a bounded disturbance to be attenuated on the estimation dynamic, however, in this article its effect is going to be compensated. Considering the mean value theorem, one can obtain that $\phi(x(t),u(t)) = \phi(x(t_k),u(t_k)) + \Delta\phi(t,t_k)$, where $\Delta\phi(t,t_k) = \bar{\phi}(.)\begin{bmatrix}\Delta x(t,t_k)\\ \Delta u(t,t_k)\end{bmatrix}$, $\bar{\phi}(.) = [\nabla_x^T\phi(x(t)-c\Delta x(t,t_k),u(t_k)) \quad \nabla_u^T\phi(x(t_k),u(t)-c\Delta u(t,t_k))]$ and $c\in(0,1)$ [28]. Accordingly, it yields that $||\Delta\phi(t,t_k)||\leq\bar{\Delta}_\phi$ where $\bar{\Delta}_\phi$ is unknown positive constant. This implies that the nonlinear function varies finitely in each sampling time.

Remark 1: In this work, we have not considered that the unknown input is bounded, in contrast to some works, e.g., [2]. Also, the full states of the system are not estimated, instead a linear combination of system states is estimated. Accordingly, the required restrictive assumptions are relaxed [29]. The full-order observer is a special case of designed scheme and we can extend this to full-state observer simply by setting $L=I_n$.

Remark 2: The bounded variation of $d(t)$ is rather a practical assumption, considering the physical sources of the unknown input that can change boundedly. Also, the assumption $q\leq p$ is a standard assumption [27]. Regarding the finite variation of states in each sampling period, it is worth noting that for the accurate estimation, if the states can vary infinitely then the observer is not able to keep estimating by use of updated measurement. A well known case is input-to-state stable systems. For many practical systems, there has been already a control with guaranteed closed-loop stability. Therefore, it is practical and reasonable to have this assumption. On the other hand, considering the unstable systems, it is practical to assume that $\lim_{t\to\infty}x(t)=\infty$, i.e., the states trajectory does not escape into infinity in the finite time. Accordingly, for unstable systems, it is reasonable to assume that the sampling period is selected small enough to have finite variation of the states trajectory [24]. Finally, considering the bounded actuator efforts in practice, the upper bound on the control variation in each time interval is reasonable.

In (1a), the nonlinear term $\phi(x,u)$ is affecting the dynamics, which is a function of the full states x. Then, the estimation of x is required. However, in this work, we aim to estimate only a linear combination of full state, i.e., z. To resolve this, let us consider the following cases.

Case I: This case corresponds to $\mathrm{rank}(O)=p+r<n$, where $O=[C^T L^T]^T$, i.e., O is a full row rank matrix. Then, we can deduce that, $x(t)=M\xi(t)+M_1\nu(t)$, where $\nu(t)$ is an arbitrary vector, $\xi=[y^T z^T]^T$, $M=O^-$ and $M_1=I_n-O^-O$. In this case, we assume that the nonlinear term $\phi(x,u)$ can be written as

$$\phi(x,u) = \phi(M\xi+M_1\nu,u) = \phi_1(\xi,u)+D\phi_2(\nu,u) \tag{3}$$

where $D\in\mathbb{R}^{n\times d_\phi}$ and $\phi_1(\xi,u)$ is Lipschitz in its argument with computable Lipschitz constant $\Lambda_\phi>0$, [30], i.e., $\forall(\xi_1,u),(\xi_2,u)\in\mathbb{R}^{p+r}\times\mathbb{R}^m$

$$||\phi_1(\xi_1,u)-\phi_1(\xi_2,u)||\leq\Lambda_\phi||\xi_1-\xi_2||. \tag{4}$$

Then, $\phi_2(\nu, u) \in \mathbb{R}^{d_\phi}$ is considered as an unknown input.

Case II: This case corresponds to $\text{rank}(O) = n$, i.e., the matrix O is full column rank. Then, one can obtain that, $x(t) = M\xi(t)$ and $\phi(x, u) = \phi(M\xi, u) = \phi_1(\xi, u)$ with $\phi_1(\xi, u)$ is Lipschitz as in (4).

Remark 3: In Case I, when $\phi(x, u)$ is Lipschitz, we have $\phi(x, u) = \phi_1(\xi, u)$ and $||\phi_1(\xi_1, u) - \phi_1(\xi_2, u)|| = ||\phi_1(M\xi_1 + M_1\nu_1, u) - \phi_1(M\xi_2 + M_1\nu_2, u)|| \leq \Lambda_1||\xi_1 - \xi_2|| + \Lambda_2||\nu_1 - \nu_2||$, with positive constants Λ_1 and Λ_2. Then, one can assume that the unknown vector $\nu(t)$ is bounded or $||\nu_1 - \nu_2|| \leq \eta$, with positive constant η.

In the sequel, we consider only case I since the case II corresponds to $D = 0$. The following lemmas are used in the sequel of this article.

Lemma 1: Let X represents an $m \times n$ matrix and Y an $n \times p$ matrix then $rank(XY) = rank(Y)$, if and only if $rank \begin{bmatrix} X \\ I_n - YY^- \end{bmatrix} = n$ [4].

Lemma 2: The consistent equation $XK = W$, with known matrices K and W, has a general solution $X = WK^- - Z(I - KK^-)$, if and only if $rank([K^T W^T]^T) = rank(K)$, where Z is an arbitrary matrix of appropriate dimension [31].

Lemma 3: Inequality $\pm(W^TY + Y^TW) \leq \beta Y^TY + W^TW/\beta$ holds for vectors W and Y and any positive scalar β [30].

