Lecture - 13
Predicate Calculus

We have discussed the propositional logic or statement logic and its theory of inference. However, sometime it is not possible to express the meaning of all the statements in mathematics and natural language by this logic. e.g.
"Every computer connected to the college network is working properly."
Truth of the above statement cannot be decided by any rules propositional logic. Therefore, to conclude the truth of a wide range of statements in mathematics and computer science we shall now introduce a powerful logic called **predicate logic.** We first discuss what is predicate and then introduce notion of quantifiers, which enables us reason with statements that assert that a certain property holds for objects of a certain type and with statements that assert the existence of an object with a particular property.

Predicate: Consider the statements

$$\text{“}x > 3,\text{”} \quad \text{“}x = y + 3,\text{”} \quad \text{“}x + y = z,$$

and "computer x is under attack by an intruder"

" computer x is working properly".
This type of statements are often occur in mathematical assertions and in computer programs. We cannot say anything about the truthness or falseness of these statements when the variables are not specified.

Consider now the statement "x is greater than 3." This statement has two parts. The first part is the variable x, which is the subject of the statement. The second part - the predicate, "is greater than 3"- and this refers to a property that the subject of the statement can have. Thus, we can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate " is greater than 3" and x is the variable. We say that $P(x)$ is the value of the propositional function P at x. Once we assign variable x is assigned the statement $P(x)$ becomes a proposition and we can talk about its truth value.

Example: If $P(x)$ denotes the statement "$x > 3$", then the truth value of $P(4)$ and $P(2)$, is true and false, respectively.

One can have statements involving more than one variable. e.g., consider the statement "$x = y + 3$." We shall denote this by $Q(x, y)$, here x and y are variables and Q is the predicate. Again, if we assign the variables x and y, we can talk about its truth value. It is immediate that $Q(1, 2)$ is false where as $Q(3, 0)$ is true.

Similarly, if we denote the statement $x + y = z$ by $R(x, y, z)$, then the truth values of $R(1, 2, 3)$ and $R(0, 0, 1)$ is true and false, respectively.

More general, we denote a statement that involves the n variables $x_1, x_2, \cdots, x_n$ by $P(x_1, x_2, \cdots, x_n)$.

A statement of the form $P(x_1, x_2, \cdots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \cdots, x_n)$.

Propositional function occur in computer programs and predicates are used to verify that computer programs always produce the desired output when given valid input. The statements that describe valid input are called as preconditions and the conditions that the output should satisfy when the program has run are called postconditions.

When we assign the values to the variables in a propositional function, the resulting statement becomes a proposition with a certain truth value. However, there is another important way to, called *quantification*, to create a proposition from a propositional function. Quantification express the extent to which a predicate is true over a range of elements.

Usually, the english words *all, many, none,* and *few* are used in quantification. Here, we shall discuss two types of quantifications: universal and existential; universal quantification tells us that a predicate is true for every element under consideration where as existential quantification tells us that there is one or more element in consideration for which predicate is true. The area of logic in which we deal with predicates and quantifiers is called the *predicate calculus.*

The Universal Quantifier: There are many mathematical statements that asserts that a property is true for all values of a variable in a particular domain, called the domain of discourse (or the universe of discourse), or simply domain. We express such statements by using universal quantifier. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain. It is important that the domain must always be specified when a universal quantifier is used otherwise quantification of a statement is not defined.

Definition 0.1. *The universal quantification of $P(x)$ is the statement*
"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here, $\forall$ is called the **universal quantifier**. *$\forall x P(x)$ is read as "for all $x P(x)$" or 'for every $x P(x)$." An element for which $P(x)$ is false is called a* **counterexample** *of $\forall x P(x)$.*

- If $P(x)$ is the statement "$x + 1 > x$", then as $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.
- If $P(x)$ is the statement "$x^2 > 0$", then the quantification $\forall x P(x)$ is false if the universe of discourse is the set of all integers.(??)

