Lecture - 10

Ordering and uniqueness of normal forms. We shall first fix the ordering in variables. If we consider n variables, then

- If the variables are denoted by capital letters, we shall arrange them in alphabetical order.
- If subscripted letters are also used to denote the variables , then we shall use following illustration;

$$A, B, \cdots, Z, A_1, B_1, \cdots, Z_1, A_2, B_2, \cdots$$

e.g. if the variables are P_1, Q, R_3, S_1, T_2 and Q_3, then we arrange them in the following order;

$$Q, P_1, S_1, T_2, Q_3, R_3.$$

Once we fix the ordering, we can assign numbering to them e.g., first variable, second variable, etc.

If we are given n variables which have been arranged according to the above ordering. There will be 2^n miniterms corresponding to these n variables and we can number these miniterms as follows;

$$m_0, m_1, \cdots, m_{2^n-1}.$$

Write the subscript of miniterms in binary and add a suitable number of zeros on the left (if necessary) so that the number of digits in the subscript is exactly n, then we get the corresponding miniterm as follows;

- If in the i-th location from the left there appears 1, then the i-th variables appears in the conjunction.
- If 0 appears in the i-th location from the left, then negation of the i-th variable appears in the conjunction forming the miniterm.

Thus, each of the $m_0, m_1, \cdots, m_{2^n-1}$ corresponds to a unique miniterm, which is determined by the binary representation of the subscript.
Conversely, for a given miniterm, it is easy to see that which of $m_0, m_1, \cdots, m_{2^n-1}$ designates it.

Consider three variables P, Q, and R arrange in this order. Let corresponding miniterms are denoted by $m_0, m_1, \cdots, m_7$.

The binary representation of 5 is 101, hence the miniterm corresponding to m_5 is $P \wedge \neg Q \wedge R$. Similarly, m_0 corresponds to the miniterm $\neg P \wedge \neg Q \wedge \neg R$.
To obtain the miniterms m_3, write 3 in binary representation which is 11. We add an extra zero to get 011 and hence the miniterm m_3 is given by $\neg P \wedge Q \wedge R$.

If we consider six variables $P_1, P_2, \cdots, P_6$ then miniterms are denoted by $m_0, m_1, cdots, m_63$. To get m_{38} write 38 in binary representation which is 100110, thus m_{38} is given by

$$P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge P_4 \wedge P_5 \wedge \neg P_6.$$

Using the above notation, we write the sum-of-products canonical form representing the disjunction of m_i, m_j, and m_k as $\sum i, j, k$.

We have seen that

$$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R) \iff (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R).$$

Thus principal disjunctive normal form of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is given by $\sum 1, 3, 6, 7$.

We now develop the similar notation for principal conjunctive normal forms.

Let us denote the maxterms associated to n variables as $M_0, M_1, \cdots, M_{2^n-1}$. Then the maxterm corresponding to M_j is obtained by expressing j in binary and adding a suitable number of zero to the left in order to get n digits.

- If 0 appears in the i-th location from the left, then the i-th variable appears in the disnjunction forming the maxterm.
- If 1 appears in the i-th location from the left, then negation of the i-th variable appears in the disnjunction forming the maxterm.

Thus, the binary representation of subscripts determine thr maxterms and conversely, every binary representation of numbers between 0 and $2^n - 1$ determines a maxterms.

Note here that the convention regarding 1 and 0 is opposite of what was used for miniterms. This is in view to connect the two principal normal forms of a given formula.

The maxterms $M_0, M_1, \cdots, M_7$ corresponding to three variables P, Q, and R is given by

$$\begin{array}{cccc} P \vee Q \vee R & P \vee Q \vee \neg R & P \vee \neg Q \vee R & P \vee \neg Q \vee \neg R \\ \neg P \vee Q \vee R & \neg P \vee Q \vee \neg R & \neg P \vee \neg Q \vee R & \neg P \vee \neg Q \vee \neg R \end{array}$$

We denote the principal conjunctive normal form by $\prod i, j, k$ which represents the conjunction of maxterms M_i, M_j, adn M_k.