III. Observer Design

The structure of functional observer is proposed as

$$\begin{aligned} \dot{w}(t) &= Nw(t) + Hu(t) + Jy(t_k) + Q\phi_1(\hat{\xi}, u) \\ \hat{z}(t) &= w(t) + Ey(t_k) \end{aligned} \tag{5}$$

for $t \in [t_k, t_{k+1})$, where $w \in \mathbb{R}^r$ is the observer state, $N \in \mathbb{R}^{r\times r}$, $H \in \mathbb{R}^{r\times m}$, $J \in \mathbb{R}^{r\times p}$, $Q \in \mathbb{R}^{r\times n}$, and $E \in \mathbb{R}^{r\times p}$ are design matrices. Also, $\hat{z}$ is estimation of z and $\hat{\xi} = \begin{bmatrix} y(t_k) \\ \hat{z} \end{bmatrix}$. As the observer is designed in $t \in [t_k, t_{k+1})$, at $t = t_{k+1}$ the updated measurement $y(t_{k+1})$ is injected into observer (5), which might lead to a discontinuity and, consequently, obstacle in the convergence of the observer. Accordingly, at $t = t_{k+1}$, the observer is corrected as

$$\hat{z}(t_{k+1}) = \hat{z}^-(t_{k+1}) \tag{6}$$

where $\hat{z}^-(t_{k+1}) = \lim_{t\to t_{k+1}} \hat{z}(t)$ for $t < t_{k+1}$. This correction term resolves the discontinuity problem, as studied in the sequel. It is worth noting that the time derivative of $y(t_k)$ is numerically zero as $y(t_k)$ is constant for $t \in [t_k, t_{k+1})$. However, the time derivative of $y(t_k)$ can be analytically obtained as

$$\dot{y}(t_k) = CAx(t_k) + CBu(t_k) + CFd(t_k) + C\phi(x(t_k), u(t_k)). \tag{7}$$

It should be noted that (7) is not accurate as $x(t_k)$, $u(t_k)$ and $d(t_k)$ vary in $t \in [t_k, t_{k+1})$, thus, are not necessarily constant. Also, $x(t_k)$ and $d(t_k)$ are not available. Therefore, (7) cannot be implemented into the observer. However, it is useful in the convergence analysis. Now, to study the convergence of the observer, define the estimation error as

$$e(t) = z(t) - \hat{z}(t). \tag{8}$$

Accordingly, it is easy to obtain that

$$w(t) = Lx(t) - e(t) - Ey(t_k). \tag{9}$$

The time derivative of $e(t)$ for $t \in [t_k, t_{k+1})$ can be written as

$$\begin{aligned} \dot{e}(t) &= Ne(t) + (PA - NP - JC)x(t) + (PB - H)u(t) \\ &\quad + PFd(t) + P\phi(x, u) - Q\phi_1(\hat{\xi}, u) + d_1(t, t_k) \end{aligned} \tag{10}$$

where $P = L - EC$ and $d_1(t, t_k) = (ECA - NEC + JC)\Delta x(t, t_k) + ECB\Delta u(t, t_k) + ECF\Delta d(t, t_k) + EC\Delta\phi(t, t_k)$. Considering Assumption 2, it is easy to show that $||d_1(t, t_k)|| \leq \bar{d}_1$, where is $\bar{d}_1$ is unknown positive constant. Now, by satisfying the following conditions:

$$PA - NP - JC = 0_{r\times n} \tag{11}$$

$$H = PB \tag{12}$$

$$PF = 0_{r\times q} \tag{13}$$

$$PD = 0_{r\times d_\phi} \tag{14}$$

$$Q = P \tag{15}$$

the estimation error dynamics (10) becomes

$$\dot{e}(t) = Ne(t) + P\tilde{\phi}_1(\bullet) + d_1(t_k) \tag{16}$$

where $\tilde{\phi}_1(\bullet) = \phi_1(\xi, u) - \phi_1(\hat{\xi}, u)$. It is readily shown that $(\xi - \hat{\xi})^T(\xi - \hat{\xi}) = ||C\Delta x(t, t_k)||^2 + ||e||^2$. The stability analysis of (16) is given by the following Lemma.

Lemma 4: Under Assumptions 1 and 2, and conditions (11)–(15), the error dynamics (16) is stable, remains in a finite bound for $||d_1|| \neq 0$, and asymptotically converges to zero for $||d_1|| = 0$, in all sampling periods, if there exist a symmetric positive definite matrix (PDM) $X_1 \in \mathbb{R}^{r\times r}$ that satisfies the following inequality for given positive constants γ_1 and β:

$$\begin{bmatrix} \iota_1 & X_1 & 0_{r\times p} \\ X_1 & -\gamma_1^2 I_r & 0_{r\times p} \\ 0_{p\times r} & 0_{p\times r} & \iota_2 \end{bmatrix} < 0 \tag{17}$$

where $\iota_1 = N^TX_1 + X_1N + \beta\Lambda_\phi^2 I_r + X_1PP^TX_1/\beta + I_r$ and $\iota_2 = (\beta\Lambda_\phi^2||C||^2 - \gamma_1^2)I_p$.

Proof: Define a positive definite Lyapunov function as

$$V_1(t) = e^T(t)X_1e(t) \tag{18}$$

where $X_1 \in \mathbb{R}^{r\times r}$ is symmetric PDM. By taking into account Lemma 3 and (4), one can obtain that

$$\dot{V}_1(t) \leq \begin{bmatrix} e^T(t) & d_1^T(t, t_k) & \Delta x^T(t, t_k) \end{bmatrix} \kappa \begin{bmatrix} e(t) \\ d_1(t, t_k) \\ \Delta x(t, t_k) \end{bmatrix} \tag{19}$$

where $\kappa = \begin{bmatrix} \iota_1 - I_r & X_1 & 0_{r\times p} \\ X_1 & 0_{r\times r} & 0_{r\times p} \\ 0_{p\times r} & 0_{p\times r} & \iota_2 + \gamma_1^2 I_p \end{bmatrix}$. Now, consider the initial sample period $t \in [t_0, t_1)$. Define

$$\begin{aligned} \bar{\Gamma}_0(t) &= \dot{V}_1(t) + e^T(t)e(t) - \gamma_1^2 d_1^T(t, t_0)d_1(t, t_0) \\ &\quad - \gamma_1^2\Delta x^T(t, t_0)\Delta x(t, t_0). \end{aligned} \tag{20}$$

