Moments, Skewness and Kurtosis

Moments: If $x_1, x_2, \cdots, x_N$ are the N values assumed by the variable X, we define the r-th moment about the origin as (about the point '0')

$$m'_r = \frac{1}{N}\sum_{j=1}^{N}(x_j - 0)^r = \frac{x_1^r + x_2^r + \cdots + x_N^r}{N} \quad \text{— (1)}$$

The r-th moment about the mean $\bar{x}$ is defined as

$$m_r = \frac{1}{N}\sum_{j=1}^{N}(x_j - \bar{x})^r \quad \text{— (2)} \quad \text{(Central moment)}.$$

From (1) for $r=1$, $m'_1 = \frac{1}{N}\sum_{j=1}^{N} x_j = \bar{x}$ which is the mean.

From (2) for $r=1$, $m_1 = \frac{1}{N}\sum_{j=1}^{N}(x_j - \bar{x}) = \frac{1}{N}\sum_{j=1}^{N} x_j - \bar{x}$

$= \bar{x} - \bar{x} = 0$.

and for $r=2$, $m_2 = \frac{1}{N}\sum_{j=1}^{N}(x_j - \bar{x})^2 = \sigma^2$ (variance)

Moments for Grouped data

If $x_1, x_2, \cdots, x_K$ occur with frequencies $f_1, f_2, f_3, \cdots, f_K$ respectively, then the above moments are given by

$$m'_r = \frac{1}{N}\sum_{j=1}^{K} f_j x_j^r$$

$$m_r = \frac{1}{N}\sum_{j=1}^{K} f_j (x_j - \bar{x})^r$$

where $N = \sum_{j=1}^{K} f_j$

Relations between moments: [About origin (m'_r) and about mean (m_r)]

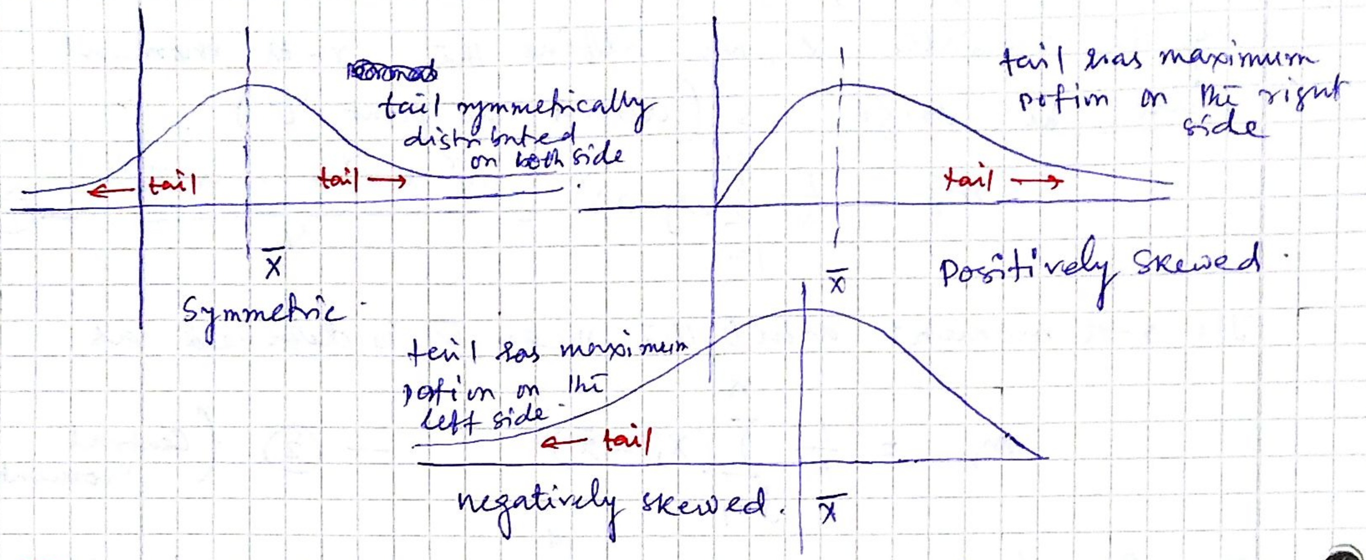
$$m_2 = m'_2 - m_1'^2$$

$$m_3 = m'_3 - 3m'_1 m'_2 + 2m_1'^3$$

$$m_4 = m'_4 - 4m'_1 m'_3 + 6m_1'^2 m'_2 - 3m_1'^4$$

and so on.

Skewness: Skewness is the degree of asymmetry, or departure from symmetry of a distribution.



(*) Pearson's first and second Coefficient of skewness:

(a) $$\text{Skewness} = \frac{\text{mean} - \text{mode}}{\text{standard deviation}} = \frac{\bar{x} - \text{mode}}{\sigma}$$

(b) $$\text{Skewness} = \frac{3\,(\text{mean} - \text{median})}{\text{standard deviation}} = \frac{3\,(\bar{x} - \text{median})}{\sigma}$$

(*) Moment coefficient of skewness.

$$a_3 = \frac{m_3}{\sigma^3} = \frac{m_3}{(\sqrt{m_2})^3}$$

Kurtosis Kurtosis is the degree of peakedness of a distribution; usually taken relative to a normal distribution.

