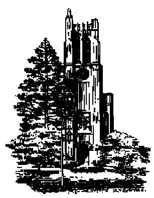
Tuesday June 28, 2016
Lecture 25

Relations



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Notables

- Homework #13
  - Page 405, Problems 2 and 6
  - Page 432, Problem 16
  - Page 581, Problems 2, 6, and 8
  - Page 606, Problem 3
  - Page 615, Problem 2
  - Page 616, Problems 24, and 36
  - Due Wednesday June 29
- Read Chapter 9

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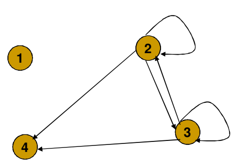
Pictorial Representation of Binary Relations

- Directed Graphs
  - Vertices, one for each element in the set
  - Arcs, one for each ordered-pair in the relation
- Example: Set $S = \{1,2,3,4\}$ and relation R on S

  $R = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$



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Equivalence Relations

- A relation R on a set A is called an *equivalence* relation if it is *reflexive*, *symmetric* and *transitive*.
- In such a relation, for each element $a \in A$, the set of all elements related to a under R is called the *equivalence class* of a, and is denoted by $[a]$.

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Example

- Determine the properties of each of the following relations defined on the set of all real numbers $\boldsymbol{R}$:
  - $R = \{(x,y) \mid x + y = 0\}$
  - $R = \{(x,y) \mid x = \pm y\}$
  - $R = \{(x,y) \mid x - y$ is a rational number$\}$
  - $R = \{(x,y) \mid x = 2y\}$
  - $R = \{(x,y) \mid x\,y \geq 0\}$
  - $R = \{(x,y) \mid x\,y = 0\}$
  - $R = \{(x,y) \mid x = 1\}$
  - $R = \{(x,y) \mid x = 1 \text{ or } y = 1\}$

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$R = \{(x,y) \mid x + y = 0\}$

- Solution:
  - It is not reflexive since, for example, $(1,1) \notin R$.
  - It is not irreflexive since, for example, $(0,0) \in R$.
  - Since $x + y = y + x$, it follows that if $x + y = 0$ then $y + x = 0$, so the relation is symmetric.
  - It is not antisymmetric since, for example, $(-1,1)$ and $(1,-1)$ are both in R, but $1 \neq -1$
  - The relation is not transitive since, for example, $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.

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$R = \{(x,y) \mid x = \pm y\}$

- Solution:
  - Since for each x, xRx then it is reflexive.
  - Since it is reflexive, it is not irreflexive.
  - Since $x = \pm y$ if and only if $y = \pm x$, then it is symmetric.
  - It is not antisymmetric since, for example, (1,–1) and (–1,1) are both in R but $1 \neq -1$
  - It is also transitive because essentially the product of ± 1 and ± 1 is ± 1.

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$R = \{(x,y) \mid x - y$ is a rational number$\}$

- Solution:
  - It is reflexive since $x - x = 0$ is a rational number.
  - Since it is reflexive, it is not irreflexive.
  - It is symmetric because if $x - y$ rational so is $-(x - y) = y - x$.
  - It is not antisymmetric because, for example, (1, –1) and (– 1,1) are both in R but $1 \neq -1$.
  - It is transitive because if $x - y$ is a rational and $y - z$ is a rational, so is $x - z$.

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$R = \{(x,y) \mid x = 2y\}$

- Solution:
  - It is not reflexive since, for example, $(1,1) \notin R$.
  - It is not irreflexive since, for example, $(0,0) \in R$.
  - It is not symmetric since, for example, $(2,1) \in R$ but $(1,2) \notin R$.
  - It is antisymmetric because $x = y = 0$ is the only time that (x,y) and (y,x) are both in R.
  - It is not transitive since, for example, $(4,2) \in R$ and $(2,1) \in R$ but $(4,1) \notin R$.

