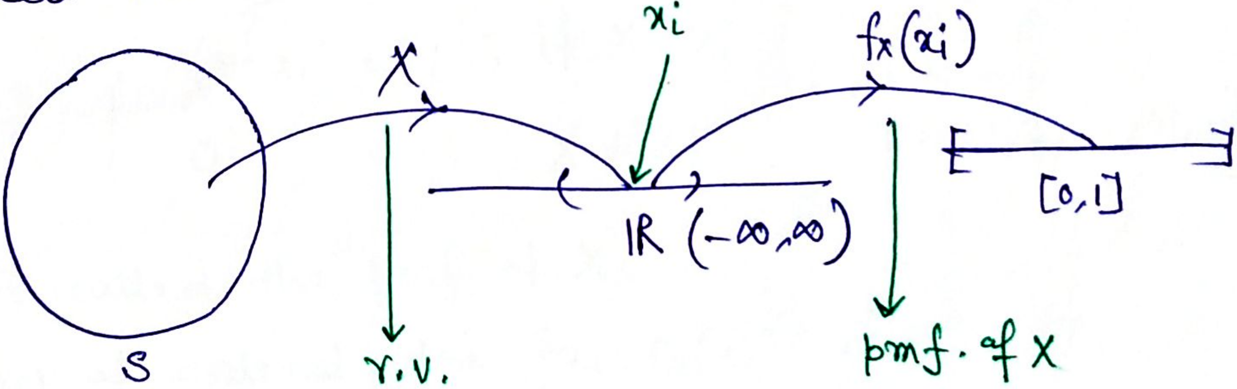
Random variable, Distribution function, Probability mass function

Random variable :

Let S be a sample space associated with a given random experiment E.

A real valued (measurable) function defined on S is called a one-dimensional random variable (r.v).

If the sample space is discrete, then the corresponding random variable is said to be discrete r.v.



Example: Let us consider the example of tossing a unbiased coin

$S = \{H, T\}$

Let us define the real valued fn. X on S as

$X = 1$, H appears

$= 0$, T appears.

$X \longrightarrow$ discrete r.v.

Probability mass function

Let X be a one-dimensional discrete random variable taking atmost countably infinite number of values $x_1, x_2, \cdots$ (i.e., $x_i, i \in I$, I being the index set) then its probabilistic behaviour at each real point is described by a function, called the probability mass function (pmf) and is defined as follows.

Definition: If X is a discrete r.v with distinct values $x_1, x_2, \cdots, x_n, \cdots$ then the function $p(x)$ or $p_X(x)$ or $f_X(x)$ is defined as follows

$$p_X(x) = \begin{cases} P(X=x_i) = p_i \ ; \text{ if } X = x_i \\ 0 \ ; \ X \neq x_i \end{cases} \ ; \ i \in I. \quad I = \{1, 2, \cdots, n, \cdots\}$$

and is called the pmf of X.

The set of ordered pairs $\{x_i, p_X(x_i)\}, i \in I$, or, $\{(x_1, p_1), (x_2, p_2), \cdots, (x_n, p_n), \cdots\}$ represents the probability distribution of the r.v X.

\# If $p_X(x_i); i \in I$ is the pmf, then it must satisfies the following two conditions.

(i) $p_X(x_i) \geq 0 \ \forall i \in I$.

(ii) $\sum_{i \in I} p_X(x_i) = 1$.

Example: Let us consider the example taken in Lecture 4 P(1)

$$\left.\begin{aligned} P[X=1] &= \tfrac{1}{2} = p_1 \\ P[X=0] &= \tfrac{1}{2} = p_2 \end{aligned}\right\} \text{pmf of } X.$$

Cumulative distribution fn. (cdf) or distribution fn. (df) of a discrete r.v. :

Let $P : \Delta \to \mathbb{R}$ be a probability function, where Δ is the class of subsets of S (S being the sample space associated with the r.e. E) or power set of S forming the class of events. Then we remember that the order 3 tuple (S, Δ, P) is called the probability space.

Let X be the random variable defined on the event space S.

The distribution fn. or cumulative distribution fn. of the r.v X w.r.t the prob. space (S, Δ, P) is a real valued fn. $F_X(x)$ of real real variable x, defined in $(-\infty, \infty)$ such that

$$F_X(x) = P(-\infty < X \le x) \quad \forall\, x \in (-\infty, \infty).$$

It is clear that the range of the fn. is a subset of $[0,1]$.

Let X be a discrete r.v.

then cdf of X is defined as

$$F_X(x) = \sum_{x_j \le x_i}^{i} P(X = x_j) = \sum_{j=-\infty}^{i} p_j \quad \text{if } x_i \le x \le x_{i+1}.$$

$$i = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Thus $F_X(x)$ is a step fn.

See the fig. in next page.

