SOLUTION OF KEPPLER PROBLEM IN DELAUNAY VARIABLES AND THE LIDOV-KOZAI MECHANISM

By Shashank Dattathri
Indian Institute of Science
CV Raman Road, Bengaluru 560012

Guided by Prof. Arun Mangalam
Indian Institute of Astrophysics
Sarjapur Main Road, 2nd Block,
Koramangala, Bengaluru 560034

Abstract

In the first part, the fundamentals of Hamiltonian mechanics are studied. We can define canonical transformations in order to transform the phase space variables into more suitable variables for a particular problem. The common method is to make the new Hamiltonian equal to zero, which gives rise to the Hamilton-Jacobi equation. We further define action-angle variables for a system, which are useful for solving systems involving periodic motion. These principles are used to solve the Kepler problem in action-angle variables, and the solution can be elegantly expressed in Delaunay variables. In the next part, the restricted three body problem (R3PB) is studied, in which two bodies move around their common centre of mass in circular orbits, and a distant third body orbits around this system and causes perturbations. Using a small perturbing function in the Hamiltonian, we can approximate the Hamiltonian for the R3BP by expanding it in terms of Legendre polynomials. The Hamiltonian is used to determine the equations of motion of the Delaunay variables and the orbit elements (i.e. time derivatives). It is found that the angular momentum perpendicular to the plane of the motion of the binary is conserved, and this results in the possibility of an exchange between the eccentricity and the inclination of the binary. If the initial inclination of the binary is lesser than a certain critical inclination, then the exchange is minimal, and the argument of pericenter of the binary will rotate continuously (circulate). If the initial inclination is greater than the critical inclination, then the eccentricity-inclination exchange is profound, and the argument of pericenter will librate around an extremum value. This mechanism provides insight to the stability of satellites around binaries and the long-term evolution of binary systems.

1 Lagrangian Mechanics

1.1 Lagrange's Equations

In a system, the Lagrangian is defined as

$$L = T - V \tag{1}$$

where T is the total kinetic energy and V is the total potential energy. [1, p. 21]
If the potential is velocity independent, then we have the Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

where the q_i's represent the generalized coordinates and the $\dot{q}_i$'s represent the generalized velocities. There will be one equation for each generalized coordinate.
The conjugate momenta for each generalized coordinate is defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3}$$

They are also related as:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{4}$$

1.2 Hamilton's Principle

Hamilton's Principle states:

The motion of the system between times t_1 and t_2 is such that the integral

$$I = \int_{t_1}^{t_2} L dt \tag{5}$$

called the action, has a stationary value. [1, p. 35]

Thus, the variation of the action integral for fixed t_1 and t_2 is zero [1, p. 35]:

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0 \tag{6}$$

Here, the Lagrangian L can depend on all the q_i's, the $\dot{q}_i$'s, and time t.

1.3 Cyclic Coordinates and Conservation Theorems

If a particular coordinate does not appear in the Lagrangian, it is said to be a *cyclic coordinate.* Thus, $\frac{\partial L}{\partial q_i} = 0$ if q_i is a cyclic coordinate. [1]
From equations 2 and 3, we see that the Lagrange equation becomes:

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$
$$p_i = constant$$

So, the conjugate momentum is conserved for every cyclic coordinate.

2 Hamiltonian Mechanics

2.1 Hamilton's Equations

The Hamiltonian $H(q, \dot{q}, t)$ is generated by the Legendre transformation [1, p. 337]:

$$H = \sum_i \dot{q}_i p_i - L \tag{7}$$

We have the relations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{8a}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{8b}$$

$$-\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t} \tag{8c}$$

The Hamiltonian is the total energy of the system if:

- The generalized coordinates q_i's do not depend on time explicitly
- All the forces are conservative

If these conditions are met, then $H = T + V$ [1, p. 345].

2.2 Conservation Theorems

From equations 3 and 8a, we see that if $\dot{p}_i = 0$, then $\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} = 0$. So, if a particular generalized coordinate is cyclic in the Lagrangian, it is also cyclic in the Hamiltonian and the conjugate momentum is conserved [1, p. 344].

2.3 Canonical Transformations

A canonical transformation is the transformation from one set of coordinates to another, which may be more suitable for solving a problem. If q_i's, p_i's, and H are the old generalized coordinates, momenta, and Hamiltonian respectively, we want new coordinates Q_i's, momenta P_i's, and Hamiltonian K such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Since both the old and new coordinates must satisfy Hamilton's principle [1, p. 370],

$$\delta \int_{t_1}^{t_2} \sum_i (\dot{q}_i p_i - H) dt = \delta \int_{t_1}^{t_2} \sum_i (\dot{Q}_i P_i - K) dt = 0$$

So, the old and new Hamiltonians are related by:

$$K = H + \frac{\partial F}{\partial t} \tag{9}$$

where F is called the generating function of the transformation [1, p. 371].

2.4 Hamilton Jacobi Equation

We want the new Hamiltonian to vanish so that:

$$\dot{Q}_i = 0 \Rightarrow Q_i = \beta_i$$
$$\dot{P}_i = 0 \Rightarrow P_i = \alpha_i$$

Let us say the generating function is a function of the old coordinates q_i, new momenta P_i, and time t, and denote it by $S(q_i, \alpha_i, t)$. It is called Hamilton's principle function [1, p. 431].
We have,

$$p_i = \frac{\partial S}{\partial q_i}$$
$$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}$$

To make the new Hamiltonian vanish, we must have

$$H + \frac{\partial S}{\partial t} = 0 \tag{10}$$

This is the Hamilton Jacobi equation [1, p. 431].

If H does not explicitly depend on t, then we can write the principle function as

$$S(q, \alpha.t) = W(q, \alpha) - at \tag{11}$$

Here, W is called the characteristic function [1, p. 434].

We have:

$$p_i = \frac{\partial W}{\partial q_i}$$
$$Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}$$

The constant a is the total energy in mechanical systems, as is seen by plugging equation 10 in equation 9:

$$H + \frac{\partial}{\partial t}(W(q, \alpha) - at) = 0 \Rightarrow H = a$$

Since the constant a can be chosen arbitrarily, we choose $a = \alpha_1$. We have :

$$\dot{q}_i = \frac{\partial K}{\partial \alpha_i} \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, \end{cases}$$

since the new Hamiltonian K only depends on α_1 among the α_i's.
This yields [1, p. 441]:

$$q_i = \frac{\partial W}{\partial \alpha_i} \begin{cases} t + \beta_i & \text{if } i = 1, \\ \beta_i & \text{if } i \neq 1, \end{cases} \tag{12}$$

2.5 Action Angle Variables

The action variable is defined as [1]:

$$J = \oint pdq \tag{13}$$

The generalized coordinate conjugate to J is called the angle variable, defined by:

$$\omega = \frac{\partial W}{\partial J} \tag{14}$$

The equation of motion for ω is:

$$\dot{\omega} = \frac{\partial H}{\partial J} = \nu \Rightarrow \omega = \nu t + \beta$$

Here, the constant ν is the frequency of the periodic motion of q [1, p. 454].

2.6 Separation of Variables

A coordinate q_i is said to be separable if Hamilton's principle function can be split as into two parts, one which depends only on q_i and the other which is independent of q_i [1, p. 457].
If all the coordinates are separable, we have:

$$S = \sum_i S_i(q_i; \alpha_1, \dots, \alpha_n)$$

If the Hamiltonian doesn't depend on time explicitly, for each S_i we have:

$$S_i = W_i - a_i t$$

The action-angle variables are then defined by [1, p. 459]:

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i}{\partial q_i} dq_i \tag{15}$$

$$\omega_i = \frac{\partial W}{\partial J_i} = \sum_j \frac{\partial W_j}{\partial J_i} \tag{16}$$

3 The Kepler Problem

3.1 General Central Force Problem

Consider two masses m_1 and m_2 which interact by a potential U which depends only on the separation vector $r = r_1 - r_2$.
Let r_1' and r_2' are the radius vectors of the particles relative to the center of mass, related by:

$$r_1' = \frac{-m_2}{m_1 + m_2} r$$

$$r_2' = \frac{m_1}{m_1 + m_2} r$$

The kinetic energy can be split into the kinetic energy of the motion of the center of mass and the motion about the center of mass [1, p. 71]:

$$T = \frac{m_1 + m_2}{2} \dot{R}^2 + (\tfrac{1}{2} m_1 \dot{r_1}^2 + \tfrac{1}{2} m_2 \dot{r_2}^2)$$
$$= \tfrac{1}{2} M \dot{R}^2 + \tfrac{1}{2} \mu \dot{r}^2$$

where M is the total mass $m_1 + m_2$ and μ is the reduced mass $\frac{m_1 m_2}{m_1 + m_2}$. The total Lagrangian is thus:

$$L = \tfrac{1}{2} M \dot{R}^2 + \tfrac{1}{2} \mu \dot{r}^2 - U(r)$$

The equation is cyclic in R, so none of the equations for r and $\dot{r}$ will involve R. We will only consider the motion about the center of mass, so the first term in the Lagrangian can be ignored. [1, p. 71]
Since the problem is spherically symmetric, the total angular momentum **L** is conserved. [1, p. 72] Expressing the Lagrangian in spherical polar coordinates, we have

$$L = \tfrac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \tag{17}$$

The first Lagrange equation gives us:

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Substituting for L and using the fact that $\frac{\partial L}{\partial \dot{\theta}} = p_\theta$,

$$\dot{p_\theta} - \frac{\partial}{\partial t} \mu r^2 \dot{\theta} = 0$$
$$p_\theta = \mu r^2 \dot{\theta} = l \tag{18}$$

where l is the magnitude of the angular momentum. [1, p. 73]
We can manipulate the previous equations to:

$$\frac{\partial}{\partial t} \tfrac{1}{2} r (r d\dot{\theta}) \Rightarrow \frac{\partial}{\partial t} \frac{dA}{dt} = 0 \tag{19}$$

Here, $dA = \frac{1}{2} r (r\theta)$ which is the differential area swept out by the radius vector in time dt. Thus, the areal velocity $\frac{dA}{dt}$ is a constant, which is Kepler's second law of planetary motion. [1, p. 73]

(From here on, we will use m in place of μ for simplicity. In fact, it is a good approximation in many systems.)

