Lecture - 8
Normal forms

Let us consider all possible truth tables that can be obtained when the formulas involve only one variable P. These possible truth tables are shown in the table below;

P	1	2	3	4
T	T	F	T	F
F	F	T	T	F

Any formula involving only one variable will have one of these four truth tables. The most simplest formulas having the truth tables $1, 2, 3$ and 4 are $P, \neg P, P \vee \neg P$ and $P \wedge \neg P$, respectively. Any other formula which involves P only will be equivalent to one of these four formulas.

Let us now consider the formulas obtained by using the two variables and any connectives, then we shall get several formulas. The number of distinct truth tables for formulas involving two variables is equals to $2^{2^2} = 16$. Since there are there are 2^2 rows in the truth tables and each row could have any of the two entries T or F, therefore, we have 2^{2^2} possible tables (given in the table below);

P	Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

Thus, any formula involving two variables will have one of these 16 truth tables. Also, formulas which have one of these truth tables are equivalent.
Similarly, a statement formula containing n variables must have as its truth table of the 2^{2^n} possible truth tables and each of them will have 2^n rows. Thus we see that there are several formulas which may look different but are equivalent.

One method to check whether two formulas A and B are equivalent is to construct their truth table and compare them. However, this method is tedious and difficult to perform even on a computer because the number of entries increase very rapidly as n increases. Therefore, another approach could be to transform A and B to some standard forms A' and B' and compare A' and B' to check whether $A \iff B$. These standard forms are called canonical or normal forms.

Let $A(P_1, P_2, \cdots, P_n)$ be a statement formula and $P_1, P_2, \cdots, P_n$ are the atomic variables. To get the truth table for the formula A, we assign all the possible truth values to the variables $P_1, P_2, \cdots, P_n$ and obtain the truth values of the formula A. Note that there will be 2^n rows in the truth table of the formula A. Now,

- A is a tautology.
- A is a contradiction.

If A has the truth value T for at least one possible combination of the truth values of the variables $P_1, P_2, \cdots, P_n$, then we say that A is *satisfiable.*

The problem of determining, in a finite number of steps, whether a formula is a tautology or a contradiction or at least satisfiable is called a *decision problem.*

Note that a decision problem in statement calculus has always a solution because the truth table for a statement formula can be obtained in finite number of steps. However, for other logical system (e.g., predicate calculus) solution to the decision problem may not be simple.

For our convenience, we shall use the word "product" (resp. "sum") for "conjunction" (resp. "disjunction").

A product of the variables and their negations (resp. a sum of the variables and their negations) in a formula is called an *elementary product* (resp. *elementary sum*).

Examples: For atomic variables P and Q;
$P, \neg P \wedge Q, \neg Q \wedge P \wedge \neg P, P \wedge \neg P, Q \wedge \neg P$ - elementary products.
$P, \neg P \vee Q, \neg Q \vee P \vee \neg P, P \vee \neg P, Q \vee \neg P$ - elementary sums.

Any part of the elementary product or sum which is itself and elementary product or sum is called a factor of the original elementary product or sum.

$\neg Q, P \wedge \neg P, \neg Q \wedge P$ are some factors of $\neg Q \wedge P \wedge \neg P$.

The following statement holds for elementary sums and products;

A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors in which one is the negation of the other.

A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

The proof of above two statements are left as exercise.

Disjunctive normal form: Let A be a given formula. A formula B which is equivalent to the formula A is called a disjunctive normal form of A, if B consists of a sum of elementary products.

Let us see how we can obtain a disjunctive normal form of a given formula.

We have seen that the connectives $\rightarrow$ and $\leftrightarrow$ can be replaced by $\wedge, \vee$, and $\neg$. Therefore, without loss of generality we may assume that the given formula contains the connectives $\wedge, \vee$, and $\neg$, only.

If we apply the negation to the formula or to a part of the formula and not to the variables appearing in it, then by using De Morgan's law we obtain an equivalent formula, in which the negation is applied to the variables only.

If the formula is disjunctive normal form of the given formula then we are done otherwise;

the repeated application of distributive law we obtained the required form.