$X \to$ discrete r.v
$\hookrightarrow$ p.m.f]
$P(X=x_i) = p_i$, $i=-2,-1,0,1,2,3$

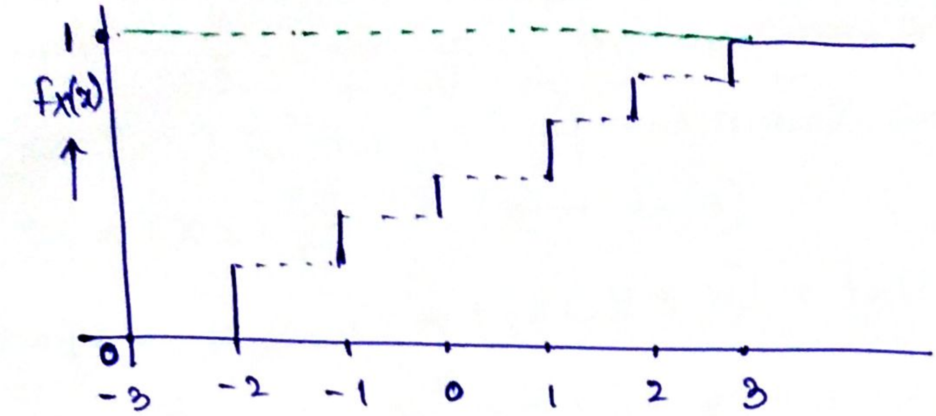


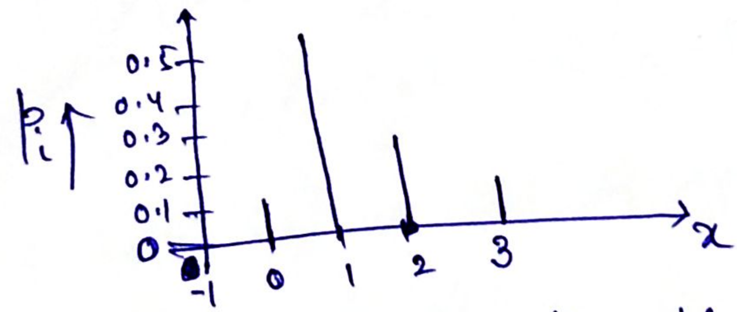
Fig. cdf of a discrete distribution

Example : Let X be a discrete r.v s.t

$P(X=0)=0.1$, $P(X=1)=0.5$, $P(X=2)=0.25$, $P(X=3)=0.15$

Hence, $\sum_{i=0}^{3} P(X=i) = 1$.

If pmf look like as given in the following fig



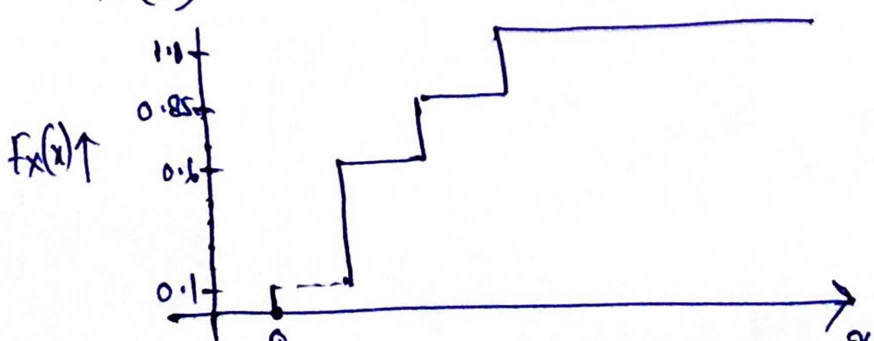
Let us ~~calculate~~ calculate its cdf.

$F_X(0) = P(X \le 0) = P(X=0) = 0.1$

$F_X(1) = P(X \le 1) = P(X=0) + P(X=1) = 0.1 + 0.5 = 0.6$

$F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 0.1 + 0.5 + 0.25 = 0.85$

$F_X(3) = P(X \le 3) = p_0 + p_1 + p_2 + p_3 = 1$.



Properties of distribution fn. or cumulative distribution fn.

1. Let $F_X(x)$ be the cdf of the r.v. X and $a<b$, then

$$P(a<X\le b) = F_X(b) - F_X(a)$$

Proof: $P(X\le a) + P(a<X\le b) = F_X(b)$, by defn.

$$\Rightarrow P(a<X\le b) = F_X(b) - P(X\le a)$$
$$= F_X(b) - F_X(a).$$

Cor. 1. $P(a\le X\le b) = P[(X=a) \cup (a<X\le b)]$

$= P(X=a) + P(a<X\le b)$ $\because$ disjoint.

$= P(X=a) + F_X(b) - F_X(a).$

Cor. 2. $P(a<X<b) + P(X=b) = P(a<X\le b)$

$\Rightarrow P(a<X<b) = F_X(b) - F_X(a) - P(X=b).$

2. If $F_X(x)$ be the cdf of one-dimensional r.v. X,

then (i) $0 \le F_X(x) \le 1$

(ii) $F_X(x) \le F_X(y)$ if $x<y$.

3. $F(-\infty) = \lim_{x\to -\infty} F_X(x) = 0$

$F(\infty) = \lim_{x\to} F_X(x) = 1.$