By satisfying (17), $\bar{\Gamma}_0(t) < 0$ is achieved. This yields that

$$\begin{aligned} V_1(t) - V_1(t_0) &< \gamma_1^2 \int_{t_0}^t d_1^T(t, t_0)d_1(t, t_0)dt + \gamma_1^2 \int_{t_0}^t \Delta x^T \\ &\quad (t, t_0)\Delta x(t, t_0)dt - \int_{t_0}^t e^T(t)e(t)dt. \end{aligned} \tag{21}$$

With zero initial conditions, since $V_1(t) > 0$, (21) yields

$$\int_{t_0}^t e^T(t)e(t)dt < \Delta_1(t - t_0) \tag{22}$$

where $\Delta_1 = \gamma_1^2(\bar{d}_1^2 + \bar{\Delta}_x^2)$ is a bounded unknown constant. Accordingly, the estimation error is bounded for $t \in [t_0, t_1)$. This bound can be made small by minimization of γ_1. To accurately find this bound from (20) one can obtain

$$\dot{V}_1(t) < -\theta_1 V_1(t) + \Delta_1 \tag{23}$$

where $\theta_1 = 1/\lambda_{\max}(X_1)$. This further yields

$$V_1(t) < \frac{\Delta_1}{\theta_1} - \frac{\Delta_1}{\theta_1} e^{-\theta_1(t-t_0)} < \frac{\Delta_1}{\theta_1}. \tag{24}$$

Accordingly, it is readily obtained that

$$||e(t)|| < \sqrt{\frac{\Delta_1}{\Theta_1}} \tag{25}$$

where $\Theta_1 = \lambda_{\min}(X_1)/\lambda_{\max}(X_1)$. Now, at $t = t_1$ by using the correction term (6), we have $e(t_1) = z(t_1) - \hat{z}(t_1) = z(t_1) - \hat{z}^-(t_1) = e^-(t_1)$. Also, $\lim_{t\to t_1} V_1(t) = V_1^-(t_1) < \Delta_1/\theta_1$. Thus

$$V_1(t_1) = e^T(t_1)X_1e(t_1) = e^{-T}(t_1)X_1e^-(t_1) = V_1^-(t_1) < \frac{\Delta_1}{\theta_1}. \tag{26}$$

Now, consider the sample period $t \in [t_k, t_{k+1})$ for $k \in \mathbb{Z}_{>0}$. Define $\bar{\Gamma}_k(t)$ as

$$\begin{aligned}\bar{\Gamma}_k(t) &= \dot{V}_1(t) + e^T(t)e(t) - \gamma_1^{2\,\sim} d_1^T(t,t_k)d_1(t,t_k)\\ &\quad - \gamma_1^2 \Delta x^T(t,t_k)\Delta x(t,t_k)\\ &\le \begin{bmatrix} e^T(t) & d_1^T(t,t_k) & \Delta x^T(t,t_k)\end{bmatrix} \bar{\kappa} \begin{bmatrix} e(t)\\ d_1(t,t_k)\\ \Delta x(t,t_k)\end{bmatrix}\end{aligned} \tag{27}$$

where $\bar{\kappa} = \begin{bmatrix} \iota_1 & X_1 & 0_{r\times p}\\ X_1 & -\gamma_1^2 I_r & 0_{r\times p}\\ 0_{p\times r} & 0_{p\times r} & \iota_2\end{bmatrix}$. Therefore, (17) leads to $\bar{\Gamma}_k(t) < 0$. It is easy to obtain that

$$V_1(t) < \frac{\Delta_1}{\theta_1} - \frac{\Delta_1}{\theta_1} e^{-\theta_1(t-t_k)} + V_1(t_k)e^{-\theta_1(t-t_k)}. \tag{28}$$

Also, one can obtain that $V_1(t_k) < \Delta_1/\theta_1$, for $t \in [t_{k-1}, t_k)$. Therefore, by taking this into account, (28) further yields

$$V_1(t) < \frac{\Delta_1}{\theta_1} \tag{29}$$

for $t \in [t_k, t_{k+1})$. Furthermore, $e(t_{k+1}) = z(t_{k+1}) - \hat{z}(t_{k+1}) = z(t_{k+1}) - \hat{z}^-(t_{k+1}) = e^-(t_{k+1})$. On the other hand, $\lim_{t\to t_{k+1}} V_1(t) = V_1^-(t_{k+1}) < \Delta_1/\theta_1$. So, $V_1(t_{k+1}) = e^T(t_{k+1})X_1e(t_{k+1}) = e^{-T}(t_{k+1})X_1e^-(t_{k+1}) = V_1^-(t_{k+1}) < \Delta_1/\theta_1$. Accordingly, it is easy to show that for $t \in [t_k, t_{k+1})$, for $k \in \mathbb{Z}_{\ge 0}$, the observer estimation error satisfies $||e(t)|| < \sqrt{\Delta_1/\Theta_1}$. Therefore, it is proven that the estimation error is always bounded and remains in a finite bound. Also, for $||d_1(t,t_k) = 0||$, by taking Schur complement into account, (17) yields $N^TX_1 + X_1N + X_1PP^TX_1/\beta \le -(\beta\Lambda_\phi^2 + 1)I_r$. Accordingly, dynamics (16) is asymptotically stable. ■

Remark 4: Consider (22), Assumptions 1 and 2, and let $\alpha > 0$. If the upper bound of the sampling interval δ_M, i.e., $\delta \le \delta_M$, satisfies $\delta_M < \alpha/\Delta_1$, then we can guarantee that $||e(t)|| < \sqrt{\alpha}$, for $t \in [t_0, t_1)$. Similar to (21), we see that $0 < V_1(t_{k+1}) < V_1(t_k) + \Delta_1\delta - \int_{t_k}^{t_{k+1}} e^T(t)e(t)dt$ which yields $\int_{t_k}^{t_{k+1}} e^T(t)e(t)dt < V_1(t_k) + \Delta_1\delta$. If $\delta_M < (\alpha - \lambda_{max}(X_1)e^T(t_k)e(t_k))/\Delta_1$, then $||e(t)|| < \sqrt{\alpha}$, for $t \in [t_k, t_{k+1})$.

