Mathematical Methods(MA-203)

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Contents

Introduction

- In mathematics, the Laplace transform, named after its inventor **Pierre-Simon Laplace**, is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency).
- The concept of Laplace transform plays a vital role in wide field of science and technology like, electric and communication engineering, quantum physics, and many others.
- Specially, it is used as a tool for solving linear differential equations arising in engineering.
- In particular, it transforms ordinary linear differential equations into algebraic equations, whose solution can be once again transformed/inverted to obtain the solution of the original problem.

Laplace Transform

Definition:

The ***Laplace transform*** of a given function $F(t)$ defined for all real $t \geq 0$ is a function of a new variable s given by

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1.1}$$

where L is a Laplace transformation operator and $f(s)$ denotes the Laplace transform of $F(t)$. s (in general complex number) is called as Laplace transform parameter

Note: The Laplace transform of $F(t)$ is said to exist if the improper integral (1.1) converges for some value of s, otherwise it does not exist.

Examples

Example 1:

Find the Laplace transform of $F(t) = 1$.

Sol: By definition, we have

$$L\{F(t)\} = L\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

$$= \underset{x\to\infty}{Lim} \int_0^x e^{-st}dt = \underset{x\to\infty}{Lim} \left[\frac{e^{-st}}{-s}\right]_0^x$$

$$= -\frac{1}{s} \underset{x\to\infty}{Lim} \left(\frac{1}{e^{sx}} - 1\right) = \frac{1}{s}, \;\; s > 0$$

Examples contd...

Example 2:

Find the Laplace transform of $F(t) = e^{at}$.

Sol: By definition, we have

$$\begin{aligned}
L\{F(t)\} = L\{e^{at}\} &= \int_0^\infty e^{-st}\left(e^{at}\right) dt \\
&= \underset{x\to\infty}{Lim} \int_0^x e^{-(s-a)t} dt = \underset{x\to\infty}{Lim}\left[-\frac{e^{-(s-a)t}}{(s-a)}\right]_0^x \\
&= -\frac{1}{(s-a)} \underset{x\to\infty}{Lim}\left[e^{-(s-a)x} - 1\right] \\
&= \frac{1}{(s-a)}, \ \ s > a
\end{aligned}$$

Examples contd...

Example 3:

Find the Laplace transform of $F(t) = t^n$, n is any real number greater than -1.

Sol: By definition, we have

$$L\{F(t)\} = \int_0^\infty e^{-st}\left(t^n\right)dt = \int_0^\infty e^{-st}t^{(n+1)-1}dt \tag{1.2}$$

From the properties of Gamma function, we know that

$$\int_0^\infty e^{-ax}x^{m-1}dx = \frac{\Gamma(m)}{a^m},\ a>0,\ m>0 \tag{1.3}$$

Replacing a by s, m by $(n+1)$, and x by t in (1.3), we have

$$\int_0^\infty e^{-st}t^{(n+1)-1}dt = \frac{\Gamma(n+1)}{s^{n+1}},\ s>0,\ n+1>0$$

$$\Rightarrow L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}},\ s>0,\ n>-1$$

Existence of Laplace transform

Existence of L$\{$F(t)$\}$ depends of two properties of F(t):

- F(t) is piecewise continuous, which means, it must be single valued but can have a finite number of finite isolated discontinuies for t$>$0.
- and F(t) is of exponential order.

Sufficient conditions for the existence of Laplace transform

Theorem:

If $F(t)$ is of some exponential order as $t \to \infty$ and is piecewise continuous over every finite interval of $t \geq 0$, then Laplace of $F(t)$ exists.

OR

If $F(t)$ is piecewise continuous over every finite interval of $t \geq 0$ and satisfies $|F(t)| < me^{\sigma t}$ for all $t \geq 0$ (σ and m are constants), then $L\{F(t)\}$ exists.

Proof: Since $F(t)$ is of exponential order, say σ, then we have

$$|F(t)| < me^{\sigma t} \quad \text{for} \quad t \geq t_0 \tag{1.4}$$

where σ is a positive constant and $m > 0$, $t_0 > 0$. Now

$$\begin{aligned} L\{F(t)\} &= \int_0^\infty e^{-st} F(t)\, dt \\ &= \int_0^{t_0} e^{-st} F(t)\, dt + \int_{t_0}^\infty e^{-st} F(t)\, dt = I_1 + I_2 \end{aligned} \tag{1.5}$$

Proof contd....