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$R = \{(x,y) \mid x\,y \geq 0\}$

- Solution:
  - It is reflexive since $x \cdot x \geq 0$.
  - Since it is reflexive, it is not irreflexive.
  - It is symmetric as the role of x and y are interchangeable.
  - It is not antisymmetric since, for example, (2,3) and (3,2) are both in R, but $2 \neq 3$.
  - It is not transitive because, for example, (1,0) and (0, – 2) are both in R but (1, – 2) is not.

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$R = \{(x,y) \mid x\,y = 0\}$

- Solution:
  - It is not reflexive since $(1,1) \notin R$.
  - It is not irreflexive since $(0,0) \in R$.
  - It is symmetric as the role of x and y are interchangeable.
  - It is not antisymmetric since, for example, (2,0) and (0,2) are both in R but $2 \neq 0$.
  - It is not transitive because, for example, (1,0) and (0, – 2) are both in R but (1, – 2) is not.

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$R = \{(x,y) \mid x = 1\}$

- Solution:
  - It is not reflexive since $(2,2) \notin R$.
  - It is not irreflexive since $(1,1) \in R$.
  - It is not symmetric since, for example, $(1,2) \in R$ but $(2,1) \notin R$.
  - It is antisymmetric because if $(x,y) \in R$ and $(y,x) \in R$ it means $x = y = 1$.
  - It is transitive since if $(1,y) \in R$ and $(y,z) \in R$, so is $(1,z) \in R$.

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$R = \{(x,y) \mid x = 1 \text{ or } y = 1\}$

- Solution:
  - It is not reflexive since $(2,2) \notin R$.
  - It is not irreflexive since $(1,1) \in R$.
  - It is symmetric as the roles of x and y are interchangeable.
  - It is not antisymmetric since, for example, (2,1) and (1,2) are both in R but $2 \neq 1$.
  - It is not transitive since, for example, (3,1) and (1,7) are both in R but $(3,7) \notin R$.

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Summary

Relation	R	IR	S	A	T	E
$R = \{(x,y) \mid x + y = 0\}$			✓			
$R = \{(x,y) \mid x = \pm y\}$	✓		✓		✓	✓
$\{(x,y) \mid x - y$ is a rational number$\}$	✓		✓		✓	✓
$R = \{(x,y) \mid x = 2y\}$				✓		
$R = \{(x,y) \mid x\,y \geq 0\}$	✓		✓			
$R = \{(x,y) \mid x\,y = 0\}$			✓			
$R = \{(x,y) \mid x = 1\}$				✓	✓	
$R = \{(x,y) \mid x = 1 \text{ or } y = 1\}$			✓			

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Combining Relations

- Since relations are special kind of sets, all of the operators we used for combining sets could be used to combine relations.

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Combining Relations – Examples

- Let $A=\{1, 2, 3\}$ and $B=\{1, 2, 3, 4\}$
- Let R_1 and R_2 be relations from A to B
  - $R_1 = \{(1,1), (2,2), (3,3)\}$
  - $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$
  - $R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$
  - $R_1 \cap R_2 = \{(1,1)\}$
  - $R_1 - R_2 = \{(2,2), (3,3)\}$

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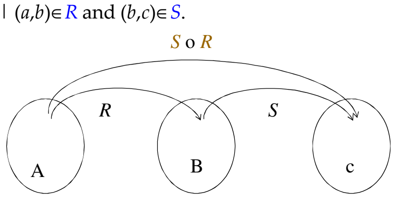
Composition of Relations

- Definition: Let R be a relation from a set A to a set B, and S be a relation from B to a set C. The *composition* of R and S, denoted by $S \circ R$, is the relation consisting of ordered pairs (a,c), where:
  - $a \in A$, $c \in C$, and
  - $\exists b \in B \mid (a,b) \in R$ and $(b,c) \in S$.



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Example

- Let $A=\{1, 2, 3\}$, $B=\{x, y, w, z\}$, $C=\{0, 1, 2\}$, R a relation from A to B:
  - $R = \{(1, x), (1, z), (2, w), (3, x), (3, z)\}$, and

  S a relation from B to C:
  - $S = \{(x,0), (y,0), (w,1), (w,2), (z,1)\}$
- What is $S \circ R$?

