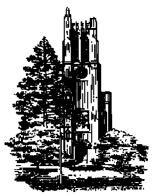
Wednesday June 8, 2016
Lecture 14

Algorithms Complexity



Notables

- Tentative Schedule for the week

Week	M	T	W	R	Topic	Section
1	6-6				Countable and uncountable sets	2.5
		6-7			Algorithms	3.1
			6-8		More on Algorithms	3.1
				6-9	Growth functions	3.2

Homework#8, Due Thursday (6/10)

- Sort the list [64, 13, 29, 32, 21, 46, 60] using
  - Exchange sort; show each pass
  - Bubble sort; show each pass
  - Merge sort; show each sublist
  - For each case, give the number of 2-number comparisons required.
- Use Newton's method to find the square root of 27 up to 2 decimal places. Use 7 as your initial guess.
- Page 204, Problem 52
- Page 216, Problem 2, 8, and 14
- Page 217, 34(a)
- Page 218, 74(a), and 74(b)

Complexity of An Algorithm

- The *complexity* of an algorithm is the cost, measured in some parameter (such as the running time, amount of storage, number of particular operations, etc) of using the algorithm.
- To express the complexity, we need to
  - Decide on the cost parameter(s).
  - Describe the cost function. How?

The Impact of Data Representation

- Consider the following tasks
  - T1: Adding two numbers.
  - T2: Deciding whether a number is even.
- Now, consider the following ways of representing numbers. Take, for example, integer twelve.
  - R1: "usual" representation: Example: 12
  - R2: "long" representation: Example: 111111111111
- Which representation is "preferred" for which task?

Testing if n is prime

1. Input: A positive integer $n > 1$.
2. For $i = 2$ to $n^{1/2}$ Do
   (a) Divide n by i, and let r be the remainder.
   (b) If $r = 0$ output "n is not prime", stop.
   End For
3. Output "n is prime", stop.

Worst-case analysis: cost = $d \cdot n^{1/2}$ where d is some constant.

Primality Testing …..

- So, if $n = 113$, the cost would be about $d \cdot n^{1/2} = d \cdot (113)^{1/2} = 11d$.
- How do we represent number n?
  - Using R1?
  - Using R2?
- The convention is to express the cost function in terms of the "minimum amount of information" (referred to as the input size) needed to describe the algorithm.

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The Growth Functions

- The goal is to be able to make quantitative assertions about algorithms for the purpose of comparing the "goodness" of algorithms, when the input size is "VERY LARGE".
- The Big-*Oh* notation, $O(\,)$
- The Big-Omega notation, $\Omega(\,)$
- The Big- Theta notation, $\Theta(\,)$

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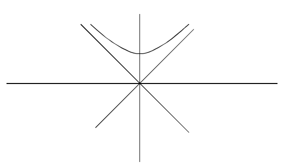
The Big-*Oh* notation, *O*( )

- Developed by Backmann, 1894, for asymptotic analysis
- The word "asymptotic" stems from a Greek root meaning "not falling together".



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Facts about log function

Logarithms and Log Properties

Definition

$y = \log_b x$ is equivalent to $x = b^y$

Example

$\log_5 125 = 3$ because $5^3 = 125$

Special Logarithms

$\ln x = \log_e x$ natural log

$\log x = \log_{10} x$ common log

where $e = 2.718281828\ldots$

Logarithm Properties

$\log_b b = 1$ $\quad \log_b 1 = 0$

$\log_b b^x = x$ $\quad b^{\log_b x} = x$

$\log_b \left(x^r\right) = r \log_b x$

$\log_b (xy) = \log_b x + \log_b y$

$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

The domain of $\log_b x$ is $x > 0$

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The Big-*Oh* notation, *O*( )

- The notation allows us to suppress unimportant detail and concentrate on the salient features; we can hide the details without losing the big picture
- Definition: A *positive* function $f(x)$ is Big O of another *positive* function $g(x)$, written as $f(x) = O(g(x))$, and read as $f(x)$ is Big-*Oh* of $g(x)$, if there are constants c and k such that

  $f(x) \le c\, g(x)$ for all $x > k$

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The Big-*Oh* notation, *O*( )

- By saying that $f(x)$ is Big-*Oh* of $g(x)$, we mean that f certainly does not "grow" at a faster rate than g; It might grow at the same rate, or it might grow more slowly.

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Example

Show that $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$ is $O(x^3)$.

Solution: $\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} \le \frac{x^3}{3} + \frac{x^3}{2} + \frac{x^3}{6} = x^3$ for $x \ge 0$

Thus, $f(x) \le cx^3$ when $c \ge 1$ $x \ge 0$

Note that we could also be sloppy and write $f(x) = O(x^{10})$. Nothing in the definition of O requires us to give a best possible bound.

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Example

Show that $\log n = O\left(\sqrt[m]{n}\right)$ $\forall m \in Z^+$

Solution:

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Example

Show that $\log n = O\left(\sqrt[m]{n}\right)$ $\forall m \in Z^+$

Solution:

First note that $n \le 2^{n-1}$ $\forall n \in Z^+$

Thus, $\log_2 n \le n - 1 \Rightarrow \log n < n$.

