Lecture - 5
Duality laws, tautological implications

Recall the previous lectures; primary statements, truth tables, compound statements, connectives, negation, conjunction, disjunction, statement formulas, conditional and biconditional statements, converse, contrapositive and inverse, well-formed formula, tautology, substitution instances, equivalence of formulas and replacement process.

We now consider the formulas which involves the connectives $\wedge, \vee$ and $\neg$ only. There is no loss of generality in assuming this and hence we can restrict ourselves to these connectives only, as we shall se that any formula involving any other connectives can be replaced by an equivalent formula which involves $\wedge, \vee$ and $\neg$ only.

Let A and A^* be two formulas, we say that A and A^* are duals of each other if one can be obtained from other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other. Furthermore, if the formula A involves the special variables $\mathbf{T}$ or $\mathbf{F}$, then its dual A^* is obtained by replacing $\mathbf{T}$ by $\mathbf{F}$ and $\mathbf{F}$ by $\mathbf{T}$, in addition to the above mentioned interchanges.

Examples: The duals of the formulas $(P \vee Q) \wedge R,\ (P \wedge Q) \vee \mathbf{T}$ and $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$ are $(P \wedge Q) \vee R,\ (P \vee Q) \wedge \mathbf{F}$ and $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$, respectively.

Let A and A^* be duals formulas and let $P_1, P_2, \cdots, P_n$ be all the atomic variables which appears in A and A^*. Then we may write A and A^* as follows

$$A(P_1, P_2, \cdots, P_n),\ A^*(P_1, P_2, \cdots, P_n).$$

Then by using De Morgan's law we see that

$$P \wedge Q \iff \neg(\neg P \vee \neg Q),\ P \vee Q \iff \neg(\neg P \wedge \neg Q)$$

and finally

$$\neg A(P_1, P_2, \cdots, P_n) \iff A^*(\neg P_1, \neg P_2, \cdots, \neg P_n)$$

and consequently

$$A(\neg P_1, \neg P_2, \cdots, \neg P_n) \iff \neg A^*(P_1, P_2, \cdots, P_n).$$

Exercise: Verify the above for $A(P, Q, R) = \neg P \wedge \neg(Q \vee R)$.

$$A^*(P, Q, R) = \neg P \vee \neg(Q \wedge R)$$

and

$$A^*(\neg P, \neg Q, \neg R) = \neg\neg P \vee \neg(\neg Q \wedge \neg R) \iff P \vee (Q \vee R).$$

Also,

$$\neg A(P, Q, R) = \neg(\neg P \wedge \neg(Q \vee R)) \iff P \vee (Q \vee R).$$

Second equivalence can be verified in a similar way.

Next we see the relation between duals and equivalence.
Let A and B be two formulas and let $P_1, P_2, \cdots, P_n$ be all the atomic variables which appears in A and B.
Assume that $A \iff B$, i.e. A is equivalent to B. This means that $A \leftrightarrow B$ is a tautology. Then

$$A(\neg P_1, \neg P_2, \cdots, \neg P_n) \leftrightarrow B(\neg P_1, \neg P_2, \cdots, \neg P_n)$$

is also a tautology.
Now, use

$$A(\neg P_1, \neg P_2, \cdots, \neg P_n) \iff \neg A^*(P_1, P_2, \cdots, P_n)$$

in above to conclude that

$$\neg A^*(P_1, P_2, \cdots, P_n) \leftrightarrow \neg B^*(P_1, P_2, \cdots, P_n)$$

is a tautology, which means

$$A^* \iff B^*.$$

Recall the list of basic equivalent formulas from the last lecture, for every pair of equivalent formulas we had a pair of equivalent formulas which was dual to the first pair.

Exercise: Prove that

(1) $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \iff (\neg P \vee Q)$.
(2) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \iff (\neg P \wedge Q)$.

Solution:
Since

$$(A \rightarrow B) \iff (\neg A \vee B),$$

therefore by taking $A = \neg(P \wedge Q)$ and $B = (\neg P \vee (\neg P \vee Q))$, we obtain

$$\begin{aligned}
\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) &\iff (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \\
&\iff (P \wedge Q) \vee (\neg P \vee Q) \\
&\iff (P \wedge Q) \vee \neg P \vee Q \\
&\iff ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q \\
&\iff (\mathbf{T} \wedge (Q \vee \neg P)) \vee Q \\
&\iff (Q \vee \neg P)) \vee Q \\
&\iff (Q \vee \neg P)
\end{aligned}$$

Now, consider the formula

$$(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)),$$

then its dual is given by

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)).$$

We have already seen that if $A \iff B$, then $A^* \iff B^*$, and we have proved that

$$A = (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \iff (Q \vee \neg P) = B.$$

Thus $A^* \iff B^*$, yields the following

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \iff (\neg P \wedge Q).$$

Tautological Implications: We know that the connectives $\vee, \wedge$ and $\leftrightarrow$ are symmetric, in the sense that

$$P \wedge Q \iff Q \wedge P, P \vee Q \iff Q \vee P, P \leftrightarrow Q \iff Q \leftrightarrow P.$$

Also, we have seen that $P \to Q$ is not equivalent to $Q \to P$.

