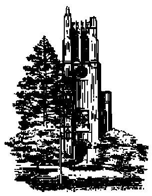
Monday June 27, 2016
Lecture 24

Basics of Counting

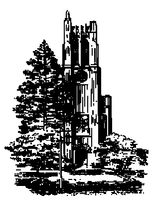


Notables

- Homework #13
  - Page 405, Problems 2 and 6
  - Page 432, Problem 16
  - Page 581, Problems 2, 6, and 8
  - Page 606, Problem 3
  - Page 615, Problem 2
  - Page 616, Problems 24, and 36
  - Due Wednesday June 29
- Read Chapter 9

The Pigeonhole Principle

Section 6.2



The Pigeonhole Principle

- Theorem: The Pigeonhole Principle: Let S and T be finite sets such that $|S| > |T|$. Then there is no one-to-one function $f: S \rightarrow T$
  - Proof: Trivial!
- Example: The decimal expansion of a rational number which is not an integer is *periodic*, that is, after a while the numbers repeat themselves.

Example: 29/54 = 0.5370370370370….

Computing 29/54

- 10*29 = 5*54 + 20
- 10*20 = 3*54 + 38
- 10*38 = 7*54 + 2
- 10*2 = 0*54 + 20
- 10*20 = 3*54 + 38
- …..
- 29/54 = 0.53703…

Consider the case $0 < m < n$

$$\frac{m}{n} = 0.d_1d_2d_3\ldots$$

Let's perform the following division now:

$$10 \times m = q \times n + r = 10 \times (n \times 0.d_1d_2d_3\ldots) = n \times (d_1 + 0.d_2d_3d_4\ldots)$$
$$= d_1 \times n + n \times 0.d_2d_3\ldots$$

Thus,

$10 \times m = d_1 \times n + r_1$ where $r_1 = n \times 0.d_2d_3d_4\ldots$ Note that $0 \le r_1 < n$

Similarly, $10 \times r_1 = d_2 \times n + r_2$ where $r_2 = n \times 0.d_3d_4d_5\ldots$

In general, we would have $10 \times r_j = d_{j+1} \times n + r_{j+1}$
where $r_{j+1} \in \{0, 1, 2, \ldots, n-1\}$

So, after a while the r's must repeat and thus the d's must repeat.

Example

- Claim: Let $d_1, d_2, \ldots, d_n$ be positive integers, not necessarily distinct. Then, some of these numbers add up to a number which is a multiple of n.
  - For example, 1, 1, 5, 5,10, 20, 36. Here $n = 7$.

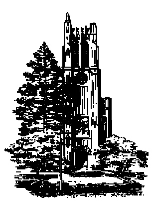
Proof: Define the following:

$$
\begin{aligned}
&b_0 = 0 && \Rightarrow\ b_0 = k_0 \times n + r_0 \\
&b_1 = d_1 && \Rightarrow\ b_1 = k_1 \times n + r_1 \\
&b_2 = d_1 + d_2 && \Rightarrow\ b_2 = k_2 \times n + r_2 \\
&b_3 = d_1 + d_2 + d_3 && \Rightarrow\ b_3 = k_3 \times n + r_3 \\
&\quad\vdots && \Rightarrow\quad \vdots \\
&b_n = d_1 + d_2 + d_3 + \cdots + d_n && \Rightarrow\ b_n = k_n \times n + r_n
\end{aligned}
$$

Note that $r_i \in \{0,1,\ldots,n-1\}$ and thus there must be b_i and b_j such that $r_i = r_j$. WLOG, assume $j > i$, $(\therefore b_j > b_i)$. Then we have $b_j - b_i = k_j \times n + r_j - k_i \times n - r_i = (k_j - k_i)n$. Therefore, $d_1 + d_2 + \cdots + d_j - (d_1 + d_2 + \cdots + d_i) = d_{i+1} + d_{i+2} + \cdots + d_j = \ell \times n$.

Relations and Their Properties

Section 9.1



Outline

- Review & Introduction
- Functions as Relations
- Relations on a Set
- Properties of Relations
  - Reflexive, symmetric, antisymmetric, transitive, equivalence
- Combining Relations
  - Union, intersection, difference
  - Composition of relations, Powers of relations

Review: Cartesian Products

- Definition: An *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element…, and a_n as its *nth* element.
- 2-tuples are called *ordered pairs*.
  - $(a,b) = (c,d)$ if and only if $a=c$ and $b=d$
  - $(a,b) \neq (b,a)$ unless $a=b$

Cartesian Product …

- Definition: The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to set A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \ldots, n\}$$

Binary Relations

- Definition: Let A and B be sets. A *binary relation* R from A to B is a subset of the Cartesian product $A \times B$.
- Notation: xRy means $(x,y) \in R$, and
  x is said to be related to y under R.
  $x\not R y$ denotes $(x,y) \notin R$.

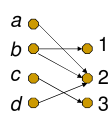
Example

- Let A be the set of all cities, and let B be the set of all states in the U.S.
  Define the relation R from A to B by:
  $(a,b) \in R$ if city a is in state b.
- Some ordered pairs belonging to relation R:

R= {(East Lansing, Michigan), (Boulder, Colorado), (Chicago, Illinois), (Columbus, Ohio), …}

Another Example

$A = \{a, b, c, d\}$ $\quad B = \{1, 2, 3\}$
$R = \{(a,2), (b,1), (b, 2), (c,3), (d,2)\}$.