Example: Consider the following formula $(P \wedge Q) \vee (\neg P \wedge R)$

$$\begin{aligned} & (P \wedge Q) \vee (\neg P \wedge R) \\ \iff & ((P \wedge Q) \vee \neg P) \wedge ((P \wedge Q) \vee R) \\ \iff & (P \vee \neg P) \wedge (Q \vee \neg P) \wedge (P \vee R) \wedge (Q \vee R) \\ \iff & (Q \vee \neg P \vee (R \wedge \neg R)) \wedge (P \vee R \vee (Q \wedge \neg Q)) \wedge (Q \vee R \vee (P \wedge \neg P)) \\ \iff & (Q \vee \neg P \vee R) \wedge (Q \vee \neg P \vee \neg R) \wedge (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (Q \vee R \vee P) \wedge (Q \vee R \vee \neg P) \\ \iff & (\neg P \vee Q \vee Q) \wedge (\neg P \vee Q \neg R) \wedge (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \end{aligned}$$

Thus in the notation above, the product-of-sums canonical form of $(P \wedge Q) \vee (\neg P \wedge R)$ is given by $\prod 0, 2, 4, 5$.

Also, its disjunctive normal form is given by

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \iff \sum 1, 3, 6, 7.$$

Theory of Inference. The main function of logic is to provide the rules of inference, or principles of reasoning. The theory associated with such rules is known as inference theory.

When we derive a conclusion from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a *deduction* or a*formal proof.*

Formal proof: Every rule of inference that is used at any stage in the derivation is acknowledged.

Mathematical proofs are, in general, informal in the sense that many steps in the derivation are either omitted or considered to be understood.

Validity using truth tables: Let A and B be two statement formulas. We say that "*B logically follows from A*" or "*B is a valid conclusion (consequence)* of the premise A" iff $A \to B$ is a tautology, i.e., $A \implies B$.

The above can be extended for a set of premises as follows; we say that from a set of premises $\{H_1, H_2, \cdots, H_m\}$ a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \implies C \tag{1}$$

Let $P_1, P_2, \cdots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \cdots, H_m$ and tge conclusion C. By looking at truth table one can immediately tell whether (**??**) holds or not.

We look for the rows in which all the premises $H_1, H_2, \cdots, H_m$ have the truth value T. If, for every such row, C also has the truth value T, then (**??**) holds.

Alternatively, we may look for rows in which the truth value of C is F. If, in every such row, at leat one of the truth values of $H_1, H_2, \cdots, H_m$ is F, then (**??**) holds.

This method of finding the validity is called "truth table technique."

Examples: Determine whether the conclusion C follows logically from the premises H_1 and H_2.

(1) $H_1 : P \to Q \quad H_2 : P \quad C : Q$
(2) $H_1 : P \to Q \quad H_2 : \neg P \quad C : Q$
(3) $H_1 : P \to Q \quad H_2 : \neg(P \wedge Q) \quad C : \neg P$
(4) $H_1 : \neg P \quad H_2 : P \leftrightarrow Q \quad C : \neg(P \wedge Q)$
(5) $H_1 : P \to Q \quad H_2 : Q \quad C : P$

We first construct the truth table for all the formulas involved here;

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \leftrightarrow Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

(1) Observe that the first row is the only row in which both the premises have the truth value T. The conclusion C also has the truth value T in that row. Therefore, it is valid.
(2) Not valid.
(3) Valid.
(4) Valid.
(5) Not Valid.

Rules of Inference: We now give a process of derivation by which one can determine whether a particular formula is a valid consequence of a given set of premises. We first give two rules of inference called rules **P** and **T.**

Rule **P**: A premise may be introduced at any point in the derivation.
Rule **T**: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation.

Before proceeding to the process of derivation, let us list some important formulas which will be used frequently.