  $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

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Powers of Relations

- Definition: Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by:
  - $R^1 = R$,
  - $R^{n+1} = R^n \circ R$, for $n \geq 1$.
  - So:
    - $R^2 = R \circ R$,
    - $R^3 = R^2 \circ R = (R \circ R) \circ R$
    - …

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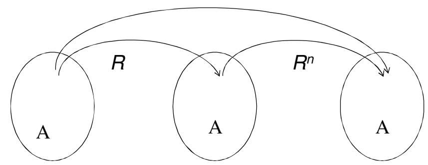
R^{n+1}

$R^{n+1} = R^n \circ R$



In the digraph representation, R^n consists of all the order pairs (x,y) where y is reachable from x using a directed path of length n.

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Example

- Consider the following relations on {1,2,3,4}.
  - $R = \{(1,2), (1,3), (3,4)\}$
  - Is R transitive?
    - No
  - Find $R^2 = R \circ R$
  - $R^2 = \{(1,4)\}$
- Consider the following relation:
  - $R = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$
  - Is R transitive?
    - Yes
  - Find $R^2 = R \circ R$
  - $R^2 = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$

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Characterizing Transitive Relations

- Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

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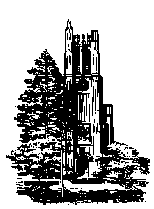
Closures of Relations

Section 9.4



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Concept of "closure"

- The natural numbers N are
  - Closed under addition: if $n, m \in N$, then $n + m \in N$,
  - Not closed under subtraction: $1, 2 \in N$, but $1 - 2 \notin N$
  - The closure of N under subtraction—i.e., smallest set containing N and closed under subtraction—is Z.
- The set of integers Z is
  - Closed under multiplication: if $n, m \in Z$, then $n \cdot m \in Z$
- Not closed under division: $1, 2 \in Z$, but $(1/2) \notin Z$
  - The closure of Z under division—i.e., smallest set containing Z and closed under division—is Q.

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Closure of a binary relation

- Definition: Let *R* be a binary relation on a set *A* and **P** be a property of binary relations (i.e., reflexive, symmetric, etc.). If there is a relation *S* such that

  1) *S* contains *R* and *S* satisfies **P**, and

  2) *S* is a subset of every other set that both contains *R* and satisfies **P**, then *S* is called *the closure of R with respect to* **P**.

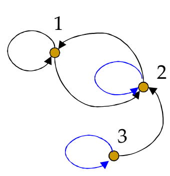
"Closing" a relation has important applications in databases, analysis of algorithms, programming and programming languages, networking, etc.

How to close a relation

- When a property does not hold for a relation, we try to *minimally* augment the relation so that the property holds.
- Involves adding pairs to the original relation
  - Just those pairs needed to make the property true.
  - No more pairs than are needed.

Reflexive Closure

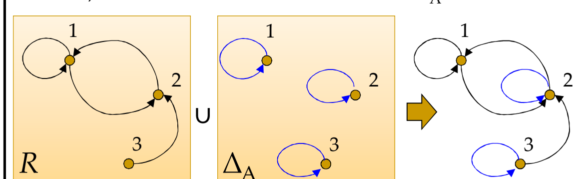
- Example: Consider the relation
  $R = \{(1,1), (1, 2), (2,1), (3, 2)\}$ on set $\{1, 2, 3\}$
  - Is it reflexive?
  - How can we produce a reflective relation containing *R* that is as small as possible?