One can see that (17) is not an LMI, due to existence of γ_1^2. The following lemma transforms (17) into an LMI form.

Lemma 5: Under Assumptions 1 and 2, and conditions (11)–(15), the estimation error dynamics (16) is stable, remains in a finite bound for $||d_1|| \neq 0$, and asymptotically converges to zero for $||d_1|| = 0$, in all sampling periods, if there exists a symmetric PDM $\bar{X}_1 \in \mathbb{R}^{r\times r}$ that satisfies the following LMI for given positive constants γ_1 and β_1:

$$\begin{bmatrix} \bar{\iota}_1 & \bar{X}_1 & 0_{r\times p} & \bar{X}_1P & I_r\\ \bar{X}_1 & -\gamma_1 I_r & 0_{r\times p} & 0_{r\times n} & 0_{r\times r}\\ 0_{p\times r} & 0_{p\times r} & \bar{\iota}_2 & 0_{p\times n} & 0_{p\times r}\\ P^T\bar{X}_1 & 0_{n\times r} & 0_{n\times p} & -\beta_1 I_n & 0_{n\times r}\\ I_r & 0_{r\times r} & 0_{r\times p} & 0_{r\times n} & -\gamma_1 I_r\end{bmatrix} < 0 \tag{30}$$

where $\bar{\iota}_1 = N^T\bar{X}_1 + \bar{X}_1N + \beta_1\Lambda_\phi^2 I_r$ and $\bar{\iota}_2 = (\beta_1\Lambda_\phi^2||C||^2 - \gamma_1)I_p$.

Proof: Using the Schur complement, (17) can be rewritten as

$$\begin{bmatrix} \iota_3 & X_1 & 0_{r\times p} & X_1P\\ X_1 & -\gamma_1^2 I_r & 0_{r\times p} & 0_{r\times n}\\ 0_{p\times r} & 0_{p\times r} & \iota_2 & 0_{p\times n}\\ P^TX_1 & 0_{n\times r} & 0_{n\times p} & -\beta I_n\end{bmatrix} < 0 \tag{31}$$

where $\iota_3 = N^TX_1 + X_1N + \beta\Lambda_\phi^2 I_r + I_r$. By multiplying (31) by $1/\gamma_1$, since $\gamma_1 \neq 0$, let $\beta_1 = \beta/\gamma_1$ and $\bar{X}_1 = X_1/\gamma_1$, then by applying the Schur complement we obtain (30). ■

To design observer (5), we write (11), (13), and (14) as

$$\begin{aligned} LA - ECA - NL - KC &= 0_{r\times n}\\ ECF &= LF\\ ECD &= LD\end{aligned} \tag{32}$$

respectively, where $K = J - NE$. Furthermore, (32) can be rewritten as

$$\begin{bmatrix} N & E & K\end{bmatrix}\Sigma_1 = \Lambda_1 \tag{33}$$

where $\Sigma_1 = \begin{bmatrix} L & 0_{r\times q} & 0_{r\times d_\phi}\\ CA & CF & CD\\ C & 0_{p\times q} & 0_{p\times d_\phi}\end{bmatrix} \in \mathbb{R}^{(r+2p)\times(n+q+d_\phi)}$ and $\Lambda_1 = \begin{bmatrix} LA & LF & LD\end{bmatrix} \in \mathbb{R}^{r\times(n+q+d_\phi)}$. The condition for the existence of a solution for (33) is given in Lemma 6.

Lemma 6: Equation (33) has a solution if and only if

$$rank\begin{bmatrix} L & 0_{r\times q} & 0_{r\times d_\phi}\\ CA & CF & CD\\ C & 0_{p\times q} & 0_{p\times d_\phi}\\ LA & LF & LD\end{bmatrix} = rank\begin{bmatrix} L & 0_{r\times q} & 0_{r\times d_\phi}\\ CA & CF & CD\\ C & 0_{p\times q} & 0_{p\times d_\phi}\end{bmatrix}. \tag{34}$$

Proof: Considering Lemma 2, the necessary and sufficient condition for the existence of a solution for (33) is $\text{rank}([\Sigma_1^T\Lambda_1^T]^T) = \text{rank}(\Sigma_1)$, which is exactly (34). ■

Now under condition (34) and using Lemma 2, the general solution of (33) is given by

$$\begin{bmatrix} N & E & K\end{bmatrix} = \Lambda_1\Sigma_1^- - Z_1(I_{(r+2p)} - \Sigma_1\Sigma_1^-) \tag{35}$$

where $Z_1 \in \mathbb{R}^{r\times(r+2p)}$ is an arbitrary matrix. From (35), we obtain

$$N = A_1 - Z_1B_1 \tag{36a}$$

$$E = A_2 - Z_1B_2 \tag{36b}$$

$$K = A_3 - Z_1B_3 \tag{36c}$$

where $A_1 = \Lambda_1\Sigma_1^- \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r}\end{bmatrix}$, $A_2 = \Lambda_1\Sigma_1^- \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p}\end{bmatrix}$, $A_3 = \Lambda_1\Sigma_1^- \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p\end{bmatrix}$, $B_1 = (I_{(r+2p)} - \Sigma_1\Sigma_1^-) \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r}\end{bmatrix}$, $B_2 = (I_{(r+2p)} - \Sigma_1\Sigma_1^-) \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p}\end{bmatrix}$, and $B_3 = (I_{(r+2p)} - \Sigma_1\Sigma_1^-) \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p\end{bmatrix}$.

