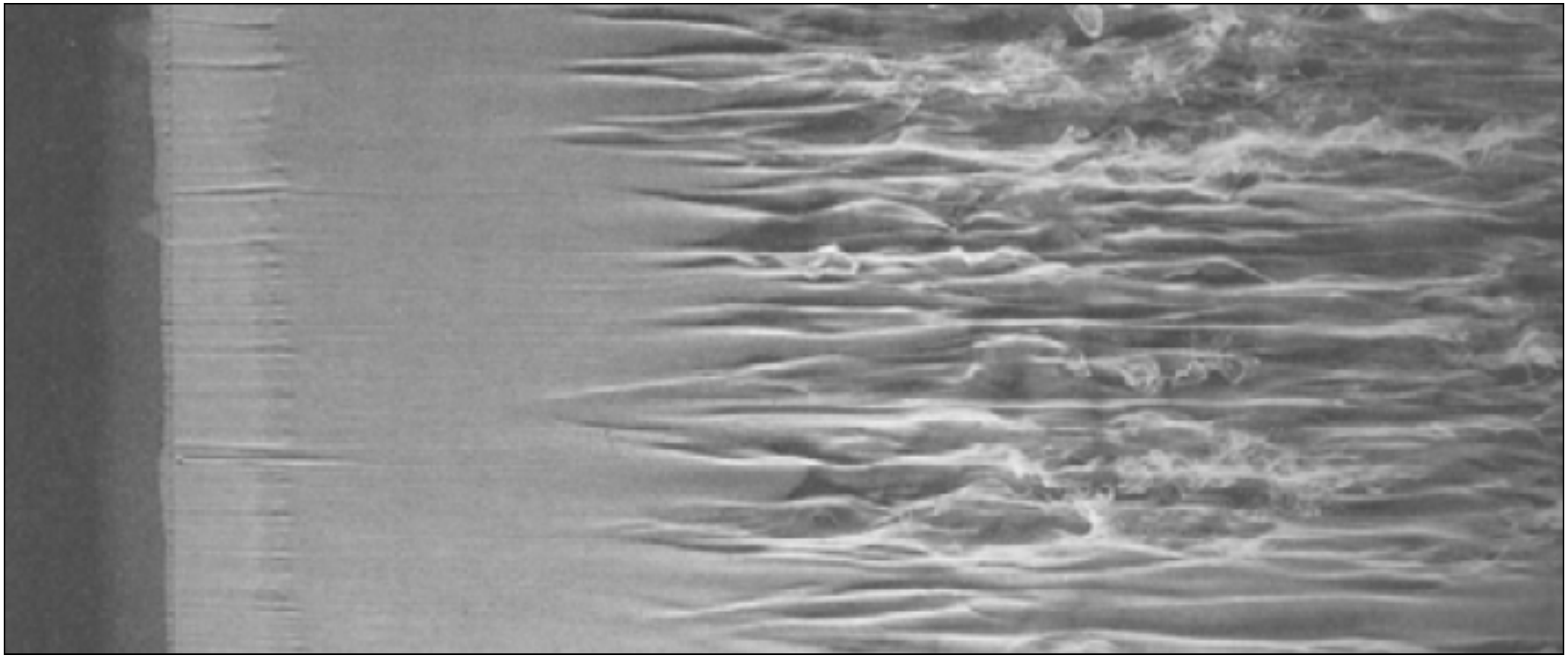
Fanno Flow



Simple frictional flow (Fanno Flow)
Adiabatic frictional flow in a constant-area duct

* The Flow of a compressible fluid in a duct is Always accompanied by :-

* Variation in the cross –sectional area of the duct
  - Heat transfer
  - Friction

The above parameters contribute to changes the flow properties.

* Although, it is difficult in many cases to separate the effects of each of these parameters , yet in order to provide an insight into the effect of friction , adiabatic flow thermally insulated in a constant area duct is analyzed in this chapter.

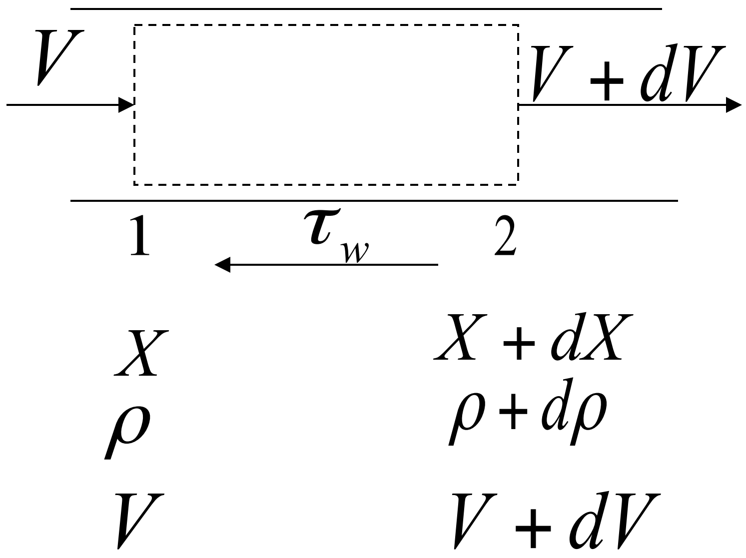
* The entropy of the system still increase because of friction ….

-Friction is associated with the turbulence and viscous shear of molecules of the gas .

-Irreversibility associated with friction causes a decrease in the stagnation pressure , steady 1-D flow as a perfect gas .

<u>Assumptions:</u>

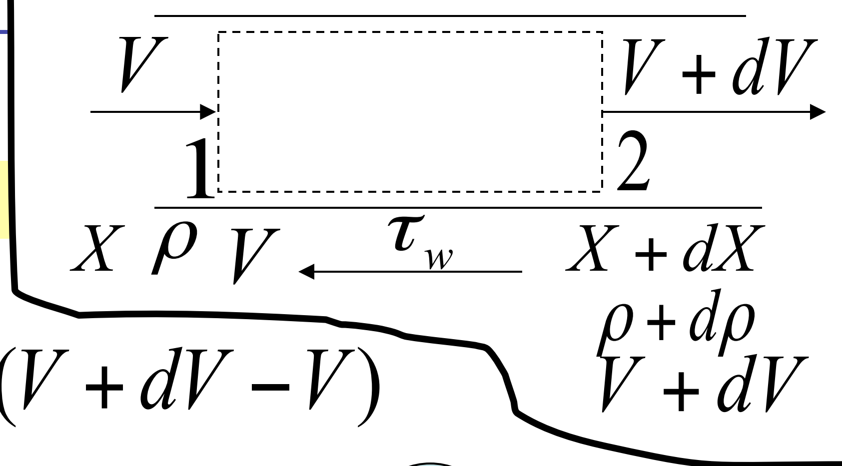
- Adiabatic frictional
- flow in a constant cross-sectional area duct .



