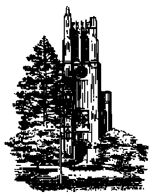
Tuesday June 21, 2016
Lecture 21

Solving recurrence relations



628

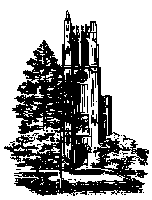
Notables

- Homework #12
  - Page 525, Problem 12
  - Page 525, Problem 14
  - Find the solution to $a_n = 2a_{n-1} + 2n^2$ with $a_1 = 4$.
  - Page 396, Problems 2, 4, 6, 12, 36
  - Page 413, Problem 6
  - Page 414, Problem 22
  - Due Thursday June 23
- Read Chapter 6

629

Solving Recurrence Relations

Section 8.2



630

Example

- The *Fibonacci* numbers, $f_0, f_1, f_2, \ldots$, are defined recursively by:
  - $f_0 = 0,$
  - $f_1 = 1,$
  - $f_n = f_{n-1} + f_{n-2} \quad n \geq 2.$
- Suppose we want to find f_6
  - $f_2 = f_1 + f_0 = 1 + 0 = 1$
  - $f_3 = f_2 + f_1 = 1 + 1 = 2$
  - $f_4 = f_3 + f_2 = 2 + 1 = 3$
  - $f_5 = f_4 + f_3 = 3 + 2 = 5$
  - $f_6 = f_5 + f_4 = 5 + 3 = 8$
- Is there a better way?
  - Can we come up with a closed-form formula for f_n ?

631

Recurrence Relations

Consider the following equation involving a recursive function f_n

$c_0 f_n + c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_r f_{n-r} = g(n)$

where each c_i is a constant. Clearly if the valuse of r consecutive $f\,'s$ in the sequence $f_{k-r}, f_{k-r+1}, \ldots, f_{k-1}$ are known for some k, then the value of f_k can be computed. The goal is to find a closed form solution for f_n. We will study how to solve such recurrence relations, in particular, when $g(n) = 0$. Example:

$$\begin{cases} f_n - f_{n-1} - f_{n-2} = 0 & \text{for } n \geq 2 \\ f_0 = 0, \ f_1 = 1 \end{cases}$$

Note that in the above example, $c_0 = 1, c_1 = -1,$ and $c_2 = -1$.

632

Solving Homogeneous Recurrence Relations

Consider the following homogeneous recurrence relation

$c_0 f_n + c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_r f_{n-r} = 0$

It is known that the solution to the above equation has the generic form α^n.

We proceed as follows:

(1) Construct the characteristic equation $c_0\alpha^r + c_1\alpha^{r-1} + c_2\alpha^{r-2} + \cdots + c_{r-1}\alpha + c_r = 0$.

(2) Find all the roots of the characteristic equation, call them $\alpha_1, \alpha_2, \ldots, \alpha_r$.

(3) If all the roots are distinct, then $f_n = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_r\alpha_r^n$. We will then use the initial conditions to find $A_1, A_2, \ldots, A_r$.

(4) If all the roots are not distinct we do the following. Suppose a root, say α_1, is a k-multiple root. Then the corresponding term for α_1 becomes:

$(A_1 n^{k-1} + A_2 n^{k-2} + \cdots + A_{k-2} n^2 + A_{k-1} n + A_k)\alpha_1^n$

633

Example

Solve the following recurrence relation:

$$\begin{cases} f_n - f_{n-1} - f_{n-2} = 0 & \text{for } n \ge 2 \\ f_0 = 0,\ f_1 = 1 \end{cases}$$

634

Solve the following recurrence relation:

$$\begin{cases} f_n - f_{n-1} - f_{n-2} = 0 & \text{for } n \ge 2 \\ f_0 = 0,\ f_1 = 1 \end{cases}$$

Solution: The charateristic equation is $\alpha^2 - \alpha - 1 = 0$ which has the distinct roots $\alpha_1 = \frac{1+\sqrt{5}}{2}$, and $\alpha_2 = \frac{1-\sqrt{5}}{2}$. So, $f_n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$.

We now use the initial conditions to find A's

$$\begin{cases} f_0 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = A_1 + A_2 = 0. \\ f_1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \end{cases}$$

These equations would give $A_1 = \frac{1}{\sqrt{5}}$, and $A_2 = -\frac{1}{\sqrt{5}}$. Thus,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

635

Solve the following recurrence relation.

$$\begin{cases} f_n + 6f_{n-1} + 12f_{n-2} + 8f_{n-3} = 0 & \text{for } n \ge 3 \\ f_0 = 1 \\ f_1 = -2 \\ f_2 = 8 \end{cases}$$

Solution: The characteristic equation is:

$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = (\alpha + 2)^3 = 0$

Its triple root is -2.

So, $f_n = (A_1 n^2 + A_2 n + A_3)(-2)^n$. Using the initial conditions we will find $A_1 = \frac{1}{2}$, $A_2 = -\frac{1}{2}$, and $A_3 = 1$.

