Special Continuous Probability Distributions

NORMAL DISTRIBUTION (Gaussian distribution)

The normal distribution was first discovered in 1733 by English mathematician De-Moivre, who obtained this continuous dist as a limiting case of the binomial dist. and applied it to problems arising in the game of chance. It was also known ~~as~~ to Laplace, no later than 1774 but through a historical error it was credited to Gauss, who first made reference to it in the beginning of 19th century (1809), as the dist. of error in Astronomy.

Definition: A r.v X is said to have a normal distribution with parameter μ (called 'mean') and σ^2 (called 'variance') if its p.d.f. is given by the probability law:

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0.$$

\#1. $X \sim N(\mu, \sigma^2)$.

\# 2. If $X \sim N(\mu,\sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma}$ is a standard normal variate with $E(Z)=0$ and $var(Z)=1$ and we write $Z \sim N(0,1)$.

\# 3. The p.d.f of standard normal variate Z is given by:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad ; -\infty < z < \infty.$$

A Banerjee

and the corresponding dist. fn., denoted by $\Phi(z)$ is given by:

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

Result 1. $\Phi(-z) = 1 - \Phi(z)$, $z > 0$

~~Result 2~~ proof. $\Phi(-z) = P(Z \le -z) = P(Z \ge z) = 1 - P(Z \le z)$
$= 1 - \Phi(z)$.

Result 2. $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

where $X \sim N(\mu, \sigma^2)$.

Proof: $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$

$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$

$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$.

Normal distribution as a Limiting form of Binomial dist.

Normal dist. is another limiting form of the binomial dist. under the following conditions:

(i) n, the no. of trials is indefinitely large, i.e., $n \to \infty$; and

(ii) neither p nor q is very small.

Normal dist. can also be obtained as a limiting case of Poisson dist. with parameter $\lambda \to \infty$.

Lecture 13 P(1.1) ABanerjee

Normal distribution / Gaussian distribution

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; \quad -\infty < x < \infty, \; -\infty < \mu < \infty$$

$\sigma > 0.$

Verification

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \cancel{\sigma}\,du$$

let $\frac{x-\mu}{\sigma} = u$

$dx = \sigma\,du$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du$$

~~let $\frac{u^2}{2} = t$~~

$$= \frac{1 \times 2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2}\,du \quad \text{(as } e^{-u^2/2} \text{ is an even fn.)}$$

let $\frac{u^2}{2} = t$

$\Rightarrow \frac{\cancel{2}u\,du}{\cancel{2}} = dt$

$\Rightarrow du = \frac{dt}{\sqrt{2t}}$

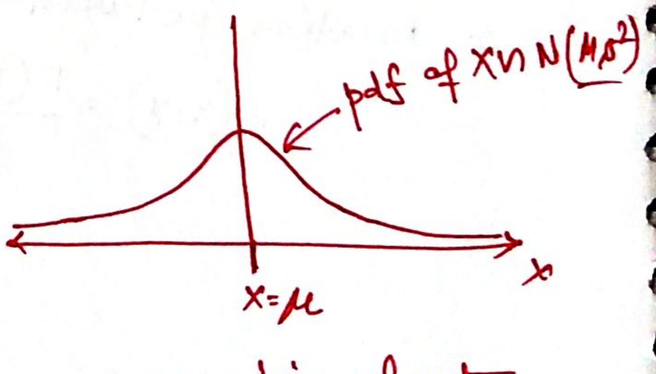
$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}\, t^{-1/2}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}\, t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1.$$

Hence proved.



symmetric about the point $X = \mu$.

ABanerjee

MGF of $X \sim N(\mu, \sigma^2)$

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)} e^{-u^2/2}\,\sigma\,du.$$

Let $\frac{x-\mu}{\sigma} = u$
$\Rightarrow dx = \sigma du$
$x = \mu + u\sigma$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma u} e^{-u^2/2}\,du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2 - 2t\sigma u)}\,du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2 - 2t\sigma u + t^2\sigma^2 - t^2\sigma^2)}\,du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-t\sigma)^2} \cdot e^{\frac{t^2\sigma^2}{2}}\,du$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-t\sigma)^2}\,du.$$