Momentum equation

$B = 4A/D_H$ = wetted perimeter

$D_H = hydraulic\ diameter$



$$AP - A(P + dP) - \tau_w B dx = \rho A V (V + dV - V)$$

$$- AdP - \tau_w B dx = \rho A V dV \longrightarrow (1)$$

$$\because \tau_w = F\left(\frac{1}{2}\rho V^2\right)$$

$\frac{1}{2}\rho V^2$ Dynamic pressure

F is the conventional Fanning friction factor (circular duct)
F for flow over as the drag coefficient

$$h_{loss\ in\ pipe} = 4F_F \frac{L}{D}\frac{V^2}{2g} = f_{D.W}\frac{L}{D}\frac{V^2}{2g}$$

F_F: Fanning coefficient of friction

$f_{D.W}$: Darcy-weisbech coefficient of friction

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From eq. 1 (eq.1 / A)

$$-AdP - \tau_w B dx = \rho A V dV \tag{1}$$

$$dP + \tau_w \frac{B}{A} dX + \rho V dV = 0$$

$$dP + \frac{4F}{D_H}\frac{\rho V^2}{2} dX + \frac{\rho V^2}{2}\frac{dV^2}{V^2} = 0 \longrightarrow \quad 2$$

$$F = f(\text{Reynold's number , roughness of boundaries})$$

VdV= dV²/2

$$\rho V dV = \frac{\rho V^2}{2}\frac{dV^2}{V^2}$$

$$\tau_w = F\frac{\rho V^2}{2}$$

$$D_H = \frac{4A}{B}$$

The friction factor for turbulent in smooth ducts is the Von-Karman Nikurdse formula

$$\frac{1}{\sqrt{4F}} = -0.8 + 2\log_{10}(\mathrm{Re}\sqrt{4f})$$

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Perfect gas law

$$P = \rho R T$$

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \longrightarrow 3$$

Continuity Eq.

$$\frac{m}{A} = \rho V = \text{Constant}$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \longrightarrow 4$$

Energy Eq.

$$h_o = h + \frac{V^2}{2} = \text{Constant}$$

$$dh + VdV = 0 \longrightarrow 5$$

Mach number

$$M^2 = \frac{V^2}{\gamma R T}$$

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \longrightarrow 6$$

2nd law of thermodynamics

$$ds \geq 0 \longrightarrow 7$$

$dP + \frac{4F}{D_H}\frac{\rho V^2}{2}dX + \frac{\rho V^2}{2}\frac{dV^2}{V^2} = 0$	2	Momentum equation
$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$	3	Perfect gas law
$\frac{d\rho}{\rho} + \frac{dV}{V} = 0$	4	Continuity Eq.
$dh + VdV = 0$	5	Energy Eq.
$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T}$	6	Mach number
$ds \geq 0$	7	2nd law of thermodynamics

$M, V, P, \rho, T, S \; \& \; \frac{4FdX}{D_H} (\tau$ Variable $)$ $\longrightarrow$ 6 equation

From eq. 2 Dividing by P

$$dP + \frac{4F}{D_H}\frac{\rho V^2}{2}dX + \frac{\rho V^2}{2}\frac{dV^2}{V^2} = 0 \quad (2)$$

$$\frac{dP}{P} + \frac{4F}{D_H}\frac{\rho V^2}{2P}dX + \frac{\rho V^2}{2P}\frac{dV^2}{V^2} = 0$$

$$\boxed{\because \rho V^2 = \frac{\gamma P V^2}{\gamma RT} = \gamma P M^2}$$

therefore

$$\frac{dP}{P} + \frac{4F}{D_H}\frac{\gamma M^2}{2}dX + \frac{\gamma M^2}{2}\frac{dV^2}{V^2} = 0 \longrightarrow \quad (8)$$

from_Eq.5 $\quad dh + VdV = 0 \quad (5)$

$$C_P dT = -\frac{dV^2}{2}$$

$$\frac{dT}{T} = \frac{-dV^2}{2C_P T} = \frac{-(\gamma-1)dV^2}{2\gamma RT} = -\frac{\gamma-1}{2}M^2\frac{dV^2}{V^2} \quad (9)$$

$$\frac{dT}{T}=\frac{-dV^2}{2C_pT}=\frac{-(\gamma-1)dV^2}{2\gamma RT}=-\frac{\gamma-1}{2}M^2\frac{dV^2}{V^2}\ (9),\quad \frac{dM^2}{M^2}=\frac{dV^2}{V^2}-\frac{dT}{T}\ (6)$$

Combining eq.(9) with eq. (6)

$$\frac{dV^2}{V^2}-\frac{dM^2}{M^2}=-\frac{\gamma-1}{2}M^2\frac{dV^2}{V^2}$$

$$\frac{dV^2}{V^2}=\frac{1}{1+\frac{\gamma-1}{2}M^2}*\frac{dM^2}{M^2}\longrightarrow 10$$

From eq. 3 & 4

$$\frac{dP}{P}=-\frac{dV}{V}+\frac{dT}{T}\ (3),\quad \frac{d\rho}{\rho}+\frac{dV}{V}=0\ \ (4)$$

From eq. 9

$$\frac{dP}{P}=-\frac{dV}{V}-\frac{\gamma-1}{2}M^2\frac{dV^2}{V^2}\qquad \frac{dV}{V}=\frac{dV^2}{2V^2}$$

$$\frac{dP}{P}=\frac{dV^2}{V^2}\left(-\frac{1}{2}-\frac{\gamma-1}{2}M^2\right) \longrightarrow 11$$

The $\frac{dP}{P}$ term and the $\frac{dV^2}{V^2}$ term in eq. 8 can be replaced to give

$$\frac{dP}{P}+\frac{4F}{D_H}\frac{\gamma M^2}{2}dX+\frac{\gamma M^2}{2}\frac{dV^2}{V^2}=0\ (8)$$

$$\frac{dV^2}{V^2}\left(-\frac{1}{2}-\frac{\gamma-1}{2}M^2\right)+\frac{\gamma M^2}{2}\frac{dV^2}{V^2}+\frac{4F}{D_H}\frac{\gamma M^2}{2}dX=0$$