The second Lagrange equation gives:

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$
$$\frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 - \frac{\partial U}{\partial r}) = 0$$

But, $\frac{\partial V}{\partial r} = -f(r)$, the central force.

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

From equation 17, we substitute for $\dot{\theta}$:

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \tag{20}$$

Now, we use the relation $l = mr^2\dot{\theta}$, so:

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta} \tag{21}$$

Substituting for $\ddot{r}$ in the previous equation, we get:

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

On substituting $u = \frac{1}{r} \Rightarrow du = -\frac{dr}{r^2}$, we have:

$$\frac{-l^2u}{m}\frac{d^2u}{d\theta^2} - \frac{l^2u^3}{m} = f(\frac{1}{u})$$

Since $f(r) = -\frac{d}{dr}V(r)$, we have $f(\frac{1}{u}) = u^2\frac{d}{du}V(\frac{1}{u})$. Substituting, we finally get:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right) \tag{22}$$

which is the orbit equation for central potentials. [1, p. 87]

From the conservation of energy, we have:

$$\begin{aligned} E =& \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \\ =& \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U(r) \end{aligned}$$

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)} \tag{23}$$

To obtain the shape of the orbit, we need r as a function of θ. We use equation 20 to eliminate time: [1, p. 88]

$$\dot{r} = \frac{l}{mr^2}\frac{dr}{d\theta} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

$$d\theta = \frac{l dr}{mr^2\sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}}$$

$$\theta = \int_{r_0}^{r} \frac{ru}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mU(r)}{l^2} - \frac{1}{r^2}}} + \theta_0$$

To perform the integration, we change variables to $u = \frac{1}{r}$, so that $du = -\frac{1}{r^2}$.

$$\theta = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mU(r)}{l^2} - u^2}} + \theta_0$$

3.1.1 Substitution of Gravitational Potential

We now substitute for $U(r) = -\frac{k}{r} \Rightarrow U(u) = -ku$.

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}$$

The integral is a standard integral, and can be easily evaluated:

$$\begin{aligned}
\int \frac{dx}{\sqrt{ax^2 + bx + c}} =& \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \left(\frac{b}{a}\right) x + \left(\frac{c}{a}\right)}} \\
=& \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \left(\frac{b}{a}\right) x + \left(\frac{b}{2c}\right)^2 - \left(\frac{b}{2c}\right)^2 + \left(\frac{c}{a}\right)}} \\
=& \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\left(\frac{b}{2c}\right)^2 - \frac{c}{a}}\right)^2}} \\
=& \frac{1}{\sqrt{-a}} \int \frac{dx}{\sqrt{\left(\sqrt{\left(\frac{b}{2c}\right)^2 - \frac{c}{a}}\right)^2 - \left(x + \frac{b}{2a}\right)^2}} \\
=& \frac{1}{\sqrt{-a}} cos^{-1} \frac{x + \frac{b}{2c}}{\sqrt{\left(\frac{b}{2c}\right)^2 - \frac{c}{a}}}
\end{aligned}$$

Substituting $a = -1, b = \frac{2mku}{l^2}$,and $c = \frac{2mE}{l^2}$, we end up with:

$$\theta = \theta_0 - cos^{-1} \frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}$$

$$\frac{1}{r} = \frac{mk}{l^2} \left(1 + \sqrt{1 + \frac{2El^2}{mk^2}} cos\left(\theta - \theta_0\right)\right) \tag{24}$$

The standard equation for a conic section with eccentricity e is [1, p. 94]:

$$\frac{1}{r} = C\left(1 + e\, cos\left(\theta - \theta_0\right)\right)$$

We will only concern ourselves with orbits with $E < 0$. These orbits will be ellipses with:

$$\text{eccentricity} = e = \sqrt{1 + \frac{2El^2}{mk^2}} \tag{25}$$

$$\text{semimajor axis} = a = \frac{-k}{2E} \tag{26}$$

3.2 The Hamiltonian Framework

Since the conditions in section 2.1 are met, the Hamiltonian is just the total energy of the system. Working in spherical polar coordinates,

$$\begin{aligned} H &= T + V = \tfrac{1}{2}mv^2 + U(r) \\ &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\phi}^2) + U(r) \end{aligned}$$

The conjugate momenta are:

$$\begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \\ p_\phi &= mr^2 sin^2\theta\dot{\phi} \end{aligned}$$

Substituting in the Hamiltonian, we have:

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}) + U(r) \tag{27}$$

Using the separation of variables, the characteristic function will be of the form:

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

[1, p. 450]
Since the ϕ coordinate is cyclic in the Hamiltonian, the conjugate momentum p_ϕ will be conserved. Since $p_\phi = \frac{\partial W_\phi}{\partial \phi}$, we have

$$W_\phi = \alpha_\phi \phi \tag{28}$$

Substituting the above expression and $p_r = \frac{\partial W_r}{\partial r}$ and $p_\theta = \frac{\partial W_\theta}{\partial \theta}$ in the Hamiltonian, we have

$$\frac{1}{2m}\left(\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{sin^2\theta}\right] + U(r)\right) = H$$

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{a_\phi^2}{sin^2\theta}\right] + 2mU(r) = 2mE \tag{29}$$

where we substituted the Hamiltonian with the total energy E.
The quantity inside the square brackets only depends on θ, and the rest of the RHS depends only on r. Together, they give a constant $2mE$. Thus, the quantity in the square brackets must be a constant: [1, p. 450]

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{sin^2\theta} = \alpha_\theta^2 \tag{30}$$

The Hamiltonian thus becomes

$$H = \frac{1}{2m}\left(p_r^2 + \frac{\alpha_\theta^2}{r^2}\right) + U(r)$$

which reveals that $\alpha_\theta = l$, the magnitude of the total angular momentum.

Substituting equation 23 in equation 22,

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m(E - U(r))$$

$$\frac{\partial W_r}{\partial r} = \sqrt{2m(E - U(r)) - \frac{\alpha_\theta^2}{r^2}}$$

$$W_r = \int \sqrt{2m(E - U(r)) - \frac{\alpha_\theta^2}{r^2}}\, dr \tag{31}$$

[1]

3.3 Action Angle Variable Setup

Since the characteristic function is split into three part, we will have three action variables:

$$J_\phi = \oint \frac{\partial W_\phi}{\partial \phi} d\phi = \oint \alpha_\phi d\phi \tag{32}$$

$$J_\theta = \oint \frac{\partial W_\theta}{\partial \theta} d\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{sin^2\theta}}\, d\theta \tag{33}$$

$$J_r = \oint \frac{\partial W_r}{\partial r} = \oint \sqrt{2m(E - U(r)) - \frac{a_\theta^2}{r^2}}\, dr \tag{34}$$

The first integral, equation 25, is trivial:

$$J_\phi = \int_0^{2\pi} \alpha_\phi d\phi = 2\pi\alpha_\phi \tag{35}$$

[1, p. 467]

3.3.1 Evaluation of J_θ

We can write equation 26 as:

$$J_\theta = \alpha_\theta \oint \sqrt{1 - \frac{\alpha_\phi^2}{\alpha_\theta^2\, sin^2\theta}}\, d\theta = \alpha_\theta \oint \sqrt{1 - cos^2 i\, cosec^2\theta}\, d\theta$$

where $cos\, i = \frac{\alpha_\phi}{\alpha_\theta}$. (We will see later why we have used i for this particular expression.)
We see that the integrand vanishes if $cos\, i\, cosec\, \theta_0 = 1$ for some value of θ_0. Hence the limits of integration are from $-\theta_0$ to $+\theta_0$ and back again.

$$\begin{aligned} J_\theta =& 2\alpha_\theta \int_{-\theta_0}^{+\theta_0} \sqrt{1 - cos^2 i\, cosec^2\theta}\, d\theta \\ =& 4\alpha_\theta \int_0^{+\theta_0} cosec\, \theta \sqrt{sin^2 i - cos^2\theta}\, d\theta \end{aligned}$$

since the integrand is even.
To evaluate the integral, we substitute $cos\, \theta = sin\, i\, sin\, \psi$, so $d\theta = \frac{sin\, i\, cos\, \psi\, d\psi}{sin\, \theta} = \frac{sin\, i\, cos\, \psi\, d\psi}{\sqrt{1 - sin^2 \imath\, sin^2\psi}}$. The limits of integration become 0 to $\frac{\pi}{2}$ since $cos\, i = sin\, \theta_0$. [1, p. 468]

$$\begin{aligned} J_\theta =& 4\alpha_\theta \int_0^{\theta_0} \sqrt{sin^2 i - sin^2 i\, sin^2\psi} \frac{d\theta}{sin\, \theta} \\ =& 4\alpha_\theta sin\, i \int_0^{\frac{\pi}{2}} \sqrt{1 - sin^2\psi} \frac{sin\, i\, cos\, \psi}{\sqrt{1 - sin^2 i\, sin^2\psi}} \frac{d\psi}{\sqrt{1 - sin^2 \imath\, sin^2\psi}} \\ =& 4\alpha_\theta sin^2 i \int_0^{\frac{\pi}{2}} \frac{cos^2\psi\, d\psi}{1 - sin^2 i\, sin^2\psi} \end{aligned}$$

The substitution $u = tan\psi$, $du = sec^2\psi\, d\psi$ changes the limits from 0 to ∞. The expression $cos^2\psi$ becomes $\frac{1}{1+u^2}$, and $\frac{d\psi}{1 - sin^2 i\, sin^2\psi} = \frac{d\psi}{1-(1-cos^2 i) sin^2\psi} = \frac{d\psi}{cos^2\psi + cos^2 i\, sin^2\psi} = \frac{sec^2\psi d\psi}{1 + tan^2\psi\, cos^2 i} = \frac{du}{1+u^2 cos^2 i}$ The integral becomes:

$$J_\theta = 4\alpha_\theta sin^2 i \int_0^\infty \frac{du}{(1+u^2)(1+u^2 cos^2 i)}$$

Writing $sin^2 i = 1 - cos^2 i$ and splitting the integral, we get:

$$J_\theta = 4\alpha_\theta \int_0^\infty du \left(\frac{1}{1+u^2} - \frac{cos^2 i}{1+u^2 cos^2 i} \right) = 4\alpha_\theta \int_0^\infty du \left(\frac{1}{1+u^2} - \frac{1}{sec^2 i + u^2} \right)$$

