Tutorial-3
MA 202

Q. Let X be a discrete r.v. with pmf.

$$P(X=x) = \frac{1}{x(1+x)} \quad \text{for } x=1,2,3,\ldots$$
$$= 0 \quad \text{elsewhere.}$$

Show that MGF of X exists however non-of its moments exist.

Soln. for any pos. int. r

$$\mu_r = E(X^r) = \sum_{x=1}^{\infty} \frac{x^r}{x(1+x)} \quad \text{——} (1).$$

Hence, μ_r exist if the series in (1) is absolutely convergent.

(1) is a series of positive term. where

Let u_x be the xth term of the series.

$$u_x = \frac{x^{r-1}}{x+1} \geq \frac{1}{x+1} \quad \text{for } x=1,2,3,\ldots$$

Now $\sum_{x=1}^{\infty} \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$\longrightarrow$ divergent series

Hence, by comparison test we can conclude that $\sum_{x=1}^{\infty} u_x$ is also divergent series.

$\Rightarrow$ $E(X^r)$ does not exist for any positive integer r.

$\Rightarrow$ Non of the moments about origin of X exist.

P.T.O.

Now $M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{x(x+1)}$ —— ②

Let us consider the nth partial sum $S_n(t)$ of the above series.

$$S_n(t) = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)} \quad —— ③.$$

$$= e^{t}\left(1-\frac{1}{2}\right) + e^{2t}\left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + e^{nt}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \left(e^{t} + \frac{e^{2t}}{2} + \cdots + \frac{e^{nt}}{n}\right) - e^{-t}\left(\frac{e^{2t}}{2} + \frac{e^{3t}}{3} + \cdots + \frac{e^{(n+1)t}}{n+1}\right)$$

$$= \left\{e^{t} + \frac{(e^t)^2}{2} + \cdots + \frac{(e^t)^n}{n}\right\} - e^{-t}\left[\left\{e^{t} + \frac{(e^t)^2}{2} + \cdots + \frac{(e^t)^{n+1}}{n+1}\right\} - e^{t}\right]$$

Now $\lim_{n\to\infty} \left\{e^{t} + \frac{(e^t)^2}{2} + \cdots + \frac{(e^t)^n}{n}\right\}$

$$= \lim_{n\to\infty} \left\{e^{t} + \frac{(e^t)^2}{2} + \cdots + \frac{(e^t)^{n+1}}{n+1}\right\}$$

$$= -\ln(1-e^{t}) \quad \text{if } e^{t} < 1.$$

Hence $\lim_{n\to\infty} S_n(t) = -\ln(1-e^t) - e^{-t}\left[-\ln(1-e^t) - e^t\right]$ if $e^t < 1$.

$$= 1 - \ln(1-e^t) + e^{-t}\ln(1-e^t), \quad e^t < 1.$$

$$= 1 + (e^{-t}-1)\ln(1-e^t), \quad e^t < 1.$$

$\therefore M_X(t) = \lim_{n\to\infty} S_n(t) = 1 + (e^{-t}-1)\ln(1-e^t)$, $e^t < 1$. i.e., $t < 0$

& $M_X(0) = E(e^{0.X}) = E(1) = 1.$

$\therefore$ MGF of X exists and is given by

$$M_X(t) = 1 + (e^{-t}-1)\ln(1-e^t), \quad e^t < 1 \Rightarrow t < 0$$
$$= 1, \quad t = 0$$

Tutorial 3. P(2) ABanerji.

Q.2. Find the mean and variance of a Poisson μ-distribution from the PGF.

Soln. Let $X \sim P(\mu)$

Then $P(x) = \frac{e^{-\mu}\mu^x}{x!}, \; x = 0, 1, 2, \cdots$

The corresponding PGF is given by

$$G(z) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} z^x \quad , |z| \le 1.$$

Note:- Let X be a discrete r.v. with pmf $P(X=x)$, $x=0,1,\cdots$

Then $G(z) = \sum_{x=0}^{\infty} P(X=x) z^x$, which is absolutely and uniformly convergent in $|z|\le 1$, is said to be the pgf of X.

Now $G(0) = P(X=0)$

$G'(0) = P(X=1)$ $\left[\because G'(z) = \sum_{x=1}^{\infty} x P(X=x) z^{x-1}\right]$

$G''(0) = 2!\,P(X=2)$ $\left[\because G''(z) = \sum_{x=2}^{\infty} x(x-1) P(X=x) z^{x-2}\right]$

$\vdots$

$G^{(k)}(0) = k!\,P(X=k)$, $k = 0, 1, 2, \cdots$

Hence, $P(X=k) = \frac{1}{k!} G^{(k)}(0)$, $k = 0, 1, 2, \cdots$

where $G^{(k)}(0)$ is the kth order derivative of $G(z)$ w.r.t z at $z=0$.

Let all the moments of the r.v. X exist.

Then $G'(1) = \sum_{x=1}^{\infty} x P(X=x) = E(X)$.

$G''(1) = \sum_{x=2}^{\infty} x(x-1) P(X=x) = E[X(X-1)]$.

P.T.O.

