Mathematical Methods(MA-203)

Lectures 3-4

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Contents

Inverse Laplace Transform

Definition:

If $f(s)$ be the Laplace transform of a function $F(t)$, i.e., if $L\{F(t)\} = f(s)$, then $F(t)$ is called the ***Inverse Laplace transform*** of $f(s)$ and is written as

$$F(t) = L^{-1}\{f(s)\}$$

where L^{-1} is the inverse Laplace transformation operator.
The inverse Laplace transform of a function is unique provided the function is not a ***Null Function***. There is one important theorem, Lerch's theorem in this respect.

Lerch's Theorem

Lerch's Theorem:
If there are two functions $F_1(t)$ and $F_2(t)$ with the same Laplace (integral) transform) i.e., if
$L\{F_1(t)\} = L\{F_2(t)\} = f(s)$
Then a Null Function can be defined by $N(t) \equiv F_1(t) - F_2(t)$ so that the integral
$\int_0^a N(t)dt$ vanishes for all a > 0.

Examples

Example 1:

Find the inverse Laplace transform of $\frac{1}{s^{n+1}}$, where n is any real number such that $n > -1$.

Sol: By definition of Laplace transform, we have

$$L\left\{t^n\right\} = \frac{\Gamma\left(n+1\right)}{s^{n+1}},\ s > 0,\ n > -1$$

Therefore, by definition of the inverse Laplace transform, we have

$$L^{-1}\left\{\frac{\Gamma\left(n+1\right)}{s^{n+1}}\right\} = t^n$$
$$\Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma\left(n+1\right)},\ s > 0,\ n > -1$$

Examples contd...

Example 2:

Find the inverse Laplace transform of $\frac{1}{s-a}$.

Sol: By definition, we have

$$L\left\{e^{at}\right\} = \frac{1}{s-a}, \, s > a$$

Thus, by definition of the inverse Laplace transform, we have

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \, s > a$$

Examples contd...

Example 3:

Find the inverse Laplace transform of $\frac{1}{s^2+a^2}$, $s > 0$.

Sol: By definition, we have

$$L\left\{sin\, at\right\} = \frac{a}{s^2+a^2},\ s > 0$$

By definition of the inverse Laplace transform, we get

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = sin\, at$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{sin\, at}{a},\ s > 0$$

Linearity Property

Linearity Property:

If $f_1(s)$ and $f_2(s)$ respectively be the Laplace transforms of $F_1(t)$ and $F_2(t)$ and C_1 and C_2 be two constants, then

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\}$$

Proof: By linearity property of Laplace transform, we have

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\} \tag{1.1}$$

But, given that

$$L\{F_1(t)\} = f_1(s), L\{F_2(t)\} = f_2(s) \tag{1.2}$$

$$\Rightarrow L^{-1}\{f_1(s)\} = F_1(t), L^{-1}\{f_2(s)\} = F_2(t)$$

From (1.1), we get

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 f_1(s) + C_2 f_2(s)$$

$$\Rightarrow C_1 F_1(t) + C_2 F_2(t) = L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\}$$

$$\Rightarrow C_1 f_1(s) + C_2 f_2(s) = L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\}, \textit{using } (1.2)$$

Linearity Properties contd...

Example:

Find the inverse Laplace transform of $\frac{3s-2}{s^{5/2}} - \frac{7}{3s+2}$.

Sol: By definition, we have

$$
\begin{aligned}
L^{-1}\left\{\frac{3s-2}{s^{5/2}} - \frac{7}{3s+2}\right\} &= L^{-1}\left\{3\frac{1}{s^{3/2}} - 2\frac{1}{s^{5/2}} + \frac{7}{3}\frac{1}{s+(2/3)}\right\} \\
&= 3L^{-1}\left\{\frac{1}{s^{3/2}}\right\} - 2L^{-1}\left\{\frac{1}{s^{5/2}}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s+(2/3)}\right\} \\
&= 3\frac{t^{1/2}}{\Gamma(3/2)} - 2\frac{t^{3/2}}{\Gamma(5/2)} - \frac{7}{3}e^{-(2t/3)} \\
&= 6\left(\frac{t}{\pi}\right)^{1/2} - \frac{8t}{3}\left(\frac{t}{\pi}\right)^{1/2} - \frac{7}{3}e^{-(2t/3)}
\end{aligned}
$$

First Shifting Theorem

First Shifting Theorem:

If $L^{-1}f(s) = F(t)$, then
$L^{-1}f(s-a) = e^{at}F(t) = e^{at}L^{-1}\{f(s)\}$.