Considering (16), the matrix N governs the estimation error dynamics $e(t)$. On the other hand, in (36a), the eigenvalues of N are placed by choice of matrix Z_1. Therefore, the pair (B_1, A_1) must be detectable and we have the following lemma.

Lemma 7: There exists a matrix parameter $Z_1 \in \mathbb{R}^{r\times(r+2p)}$ such that the matrix N is Hurwitz if and only if

$$\mathit{rank} \begin{bmatrix} sL - LA & -LF & -LD \\ CA & CF & CD \\ C & 0_{p\times q} & 0_{p\times d_\phi}\end{bmatrix} = \mathit{rank}(\Sigma_1) \tag{37}$$

for all $s \in \mathbb{C}$ and $0 \le Re(s)$.

Proof: There exists a matrix Z_1 such that the matrix N is Hurwitz if and only if, the pair (B_1, A_1) is detectable, i.e., rank $\begin{bmatrix} sI_r - A_1 \\ B_1\end{bmatrix} = r$, for all $s \in \mathbb{C}$ and $0 \le Re(s)$. On the other hand, by using the fact that $\Lambda_1\Sigma_1^-\Sigma_1 = \Lambda_1$, one can obtain that

$$\begin{bmatrix} sL - LA & -LF & -LD \\ CA & CF & CD \\ C & 0_{p\times q} & 0_{p\times d_\phi}\end{bmatrix} = W_1\Sigma_1$$

where $W_1 = \begin{bmatrix} sI_r & 0_{r\times p} & 0_{r\times p} \\ 0_{p\times r} & I_p & 0_{p\times p} \\ 0_{p\times r} & 0_{p\times p} & I_p\end{bmatrix} - \begin{bmatrix} \Lambda_1\Sigma_1^- \\ 0_{p\times(r+2p)} \\ 0_{p\times(r+2p)}\end{bmatrix}$.

Therefore, from (37), it is required that $\mathrm{rank}(W_1\Sigma_1) = \mathrm{rank}(\Sigma_1)$, for all $s \in \mathbb{C}$ and $0 \le Re(s)$. Using Lemma 1, it is equivalent to $\begin{bmatrix} W_1 \\ I_{(r+2p)} - \Sigma_1\Sigma_1^-\end{bmatrix}$ to be of full column rank. This, in turn, yields that the matrix $\begin{bmatrix} sI_r - A_1 & -A_2 & -A_3 \\ 0_{p\times r} & I_p & 0_{p\times p} \\ 0_{p\times r} & 0_{p\times p} & I_p \\ B_1 & B_2 & B_3\end{bmatrix}$ to be full column rank. Consequently, the matrix $\begin{bmatrix} sI_r - A_1 \\ B_1\end{bmatrix}$ is to be of rank r or of full column rank. ∎

Considering (25), the upper bound of $e(t)$ is a function of sampling time δ, parameter γ_1, and constant $\bar{d}_1$. Considering different practical systems, it might be impossible to reduce the sampling time. Also, $\bar{d}_1$ depends on the system dynamics characteristics and sampling time. Therefore, to make the error bound small, we minimize the parameter γ_1 and we have the following Theorem.

Theorem 1: Under Assumptions 1 and 2, and conditions (34) and (37), the estimation error dynamics (16) is stable and bounded as in (25) with minimized bound, if there exist a symmetric PDM $\bar{X}_1 \in \mathbb{R}^{r\times r}$, matrix $R_1 \in \mathbb{R}^{r\times(r+2p)}$ and positive constants γ_1 and β_1, solution to the following optimization problem. Minimize γ_1, subject to

$$\begin{bmatrix} \chi_1 & \bar{X}_1 & 0_{r\times p} & \chi_2 & I_r \\ \bar{X}_1 & -\gamma_1 I_r & 0_{r\times p} & 0_{r\times n} & 0_{r\times r} \\ 0_{p\times r} & 0_{p\times r} & \bar{\iota}_2 & 0_{p\times n} & 0_{p\times r} \\ \chi_2^T & 0_{n\times r} & 0_{n\times p} & -\beta_1 I_n & 0_{n\times r} \\ I_r & 0_{r\times r} & 0_{r\times p} & 0_{r\times n} & -\gamma_1 I_r\end{bmatrix} < 0 \tag{38}$$

where $\chi_1 = A_1^T\bar{X}_1 + \bar{X}_1 A_1 - B_1^T R_1^T - R_1 B_1 + \beta_1\Lambda_\phi^2 I_r$ and $\chi_2 = \bar{X}_1 L - \bar{X}_1 A_2 C + R_1 B_2\,\tilde{}\,C$. Then, the parameter matrix Z_1 is given by $Z_1 = \bar{X}_1^{-1} R_1$.

Proof: The stability and boundedness of the estimation error is proven in Lemma 4 by solving (17). Replacing (36) in (30) with $\bar{X}_1 Z_1 = R_1$, the LMI (38) is obtained. Also, minimization of γ_1 leads to minimization of the error upper bound. ∎

By solving (38), Z_1 is obtained. Then, N, E, and K are computed using (36) and $J = K + NE$.

IV. Special Cases

In this section, the matrices of observer (5) are determined for some special cases. The proofs of following theorems are straightforward and, therefore, omitted.