Both these integrals are standard integrals.

$$\begin{aligned} J_\theta =& 4\alpha_\theta \left(tan^{-1} u - \frac{1}{sec\, i} tan^{-1}\left(\frac{u}{sec\, i} \right) \right) \Big|_0^\infty \\ =& 4\alpha_\theta (\frac{\pi}{2} - \frac{\pi}{2} cos\, i) \end{aligned}$$

Substituting $cos\, i = \frac{\alpha_\phi}{\alpha_\theta}$, we finally get:

$$J_\theta = 2\pi(\alpha_\theta - \alpha_\phi) \tag{36}$$

3.3.2 Evaluation of J_r

We can use equations 28 and 29 to express α_θ in terms of the J's:

$$\begin{aligned} \alpha_\theta =& \frac{J_\theta}{2\pi} + \alpha_\phi \\ =& \frac{J_\theta + J_\phi}{2\pi} \end{aligned}$$

Sso, equation 27 becomes

$$J_r = \oint \sqrt{2m(E - U(r)) - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2}}\, dr$$

Note that J_θ and J_ϕ occur in the expression for E only in the combination of $J_\theta + J_\phi$. Thus, central force motion is at least singly degenerate. This was found without invoking any specific form of $U(r)$; it is a property of central force motion due to the conservation of angular momentum. The frequencies in θ and ϕ must be equal, implying that the motion is planar. This is expected, from the conservation of angular momentum. [1, p. 468]

We now substitute for the gravitational potential $U(r) = -\frac{k}{r}$.

$$J_r = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2}}\, dr \tag{37}$$

This integral is best solved by the method of contour integrals. Since it is an important topic, we shall go through the method in detail.

The **Laurent Theorem** states that if a function f is analytic (infinitely differentiable) within the annulus $r_1 < |z - z_0| < r_2$, then $f(z)$ can be represented by the series

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k$$

This is similar to the Taylor expansion of real valued functions, but the Laurent series can be applied to complex valued functions as well. The important difference is that the summation goes from $-\infty$ to $+\infty$, instead of 0 to $+\infty$ as in the case of Taylor series. [2]

If the function $f(z)$ has a term $\frac{a_{-1}}{z-z_0}$, then it is said to have a pole at the point $z = z_0$, and the coefficient a_{-1} is called the residue of $f(z)$ at z_0, denoted

by $Res\,(f(z), z_k)$.
The residue of a function at a pole of order m can be evaluated by the formula:

$$Res\,(f(z), z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]_{z=z_0}$$

[5]
The **Residue Theorem** states that if a function f is analytic everywhere in a closed contour C except at a finite number of isolated singularities within C (called z_1, z_2, $\ldots z_n$), then:

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res\,(f(z), z_k)$$

In other words, the integral of the function over the closed contour is $2\pi i$ times the summation of the residues of the function at the singularities within the contour. [**?**]

In order to perform the integration in equation 36, we first realize that the integration is from r_1 to r_2 and back again, where r_1 and r_2 are the roots of the integrand. In the first half from r_1 to r_2, we must take the positive square root, while from r_2 to r_1 we must take the negative square root.
Let the integrand be represented by

$$f(r) = \sqrt{A + \frac{B}{r} + \frac{C}{r^2}} = \frac{1}{r}\sqrt{Ar^2 + Br + C}$$

There are only two singular points, 0 and ∞. For the evaluation of the residue at $r = 0$, we must take the positive square root. It is clear that the function has a simple pole (pole of order 1) at $r = 0$, so we can evaluate the residue by:

$$\begin{aligned} Res(f(r), 0) &= \lim_{r \to 0} (r)\, f(r) \\ &= \lim_{r \to 0} \sqrt{Ar^2 + Br + C} \\ &= \sqrt{C} \end{aligned}$$

To evaluate the residue at ∞, we must take the negative square root. The variable of integration is changed to $z = \frac{1}{r} \Rightarrow dz = -\frac{1}{r^2} \Rightarrow dr = -\frac{1}{z^2}$ and the residue is evaluated at $z = 0$. So, the integrand becomes:

$$f(z) = \frac{1}{z^2}\sqrt{A + Bz + Cz^2}$$

Note that the function now has a pole of order 2 at $z = 0$. The residue at 0 is:

$$\begin{aligned} Res(f(z), 0) =& \frac{d}{dz} \left(z^2 f(z)\right)_{z=0} \\ =& \frac{d}{dz} \left(\sqrt{A + Bz + Cz^2}\right)_{z=0} \\ =& \frac{1}{2\sqrt{A + Bz + Cz^2}} \left(B + 2Cz\right)_{z=0} \\ =& \frac{B}{2\sqrt{A}} \end{aligned}$$

So, the value of the integral over the contour is

$$J_r = 2\pi i \left(\sqrt{C} + \frac{B}{2\sqrt{A}}\right)$$

Substituting $A = 2mE$, $B = 2mk$, and $C = -\frac{(J_\theta + J_\phi)^2}{4\pi^2}$, we finally get:

$$J_r = -(J_\theta + J_\phi) + \pi k \sqrt{\frac{2m}{-E}} \tag{38}$$

or,

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2} = H$$

We see that all three of the action angle variables occur only together, in the form $J_r + J_\theta + J_\phi$. Thus, motion under the gravitational potential is completely degenerate. If the motion is completely degenerate, it takes the same time for all the variables to return to their initial values. Thus, a completely degenerate orbit (as in the case of the Kepler problem) corresponds to a closed planar orbit. [1, p. 470] There is only one frequency of motion:

$$\nu_r = \nu_\theta = \nu_\phi = \frac{\partial J_r}{\partial r} = \frac{\partial J_\theta}{\partial \theta} = \frac{\partial J_\phi}{\partial \phi} = \frac{4\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^3} \tag{39}$$

Substituting for $J_r + J_\theta + J_\phi$,

$$\nu = \frac{1}{\pi k}\sqrt{\frac{-2E^3}{m}} \Rightarrow \tau = \pi k \sqrt{\frac{m}{-2E^3}}$$

where τ is the time period. Recognising that $E = \frac{-k}{2a}$ from equation 24, where a is the semimajor axis, we get

$$\tau = 2\pi \sqrt{\frac{ma^3}{k}} \tag{40}$$

which is Kepler's third law of planetary motion.

3.4 The Orbit Elements

Since the Kepler problem has three degrees of freedom, there must be six constants of motion [1, p. 473].

- The semimajor axis a
- The eccentricity e
- The longitude of ascending node Ω
- The argument of perihelion ω
- The inclination i
- A phase constant T

The first two determine the shape and size of the ellipse, while the next three determine the orientation of the ellipse in space. The last constant is simply the initial moment of time at which the body is at the point of perihelion, and is not related to the action angle variables.
The parameters specifying the orientation of the orbit are shown in following diagram:

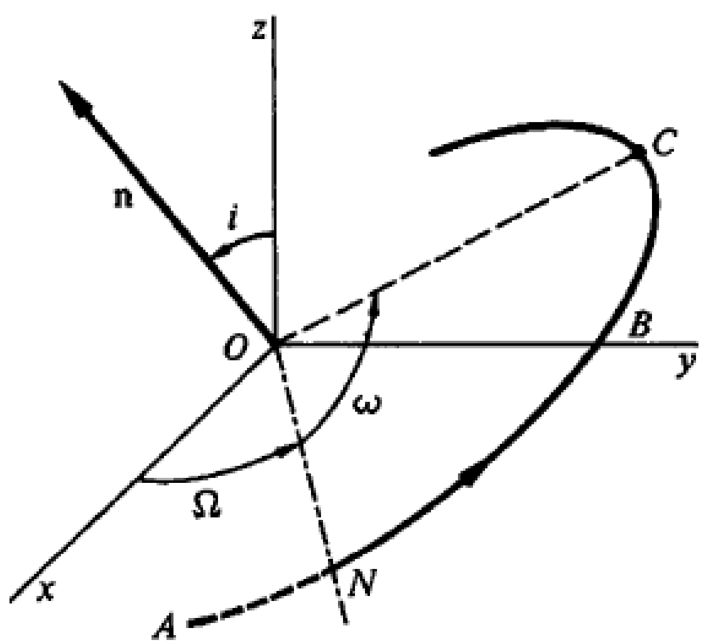

Figure 1: Angular elements in the Kepler problem (figure taken from [1, p. 474])

3.5 Constants of Motion

Equation 31 gave us degenerate frequencies. To lift the degeneracy, we use a canonical transformation.
We can express the degeneracy condition in section 3.3 as $\nu_\phi - \nu_\theta = 0$ and $\nu_\theta - \nu_r = 0$. We now define new angle variables as $\omega_1 = \omega_\phi - \omega_\theta$, $\omega_2 = \omega_\theta - \omega_r$, and $\omega_3 = \omega_r$. These angle variables will have corresponding action variables J_1, J_2, and J_3.
(The reason for using these particular angle variables is that, as we will see earlier, the polar angular momentum, total angular momentum, and total energy can be conveniently expressed using J_1, J_2, and J_3 respectively.)

A generating function in action angle variables will have the form $F = \omega_1 J_1 + \omega_2 J_2 + \ldots + \omega_n J_n$ so that, according to equation 13, we will have $\omega_i = \frac{\partial F}{\partial J_i}$. Applying this to the Kepler problem, we obtain the generating function:

$$F = (\omega_\phi - \omega_\theta)J_1 + (\omega_\theta - \omega_r)J_2 + \omega_r J_3$$

[1, p. 471]
This gives:

$$\begin{aligned}
\omega_1 =& \frac{\partial F}{\partial J_1} = \omega_\phi - \omega_\theta \\
\omega_2 =& \frac{\partial F}{\partial J_2} = \omega_\theta - \omega_r \\
\omega_3 =& \frac{\partial F}{\partial J_3} = \omega_r
\end{aligned}$$

Using equation 14, we obtain expressions for the old angle variables in terms of the new angle variables:

$$\begin{aligned}
J_\phi =& \frac{\partial F}{\partial \omega_\phi} = J_1 \\
J_\theta =& \frac{\partial F}{\partial \omega_\theta} = J_2 - J_1 \\
J_r =& \frac{\partial F}{\partial \omega_r} = J_3 - J_2
\end{aligned}$$

which can be inverted to give:

$$\begin{aligned}
J_1 =& J_\phi \\
J_2 =& J_\theta + J_\phi \\
J_3 =& J_r + J_\theta + J_\phi
\end{aligned}$$

[1, p. 471]
We see that $\dot{\omega_1} = \dot{\omega_\phi} - \dot{\omega_\theta} = \nu_\phi - \nu_\theta = 0$, thus ω_1 is a constant. Similarly, ω_2 is

also a constant.
It is also noted that $J_1 = J_\phi = 2\pi p_\phi$, so J_1 is a constant. Similarly, $J_2 = J_\phi + J_\theta = 2\pi p_\phi + 2\pi(p_\theta - p_\phi) = 2\pi p_\theta$, so J_2 is also a constant (since p_ϕ and p_θ are constants).
The Hamiltonian is given by:

$$H = -\frac{2\pi^2 mk^2}{J_3^2} = E$$

so, J_3 is also a constant.