Now $G(z) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^{x}}{x!} z^{x}$

$$= \sum_{x=0}^{\infty} \frac{e^{-\mu}(\mu z)^{x}}{x!}$$

$$= e^{-\mu} e^{\mu z}.$$

$$\boxed{G(z) = e^{\mu(z-1)}}$$

$\therefore \quad G'(z) = \mu e^{\mu(z-1)}$

$G''(z) = \mu^{2} e^{\mu(z-1)}$

$\therefore E(X) = G'(1) = \mu.$

$E[X(X-1)] = G''(1) = \mu^{2}$

$\therefore Var(X) = E[X(X-1)] - \mu(\mu-1)$

$= \mu^{2} - \mu^{2} + \mu = \mu.$

Ans.

$E[X(X-1)] - E(X)[E(X)-1]$

$\Rightarrow E(X^{2}-X) - \{E(X)\}^{2} + E(X)$

$\Rightarrow E(X^{2}) - \cancel{E(X)} - \{E(X)\}^{2} + \cancel{E(X)}$

$\Rightarrow E(X^{2}) - \{E(X)\}^{2}$

$= Var(X).$

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Tutorial 3 P(4)

Q.3. If a person gets Rs $(2x+5)$ where x denotes the number appearing when a balanced die is rolled once, then how much money can be expected in the long run per game?

Soln. Let X be the r.v. denoting the no. appearing on the die.

Then $P(X=x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$.

Then required expectation is

$$E(2X+5) = \sum_{x=1}^{6} (2x+5) P(X=x)$$

$$= 12$$

Ans. Rs 12.

Tutorial 3 P(5) ABanerjee.

Q.4. Find the median of binomial $(5, \frac{1}{2})$ distribution.

Soln. If $X \sim B(n, p)$ then its pmf is given by

$$P(X=x) = {}^{n}C_x \, p^x (1-p)^{n-x} \quad , x = 0, 1, 2, \ldots, n.$$

For the given problem

$$P(X=x) = {}^{5}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} \quad , x = 0, 1, 2, 3, 4, 5.$$

The cdf is writen as.

$$F_X(x) = 0 \quad \text{if } -\infty < x < 0$$

$$= \left(\frac{1}{2}\right)^5 \quad \text{if } 0 \le x < 1$$

$$= \left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 \quad \text{if } 1 \le x < 2$$

$$= \left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 \quad \text{if } 2 \le x < 3$$

$$= \left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5 \quad \text{if } 3 \le x < 4$$

$$= \left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 \quad \text{if } 4 \le x < 5$$

$$\text{if } x \geq 5.$$

$$= 1$$

Here, $F(2) = \frac{1}{2}$. $F(3) = \frac{13}{16} > \frac{1}{2}$ & $F(3-0) = 1/2$

$F(2-0) = \frac{6}{32} < 2.$

$\Rightarrow$ The median belongs to $[2, 3)$. Ans.

~~Let it be~~ $\frac{1}{2}(2+3) = 2.5$.

Tutorial 3 P(6)

Q.5. Find the mode or modes of binomial (n, p) distribution.

Soln. Let $X \sim B(n, p)$. Then its p.m.f. is given by

$$p_X(x) = {}^nC_x\, p^x (1-p)^{n-x} \;;\; x = 0, 1, 2, \ldots$$

Now $p_X(x) \geq p_X(x+1)$ if

$$ {}^nC_x\, p^x (1-p)^{n-x} \geq {}^nC_{x+1}\, p^{x+1} (1-p)^{n-x-1}$$

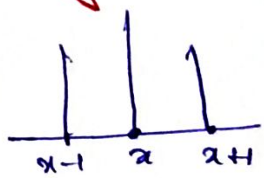
do the stepts by yourself

$$\Rightarrow \frac{1-p}{n-x} \geq \frac{p}{x+1} \quad (x \neq n).$$

$$\Rightarrow \boxed{x \geq (n+1)p - 1.}$$

Similarly

$p_X(x) \geq p_X(x-1)$ if $\boxed{x \leq (n+1)p}$ → do it by yourself.



Case I. Let $(n+1)p$ is an integer.

Let $M = (n+1)p$.

Then $p_X(x) \geq p_X(x+1)$; $x \geq M-1$ (at $x = M-1$, $p_X(M-1) \geq p_X(M)$)

$p_X(x) \geq p_X(x-1)$; $x \leq M$ (at $x = M$, $p_X(M) \geq p_X(M-1)$)

$\Rightarrow p_X(M) = p_X(M-1)$

Hence, $p_X(M-2) < p_X(M-1) = p_X(M) > p_X(M+1)$.

$\Rightarrow (n+1)p - 1$ & $(n+1)p$ are two modes of $X \sim B(n, p)$.

Case II, P.T.O.

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Case II Let $M = (n+1)p$ is not an integer

Then we have $M-1 < [M] < M$,

where $[M]$ is the greatest integer not greater than M.

Then $f([M]) > f([M]+1)$

& $f([M]) > f([M]-1)$

$\Rightarrow [M] = [(n+1)p]$ is the unique mode of

$X \sim B(n, p)$.

———— x ————

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Tutorial 3 P(8)