Lecture - 4
Equivalence of formulas

Recall the previous lectures; primary statements, truth tables, compound statements, connectives, negation, conjunction, disjunction, statement formulas, conditional and biconditional statements, converse, contrapositive and inverse, well-formed formula, tautology, substitution instances.

Let A and B be two statement formulas and let $P_1, P_2, \cdots, P_n$ denote all the variables which are occurring in both A and B. If the truth value of A is equal to the truth value of B for every one of the 2^n possible sets of truth values assigned to $P_1, P_2, \cdots, P_n$, then we say that A is equivalent to B.

If we assume that the variables and assignment of the truth values to the variables appear in the same order in the truth tables of A and B, then the final columns in the truth tables of A and B are identical, if A and B are equivalent.

Examples:

(1) $\neg\neg P$ is equivalent to P.
(2) $P \vee P$ is equivalent to P.
(3) $(P \wedge \neg P) \vee Q$ is equivalent to Q.
(4) $P \vee \neg P$ is equivalent to $Q \vee \neg Q$.

Note that it is not necessary that both A and B contain the same variables. e.g. (3) and (4). However, if two formulas are equivalent and a variable occurs in only one of them, then the truth value of this formula is independent of this variable. e.g. in (3), the truth value of $(P \wedge \neg P) \vee Q$ is independent of the truth value of P. Similarly, in (4), the truth value of $(P \vee \neg P)$ and $Q \vee \neg Q$ is independent of the truth value of P and Q.

Consider the truth table for conditional and biconditional statements;

P	Q	$P \to Q$	$P \leftrightarrow Q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

We see that $P \leftrightarrow Q$ is T whenever both P and Q have the same truth values. Thus, the statement formulas A and B are equivalent provided $A \leftrightarrow B$ is a tautology and conversely, if $A \leftrightarrow B$ is a tautology then A is equivalent to B. We use the notation $A \iff B$ to say that A is equivalent to B.

It is easy to see that A is equivalent to B or $A \iff B$ is an equivalence relation.

Binary relation: A binary relation over sets X and Y is a subset of the cartesian product $X \times Y$; that is, it is a set of ordered pairs (x, y) consisting of elements $x \in X$ and $y \in Y$. An element x is related to an element y, if and only if the pair (x, y) belongs to the set. We say x is related to y denoted by xRy.
Consider a binary relation on X, i.e. consider a subset of $X \times X$.

(1) **Reflexive:** x is related to x, $\forall x \in X$, i.e. xRx, $\forall x \in X$ or $(x, x) \in X \times X$, $\forall x \in X$.
(2) **Symmetric:** If x is related to y, the y is related to x, i.e. if xRy, then yRx, or if $(x, y) \in X \times X$, then $(y, x) \in X \times X$.
(3) **Transitive:** If xRy and yRz, then xRz.

Example: Let $X = \mathbb{N}$, for $m, n \in \mathbb{N}$ define mRn if $m|n$.

Example: Let $S \neq \phi$ and $X = \mathcal{P}(S)$. For $A, B \subseteq S$ define ARB if $A \subseteq B$.

Example: Let $X = M_{n\times n}(\mathbb{R})$, for matrices $A, B \in M_{n\times n}(\mathbb{R})$ define a relation ARB if there exists an invertible matrix $P \in M_{n\times n(}\mathbb{R})$ such that $B = P^{-1}AP$.

As we have seen in the case of tautologies, one method to determine whether given two formulas are equivalent is to construct their truth table. Write all the possible combinations of the truth values of the variables present in both formulas and then compare the final column.

Example (1): Prove that $(P \to Q) \iff (\neg P \vee Q)$.

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \iff (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

We now give a list of equivalent formulas;

$$
\begin{array}{lcc}
(1) & P \vee P \iff P & P \wedge P \iff P \\
(2) & (P \vee Q) \vee R \iff P \vee (Q \vee R), & (P \wedge Q) \wedge R \iff P \wedge (Q \wedge R) \\
(3) & P \vee Q \iff Q \vee P, & P \wedge Q \iff Q \wedge P \\
(4) & P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R), & P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R) \\
(5) & P \vee (P \wedge Q) \iff P, & P \wedge (P \vee Q) \iff P \\
(6) & \neg(P \vee Q) \iff \neg P \wedge \neg Q, & \neg(P \wedge Q) \iff \neg P \vee \neg Q \\
(7) & P \vee \mathbf{F} \iff P & P \wedge \mathbf{T} \iff P \\
(8) & P \vee \mathbf{T} \iff \mathbf{T} & P \wedge \mathbf{F} \iff \mathbf{F} \\
(9) & P \vee \neg P \iff \mathbf{T} & P \wedge \neg P \iff \mathbf{F}
\end{array}
$$

In the above list of formulas, the symbols **T** and **F** is used in the sense that **T** can be replaced only by a tautology and **F** by a contradiction.

