Quantum Cloning:

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In[•]:= (*Partial trace function*)
SwapParts[expr_, pos1_, pos2_] := ReplacePart[#, #, {pos1, pos2}, {pos2, pos1}] &[expr]
TraceSystem[D_, s_] := (Qubits = Reverse[Sort[s]];
   TrkM = D;
   z = (Dimensions[Qubits][[1]] + 1);
   For[q = 1, q < z, q++, n = Log[2, (Dimensions[TrkM][[1]])];
    M = TrkM;
    k = Qubits[[q]];
    If[k == n, TrkM = {};
     For[p = 1, p < 2^n + 1, p = p + 2,
      TrkM = Append[TrkM, Take[M[[p, All]], {1, 2^n, 2}] + Take[M[[p + 1, All]], {2, 2^n, 2}]];],
     For[j = 0, j < (n - k), j++, b = {0};
       For[i = 1, i < 2^n + 1, i++, If[(Mod[(IntegerDigits[i - 1, 2, n][[n]] +
                 IntegerDigits[i - 1, 2, n][[n - j - 1]]), 2]) == 1 && Count[b, i] == 0, Permut = {i,
            (FromDigits[SwapParts[(IntegerDigits[i - 1, 2, n]), {n}, {n - j - 1}], 2] + 1)};
          b = Append[b, (FromDigits[SwapParts[(IntegerDigits[i - 1,
                    2, n]), {n}, {n - j - 1}], 2] + 1)];
          c = Range[2^n];
          perm = SwapParts[c, {i}, {(FromDigits[SwapParts[
                   (IntegerDigits[i - 1, 2, n]), {n}, {n - j - 1}], 2] + 1)}];
          M = M[[perm, perm]];]];
       TrkM = {};
       For[p = 1, p < 2^n + 1, p = p + 2, TrkM = Append[TrkM,
            Take[M[[p, All]], {1, 2^n, 2}] + Take[M[[p + 1, All]], {2, 2^n, 2}]];]];]];
   Return[TrkM]);
```

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In[•]:= (*Definitions*)
u = {{1}, {0}};
d = {{0}, {1}};
(*Position dependent spin initialisations*)
sp[i_, 0] := u;
sp[i_, 1] := d;
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In[ ]:=
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(*Spin configurations*)

(*Starting from the state (α|01⟩+β|10⟩) (|01⟩- |10⟩)/√2 (|01⟩- |10⟩)/√2,
we obtain 8 possible spin configurations.The first 4 correspond to the α branch and the
 last 4 to the β branch,following the (+--+) and (+--+) order respectively.*)

spins = {{0, 1, 0, 1, 0, 1}, {0, 1, 1, 0, 0, 1}, {0, 1, 0, 1, 1, 0}, {0, 1, 1, 0, 1, 0},
   {1, 0, 0, 1, 0, 1}, {1, 0, 1, 0, 0, 1}, {1, 0, 0, 1, 1, 0}, {1, 0, 1, 0, 1, 0}};

(*State based on position list*)
State[spinConfig_List, pos_List] :=
  KroneckerProduct @@ Table[sp[i, spinConfig[[pos[[i]]]]], {i, 6}];

(*Position order*)
pos = {1, 2, 3, 4, 5, 6};

(*initial state*)
ξ = α/2 (State[spins[[1]], pos] - State[spins[[2]], pos] -
      State[spins[[3]], pos] + State[spins[[4]], pos]) + β/2 (State[spins[[5]], pos] -
      State[spins[[6]], pos] - State[spins[[7]], pos] + State[spins[[8]], pos]);
```

In[ ]:=
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(*Define the Beam Splitter action;|s_1> →Sqrt[1-R]|s_1'>+i Sqrt[R]|s_3'>;|s_3> →
       Sqrt[1-R]|s_3'>+i Sqrt[R]|s_1'>;|s_2> →
     Sqrt[1-R]|s_2'>+i Sqrt[R]|s_5'>;|s_5> →Sqrt[1-R]|s_5'>+i Sqrt[R]|s_2'>;
We have a six-qubit state,so we can write it as:
  |s_1>|s_2>|s_3>|s_4>|s_5>|s_6>,
where each position s_1 to s_6 may take a different spin
   value.After the action of the BS unitary:
    |s_1>|s_2>|s_3>|s_4>|s_5>|s_6> →
  (Sqrt[1-R]|s_1'>+i Sqrt[R]|s_3'>) (Sqrt[1-R]|s_2'>+i Sqrt[R]|s_5'>)
       (Sqrt[1-R]|s_3'>+i Sqrt[R]|s_1'>)... (and similarly for s_4,s_5,s_6)
    |s_1>|s_2>|s_3>|s_4>|s_5>|s_6> →(1-R)^2|s_1'>|s_2'>|s_3'>|s_4'>|
    s_5'>|s_6'>-R (1-R)|s_3'>|s_2'>|s_1'>|s_4'>|s_5'>|
    s_6'>+R^2|s_3'>|s_5'>|s_1'>|s_4'>|s_2'>|s_6'>;*)