Let us apply this process to the following formulas;

$$(1)\ P \wedge (P \rightarrow Q)\ (2)\ \neg(P \vee Q) \leftrightarrow (P \wedge Q).$$

$$(1)\ P \wedge (P \rightarrow Q) \iff P \wedge (\neg P \vee Q) \iff (P \wedge \neg P) \vee (P \wedge Q).$$

$$\begin{aligned}(2)\ \neg(P \vee Q) \leftrightarrow (P \wedge Q) &\iff (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))\\ &\iff (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge Q)\\ &\iff (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)\end{aligned}$$

which is the required disjunctive normal form.

The disjunctive normal form of a given formula is not necessarily unique, because one can apply the distributive law in a different way. e.g., consider the formula $P \vee (Q \wedge R)$ (which is already in disjunctive normal form) can also be written as

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R) \iff (P \wedge P) \vee (P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$$

In any case all the disjunctive normal form of a given formula are equivalent.

Note that, a given formula is identically false if every elementary product appearing in its disjunctive normal form is identically false.

A given formula is identically true if every elementary product have at least two factors, of which one is the negation of the other.

Conjunctive normal form: Let A be a given formula. A formula B which is equivalent to the formula A is called a conjunctive normal form of A, if B consists of a product of elementary sums.

The process for obtaining a conjunctive normal form is similar to the process as discussed for disjunctive normal form. Like disjunctive normal forms a conjunctive normal form of a formula is not necessarily unique.

A given formula is identically true if every elementary sum in its conjunctive normal form is identically true (when is this true?).

Examples: Find a conjunctive normal form of the following formulas;

$$(1)\ P \wedge (P \to Q)\ (2)\ \neg(P \vee Q) \leftrightarrow (P \wedge Q).$$

$$(1)\ P \wedge (P \to Q) \iff P \wedge (\neg P \vee Q).$$

$$\begin{aligned}
(2)\ \neg(P \vee Q) \leftrightarrow (P \wedge Q) &\iff (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q)) \\
&\iff ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q)) \\
&\iff ((P \vee Q \vee P) \wedge (P \vee Q \vee Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)) \\
&\iff (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q).
\end{aligned}$$

Examples: Prove that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Try to write a conjunctive normal form of the given formula (it is given by)

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \iff (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Note here that each of the elementary sums is a tautology and so is the given formula (why?).

Principal disjunctive normal forms: Let P and Q be two statement variables. Consider all the formulas which consist of conjunctions of P or its negation and Q or its negation. No formula should contain both a variable and its negation. Furthermore, a formula which is obtained by commuting the formulas in the conjunction is not included in the list because such formula will be equivalent to the one which is included in the list. e.g., either $P \wedge Q$ or $Q \wedge P$ is included, but not both.

In this case (case of two variables P and Q), there are 4 such formulas given by

$$P \wedge Q,\ P \wedge \neg Q,\ \neg P \wedge Q,\ \text{and}\ \neg P \wedge \neg Q.$$

These formulas are called *miniterms* or Boolean conjunctions of P and Q.

It is immediate to see that no two miniterms are equivalent.

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Observe that each each of the miniterms has the truth value T for exactly one combination of the truth values of the variables P and Q.

Now, if the truth table of any formula containing only the variables P and Q is known, then it is easy to obtain an equivalent formula involving a disjunction of some of the miniterms.

For every truth value T in the truth table of the given formula, select the miniterm with truth value T for the same combination of truth values of P and Q. The disjunction of these miniterms will then be equivalent to the given formula.

Let A be a given formula, an equivalent formula involving the disjunctions of miniterms only is called the *principal disjunctive normal form* of the formula A.

Examples: Find the principal disjunctive normal form of the following formulas

$$P \to Q,\ P \vee Q,\ \neg(P \wedge Q).$$

P	Q	$P \to Q$	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

$$\begin{aligned} P \to Q &\iff (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \\ P \vee Q &\iff (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \\ \neg(P \wedge Q) &\iff (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$