Lecture - 2
Statement formula, truth tables, conditional and bi-conditional

Recall the previous lecture; Primary statements, truth tables, compound statements, connectives, negation, conjunction, disjunction, statement formulas.

- Those statements which do not contain any connectives are called atomic or primary or simple statements.
- Those statements which contain one or more connectives are called molecular or composite or compound statements.
- Let P and Q be two statements. Few examples of compound statements are;

$$\neg P, P \vee Q, (P \wedge Q) \vee (\neg P), P \wedge (\neg Q).$$

- The parrntheses are used in the same sense in which they are used in arithmetic or algebra, which means that expressions in the innermost parentheses are simplified first.
- $\neg(P \wedge Q), (P \wedge Q) \vee (Q \wedge R), ((P \wedge Q) \vee R) \wedge (\neg P)$.
- Truth tables for $P \vee \neg Q, P \wedge \neg P, (P \vee Q) \wedge \neg P$.

We now determine the truth value of a statement formula for each possible combination of the truth values of the component statements. A table which shows such truth value is called the truth table of the statement formula.

- One component - two rows, e.g. negation.
- Two components - four rows, e.g. $\vee$ and $\wedge$.
- n - components - 2^n rows.

There are two methods of constructing the truth table of a given statement formula, which is illustrated by following example;

Find the truth table for the statements formula $P \vee \neg Q$:

Method 1: We shall consider all the possible truth values of the statements P and Q, and these values are entered in first two columns.

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

In the third column the truth values of $\neg Q$ is entered and finally the truth values of $P \vee \neg Q$ is entered in the fourth column.

Method 2: In this method a column is drawn for each statement as well as for the connectives that appear. The truth values are entered step by step. The step number at the last row of the table show the sequence followed in arriving at the final step.

P	Q	P	$\vee$	$\neg$	Q
T	T	T	T	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	T	T	F
Step number		1	3	2	1

Exercise: Find the truth table for the statement formulas $P \wedge \neg P$ and $(P \vee Q) \wedge \neg P$.

Conditional and Biconditional: Let P and Q be two statements. Then statement $P \to Q$ which is read as "If P, then Q" is called a *conditional* statement. The statement $P \to Q$ has a truth value F when Q has the truth value F and P the truth value T; otherwise it has the truth value T.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $P \to Q$ is called conditional because the truth value of Q is T on the condition that P holds. Note that, in the truth table of $P \to Q$, the truth value of $P \to Q$ is T when truth values of both P and Q is T, and when the truth value of P is F (in this case it does not matter what is the truth value of Q.)

Example: Let (x_n) be a sequence of real numbers. If the series $\sum\limits_{n=1}^{\infty} x_n$ converges, then $\lim\limits_{n\to\infty} x_n = 0$.

Problem: Express in english the statement $P \to Q$ where
P : The sun is shining today.
$Q : 2 + 7 > 4$.
Solution: If the sun is shining today, then $2 + 7 > 4$.

The following expressions are represented by $P \to Q$;

- Q is necessary for P.
- P is sufficient for Q.
- P implies Q.
- Q if P.
- P only if Q.

Problem: Write the following statement in symbolic form; If either Jak takes Mathematics or Michael takes Physics, then Raj will take Chemistry.
Solution: P : Jak takes Mathematics.

Q : Michael takes Physics.
R : Raj takes Chemistry.
The given statement is $(P \vee Q) \to R$.

Exercise: Write the truth table for $(P \to Q) \wedge (Q \to P)$.

Let P and Q be two statements. Then statement $P \iff Q$, which is read as "P if and only if Q" and abbreviated as "P iff Q" is called a *biconditional* statement. The statement $P \iff Q$ has a truth value T whenever both P and Q have identical truth values. This is translated as" P is necessary and sufficient for Q."

Exercise: Write the truth table for $P \iff Q$.
Exercise: Write the truth table for the formula $\neg(P \wedge Q) \iff (\neg P \vee \neg Q.)$

Converse, Contrapositive and Inverse: We now look at some new conditional statements obtained from $P \to Q$.

- $Q \to P$ is called the *converse* of $P \to Q$.
- $\neg Q \to \neg P$ is called the *contrapositive* of $P \to Q$.
- $\neg P \to \neg Q$ is called the *inverse* of $P \to Q$.

The truth value of the contrapositive $\neg Q \to \neg P$ of a conditional statement $P \to Q$ is same as of the statement $P \to Q$. Note that the truth value of contrapositive is F only when the truth value of $\neg P$ is F and that of $\neg Q$ is T. Which means the truth value of contrapositive is F only when the truth value of P is T and that of Q is F.

Note also that neither the converse $Q \to P$ nor the inverse $\neg P \to \neg Q$ has same truth values as $P \to Q$ for all possible truth values of P and Q. Because when the truth values of P and Q are T and F, respectively, then the truth value of $p \to Q$ is F. However, in this case the converse and the inverse have truth value T (why?).

Exercise: Write the truth table for the formula $(P \vee \neg Q) \to (P \wedge Q.)$