In view of the formula (2) which is called associativity, the formulas $(P \vee Q) \vee R$ and $(P \wedge Q) \wedge R$ can be written as $P \vee Q \vee R$ and $P \wedge Q \wedge R$, respectively.

Also, note that the formulas listed above is written as a pair of formula;

$$A_1 \iff B_1,\ A_2 \iff B_2$$

and for each pair A_1, B_1 there is a pair A_2, B_2 in which $\vee, \wedge, \mathbf{T}$ and $\mathbf{F}$ is replaced by $\wedge, \vee, \mathbf{F}$ and $\mathbf{T}$, respectively. A_1 and A_2 are called duals of each other, and so are B_1 and B_2.

Recall that, while constructing substitution instances we only substituted for the variables which appeared in the formula. Also, the same formula (which we substituted) replace that particular variable everywhere in the formula. This ensures that a substitution instance of a tautology is again a tautology.
We shall now introduce another substitution process, called a *replacement process.* In replacement process any part of the formula (which is itself a formula, atomic, molecular) can be replaced by any other formula.

Consider the formula $(P \vee Q) \to P$, replace $(P \vee Q)$ by $R \to (S \wedge \neg M)$ and also replace the second P by $(P \wedge R) \to (\neg S \vee M)$, with the above replacement we obtain the following formula;

$$(R \to (S \wedge \neg M) \vee Q) \to ((P \wedge R) \to (\neg S \vee M)).$$

In general, the replacement process gives a new formula, however, the resulting formula may not always be "interesting."

If we add an extra condition that any part of a given formula that is to be replaced by another formula must be equivalent to other formula, then the resulting formula is equivalent to the original formula. Thus, using this process we can obtain new formulas equivalent to the original one.
For example, in the formula $P \wedge Q$ we can replace P by the formula $P \vee P$ as $P \vee P \iff P$, to obtain $(P \vee P) \wedge Q$ and it is easy to see that $(P \vee P) \wedge Q \iff P \wedge Q$.

Also, if we replace any part of a tautology by formulas which is equivalent to this part, the resulting formula will also be a tautology.

Exercise: Show that $P \to (Q \to R) \iff P \to (\neg Q \vee R) \iff (P \wedge Q) \to R$.
Hint: Use example 1 and the equivalent formulas (associativity of $\vee$, De Morgan's law).

Example: Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \iff R$.

$$
\begin{gathered}
(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \\
\iff (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R) \text{ (by distributive property)}, \\
\iff ((\neg P \wedge \neg Q) \wedge R)) \vee ((Q \vee P) \wedge R) \text{ (by associative property)}, \\
\iff ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R) \text{ (by distributive property)}, \\
\iff ((\neg P \wedge \neg Q) \vee (P \vee Q)) \wedge R) \text{ (by commutative property)}, \\
\iff (\neg(P \vee Q) \vee (P \vee Q)) \wedge R \text{ (by De Morgan's law)}, \\
\iff \mathbf{T} \wedge R \text{ (by (9))}, \\
\iff R \text{ (by (7))}.
\end{gathered}
$$

Example: Show that $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology. We have

$$\neg(\neg P \wedge (\neg Q \vee \neg R) \iff \neg(\neg P \wedge \neg(Q \wedge R)) \iff P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R),$$

and

$$(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R)) \iff (P \vee Q) \wedge (P \vee R).$$

Now,

$$\neg P \wedge \neg Q \iff \neg(P \vee Q), \ \neg P \wedge \neg R \iff \neg(P \vee R).$$

Therefore,

$$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \iff \neg(P \vee Q) \vee \neg(P \vee R) \iff \neg((P \vee Q) \wedge (P \vee R)).$$

Finally, the given formula is equivalent to

$$((P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R)).$$

The above formula clearly is a substitution instance of the formula $P \vee \neg P$ (with P replaced by $((P \vee Q) \wedge (P \vee R))$ which is a tautology. Hence, the given formula is a tautology.