From Eq. 10

$$\frac{dV^2}{V^2}=\frac{1}{1+\frac{\gamma-1}{2}M^2}*\frac{dM^2}{M^2}\quad(10)$$

$$\frac{1}{2}\left[\frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2}\right](-1-(\gamma-1)M^2+\gamma M^2)+\frac{4F}{D_H}\frac{\gamma M^2}{2}dX=0$$

$$\frac{1}{2}\left[\frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2}\right](-1-(\gamma-1)M^2+\gamma M^2)+\frac{4F}{D_H}\frac{\gamma M^2}{2}dX=0$$

$$\frac{4FdX}{D_H}=\frac{(1-M^2)}{\gamma M^2(1+\frac{\gamma-1}{2}M^2)}*\frac{dM^2}{M^2}$$

$$\frac{dM^2}{M^2}=\frac{\gamma M^2(1+\frac{\gamma-1}{2}M^2)}{(1-M^2)}\cdot\frac{4FdX}{D_H} \rightarrow 12$$

M<1

dM +ve

M>1

dM -ve

From continuity eq. (eq. 4) & from eqs. 10 & 12

$$\frac{dV^2}{V^2} = \frac{1}{1+\frac{\gamma-1}{2}M^2} * \frac{dM^2}{M^2} \quad (10)$$

$$\frac{dV^2}{V^2} = \frac{2VdV}{V^2} = \frac{2dV}{V} \quad (11)$$

$$\frac{dM^2}{M^2} = \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)} \cdot \frac{4FdX}{D_H} \quad 12$$

$$\frac{dV}{V} = -\frac{d\rho}{\rho} = \frac{\gamma M^2}{2(1-M^2)} \cdot \frac{4FdX}{D_H} = \frac{dM}{M\left(1+\frac{\gamma-1}{2}M^2\right)} \quad 13$$

M<1

dV +ve
ρ -ve

M>1

dV -ve
ρ +ve

From eqs. 11 & 13

$$\frac{dP}{P} = -\frac{\gamma M^2(1+(\gamma-1)M^2)}{2(1-M^2)} \cdot \frac{4FdX}{D_H} = -\left[\frac{(1+(\gamma-1)M^2)}{M(1+\frac{(\gamma-1)}{2}M^2)}\right]dM$$

14

From eqs. 3,13& 14

$$\frac{dT}{T} = -\frac{\gamma(\gamma-1)M^4}{2(1-M^2)} \cdot \frac{4FdX}{D_H} = -(\gamma-1)\left(\frac{(MdM)}{(1+\frac{\gamma-1}{2}M^2)}\right.$$

15

Entropy changes are determined from

$$ds = C_P \frac{dT}{T} - R\frac{dP}{P}$$

From eqs. 14 & 15 & $C_P = \dfrac{\gamma R}{\gamma - 1}$

$$\frac{dP}{P} = -\frac{\gamma M^2(1+(\gamma-1)M^2)}{2(1-M^2)} \cdot \frac{4FdX}{D_H} = -\left[\frac{(1+(\gamma-1)M^2)}{M(1+\frac{(\gamma-1)}{2}M^2)}\right]dM \qquad (14)$$

$$\frac{dT}{T} = -\frac{\gamma(\gamma-1)M^4}{2(1-M^2)} \cdot \frac{4FdX}{D_H} = -(\gamma-1)\left(\frac{(MdM)}{(1+\frac{\gamma-1}{2}M^2)}\right. \qquad (15)$$

Then

$$ds = \frac{\gamma R M^2}{2} \cdot \frac{4FdX}{D_H} = \frac{R(1-M^2)}{M(1+\frac{\gamma-1}{2}M^2)} \frac{dM}{M} \qquad 16$$

$\because ds \geq 0 \qquad \therefore F$ Is positive

$$\frac{ds}{C_P} = \frac{(\gamma-1)M^2}{2} \cdot \frac{4FdX}{D_H}$$

$$\because ds = \frac{R(1-M^2)}{M(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M} \qquad (16)$$

Also as a result of frictional flow

- For subsonic flow the Mach number increases

$$M < 1 \longrightarrow (1-M^2)\ +ve$$

$$\because ds = +ve \qquad \therefore dM = +ve$$

- For supersonic flow the Mach number decreases

$$M > 1 \longrightarrow (1-M^2)\ -ve$$

$$\because ds = +ve \qquad \therefore dM = -ve$$

The stagnation pressure can be calculated from

$$P_0 = P\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\therefore \frac{dP_O}{P_0} = \frac{dP}{P} + \frac{\gamma M^2/2}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2}$$

From eqs. 12 & 14 then

$$\frac{dM^2}{M^2} = \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)}\cdot\frac{4FdX}{D_H} \quad (12),$$

$$\frac{dP}{P} = -\frac{\gamma M^2(1+(\gamma-1)M^2)}{2(1-M^2)}\cdot\frac{4FdX}{D_H} = -\left[\frac{(1+(\gamma-1)M^2)}{M\left(1+\frac{(\gamma-1)}{2}M^2\right)}\right]dM \quad (14)$$

$$\frac{dP_0}{P_0} = \frac{-\gamma M^2}{2}\cdot\frac{4FdX}{D_H} = \frac{-(1-M^2)}{M\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

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$$\frac{dP_0}{P_0} = \frac{-\gamma M^2}{2} \cdot \frac{4FdX}{D_H} = \frac{-(1-M^2)}{M(1+\frac{\gamma-1}{2}M^2)} \frac{dM}{M} \quad (17)$$

- For subsonic flow

$$M < 1 \longrightarrow (1-M^2)\ +ve$$

$$\because dM = +ve \qquad \therefore dP_o = -ve$$

$$ds = +ve$$

- For supersonic flow

$$M > 1 \longrightarrow (1-M^2)\ -ve$$

$$\because dM = -ve \qquad \therefore dP_o = -ve$$

$$ds = +ve$$

$$\frac{dM}{M} = \frac{\gamma M^2 \left(1 + \frac{\gamma - 1}{2} M^2\right)}{2(1 - M^2)} \cdot \frac{4FdX}{D_H}$$

$$\frac{dV}{V} = -\frac{d\rho}{\rho} = \frac{\gamma M^2}{2(1 - M^2)} \cdot \frac{4FdX}{D_H}$$

$$\frac{dP}{P} = -\frac{\gamma M^2 (1 + (\gamma - 1) M^2)}{2(1 - M^2)} \cdot \frac{4FdX}{D_H}$$

$$\frac{dT}{T} = -\frac{\gamma (\gamma - 1) M^4}{2(1 - M^2)} \cdot \frac{4FdX}{D_H}$$

$$ds = \frac{\gamma R M^2}{2} \cdot \frac{4FdX}{D_H} = \frac{R(1 - M^2)}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} \frac{dM}{M}$$

$$\frac{dP_O}{P_0} = \frac{-\gamma M^2}{2} \cdot \frac{4FdX}{D_H} = \frac{-(1 - M^2)}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} \frac{dM}{M}$$