Since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$, therefore I_1 exists. Again

$$|I_2| = \left|\int_{t_0}^{\infty} e^{-st} F(t)\, dt\right| \leq \int_{t_0}^{\infty} e^{-st} |F(t)|\, dt < m \int_{t_0}^{\infty} e^{-st} e^{\sigma t} dt, \ using\, (1.4$$

Thus,

$$|I_2| < m\left[-\frac{e^{-(s-\sigma)t}}{(s-\sigma)}\right]_{t_0}^{\infty} = \frac{me^{-(s-\sigma)t_0}}{s-\sigma} \tag{1.6}$$

Now, when $s > \sigma$, then $e^{-(s-\sigma)t_0} \to 0$ as $t \to \infty$. Hence, (1.6) shows that $|I_2|$ is finite for all $t_0 > 0$ when $s > \sigma$ and hence, I_2 is also convergent. Then from (1.5), it follows that $L\{F(t)\}$ exists for all $s > \sigma$.

Note: The conditions stated in the above theorem are sufficient to ensure the existence of Laplace transform of $F(t)$. These are not necessary conditions for the existence of $L\{F(t)\}$.

Counter example

Counter Example:

Consider the function $F(t) = 1/\sqrt{t}$.

The function $F(t)$ is not piecewise continuous on every finite interval in the range $t \geq 0$ as $t \to 0$, $F(t) \to \infty$.
Now, taking Laplace transform of $F(t)$, we have

$$L\{F(t)\} = \int_0^\infty e^{-st}\left(1/\sqrt{t}\right)dt = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}dx$$
$$= \frac{2}{\sqrt{s}}.\frac{\sqrt{\pi}}{2}, \quad s > 0$$

Hence, $L\{F(t)\}$ exists for $s > 0$ even if limit $F(t)$ does not exist at $t = 0$.
Note: $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Example

Task-1: Find Laplace transform of $H(t\text{-}a)$, where

$$H(t-a) = 0, t < a,$$

$$= 1, t \geq a$$

H(t) is heaviside unit step function.
Task-2: Find Laplace transform of Dirac Delta function: $\delta(t)$,where

$$\delta(t) = \infty, t = 0$$

$$= 0, t \neq 0$$

Ans: Task-1: $1/s$
Task-2: 1

Properties of Laplace transform

Linearity Property:

If C_1 and C_2 be constants, then

$$L\{C_1F_1(t)+C_2F_2(t)\}=C_1L\{F_1(t)\}+C_2L\{F_2(t)\}$$

Proof. By definition, we have

$$\begin{aligned}
L\{C_1F_1(t)+C_2F_2(t)\} &= \int_0^\infty e^{-st}\{C_1F_1(t)+C_2F_2(t)\}\,dt \\
&= \int_0^\infty e^{-st}\{C_1F_1(t)\}\,dt+\int_0^\infty e^{-st}\{F_2(t)\}\,dt \\
&= C_1\int_0^\infty e^{-st}F_1(t)\,dt+C_2\int_0^\infty e^{-st}F_2(t)\,dt \\
&= C_1L\{F_1(t)\}+C_2L\{F_2(t)\}
\end{aligned}$$

Properties contd...

Example:

Find the Laplace transform of $F(t) = sinh(at)$.

Sol: We know that $sinh(at) = \left(e^{at} - e^{-at}\right)/2$, we have

$$L\{sinh(at)\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = L\left\{\frac{e^{at}}{2} - \frac{e^{-at}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2 - a^2}, \quad |s| > a$$

First shifting theorem:

If $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$, where a is any real or complex constant.

Proof. By definition, we have

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)\,dt \tag{2.1}$$

Replacing s by $(s-a)$ on both sides of (2.1), we get

$$\begin{aligned} f(s-a) &= \int_0^\infty e^{-(s-a)t}F(t)\,dt \\ &= \int_0^\infty e^{-st}\left\{e^{at}F(t)\right\}dt \\ &= L\left\{e^{at}F(t)\right\} \end{aligned}$$

Properties contd...

Example:

Find the Laplace transform of $F(t) = e^{at} cos(bt)$.

Sol: Since

$$L\{cos(bt)\} = \frac{s}{s^2+b^2} = f(s)$$

Hence, by first shifting theorem, we have

$$L\{e^{at}cos(bt)\} = f(s-a)$$
$$= \frac{(s-a)}{(s-a)^2+b^2}$$

Properties contd...

Second shifting theorem:

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t \geq a \\ 0 \quad , & t < a \end{cases}$, then $L\{G(t)\} = e^{-as}f(s)$.

Proof: By definition of Laplace transform, we have

$$\begin{aligned} L\{G(t)\} &= \int_0^\infty e^{-st}G(t)\,dt \\ &= \int_0^a e^{-st}G(t)\,dt + \int_a^\infty e^{-st}G(t)\,dt, \; where\, 0 < a < \infty \\ &= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}F(t-a)\,dt \\ &= \int_a^\infty e^{-st}F(t-a)\,dt \\ &= \int_0^\infty e^{-s(a+u)}F(u)\,du = e^{-sa}\int_0^\infty e^{-su}F(u)\,du \\ &= e^{-sa}L\{F(t)\} = e^{-sa}f(s) \end{aligned}$$

Example:

Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a} & , t > a \\ 0 & t < a \end{cases}$.