List of Implications

I_1	$P \wedge Q \implies P$	(simplification)
I_2	$P \wedge Q \implies Q$	(simplification)
I_3	$P \implies P \wedge Q$	(addition)
I_4	$Q \implies P \wedge Q$	(addition)
I_5	$\neg P \implies P \to Q$	
I_6	$Q \implies P \to Q$	
I_7	$\neg(P \to Q) \implies P$	
I_8	$\neg(P \to Q) \implies \neg Q$	
I_9	$P, Q \implies P \wedge Q$	
I_{10}	$\neg P, P \vee Q \implies Q$	(disjunctive syllogism)
I_{11}	$P, P \to Q \implies Q$	(modus ponens)
I_{12}	$\neg Q, P \to Q \implies \neg P$	(modus tollens)
I_{13}	$P \to Q, Q \to R \implies P \to R$	(hypothetical syllogism)
I_{14}	$P \vee Q, P \to R, Q \to R \implies R$	(dilemma)

List of Equivalences

E_1	$\neg\neg P \iff P$	(double negation)
E_2	$P \wedge Q \iff Q \wedge P$	(commutativity)
E_3	$P \vee Q \iff Q \vee P$	(commutativity)
E_4	$(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$	(associativity)
E_5	$(P \vee Q) \vee R \iff P \vee (Q \vee R)$	(associativity)
E_6	$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (Q \wedge R)$	(distributive law)
E_7	$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (Q \vee R)$	(distributive law)
E_8	$\neg(P \wedge Q) \iff \neg P \vee \neg Q$	(De Morgan's law)
E_9	$\neg(P \vee Q) \iff \neg P \wedge \neg Q$	(De Morgan's law)
E_{10}	$P \vee P \iff P$	
E_{11}	$P \wedge P \iff P$	
E_{12}	$R \vee (P \wedge \neg P) \iff R$	
E_{13}	$R \wedge (P \vee \neg P) \iff R$	
E_{14}	$R \vee (P \vee \neg P) \iff \mathbf{T}$	
E_{15}	$R \wedge (P \wedge \neg P) \iff \mathbf{F}$	
E_{16}	$P \to Q \iff \neg P \vee Q$	
E_{17}	$\neg(P \to Q) \iff P \wedge \neg Q$	
E_{18}	$P \to Q \iff \neg Q \to \neg P$	
E_{19}	$P \to (Q \to R) \iff (P \wedge Q) \to R$	
E_{20}	$\neg(P \leftrightarrow Q) \iff P \leftrightarrow \neg Q$	
E_{21}	$P \leftrightarrow Q \iff (P \to Q) \wedge (Q \to P)$	
E_{22}	$P \leftrightarrow Q \iff (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

Example 1: Demonstrate that R is a valid inference from the premises $P \to Q, Q \to R$, and P.

$\{1\}$	(1)	$P \to Q$	Rule **P**
$\{2\}$	(2)	P	Rule **P**
$\{1,2\}$	(3)	Q	Rule **T**, (1), (2), and I_{11}
$\{4\}$	(4)	$Q \to R$	Rule **P**
$\{1,2,4\}$	(5)	R	Rule **T**, (3), (4), and I_{11}

- The first column for each line shows the premises on which the formula in the line depends.
- The second column of numbers designates the formula as well as the line of derivation in which it occurs.
- On the right, **P** or **T** represents the rule of inference, followed by a comment showing from which formulas and tautology that particular formula has been obtained.

e.g., if we follow this notation, the third line shows that formula in this line is numbered (3) and has been obtained from premises (1) and (2). The comment on the right says that the formula Q has been introduced using rule **T** and also indicates the details of the application of rule **T**.

Example 2: Prove that $R \vee S$ follows logically from the premises $C \vee D, (C \vee D) \rightarrow \neg H, \neg H \rightarrow (A \wedge B)$, and $(A \wedge \neg B) \rightarrow (R \vee S)$.