Thus, we have derived five constants of motion in terms of the action angle variables: J_1, J_2, J_3, ω_1, and ω_2. We now relate these constants to the orbit elements given in section 3.4.

3.5.1 Relations of Action Variables

We know that $J_2 = J_\theta + J_\phi = 2\pi(\alpha_\theta + \alpha_\phi) + 2\pi(\alpha_\phi) = 2\pi\alpha_\theta = 2\pi l$. So, we have:

$$\frac{J_1}{J_2} = \frac{\alpha_\phi}{\alpha_\theta} = cos\, i \tag{41}$$

We see that this physically makes sense, as $\frac{\alpha_\phi}{\alpha_\theta}$ is the ratio of the polar angular momentum to the total angular momentum, which will give the cosine of the inclination of the orbit.

As we saw earlier, the total energy can be expressed as a function of J_3 alone. Using this, and equation 24, we get:

$$a = -\frac{k}{2E} = -\frac{k}{2(-\frac{2\pi^2 mk^2}{J_3^2})}$$

$$a = \frac{J_3^2}{4\pi^2 mk^2} \tag{42}$$

Writing the total angular momentum in terms of J_2, we substitute in equation 23 to get:

$$e = \sqrt{1 - \frac{J_2^2}{4\pi^2 mka}}$$

Using equation 33, we substitute for a to get:

$$e = \sqrt{1 - \left(\frac{J_2}{J_3}\right)^2} \tag{43}$$

[1, p. 475]

3.5.2 Relation of ω_1

As per the definition of angle variables,

$$\omega_1 = \frac{\partial W}{\partial J_1}$$

We separate the variables to write W as

$$W = \int p_\phi d\phi + \int p_\theta d\theta + \int p_r dr$$

We know that the radial momentum only involves the total angular momentum (J_2) and the energy (J_3), hence $\frac{\partial p_r}{\partial J_1} = 0$.
Next, we have:

$$p_\phi = \alpha_\phi = \frac{J_1}{2\pi}$$

$$\frac{\partial}{\partial J_1} \int p_\phi d\phi = \frac{1}{2\pi} \int d\phi = \frac{\phi}{2\pi}$$

Finally, we have

$$p_\theta = \pm\sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{sin^2\theta}} = \pm\frac{1}{2\pi}\sqrt{J_2^2 - \frac{J_1^2}{sin^2\theta}}$$

It will turn out that we must select the negative root for physical significance.

$$\begin{aligned}
\frac{\partial}{\partial J_1} \int p_\theta d\theta = &- \frac{1}{2\pi} \int \frac{1}{2\sqrt{J_2^2 - \frac{J_1^2}{sin^2\theta}}} \frac{-2J_1}{sin^2\theta} \, d\theta \\
=&\frac{1}{2\pi}\frac{J_1}{J_2} \int \frac{d\theta}{sin^2\theta\sqrt{1 - \left(\frac{J_1}{J_2}\right)^2 cosec^2\theta}} \\
=&\frac{1}{2\pi} cos\, i \int \frac{d\theta}{sin^2\theta\sqrt{1 - cos^2 i \, cosec^2\theta}} \\
=&\frac{1}{2\pi} cot\, i \int \frac{cosec^2\theta \, d\theta}{\sqrt{cosec^2 i - cot^2 i \, cosec^2\theta}} \\
&\text{(dividing the numerator and denominator by } sin\, i\text{)} \\
=&\frac{1}{2\pi} cot\, i \int \frac{cosec^2\theta \, d\theta}{\sqrt{cosec^2 i - cot^2 i \ (cot^2\theta + 1)}} \\
=&\frac{1}{2\pi} cot\, i \int \frac{cosec^2\theta \, d\theta}{\sqrt{1 - cot^2 i \, cot^2\theta}}
\end{aligned}$$

We change the independent variable from θ to u, defined by $sin\, u = cot\, i\, cot\, \theta \Rightarrow cos\, u\, du = -cot\, i\, cosec^2\theta$. [1, p. 476]

$$\begin{aligned}\frac{\partial}{\partial J_1}\int p_\theta d\theta =& \frac{1}{2\pi}\int \frac{-cos\, u\, du}{\sqrt{1 - sin^2 u}} \\ =& -\frac{1}{2\pi}\int du \\ =& -\frac{u}{2\pi}\end{aligned}$$

So, finally we have:

$$\begin{aligned}\omega_1 &= \frac{\phi}{2\pi} - \frac{u}{2\pi} \\ 2\pi\omega_1 &= \phi - u\end{aligned}$$

We now investigate the physical significance of this relationship. The detailed orbit diagram is shown on the next page.

Clearly, $BC = OB\, cos\, \theta$ and $AC = BC\, cot\, i = OB\, cos\, \theta\, cot\, i$. On the other

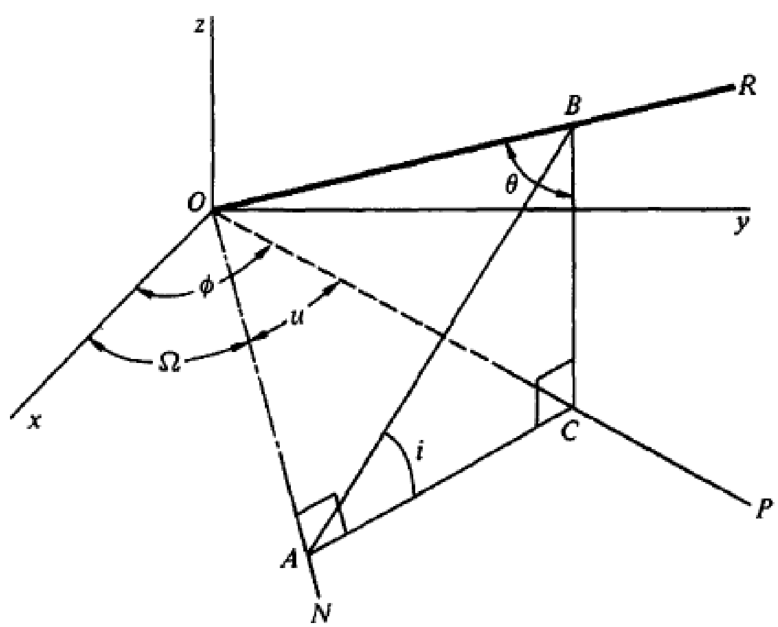

Figure 2: Relations between angle variables (figure taken from [1, p. 477])

hand, $OC = OB\, sin\, \theta$ so $AC = OC\, sin\, u = OB\, sin\, u\, sin\, \theta$. Equating these two expressions for AC, we get $sin\, u = cot\, i\, cot\, \theta$. From the figure, we see that $\phi - u = \Omega$, so we have:

$$2\pi\omega_1 = \Omega \tag{44}$$

[1]

3.5.3 Relation of ω_2

We have

$$\omega_2 = \frac{\partial W}{\partial J_2} = \frac{\partial}{\partial J_2}\left(\int p_\phi d\phi + \int p_\theta d\theta + \int p_r dr\right)$$

We know that p_ϕ only involves J_1, so that term will not contribute.
The second term yields:

$$\frac{\partial}{\partial J_2}\int p_\theta d\theta = \frac{\partial}{\partial J_2}\int \pm\frac{1}{2\pi}\sqrt{J_2^2 - \frac{J_1^2}{sin^2\theta}}d\theta$$

We must choose the negative sign for physical significance.

$$= -\frac{1}{2\pi}\int \frac{1}{2\sqrt{J_2^2 - \frac{J_1^2}{sin^2\theta}}}\,2J_2 d\theta$$

$$= -\frac{1}{2\pi}\int \frac{d\theta}{\sqrt{1 - cos^2 i\, cosec^2\theta}}$$

$$= -\frac{1}{2\pi}\int \frac{sin\,\theta\; d\theta}{\sqrt{sin^2\theta - cos^2 i}}$$

$$= -\frac{1}{2\pi}\int \frac{sin\,\theta\; d\theta}{\sqrt{1 - cos^2\theta - 1 + sin^2 i}}$$

$$= -\frac{1}{2\pi}\int \frac{sin\,\theta\; d\theta}{\sqrt{sin^2 i - cos^2\theta}}$$

Subsituting $cos\theta = u \Rightarrow -sin\theta\; d\theta = du$ we get:

$$= \frac{1}{2\pi}\int \frac{du}{\sqrt{sin^2 i - u^2}}$$

$$= \frac{1}{2\pi} sin^{-1}\left(\frac{u}{sin\,i}\right)$$

$$= \frac{1}{2\pi} sin^{-1}\left(\frac{cos\,\theta}{sin\,i}\right)$$

We see from the previous figure that $cos\,\theta = \frac{BC}{OB}$ and $sin\,i = \frac{BC}{AB}$, so $\frac{cos\,\theta}{sin\,i} = \frac{AB}{OB}$.
The sine inverse of this gives the angle between the radius vector OR and the line of nodes OA.