Proof: By definition of Laplace transform, we have

$$f(s) = \int_0^\infty e^{-st}F(t)\,dt \tag{1.3}$$

Replacing s by $(s-a)$ on both sides of (1.3), we get

$$\begin{aligned} f(s-a) &= \int_0^\infty e^{-(s-a)t}F(t)\,dt \\ &= \int_0^\infty e^{-st}\left\{e^{at}F(t)\right\}dt \\ &= L\left\{e^{at}F(t)\right\} \end{aligned}$$

Hence, by definition of inverse Laplace transform, we have

$$L^{-1}\{f(s-a)\} = e^{at}F(t) = e^{at}L^{-1}\{f(s)\}$$

First shifting theorem contd...

Example:

Evaluate the inverse Laplace transform of $\frac{s}{s^2+2s+5}$.

Sol: By definition of inverse Laplace transform, we have

$$
\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+2s+5}\right\} &= L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+2^2}\right\} \\
&= e^{-t}L^{-1}\left\{\frac{s-1}{s^2+2^2}\right\} \\
&= e^{-t}\left[L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right] \\
&= e^{-t}\left(cos2t - \frac{sin2t}{2}\right)
\end{aligned}
$$

Second Shifting Theorem

Second shifting theorem:

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\left\{e^{-as}f(s)\right\} = G(t)$, where

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0 \quad , & t < a \end{cases}$$

Proof: By definition of Laplace transform, we have

$$\begin{aligned}
L\{G(t)\} &= \int_0^\infty e^{-st}G(t)\,dt \\
&= \int_0^a e^{-st}G(t)\,dt + \int_a^\infty e^{-st}G(t)\,dt, \text{ where } 0 < a < \infty \\
&= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}F(t-a)\,dt \\
&= \int_a^\infty e^{-st}F(t-a)\,dt \\
&= \int_0^\infty e^{-s(a+u)}F(u)\,du = e^{-sa}\int_0^\infty e^{-su}F(u)\,du \\
&= e^{-sa}L\{F(t)\} = e^{-sa}f(s)
\end{aligned}$$

$$\Rightarrow L^{-1}\left\{e^{-as}f(s)\right\} = G(t)$$

Second Shifting Theorem contd...

Example:

Evaluate $L^{-1}\left\{\frac{e^{-4s}}{(s-3)^4}\right\}$.

Sol: Let $f(s) = \frac{1}{(s-3)^4}$ and $F(t) = L^{-1}\{f(s)\}$. Therefore, we have

$$\begin{aligned} F(t) = L^{-1}\{f(s)\} &= L^{-1}\left\{\frac{1}{(s-3)^4}\right\} \\ &= e^{3t}L^{-1}\left\{\frac{1}{s^4}\right\}, \textit{ by first shifting theorem} \\ &= e^{3t}.\frac{t^3}{3!} = \frac{1}{6}t^3e^{3t} \end{aligned}$$

Hence, by second shifting theorem,

$$L^{-1}\left\{e^{-4s}f(s)\right\} = \begin{cases} F(t-4), & t > 4 \\ 0, & t < 4 \end{cases}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-4s}}{(s-3)^4}\right\} = \begin{cases} \frac{1}{6}(t-4)^3 e^{3(t-4)}, & t > 4 \\ 0, & t < 4 \end{cases}$$

Change of scale property

Change of scale property:

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$.

Proof: Given that $L^{-1}\{f(s)\} = F(t)$ so that $f(s) = L\{F(t)\}$. By definition

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Replacing s by as on both sides, we get

$$\begin{aligned} f(as) &= \int_0^\infty e^{-ast} F(t)\, dt = \frac{1}{a}\int_0^\infty e^{-sx} F\left(\frac{x}{a}\right) dx, (put\ at = x) \\ &= \frac{1}{a}\int_0^\infty e^{-st} F\left(\frac{t}{a}\right) dt = \frac{1}{a} L\left\{F\left(\frac{t}{a}\right)\right\} \\ \Rightarrow f(as) &= L\left\{\frac{1}{a}F\left(\frac{t}{a}\right)\right\} \\ \Rightarrow L^{-1}\{f(as)\} &= \frac{1}{a}F\left(\frac{t}{a}\right) \end{aligned}$$

Change of scale property contd...