Let $u - t\sigma = z$
$\Rightarrow du = dz.$

$$= \frac{2 \times e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\,dz.$$

Let $\frac{z^2}{2} = y$
$\Rightarrow z\,dz = dy$
$\Rightarrow dz = \frac{dy}{\sqrt{2y}}$

$$= \frac{2 \times e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{2 \times e^{\mu t + \frac{t^2\sigma^2}{2}}}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{-1/2}\,dy$$

$$= \frac{\cancel{2} \times e^{\mu t + \frac{t^2\sigma^2}{2}}}{\cancel{2}\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}-1}\,dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \Gamma\left(\tfrac{1}{2}\right) = \frac{1}{\cancel{\sqrt{\pi}}} e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \cancel{\sqrt{\pi}}$$

$$\boxed{M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}}$$

ABanerji

Lecture 13P(2)

$$M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$\frac{d}{dt} M_X(t) = \left(\mu + \frac{2t\sigma^2}{2}\right) e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$= \left(\mu + t\sigma^2\right) e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$\therefore E(X) = \frac{d}{dt} M_X(t)\Big]_{t=0} = \mu. \Rightarrow \boxed{E(X) = \mu}$$

$$\frac{d^2}{dt^2} M_X(t) = \sigma^2 e^{\mu t + \frac{t^2\sigma^2}{2}} + \left(\mu + t\sigma^2\right)\left(\mu + t\sigma^2\right) e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$\Rightarrow \frac{d^2}{dt^2} M_X(t)\Big]_{t=0} = \sigma^2 + \mu^2 = E(X^2)$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2$$

$$= \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

$$\therefore \boxed{Var(X) = \sigma^2}$$

Let $X \sim N(\mu, \sigma^2)$

Let $Z = \frac{X - \mu}{\sigma}$

Then $Z \sim N(0,1)$

Hence, $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$

$$M_Z(t) = E(e^{Zt}) = \int_{-\infty}^{\infty} e^{zt} f_Z(z)\, dz$$

$$= \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2zt)}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2zt + t^2 - t^2)}\, dz.$$

$$M_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} \cdot e^{t^2/2} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

Let $\cancel{\frac{(z-t)^2}{2}}$ $= u$

$\Rightarrow z - t = u$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$= 2\frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2} du$$

Let $\frac{u^2}{2} = y$

$\Rightarrow u\,du = dy$

$\Rightarrow du = \frac{dy}{\sqrt{2y}}$

$$= \sqrt{\frac{2}{\pi}}\, e^{t^2/2} \int_0^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{1}{\sqrt{\pi}}\, e^{t^2/2} \int_0^{\infty} e^{-y} y^{-1/2} du$$

$$= \frac{1}{\sqrt{\pi}}\, e^{t^2/2}\, \Gamma(1/2) = \frac{1}{\cancel{\sqrt{\pi}}} e^{t^2/2} \cdot \cancel{\sqrt{\pi}}$$

$$\boxed{M_Z(t) = e^{t^2/2}}$$

$$\frac{d}{dt} M_Z(t) = t e^{t^2/2} \Rightarrow E(Z) = \frac{d}{dt} M_Z(t)\Big]_{t=0} = 0$$

$$\therefore \boxed{E(Z) = 0}$$

$$\frac{d^2}{dt^2} M_Z(t) = e^{t^2/2} + t^2 e^{t^2/2} \Rightarrow E(Z^2) = \frac{d^2}{dt^2} M_Z(t)\Big]_{t=0} = 1.$$

$$\therefore Var(Z) = E(Z^2) - \{E(Z)\}^2 = 1 - 0 = 1.$$

$$\therefore \boxed{Var(Z) = 1}$$

ABanerji

Lecture 13 P(7)

Q. Let independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ $i = 1, 2, \ldots, n$.

$$M_{X_i}(t) = e^{\mu_i t + \frac{t^2 \sigma_i^2}{2}}, \quad i = 1, 2, \ldots, n$$

$$Y = \sum_{i=1}^{n} a_i X_i$$

Then $M_Y(t) = \prod_{i=1}^{n} M_{a_i X_i}(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$

$$= \prod_{i=1}^{n} e^{\mu_i a_i t + \frac{a_i^2 t^2 \sigma_i^2}{2}}$$

$$= e^{\left(\sum_i \mu_i a_i\right) t + \frac{t^2}{2} \left(\sum_i a_i^2 \sigma_i^2\right)}$$

$$\Rightarrow Y \sim N\left(\sum_{i=1}^{n} \mu_i a_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Area property (Normal Probability Integral)

Let $X \sim N(\mu, \sigma^2)$.

