Mathematical Methods(MA-203)

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Contents

Laplace transform of derivatives

Laplace transform of $F'(t)$:

Let $F(t)$ be continuous function fol all $t \geq 0$ and limit $(e^{-st}F(t)) \to 0$ as $t \to \infty$,
then Laplace transform of the derivative $F'(t)$ exists and

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Proof:continue

Contd

Proof:

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= e^{-st} F(t)\Big|_{t=0}^{t=\infty} + s \int_0^\infty e^{-st} F(t)\, dt = -F(0) + sf(s)$$

Laplace transform of derivatives contd...

Laplace transform of $F''(t)$:

Let $F(t)$ and $F'(t)$ be continuous functions fol all $t \geq 0$ and Limit $e^{-st}F(t) \to 0, e^{-st}F''(t) \to 0$ as $t \to \infty$, then Laplace transform of $F''(t)$ exists and is defined by

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$$

Proof: Exercise

Laplace transform of derivatives contd...

Laplace transform of $F^{(n)}(t)$:

Let $F(t)$ and $F'(t), F''(t), ..., F^{(n-1)}(t)$ be continuous functions fol all $t \geq 0$
and be of exponential order s as $t \to \infty$, i.e., Limit $e^{-st}F^{(i)}(t) \to 0$, as $t \to \infty$, for i=0,1,2,...n-1
then Laplace transform of $F^{(n)}(t)$ exists and is defined by

$$L\left\{F^{(n)}(t)\right\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \dots - F^{(n-1)}(0)$$

Proof: Exercise

Laplace transform of integrals

Theorem:

If $L\{F(t)\} = f(s)$, then $L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$.

This implies: $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(x)\,dx$

Proof contd.

Proof: Let $G(t) = \int_0^t F(x)\,dx$, then

$$G'(t) = F(t), \qquad G(0) = 0 \tag{2.1}$$

Now,

$$L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\}$$
$$\Rightarrow L\{F(t)\} = sL\{G(t)\}, \ using\ (3.1)$$
$$\Rightarrow L\{G(t)\} = \frac{L\{F(t)\}}{s} = \frac{f(s)}{s}$$
$$\Rightarrow L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$$

Laplace transform of integrals contd...

Examples:

Evaluate (i) $L\left\{\int_0^t \frac{\sin x}{x} dx\right\}$. Ans: $\frac{\cot^{-1} s}{s}$

(ii) $L\left\{\int_0^t \frac{e^t \sin t}{t} dt\right\}$:

Solution

Sol: (i) We have $F(t) = \sin t$, therefore

$$L\{F(t)\} = L\{\sin t\} = \frac{1}{s^2+1} = f(s) \tag{2.2}$$

Now,

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{1}{s^2+1}ds, \text{ using } (3.2)$$
$$= \left[\tan^{-1} s\right]_s^\infty = \cot^{-1} s$$

By definition of integral transform,

$$L\left\{\int_0^t \frac{\sin x}{x}dx\right\} = \frac{\cot^{-1} s}{s}$$

(ii) Ans: $\frac{\cot^{-1}(s-1)}{s}$

Inverse Laplace transform of $s^n f(s)$

Theorem:

If $F(0) = F'(0) = F''(0) = \ldots F^{(n-1)}(0) = 0$, then

$$L^{-1}\{s^n f(s)\} = F^{(n)}(t)$$

where $F^{(n)}(t) = \frac{d^n}{dt^n} F(t)$ and $F^{(n-1)}(0) = \left[\frac{d^{n-1}}{dt^{n-1}} F(t)\right]_{t=0}$.

Proof: Exercise

Inverse Laplace transform of $\frac{f(s)}{s^n}$

Theorem:

If $L^{-1}\left\{f(s)\right\} = F(t)$, then

$$L^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \int_0^t \ldots\ldots \int_0^t F(t)\, dt^n$$

Proof: Exercise

Questions

Example 1:

Evaluate $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$.

Sol: Let $f(s) = \frac{1}{(s^2+4)}$. Then $L^{-1}\{f(s)\} = F(t) = \frac{sin\,2t}{2}$ and $\frac{d}{ds}f(s) = (-2)\frac{s}{(s^2+4)^2}$. Therefore,

$$L^{-1}\left\{\frac{d}{ds}f(s)\right\} = -2L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

$$\Rightarrow (-1)\,tF(t) = 2L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\,sin\,2t$$

Contd

Now Let $g(s) = \frac{s}{(s^2+4)^2}$ and $G(t) = \frac{t}{4} sin\, 2t$. Here $G(0) = 0$, therefore we have

$$L^{-1}\{sg(s)\} = G'(t)$$

$$\Rightarrow L^{-1}\left\{s.\frac{s}{(s^2+4)^2}\right\} = \frac{d}{dt}\left(\frac{t}{4} sin\, 2t\right)$$

$$\Rightarrow L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \frac{1}{4}(sin\, 2t + 2t\, cos\, 2t)$$

Question

Example 2:

Evaluate $L^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\}$.

