Mathematical Methods(MA-203)

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Contents

The convolution theorem and its applications

The convolution theorem:

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F * G$$

$$OR$$

The convolution theorem can be re-written as

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = L\{F(t) * G(t)\} = L\{F(t)\}\,.\,L\{G(t)\}$$

Exercise

1. Find $L^{-1}[log\frac{s+1}{s+2}]$
2. Find $L^{-1}[log\frac{s^2+1}{s(s+1)}]$
3. **Find** $L^{-1}[tan^{-1}\frac{2}{s^2}]$
4. Find $L^{-1}[\frac{1}{(s^2+a^2)^2}]$

Solution of IVP(ODE) using Laplace transform

Laplace transfrm technique is a very useful and simple tool to solve intial value problems, that is to obtain the solution of differential equation subjected to intial conditions. It transforms the ODE in t domain to an algebraic eqution in Laplace transform domain which is solved by straight forward manipulations. The initial conditions are also used while taking Laplac transform. Then finally taking inversion of the Laplace transforms, we can get the olution to the original problem in t domain.

Solution of IVP(ODE) using Laplace transform

Example 1:

Solve $\left(D^2+4\right) y = t$ given that $y\left(0\right) = y'\left(0\right) = 0$ and $D \equiv \frac{d}{dt}$.

Sol: Given that

$$y''(t) + 4y(t) = t$$

and $y(0) = y'(0) = 0$.

Taking Laplace transform, we get

$$\begin{aligned} L\left\{y''(t)\right\} + 4L\left\{y(t)\right\} &= L\left\{t\right\} \\ \Rightarrow s^2 L\left\{y(t)\right\} - sy(0) - y'(0) + 4L\left\{y(t)\right\} &= 1/s^2 \\ \Rightarrow (s^2+4)L\left\{y(t)\right\} &= 1/s^2 \end{aligned}$$

$$\Rightarrow L\left\{y(t)\right\} = \frac{1}{s^2\left(s^2+4\right)}$$

Taking inverse Laplace transform, **we get final solution as**

$$y(t) = L^{-1}\left\{\frac{1}{s^2\left(s^2+4\right)}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+4}\right\} = \frac{1}{4}\left(t - \frac{sin\,2t}{2}\right)$$

Solution of ODE using Laplace transform contd...

Example 2:

Solve $\left(D^2+1\right)y = 6\,cos\,2t$, if $y=3,\ Dy=1,$ when $t=0$.

Ans: ????

Sol: We have

$$y''(t) + y(t) = 6\,cos\,2t$$

and $y(0) = 3,\ y'(0) = 1.$
Taking Laplace transform of (7.3), we get

$$\begin{aligned} L\left\{y''(t)\right\} + L\{y(t)\} &= 6L\{cos\,2t\} \\ \Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + L\{y(t)\} &= \frac{6s}{(s^2+4)} \\ \Rightarrow s^2L\{y(t)\} - 3s - 1 + L\{y(t)\} &= \frac{6s}{(s^2+4)} \end{aligned}$$

Solution of ODE contd...

$$L\{y(t)\} = \frac{3s+1}{(s^2+1)} + \frac{6s}{(s^2+4)(s^2+1)}$$

$$\Rightarrow L\{y(t)\} = 5\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\frac{s}{s^2+4}$$

$$\Rightarrow y(t) = L^{-1}\left\{5\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\frac{s}{s^2+4}\right\}$$

$$\Rightarrow y(t) = 5\cos t + \sin t - 2\cos 2t$$

This is the final solution of the Initial value problem.

Solution of ODE using Laplace transform contd...

Example 3:

Solve $\left[tD^2 + (1-2t)\,D - 2\right] y = 0$ when $y\,(0) = 0$ and $y'\,(0) = 2.$

Sol: Given that

$$ty''\,(t) + y'\,(t) - 2ty'\,(t) - 2y\,(t) = 0y\,(0) = 1,\ y'\,(0) = 2$$

Taking Laplace transform of both sides, we get

$$L\left\{ty''\right\} + L\left\{y'\right\} - 2L\left\{ty'\right\} - 2L\,\{y\} = 0$$

$$i.e., -\frac{d}{ds}L\left\{y''\right\} + L\left\{y'\right\} + 2\frac{d}{ds}L\left\{y'\right\} - 2L\,\{y\} = 0$$

$$i.e., -\frac{d}{ds}\left[s^2L\,\{y\} - sy\,(0) - y'\,(0)\right] + sL\,\{y\}$$

$$-y\,(0) + 2\frac{d}{ds}\left[s\bar{y} - y\,(0)\right] - 2\bar{y} = 0$$

$$i.e., -\frac{d}{ds}\left[s^2\bar{y} - s - 2\right] + s\bar{y} - 1 + 2\frac{d}{ds}\left[s\bar{y} - 1\right] - 2\bar{y} = 0,$$

where $L\,\{y\} = \bar{y}\,(s)$.

Solution of ODE using Laplace transform contd...

Example 3 contd...

Hence, we get

$$\begin{aligned}
\left(s^2 - 2s\right)\frac{d\bar{y}}{ds} &= s\bar{y} \\
\Rightarrow \frac{d\bar{y}}{\bar{y}} + \frac{ds}{(s-2)} &= 0 \\
\Rightarrow log\bar{y} + log\,(s-2) &= logC \\
\Rightarrow \bar{y} &= \frac{C}{(s-2)} \qquad (3.1)
\end{aligned}$$

Taking inverse Laplace transform of both sides, **we get the solution as**

$$\begin{aligned}
y\,(t) &= CL^{-1}\left\{\frac{1}{(s-2)}\right\} = Ce^{2t} \\
\Rightarrow y\,(t) &= e^{2t}, (as \;\; y(0) = 1))
\end{aligned}$$

Exercise

Exercise:

1. Solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^t$ with $y(0) = y'(0) = 0$.
2. Solve: $t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = cost$ with $y(0) = 1$.

ANS?????