Q.

X \ Y	0	1	2
1	0.3	0.2	0.1
2	0.1	0.0	0.3

(i) Are X and Y independent?
(ii) Determine the correlation coefficient between X and Y.

Soln.

X \ Y	0	1	2	Total
1	0.3	0.2	0.1	0.6
2	0.1	0.0	0.3	0.4
Total	0.4	0.2	0.4	1

$p_{10} = 0.3$, $p_{11} = 0.2$,
$p_{12} = 0.1$, $p_{20} = 0.1$,
$p_{21} = 0.0$, $p_{22} = 0.3$
$p_{1.} = 0.6$, $p_{2.} = 0.4$
$p_{.0} = 0.4$, $p_{.1} = 0.2$
$p_{.2} = 0.4$

Now $p_{10} = P(X=1, Y=0) = 0.3$

$p_{1.} \times p_{.0} = 0.6 \times 0.4 = 0.24$

Hence, $P(X=1, Y=0) \neq P(X=1)\,P(Y=0)$.

Hence, X and Y are not independent random variables.

$$E(X) = \sum_{i=1}^{2} x_i p_{i.} = 1.p_{1.} + 2.p_{2.} = 1 \times 0.6 + 2 \times 0.4$$
$$= 0.6 + 0.8 = 1.4.$$

$$E(Y) = \sum_{j=0}^{2} y_j p_{.j} = y_0 p_{.0} + y_1 p_{.1} + y_2 p_{.2}$$
$$= 0 \times 0.4 + 1 \times 0.2 + 2 \times 0.4 = 0 + 0.2 + 0.8$$
$$= 1.0$$

$$E(XY) = \sum_{i=1}^{2}\sum_{j=0}^{3} x_i y_j p_{ij} = 1 \times 0 \times 0.3 + 1 \times 1 \times 0.2 + 1 \times 2 \times 0.1$$
$$+ 2 \times 0 \times 0.1 + 2 \times 1 \times 0.0 + 2 \times 2 \times 0.3$$
$$= 0 + 0.2 + 0.2 + 0 + 0 + 1.2$$
$$= 1.6.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1.6 - 1.4 \times 1.0 = 0.2.$$

$$\therefore \boxed{Cov(X,Y) = 0.2}$$

$$E(X^2) = 1^2 \times 0.6 + 2^2 \times 0.4 = 0.6 + 1.6 = 2.2$$

$$E(Y^2) = 0^2 \times 0.4 + 1^2 \times 0.2 + 2^2 \times 0.4 = 0.2 + 1.6 = 1.8$$

$$\therefore \sigma_X^2 = 2.2 - (1.4)^2 = 0.24, \quad \sigma_X = \sqrt{0.24}$$

$\sigma_y^2 = \cancel{0.24}\ 1.8 - (1.0)^2 = 0.8$, $\sigma_y = \sqrt{0.8}$

$\therefore \rho_{xy} = \dfrac{0.2}{\sqrt{0.24}\sqrt{0.8}} = 0.46.$

Q. The marginal distributions of X and Y are given in the following table

X \ Y	5	7	Total
3	p	$\frac{1}{3}-p$	$1/3$
6	$\frac{1}{2}-p$	$\frac{1}{6}+p$	$2/3$
Total	$1/2$	$1/2$	1

If $\text{cov}(X,Y) = -1/2$ then obtain p_{ij} for $i=3,6$ & $j=5,7$. Hence calculate $P(\cancel{X=5} \,Y=5 \mid \cancel{Y=6}\, X=6)$. and $P(Y>X)$.

From the given informations, we can calculate

$E(X) = 3\times\frac{1}{3} + 6\times\frac{2}{3} = 1+4 = 5$

$E(Y) = 5\times\frac{1}{2} + 7\times\frac{1}{2} = 2.5+3.5 = 6.$

as $\text{cov}(X,Y) = -\frac{1}{2} \Rightarrow E(XY) = -\frac{1}{2} + E(X)E(Y) = 30 - \frac{1}{2} = 29.5$

$E(XY) = 15p + 21\left(\frac{1}{3}-p\right) + 30\left(\frac{1}{2}-p\right) + 42\left(\frac{1}{6}+p\right)$

$= 15p + 7(1-3p) + 15(1-2p) + 7(1+6p)$

$= 15p + 7 - 21p + 15 - 30p + 7 + 42p$

$= (15-21-30+42)p + 29$

$= 6p + 29 = 29.5$

$$\begin{array}{r} 42 \\ 15 \\ \hline 57 \\ -51 \\ \hline 6 \end{array}$$

$\Rightarrow 6p = 0.5$

$\Rightarrow p = \frac{0.5}{6} = \boxed{\frac{1}{12} = p}$

$\therefore p_{35} = \frac{1}{12}$, $p_{37} = \frac{1}{3} - \frac{1}{12} = \frac{4-1}{12} = \frac{1}{4}$