Reflexive Closure

- Given a set *A*, the *diagonal relation*, denoted Δ_A, on *A* is defined as follows:

  $\Delta_A = \{ (a, a) \mid a \in A \}$
- The reflexive closure of a relation *R* on a set *A* can be formed by adding Δ_A to *R*.
- I.e., the reflexive closer of *R* is: $R \cup \Delta_A$



Reflexive Closure

- The reflexive closure of a relation *R* on a set *A* is $R \cup \Delta_A$
- <u>Properties:</u>
  - $R \subseteq (R \cup \Delta_A)$;
  - $R \cup \Delta_A$ is reflexive;
  - $\forall S \quad (R \subseteq S \;\wedge\; S \text{ is reflexive}) \;\rightarrow\; (R \cup \Delta_A) \subseteq S$.
- In zero-one matrix notation:
  $M_{R\cup\Delta} = M_R \vee M_\Delta = M_R \vee I_A$
- So, if *A* is finite, to find the matrix for the reflexive transitive closure of *R*, just turn on the diagonal bits of the matrix for *R*

Exercise

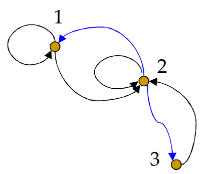
- Consider the "less than" relation, "<", on **Z**
- What well-know relation is the reflexive closure of the "less than" relation?

$\{ (a,b) \mid (a,b) \in \mathbf{Z}\times\mathbf{Z} \wedge a < b\} \cup \{(a,a) \mid a \in \mathbf{Z}\}$

$= \{ (a,b) \mid (a,b) \in \mathbf{Z}\times\mathbf{Z} \wedge a \le b\}$

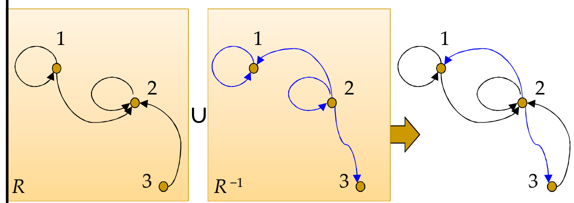
Symmetric Closure

- Example: Consider
  $R = \{(1,1), (1,2), (2,2), (3,2)\}$
  - R is not symmetric
  - How can we produce a symmetric relation containing R that is as small as possible?



Symmetric Closure

- Let R be a binary relation on a set A. The *inverse* of R, denoted R^{-1}, is defined:
  $$R^{-1} = \{(b,a) \mid (a, b) \in R\}.$$
- The symmetric closure of R is the relation $R \cup R^{-1}$.



Symmetric Closure

- The symmetric closure of relation R is the relation
  $R \cup R^{-1}$.
- Properties:
  - $R \subseteq (R \cup R^{-1})$;
  - $R \cup R^{-1}$ is symmetric;
  - $\forall S \quad (R \subseteq S \ \wedge \ S \text{ is symmetric}) \ \rightarrow \ (R \cup R^{-1}) \subseteq S$.
- In zero-one matrix notation: $M_R \vee M_R^t$

Exercise

- Consider the "less than" relation on $\mathbf{Z}$
- What well-known relation is the symmetric closure of "less than" on $\mathbf{Z}$?

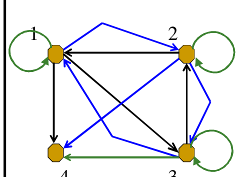
$\{ (a,b) \in \mathbf{Z{\times}Z} \mid a < b\} \cup \{ (b,a) \in \mathbf{Z{\times}Z} \mid a < b\}$

$= \{(a,b) \in \mathbf{Z{\times}Z} \mid a < b \vee b < a\}$

$= \{(a,b) \in \mathbf{Z{\times}Z} \mid a \neq b\}$

Transitive Closures

- Consider $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$.
  - R is not transitive.
  - What edge(s) are missing?
  - Is our new relation transitive?
  - Keep going until it is …



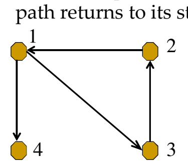
Transitive Closure: formalizing this process …

- When a relation R on a set A is not transitive:
  - We want to minimally augment R (adding the minimum number of ordered pairs) to obtain a transitive relation.
  - To see how this can be done, we formalize the concept of a *path in a relation R*

Paths in Directed Graphs

- Definition: A *path* from a to b in a digraph G is a sequence of one or more adjacent edges

  $(a, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, b)$.
  - Denoted: $a, x_1, x_2, x_3, \ldots, x_{n-1}, b$
  - Has *length* n.
  - If $a = b$, the path is called a *circuit* or a *cycle,* since the path returns to its start.