The final term is:

$$\begin{aligned}
\frac{\partial}{\partial J_2}\int p_r dr =& \frac{\partial}{\partial J_2}\int \frac{\partial J_r}{\partial r} dr \\
=& \frac{\partial}{\partial J_2}\int \sqrt{2mE + \frac{2mk}{r} - \frac{J_2^2}{4\pi^2 r^2}} dr \\
=& \int \frac{1}{2\sqrt{2mE + \frac{2mk}{r} - \frac{J_2^2}{4\pi^2 r^2}}} \frac{-2J_2}{4\pi^2 r^2} dr
\end{aligned}$$

Changing variables to $u = \dfrac{1}{r}$ and using $J_2 = 2\pi l$,

$$\begin{aligned}
=& \frac{1}{2\pi}\int \frac{ldu}{\sqrt{2mE + 2mku - l^2u^2}} \\
=& \frac{1}{2\pi}\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}
\end{aligned}$$

This integral is in the same form as in section 3.1, and we directly read off the result:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-a}} cos^{-1} \frac{x + \frac{b}{2c}}{\sqrt{\left(\frac{b}{2c}\right)^2 - \frac{c}{a}}}$$

Thus, we have:

$$\begin{aligned}
\frac{\partial}{\partial J_2}\int p_r dr =& \frac{1}{2\pi} cos^{-1} \frac{\frac{l^2u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}} \\
=& \frac{1}{2\pi} cos^{-1} \frac{au(1-e^2) - 1}{e}
\end{aligned}$$

We see from the diagram that this is $-\frac{1}{2\pi}$ times the angle between the radius vector and line of periapsis. So we have:

$$\begin{aligned}
2\pi\omega_2 =& \text{ angle between radius vector and line of nodes}- \\
& \text{angle between radius vector and line of periapsis} \\
2\pi\omega_2 =& \text{ angle between line of nodes and line of periapsis}
\end{aligned}$$

$$2\pi\omega_2 = \omega \tag{45}$$

So, we have related all the constants of motion with the orbit elements.

3.6 The Delaunay Variables

As we saw earlier, there are six orbital elements in the two body Kepler problem: T, Ω, ω, i, a, and e. Using these, we can define new variables:

The gravitational potential is directly proportional to the mass of the body, so we can express it as $U(r) = -\frac{k}{r} = -\frac{\mu m}{r}$. The total energy becomes $E = -\frac{\mu m}{2a}$, and this is conjugate to the phase constant T.
The eccentricity can be expressed in terms of the total angular momentum as:

$$e = \sqrt{1 - \frac{l^2}{mka}}$$

which gives us

$$l = \sqrt{mka(1-e^2)} = \sqrt{m^2\mu a(1-e^2)} = G$$

The polar angular momentum can now be expressed as

$$H = l\,cos\,i = \sqrt{m^2\mu a(1-e^2)}\,cos\,i$$

The mass of the body m is just a multiplicative constant which appears in the momenta, and can be omitted. Thus, we now have the elements:

$$\begin{aligned} l' &= -T \\ h &= \Omega \\ g &= \omega \\ L' &= -\frac{\mu}{2a} \\ H &= \sqrt{\mu a(1-e^2)}\,cos\,i \\ G &= \sqrt{\mu a(1-e^2)} \end{aligned}$$

[7, p. 108]
Since both Ω and ω are angles, we attempt to transform T to an angle as well, and find a conjugate momentum.
The mean anomaly M is defined as the angle between the perihelion and the radius vector. assuming that the body moves with constant angular velocity, we have

$$M = 2\pi\frac{t-T}{\tau}$$

where τ is the time period of revolution. [7, p. 60]

We see that H is the conjugate momentum to Ω and G is the conjugate momentum to ω. We now attempt to find a momentum L conjugate to M. Similar to the previous section, a generating function for this transformation must have the form $F = \omega_1\,J_1 + \omega_2\,J_2 + \ldots + \omega_n\,J_n$, so that $J_1 = \frac{\partial F}{\partial \omega_1}$. The generating function which will yield this is:

$$F = \left(\frac{2\pi}{\tau}L - \frac{3\mu}{2a}\right)(t+l) + \Omega H + \omega G$$

[7, p. 109]

$$
\begin{aligned}
L' =& \frac{\partial F}{\partial l} \\
-\frac{\mu}{2a} =& \frac{2\pi}{\tau} L - \frac{3\mu}{2a} \\
L =& \frac{\tau}{2\pi} \left(-\frac{\mu}{2a} + \frac{3\mu}{2a} \right) \\
=& \frac{\tau}{2\pi} \frac{\mu}{a}
\end{aligned}
$$

Substituting for τ from equation 39,

$$
\begin{aligned}
L =& \frac{\mu}{a\sqrt{\mu} a^{-\frac{3}{2}}} \\
=& \sqrt{a\mu}
\end{aligned}
$$

In terms of L, the new Hamiltonian is given by equation 9 (section 2.3).

$$
\begin{aligned}
K =& H + \frac{\partial F}{\partial t} \\
=& 0 + \left(\frac{2\pi}{\tau} L - \frac{3\mu}{2a} \right) \\
=& \frac{\mu}{a} - \frac{3\mu}{2a} = -\frac{\mu}{2a} \\
=& -\frac{\mu^2}{2L^2}
\end{aligned}
$$

(The old Hamiltonian H was constructed to be 0)

So, we have obtained the set of quantities:

$$
\begin{aligned}
l =& M \\
h =& \Omega \\
g =& \omega \\
L =& \sqrt{a\mu} \\
H =& \sqrt{\mu a(1-e^2)}\, cos\, i \\
G =& \sqrt{\mu a(1-e^2)} \\
K =& -\frac{\mu^2}{2L^2}
\end{aligned}
$$

These are called **Delaunay's variables** and are frequently used in celestial mechanics. [7, p. 109]

4 The Lidov-Kozai Mechanism

The Lidov-Kozai mechanism describes the system in which there is a central binary which is perturbed by a distant third body. The general method is to introduce a small perturbing function R in the Hamiltonian.

$$\mathcal{H} = \mathcal{H}_0 - R$$

4.1 The Three-Body Perturbing Function

Let the binary system be comprised of masses m_1 and m_2, which is perturbed by a distant body, called the third body or perturbing body, of mass m_3. The diagram is shown below:

The Hamiltonian of the system is given by

$$\mathcal{H} = -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3}$$

where the potential energy is

$$\begin{aligned} V &= -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \\ &= -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{R_3 + \frac{m_2}{m_1+m_2}r} - \frac{Gm_2m_3}{R_3 - \frac{m_1}{m_1+m_2}r} \end{aligned}$$

We now evaluate the last two terms using multipole expansion. The method is outlined below:

$$\begin{aligned} \frac{1}{|r-r'|} &= \left(r^2 + r'^2 - 2rr'\cos\psi\right)^{-\frac{1}{2}} \\ &= \frac{1}{r}\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\psi\right)^{-\frac{1}{2}} \\ &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\psi) \end{aligned}$$

where ψ is the angle between r and r', and P_n is the n^{th} Legendre polynomial. To quadrapole approximation, the expression for potential energy becomes

$$\begin{aligned}
V =& -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{R_3}\sum_{n=2}^{\infty}\left(-\frac{m_2}{m_1+m_2}\right)^2\left(\frac{r}{R_3}\right)^n P_n(\cos\psi) \\
& - \frac{Gm_2m_3}{R_3}\sum_{n=2}^{\infty}\left(\frac{m_1}{m_1+m_2}\right)^2\left(\frac{r}{R_3}\right)^n P_n(\cos\psi) \\
=& \frac{Gm_1m_2}{r} - \frac{Gm_3}{R_2}\Bigg[(m_1+m_2) + \left(-\frac{m_1m_2}{m_1+m_2} + \frac{m_1m_2}{m_1+m_2}\right)\frac{r}{R_3}\cos\psi \\
& + \left(\frac{m_1m_2^2}{(m_1+m_2)^2} + \frac{m_2m_1^2}{(m_1+m_2)^2}\right)\left(\frac{r}{R_3}\right)^2\left(\frac{3\cos^2\psi - 1}{2}\right)\Bigg] \\
=& \frac{Gm_1m_2}{r} - \frac{Gm_3(m_1+m_2)}{R_3} - \frac{Gm_1m_2m_3}{2(m_1+m_2)R_3}\left(\frac{r}{R_3}\right)^2(3\cos^2\psi - 1)
\end{aligned}$$

[7, p. 226] The first term corresponds to the potential energy of the binary, and the second corresponds to the potential energy of the binary-perturber system. The last term is the perturbing function (to quadrapole order).

$$R = \frac{Gm_1m_2m_3}{2(m_1+m_2)R_3}\left(\frac{r}{R_3}\right)^2(3\cos^2\psi - 1) \tag{46}$$

4.2 The Lagrangian Planetary Equations

In the restricted three-body problem, we assume that the Hamiltonian of the system is nearly the same as the Hamiltonian of the two body problem, but perturbed by a small perturbing function (this was calculated in the previous section). Thus, all the Delaunay variables that we calculated in the previous chapter will also be applicable for the central binary of the three body problem as well, but they will have small perturbations due to the third body.

In the previous chapter, we found that for the two body problem the Delaunay variables are constants for the system. Once we introduce the perturbing function, they will no longer be constants. We now attempt to find how the Delaunay variables evolve with time.
In terms of Delaunay variables, the Hamiltonian can be expressed as:

$$\mathcal{H} = -\frac{\mu^2}{2L^2} - R \tag{47}$$

Assuming that all the orbit elements (section 3.4) depend on time, the time derivatives of the Delaunay variables are:

$$
\begin{aligned}
\dot{L} =& \frac{d}{dt}\sqrt{\mu a} = \frac{1}{2}\sqrt{\frac{\mu}{a}}\dot{a} \\
\dot{G} =& \frac{d}{dt}\sqrt{\mu a(1-e^2)} = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\dot{a} - \frac{\sqrt{\mu a}e}{\sqrt{(1-e^2)}}\dot{e} \\
\dot{H} =& \frac{d}{dt}\sqrt{\mu a(1-e^2)}\cos\iota = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\cos\iota\,\dot{a} \\
& - \frac{\sqrt{\mu a}e}{\sqrt{(1-e^2)}}\cos\iota\,\dot{e} - \sqrt{\mu a(1-e^2)}\sin\iota\,\dot{\iota} \\
\dot{l} =& \dot{M} \\
\dot{g} =& \dot{\omega} \\
\dot{h} =& \dot{\Omega}
\end{aligned}
$$