Solution

Let $f(s) = \frac{1}{(s^2+1)}$. We know that $L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = sin\, t = F(t)$. Therefore,

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t F(t)\, dt = \int_0^t sin\, t\, dt = 1 - cos\, t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - cos\, t)\, dt = t - sin\, t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (t - sin\, t)\, dt = \frac{t^2}{2} + cos\, t - 1$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = \int_0^t \left(\frac{t^2}{2} + cos\, t - 1\right) dt$$

$$= \frac{t^3}{6} + sint - t$$

Exercise

Exercises:

1. Find $L^{-1}[log\frac{s+2}{s+3}]$
2. Find $L^{-1}[\frac{1}{(s^2+a^2)^2}]$

Solution of Exercise

1. Find $L^{-1}[log\frac{s+2}{s+3}]$

Hints:If L[F(t)]= f(s), then $tF(t) = -L^{-1}[\frac{d}{ds}(f(s)]$

Now, $\frac{d}{ds}(log\frac{s+2}{s+3}) = \frac{d}{ds}(log(s+2) - log(s+3)) = \frac{1}{s+2} - \frac{1}{s+3}$

Hence, $tF(t) = -L^{-1}(\frac{1}{s+2}) + L^{-1}(\frac{1}{s+3}) = e^{-3t} - e^{-2t}$

Solution of Exercise

2. Find $L^{-1}[\frac{1}{(s^2+a^2)^2}]$

Hints: $L^{-1}[\frac{s}{(s^2+a^2)^2}) = \frac{1}{2a} tsin(at) = F(t)$,say

Hence,

$$L^{-1}[\frac{1}{(s^2+a^2)^2}) = L^{-1}[\frac{1}{s}(\frac{s}{(s^2+a^2)^2})]$$
$$= L^{-1}[\frac{f(s)}{s})] = \int_0^t F(t)\, dt = \int_0^t \frac{1}{2a} tsin(at)\, dt$$
$$= \frac{1}{2a^3}(sin(at) - atcos(at)),$$

Laplace transform of periodic functions

Theorem:

If $F(t)$ be a periodic function with period T. Then $L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$.

Proof: By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^T e^{-st} F(t)\, dt + \int_T^\infty e^{-st} F(t)\, d$$

$$= \int_0^T e^{-st} F(t)\, dt + \int_0^\infty e^{-s(z+T)} F(z+T)\, dz, (put\ t =$$

$$= \int_0^T e^{-st} F(t)\, dt + e^{-sT} \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^T e^{-st} F(t)\, dt + e^{-sT} L\{F(t)\}$$

Laplace transform of periodic functions

Hence we get

$$\Rightarrow \left(1 - e^{-sT}\right) L\left\{F(t)\right\} = \int_0^T e^{-st} F(t)\, dt$$

$$\Rightarrow L\left\{F(t)\right\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

Laplace transform of periodic functions contd...

Example:

If $F(t) = t^2, 0 < t < 2$ and $F(t+2) = F(t)$, find $L\{F(t)\}$.

Sol: $F(t+2) = F(t) \Rightarrow F(t)$ is periodic function with period $T = 2$. Therefore,

$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} t^2 dt \tag{3.1}$$

Since, $\int_0^2 e^{-st} t^2 dt = \left\{2 - \left(4s^2 + 4s + 2\right) e^{-2s}\right\} / s^3$, therefore from (4.1)

$$L\{F(t)\} = \frac{2 - \left(4s^2 + 4s + 2\right) e^{-2s}}{s^3\left(1 - e^{-2s}\right)}$$

Laplace transform of the unit step function

Definition:

The unit step function is denoted and defined by

$$u_a(t) = H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \tag{4.1}$$

$$\begin{aligned} L\{H(t-a)\} &= \int_0^\infty e^{-st} H(t-a)\, dt \\ &= \int_0^a e^{-st} H(t-a)\, dt \\ &\quad + \int_a^\infty e^{-st} H(t-a)\, dt \\ &= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt, \\ \Rightarrow L\{H(t-a)\} &= \frac{e^{-as}}{s} \end{aligned}$$

Laplace transform of the unit step function contd...