Now, $\log n = \frac{m}{m} \log n = m \log n^{1/m} = m \log \sqrt[m]{n} < m\sqrt[m]{n}$

$\Rightarrow \log n = O\left(\sqrt[m]{n}\right)$

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Example

- Determine whether each of these functions is $O(x)$. For each possible case, give a c and a k.
  1. $f(x) = 10$
  2. $f(x) = 3x + 7$
  3. $f(x) = x^2 + x + 1$
  4. $f(x) = 5\log x$
  5. $f(x) = \lfloor x \rfloor$
  6. $f(x) = \lceil x/2 \rceil$

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Example

- Determine whether each of these functions is $O(x)$. For each possible case, give a c and a k.
  1. $f(x) = 10$
     - $f(x) \le cx$, $c = 1$ an $k = 10$, or $c = 0.5$ and $k = 20$, etc
  2. $f(x) = 3x+7$
     - $f(x) = 3x+7 \le cx$, $c = 4$, $k = 7$, or $c = 5$ and $k = 4$, etc
  3. $f(x) = x^2 + x + 1$
     - No such c and k exist
  4. $f(x) = 5\log x$
     - $f(x) = 5\log x \le cx$, $c = 5$ an $k = 1$, etc
  5. $f(x) = \lfloor x \rfloor$
     - $f(x) = \lfloor x \rfloor \le cx$, $c = 1$ an $k = 0$, etc
  6. $f(x) = \lceil x/2 \rceil$
     - $f(x) = \lceil x/2 \rceil \le cx$, $c = 1$ an $k = 0$, etc

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Example

- Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.
  1. $f(x) = 2x^3 + x^2 \log x$
  2. $f(x) = 3x^3 + (\log x)^4$
  3. $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
  4. $f(x) = (x^4 + 5 \log x)/(x^4 + 1)$

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Example

- Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.
  1. $f(x) = 2x^3 + x^2 \log x$
     - $O(x^3)$
  2. $f(x) = 3x^3 + (\log x)^4$
     - $O(x^3)$
  3. $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
     - $O(x^1)$
  4. $f(x) = (x^4 + 5 \log x)/(x^4 + 1)$
     - $O(x^0)$
       - Use $c = 2$, and $k = 1$ (for example)

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Example

Show that $n^m = O(2^n) \quad \forall m \in Z^+$

For example, taking $m = 1000$,

we would have $n^{1000} = O(2^n)$

Solution hint:

Knowing that $\log n < n$ could be helpful.

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Example

Show that $n^m = O(2^n) \quad \forall m \in Z^+$

Solution:

Remember that $\log n < n$.

$\log_2 n^m = m \log_2 n = 2m \log \sqrt{n} < 2m\sqrt{n}$

$\log_2 n^m < 2m\sqrt{n} = \dfrac{2m \cdot n}{\sqrt{n}}$ But $\dfrac{2m}{\sqrt{n}} \le 1 \quad \forall n \ge 4m^2$

Therefore, $\log_2 n^m < n \qquad \forall n \ge 4m^2$

$\Rightarrow \qquad n^m < 2^n \qquad \forall n \ge 4m^2$

$\Rightarrow \qquad n^m = O(2^n)$

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Big-*Oh* Hierarchy

$1, \log n, \ldots, \sqrt[4]{n}, \sqrt[3]{n}, \sqrt[2]{n}, \sqrt{n}, n,$

$n \log n, n\sqrt{n}, n^2, n^3, n^4, \ldots, 2^n, n!, n^n$

Each term is the Big - *Oh* of any term to the right of it.

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The Big-Omega Notation, Ω( )

- Definition: A positive function $f(x)$ is Big-Omega of another positive function $g(x)$, written as $f(x) = \Omega(g(x))$, if and only if
  $g(x) = O(f(x))$.
- Ω-notation is used to express "lower bound". For example, finding a maximum element in a list of n elements would require $\Omega(n)$ two-number comparisons.

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The Big-Omega notation, Ω( )…

- An algorithm whose running time is $\Omega(n^2)$ is inefficient compared with another algorithm whose running time is $O(n\log n)$.

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The Big-Theta Notation, $\Theta(\,)$

- Definition: A positive function $f(x)$ is Big Θ of another positive function $g(x)$, written as $f(x) = \Theta(g(x))$, if there are constants c_1, c_2, and k such that

  $c_1\, g(x) \le f(x) \le c_2\, g(x)$ for all $x > k$
- Equivalently,

  $f(x) = \Theta(g(x)) \Leftrightarrow f(x) = O(g(x)) \wedge f(x) = \Omega(g(x))$
- Equivalently,

  $f(x) = \Theta(g(x)) \Leftrightarrow f(x) = O(g(x)) \wedge g(x) = O(f(x))$
- Example: $(x + 1)^2 = \Theta(3x^2)$

O, Ω, Θ, Recap, interpretation

- $f(x) = O(g(x))$
  - Function f grows either slower than function g or at the same rate as function g, but not faster than g.
- $f(x) = \Omega(g(x))$
  - Function f grows at least as fast as g; it can even grow faster.
- $f(x) = \Theta(g(x))$
  - Functions f and g grow at the same rate.

Using Limit to Express O, Ω, Θ

$$\text{If } \lim \frac{f(x)}{g(x)} = 0 \quad \Rightarrow \quad f(x) = O(g(x))$$
$$g(x) \ne O(f(x))$$
$$f(x) \ne \Theta(g(x))$$

$$\text{If } \lim \frac{f(x)}{g(x)} \in R^{+} \quad \Rightarrow \quad f(x) = O(g(x))$$
$$g(x) = O(f(x))$$
$$f(x) = \Theta(g(x))$$

$$\text{If } \lim \frac{f(x)}{g(x)} = +\infty \quad \Rightarrow \quad f(x) \ne O(g(x))$$
$$g(x) = O(f(x))$$
$$f(x) = \Omega(g(x))$$