A. Case 1. Nonlinear Systems Without Unknown Inputs

This case corresponds to $F = 0$. Accordingly, conditions (34) and (37) become

$$\mathrm{rank} \begin{bmatrix} L & 0_{r\times d_\phi} \\ CA & CD \\ C & 0_{p\times d_\phi} \\ LA & LD\end{bmatrix} = \mathrm{rank} \begin{bmatrix} L & 0_{r\times d_\phi} \\ CA & CD \\ C & 0_{p\times d_\phi}\end{bmatrix} \tag{39}$$

$$\mathrm{rank} \begin{bmatrix} sL - LA & -LD \\ CA & CD \\ C & 0_{p\times d_\phi}\end{bmatrix} = \mathrm{rank}(\Sigma_2) \tag{40}$$

for all $s \in \mathbb{C}$ and $0 \le Re(s)$, respectively, where $\Sigma_2 = \begin{bmatrix} L & 0_{r\times d_\phi} \\ CA & CD \\ C & 0_{p\times d_\phi}\end{bmatrix} \in \mathbb{R}^{(r+2p)\times(n+d_\phi)}$. Define $\Lambda_2 = \begin{bmatrix} LA & LD\end{bmatrix} \in \mathbb{R}^{r\times(n+d_\phi)}$, $A_4 = \Lambda_2\Sigma_2^- \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r}\end{bmatrix}$, $A_5 = \Lambda_2\Sigma_2^- \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p}\end{bmatrix}$, $A_6 = \Lambda_2\Sigma_2^- \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p\end{bmatrix}$, $B_4 = (I_{(r+2p)} - \Sigma_2\Sigma_2^-) \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r}\end{bmatrix}$, $B_5 = (I_{(r+2p)} - \Sigma_2\Sigma_2^-) \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p}\end{bmatrix}$, and $B_6 = (I_{(r+2p)} - \Sigma_2\Sigma_2^-) \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p\end{bmatrix}$.

Then, observer matrices are obtained as, $N = A_4 - Z_2 B_4$, $E = A_5 - Z_2 B_5$, $K = A_6 - Z_2 B_6$, and $J = K + NE$. The stability analysis is given by the following Theorem and Z_2 is obtained.

Theorem 2: Under Assumptions 1 and 2, and conditions (39) and (40), for a nonlinear systems represented by (1) without unknown inputs, the estimation error dynamics is stable and bounded as $||e(t)|| \le \sqrt{\Delta_2/\Theta_2}$ with minimized bound, where $\Delta_2 = \gamma_2^2(\bar{d}_2^2 + \bar{\Delta}_x^2)$, $d_2(t, t_k) = (ECA - NEC + JC)\Delta x(t, t_k) + ECB\Delta u(t, t_k) + EC\Delta\phi(t, t_k), ||d_2(.)|| \le \bar{d}_2^2$, and $\Theta_2 = \lambda_{min}(X_2)/\lambda_{max}(X_2)$, if there exist a symmetric PDM $X_2 \in \mathbb{R}^{r\times r}$, matrix $R_2 \in \mathbb{R}^{r\times(r+2p)}$ and positive constants γ_2 and β_2, solution to the following optimization problem.

Minimize γ_2, subject to

$$\begin{bmatrix} \psi_1 & X_2 & 0_{r\times p} & \psi_2 & I_r \\ X_2 & -\gamma_2 I_r & 0_{r\times p} & 0_{r\times n} & 0_{r\times r} \\ 0_{p\times r} & 0_{p\times r} & \psi_3 & 0_{p\times n} & 0_{p\times r} \\ \psi_2^T & 0_{n\times r} & 0_{n\times p} & -\beta_2 I_n & 0_{n\times r} \\ I_r & 0_{r\times r} & 0_{r\times p} & 0_{r\times n} & -\gamma_2 I_r \end{bmatrix} < 0 \quad (41)$$

where $\psi_1 = A_4^T X_2 + X_2 A_4 - B_4^T R_2^T - R_2 B_4 + \beta_2 \Lambda_\phi^2 I_r$, $\psi_2 = X_2 L - X_2 A_5 C + R_2 B_5 \tilde{}C$, and $\psi_3 = (\beta_2 \Lambda_\phi^2 ||C||^2 - \gamma_2) I_p$. Then, $Z_2 = X_2^{-1} R_2$.

B. Case 2. Full- and Reduced-Order Observers for Nonlinear Systems Under Unknown Inputs

The both cases can be readily obtained by solving the optimization problem (38). For the full-order observer, i.e., $L = I_n$, condition (34) becomes rank $\begin{bmatrix} CF & CD \end{bmatrix} = \text{rank}(CD) + \text{rank}(F)$, which is equivalent to $\text{rank}(CF) = \text{rank}(F)$. Also, condition (37) yields rank $\begin{bmatrix} sI_n - A & -F \\ C & 0_{p\times q} \end{bmatrix} = n + \text{rank}(F)$ for all $s \in \mathbb{C}$ and $0 \le Re(s)$. Also, for the reduced-order observer, i.e. $r = n - p$ and L is full row rank, the conditions to be satisfied are rank $\begin{bmatrix} L \\ C \end{bmatrix} = n$ and $\text{rank}(CF) = \text{rank}(F)$.

C. Case 3. Linear System Under Unknown Inputs

This case corresponds to $\phi(x, u) = 0$ in (1). Accordingly, we choose $Q = 0_{r\times n}$ in observer (5). Consequently, estimation error $e(t)$, given in (10) for $t \in [t_k, t_{k+1})$, becomes

$$\begin{aligned} \dot{e}(t) &= Ne(t) + (PA - NP - JC)x(t) + (PB - H)u(t) \\ &\quad + PFd(t) + d_3(t, t_k) \end{aligned} \quad (42)$$

where $d_3(t, t_k) = (ECA - NEC + JC)\Delta x(t, t_k) + ECB\Delta u(t, t_k) + ECF\Delta d(t, t_k)$. Considering Assumption 2, it is easy to show $||d_3(t, t_k)|| \le \bar{d}_3$, where is $\bar{d}_3$ is unknown positive constant. Now, by satisfying the following conditions:

$$PA - NP - JC = 0_{r\times n} \quad (43)$$

$$H = PB \quad (44)$$

$$PF = 0_{r\times q} \quad (45)$$

the estimation error dynamics (42) becomes

$$\dot{e}(t) = Ne(t) + d_3(t, t_k). \quad (46)$$

In this case, conditions (34) and (37) become

$$\text{rank} \begin{bmatrix} L & 0_{r\times q} \\ CA & CF \\ C & 0_{p\times q} \\ LA & LF \end{bmatrix} = \text{rank} \begin{bmatrix} L & 0_{r\times q} \\ CA & CF \\ C & 0_{p\times q} \end{bmatrix} \quad (47)$$

$$\text{rank} \begin{bmatrix} sL - LA & -LF \\ CA & CF \\ C & 0_{p\times q} \end{bmatrix} = \text{rank}(\Sigma_3). \quad (48)$$

for all $s \in \mathbb{C}$ and $0 \le Re(s)$, respectively, where $\Sigma_3 = \begin{bmatrix} L & 0_{r\times q} \\ CA & CF \\ C & 0_{p\times q} \end{bmatrix} \in \mathbb{R}^{(r+2p)\times(n+q)}$. Define $\Lambda_3 = \begin{bmatrix} LA & LF \end{bmatrix} \in \mathbb{R}^{r\times(n+q)}$, $A_7 = \Lambda_3 \Sigma_3^- \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r} \end{bmatrix}$, $A_8 = \Lambda_3 \Sigma_3^- \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p} \end{bmatrix}$, $A_9 = \Lambda_3 \Sigma_3^- \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p \end{bmatrix}$, $B_7 = (I_{(r+2p)} - \Sigma_3 \Sigma_3^-) \begin{bmatrix} I_r \\ 0_{p\times r} \\ 0_{p\times r} \end{bmatrix}$, $B_8 = (I_{(r+2p)} - \Sigma_3 \Sigma_3^-) \begin{bmatrix} 0_{r\times p} \\ I_p \\ 0_{p\times p} \end{bmatrix}$, and $B_9 = (I_{(r+2p)} - \Sigma_3 \Sigma_3^-) \begin{bmatrix} 0_{r\times p} \\ 0_{p\times p} \\ I_p \end{bmatrix}$. Then, observer matrices are obtained as $N = A_7 - Z_3 B_7$, $E = A_8 - Z_3 B_8$ and $K = A_9 - Z_3 B_9$, $J = K + NE$, and $Q = 0_{r\times n}$. The stability analysis is given by the following Theorem and Z_3 is obtained.

Theorem 3: Under Assumptions 1 and 2, and conditions (47) and (48), for a linear systems represented by (1) with $\phi(x, u) = 0$, there exists a functional observer in the form of (5), satisfying (43)–(45) and $Q = 0_{r\times n}$, such that the estimation error dynamics is stable and remains in a finite bound $||e(t)|| < \sqrt{\Delta_3/\Theta_3}$ with minimized bound, where $\Delta_3 = \gamma_3^2 \bar{d}_3^2$, $\Theta_3 = \lambda_{min}(X_3)/\lambda_{max}(X_3)$ and $||e(t)||/||d_3(t, t_k)|| \le \gamma_3$ for $||d_3|| \ne 0$, and asymptotically converges to zero for $||d_3|| = 0$, in all sampling periods, if there exist a symmetric PDM $X_3 \in \mathbb{R}^{r\times r}$, matrix $R_3 \in \mathbb{R}^{r\times(r+2p)}$ and a positive constant γ_3, solution to the following optimization problem.

Minimize γ_3, subject to

$$\begin{bmatrix} \Omega_1 & X_3 & I_r \\ X_3 & -\gamma_3 I_r & 0_{r\times r} \\ I_r & 0_{r\times r} & -\gamma_3 I_r \end{bmatrix} < 0 \quad (49)$$

where $\Omega_1 = A_7^T X_3 + X_3 A_7 - R_3 B_7 - B_7^T R_3^T$ Then, $Z_3 = X_3^{-1} R_3$.

Remark 5: There are some other special cases for linear systems, such as full-order, reduced-order observers with or without unknown inputs, which can be readily found using Theorem 3, and there are several works on them, and therefore not considered here (see [4] and references therein). We can remark that the existence conditions of the functional observer compared to those of full-order observer are less restrictive. In fact if $F = 0$ and $D = 0$ and $L = I_n$ condition (34) and (37) reduce to the detectability of the pair (C, A) since condition (34) is always satisfied and condition (37) becomes $rank \begin{bmatrix} sI_n - A \\ CA \\ C \end{bmatrix} = n$, or equivalently, $rank \begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n$. Then, we can see that the condition of the detectability of the pair (C, A) is not needed for the design of the functional observer.

Remark 6: To make a tighter bound on the $e(t)$, as given in (25), we may make $\bar{d}_1$ as small as possible. One way is to remove the effect $\Delta x(t, t_k)$ from $d_1(t, t_k)$ by satisfying $ECA - NEC + JC = 0$. Using (11), this leads to $NL = LA$. However, by taking Sylvester equation into account, $NL = LA$ implies that the matrices N and A must have some common eigenvalues. According to (16), matrix N must be Hurwitz. Therefore, this imposes restriction on the system matrix A. On the other hand, according to Lemma 2, the unique solution $NL = LA$ is $N = LAL^-$, satisfying $rank([(LA)^T A^T]^T) = rank(L)$, which itself is restrictive. Therefore, in this article, we assume that $\Delta x(t, t_k)$ is finite and bounded for each sampling time.

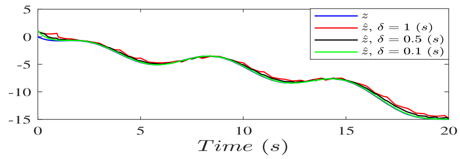

Fig. 1. $z(t)$ and $\hat{z}(t)$, for linear system with unknown input.