Then $P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-(x-\mu)^2/2\sigma^2}\,dx$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$

$\therefore P(\mu < X < x_1) = \int_0^{z_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\,dz$, where $z_1 = \frac{x_1-\mu}{\sigma}$

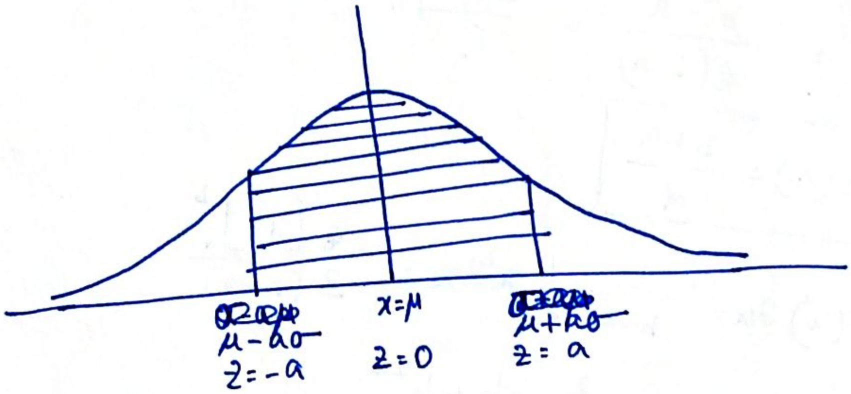
$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2}\,dz$

$\int_0^{z_1} \phi(z)\,dz$ is the area under standard normal curve between the ordinate $z=0$ and $z=z_1$. Given in normal table

$= P(0 < Z < z_1)$.

where $Z \sim N(0,1)$. & $z_1 = \frac{x_1-\mu}{\sigma}$

$Z = \frac{X-\mu}{\sigma}$



~~$P(\sigma - a\mu < X < \sigma + a\mu) = 2P(\mu < X < \sigma + a\mu)$~~ due to symmetry

$P(\mu - a\sigma < X < \mu + a\sigma) = 2P(\mu < X < \mu + a\sigma)$

$= 2P\left(\frac{\mu-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu + a\sigma - \mu}{\sigma}\right)$

$= 2P(0 < Z < a)$.

Characteristics of the normal distribution and normal probability curve :-

Let $X \sim N(\mu, \sigma^2)$.

then its pdf is given by $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

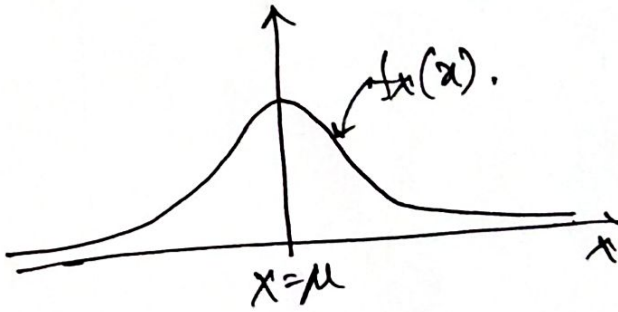
$-\infty < x < \infty$

$-\infty < \mu < \infty, \sigma > 0.$

The normal curve has the following properties

(i) The curve is bell shaped and symmetric about the line $x = \mu$.

(ii) Mean, median and mode of the distribution coincide at $x = \mu$.



$$P(a \le X \le b) = \int_a^b f_X(x)\, dx.$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-u^2/2} \cancel{\sigma}\, du \qquad \text{Let } \frac{x-\mu}{\sigma} = u.$$

$$= \int_{-\infty}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du - \int_{-\infty}^{(a-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

where $\Phi \rightarrow$ cdf of standard normal variate

Example :- Let X be a normally distributed r.v. with mean 12 and standard deviation 4.

Find (a) (i) $P(X \geq 20)$, (ii) $P(X \leq 20)$, (iii) $P(0 \leq X \leq 12)$

(b) Find x' such that $P(X > x') = 0.24$.

(c) Find x_0' and x_1' s.t. $P(x_0' < X < x_1') = 0.50$

and $P(X > x_1') = \cancel{0.25}\ 0.24$.

Given that $f_1(t) = \int_{-\infty}^{t} \phi(z)\,dz$ and $f_2(t) = \int_{0}^{t} \phi(z)\,dz$.