- 1, 3, 2 denotes a path of length 2 from 1 to 2, consisting of edges (1,3), (3,2).
- 1, 3, 2, 1 denotes a cycle of length 3: consisting of edges (1,3), (3,2), (2.1).
- There are no paths from 4 to 1, or 4 to 2, or 4 to 3

Exercise

- Given a digraph G, let the "path" relation P on the vertices of G be defined as follows:
  - aPb if and only if there is a path in G from a to b.
- Is this relation transitive?
  - Yes

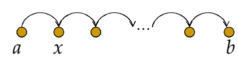
Paths in Relations

- A *path* from a to b in relation R is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R, \ldots,$ $(x_{n-1}, b) \in R$ — that is, with aRx_1, x_1Rx_2, …, $x_{n-1}Rb$.
- If $a = b$, the path is called a *cycle*.
- Example. Let $R = \{(1,3), (1,4), (2,1), (3,2)\}$
  - 1, 3, 2 is a path in R from 1 to 2 (of length 2): $1R3$ and $3R2$.
  - 1, 3, 2, 1 is a path in R from 1 to 1 (of length 3): $3R2$, $2R1$, and $1R4$.
  - There is no path in R from 4 to 1, nor to 2, nor to 3.

Theorem 1

Theorem: Let R be a relation on a set A.
There is a path in R of length n from a to b if and only if $(a, b) \in R^n$.

Proof: Induct on n.
- Basis step: There is a path in R of length 1 from a to b iff $(a,b) \in R = R^1$. This establishes the basis step.
- Induction step: Assume $n \geq 1$ and that the theorem is true for n. (IH)
  There is a path of length n+1 from a to b iff there is a path from a to x of length 1 and a path from x to b of length n, for some $x \in A$.



But this latter statement is true iff $(a,x) \in R$ and (by the IH) $(x,b) \in R^n$; which, in turn, is true iff $(a,b) \in R \circ R^n = R^{n+1}$.

This establishes the induction step. Thus, the theorem holds for all $n \geq 1$.

Transitive Closure

- Definition: Let R be a relation on a set A.

  The *connectivity relation* is the relation R* defined as:

  $R^* = \{(a,b) \mid \exists$ a path in R from a to $b\}$.

- From this definition and the previous theorem, we conclude that:

  $R^* = \bigcup_{k=1}^{\infty} R^k$.

We used this notion of "connectivity" in the airplane flights example.

Transitive Closure.

- **Theorem 2** Let R be a relation on a set A. Then the relation R^* is the transitive closure of R.
- Proof:
  1. $R \subseteq R^*$, because $R^* = \cup_{k=1, 2, \ldots\infty} R^k$..
  2. R^* is transitive, because its definition implies $(a, b) \in R^*$ and $(b, c) \in R^*$ only if $(a, b) \in R^n$ and $(b, c) \in R^m$, for some $n \geq 1$ and $m \geq 1$; which implies $(a, c) \in R^{n+m} \subseteq R^*$.
  3. $\forall S\ ((R \subseteq S \wedge S$ is transitive$) \rightarrow R^* \subseteq S)$.

     Proof. Let S be a transitive relation, and assume $R \subseteq S$.

     A simple induction argument shows that $R^k \subseteq S^k$ follows from the assumption $R \subseteq S$, for all positive k . (Exercise)

     It follow that $R^* \subseteq S^*$.

     Moreover, because S is transitive, $S^k \subseteq S$ (see last theorem of sect. 6.1), for all positive k.

     This, in turn, implies $S^* \subseteq S$, by definition of S^*.

     Therefore, $R^* \subseteq S$.