Is this a function?

R	1	2	3
a		x	
b	x	x	
c			x
d		x	

Functions As Relations

- Recall that: Let $f: A \rightarrow B$ be a function. The *graph* of f is the set of ordered pairs
  $G_f = \{(x, f(x)) \mid x \in A\}$.
- The graph of a function from A to B is a relation from A to B.
- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly <u>one</u> ordered pair of R, then a function can be defined with R as its graph.

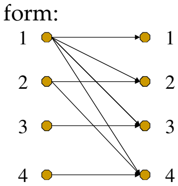
Relations on a Set

- Definition: A relation on a set A is a relation from A to A, that is, a subset of $A \times A$.
- Example: Let $A = \{1, 2, 3, 4\}$, and
  $R = \{(a,b) \mid a \text{ evenly divides } b\}$.
  - What are the elements of R?

  $R = \{(1,1), (1,2), (1,3), (1,4),$
  $(2,2), (2,4), (3,3), (4,4)\}$

Example

- R can be displayed graphically and in tabular form:



R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relations on a Set…

- Let A be a set with n elements. How many distinct relations are there on A?
  - Answer: The number of relations is the same as the number of subsets of $A \times A$. There are n^2 elements in $A \times A$. Thus, $P(A \times A)$, the power set of $A \times A$, has 2^{n^2} elements. Therefore, there are 2^{n^2} relations on a set with n elements.

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Types of Relations

- Definition: A relation R on a set A is called *reflexive* if $(a,a) \in R$ for every element $a \in A$, that is, $\forall a(a \in A \rightarrow (a,a) \in R)$
- Definition: A relation R on a set A is called *irreflexive* if $(a,a) \notin R$ for every element $a \in A$, that is, $\forall a(a \in A \rightarrow (a,a) \notin R)$
- Note that a relation can be both not reflexive and not irreflexive.

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Example

- Consider the following relations on {1,2,3,4}. Which are reflexive, irreflexive, neither?
  - R_1 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
    - Reflexive
  - R_2 = {(2,4), (4,2)}
    - Irreflexive
  - R_3 = {(1,2), (2,3), (3,4)}
    - Irreflexive
  - R_4 = {(1,1), (2,2), (3,3), (4,4)}
    - Reflexive
  - R_5 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}
    - Irreflexive
  - R_6 = {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
    - Neither

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Symmetric Relations

- Definition: A relation R on a set A is called *symmetric* if $(b,a) \in R$ whenever $(a,b) \in R$ for $a, b \in A$, that is,

$$\forall a \forall b (a \in A \wedge b \in A \wedge aRb \rightarrow bRa)$$

- Definition A relation R on a set A is called *antisymmetric* if whenever $(a,b) \in R$ and $(b,a) \in R$, then $a = b$, for a, b in A, that is,

$$\forall a \forall b (a \in A \wedge b \in A \wedge aRb \wedge bRa \rightarrow a = b)$$

- Note that these definitions are not complementary.

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Example

- Consider the following relations on {1,2,3,4}. Which are symmetric, antisymmetric, both, or neither?
  - R_1 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
    - Symmetric
  - R_2 = {(2,4), (4,2)}
    - Symmetric
  - R_3 = {(1,2), (2,3), (3,4)}
    - Antisymmetric
  - R_4 = {(1,1), (2,2), (3,3), (4,4)}
    - Both
  - R_5 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}
    - Neither
  - The "divides" relation on the set of positive integers
    - Antisymmetric
  - R_6 = {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
    - Neither

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Transitive Relations

- Definition: A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for a, b, c in A, that is,

$$\forall a \forall b \forall c (a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc \rightarrow aRc)$$

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Example

- Consider the following relations on {1,2,3,4}. Which are transitive?
  - R_1 = {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
    - Transitive
  - R_2 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
    - Transitive
  - R_3 = {(2,4), (4,2)}
    - Not Transitive
  - R_4 = {(1,2), (2,3), (3,4)}
    - Not Transitive
  - R_5 = {(1,1), (2,2), (3,3), (4,4)}
    - Transitive
  - R_6 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}
    - Not Transitive
  - The "divides" relation on the set of positive integers
    - Transitive

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Example Summary

- Consider the following relations on {1,2,3,4}. Name its properties?
  - R_1 = {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
  - R_2 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
  - R_3 = {(2,4), (4,2)}
  - R_4 = {(1,2), (2,3), (3,4)}
  - R_5 = {(1,1), (2,2), (3,3), (4,4)}
  - R_6 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}

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Example Summary

- Consider the following relations on {1,2,3,4}. Name its properties?
  - R_1 = {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
    - T
  - R_2 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
    - R/S/T
  - R_3 = {(2,4), (4,2)}
    - S
  - R_4 = {(1,2), (2,3), (3,4)}
    - A
  - R_5 = {(1,1), (2,2), (3,3), (4,4)}
    - R/S/A/T
  - R_6 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}
    - None
  - The "divides" relation on the set of positive integers
    - R/A/T
- Keys: R = Reflexive, S = Symmetric, A = Antisymmetric, T=Transitive

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