Example:

If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\, sin\, t$, then find $L^{-1}\left\{\frac{32\, s}{(16s^2+1)^2}\right\}$.

Sol: Given that

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\, sin\, t$$

Replacing s by as and using change of scale property, we have

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}.\frac{1}{a}.\frac{t}{a}sin\frac{t}{a}$$

Putting $a = 4$, we get

$$L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = \frac{1}{32}sin\frac{t}{4}$$

$$\Rightarrow L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{1}{4}sin\frac{t}{4}$$

Questions

Find Laplace inverse of the following functions:

(i) $\frac{s^2}{(s-2)^3}$

(ii) $\frac{s+3}{(s^2-4s+13)}$

(iii) $\frac{1}{(s^2+1)(s+1)}$

(iv) $\frac{s+2}{(s^2+4s+5)^2)}$

Ans;?????: Discussed in class

Laplace transform of derivatives

Laplace transform of $F'(t)$:

Let $F(t)$ be continuous function fol all $t \geq 0$ and limit $(e^{-st}F(t)) \to 0$ as $t \to \infty$,
then Laplace transform of the derivative $F'(t)$ exists and

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Proof: Exercise

Contd

Proof:

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= e^{-st} F(t)\big|_{t=0}^{t=\infty} + s \int_0^\infty e^{-st} F(t)\, dt = -F(0) + sf(s)$$

Laplace transform of derivatives contd...

Laplace transform of $F''(t)$:

Let $F(t)$ and $F'(t)$ be continuous functions fol all $t \geq 0$ and Limit $e^{-st}F(t) \to 0, e^{-st}F''(t) \to 0$ as $t \to \infty$, then Laplace transform of $F''(t)$ exists and is defined by

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$$

Proof: Exercise

Laplace transform of derivatives contd...

Laplace transform of $F^{(n)}(t)$:

Let $F(t)$ and $F'(t), F''(t), ..., F^{(n-1)}(t)$ be continuous functions fol all $t \geq 0$
and be of exponential order s as $t \to \infty$, i.e., Limit $e^{-st}F^{(i)}(t) \to 0$, as $t \to \infty$, for i=0,1,2,...n-1
then Laplace transform of $F^{(n)}(t)$ exists and is defined by

$$L\left\{F^{(n)}(t)\right\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \ldots - F^{(n-1)}(0)$$

Proof: Exercise

Laplace transform of integrals

Theorem:

If $L\{F(t)\} = f(s)$, then $L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$.

Proof contd.

Proof: Let $G(t) = \int_0^t F(x)\,dx$, then

$$G'(t) = F(t), \qquad G(0) = 0 \tag{3.1}$$

Now,

$$L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\}$$
$$\Rightarrow L\{F(t)\} = sL\{G(t)\},\ using\ (3.1)$$
$$\Rightarrow L\{G(t)\} = \frac{L\{F(t)\}}{s} = \frac{f(s)}{s}$$
$$\Rightarrow L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$$

Laplace transform of integrals contd...

Task:

Evaluate (i) $L\left\{\int_0^t \frac{\sin x}{x} dx\right\}$. Ans: $\frac{\cot^{-1} s}{s}$

(ii) $L\left\{\int_0^t \frac{e^t \sin t}{t} dt\right\}$: Ans: $\frac{\cot^{-1}(s-1)}{s}$

Solution

Sol: (i) We have $F(t) = sin\, t$, therefore

$$L\{F(t)\} = L\{sin\, t\} = \frac{1}{s^2+1} = f(s) \tag{3.2}$$

Now,

$$L\left\{\frac{sin\, t}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty \frac{1}{s^2+1} ds, using\ (3.2)$$
$$= \left[tan^{-1} s\right]_s^\infty = cot^{-1} s$$

By definition of integral transform,

$$L\left\{\int_0^t \frac{sin\, x}{x} dx\right\} = \frac{cot^{-1} s}{s}$$

(ii) Ans: ????

Exercise

*****END*****