Lecture - 9
Normal forms

Principal disjunctive normal forms: Let P and Q be two statement variables. Consider all the formulas which consist of conjunctions of P or its negation and Q or its negation. No formula should contain both a variable and its negation. Furthermore, a formula which is obtained by commuting the formulas in the conjunction is not included in the list because such formula will be equivalent to the one which is included in the list. e.g., either $P \wedge Q$ or $Q \wedge P$ is included, but not both.

In this case (case of two variables P and Q), there are 4 such formulas given by

$$P \wedge Q,\ P \wedge \neg Q,\ \neg P \wedge Q,\ \text{and}\ \neg P \wedge \neg Q.$$

These formulas are called *miniterms* or Boolean conjunctions of P and Q.

It is immediate to see that no two miniterms are equivalent.

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Observe that each each of the miniterms has the truth value T for exactly one combination of the truth values of the variables P and Q.

Now, if the truth table of any formula containing only the variables P and Q is known, then it is easy to obtain an equivalent formula involving a disjunction of some of the miniterms.

For every truth value T in the truth table of the given formula, select the miniterm with truth value T for the same combination of truth values of P and Q. The disjunction of these miniterms will then be equivalent to the given formula.

Let A be a given formula, an equivalent formula involving the disjunctions of miniterms only is called the *principal disjunctive normal form* of the formula A.

Examples: Find the principal disjunctive normal form of the following formulas

$$P \to Q,\ P \vee Q,\ \neg(P \wedge Q).$$

P	Q	$P \to Q$	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

$$\begin{aligned} P \to Q &\iff (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \\ P \vee Q &\iff (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \\ \neg(P \wedge Q) &\iff (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$

Note that the number of miniterms appearing in the normal form is the same as the number of entries with the truth value T in the truth table of the formula.

Thus every formula which is not a contradiction has an equivalent principal disjunctive normal form. Also, it is unique up to the rearrangement of the factors as well as in each miniterms.

If we impose certain ordering in which the the variables appear in the miniterms appear as well as the in which order the miniterms appear in the disjunction, then we obtain the unique normal form. Thus, if two formulas are equivalent then they must have identical principal disjunctive normal forms. Hence, it is sufficient to know the principal disjunctive normal form to know the equivalent formulas.

We have seen the construction of principal disjunctive normal for the formulas involving two variables P and Q. It is possible to extend this for more than two variables;

Let us first define the miniterms for three variables $P,\ Q$ and R.

$$P \wedge Q \wedge R,\ P \wedge Q \wedge \neg R,\ P \wedge \neg Q \wedge R,\ P \wedge \neg Q \wedge \neg R,$$

$$\neg P \wedge Q \wedge R,\ \neg P \wedge \neg Q \wedge R,\ \neg P \wedge Q \wedge \neg R,\ \neg P \wedge \neg Q \wedge \neg R.$$

These miniterms satisfy properties similar to the miniterms for two variables. Using these miniterms one can obtain an equivalent principal disjunctive normal form of a given formula in three variables $P,\ Q$ and R.

Note here that, there are 2^3 miniterms in this case. More generally, if we consider n variables, say $P_1, P_2, \cdots, P_n$ there will be 2^n miniterms.

In order to obtain the principal disjunctive normal form of a given formula, first replace the conditionals and biconditionals by using the connectives $\wedge, \vee$ and $\neg$. Then use the De Morgan's law and distributive law as we have done to obtain disjunctive normal form. If an elementary product is contradiction then remove it.

We now illustrate this process by examples

Example: Obtain the principal disjunctive normal forms of the following formulas

$$\neg P \vee Q,\ (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$$

$$\begin{aligned} \neg P \vee Q &\iff (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)),\ (A \wedge \mathbf{T} \iff A) \\ &\iff (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P),\ \text{(distributive law)} \\ &\iff (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q),\ \text{(commuativity)} \end{aligned}$$

$$
\begin{aligned}
& (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R) \\
\iff\; & (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P)) \\
\iff\; & (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)
\end{aligned}
$$