$p_{65} = \frac{1}{2} - \frac{1}{12} = \frac{6-1}{12} = 5/12$, $p_{67} = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

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Tutorial 4 P(2)

X \ Y	5	7	
3	1/12	1/4	1/3
6	5/12	1/4	2/3
Total	1/2	1/2	1

Now $P(X > Y)$

$= P_{65} = 5/12$

$P(Y > X) = P_{35} + P_{37} + P_{67}$

$= \frac{1}{12} + \frac{1}{4} + \frac{1}{4} =$

$= \frac{1+3+3}{12} = 7/12.$

$$P(Y=5 \mid X=6) = \frac{5/12}{2/3} = \frac{5}{\cancel{12}_4} \times \frac{\cancel{3}}{2}$$

$= 5/8$. Ans.

Tutorial 4 P(3)

Q. Ten coins are thrown simultaneously. Find the prob. of getting at least seven heads.

soln. p = prob. of getting a head $= 1/2$.

q = prob. of not getting a head $= 1/2$.

The prob. of getting x heads in a random throw of 10 coins is :

$$P(x) = {}^{10}C_x\, p^x\, q^{n-x} = {}^{10}C_x\, p^x\, q^{10-x} \; ; \; x = 0, 1, 2, \ldots, 10$$

$$P(X \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$= {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + {}^{10}C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}C_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right)$$

$$+ {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10}$$

$$= \left(\tfrac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right] = \frac{176}{1024}$$

Q. A and B play a game in which their chances of winning are in ratio 3:2. Find A's chance of winning at least three games out of the five games played.

$p \rightarrow$ prob. that 'A' wins the game

$n = 5$, $p = \frac{3}{5}$, $q = \frac{2}{5}$

$$P(x) = {}^nC_x\, p^x\, q^{n-x} = {}^5C_x \left(\tfrac{3}{5}\right)^x \left(\tfrac{2}{5}\right)^{5-x}, \quad x = 0, 1, 2, 3, 4, 5$$

$$P(X \geq 3) = P(3) + P(4) + P(5)$$

$$= {}^5C_3 \left(\tfrac{3}{5}\right)^3 \left(\tfrac{2}{5}\right)^2 + {}^5C_4 \left(\tfrac{3}{5}\right)^4 \left(\tfrac{2}{5}\right)^1 + {}^5C_5 \left(\tfrac{3}{5}\right)^5.$$

$$= \frac{1}{5^5} \left[3^3 . 2^2\, {}^5C_3 + 3^4 . 2\, {}^5C_4 + 3^5\, {}^5C_5 \right]$$

$$= 0.68.$$

Q. A multiple-choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the 3rd answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the prob. that the student secure a distinction?

Soln. Since there are three answers to each qn., out of which only one is correct, the prob. of getting an ans. to a qn. correctly is given by

$p = 1/3$, so $q = 2/3$.

The prob. of getting r correct ans. in a 8-qn. test is given by (By Binomial Law)

$$P(X=x) = {}^8C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{8-x}, \quad x = 0, 1, \ldots, 8$$

Now 75% of 8 qn. $= \frac{\cancel{75}\,3}{\cancel{100}\,4} \times \cancel{8}\,2 = 6$

∴ Required prob = getting at least 6 correct ans. to at least 6 out of 8 qns.

$= p(6) + p(7) + p(8) = 0.0197.$

Q. In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the no. of errors per page, find the prob. that a random sample of 5 pages will contain no error.

Soln. The average no. of typographical errors per page in the book is given by

$$\lambda = (390/520) = 0.75.$$

Hence using Poisson prob. law, the prob. of x errors per page is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.75}(0.75)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

The required prob. that a random sample of 5 pages will contain no error is given by $[P(X=0)]^5 = (e^{-0.75})^5 = e^{-3.75}$.

Q. A manger accepts the work submitted by his typist only when there is no mistakes in the work. The typist has to type on an average 20 letters per day of about 200 words each. Find the chance of her making a mistake: (i) If less than 1% of the letters submitted by here are rejected, (ii) if on 90% days all the letter submitted by heres are accepted.

[As the prob. of making a mistake is small, you may use Poisson dist. Take $e = 2.72$]

Tutorial 4 P(6)