	M<1	M>1
dM	+ve	-ve
dV	+ve	-ve
dρ	-ve	+ve
dP	-ve	+ve
dT	-ve	+ve
dS	+ve	+ve
dP_o	-ve	-ve
h_o	Constant	

Fanno table relations.
M, 4FL*/D, V/V*, P/P*, T/T*, ρ/ρ*, P_o/P_o^*, $(S\text{-}S^*)/C_p$

By separating the variables and by establishing the limits
$M=M_1$ at the duct entrance.
$M=M_2$ at distance L.
If $M_2=1$, then $L_2=L^*$, $V_2=V^*$, $P_2=P^*$, $T_2=T^*$, $\rho_2=\rho^*$, $P_{o2}=P_o^*$, and $S_2=S^*$

$\frac{dM}{M}=\frac{\gamma M^2(1+\frac{\gamma-1}{2}M^2)}{2(1-M^2)}\cdot\frac{4FdX}{D_H}$	$\frac{dV}{V}=-\frac{d\rho}{\rho}=\frac{\gamma M^2}{2(1-M^2)}\cdot\frac{4FdX}{D_H}$
$\frac{dP}{P}=-\frac{\gamma M^2(1+(\gamma-1)M^2)}{2(1-M^2)}\cdot\frac{4FdX}{D_H}$	$\frac{dT}{T}=-\frac{\gamma(\gamma-1)M^4}{2(1-M^2)}\cdot\frac{4FdX}{D_H}$
$ds=\frac{\gamma RM^2}{2}\cdot\frac{4FdX}{D_H}=\frac{R(1-M^2)}{M(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M}$	$\frac{dP_o}{P_0}=\frac{-\gamma M^2}{2}\cdot\frac{4FdX}{D_H}=\frac{-(1-M^2)}{M(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M}$

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The Fanno Line

* It is defined by the

- Continuity eq.
- Energy eq.
- Eq. of state

From Continuity eq. (4)

$$\frac{d\rho}{\rho} = -\frac{dV}{V}$$

From Energy eq. (5)

$$V = \sqrt{2C_p(T_0 - T)} \qquad \therefore \frac{dV}{V} = \frac{d(T_o - T)}{2(T_o - T)}$$

The change of entropy for the perfect gas

$$ds = C_v \frac{dT}{T} - R\frac{d\rho}{\rho}$$

$$ds = C_v \frac{dT}{T} - R\frac{d(T_o - T)}{2(T_o - T)} \qquad \because \frac{R}{C_v} = \gamma - 1$$

$$\therefore \frac{ds}{C_v} = \frac{dT}{T} - \frac{\gamma - 1}{2}\frac{d(T_o - T)}{2(T_o - T)}$$

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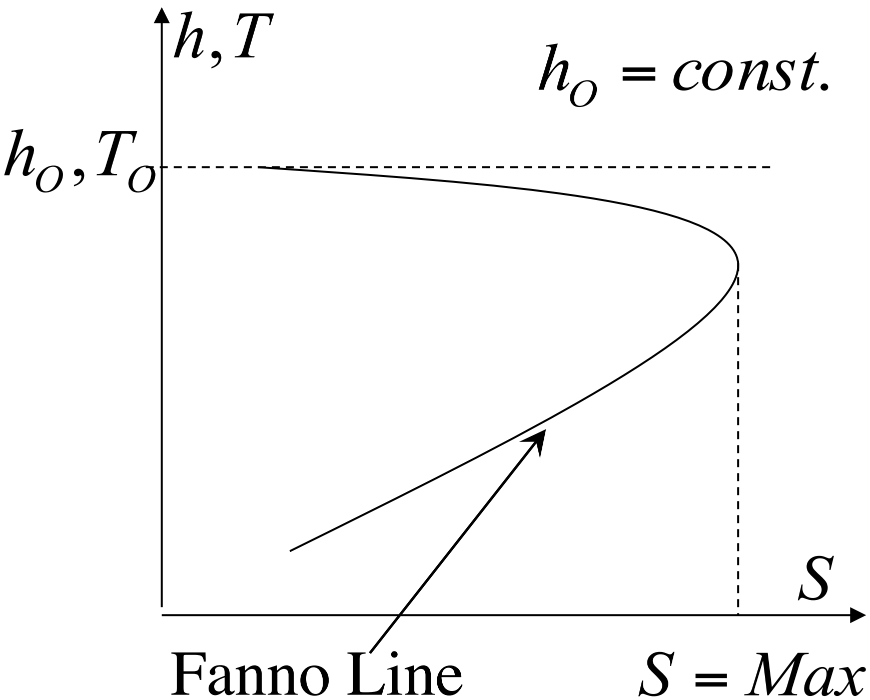
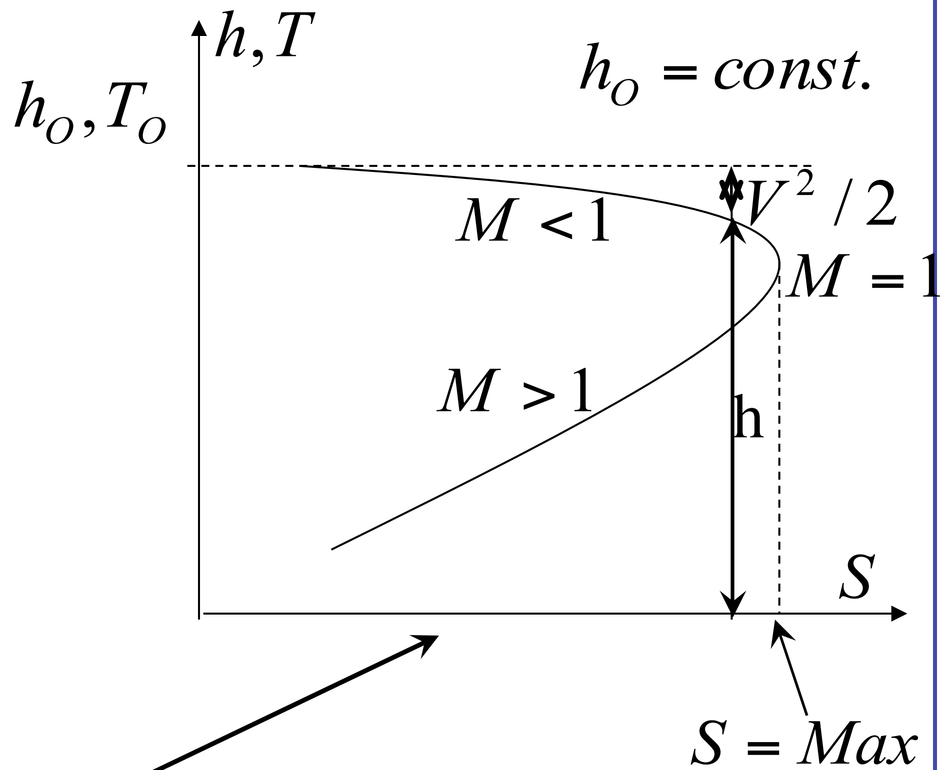
$$\therefore \frac{ds}{C_v} = \frac{dT}{T} - \frac{\gamma - 1}{2} \frac{d(T_o - T)}{2(T_o - T)} \qquad (19)$$