[7, p. 223] (We have used ι (iota) for inclination instead of the English letter i, so that the time derivative will not look obnoxious.)
The definition of action-angle variables gives us $\dot{J} = -\frac{\partial \mathcal{H}}{\partial \omega}$ and $\dot{\omega} = \frac{\partial \mathcal{H}}{\partial J}$. So, we can also express the time derivatives as:

$$
\begin{aligned}
\dot{L} &= -\frac{\partial \mathcal{H}}{\partial l} = \frac{\partial R}{\partial l} \\
\dot{G} &= -\frac{\partial \mathcal{H}}{\partial g} = \frac{\partial R}{\partial g} \\
\dot{H} &= -\frac{\partial \mathcal{H}}{\partial h} = \frac{\partial R}{\partial h} \\
\dot{l} &= \frac{\partial \mathcal{H}}{\partial l} = \frac{\mu^2}{L^3} - \frac{\partial R}{\partial L} \\
\dot{g} &= \frac{\partial \mathcal{H}}{\partial G} = \frac{\partial R}{\partial G} \\
\dot{h} &= \frac{\partial \mathcal{H}}{\partial H} = \frac{\partial R}{\partial H}
\end{aligned}
$$

[7, p. 223] To compare the two sets of equations, we must express the orbit elements in terms of the Delaunay variables. Inverting the set of equations in section 3.6, we have

$$\begin{aligned}
a =& \frac{L^2}{\mu} \\
e =& \sqrt{1 - \frac{G^2}{L^2}} \\
\cos\iota =& \frac{H}{G} \\
l =& M \quad g = \omega \quad h = \Omega
\end{aligned}$$

The various derivatives which will be used are:

$$\begin{aligned}
\frac{\partial a}{\partial L} =& \frac{2L}{\mu} \\
\frac{\partial e}{\partial L} =& \frac{\frac{G^2}{L^3}}{\sqrt{1 - \frac{G^2}{L^2}}} \\
\frac{\partial e}{\partial G} =& \frac{-\frac{G}{L^2}}{1 - \frac{G^2}{L^2}} \\
\frac{\partial \iota}{\partial G} =& \frac{H}{G^2 \sin\iota} \\
\frac{\partial \iota}{\partial H} =& \frac{-1}{G \sin\iota}
\end{aligned}$$

Using this, we can evaluate the necessary derivates of R:

$$\begin{aligned}
\frac{\partial R}{\partial L} =& \frac{\partial R}{\partial a}\frac{\partial a}{\partial L} + \frac{\partial R}{\partial e}\frac{\partial e}{\partial L} = \frac{\partial R}{\partial a}\frac{2L}{\mu} + \frac{\partial R}{\partial e}\frac{\frac{G^2}{L^3}}{\sqrt{1 - \frac{G^2}{L^2}}} \\
=& \frac{\partial R}{\partial a} 2\sqrt{\frac{a}{\mu}} + \frac{\partial R}{\partial e}\frac{1 - e^2}{e\sqrt{\mu a}} \quad \text{since } L = \sqrt{\mu a} \\
\frac{\partial R}{\partial G} =& \frac{\partial R}{\partial e}\frac{\partial e}{\partial G} + \frac{\partial R}{\partial \iota}\frac{\partial \iota}{\partial G} = \frac{\partial R}{\partial e}\frac{-\frac{G}{L^2}}{1 - \frac{G^2}{L^2}} + \frac{\partial R}{\partial \iota}\frac{H}{G^2 \sin\iota} \\
=& \frac{\partial R}{\partial e}\left(-\frac{\sqrt{1 - e^2}}{e\sqrt{\mu a}}\right) + \frac{\partial R}{\partial \iota}\frac{\cos\iota}{\sqrt{\mu a(1 - e^2)}\sin\iota} \\
\frac{\partial R}{\partial H} =& \frac{\partial R}{\partial \iota}\frac{\partial \iota}{\partial H} = \frac{\partial R}{\partial \iota}\frac{-1}{G \sin\iota} = \frac{\partial R}{\partial \iota}\frac{-1}{\sqrt{\mu a(1 - e^2)}\sin\iota}
\end{aligned}$$

Substituting back in the previous equations, we have:

$$
\begin{aligned}
\frac{\partial R}{\partial M} &= \frac{1}{2}\sqrt{\frac{\mu}{a}}\dot{a} \\
\frac{\partial R}{\partial \omega} &= \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\dot{a} - \frac{\sqrt{\mu a}e}{\sqrt{(1-e^2)}}\dot{e} \\
\frac{\partial R}{\partial \Omega} &= \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\cos\iota\,\dot{a} - \frac{\sqrt{\mu a}e}{\sqrt{(1-e^2)}}\cos\iota\,\dot{e} - \sqrt{\mu a(1-e^2)}\sin\iota\,\dot{\iota} \\
\dot{M} &= \frac{\mu^2}{L^3} - \frac{\partial R}{\partial a}2\sqrt{\frac{a}{\mu}} - \frac{\partial R}{\partial e}\frac{1-e^2}{e\sqrt{\mu a}} \\
\dot{\omega} &= \frac{\partial R}{\partial e}\left(\frac{\sqrt{1-e^2}}{e\sqrt{\mu a}}\right) - \frac{\partial R}{\partial \iota}\frac{\cos\iota}{\sqrt{\mu a(1-e^2)}\sin\iota} \\
\dot{\Omega} &= \frac{\partial R}{\partial \iota}\frac{1}{\sqrt{\mu a(1-e^2)}\sin\iota}
\end{aligned}
$$

[7, p. 224] We can invert these equations to obtain the time derivatives of the orbital elements in terms of the perturbing function.
The first equation gives:

$$\dot{a} = 2\sqrt{\frac{a}{\mu}}\frac{\partial R}{\partial M}$$

Substituting this in the second equation, we get

$$\frac{\partial R}{\partial \omega} = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\;2\sqrt{\frac{a}{\mu}}\frac{\partial R}{\partial M} - \frac{\sqrt{\mu a}e}{\sqrt{(1-e^2)}}\dot{e}$$

Solving for $\dot{e}$,

$$\dot{e} = -\frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{e}\sqrt{\frac{1}{\mu a}}\frac{\partial R}{\partial M}$$

Finally, we substitute for $\dot{a}$ and $\dot{e}$ in the third equation and solve for $\dot{\iota}$:

$$\dot{\iota} = -\frac{1}{\sqrt{\mu a(1-e^2)}\sin\iota}\frac{\partial R}{\partial \Omega} + \frac{\cos\iota}{\sqrt{\mu a(1-e^2)}\sin\iota}\frac{\partial R}{\partial \omega}$$

To tidy things up, we substitute for the frequency of motion:

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$\sqrt{\mu a} = a^2 n \qquad \sqrt{\frac{\mu}{a}} = an$$

Finally, we have:

$$
\begin{aligned}
\dot{a} =& \frac{2}{na}\frac{\partial R}{\partial M} \\
\dot{e} =& -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2e}\frac{\partial R}{\partial M} \\
\dot{\imath} =& -\frac{1}{na^2\sqrt{1-e^2}\sin\iota}\frac{\partial R}{\partial \Omega} + \frac{\cos\iota}{na^2\sqrt{1-e^2}\sin\iota}\frac{\partial R}{\partial \omega} \\
\dot{M} =& n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} \\
\dot{\omega} =& \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cos\iota}{na^2\sqrt{1-e^2}\sin\iota}\frac{\partial R}{\partial \iota} \\
\dot{\Omega} =& \frac{1}{na^2\sqrt{1-e^2}\sin\iota}\frac{\partial R}{\partial \iota}
\end{aligned}
$$

These are the **Lagrangian planetary equations**. [7, p. 225]

4.3 Equations of Motion

The expression for the perturbing function R is given by equation 46. However, we do not use this directly in the Lagrangian planetary equations. In the doubly averaged restricted three-body model, we calculate the time average of R, by first integrating over one complete inner cycle (the binary) and one outer cycle (binary-perturber system). The final result is:

$$
\langle\langle R \rangle\rangle = \frac{Gm_1m_2m_3a_i^2}{8m_Ba_p^3(1-e_p^2)^{\frac{3}{2}}}\left[2 + 3e_i^2 - 3\sin^2\iota\left(5e_i^2\sin^2\omega_i + 1 - e_i^2\right)\right] \tag{48}
$$

where

m_1 and m_2 = masses of the binary bodies
m_B = total mass of binary = $m_1 + m_2$
m_3 = mass of perturber
e_i = eccentricity of binary
e_p = eccentricity of perturber
a_i = semimajor axis of binary
a_p = semimajor axis of perturber
ω_i = argument of pericenter of binary

[7, p. 231] (In general, the subscript i means inner binary, while p means perturber.)

Since we have averaged R over both cycles by integrating it over the mean anomalies, it is obvious that both l_i and l_p do not appear in equation 49. Thus,

it is cyclic in these variables and the corresponding canonical momenta L_i and L_p are constants. Since $L = \sqrt{\mu a}$, we see that both the semimajor axes remain constant over the motions.
We also see that the argument of pericenter of the perturber ω_p is absent in equation 48, so the corresponding momentum G_p is constant. Since $e_p = \sqrt{1 - \frac{G_p^2}{L_p^2}}$, we have that the eccentricity of the perturber is also a constant. Since $H = \sqrt{\mu a(1-e^2)}\, cos\, \iota$ is a constant of motion for both the outer and inner orbits, we have that ι_p is also a constant.
Turning our attention to the inner orbit, we see that H_i being a constant implies

$$\sqrt{1-e^2}\ \cos\iota = constant = c_1 \tag{49}$$

[6, p. 35]
Let us now calculate the various derivatives of $\langle\langle R\rangle\rangle$ required for the Lagrangian planetary equations. Just to be explicitly clear, all the orbit elements in the planetary equations are the elements of the inner (binary) orbit only.

$$\begin{aligned}
\frac{\partial\langle\langle R\rangle\rangle}{\partial M_i} =& \frac{\partial\langle\langle R\rangle\rangle}{\partial \Omega_i} = 0 \\
\frac{\partial\langle\langle R\rangle\rangle}{\partial \omega_i} =& -\frac{15}{8} e_i^2 \sin 2\omega_i \sin^2 \iota_i \frac{G m_1 m_2 m_3 a_i^2}{m_B a_p^3 (1-e_p^2)^{\frac{3}{2}}} \\
\frac{\partial\langle\langle R\rangle\rangle}{\partial e_i} =& \frac{3}{4} e_i \left(1 + \sin^2 \iota_i - 5\sin^2 \iota_i \sin^2 \omega_i\right) \frac{G m_1 m_2 m_3 a_i^2}{m_B a_p^3 (1-e_p^2)^{\frac{3}{2}}} \\
\frac{\partial\langle\langle R\rangle\rangle}{\partial \iota_i} =& -\frac{3}{4} \sin\iota_i \cos\iota_i \left(1 - e_i^2 + 5 e_i^2 \sin^2 \omega_i\right) \frac{G m_1 m_2 m_3 a_i^2}{m_B a_p^3 (1-e_p^2)^{\frac{3}{2}}}
\end{aligned}$$

We see that there is a common factor in all these equations, which we denote as:

$$A = \frac{G m_1 m_2 m_3 a_i^2}{m_B a_p^3 (1-e_p^2)^{\frac{3}{2}}}$$

[7, p. 232] Except for this common factor A, all the variables are only variables of the inner orbit. So, we drop the subscript i, declaring that all expressions henceforth only refer to the inner orbit elements.