Example: Find the principal disjunctive normal form of the formula

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P)).$$

$$
\begin{aligned}
& P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))) \\
\iff\; & \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P)) \\
\iff\; & \neg P \vee (\neg P \wedge (Q \wedge P) \vee (Q \wedge (Q \wedge P))) \\
\iff\; & \neg P \vee (Q \wedge P) \\
\iff\; & (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge P) \\
\iff\; & (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q).
\end{aligned}
$$

Example: Show that the following formulas are equivalent;

(1) $P \vee (P \wedge Q) \iff P$

(2) $P \vee (\neg P \wedge Q) \iff P \vee Q$

Solution:

(1) $P \vee (P \wedge Q) \iff (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \iff (P \wedge Q) \vee (P \wedge \neg Q)$

$$P \iff P \wedge (Q \vee \neg Q) \iff (P \wedge Q) \vee (P \wedge \neg Q).$$

(2) $P \vee (\neg P \wedge Q) \iff (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q) \iff (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q$

$$P \vee Q \iff (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)) \iff (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q).$$

Principal conjunctive normal forms: For a given number of variables, we define *maxterm* which consists of disjunctions and in which each variable or its negation, but not both, appears only once. Maxterms are dual of miniterms.

Either by using duality or truth table we see that each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Also, different maxterms have the truth value F for different combinations of the truth values of the variables.

For a given formula, an equivalent formula consisting of conjunctions of the maxterms only is called its *principal conjunctive normal form.*

A formula which is not a tautology has an equivalent principal conjunctive normal form. A principal conjunctive normal form of a given formula is unique up to rearrangements of the factors in maxterms as well as in their conjunctions.

The method to obtain the principal conjunctive normal form for a given formula is similar to the one for obtaining principal disjunctive normal form.
In fact all the assertion which is true for principal disjunctive normal form is also true for

principal conjunctive normal form in view of the duality laws.

By using repeated applications of De Morgan's law and the equivalence $A \iff \neg\neg A$ one can obtain the principal conjunctive (disjunctive) normal form.

Example: Obtain the principal conjunctive normal form of the formula

$$A := (\neg P \to R) \wedge (Q \leftrightarrow P).$$

$$\begin{aligned}
& (\neg P \to R) \wedge (Q \leftrightarrow P) \\
\iff\; & (P \vee R) \wedge ((Q \to P) \wedge (P \to Q)) \\
\iff\; & (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q)) \\
\iff\; & (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \wedge (\neg P \vee Q \vee (R \wedge \neg R)) \\
\iff\; & (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \neg R)
\end{aligned}$$

Now, we can find the principal conjunctive normal form of $\neg S$ by writing the conjunction of remaining maxterms; thus the principal conjunctive normal form of $\neg A$ is given by

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R).$$

Consider now, $\neg\neg A$,

$$\neg(P \vee Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R)$$

$$\iff (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R),$$

which is the principal disjunctive normal form of A.

Example: Let A be a given formula. Below is the truth table for A. Find its disjunctive and conjunctive normal forms.

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

We now choose the miniterms corresponding to each T value of A to obtain

$$A \iff (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

Similarly,

$$A \iff (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \neg Q \wedge R).$$

Ordering and uniqueness of normal forms. We shall first fix the ordering in variables. If we consider n variables, then

- If the variables are denoted by capital letters, we shall arrange them in alphabetical order.
- If subscripted letters are also used to denote the variables , then we shall use following illustration;

$$A, B, \cdots, Z, A_1, B_1, \cdots, Z_1, A_2, B_2, \cdots$$

e.g. if the variables are P_1, Q, R_3, S_1, T_2 and Q_3, then we arrange them in the following order;

$$Q, P_1, S_1, T_2, Q_3, R_3.$$

Once we fix the ordering, we can assign numbering to them e.g., first variable, second variable, etc.

If we are given n variables which have been arranged according to the above ordering. There will be 2^n miniterms corresponding to these n variables and we can number these miniterms as follows;

$$m_0, m_1, \cdots, m_{2^n-1}.$$

Write the subscript of miniterms in binary and add a suitable number of zeros on the left (if necessary) so that the number of digits in the subscript is exactly n, then we get the corresponding miniterm as follows;

- If in the i-th location from the left there appears 1, then the i-th variables appears in the conjunction.
- If 0 appears in the i-th location from the left, then negation of the i-th variable appears in the conjunction forming the miniterm.