(*Define the Beam splitter action;
 |s>₁→ √(1-R) |s>₁'+i √R |s>₃';|s>₃→ √(1-R) |s>₃'+i √R |s>₁';|s>₂→ √(1-R) |s>₂'+i √R |s>₅';|
   s>₅'→ √(1-R) |s>₅'+i √R |s>₂';

We have a six qubit state, so we can write this like
    |s₁>₁|s₂>₂|s₃>₃|s₄>₄|s₅>₅|s₆>₆;
  each position can take different spin so in general s₁to s₆ may be different;
```

After the action of BS\unitary ;

$|s_1\rangle_1 |s_2\rangle_2 |s_3\rangle_3 |s_4\rangle_4 |s_5\rangle_5 |s_6\rangle_6 \rightarrow \left(\left|\sqrt{1-R}\right|s_1\rangle_{1'} + i\sqrt{R}\,|s_1\rangle_{3'}\right)$
$\left(\left|\sqrt{1-R}\right|s_2\rangle_{2'} + i\sqrt{R}\,|s_2\rangle_{5'}\right)\left(\left|\sqrt{1-R}\right|s_3\rangle_{3'} + i\sqrt{R}\,|s_3\rangle_{1'}\right) s_4 \left(\left|\sqrt{1-R}\right|s_5\rangle_{5'} + i\sqrt{R}\,|s_5\rangle_{2'}\right) s_6$;

Some terms like $i\sqrt{R}\ \sqrt{1-R}\,|s_1\rangle_{1'}|s_1\rangle_{1'}$ and other combinations occur,
where two photons appear in a single mode.

In our calculation, we consider only single-photon-per-mode terms, discarding all terms where multiple photons occupy the same mode.This is equivalent to post-selecting our system to states where each mode contains exactly one photon.Then the unnormalized post-selected state is given by:

$|s_1\rangle_1 |s_2\rangle_2 |s_3\rangle_3 |s_4\rangle_4 |s_5\rangle_5 |s_6\rangle_6 \rightarrow$
$(1-R)^2 |s_1\rangle_{1'} |s_2\rangle_{2'} |s_3\rangle_{3'} |s_4\rangle_4 |s_5\rangle_{5'} |s_6\rangle_6 \ -R(1-R)\,|s_3\rangle_{1'} |s_2\rangle_{2'} |s_1\rangle_{3'} |s_4\rangle_4 |s_5\rangle_{5'}|$
$s_6\rangle_6 \ -R(1-R)\,|s_1\rangle_{1'} |s_5\rangle_{2'} |s_3\rangle_{3'} |s_4\rangle_4 |s_2\rangle_{5'} |s_6\rangle_6 + R^2 |s_3\rangle_{1'} |s_5\rangle_{2'} |s_1\rangle_{3'} |s_4\rangle_4 |s_2\rangle_{5'} |s_6\rangle_6$;

*)