Substituting these expressions in the Lagrangian planetary equations, we get:

$$\dot{a} = \frac{2}{na}(0) = 0$$

$$\begin{aligned}\dot{e} =& -\frac{\sqrt{1-e^2}}{na^2 e}\left(-\frac{15}{8}e_i^2 \sin 2\omega_i \sin^2 \iota_i A\right) + \frac{1-e^2}{na^2 e}(0)\\ =& \frac{15}{8} e\sqrt{1-e^2}\sin 2\omega \sin^2 \iota \frac{A}{n}\end{aligned}$$

$$\begin{aligned}\dot{\iota} =& -\frac{1}{na^2\sqrt{1-e^2}\sin\iota}(0) + \frac{\cos\iota}{na^2\sqrt{1-e^2}\sin\iota}\left(-\frac{15}{8}e_i^2 \sin 2\omega_i \sin^2 \iota_i A\right)\\ =& -\frac{15}{8}\frac{e^2}{\sqrt{1-e^2}}\sin 2\omega \sin\iota\cos\iota\frac{A}{n}\end{aligned}$$

$$\begin{aligned}\dot{\omega} =& \frac{\sqrt{1-e^2}}{na^2 e}\left(\frac{3}{4}e_i\left(1+\sin^2\iota_i - 5\sin^2\iota_i \sin^2\omega_i\right)A\right) -\\ & \frac{\cos\iota}{na^2\sqrt{1-e^2}\sin\iota}\left(-\frac{3}{4}\sin\iota_i\cos\iota_i\left(1-e_i^2+5e_i^2\sin^2\omega_i\right)A\right)\\ =& \frac{3}{4}\frac{1}{\sqrt{1-e^2}}\left[2(1-e^2)+5\sin^2\omega(e^2-\sin^2\iota)\right]\frac{A}{n}\end{aligned}$$

$$\begin{aligned}\dot{\Omega} =& \frac{1}{na^2\sqrt{1-e^2}\sin\iota}\left(-\frac{3}{4}\sin\iota_i\cos\iota_i\left(1-e_i^2+5e_i^2\sin^2\omega_i\right)A\right)\\ =& -\frac{\cos\iota}{4\sqrt{1-e^2}}\left(3+12e^2-15e^2\cos^2\omega\right)\frac{A}{n}\end{aligned}$$

Since the expression $\frac{A}{n}$ appears in all these equations, we can again factor this out. We recognize that $\frac{A}{n}$ is unitless, so we can define a normalised time: $\tau = \frac{A}{n}t$ to simplify the equations. We have $\frac{d}{dt}\frac{A}{n} = \frac{d}{d\tau}$.[7, p. 232]
Furthermore, we consider the case in which the eccentricity of the binary is small enough so that terms containing e^2 and high order can be neglected. The equations now become:

$$\frac{d\iota}{d\tau} = 0 \tag{50}$$

$$\frac{de}{d\tau} = \frac{15}{8} e\sin 2\omega \sin^2\iota \tag{51}$$

$$\frac{d\omega}{d\tau} = \frac{3}{4}\left(2-5\sin^2\omega\sin^2\iota\right) \tag{52}$$

$$\frac{d\Omega}{d\tau} = -\frac{3}{4}\cos\iota \tag{53}$$

The equation 50 tells us that inclination is a constant (under the approximation that the eccentricity is small). So, we can integrate equation 52 to find ω as a function of time.

If $\sin^2\iota < 0.4 \Rightarrow \iota < 39.2^o$ then the solution is given by

$$\omega = \tan^{-1}\left(\sqrt{\frac{2}{2-5\sin^2\iota}}\tan\left(\frac{3}{4}\sqrt{4-10\sin^2\iota}\,\tau\right)\right)$$

We see that if that as time evolves, i.e. $\tau \to \infty$, ω will still remain periodic. We then say that ω will continuously rotate around, or that it will **circulate**. [6, p. 38]
As an example, let us set $\iota = 33^o \Rightarrow 2 - 5\sin^2\iota = 0.5$. Then, we have approximately $\omega = \frac{3}{2}\tau$[7, p. 235]. Substituting in equation 51, we get

$$\frac{de}{e} = \frac{6}{16}\sin 2\omega d\omega$$
$$e = e_0 \exp\left(-\frac{3}{16}\cos 2\omega\right)$$

Since $-1 \leq \cos 2\omega \leq 1$, e varies sinusoidally between $0.83e_0$ and $1.21e_0$. The variation in e is not very significant.

The more interesting care arises if $\sin^2\iota > 0.4 \Rightarrow \iota > 39.2^o$. Then the solution is given by

$$\omega = tan^{-1}\left(\sqrt{\frac{2}{-A}}\frac{e^{\frac{3}{2}\sqrt{(10\sin^2\iota-4)}\tau}+1}{e^{\frac{3}{2}\sqrt{(10\sin^2\iota-4)}\tau}-1}\right)$$

We see that if that as time evolves, i.e. $\tau \to \infty$, ω will approach a constant value. So, we can set $\frac{d\omega}{d\tau} = 0$ in the previous set of equations to get $5\sin^2\omega\sin^2\iota = 2$. Substituting this in equation 51, we get

$$\frac{de}{d\tau} =\frac{15}{4}e\sqrt{\frac{2}{5}\left(\sin^2\iota - \frac{2}{5}\right)}$$
$$e =e_0 \exp\left(\frac{\frac{4}{15}\sqrt{\frac{5}{2}}\tau}{\sqrt{\sin^2\iota - 0.4}}\right)$$

This shows that the eccentricity will exponentially increase with time. However, this is false, because if eccentricity is large, then the approximation we used to get equations 50-53 will no longer be valid.
On the other hand, the equation $c_1 = \sqrt{1-e^2}\cos\iota$ did not use this approximation, and is always valid. As eccentricity increases, the inclination will decrease so that c_1 is a constant. Since the minimum value of ι for this case is 39.2^o, the

maximum value of eccentricity can be calculated:

$$\sqrt{1-e_0^2}\cos\iota_0 = \sqrt{1-e_{max}^2}\cos\iota_{min} = \sqrt{1-e_{max}^2}\sqrt{\frac{3}{5}}$$
$$e_{max} = \sqrt{1-\frac{5}{3}\cos^2\iota_0}$$

The big idea here is that there is an exchange between the eccentricity and the inclination of the binary orbit. This is a cycle: once e reaches e_{max}, it will start to decrease again until the orbit becomes nearly circular ($e = 0$), while the inclination will increase. The process will repeat again, with eccentricity increasing and inclination decreasing. This is called the **Kozai Cycle**. [7, p. 235]

We can approximate the time taken for the eccentrity to reach e_{max} from e_o:

$$\tau = \frac{4}{15}\sqrt{\frac{5}{2}}\log\frac{\frac{e_{max}}{e_0}}{\sqrt{\sin^2\iota - 0.4}}$$

Let us substitute some typical values: $e_0 = 0.05$ and $\iota = \iota_0 = 60^o$. Then we get $e_{max} = 0.76$. So, substituting these values, we have

$$\tau = 0.518 = \frac{A}{n}t$$
$$t = 1.93\frac{na_p^3 m_B}{Gm_1m_2m_3a_i^3}$$

(as we are ignoring all higher order terms in e_p)
As a final approximation, we consider the case in which the binary consists of one large mass and one small (unit) mass. Then, the expression $\frac{m_B}{m_1m_2}$ is approximately equal to unity.
Substituting for n, we finally have:

$$t \approx 0.3\left(\frac{a_p}{a_i}\right)^3\frac{m_B}{m_3}P$$

where $P = 2\pi\sqrt{\frac{a_i^3}{Gm_B}}$ is the time period of orbit of the inner binary.
The total time period of the Kozai cycle will be four times the time to go from e_0 to e_{max}.

$$P_{Kozai} \approx \left(\frac{a_p}{a_i}\right)^3\frac{m_B}{m_3}P \tag{54}$$

[7, p. 235]

4.4 Classification of Orbits

We saw that one of the constants of motion is $c_1 = \sqrt{1-e^2}\cos\iota$. We can get another constant of motion by using the fact that $\langle\langle R\rangle\rangle$ is time-independent, and therefore a constant:

$$\begin{aligned}\langle\langle R\rangle\rangle =& \frac{Gm_1m_2m_3a_i^2}{8m_Ba_p^3(1-e_p^2)^{\frac{3}{2}}}\left[2+3e_i^2-3\sin^2\iota\left(5e_i^2\sin^2\omega_i+1-e_i^2\right)\right]\\ &= C\left[2+3e_i^2-3\sin^2\iota\left(5e_i^2\sin^2\omega_i+1-e_i^2\right)\right]\end{aligned}$$

since both the semimajor axes a_i and a_p are constants, and we are ignoring all higher order terms in e_p.

$$\begin{aligned}C\left[2+3e^2-3e^2\sin^2\iota-3\sin^2\iota-15e^2\sin^2\iota\sin^2\omega\right] &= constant\\ C\left[2+3e^2+3e^2-3e^2\cos^2\iota-3+3\cos^2\iota-15e^2\sin^2\iota\sin^2\omega\right] &= constant\\ C\left[-1+3\cos^2\iota(1-e^2)+6e^2-15e^2\sin^2\iota\sin^2\omega\right] &= constant\\ C\left[-1+3c_1^2+15e^2(\frac{2}{5}-\sin^2\iota\sin^2\omega)\right] &= constant\end{aligned}$$

The only way this is possible is if we have:

$$e^2\left(\frac{2}{5}-\sin^2\iota\sin^2\omega\right) = c_2 = constant \tag{55}$$

Let us consider $c_1' = c_1^2 = (1-e^2)\cos^2\iota$. Using these constants c_1' and c_2, we can classify various systems. Figure 3 elegantly shows the various possible values of c_1' and c_2.