Example:

If $L\{F(t)\} = f(s)$ and $a > 0$, then

$$L\{F(t-a)H(t-a)\} = e^{-as}f(s), \text{ where } H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}.$$

Sol: By definition

$$\begin{aligned} L\{F(t-a)H(t-a)\} &= \int_0^\infty e^{-st}F(t-a)H(t-a)\,dt \\ &= \int_0^a e^{-st}F(t-a)H(t-a)\,dt + \int_a^\infty e^{-} \end{aligned}$$

Laplace transform of the unit step function contd...

$$= \int_0^a e^{-st} F(t-a)(0)\,dt + \int_a^\infty e^{-st} F(t-a)(1)\,d$$

$$t = \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= e^{-as} \int_0^\infty e^{-su} F(u)\,du,$$

$$(byputing\; t-a=u)$$

$$= e^{-as} L\{F(t)\}$$

$$= e^{-as} f(s)$$

The convolution theorem and its applications

The convolution theorem:

If $L^{-1}\left\{f(s)\right\} = F(t)$ and $L^{-1}\left\{g(s)\right\} = G(t)$, then

$$L^{-1}\left\{f(s)\,g(s)\right\} = \int_0^t F(u)\,G(t-u)\,du = F * G$$

$$OR$$

The convolution theorem can be re-written as

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = L\left\{F(t) * G(t)\right\} = L\left\{F(t)\right\}.L\left\{G(t)\right\}$$

The convolution theorem and its applications contd...

Proof:

Let $H(t) = \int_{u=0}^{t} F(u)\, G(t-u)\, du = F(t) * G(t)$. Now,

$$L\{H(t)\} = \int_{t=0}^{\infty} e^{-st} H(t)\, dt = \int_{t=0}^{\infty} e^{-st} \left\{ \int_{u=0}^{t} F(u)\, G(t-u)\, du \right\} dt$$

$$= \int_{u=0}^{\infty} F(u) \left\{ \int_{t=u}^{\infty} e^{-st} G(t-u)\, dt \right\} du$$

$$= \int_{u=0}^{\infty} e^{-su} F(u) \left\{ \int_{t=u}^{\infty} e^{-s(t-u)} G(t-u)\, dt \right\} du$$

Proof of convolution theorem contd...

$$i.e., LH(t) = \int_{u=0}^{\infty} e^{-su} F(u) \left\{ \int_{v=0}^{\infty} e^{-sv} G(v)\, dv \right\} du,$$

$$(by putting\ t - u = v)$$

$$= \int_{u=0}^{\infty} e^{-su} F(u) \{g(s)\}\, du$$

$$= g(s) \int_{u=0}^{\infty} e^{-su} F(u)\, du = g(s) f(s)$$

$$\Rightarrow L^{-1}\{f(s)\} g(s) = H(t)$$

$$= \int_{u=0}^{t} F(u) G(t-u)\, du = F * G$$

Applications of The convolution theorem ..

Example 1:

Evaluate $L^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}$.

Sol: Let $f(s) = \frac{1}{(s^2+1)}$ and $g(s) = \frac{1}{(s+1)}$. Therefore,

$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = sin\, t = F(t)$

and $L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{(s+1)}\right\} = e^{-t} = G(t)$.

By convolution theorem, we have

$$
\begin{aligned}
L^{-1}\{f(s)\,g(s)\} &= \int_0^t F(u)\,G(t-u)\,du = \int_0^t \sin u\, e^{-(t-u)}du \\
&= e^{-t}\int_0^t e^u \sin u\,du = \frac{1}{2}\left(\sin t - \cos t + e^{-t}\right) \\
\Rightarrow L^{-1}&\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{1}{2}\left(\sin t - \cos t + e^{-t}\right)
\end{aligned}
$$

Note: $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$.

The convolution theorem and its applications contd...

Example 2:

Show that $\int_0^t sin\, u\, cos\,(t-u)\, du = \frac{1}{2} t\, sin\, t$.

Sol: By convolution theorem, we have

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\,.\,L\{G(t)\} \tag{5.1}$$

The convolution theorem and its applications contd...

Take $F(t) = sin\, t$ and $G(t) = cos\, t$. Then (6.1) reduces to

$$L\left\{\int_0^t sin\, u\, cos\,(t-u)\, du\right\} = L\{sin\, u\}\,.\,L\{cos\, u\}$$

$$= \frac{1}{(s^2+1)}.\frac{s}{(s^2+1)} = \frac{s}{(s^2+1)^2}$$

$$\Rightarrow \int_0^t sin\, u\, cos\,(t-u)\, du = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\, sin\, t$$

Note: $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\, sin\, t$. Prove it by convolution theorem taking $f(s) = \frac{s}{s^2+1}$ and $g(s) = \frac{1}{s^2+1}$.

Exercise:

1. Evaluate $L^{-1}\left\{\frac{3s-2}{s^{5/2}} - \frac{7}{3s+2}\right\}$.
2. Show that $\int_0^\infty e^{-tx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ and hence, deduce $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.
3. Evaluate $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$.
4. Evaluate $L^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$.
5. Evaluate $L^{-1}\left\{\frac{e^{(4-3s)}}{(s+4)^{5/2}}\right\}$.
6. If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, then find $L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$, where $a > 0$.
7. Evaluate $L^{-1}\left\{log\frac{1+s}{s}\right\}$.
8. Evaluate $L^{-1}\left\{\frac{1}{s}log\frac{s+2}{s+1}\right\}$.
9. Evaluate $L^{-1}\left\{\frac{1}{(s-2)(s^2+)}\right\}$.