Transitive Closure

- Lemma: Let A be a set with n elements, and R be a relation on A. If there is a path in R from a to b, where $a \neq b$, then the shortest path from a to b has length less than n.
  - Proof: The shortest path cannot have any repeat vertices on it, or there would be a cycle that could be removed to produce a shorter path from a to b. Since there are no repeated vertices, the number of vertices is at most n and the path's length is at most $n - 1$.
- Corollary: If A has n elements and R is a relation on A, then $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Transitive Closure – Cont.

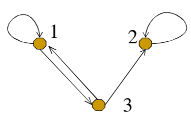
Corollary: Let M_R be the incidence matrix of relation R, where R is defined on a set with n elements.

The incidence matrix of the transitive closure R^* is:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}.$$

Example

- Find the matrix $\mathbf{M}_R^*$ of the transitive closure of R:



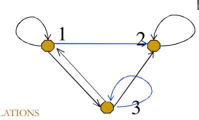
$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_R^* = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Algorithm 1 for TC

- Procedure for computing <u>transitive closure</u>:

  $\mathbf{A} = \mathbf{M}_R$;

  $\mathbf{B} = \mathbf{A}$;

  For i=2 to n do the following:

  $\mathbf{A} = \mathbf{A} \otimes \mathbf{M}_R$; // Boolean product

  $\mathbf{B} = \mathbf{B} \vee \mathbf{A}$; // Join operation

Transitive Closure: Improving the algorithm

- Lemma: Given a relation R on a set A and a positive integer k, if $\cup_{n=1,\ldots,k} R^n = \cup_{n=1,\ldots,k+1} R^n$, then $R^{k+n} = R^k$, for all $n \geq 1$.
  - Proof: Exercise (hint: induct on n).
- Theorem: Given a relation R on a set A and a positive integer k, if $\cup_{n=1,\ldots,k} R^n = \cup_{n=1,\ldots,k+1} R^n$, then

  $R^* = R \cup R^2 \cup \ldots \cup R^k$.
  - Proof: The previous lemma implies that $\cup_{n=k,k+1,k+2,\ldots,\infty} R^n = R^k$.

    Thus, $R^* = \cup_{n=1,2,\ldots,k-1,k,k+1,\ldots,\infty} R^n$

    $= (\cup_{n=1,2,\ldots,k-1} R^n) \cup (R^k)$

    $= (\cup_{n=1,2,\ldots,k} R^n)$

Transitive Closure: Improving the algorithm

- Lemma: If

  $R^1 \cup R^2 \cup \cdots \cup R^k = R^1 \cup R^2 \cup \cdots \cup R^{k+1}$,

  then

  $R^1 \cup R^2 \cup \cdots \cup R^k = R^1 \cup R^2 \cup \cdots \cup R^n$,

  for all $n \geq k$.
  - Proof: Exercise (hint: induct on n).
- Theorem: If

  $R^1 \cup R^2 \cup \cdots \cup R^k = R^1 \cup R^2 \cup \cdots \cup R^{k+1}$,

  then $R^* = R \cup R^2 \cup \ldots \cup R^k$.
  - Proof: The previous lemma implies that $\cup_{n=k,k+1,k+2,\ldots,\infty} R^n = R^k$.

    Thus, $R^* = \cup_{n=1,2,\ldots,k-1,k,k+1,\ldots,\infty} R^n$

    $= (\cup_{n=1,2,\ldots,k-1} R^n) \cup (R^k)$

    $= (\cup_{n=1,2,\ldots,k} R^n)$

Algorithm 2 for TC

- Procedure for computing <u>transitive closure</u>:

  $\mathbf{A} = \mathbf{M}_R$;

  $\mathbf{B} = \mathbf{A}$;

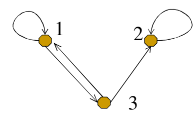
  While $\mathbf{A} \neq \mathbf{A} \otimes \mathbf{M}_R$ do the following:

  $\mathbf{A} = \mathbf{A} \otimes \mathbf{M}_R$; // Boolean product

  $\mathbf{B} = \mathbf{B} \vee \mathbf{A}$; // Join operation

Example

- Find the matrix $\mathbf{M}_R{}^*$ of the transitive closure of R:



$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{M}_R{}^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_R{}^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_R{}^* = \mathbf{M}_R \vee \mathbf{M}_R{}^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Equivalence Relations

- A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric and transitive.
- In such a relation, for each element $a \in A$, the set of all elements related to a under R is called the *equivalence class of a*, and is denoted by $[a]_R$.

$$[a]_R = \{\, b \mid b \in A \wedge aRb \,\}$$

Examples

Suppose that R …

- relate two cities iff they are connected by roads.
  - All cities that you can drive between form an equivalence class
  - e.g., cities on Mackinaw Island form one equivalence class.
- relates two people iff they have the same last name.
  - All people who have any given last name form an equivalence class
  - e.g., all people whose last name is "Smith" are one equivalence class; all people whose last name is "Brown" are another; etc.
- relates two real numbers iff they have the same absolute value
  - Some equivalence classes: {0}, {5, -5}, {-π, π}, {0.333…, -0.333…}, etc.

Example: the integers mod m

- a R b = { (a, b) | a ≡ b (mod m) }
- If m=5 then we have 5 classes induced by R
- $[0]_R = \{ 0, 5, -5, 10, -10, 15, -15, \ldots\}$
- $[1]_R = \{ 1, -1, 6, -6, 11, -11, \ldots\}$
- $[2]_R = \{ 2, -2, 7, -7, 12, -12, \ldots\}$
- $[3]_R = \{ 3, -3, 8, -8, 13, -13, \ldots\}$
- $[4]_R = \{ 4, -4, 9, -9, 14, -14, \ldots\}$
- Note that every integer belongs to one and only one of these classes – ***the R-equivalence classes partition Z***.

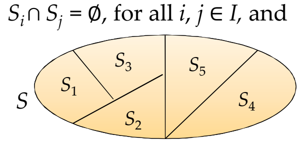
Equivalence classes & partitions

- Definition: A *partition of a set S* is a collection of pair-wise disjoint, non-empty subsets of S whose union is S.
- That is, a collection of subsets, $\{ S_k \mid k \in I \}$, forms a partition of a set S iff

  $S_k \subseteq S$ and $S_k \neq \emptyset$, for all $k \in I$;

  $S_i \neq S_j$ implies $S_i \cap S_j = \emptyset$, for all $i, j \in I$, and

  $S = \cup_{k \in I} S_k$



Equivalence classes & partitions

Theorem:
Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.

Proof:

To show: $[a] \neq \emptyset$, for $a \in S$. This follows b/c $a \in [a]$. Why??

To show: $[a] \neq [b]$ implies $[a] \cap [b] = \emptyset$, for $a, b \in S$.

We show that $[a] \cap [b] \neq \emptyset$ implies $[a] = [b]$. Assume $[a] \cap [b] \neq \emptyset$.
Then aRc and bRc for some c. In turn, symmetry & transitivity imply aRb.
Suppose $x \in [a]$. Then xRa. and aRb imply xRb, and so $x \in [b]$.
This shows $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar (Exercise).
Thus, $[a] = [b]$.

To show: $S = \cup_{a \in S} [a]$. Exercise.

Equivalence classes & partitions

Theorem.
Conversely, given any partition, $\{ S_k \mid k \in I \}$, of the set S, there is an equivalence relation R that has the sets S_k, $k \in I$, as its equivalence classes.

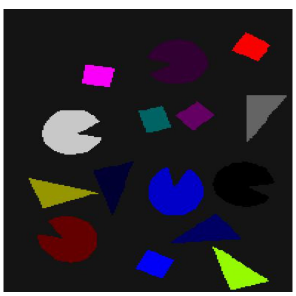
Proof:

For $a, b \in S$, define aRb iff $a \in S_j$ and $b \in S_j$ for some $j \in I$.

