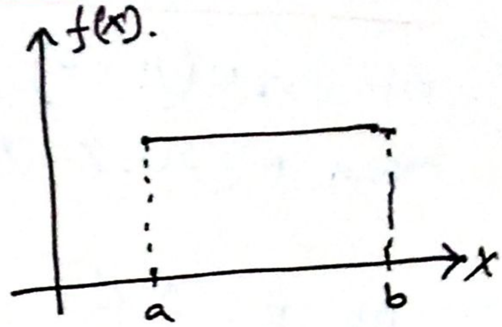
Rectangular (or Uniform) distribution:

PDF :-



$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; \ a < x < b \\ 0 & ; \ \text{otherwise} \end{cases}$$

To verify pdf.

$$\int_a^b f_X(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{b-a}{b-a} = 1.$$

cdf

$$F_X(x) = \begin{cases} 0 & ; \ x \leq a \\ \frac{x-a}{b-a} & ; \ a < x < b \\ 1 & ; \ x \geq b. \end{cases}$$

Moments

$$E(X) = \int_a^b x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$\therefore \boxed{E(X) = \frac{b+a}{2}}$$

$$\mu_2' = E(X^2) = \int_a^b x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{[x^3]_a^b}{3(b-a)}$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2$$

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Lecture 14 P(1)

MGF:

$$E(e^{tx}) = \int_a^b e^{tx} f(x)\, dx = \int_a^b \frac{e^{tx}}{b-a}\, dx = \frac{[e^{tx}]_a^b}{t(b-a)}$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)} \quad ; \; b \neq a \;\&\; t \neq 0.$$

$$\therefore \frac{d}{dt} M_X(t) = \frac{be^{tb} - ae^{ta}}{t(b-a)} - \frac{e^{tb} - e^{ta}}{t^2(b-a)}$$

Here you cannot use MGF to find moments. ~~as MGF~~

Ch. fn.

$$E(e^{itx}) = \int_a^b e^{itx} \frac{1}{b-a}\, dx = \frac{e^{itb} - e^{ita}}{t(b-a)} \quad ; t \neq 0.$$

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Lecture 14 P(2)

GAMMA DISTRIBUTION

A r.v X is said to follow gamma dist. with parameter $\lambda > 0$, if its pdf. is given by $(X \sim \gamma(\lambda))$

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)} & ; \lambda > 0, \ 0 < x < \infty \\ 0 & ; \text{otherwise}. \end{cases}$$

$\displaystyle\int_0^\infty f(x)\,dx = \int_0^\infty \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}\,dx = \frac{\Gamma(\lambda)}{\Gamma(\lambda)} = 1.$

MGF

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x(1-t)} x^{\lambda-1}\,dx$$

let $x(1-t) = u$, $dx = \dfrac{du}{1-t}$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{\lambda-1} \frac{du}{1-t}$$

$$= \frac{1}{\Gamma(\lambda)(1-t)^\lambda} \int_0^\infty e^{-u} u^{\lambda-1}\,du$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\lambda)(1-t)^\lambda} = (1-t)^{-\lambda}$$

$$\therefore \ M_X(t) = (1-t)^{-\lambda} \ ; \ |t| < 1.$$

Moments

$$\mu_r' = E(X^r) = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^r e^{-x} x^{\lambda-1}\,dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x} x^{\lambda+r-1}\,dx$$

$$= \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}. \quad r = 1, 2, \dots$$

Hence $E(X) = \mu_1' = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda)} = \dfrac{\lambda\Gamma(\lambda)}{\Gamma(\lambda)} = \lambda.$

$$E(X^2) = \mu_2' = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} = \frac{(\lambda+1)\Gamma(\lambda+1)}{\Gamma(\lambda)} = \frac{(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)} = \lambda(\lambda+1)$$

$\mathrm{Var}(X) = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$ $\therefore$ $\boxed{\text{Mean} = \text{Variance}}$

Additive Property of Gamma Distribution :

Let X_i's $(i=1,2,\dots,n)$ are n independent r.vs and $X_i \sim \gamma(\lambda_i)$; $i=1,2,\dots,n$; then $\sum_{i=1}^{n} X_i \sim \gamma\left(\sum_{i=1}^{n}\lambda_i\right)$

Proof :- $X_i \sim \gamma(\lambda_i)$; $i=1,2,\dots,n$

$$\Rightarrow M_{X_i}(t) = (1-t)^{-\lambda_i} \;;\; |t|<1.$$

$$\therefore M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= (1-t)^{-\lambda_1}(1-t)^{-\lambda_2}\cdots(1-t)^{-\lambda_n} \;;\; |t|<1$$

$$= (1-t)^{-(\lambda_1+\dots+\lambda_n)}$$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim \gamma\left(\sum_{i=1}^{n}\lambda_i\right).$$