$f_1(2) = 0.97725$, $f_2(2) = 0.4772$, $f_2(3) = 0.4987$, $f_2(0.67) = 0.25$

$f_1(3) = 0.9987$, $f_2(0.71) = 0.26$,

Soln. $X \sim N(12, 4^2)$, i.e., $X \sim N(12, 16)$. $\therefore \mu = 12, \sigma = 4$.

(a)(i) $P(X \geq 20) = P\left(\frac{X-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}\right)$

$= P(Z \geq 2) =$

$= 0.5 - f_2(2)$ [OR, $1 - f_1(2)$]

$= 0.5 - 0.4772$ [OR, $1 - 0.97725$]

$= 0.0228$ [OR, 0.0228].

(ii) $P(X \leq 20) = 1 - P(X \geq 20) = 1 - 0.0228 = 0.9772$

OR $P(X \leq 20) = P\left(\frac{X-\mu}{\sigma} \leq \frac{20-\mu}{\sigma}\right) = P(Z \leq 2) = f_1(2)$ (OR $f_2(2)$)

(iii) $P(0 \leq X \leq 12) = P\left(\frac{0-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{12-\mu}{\sigma}\right)$

$= P(-3 \leq Z \leq 0)$

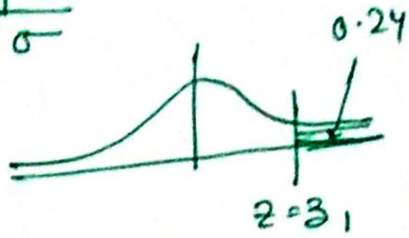
$= P(0 \leq Z \leq 3)$ [due to symmetry]

$= f_2(3) = 0.4987$.

(b) $P(X > x_1') = 0.24$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{x_1'-\mu}{\sigma}\right) = 0.24$

$\Rightarrow P(Z > z_1) = 0.24$, where $z_1 = \frac{x_1'-\mu}{\sigma}$



$\Rightarrow 0.5 - P(0 < Z < z_1) = 0.24$

$\Rightarrow P(0 < Z < z_1) = 0.26 = f_2(0.71)$ [Given]

$\Rightarrow z_1 = 0.71$

$\Rightarrow \frac{x_1'-12}{4} = 0.71$

$\Rightarrow x_1' = 0.71 \times 4 + 12 = 14.84$

(c) $P(x_0' < X < x_1') = 0.50$ & $P(X > x_1') = 0.25$

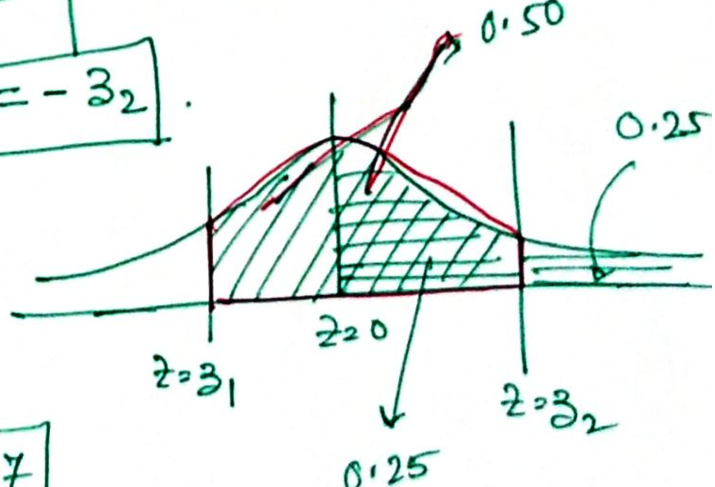
$\Rightarrow P\left(\frac{x_0'-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x_1'-\mu}{\sigma}\right) = 0.50$

$\Rightarrow P(z_1 < Z < z_2) = 0.50$ — (i)

$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{x_1'-\mu}{\sigma}\right) = 0.25$

$\Rightarrow P(Z > z_2) = 0.25$ — (ii)

From fig it is clear that $\boxed{z_1 = -z_2}$.



Now from (ii)

$\Rightarrow 0.5 - P(0 \le Z \le z_2) = 0.25$

$\Rightarrow P(0 \le Z \le z_2) = 0.25$

$\Rightarrow \boxed{z_2 = 0.67.}$ $\Rightarrow \boxed{z_1 = -0.67}$

Hence, $\boxed{x_1' = \sigma z_2 + \mu = 14.68}$

& $\boxed{x_0' = \sigma z_1 + \mu = 9.32}$ Ans.

Lecture 13 P(9). ABanerji.