Recall that for a given conditional statement formula $P \to Q, Q \to P$ the formulas $\neg P \to \neg Q$ and $\neg Q \to \neg P$ is called its converse, inverse and contrapositive, respectively. We also saw that the following holds;

$$P \to Q \iff \neg Q \to \neg P, \; Q \to P \iff \neg P \to \neg Q.$$

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We now define tautological implication as follows;

A statement formula A is said to *tautologically imply* a statement B if and only if $A \to B$ is a tautology. It is denoted by $A \implies B$ and read as "A implies B."

We now list some basic implications and they can be proved by truth tables.

(1) $P \wedge Q \implies P$
(2) $P \wedge Q \implies Q$
(3) $P \implies P \vee Q$
(4) $\neg P \implies P \to Q$
(5) $Q \implies P \to Q$
(6) $\neg(P \to Q) \implies P$
(7) $\neg(P \to Q) \implies \neg Q$
(8) $P \wedge (P \to Q) \implies Q$
(9) $\neg Q \wedge (P \to Q) \implies \neg P$
(10) $\neg P \wedge (P \vee Q) \implies Q$
(11) $(P \to Q) \wedge (Q \to R) \implies P \to R$
(12) $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \implies R$

Recall that in $P \to Q$, the statement P and Q are called antecedent and consequent, respectively.

To prove a given implication, it suffices to prove that if we assign the truth value T to the antecedent then it gives the truth value T for the consequent, which means that the given conditional statement is a tautology an hence the implication follows.

Examples: Consider the implication (9), assume that $\neg Q \wedge (P \to Q)$ has the truth value T, then both $\neg Q$ and $P \to Q$ have the truth values T, which means that the truth value of

Q is F. Now, the truth value of $P \to Q$ is T, therefore the truth value of P must be F (as the truth value of Q is F). Hence, the truth value of the consequent $\neg P$ is T.

Next consider the implication (12), assume that antecedent is true. This means that $P \vee Q, P \to R$ and $Q \to R$ all are true. Now, if P is true then R must be true because $P \to R$ is true. Similarly, if Q is true then R must be true. Also, either P or Q is true by the assumption that $P \vee Q$ is true. In any case, we conclude that R is true and hence the implication (12) follows.

Another method to prove an implication $P \implies Q$ is that, assume that the consequent is false and show that the antecedent is also false and this means that $P \to Q$ is true. Try this method to prove the implication (9)(exercise).

Exercise: Show that ($P \implies Q$ and $Q \implies P$) iff $P \iff Q$.

The above can be taken as an alternative definition of the equivalence of the two formulas. More precisely, if each of the two formulas A and B implies other, then A and B are equivalent.

Recall that, if a formula is equivalent to a tautology, then it must be a tautology. Similarly, if a formula is implied by a tautology, then it is a tautology (why?).

We now prove that the implication is transitive. Assume that $A \implies B$ and $B \implies C$. This means that $A \to B$ and $B \to C$ is a tautology and so is their conjunction $(A \to B) \wedge (B \to C)$. Now, the implication (11) says that $(A \to B) \wedge (B \to C) \implies A \to C$. Hence $A \to C$ is a tautology and this means that $A \implies C$. This proves the transitivity of implication.

Transitivity of implication can be generalized for many formulas.

Another important property of implication is the following (which is easy to see);
if $A \implies B$ and $A \implies C$, then $A \implies B \wedge C$.

Also, we can extend the notion of implication $P \implies Q$ to several formulas, say $H_1, H_2, \cdots, H_m$ jointly imply a formula Q, i.e. $H_1, H_2, \cdots, H_m \implies Q$ means

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m) \implies Q.$$

Theorem 0.1. *If $H_1, H_2, \cdots, H_m$ and P imply Q, then $H_1, H_2, \cdots, H_m$ imply $P \to Q$.*

Proof. Given that

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m \wedge P) \implies Q.$$

This means that

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m \wedge P) \to Q$$

is a tautology.

We have seen that

$$P \to (Q \to R) \iff (P \wedge Q) \to R.$$

Take $P = (H_1 \wedge H_2 \wedge \cdots \wedge H_m,\ Q = P$ and $R = Q$, then

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m) \to (P \to Q)$$

is a tautology and we are done. $\square$