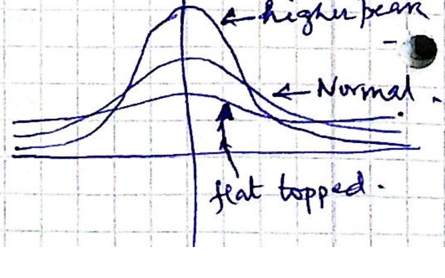
(*) A distribution having relatively high peak called leptokurtic

(*) A distribution having relatively flat topped, is called platykurtic.

(*) A normal distribution, which is not very peaked or very flat topped, is called mesokurtic.

(*) Moment coefficient of kurtosis

$$a_4 = \frac{m_4}{\sigma^4} = \frac{m_4}{m_2^2}$$



Example 1

Find the m_i' and m_i for $i = 1, 2, 3, 4$ for the given data set.

2, 3, 7, 8, 10.

Ans:

$$m_1' = \frac{\sum x_i}{N} = \frac{2+3+7+8+10}{5} = 6$$

$$m_2' = \frac{\sum x_i^2}{N} = \frac{2^2+3^2+7^2+8^2+10^2}{5} = 45.2$$

$$m_3' = \frac{\sum x_i^3}{N} = \frac{2^3+3^3+7^3+8^3+10^3}{5} = 378$$

$$m_4' = \frac{\sum x_i^4}{N} = \frac{2^4+3^4+7^4+8^4+10^4}{5} = 3318.8$$

~~Central~~ moment about the origin '0'

$$m_1 = \frac{\sum x_i}{N} - m_1' = 0 \quad . \text{ Since } \bar{x} = m_1' = 6 .$$

$$m_2 = \frac{\sum (x_i - \bar{x})^2}{N} = \frac{4^2+3^2+1^2+2^2+4^2}{5} = 9.2$$

$$m_3 = \frac{\sum (x_i - \bar{x})^3}{N} = -3.6 .$$

$$m_4 = \frac{\sum (x_i - \bar{x})^4}{N} = 122 .$$

Central moments about $\bar{x}$.

Example 2

Find (a) Pearson's first, second and moment coefficient of skewness

(b) Kurtosis

for the data given in Example 1.

Ans:

(a) First Coeff. $= \frac{\text{mean} - \text{mode}}{\text{s.d.}}$, undefined.

Since there mode is undefined.

$$\text{Second Coeff} = \frac{3(\text{mean} - \text{median})}{\text{s.d.}} = \frac{3.(m_1' - 7)}{\sqrt{m_2}} = \frac{3(6-7)}{\sqrt{9.2}} = \frac{-3}{\sqrt{9.2}}$$

$$\text{moment coefficient} = \frac{m_3}{(\sqrt{m_2})^3} = \frac{-3.6}{(\sqrt{9.2})^3} .$$

(b) $$\text{Kurtosis} = \frac{m_4}{m_2^2} = \frac{122}{(9.2)^2}$$

Example 3

(a) Find the first four central moments for the given data:

x_i	61	64	67	70	73
f_i	5	18	42	27	8

(b) Then find the moment coefficients of skewness and kurtosis.

Ans: If we scale the data by the formula

$$u = \frac{X-A}{c}$$, then the moments about the origin can be given as

$$m_j' = c^j \frac{\sum_i f_i u_i^j}{\sum_i f_i}, \text{ where } j = 1, 2, 3, \ldots$$

For the above data

X	$u = \frac{X-67}{3}$	f	fu	fu^2	fu^3	fu^4
61	−2	5	−10	20	−40	80
64	−1	18	−18	18	−18	18
A → 67	0	42	0	0	0	0
70	1	27	27	27	27	27
73	2	8	16	32	64	128
		$N = \sum f = 10$	$\sum fu = 15$	$\sum fu^2 = 97$	$\sum fu^3 = 33$	$\sum fu^4 = 253$

(a)

$$m_1' = c\frac{\sum fu}{N} = 3 \cdot \frac{15}{100} = 0.45$$

$$m_2' = c^2 \frac{\sum fu^2}{N} = 3^2 \cdot \frac{97}{100} = 8.73$$

$$m_3' = c^3 \frac{\sum fu^3}{N} = 3^3 \cdot \frac{33}{100} = 8.91$$

$$m_4' = c^4 \cdot \frac{\sum fu^4}{N} = 3^4 \cdot \frac{253}{100} = 204.93 .$$

Thus

$$m_1 = 0$$

$$m_2 = m_2' - m_1'^2 = 8.5275$$

$$m_3 = m_3' - 3m_1'm_2' + m_1'^3 = -2.6932$$

$$m_4 = m_4' - 4m_1'm_3' + 6m_1'^2 m_2' - 3m_1'^4 = 199.3759$$

(b)

$$\text{Skewness} = \frac{m_3}{(\sqrt{m_2})^3} = -\frac{2.6932}{(\sqrt{8.53})^3}$$

$$\text{Kurtosis} = \frac{m_4}{m_2^2} = \frac{199.3759}{(8.5275)^2}$$