In the region $c_2 < 0$, the LK resonance is possible, and the argument of pericenter is librating, while if $c_2 > 0$, the argument of pericenter is circulating.

The segment AB corresponds to $\iota = 0$, which are planar orbits.

The segment BE corresponds to $c_1 = 0$, which means either $\cos\iota = 0$ (polar orbits) or $e = 0$ (parabolic orbits).

The segment ED corresponds to both e and ι being constants.

Segment DA corresponds to $e = 0$, which are circular orbits.

Point A, at which $c_1' = 1$ and $c_2 = 0$, corresponds to $e = 0$ (circular orbit) and $\iota = 0$ (all three bodies in the same plane).

Point B, at which $c_1' = 0$ and $c_2 = \frac{2}{5}$, corresponds to $e = 1$ (parabolic orbit) and $\sin\omega = 0$, and the binary is arbitrarily inclined.

Point E, at which $c_1' = 0$ and $c_2 = -\frac{3}{5}$, corresponds to $\iota = 90^o$ (polar orbit).

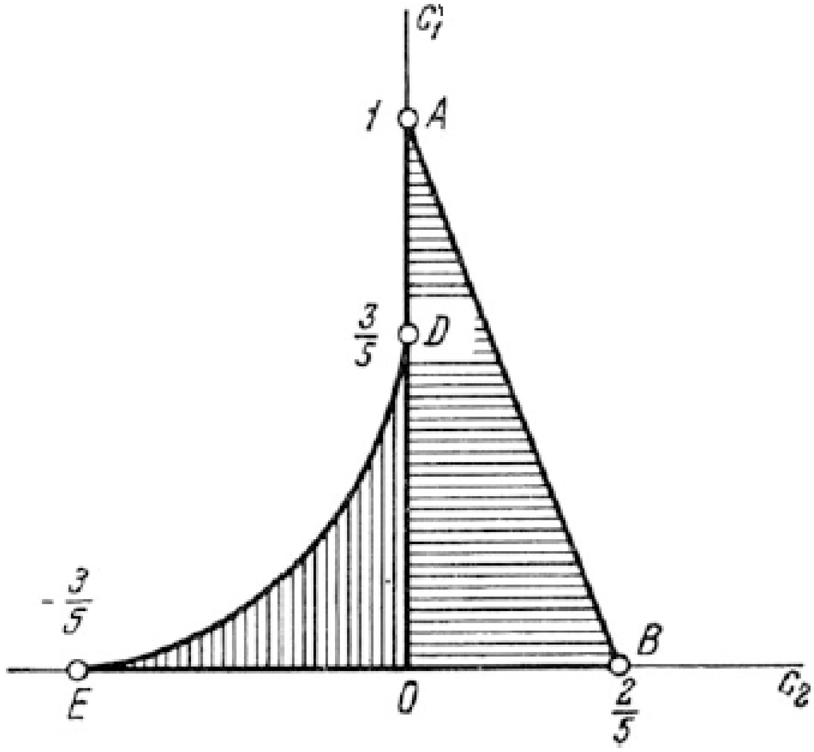

Figure 3: The possible values of c_1 and c_2 (from [6])

Point D, at which $c_1' = \frac{3}{5}$ and $c_2 = 0$, corresponds to $e = 0$ (circular orbit) with $\cos^2 \iota = \frac{3}{5} \Rightarrow \iota = 39.32^o$. This is the point where LK resonance becomes possible.

We can also eliminate ι from c_2 so that we can express c_2 in terms of only e, ω, and c_1':

$$
\begin{aligned}
\cos^2 \iota =& \frac{c_1'}{1-e^2} \\
\sin^2 \iota =& 1 - \frac{c_1'}{1-e^2} \\
c_2(e, \omega, c_1) =& e^2 \left(\frac{2}{5} - \left(1 - \frac{c_1'}{1-e^2} \right) \sin^2 \omega \right)
\end{aligned}
$$

Since c_1' and c_2 are both constants for a given system, we can generate a contour plot of c_2 in the e and ω, for a given value of c_1'.
Figure 4 shows the contour plot for c_2 for $c_1' = 0.25$. It is noted that we have

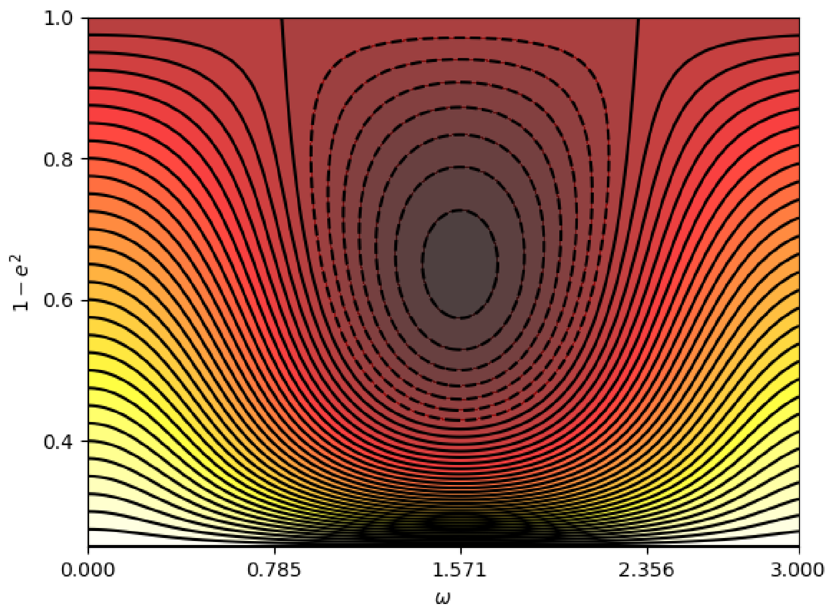

Figure 4: Lidov-Kozai Diagram for $c_1' = 0.25$ (figure inspired from [4]). The solid lines represent circulation cases, while the dashed lies represent libration cases.

libration only if $1 - e^2 > \frac{c_1'}{\cos^2 39.32^o} \Rightarrow 1 - e^2 > 0.4167$. It is also noted that the libration is always centred around $\frac{\pi}{2}$.

4.5 Stability of Orbits

We saw that if the inclination of the orbit is greater than the critical inclination, then there is a large oscillation in the eccentricity. However, the semimajor axis of the binary will remain constant. If the eccentricity increases while the semimajor axis is the same, the distance of perhilion will decrease. When the eccentricity approaches 1, the bodies will come very close to each other, and tidal forces will take over. If one of the bodies is much bigger and massive than the other (like a planet and a satellite), then the smaller body can get ripped apart from the strong tidal forces.
However, we have experimentally observed that there are indeed stable orbits around binaries, even when the inclination is large. This is due to the fact

that there are many factors which stabilise the orbit and prevent it from LK oscillations:

- **Perturbations from other bodies**: In the N-body system with $N \geq 4$, the additional bodies can suppress the LK oscillations, stabilising the orbit.
- **Tidal forces**: Tidal precession can lead to dissipation of the LK phenomenon.
- **Non-sphericity of the central body** Correction factors due to the oblateness of planets lead to orbital precession, which can dominate over LK effects. The oblateness is usually due to the body's rotation about its own axis.
- **General relativity** Due to general relativity, planets revolving around a star also undergo apsidal precession, which can dampen the LK effect. The apsidal precession is more pronounced for bodies with small perihelion distance.

Due to these perturbations, there exists a *critical radius* above which only the LK effect becomes operational. If we consider a sun-planet system with a satellite orbiting around the planet, then this critical radius is given by:

$$a_{crit} = \left[\frac{3m_1 m_2 (1-e_p^2)^{\frac{3}{2}}}{8m_3(m_1+m_2)(1-e_i^2)^{\frac{3}{2}}} \left(5\cos^2\iota - 1\right) \right]^{\frac{1}{5}} \left(a_p^3 a_i^2\right)^{\frac{1}{5}} \tag{56}$$

[6] Here, m_1 and m_2 are the masses of the planet and satellite respectively, m_3 is the mass of the perturber (sun), a_i and a_p are semimajor axes of the satellite-planet and sun-planet systems respectively, e_i and e_p are eccentricities of the satellite-planet and sun-planet systems respectively, and ι is the inclination of the satellite's orbit with respect to the orbital plane of the sun-planet system.

A real example of this is one of Pluto's moons, Charon. The Pluto-Charon system is perturbed by the sun. We substitute in equation 56, $m_1 = 1.3 \times 10^22$ kg (mass of Pluto), $m_2 = 1.7 \times 10^21$ kg (mass of Charon), $m_3 = 2 \times 10^30$ kg (mass of the sun), $e_p = 0.25$ (eccentricity of Pluto's orbit around the sun, $e_i = 0$ (eccentricity of Pluto-Charon system), $\iota = 119^o$ (inclination of Pluto-Charon system with respect to the plane of Pluto's orbit around the sun), $a_p = 6 \times 10^9$ km (semi-major axis of Pluto), $a_i = 2 \times 10^4$ km (semi-major axis of Pluto-Charon system). We find that $a_{crit} = 6 \times 10^5$ km, which is much greater than the Pluto-Charon distance. Thus, any satellite within this distance is stable. [6]

We now turn our attention to our own solar system. What if our sun actually is part of a binary and has a companion star? If the companion's inclination is greater than the critical inclination, then by the LK effect, the solar system should destabilize.

It is found that if the companion's mass is 0.4 times the mass of the sun, and at a distance of 400 AU from the sun (10 times the distance of Pluto to the sun), then the LK effect cannot destabilize the solar system at inclinations below 45^o. If the companion's mass is decreased, then the critical inclination increases, up until 0.05 solar masses, where destabilization is not possible for any inclination. The main reason for the stability of the system is the gravitational attraction between the planets themselves, which have not been considered in the calculations in this chapter. Although the sun does not have a companion star, the question of orbit stability is important for binary systems.

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