

## Compound or Joint Experiment

Let  $E$  and  $E'$  be two random experiment.

Let  $S$  and  $S'$  be the respective event spaces.

Let us now consider the random experiment  $E''$  which represents the occurrence of  $E$  first and then  $E'$ .

Then  $E''$  is called the compound random experiment of  $E$  and  $E'$ .

Example :-

Let  $E \rightarrow$  tossing an unbiased coin

$E' \rightarrow$  throwing an unbiased die.

Then  $S = \{H, T\}$ ,  $S' = \{1, 2, 3, 4, 5, 6\}$

Let  $S''$  be the sample space corresponding to the r.e.  $E''$ .

Then  $S'' = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),$   
 $(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

# Let us assume that  $S$  and  $S'$  are mostly enumerable.

Then two r.e.  $E$  and  $E'$  are said to be independent

if for any simple event  $(u_i, v_j)$  connected to  $E''$ .

$$P(u_i, v_j) = P(u_i) P(v_j)$$

Theorem :- Let  $A$  and  $B$  are two events connected to the random experiment  $E$  and  $E'$ , respectively.

If  $E$  and  $E'$  are independent, then

$$P(A, B) = P(A) P(B).$$

Asamey's

### Independent trials :-

Let  $E$  be a given random experiment with sample space  $S$ . Let  $E$  be repeated  $n$  times, where  $n$  is the positive integer, then resulting experiment gives a compound experiment  $E_n$  with sample space  $S^n = S \times S \times \dots \times S$  ( $n$  factors).

This compound experiment  $E_n$  results in  $n$  trials of  $E$ .

These  $n$  trials are said to be independent if for any simple event  $(x_1, x_2, \dots, x_n)$  connected to  $E_n$ ,

$$P(x_1, x_2, \dots, x_n) = P(x_1) P(x_2) \dots P(x_n).$$

### Bernoulli trials :-

Let  $E$  be a random experiment with the sample space  $S$ . Let  $S$  contains two distinct outcomes called 'success' (denoted by 's') and 'failure' (denoted by 'f'). If  $E$  be repeated  $n$  times under identical conditions, then we get  $n$  independent trials of  $E$ . These trials are called Bernoulli sequence of trials if the probability of 's' (or 'f') remains constant in each trial of  $E$ .

Abanayya



Example:- If an unbiased coin is tossed  $n$  times under identical condition we get a Bernoulli sequence in which the probability of getting a 'head' (i.e., success) in each trial is  $\frac{1}{2}$  and that of getting a 'tail' (i.e., failure) is also  $\frac{1}{2}$ .

### Bernoulli distribution

Let us consider a Bernoulli sequence of trials  
Let  $X$  be the random variable corresponding to the Bernoulli sequence, s.t.

$$X = 0 \quad ; \quad \text{when 'f' occurs} \\ = 1 \quad ; \quad \text{when 's' occurs.}$$

Let  $p$  be the probability of 's' and  $1-p=q$  be the probability of 'f'.

$$\text{Then } P(X=0) = 1-p = q$$

$$P(X=1) = p$$

$$\text{Hence, } \boxed{P(X=x) = p^x (1-p)^{1-x} \quad ; \quad x=0,1}$$

$X \rightarrow$  said to follow Bernoulli distribution  
with pmf  $P(X=x) = p^x q^{1-x} \quad ; \quad x=0,1.$

Asanayi

$$E(X) = \sum_x x p(x) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$E(X^2) = \sum_x x^2 p(x) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

$$\therefore \text{Var}(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p) = pq.$$

The MGF of  $X$  is given by

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$

$$= e^{t \cdot 0} \cdot q + e^{t \cdot 1} \cdot p = q + pe^t.$$

$$\therefore \boxed{M_X(t) = q + pe^t}$$

$$\frac{d}{dt} M_X(t) = pe^t \Rightarrow \left. \frac{d}{dt} M_X(t) \right|_{t=0} = p$$

$$\Rightarrow \boxed{E(X) = p}$$

$$\frac{d^2}{dt^2} M_X(t) = pe^t \Rightarrow \left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} = p.$$

$$\therefore \boxed{\text{Var}(X) = pq}$$



# BINOMIAL DISTRIBUTION

Discovered by James Bernoulli (1654-1705) in the year 1700.

- # let us consider a sequence of  $n$  independent Bernoulli trials.
- # let us repeat the Bernoulli trial  $n$  times.
- # let us consider a set of  $n$  independent Bernoulli trials.

$n \rightarrow$  finite no.

$p \rightarrow$  prob. of 'success' in each trial.

$q = 1 - p \rightarrow$  prob. of 'failure' in each trial

$p$  and  $q$  are remain constant in each trials.

let us try to find out the probability of getting  $r$  success in  $n$  independent trial.