A r.v X is said to follow gamma dist. with parameter $\lambda(>0)$ and $a(>0)$ if pdf. is given by $(X \sim \gamma(a,\lambda))$.

$$f_X(x) = \begin{cases} \dfrac{a^{\lambda} e^{-ax} x^{\lambda-1}}{\Gamma(\lambda)} & ; a>0 ; \lambda>0 ; 0<x<\infty \\ 0 & ; \text{otherwise} \end{cases}$$

MGF :

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\,dx$$

$$= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-ax} e^{tx} x^{\lambda-1}\,dx$$

Let $(a-t)x = u$

$\Rightarrow dx = \dfrac{du}{a-t}$

$$= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(a-t)x} x^{\lambda-1}\,dx.$$

$$= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-u} \left(\frac{u}{a-t}\right)^{\lambda-1} \frac{du}{a-t}$$

$$= \frac{a^\lambda}{\Gamma(\lambda)(a-t)^\lambda} \int_0^\infty e^{-u} u^{\lambda-1}\,du = \frac{a^\lambda}{(a-t)^\lambda} \cdot \frac{\Gamma(\lambda)}{\Gamma(\lambda)}$$

$$= \frac{1}{\left(1-\frac{t}{a}\right)^\lambda} = \left(1-\frac{1}{a}t\right)^{-\lambda}$$

$$\therefore M_X(t) = \left(1-\frac{1}{a}t\right)^{-\lambda} ; \frac{|t|}{|a|}<1 \text{ , i.e, } |t|<|a|.$$

Lecture 14 P(4) ABanerji

BETA DISTRIBUTION OF FIRST ORDER

A r.v X is said to follow beta dist. of first kind with parameters μ and ν $(\mu > 0, \nu > 0)$ if its pdf. is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\beta(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1} & ; (\mu,\nu) > 0 ; 0 < x < 1 \\ 0 & ; \text{otherwise.} \end{cases}$$

where $\beta(\mu,\nu) \rightarrow$ beta fn.

$$\mu_r' = \int_0^1 x^r \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} dx = \frac{1}{\beta(m,n)} \int_0^1 x^{m+r-1}(1-x)^{n-1} dx$$

$$= \frac{\beta(m+r, n)}{\beta(m,n)} = \frac{\Gamma(m+r)\cancel{\Gamma(n)}}{\Gamma(m+r+n)} \times \frac{\Gamma(m+n)}{\Gamma(m)\cancel{\Gamma(n)}}$$

$$= \frac{\Gamma(m+r)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n+r)}$$

$$\therefore E(X) = \mu_1' = \frac{\Gamma(m+1)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n+1)} = \frac{m\cancel{\Gamma(m)}\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}(m+n)\cancel{\Gamma(m+n)}}$$

$$\boxed{E(X) = \frac{m}{m+n}}$$

$$E(X^2) = \mu_2' = \frac{\Gamma(m+2)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n+2)} = \frac{(m+1)m\cancel{\Gamma(m)}\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}(m+n)(m+n+1)\cancel{\Gamma(m+n)}}$$

$$= \cancel{\frac{(m+2)(m+1)}{(m+n+1)(m+n+2)}} \quad \frac{m(m+1)}{(m+n+1)(m+n)}$$

$$\therefore Var(X) = \mu_2' - \mu_1'^2 = \cancel{\frac{(m+2)(m+1)}{(m+n+1)(m+n+2)}} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m}{m+n}\left(\frac{m+n}{m+n+1} - \frac{m}{m+n}\right) = \frac{mn}{(m+n)^2(m+n+1)}$$

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Exponential distribution

A continuous randomvariable with positive, i.e.,
$\mathcal{A} = \{x \mid x > 0\}$ is useful in a variety of applications

e.g.,

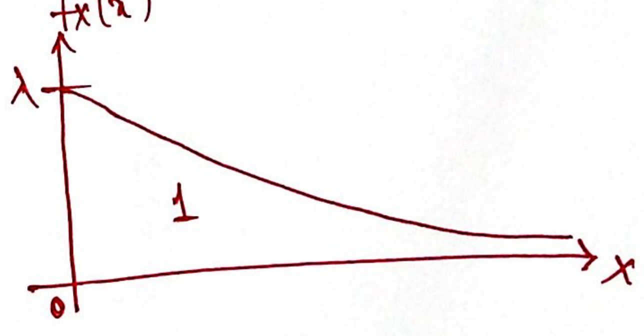
- ✓ patient survival time after the diagnosis of a particular cancer
- ✓ the life time of a light bulb.
- ? the sojourn time (waiting time + service time) for a customer purchasing a ticket at a box office
- ✓ the time between fires in a city.
- ✗ the annual rainfall at a particular location