```
BSUpdatedState[spinConfig_, pos_, R_] :=
 Module[{s}, s = Table[sp[i, spinConfig[[pos[[i]]]]], {i, 6}];
  (1 - R)^2 * KroneckerProduct[s[[1]], s[[2]], s[[3]], s[[4]], s[[5]], s[[6]]] -
   (1 - R) R * KroneckerProduct[s[[3]], s[[2]], s[[1]], s[[4]], s[[5]], s[[6]]] -
   (1 - R) R * KroneckerProduct[s[[1]], s[[5]], s[[3]], s[[4]], s[[2]], s[[6]]] +
   R^2 * KroneckerProduct[s[[3]], s[[5]], s[[1]], s[[4]], s[[2]], s[[6]]]]
```

In[∘]:=
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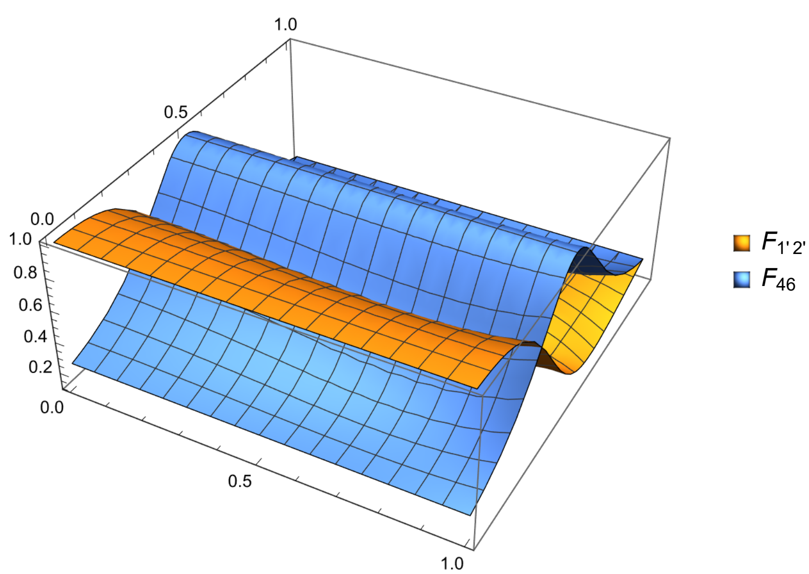
(*Final state in 1'2'3'45'6*)
ξfinal = (α / 2) (BSUpdatedState[spins[[1]], pos, R] - BSUpdatedState[spins[[2]], pos, R] -
      BSUpdatedState[spins[[3]], pos, R] + BSUpdatedState[spins[[4]], pos, R]) +
   (β / 2) (BSUpdatedState[spins[[5]], pos, R] - BSUpdatedState[spins[[6]], pos, R] -
      BSUpdatedState[spins[[7]], pos, R] + BSUpdatedState[spins[[8]], pos, R]);
ξfinalT = Transpose[ξfinal] (*ConjugateTranspose[ξfinal]*);
ρξfinal  = ξfinal.ξfinalT;
ρξfinal = ρξfinal / Tr[ρξfinal];

ρ12 = FullSimplify[TraceSystem[ρξfinal, {3, 4, 5, 6}]];
ρ46  = FullSimplify[TraceSystem[ρξfinal, {1, 2, 3, 5}]];
β = Sqrt[1 - α^2];
(*Maximally entangled state*)
ϕ = FullSimplify[α * KroneckerProduct[u, d] + Sqrt[1 - α^2] * KroneckerProduct[d, u], 0 ≤ α ≤ 1];
ϕT  = FullSimplify[ConjugateTranspose[ϕ], 0 ≤ α ≤ 1];
ρϕ  = FullSimplify[ϕ . ϕT];
ρϕ  = ρϕ / Tr[ρϕ];

F12[α_, R_] := FullSimplify[Tr[ρϕ.ρ12]];

F46[α_, R_] := FullSimplify[Tr[ρϕ.ρ46]];
Plot3D[{F12[α, R], F46[α, R]}, {α, 0, 1},
 {R, 0, 1}, PlotRange → All, PlotLegends → {"F1'2'", "F46"}]
```

Out[∘]=



In[•]:=
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(*Analytical expression form paper for α=1/√2 case*)
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$$F12ana = \frac{\left(2\ (1-2\,R)^2\ (1-R)^2 + \left(2-12\,R+28\,R^2-30\,R^3+13\,R^4\right)\right)}{\left(4\ \left(3\,R^2-3\,R+1\right)^2\right)};$$

$$F46ana = \frac{\left(2\,R^2\ (1-R)^2 + \left(1-6\,R+16\,R^2-20\,R^3+10\,R^4\right)\right)}{\left(4\ \left(3\,R^2-3\,R+1\right)^2\right)};$$

```
Plot[{F12[1/√2, R], F12ana, F46[1/√2, R], F46ana}, {R, 0, 1},
 PlotRange → All, PlotLegends → {"F1'2'", "Fana1'2'", "F46", "Fana46"}]
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Out[•]=