Eq. 19 is integrated , the change of entropy may be described in terms Of temperature for a given value of T_0 (T_o = constant) and m·/ A=constant .





The Fanno Line equation is explained as a function of Mach

Prove that, this curve is right.

The Fanno Line equation is a function of Mach

From the 1st & 2nd law of thermodynamics

$$ds = \frac{dh}{T} - R\frac{dP}{P}$$

$$\frac{dh}{ds} = \frac{1}{\frac{1}{T} - \frac{R}{P}\frac{dP}{dh}} \longrightarrow 20$$

$$\therefore dh \propto dT$$

Continuity $\frac{d\rho}{\rho} + \frac{dV}{V} = 0,$ Energy $dh + VdV = 0 \quad dh = -VdV$

$$dh = V^2\frac{d\rho}{\rho}$$

Eq. of state $\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} \quad \therefore dh = V^2\left(\frac{dP}{P} - \frac{dT}{T}\right)$

$$\because dh = V^2\left(\frac{dP}{P} - \frac{C_P dT}{C_P T}\right) \qquad \left(1 + \frac{V^2}{C_P T}\right)dh = V^2\frac{dP}{P}$$

$$\frac{dP}{dh} = \frac{P}{V^2} + \frac{P}{C_P T} \longrightarrow 21$$

From 20 & 21

$$\frac{dh}{dS} = \frac{1}{\frac{1}{T} - \frac{R}{V^2} - \frac{R}{C_P T}}$$

$$C_P = \frac{\gamma R}{\gamma - 1}$$

$$\frac{dh}{dS} = \frac{1}{\frac{1}{T} - \frac{R}{V^2} - \frac{R}{C_P T}} \qquad C_P = \frac{\gamma R}{\gamma - 1} \qquad V^2 = M^2 \gamma R T$$

$$\frac{dh}{dS} = \frac{V^2 C_P T}{V^2 C_P - R C_P T - V^2 R} = \frac{V^2 \gamma R T}{(\gamma - 1)[V^2 C_P - R C_P T - V^2 R]}$$