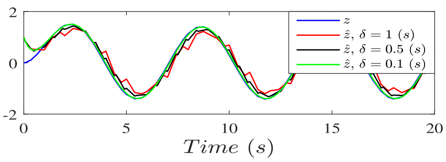

Fig. 2. $z(t)$ and $\hat{z}(t)$, for linear system without unknown input.

V. Simulation Results

In this section, the numerical simulation results are studied for both linear and nonlinear unstable systems, with the unknown unbounded input and different sampling periods. The system does not exhibit the finite escape time.

Consider system (1), with the following matrices: $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $L = [1 \quad 1 \quad 0]$, $F = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $x(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The unknown input is as $d(t) = e^{0.1t}(3\sin(0.001t + \frac{\pi}{6}) + 1)$. The nonlinear function is as $\phi(x,u) = \begin{bmatrix} 0 \\ 0.75\cos^2(x_1 + x_2) \\ 0.1e^{x_1 x_2} \end{bmatrix}$ with $\phi_1(\xi, u) = \begin{bmatrix} 0 \\ 0.75\cos^2(x_1 + x_2) \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$, $\phi_1(\hat{\xi}, u) = \begin{bmatrix} 0 \\ 0.75\cos^2(\hat{z}) \\ 0 \end{bmatrix}$, $\phi_2(x,u) = e^{x_1 x_2}$, and $\Lambda_\phi = 0.75$. Also, The initial value of observer (5) is selected as $w(t_0) = 1$.

A. Linear System Result

Here, $\phi(x,u)$ is assumed to be zero. By solving (49), the design matrices are obtained as, $N = -2.4388$, $H = 2.4148$, $J = \begin{bmatrix} 1.0597 & 2.41487 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} -1.4148 & 1 \end{bmatrix}$, and $P = \begin{bmatrix} 2.4148 & 0 & 0 \end{bmatrix}$. The $z(t)$ along with $\hat{z}(t)$ are illustrated in Figs. 1 and 2, with and without unknown input $d(t)$, respectively, for different sampling times. First, it is clear that the system is unstable in the presence of $d(t)$. However, the designed observer is able to estimate $z(t)$ and the estimation error is kept bounded, for various sampling times. More importantly, the exponential decay of $\hat{z}(t)$ toward $z(t)$ is seen in each sampling period. Also, the estimation error bound is smaller for small sampling time. The performance of the reduced-order observer to estimate the third state, i.e., $L = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is shown in Fig. 3, for the linear system without the unknown input, as studied in Case 2 in Section IV.

B. Nonlinear System Result

By solving (38), the design matrices are obtained as, $N = -7.296 \times 10^4$, $H = 3.587 \times 10^{-4}$, $J = \begin{bmatrix} 26.1697 & 0.0004 \end{bmatrix}$, $Q = P = \begin{bmatrix} 0.3587 & 0 & 0 \end{bmatrix} \times 10^{-3}$, $E = \begin{bmatrix} 0.9996 & 1 \end{bmatrix}$. $z(t)$ along with $\hat{z}(t)$ are illustrated in Figs. 4 and 5 with and without unknown input $d(t)$, respectively, for different sampling times. The system is unstable in the presence of $d(t)$. The designed observer is able to estimate $z(t)$ with sampled measurements of this unstable system. In both cases, the estimation error is kept bounded. The estimation error bound has been smaller for smaller sampling time δ. Finally, for $\delta = 1(s)$ in the presence of $d(t)$, the effect of correction term (6) is shown in Fig. 6. It is evident, the last estimated value is retained, using the correction term at each sampling time. However, without using it, the estimated value is forced towards the new measurement. This might leads to instability of numerical integrator.

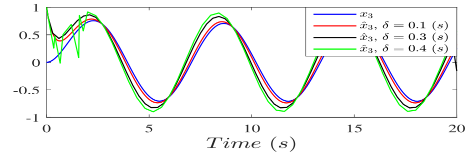

Fig. 3. $x_3(t)$ and $\hat{x}_3(t)$, for linear system without unknown input.

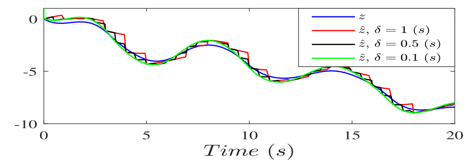

Fig. 4. $z(t)$ and $\hat{z}(t)$, for nonlinear system with unknown input.

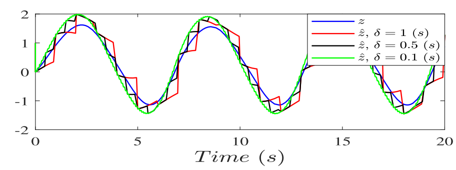

Fig. 5. $z(t)$ and $\hat{z}(t)$, for nonlinear system without unknown input.

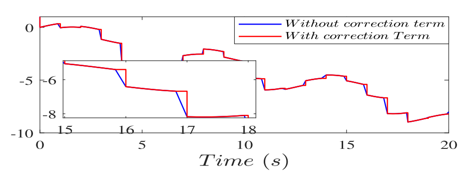

Fig. 6. Effect of correction term (6), for nonlinear system with unknown input and $\delta = 1(s)$.

VI. Conclusion

In this article, we studied the design of a functional observer for continuous systems with sampled measurement and unknown inputs. The existence conditions were derived for a class of nonlinear systems. Also, a correction term was incorporated to overcome the discontinuity due to the injection of the updated measurements. Moreover, the nonlinearity term was decomposed into two terms, including Lipschitz and unknown ones. Furthermore, the stability and convergence of the estimation were analyzed for some special cases. The designed procedure was proposed in an LMI formulation. The numerical simulations were given to evaluate the effectiveness of the proposed approach. The extension of this work is to design the control for the systems with sampled measurement, using the proposed scheme.

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