If the domain is finite and the elements are listed as; $x_1, x_2, \cdots, x_n$, then the universal quantification is same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because it is immediately clear that the above conjunction is true if and only if $P(x_1), P(x_2), \cdots, P(x_n)$ are all true.

Consider $\forall x P(x)$, where $P(x)$ is the statement "$x^2 \geq x$." If the domain under consideration is $\mathbb{R}$ - the set of real numbers, then $\forall x P(x)$ is false, e.g. $\left(\frac{1}{2}\right)^2 \ngeq \frac{1}{2}$. However, if we consider $\mathbb{Z}$ as the domain then $\forall x P(x)$ is true.

The above example shows the dependency on the domain of consideration.

The Existential Quantifier:

Definition 0.2. *The existential quantification of $P(x)$ is the statement " There exists an element x in the domain such that $P(x)$."*

We use the notation $\exists x P(x)$ for the existtential quantification of $P(x)$. Here, $\exists$ is called the **existential quantifier**.

If the domain is finite and the elements are listed as; $x_1, x_2, \cdots, x_n$, then the existential quantification is same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because it is immediately clear that the above disjunction is true if and only if at least one of the $P(x_1), P(x_2), \cdots, P(x_n)$ is true.

Statement	True	False
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Uniqueness quantifier, denoted by $\exists!$ or the notation $\exists! x P(x)$ is the statement

" There exists a unique x in the domain such that $P(x)$ is true."

e.g., every positive real number (resp. positive definite matrix) has a positive (positive definite) square root.

Example: Consider the statements $\forall x < 0(x^2 > 0), \forall y(y^3 \neq 0)$, and $\exists z(z^2 = 2)$ with $\mathbb{R}$ as the domain in each cases.

Precedence of Quantifiers: The quantifier $\forall$ and $\exists$ have higher precedence than all the logical operators from propositional calculus/statement calculus. e.g., $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$, i.e. $(\forall x P(x)) \vee Q(x)$ and not $\forall(x P(x) \vee Q(x))$.

Binding variables: When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.

An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is called **free**.

All the variables that occur in a propositional function must be bound or set equal to a particular value to turn into a proposition. This is done by using a combination of universal quantifier, existential quantifier, and the value assignments.

The part of logical expression to which a quantifier is applied is called the **scope** of this quantifier. Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specifies this variable.

Example: Consider the statement $\exists x(x + y = 1)$. Here, the variable x is bound by the existential quantification $\exists x$, and the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. Thus, x is bound and y is free.

Consider the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$. Here, all variables are bound. The scope of first quantifier, $\exists x$, is $P(x) \wedge Q(x)$ as $\exists x$ is applied only to $P(x) \wedge Q(x)$. Similarly, the scope of second quantifier, $\forall x$, is $R(x)$.

Definition 0.3 (Equivalence involving Quantifiers:)**.** *Two statements S and T involving predicates and quantifiers are said to be logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to denote that S and T are logically equivalent.*

Example: Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent.

This means that a universal quantifier can be distributed over conjunction.

The proof is easy and left as an exercise.

We now discuss the negation of a quantified expression. Consider the statement

“Every student in your class has taken a course in calculus.”

This is universal quantification, viz., $\forall x P(x)$, where $P(x)$ is the statement “x has taken a course in calculus” and the domain is the students in your class. The negation of the above statement is

“It is not the case that every student in your class has taken a course in calculus” or equivalently, “There is a student a in your class who has not taken a course in calculus.” Which is the existential quantification of the negation of the original propositional functional, viz., $\exists x \neg P(x)$.

Prove that $\neg \forall x P(x) \equiv \exists x \neg P(x)$.
Proof:

- No matter what the propositional function $P(x)$ is what the domain of consideration is, note that $\neg \forall x P(x)$ is true if an only if $\forall x P(x)$ is false.
- $\forall x P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false and which holds if and only if there is an element x in the domain for which $\neg P(x)$ is true.