$$\frac{dh}{dS} = \frac{\gamma T M^2}{M^2 - 1} \rightarrow 22$$

This equation describe fanno line as a function on Mach

At M<1 dh/dS -ve dS ≥ 0

∴ dh -ve dT –ve ∴ dV +ve

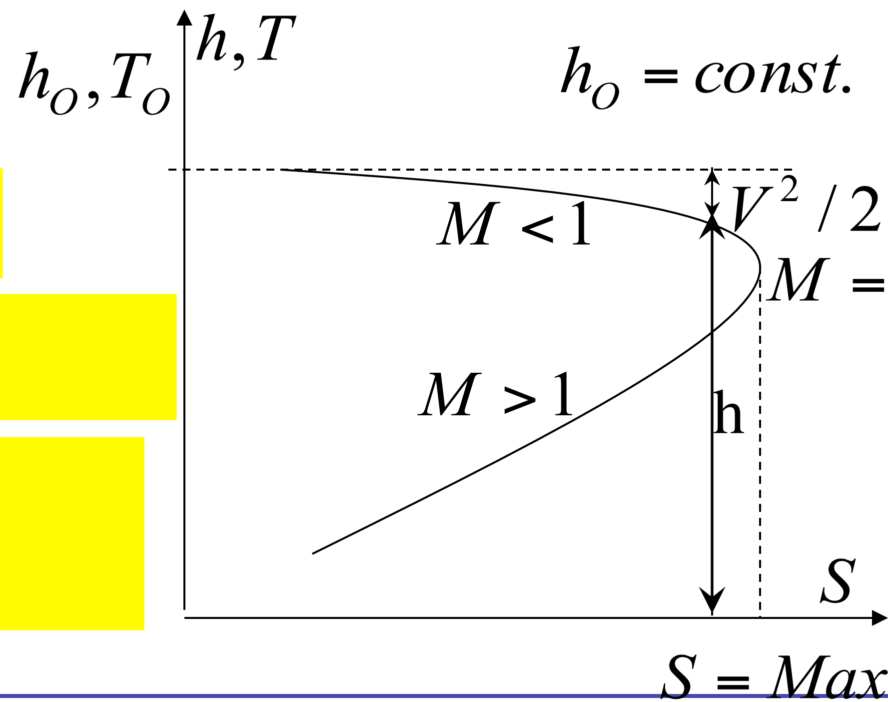
At M>1 dh/dS +ve dS ≥ 0

∴ dh +ve dT +ve ∴ dV -ve

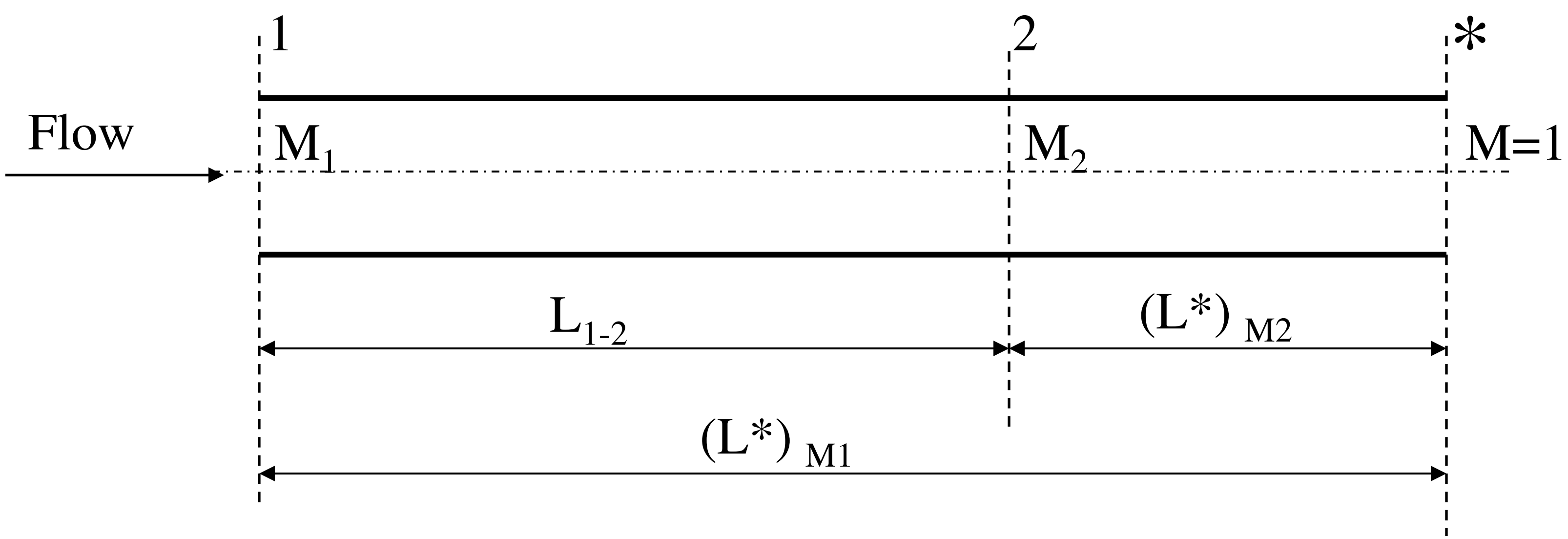
At M=1 dh/dS h which corresponding to the state of maximum entropy

This means that, A Fanno line moving only in the direction from left to right $M_1 > 1$, or $M_1 < 1$ then P_o -ve.

If m·/A increase S_{max} decease.



It was noted that L^*, the maximum length of duct which doesn't cause choking.



$$L_{1-2} = (L^*)_{M1} - (L^*)_{M2}$$

Multiply the equation by $(4F / D_H)$

$$(4FL_{1-2})/D_H = ((4F\,L^*)/ D_H)_{M1} - ((4F\,L^*)/ D_H)_{M2}$$

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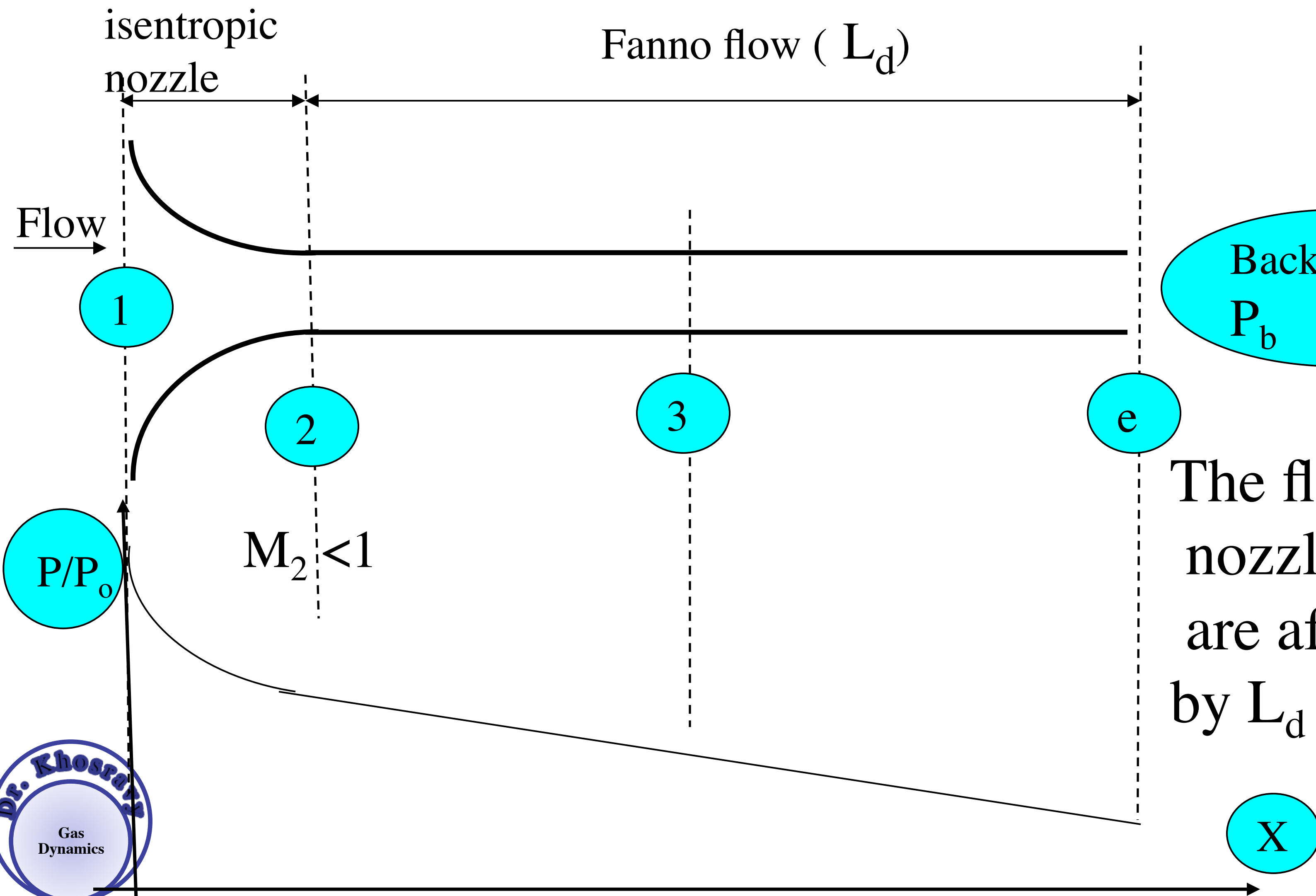
Fanno flow in a constant cross-sectional area duct proceeded by an isentropic nozzle



The flow in nozzle& duct are affected by L_d & P_b

$M_2<1$ the gas will accelerate in the duct owing to friction and pressure decrease

$M_e<1$ or 1 depend on P_b & L_d

If $M_e=1$ $(\dot{m}/A)_{max}$ choked condition

$$G^*=(\dot{m}/A)_{max} = \frac{Po}{\sqrt{To}}\sqrt{\frac{\gamma}{R}}\,[2/(\gamma+1)]^{(\gamma+1)/(2(\gamma-1))}$$