SS SFF SSFFFF SFSSF ..... FFFFS

Here  $S \rightarrow$  success  
 $F \rightarrow$  failure

$$P(SSSFFSSFFFFSFSSF \dots FFFFS)$$

$$= P(S)P(S)P(S)P(F)P(F) \dots P(S)$$

$$= \underbrace{P(S)P(S) \dots P(S)}_{r \text{ factors}} \underbrace{P(F)P(F) \dots P(F)}_{(n-r) \text{ factor}}$$

$$= p^r q^{n-r}$$

Now  $r$  success in  $n$  trials can occur in  ${}^nC_r$  ways.

$\therefore$  Probability of getting  $r$  success in  $n$  independent sequence of Bernoulli trials is given by

$$P(X=r) = {}^nC_r p^r q^{n-r} ; r = 0, 1, 2, \dots, n.$$

$$q = 1 - p$$

$$= 0$$

; otherwise  
Lecture 8 P(5) Asameyo.

Then  $X$  is said to follow Binomial distribution  
 with pmf  $P(X=r) = {}^n C_r p^r q^{n-r}$ ;  $r=0, 1, \dots, n$   
 $q=1-p$   
 $= 0$ ; otherwise

$X \rightarrow$  Binomial variable  
 $n$  &  $p \rightarrow$  parameters of Binomial distribution  
 $X \sim B(n, p)$ .

MGF

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \sum_{r=0}^{\infty} e^{tr} {}^n C_r p^r q^{n-r} \\ &= \sum_{r=0}^{\infty} {}^n C_r (pe^t)^r q^{n-r} \end{aligned}$$

$$M_X(t) = (pe^t + q)^n$$

$$\frac{d}{dt} M_X(t) = n(pe^t + q)^{n-1} pe^t.$$

$$\text{Now } \left. \frac{d}{dt} M_X(t) \right|_{t=0} = n(p+q)^{n-1} p = np.$$

$$\therefore E(X) = np$$

$$\frac{d^2}{dt^2} M_X(t) = np e^t (pe^t + q)^{n-1} + n(n-1)(pe^t + q)^{n-2} p^2 e^{2t}$$

$$\therefore \left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} = np + n(n-1)p^2 = np + n(n-1)p^2$$

$$\therefore E(X^2) = np + n(n-1)p^2$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = np + n(n-1)p^2 - n^2 p^2$$

$$= np + n^2 p^2 - np^2 - n^2 p^2 = npq$$

$$= npq = \text{Var}(X)$$



Let  $X_1 \sim B(n_1, p_1)$  &  $X_2 \sim B(n_2, p_2)$

$X_1$  and  $X_2$  are two independent random variables.

Then what is the distribution of  $X_1 + X_2$

$$\text{As } X_1 \sim B(n_1, p_1) \Rightarrow M_{X_1}(t) = (p_1 e^t + q_1)^{n_1}$$

$$X_2 \sim B(n_2, p_2) \Rightarrow M_{X_2}(t) = (p_2 e^t + q_2)^{n_2}$$

$$\text{Now } M_{X_1+X_2}(t) = (p_1 e^t + q_1)^{n_1} (p_2 e^t + q_2)^{n_2}$$

$$\text{Let } p_1 = p_2 = p$$

$$\text{& } q_1 = q_2 = q = 1 - p.$$

$$\text{Then } M_{X_1+X_2}(t) = (p e^t + q)^{n_1+n_2}$$

$$\Rightarrow X_1 + X_2 \sim B(n_1 + n_2, p).$$

Let  $X \sim B(n, p)$ . Then what is the distribution of  $Y = n - X$ .

$$\text{As } X \sim B(n, p) \text{ then } M_X(t) = (p e^t + q)^n$$

$$\therefore M_Y(t) = E(e^{tY}) = E(e^{t(n-X)}) = E(e^{tn} \cdot e^{-tX})$$

$$= e^{tn} E(e^{-tX}) = e^{tn} M_X(-t)$$

$$= e^{tn} (p e^{-t} + q)^n = (p + q e^t)^n$$

$$\Rightarrow Y \sim B(n, q).$$

### Characteristic function

$$\phi_X(t) = E(e^{itX})$$

$$= \sum_{r=0}^n e^{itr} {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^n {}^nC_r (e^{it}p)^r q^{n-r}$$

$$= (pe^{it} + q)^n$$

### Probability Generating function:

$$P(z) = \sum_{r=0}^n P(X=r) z^r = \sum_{r=0}^n {}^nC_r p^r q^{n-r} z^r$$

$$= \sum_{r=0}^n {}^nC_r (pz)^r q^{n-r}$$

$$= (\cancel{pz}) (pz + q)^n$$