- Finally, note that there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true.

  All together implies that $\neg\forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true. Which means that these two are logically equivalent.

We now negate an existential qualification. Consider the statement

"There is a student in this class who has taken a course in calculus." This is an existential quantification $\exists x Q(x)$, where $Q(x)$ is the statement "x has taken a course in calculus."

The negation of the above statement is "It is not the case that there is a student in this class who has taken a course in calculus" or equivalently"Every student in this class has not taken a course in calculus." Which is the universal quantification of the negation of the original propositional function, i.e. $\forall x \neg Q(x)$.

Prove that $\neg\exists x Q(x) \equiv \forall x \neg Q(x)$.
Proof:

- No matter what the propositional function $Q(x)$ is what the domain of consideration is, note that $\neg\exists x Q(x)$ is true if an only if $\exists x P(x)$ is false. Which is true if an only if there is no x in the domain for which $Q(x)$ is true.
- There is no x in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- Finally, note that $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for every x in the domain which holds if and only if $\forall x \neg Q(x)$ is true.

  All together implies that $\neg\exists x Q(x)$ is true if and only if $\forall x \neg Q(x)$ is true. Which means that these two are logically equivalent.

Negation	Equivalent	When is Negation True?	When False?
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every $x, P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

When the domain of the predicate $P(x)$ has finite number of elements, say n listed as follows; $x_1, x_2, \cdots, x_n$, then the rule of negation is exactly same as De Morgan's law;

$\neg\forall x P(x)$ is same as $\neg(P(x_1)\wedge P(x_2)\wedge\cdots\wedge P(x_n))$ which is equivalent to $\neg P(x_1)\vee\neg P(x_2)\vee \cdots \vee P(x_n)$ by De Morgan's law, and this is same as $\exists x \neg P(x)$.

Similarly, $\neg\exists x P(x)$ is same as $\neg(P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n))$ which is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge P(x_n)$ by De Morgan's law, and this is same as $\forall x \neg P(x)$.

Example: Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

From the previous table we see that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(\neg(P(x) \to Q(x)))$ are logically equivalent. We also know that $\neg(P(x) \to Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent. Now the claim follows immediately, by using transitivity of equivalence.

Nested Quantifiers: We now discuss nested quantifiers. Two quantifiers are nested if one is within scope of others, such as

$$\forall x \exists y(x+y=0).$$

Everything within the scope of of a quantifier can be thought of as a propositional function. e.g., $\forall x \exists y(x+y=0)$ is same as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x,y)$, where $P(x,y)$ is $x+y=0$.

- $\forall x \forall y(x+y=y+x)$ - addition is commutative.
- $\forall x \exists y(x+y=0)$ - existence of additive inverse.
- $\forall x \forall y \forall z(x(y+z)=(x+y)+z)$ - associativity in addition.

It is sometimes helpful to think in terms of loop while working with quantifications of more than one variables. e.g. to see whether $\forall x \forall y P(x,y)$ is true, we loop the values of x, and for each x we loop the values for y. If we find that $P(x,y)$ is true for all value of x and y, we have determined that $\forall x \forall y P(x,y)$ is true.

If we ever hit a value of x for which we have a value y for which $P(x,y)$ is false, we have shown that $\forall x \forall y P(x,y)$ is false.

Similar thing can be done for $\forall x \exists P(x,y), \exists x \exists y P(x,y)$.

- If $P(x,y)$ denotes the statement $x+y=y+x$, then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ are same.
- Let $P(x,y)$ be the statement $x+y=0$. Then the quantifications $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$ are not the same.

Statement	When True?	When False?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y	There is a pair x,y for. which $P(x,y)$ is false
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y.

- If $P(x,y,z)$ denotes the statement $x+y=z$, then what is the truth value of $\forall x \forall y \exists P(x,y,z)$ and $\exists z \forall x \forall y P(x,y,z)$.