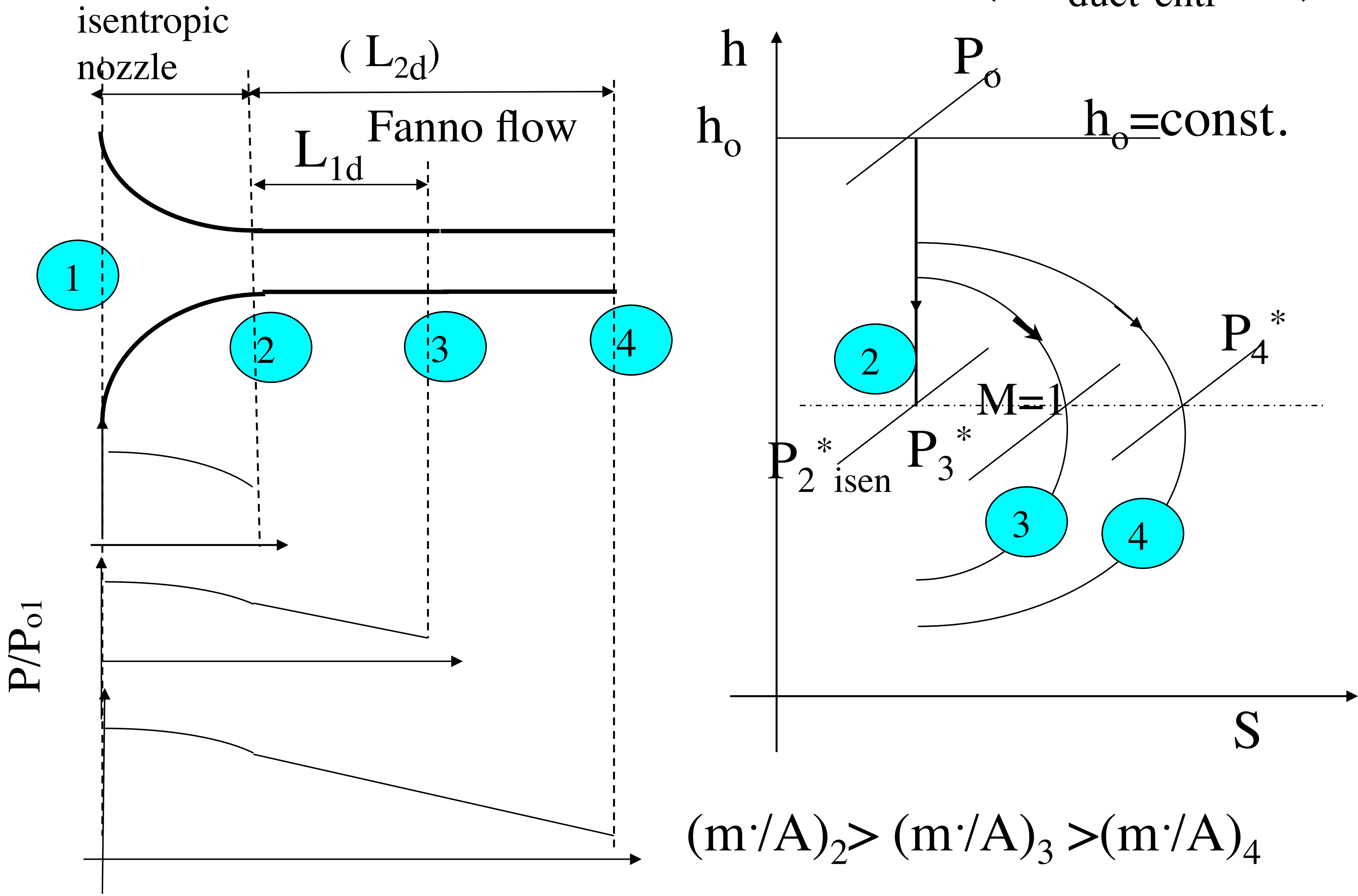
To increase mass, by decreasing T_o and/or increasing P_o at the nozzle inlet.
In this case $M_e=1$, but P_e would be higher.

Changes in duct length do not affect the mass flow rate

$M_2 > 1$, the gas will decelerate in the duct due to friction, if no shock occur, the flow will approach $M_e = 1$ directly.

In this case ($M_e = 1$), $\dot{m}_{max}$ is determined by the throat area of the nozzle and also by the area of the duct

Effect of increasing duct length (M $_{duct\ entr}$ <1)



$(m^{\cdot}/A)_2 > (m^{\cdot}/A)_3 > (m^{\cdot}/A)_4$

($M_{\text{duct entr}} > 1$)

