Lecture - 1: Statements, Notation and Connectives

Our aim is to develop units for our object language called primary (primitive or atomic) statements.
We assume that object language contains a set of declarative sentences which cannot be broken down into simpler statements. These statements are called primary statements.

Declarative sentence means; a sentence that declares a fact, i.e, is either true or false, but not both. Therefore, we only admit those declarative sentence which have one and only one of two possible values called truth values. The truth values are true and false and are denoted by symbols T and F (1 and 0), respectively.

Therefore, we assume that it is possible to assign one and only one of the two possible values to a declarative sentence. As, we admit only two possible truth values, this logic is called *two-valued* logic.

We shall develop mechanism by which we should be able to construct other declarative sentences having one of the two possible truth values.

There are two types of declarative sentences in our object language;
the first types of declarative sentences are primitive and is denoted by distinct symbols $A, B, C, \cdots, P, Q, R, \cdots$,
and the other types are those sentences which are obtained from the primitive ones by using certain symbols called connectives, and certain punctuation marks, such as parentheses to join the primitive sentences. However, in any case, all the declarative sentences to which it is possible to assign one and only one of the possible truth values are called *statements.*

Examples: We now look at few examples;

(1) Canada is a country.
(2) Moscow is the capital of Spain.
(3) This statement is false.
(4) $1 + 101 = 110$.
(5) Close the door.
(6) Toronto is an old city.
(7) Man will reach Mars by 1980.

Connectives: The truth values for simple statements are obvious. We shall construct (complicated) statements from simple statements by using certain connecting words called sentential connectives.
We shall define these connectives and try to develop methods to determine the truth values of statements that are formed by using them. Also, we shall look at properties of these statements and relationship between them (if any).
Also, we shall see how the statements along with the connectives forms an algebra satisfying

certain properties and using these properties we shall be able to perform some calculations using statements as objects. These algebras has many interesting and important applications in the field of switching theory and logical design of computers.
The new statements which is formed from primary (primitive or atomic) statements are called compound or molecular statements. Thus, primary statements are those which do not have any connectives.
Any arbitrary sentence will be denoted by $P, Q, R, \cdots, P_1, Q_1, R_1, \cdots$.
The truth value of "P" is the truth value of the statement which it represents.

Negation: The negation of a statement is generally formed by introducing the word "not" at proper place in the statement or by prefixing the statement with the phrase "It is not the case that".

- If P denotes a statement, then the negation of "P" is written as "$\neg P$" and read as " not P."
- If the truth value of P is T, then the truth value of $\neg P$ is F, and vice-versa.

Below is the truth table for negation;

P	$\neg P$
T	F
F	T

Example 1: Consider the statement P : London is a city.
Then the negation of P, $\neg P$ is a statement $\neg P$: It is not the case that London is a city.
It can also be written as $\neg P$: London is not a city.

The above two statements are not identical but have the same meaning in english language. Thus a given statement in the object language is denoted by a symbol, and it may correspond to several statements (as in the previous example) in english language.

Example 2: Consider the statement P : I went to my class yesterday, then $\neg P$ is any one of the following statements

(1) It is not the case that I went to my class yesterday.
(2) I did not go to my class yesterday.
(3) I was absent from my class yesterday.

Example 3: P : At least 10 inches of rain fell today in Mumbai.
$\neg P$: It is not the case that at least 10 inches of rain fell today in Mumbai.
$\neg P$: Less than 10 inches of rain fell today in Mumbai.

Note that negation is connective although it only modifies a statement. Thus negation is unary operation which operates on a single statement.
The other symbol for negation are $\tilde{P}, \overline{P}, NOT\ P$.

Conjunction: Let P and Q be two statements. The conjunction of P and Q is the statement $P \wedge Q$ which is read as "P and Q." The statement $P \wedge Q$ has truth value T whenever both P and Q have the truth value T, and otherwise it has the truth value F.
Below is the truth table for conjunction;

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1: Consider the statements;

P : It is raining today.
Q : There are 20 tables in this room.
$P \wedge Q$: It is raining today and there are 20 tables in this room.

The above two statements are not related still we could form a statement by adding the conjunction "and" between them and is perfectly acceptable in our logic. However, in english the conjunction "and" is used between two statements which are related in some way.

Example 2: X and Y got A+ in this course.
X got A+ in this course and Y got A+ in this course.
P : X got A+ in this course.
Q : Y got A+ in this course.
$P \wedge Q$: X and Y got A+ in this course.

We have seen that the symbol $\wedge$ is used as translation of the connective "and" in english. However, the connective "and" is sometimes used in a different sense, and in such cases it cannot be translated by symbol $\wedge$ defined above. For illustration we shall look at the following examples;

Roses are red and violets are blue.
Here, the conjunction "and" is used in the same sense as the symbol $\wedge$ defined above.

He opened the book and started to read.
Here, "and" is used in the sense of "and then."

Ram and Sam are cousins.
Here, "and" is not a conjunction.

It is easy to observe that $\wedge$ is symmetric, i.e. $P \wedge Q = Q \wedge P$.
The other symbol for conjunction are $P\&Q, P \cdot Q, P\ AND\ Q$.

Disjunction: Let P and Q be two statements. The disjunction of P and Q is the statement $P \vee Q$ which is read as "P or Q." The statement $P \vee Q$ has truth value F whenever both P

and Q have the truth value F, and otherwise it has the truth value T.
Below is the truth table for conjunction;

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The connective $\vee$ is not always the same as the word "or" because in english the word "or" is used both as an "exclusive OR" and as an "inclusive OR";

Examples:

(1) I shall watch the game on television or go to the game.
(2) There is something wrong in the bulb or with the wiring.
(3) Twenty or thirty animals were killed in the fire today.

In (1), the connective "or" is used in the exclusive sense; i.e. one or other possibility exists but not both.
In (2), the connective "or" is used is "inclusive OR," i.e. one or other or both.
In (3), "or" is used to for an approximate number of animals and not as connective.

Note, that in the definition of $\vee$ it is clear that it is "inclusive OR."

Statement Formulas and Truth Tables: We have defined the connectives $\neg, \wedge$, and $\vee$.

- Those statements which do not contain any connectives are called atomic or primary or simple statements.
- Those statements which contain one or more connectives are called molecular or composite or compound statements.
- Let P and Q be two statements. Few examples of compound statements are;
$$\neg P, P \vee Q, (P \wedge Q) \vee (\neg P), P \wedge (\neg Q).$$
- The parrntheses are used in the same sense in which they are used in arithmetic or algebra, which means that expressions in the innermost parentheses are simplified first.
- $\neg(P \wedge Q), (P \wedge Q) \vee (Q \wedge R), ((P \wedge Q) \vee R) \wedge (\neg P)$.
- Truth tables for $P \vee \neg Q, P \wedge \neg P, (P \vee Q) \wedge \neg P$.