Thus, each of the $m_0, m_1, \cdots, m_{2^n-1}$ corresponds to a unique miniterm, which is determined by the binary representation of the subscript.

Conversely, for a given miniterm, it is easy to see that which of $m_0, m_1, \cdots, m_{2^n-1}$ designates it.

Consider three variables P, Q, and R arrange in this order. Let corresponding miniterms are denoted by $m_0, m_1, \cdots, m_7$.

The binary representation of 5 is 101, hence the miniterm corresponding to m_5 is $P \wedge \neg Q \wedge R$. Similarly, m_0 corresponds to the miniterm $\neg P \wedge \neg Q \wedge \neg R$.
To obtain the miniterms m_3, write 3 in binary representation which is 11. We add an extra zero to get 011 and hence the miniterm m_3 is given by $\neg P \wedge Q \wedge R$.

If we consider six variables $P_1, P_2, \cdots, P_6$ then miniterms are denoted by $m_0, m_1, cdots, m_6 3$. To get m_{38} write 38 in binary representation which is 100110, thus m_{38} is given by

$$P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge P_4 \wedge P_5 \wedge \neg P_6.$$

Using the above notation, we write the sum-of-products canonical form representing the disjunction of m_i, m_j, and m_k as $\sum i, j, k$.

We have seen that

$$(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R) \iff (\neg P\wedge\neg Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(P\wedge Q\wedge R).$$

Thus principal disjunctive normal form of $(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$ is given by $\sum 1,3,6,7$.

We now develop the similar notation for principal conjunctive normal forms.

Let us denote the maxterms associated to n variables as $M_0, M_1, \cdots, M_{2^n-1}$. Then the maxterm corresponding to M_j is obtained by expressing j in binary and adding a suitable number of zero to the left in order to get n digits.

- If 0 appears in the i-th location from the left, then the i-th variable appears in the disnjunction forming the maxterm.
- If 1 appears in the i-th location from the left, then negation of the i-th variable appears in the disnjunction forming the maxterm.

Thus, the binary representation of subscripts determine thr maxterms and conversely, every binary representation of numbers between 0 and $2^n - 1$ determines a maxterms.

Note here that the convention regarding 1 and 0 is opposite of what was used for miniterms. This is in view to connect the two principal normal forms of a given formula.

The maxterms $M_0, M_1, \cdots, M_7$ corresponding to three variables P, Q, and R is given by

$$\begin{array}{cccc} P\vee Q\vee R & P\vee Q\vee\neg R & P\vee\neg Q\vee R & P\vee\neg Q\vee\neg R \\ \neg P\vee Q\vee R & \neg P\vee Q\vee\neg R & \neg P\vee\neg Q\vee R & \neg P\vee\neg Q\vee\neg R \end{array}$$

We denote the principal conjunctive normal form by $\prod i,j,k$ which represents the conjunction of maxterms M_i, M_j, adn M_k.

Example: Consider the following formula $(P\wedge Q)\vee(\neg P\wedge R)$

$$\begin{aligned} & (P\wedge Q)\vee(\neg P\wedge R) \\ \iff & ((P\wedge Q)\vee\neg P)\wedge((P\wedge Q)\vee R) \\ \iff & (P\vee\neg P)\wedge(Q\vee\neg P)\wedge(P\vee R)\wedge(Q\vee R) \\ \iff & (Q\vee\neg P\vee(R\wedge\neg R))\wedge(P\vee R\vee(Q\wedge\neg Q))\wedge(Q\vee R\vee(P\wedge\neg P)) \\ \iff & (Q\vee\neg P\vee R)\wedge(Q\vee\neg P\vee\neg R)\wedge(P\vee R\vee Q)\wedge(P\vee R\vee\neg Q)\wedge(Q\vee R\vee P)\wedge(Q\vee R\vee\neg P) \\ \iff & (\neg P\vee Q\vee Q)\wedge(\neg P\vee Q\neg R)\wedge(P\vee Q\vee R)\wedge(P\vee\neg Q\vee R) \end{aligned}$$

Thus in the notation above, the product-of-sums canonical form of $(P\wedge Q)\vee(\neg P\wedge R)$ is given by $\prod 0,2,4,5$.

Also, its disjunctive normal form is given by

$$(P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R) \iff \sum 1,3,6,7.$$