636

Example

Recall that

$$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Let's formulate the above as a recurrence relation:

$$\begin{cases} S_n = S_{n-1} + n & \text{for } n \ge 1 \\ S_0 = 0 \end{cases}$$

Can we solve for S_n using the method we just learned?

637

Solve the following recurrence relation:

$$\begin{cases} S_n = S_{n-1} + n & \text{for } n \ge 1 \\ S_0 = 0 \end{cases}$$

Solution:

$S_n - S_{n-1} = n$

$\underline{S_{n+1} - S_n = n + 1}$

$S_{n+1} - 2S_n + S_{n-1} = 1$

$\underline{S_{n+2} - 2S_{n+1} + S_n = 1}$

$S_{n+2} - 3S_{n+1} + 3S_n - S_{n-1} = 0$

Characteristic equation is: $\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0 \Rightarrow (\alpha - 1)^3 = 0$.

So, we have a triple root, $\alpha = 1$. Thus, $S_n = (A_2 n^2 + A_1 n + A_0)(1)^n$

Using the initial conditions we will find: $A_2 = A_1 = \frac{1}{2}$, and $A_0 = 0$.

$$S_n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}$$

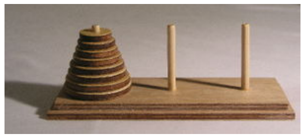
638

Thinking Recursively; Applications of recurrence relations

- Tower of Hanoi
- Binary Search
- Gossiping Problem
- Quick Sort

639

Tower of Hanoi



- Let T_n be the minimum number of moves that will transfer n disks from one peg to another, according to the game rules. We have:
  - $T_1 = 1$
  - $T_2 = 3$
  - …
  - $T_n = 2\ T_{n-1} + 1$

640

Binary Search

- Given an ordered list of items, the objective is to determine if the list contains a given item:
  - List: 3, 7, 8, 10, 14, 18, 22, 34
  - Given item: 25
- Algorithms:
  - Linear Search
  - Binary Search
- Let $C(n)$ be the number of two-item comparisons required to determine if the list contains the given item. If we do it via binary search, we have (worst case scenario) :
  - $C(1) = 1$
  - $C(n) = 1 + C(n/2)$

641

Gossiping Problem

- Consider a set of n people, where each person has a gossip to share with all others as follows. This is done by placing person-to-person "calls" whereby the two parties share all their gossips. What is the least number of calls, denoted $G(n)$, required so that everyone knows all the gossips?
  - $G(1) = 0$
  - $G(2) = 1$
  - $G(3) = 3$
  - …
  - $G(n) \le 1 + G(n-1) + 1$

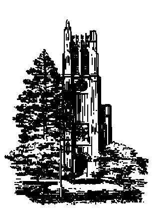
642

Merge sort analysis

- Let $M(n)$ be the cost of doing a merge sort on n numbers.
- $M(n) = 2M\left(\frac{n}{2}\right) + n, \quad M(1) = 0$
- Verify that $f(n) = n \log n$ is indeed a solution

643

The Basics of Counting



Read Chapter 6

Sections 6.1 – 6.4

644

Sample Counting Problems

- How many different passwords are there of length 5?
- How many different IP addresses are there for use on the Internet?
- How many one-to-one functions are there from a set of m elements to one with $n \ge m$ elements?
- In how many ways can we select a person from a group of 35 students and 24 faculty?
- How many non-negative integer solutions are there to the equation: $a + b + c + d = 17$?
- What size of a group does it take to have a better than 50% chance of two people having the same birthday?

645

Counting Principles

- The Product Rule:
  - If a procedure can be carried out by performing tasks $T_1, T_2, \ldots, T_n$ in sequence, where task T_i can be done in k_i ways after tasks $T_1, T_2, \ldots, T_{i-1}$ are done, then there are $k_1 \times k_2 \times \cdots \times k_n$ ways to carry out the procedure.
- The Sum Rule:
  - If there are n independent events $E_1, E_2, \ldots, E_n$ which can occur in $k_1, k_2, \ldots, k_n$ ways, respectively, then there are $k_1 + k_2 + \cdots + k_n$ ways in which *exactly one* of these events can occur.

646

Selection/Arrangement of Objects

- Many counting problems involve *selection* and/or *arrangement* of objects. Issues in solving such problems include:
  - Are objects unique?
  - Does the order matter?
  - Is replacement allowed?

647

Combinations & Permutations

- Definition: An r-*combination* of n objects is an *unordered* selection of r of these objects where replacement is not allowed. If the objects are unique, then the r-combination is just an r-element subset of the set of objects.
  - Example: From the set $S = \{1,2,3,4\}$ we can choose two elements in six different ways, namely, $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$
- The notation $C(n,r)$ denotes the number of r-combinations of n *distinct* objects.
  - Alternative notation is $\binom{n}{r}$

648