At certain length, the M=1 at exit duct cross-sectional area

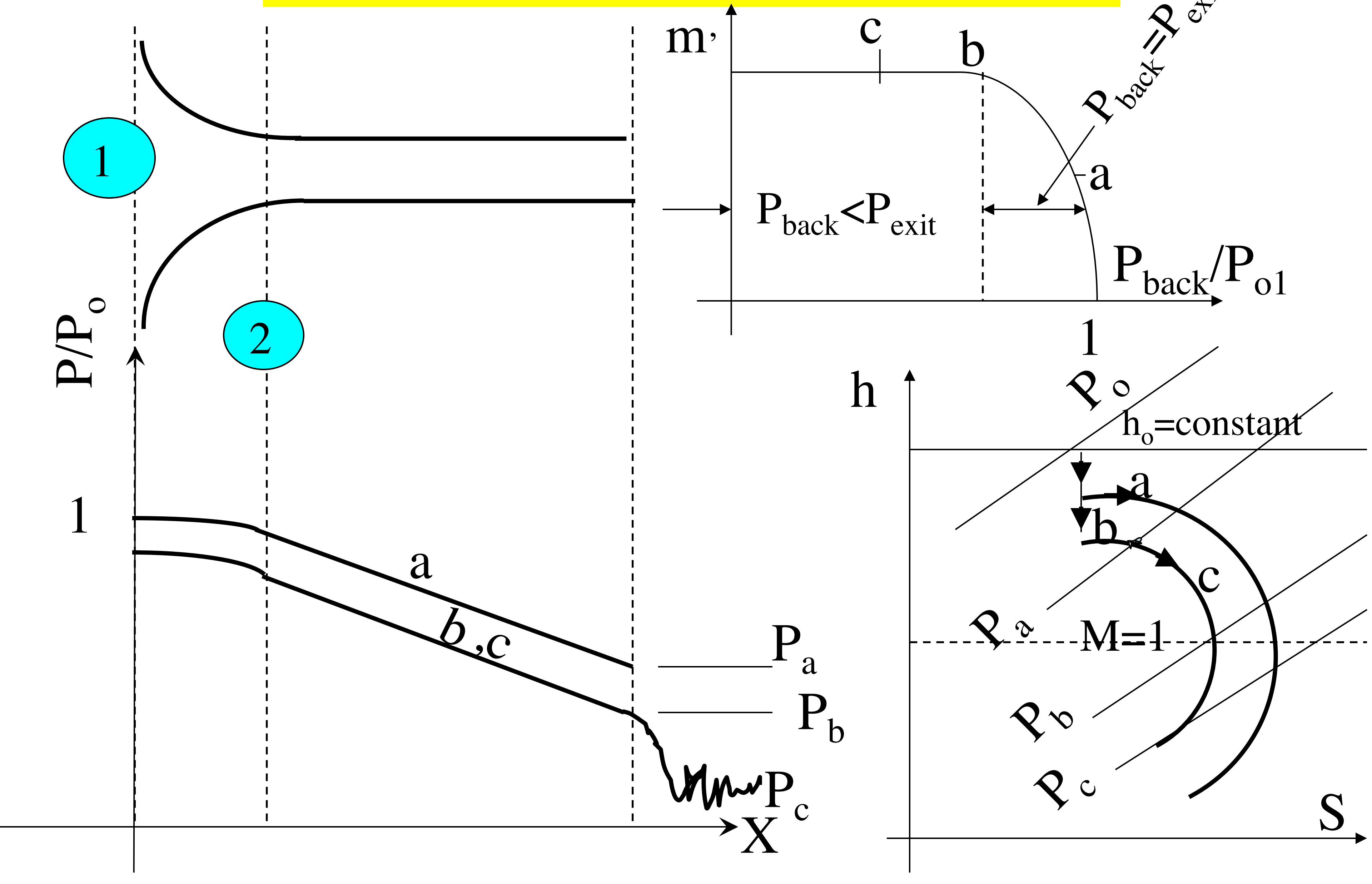
Subsonic conditions can also be attained at exit area, if discontinuity occurs.

If [$L_d > L_d^*$ at $M_e = 1$] , a shock appears in the duct

If the duct length is increased further, the shock will position itself further upstream.

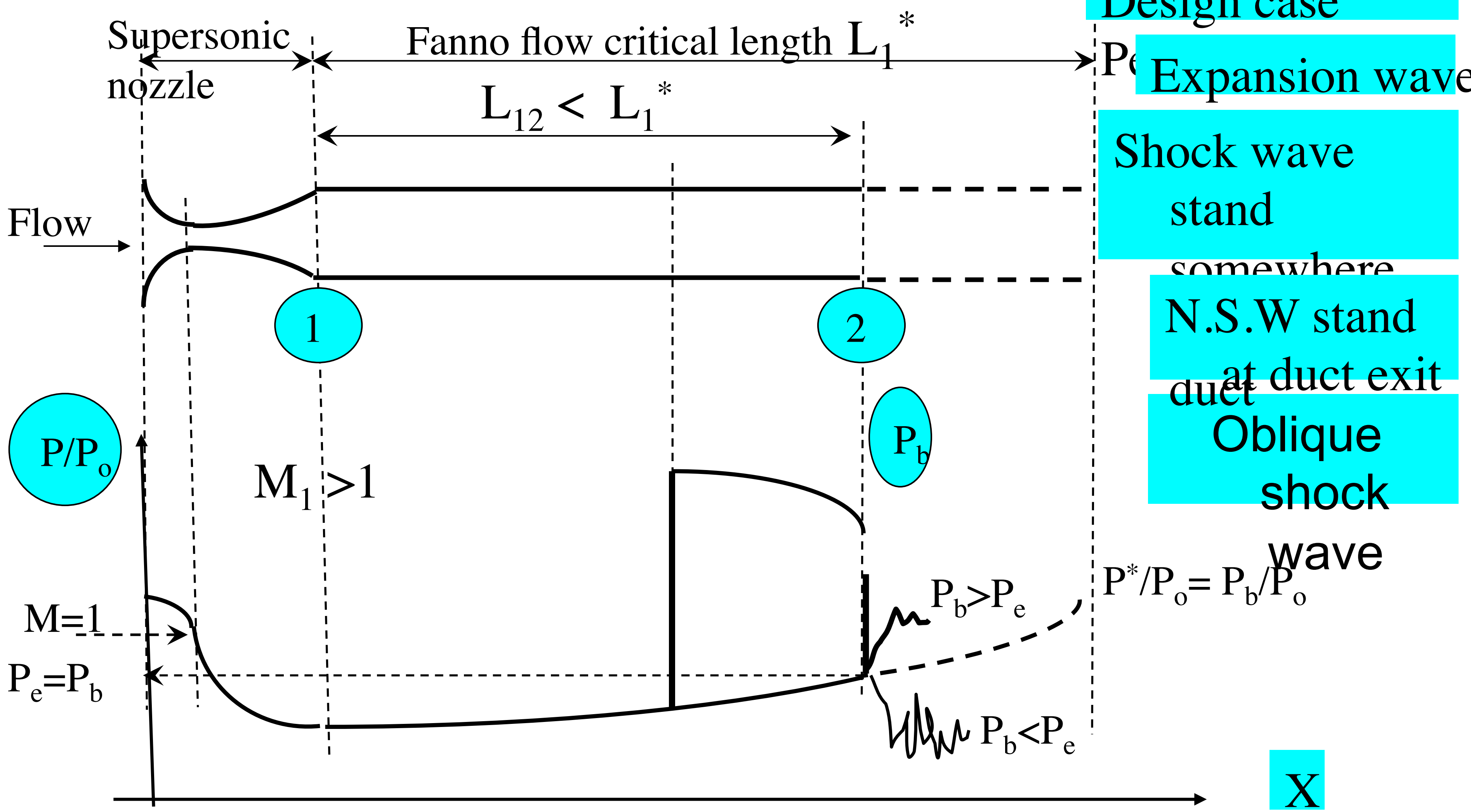
If the duct is very long, the shock will be at the throat of the nozzle. Beyond that length, there will be no shock at all. Then, the flow is subsonic at all point.

Effect of reducing back pressure (Mach at duct inlet ≤1)



Effect of reducing back pressure
(Mach at duct inlet >1)

In the duct $L_{12} < L^*$, there are 5 cases:-



The flow in nozzle& duct are affected by L_d & P_b