Lecture - 3
Well-formed formula and tautologies

Recall the previous lectures; primary statements, truth tables, compound statements, connectives, negation, conjunction, disjunction, statement formulas, conditional and biconditional statements, converse, contrapositive and inverse.

A "*statement formula*" is an expression which is a string consisting of variables, parentheses, and connective symbols. However, not every string of these symbols is a formula. We give a recursive definition of a statement formula, called well-formed formula.

A *well-formed formula* can be constructed by the following rules:

(1) A statement variable standing alone is a well-formed formula.
(2) If A is a well-formed formula, then $\neg A$ is a well-formed formula.
(3) If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, and $(A \leftrightarrow B)$ are all well-formed formulas.
(4) A string of symbols containing the statement variables, connectives, and parentheses is a well-formed formula, if and only if it can be obtained by finitely many applications of the rules $1, 2$, and 3.

Examples: The following are the examples of well-formed formulas:

$$\neg(P \wedge Q), \neg(P \vee Q), (P \to (P \vee Q)), (P \to (Q \to R)), (((P \to Q) \wedge (Q \to R)) \leftrightarrow (P \to R))$$

Non-examples: The following are not well-formed formulas:

(1) $\neg P \wedge Q$. P and Q are well-formed formulas. A well-formed formula would be either $(\neg P \wedge Q)$ or $\neg(P \wedge Q)$.
(2) $(P \to Q) \to (\wedge Q)$ is not a well-formed formula as $\wedge Q$ is not.
(3) $(P \to Q$ is not a well-formed formula. $(P \to Q)$ is a well-formed formula.
(4) $(P \wedge Q) \to Q)$ is not a well-formed formula as one of the parentheses in the beginning is missing. A well-formed formula would be $((P \wedge Q) \to Q)$. Note that $(P \wedge Q) \to Q$ is still not a well-formed formula.

Sometime to reduced the number of parentheses, we shall omit the outer parentheses. Also, we only encounter well-formed formulas, we refer to well-formed formulas as formulas.

Tautologies: In general, the final column of the truth tables of a given formula contains both the truth values T and F. However, there are some formulas whose truth values are always T or always F regardless of the truth value assigned to the variables. e.g., $P \vee \neg P$ - always T, and $P \wedge \neg P$ - always F.

A statement formula which is true regardless of the truth values of the statements which occur in in it is called a *universally valid formula* or a *tautology* or a *logical truth.*

A statement formula which is false regardless of the truth values of the statements which occur in in it is called a *contradiction.*

It is to see that the negation of a contradiction is a tautology.

In other words, a statement formula which is a tautology is identically true and a statement formula which is a contradiction is identically false.

One can determine whether a statement formula is a tautology or not by constructing its truth table. However, if the number of variables is large or when formula is complicated, constructing truth table become tedious (if there are n variables then there will be 2^n rows in the truth table).

The conjunction of two tautologies is again a tautology. To see this, let A and B be two statements formula which are tautologies. If we assign any truth values to the variables of A and B, then the truth values of A and B will always be T and hence the truth value of $A \wedge B$ will always be T. Thus $A \wedge B$ is a tautology.

A formula A is called a *substitution instance* of another formula B if A it can be obtained from B by substituting formulas for some variables of B, with the condition that that same formula is substituted for the same variable each time it occurs.

Let $B : P \to (J \wedge P)$, substitute $R \leftrightarrow S$ for P in B, we obtain

$$A : (R \leftrightarrow S) \to (J \wedge (R \leftrightarrow S)).$$

Then A is a substitution instance of B. Note that

$$A : (R \leftrightarrow S) \to (J \wedge P)$$

is not a substitution instance of B, because the variable P in $J \wedge P$ is not replaced by $R \leftrightarrow S$.

One can substitute more than one variable by other formulas, provided all the substitution is done simultaneously.

We now see examples of substitution instances of $P \to \neg Q$:

(1) $(R \wedge \neg S) \to \neg(J \vee M)$
(2) $(R \wedge \neg S) \to \neg(R \wedge \neg S)$
(3) $(R \wedge \neg S) \to \neg P$
(4) $Q \to \neg(P \wedge \neg Q)$

We now consider the following formulas which comes from $P \to \neg Q$.

(1) Substitute $P \vee Q$ for P and R for Q to get the substitution instance $(P \vee Q) \to \neg R$.
(2) First substitute $P \vee Q$ for P to obtain the substitution instance $(P \vee Q) \to \neg Q$. Next, substitute R for Q in $(P \vee Q) \to \neg Q$ and we get $(P \vee R) \to \neg R$. Note that $(P \vee R) \to \neg R$ is a substitution instance for $(P \vee Q) \to \neg Q$ with Q replaced by R but not a substitution instance of $P \to \neg Q$ under. the substitution $(P \vee Q)$ for P and R for Q. This is because we did not substitute simultaneously as done in 1.

Observe that, while constructing substitution instances of a formula, substitutions are made for the atomic formula and never for the molecular formula. e.g. $P \to Q$ is not a substitution instance $P \to \neg R$, because R should be replaced not $\neg R$.

Any substitution instance of a tautology is again a tautology. e.g. consider the tautology $P \vee \neg P$. The truth value of $P \vee \neg P$ is always T, irrespective of the truth value of P. Thus, if we substitute any formula for P, the resulting formula will be a tautology.
The following substitution instances of $P \vee \neg P$ are tautologies.

$$(R \to S) \vee \neg(R \to S)$$
$$((P \vee S) \wedge R) \vee \neg((P \vee S) \wedge R)$$
$$(((P \vee \neg Q) \to R) \leftrightarrow S) \vee \neg(((P \vee \neg Q) \to R) \leftrightarrow S)$$

Therefore, If we are able to determine that whether a given formula is a substitution instance of a tautology, then it is clear that the given formula is also a tautology.

One can construct a number of formulas which are tautologies by writing several substitution instances of a tautology.

Note the change in symbol for biconditional statement from $\iff$ to $\leftrightarrow$ in this lecture.