A continuous random variable X with pdf

$$f_X(x) = \lambda e^{-\lambda x} \quad ; x > 0$$

for some real constant $\lambda > 0$ is an exponential(

random variable



To find CDF

$$F_X(x) = \int_0^x f(u)\,du = \int_0^x \lambda e^{-\lambda u}\,du$$

$$= \lambda \left[\frac{e^{-\lambda u}}{-\lambda} \right]_0^x = 1 - e^{-\lambda x}$$

$$\therefore \quad F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

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Lecture 14 P(6)

To find MGF

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{(t-\lambda)x}\,dx$$

$$= \lambda\int_0^\infty e^{-(\lambda-t)x}\,dx = \frac{\lambda}{t-\lambda}\, e^{-(\lambda-t)}\Big|_0^\infty$$

$$= \frac{\lambda}{t-\lambda}[0-1] = \frac{\lambda}{\lambda-t} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= \sum_{r=0}^{\infty}\left(\frac{t}{\lambda}\right)^r \;;\; \text{if } \left|\frac{t}{\lambda}\right| < 1.$$

$$\mu_r' = E(X^r) = \text{coff. of } \frac{t^r}{r!} \text{ in } M_X(t)$$

$$= \frac{r!}{\lambda^r}, \quad r = 1, 2, \ldots$$

$$\therefore E(X) = \frac{1}{\lambda}$$

$$E(X^2) = \frac{2!}{\lambda^2}$$

$$\therefore Var(X) = \frac{2!}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Q. Let $X_i \sim Exp(\lambda_i)$, $i = 1, 2, 3, \ldots, n$ where each X_i's are independent random variable. Then find the dist of $\sum_{i=1}^{n} X_i$

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(1-\frac{t}{\lambda_1}\right)^{-1}\left(1-\frac{t}{\lambda_2}\right)^{-1}\cdots\left(1-\frac{t}{\lambda_n}\right)^{-1}$$

Now if $\lambda_i = \lambda \;\forall i = 1, 2, \ldots, n$, then

$$M_{\sum_{i=1}^n X_i}(t) = \left(1-\frac{t}{\lambda}\right)^{-n}$$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim \gamma(\lambda, n).$$

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Memoryless property of Exponential distribution :-

Theorem :-

Let $X \sim$ exponential (λ) and for any two positive real numbers x and y

$$P(X \geq x+y \mid X \geq x) = P(X \geq y)$$

Proof :- $P(X \geq x+y \mid X \geq x) = \dfrac{P(X \geq x+y,\ X \geq x)}{P(X \geq x)}$

$$= \frac{P(X \geq x+y)}{P(X \geq x)}$$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}}$$

$$= e^{-\lambda y}$$

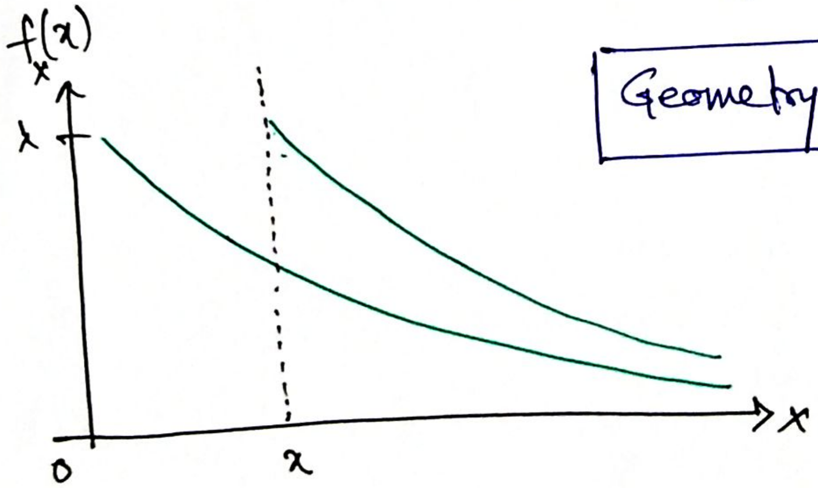
$$= P(X \geq y)$$

cdf of $X \sim \exp(\lambda)$ is. $1-e^{-\lambda x}$, i.e.,

$P(X \leq x) = 1-e^{-\lambda x}$

$\Rightarrow P(X \geq x) = 1 - P(X \leq x)$

$= e^{-\lambda x}$



Geometry of this property

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Lecture 14 P(8)