Sol: By definition of Laplace transform, we have

$$\begin{aligned}
L\{G(t)\} &= \int_0^\infty e^{-st} G(t)\, dt \\
&= \int_0^a e^{-st} G(t)\, dt + \int_a^\infty e^{-st} G(t)\, dt \\
&= \int_0^a e^{-st} .0 dt + \int_a^\infty e^{-st} e^{t-a} dt \\
&= e^{-a} \int_0^\infty e^{-(s-1)t} dt = e^{-a} \left[-\frac{e^{-(s-1)t}}{s-1} \right]_a^\infty, \; s > 1 \\
&= \frac{e^{-sa}}{s-1}, \; s > 1
\end{aligned}$$

Change of scale property:

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

Proof: By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s) \qquad (2.2)$$

Now,

$$\begin{aligned} L\{F(at)\} &= \int_0^\infty e^{-st} F(at)\, dt = \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} F(u)\, du \\ &= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} F(u)\, du \\ &= \frac{1}{a} f\left(\frac{s}{a}\right), \ using\ (2.2) \end{aligned}$$

Properties contd...

Example:

If $L\{F(t)\} = \frac{1}{s}e^{-1/s}$, then find $L\{e^{-t}F(3t)\}$.

Sol: Given that

$$L\{F(t)\} = \frac{1}{s}e^{-1/s} = f(s)\,, \ say \qquad (2.3)$$

By change of scale property, we have

$$L\{F(3t)\} = \frac{1}{3}f\left(\frac{s}{3}\right) = \frac{1}{3}.\frac{3}{s}.e^{-3/s}, \ using\,(2.3) \qquad (2.4)$$
$$= \frac{1}{s}e^{-3/s} = g(s)\,, \ say$$

By the first shifting theorem, we get

$$L\{e^{-t}F(3t)\} = g(s+1) = \frac{1}{s+1}e^{-3/(s+1)}, \ using\,(2.4)$$

Laplace transform of $\{tF(t)\}$

Laplace transform of $\{tF(t)\}$:

Let $F(t)$ be a real function of class A i.e.

(1) $F(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.

(2) There exists constants σ and m such that $|F(t)| \leq me^{\sigma t}$ for $t \geq 0$, then

$$L\{tF(t)\} = -f'(s)$$

where $L\{F(t)\} = f(s)$.

Laplace transform of $\{tF(t)\}$ contd...

Proof:

By definition

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{3.1}$$

Since $F(t)$ belongs to class A, therefore Leibnitz's rule for differentiation under the integral sign is justified. From (3.1), we have

$$\begin{aligned}\frac{d}{ds} f(s) &= \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty \frac{d}{ds}\left\{e^{-st} F(t)\right\} dt \\ &= \int_0^\infty (-t)\, e^{-st} F(t)\, dt = -\int_0^\infty e^{-st}\{tF(t)\}\, dt \\ &= -L\{tF(t)\}\end{aligned}$$

$$\Rightarrow L\{tF(t)\} = -f'(s)$$

Example:

Find $L\{t\,sin\,(at)\}$.

Sol: We know that

$$L\{sin\,(at)\} = \frac{a}{s^2 + a^2},\ s > a$$

Now,

$$L\{t\,sin\,(at)\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{\left(s^2+a^2\right)^2}$$

Laplace transform of $\left\{\frac{F(t)}{t}\right\}$

Laplace transform of $\left\{\frac{F(t)}{t}\right\}$:

If $L\{F(t)\} = f(s)$, then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds$, provided the integral exists.

Proof: By definition, we have

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt \tag{3.2}$$

Integrating (3.2) w.r.t s from $s = s$ to $s = \infty$, we get

$$\begin{aligned}
\int_s^\infty f(s)\,ds &= \int_s^\infty \left\{\int_0^\infty e^{-st} F(t)\,dt\right\} ds = \int_0^\infty \left\{\int_s^\infty e^{-st}\,ds\right\} F(t)\,dt \\
&= \int_0^\infty \left[-\frac{e^{-st}}{t}\right]_s^\infty F(t)\,dt = \int_0^\infty e^{-st} \frac{F(t)}{t} dt \\
&= L\left\{\frac{F(t)}{t}\right\}
\end{aligned}$$

Laplace transform of $\left\{\frac{F(t)}{t}\right\}$ contd...

Example:

Evaluate $L\left\{\frac{sin(at)}{t}\right\}$.