$\{1\}$	(1)	$(C \vee D) \rightarrow \neg H$	**P**
$\{2\}$	(2)	$\neg H \rightarrow (A \wedge \neg B)$	**P**
$\{1, 2\}$	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	**T**, (1), (2), and I_{13}
$\{4\}$	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	**P**
$\{1, 2, 4\}$	(5)	$(C \vee D) \rightarrow (R \vee S)$	**T**, (3), (4), and I_{13}
$\{6\}$	(6)	$C \vee D$	**P**
$\{1, 2, 4, 6\}$	(7)	$R \vee S$	**T**, (5), (6), and I_{11}

Example 3: Prove that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

$\{1\}$	(1)	$P \vee Q$	**P**
$\{1\}$	(2)	$\neg P \rightarrow Q$	**T**, (1), E_1, and E_{16}
$\{3\}$	(3)	$Q \rightarrow S$	**P**
$\{1, 3\}$	(4)	$\neg P \rightarrow S$	**T**, (2), (3), and I_{13}
$\{1, 3\}$	(5)	$\neg S \rightarrow P$	**T**, (4), E_{18}, and E_1
$\{6\}$	(6)	$P \rightarrow R$	**P**
$\{1, 3, 6\}$	(7)	$\neg S \rightarrow R$	**T**, (5), (6), and I_{13}
$\{1, 3, 6\}$	(8)	$S \vee R$	**T**, (7), E_{16}, and E_1

Example 4: Prove that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q, Q \rightarrow R, P \rightarrow M$, and $\neg M$.

$\{1\}$	(1)	$P \rightarrow M$	**P**
$\{2\}$	(2)	$\neg M$	**P**
$\{1, 2\}$	(3)	$\neg P$	**T**, (1), (2), and I_{12}
$\{4\}$	(4)	$P \vee Q$	**P**
$\{1, 2, 4\}$	(5)	Q	**T**, (3), (4), and I_{10}
$\{6\}$	(6)	$Q \rightarrow R$	**P**
$\{1, 2, 4, 6\}$	(7)	R	**T**, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8)	$R \wedge (P \vee Q)$	**T**, (4), (7), and I_9

Example 5: Prove that $\neg Q, P \rightarrow Q \implies \neg P$.

$\{1\}$	(1)	$P \rightarrow Q$	**P**
$\{1\}$	(2)	$\neg Q \rightarrow \neg P$	**T**, (1), and E_{18}
$\{3\}$	(3)	$\neg Q$	**P**
$\{1, 3\}$	(4)	$\neg P$	**T**, (2), (3), and I_{11}

We now introduce a third rule, called rule **CP** or rule of conditional proof.

Rule **CP**: If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

Rule **CP** follows also from the equivalence $(P \wedge R) \to S \iff P \to (R \to S)$.

Let P be the conjunction of the set of premises and let R be any formula. Then the above equivalence says that if we include R as an additional premise and if S is derived from $P \wedge R$, then $R \to S$ can be derived from the premise P alone.

Rule **CP** is also called the deduction theorem and is generally used if conclusion is of the form $R \to S$. In such cases, R is taken as an additional premise and S is derived from the given premise and R.

Example 6: Prove that $R \to S$ can be derived from the premise $P \to (Q \to S), \neg R \vee P$, and Q.

$\{1\}$	(1)	$\neg R \vee P$	**P**
$\{2\}$	(2)	R	**P** (assumed premise)
$\{1,2\}$	(3)	P	**T**, (1), (2), and I_{10}
$\{4\}$	(4)	$P \to (Q \to S)$	**P**
$\{1,2,4\}$	(5)	$Q \to S$	**T**, (3), (4), and I_{11}
$\{6\}$	(6)	$Q \to R$	**P**
$\{1,2,4,6\}$	(7)	S	**T**, (5), (6), and I_{11}
$\{1,4,6\}$	(8)	$R \to S$	**CP**

Thus, we have seen that a derivation consists of a sequence of formulas, each formula in the sequence being either a premise or tautologically implied by formula appearing before.

We have discussed about the decision problem, which is to determine in a finite number of steps that a given formula is tautology. Similarly, if one can determine in a finite number of steps whether an argument is valid, then the decision problem for validity is solvable.