Sol: We have

$$L\{sin(at)\} = \frac{a}{s^2+a^2} = f(s)\,,\ say$$

Now,

$$\begin{aligned} L\left\{\frac{sin(at)}{t}\right\} &= \int_s^{\infty} f(s)\,ds = \int_s^{\infty} \frac{a}{s^2+a^2}ds \\ &= a.\frac{1}{a}\left[tan^{-1}\frac{s}{a}\right]_s^{\infty} = \frac{\pi}{2} - tan^{-1}\frac{s}{a} \\ &= cot^{-1}\frac{s}{a} \end{aligned}$$

Laplace transform of $erf\left(\sqrt{t}\right)$

The "error function" is defined as

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

and the "complement of error function" is defined as

$$erfc(t) = 1 - erf(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du$$

Now,

$$L\left\{erf\left(\sqrt{t}\right)\right\} = \frac{1}{s\sqrt{s+1}}$$

Laplace transform of $erf\left(\sqrt{t}\right)$ contd...

Proof:

We have, $erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$, therefore by definition

$$erf\left(\sqrt{t}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + ...\right) du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{5.2!} - \frac{u^7}{7.3!} + ...\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + ...\right]$$

Now,

$$L\left\{erf\left(\sqrt{t}\right)\right\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5(2!s^{7/2})} - \frac{\Gamma(9/2)}{7(3!s^{9/2})} + ...\right\}$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2}.\frac{1}{s} + \frac{1.3}{2.4}\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\frac{1}{s^3} + ...\right\}$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}}$$

Examples

Example 1:

Find the Laplace transform of $f(x) = \begin{cases} x/a & , 0 < x < a \\ 1 & , x > a \end{cases}.$

Sol: By definition, we have

$$\begin{aligned} L\{f(x)\} &= \int_0^\infty e^{-sx} f(x)\, dx \\ &= \int_0^a e^{-sx} f(x)\, dx + \int_a^\infty e^{-sx} f(x)\, dx \\ &= \int_0^a e^{-sx}\left(\frac{x}{a}\right) dx + \int_a^\infty e^{-sx}(1)\, dx \\ &= \left[\left(\frac{x}{a}\right)\left(-\frac{e^{-sx}}{s}\right) - \left(\frac{1}{a}\right)\left(-\frac{e^{-sx}}{s^2}\right)\right]_0^a + \left[-\frac{e^{-sx}}{s}\right]_a^\infty, s > 0 \\ &= \frac{\left(1 - e^{-sa}\right)}{a\, s^2}, \; s > 0 \end{aligned}$$

Examples contd...

Example 2:

Show that $L\left\{sin\left(\sqrt{t}\right)\right\} = \frac{\sqrt{\pi}}{2}\frac{e^{-1/4s}}{s^{3/2}}$.

Sol. We have

$$sin\left(\sqrt{t}\right) = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \frac{\left(\sqrt{t}\right)^7}{7!} + \ldots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} -$$

$$\Rightarrow L\left\{sin\left(\sqrt{t}\right)\right\} = L\left\{t^{1/2}\right\} - \frac{1}{3!}L\left\{t^{3/2}\right\} + \frac{1}{5!}L\left\{t^{5/2}\right\} - \frac{1}{7!}L\left\{t^{7/2}\right\} + \ldots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!s^{5/2}} + \frac{\Gamma(7/2)}{5!s^{7/2}} - \frac{\Gamma(9/2)}{7!s^{9/2}} + \ldots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{7/2}} - \ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \ldots\right\}$$

$$= \frac{\sqrt{\pi}}{2}\frac{e^{-1/4s}}{s^{3/2}}$$

Examples contd...

Example 3:

Evaluate $L\left\{\frac{e^{-at}t^{n-1}}{(n-1)!}\right\}$.

Sol: By definition of Laplace transform, we have

$$\begin{aligned} L\left\{\frac{t^{n-1}}{(n-1)!}\right\} &= \frac{1}{(n-1)!}L\left\{t^{n-1}\right\} \\ &= \frac{1}{(n-1)!}.\frac{\Gamma n}{s^n} = \frac{1}{(n-1)!}.\frac{(n-1)!}{s^n} \\ &= \frac{1}{s^n} = f(s) \end{aligned}$$

Hence, by first shifting theorem, we get

$$L\left\{\frac{e^{-at}t^{n-1}}{(n-1)!}\right\} = f(s+a) = \frac{1}{(s+a)^n}$$

Exercise

Example 3:

1. Evaluate $L\left\{2^t\right\}$.
2. Evaluate $L\left\{t^2 sin^2 t\right\}$.
3. Evaluate $L\left\{t^2 e^t sin4t\right\}$.
4. Evaluate $\int_0^{\infty} te^{-3t} sint\, dt$.
5. Evaluate $L\left\{e^{3t} erf\left(\sqrt{t}\right)\right\}$.
6. Show that $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.
7. Show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4s}$.
8. Find $L\{G(t)\}$, where $G(t) = \begin{cases} sin(t - \pi/3) & , t > \pi/3 \\ 0 & , t < \pi/3 \end{cases}$

*******END**************