To show: R is an equivalence relation.

Reflexivity and symmetry follow trivially from the definitions. (Exercise)
For transitivity: Assume aRb and bRc, where $a, b, c \in S$.
Then $a \in S_j$ and $b \in S_j$ for some $j \in I$ and $b \in S_k$ and $c \in S_k$ for some $k \in I$.
Since $b \in S_j \cap S_k$ and $\{ S_k \mid k \in I \}$ is a partition, we conclude $S_j = S_k$.
Thus, $a \in S_j$ and $c \in S_j$, which implies aRc.

Example: pixel adjacency partitions a binary image into "blobs" or "objects"



15 subsets of pixels. Within each subset, pixels are connected by a path through neighbors. Pixels in each subset are not connected to pixels in any other subset. (Connecting paths are not allowed to go through background pixels.)

Transitive Closure – optional

- Toward a more efficient algorithm (Warshall's)
- Definition: Let R be a relation on $S=\{v_1,v_2,\ldots, v_n\}$.
  The *interior vertices* of a path of length m from a to b: $a, x_1, x_2, x_3, \ldots, x_{m-1}, b$ are:
  $x_1, x_2, x_3, \ldots, x_{m-1}$.

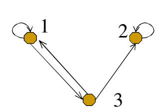
Transitive Closure – optional

- Warshall's Alg. iteratively constructs 0-1 matrices:
  $\mathbf{W}_0 = \mathbf{M}_R$;
  $\mathbf{W}_1=[w^{[1]}_{ij}]$, where $w^{[1]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j with interior vertices in $\{v_1\}$;
  $\mathbf{W}_2=[w^{[2]}_{ij}]$, where $w^{[2]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j with interior vertices in $\{v_1,v_2\}\ldots$
  $\mathbf{W}_k=[w^{[k]}_{ij}]$, where $w^{[k]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j with interior vertices in $\{v_1,v_2, \ldots, v_k\}\ldots$
  $\mathbf{M}_{R^*} = \mathbf{W}_n$.

Transitive Closure - Example

- Find the matrix $\mathbf{M}_R{}^*$ of the transitive closure of R:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



- Solution: $\mathbf{W}_0 = \mathbf{M}_R$

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & (1) \end{bmatrix}$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_R{}^* = \mathbf{W}_3 = \begin{bmatrix} 1 & (1) & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Transitive Closure – optional

- Lemma: Let R be a relation on $S=\{v_1,v_2,\ldots,v_n\}$, and let $\mathbf{W}_k=[w^{[k]}_{ij}]$ be the 0-1 matrix | $w^{[k]}_{ij}=1$ $\leftrightarrow \exists$ a path from v_i to v_j with interior vertices in $\{v_1, v_2, \ldots, v_k\}$.
  Then
  $$\forall i,j,k \le n \;\; w^{[k]}_{ij} = w^{[k-1]}_{ij} \vee (w^{[k-1]}_{ik} \wedge w^{[k-1]}_{kj}).$$

Algorithm 2 - Warshall's Algorithm

- Procedure for computing transitive closure:
  - $\mathbf{W} = \mathbf{M}_R$;
  - For k=1 to n do the following:
    - For i=1 to n do the following:
      - For j=1 to n do the following:
        - $w_{ij} = w_{ij} \vee (w_{ik} \wedge w_{kj})$;

  // That is, at step k: add row k to all other rows which have 1 as intersection with kth column.

Warshall's Algorithm - Example

- Find the matrix $\mathbf{M}_R$* of the transitive closure of R:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- Solution: $\mathbf{W}_0 = \mathbf{M}_R$

Add row 1 to row 3:

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & (1) \end{bmatrix}$$

Add row 2 to row 3:

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Add row 3 to row 1:

$$\mathbf{M}_R^* = \mathbf{W}_3 = \begin{bmatrix} 1 & (1) & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Conclusions: transitive closure

- Computing the transitive closure of a digraph is an important problem in many computer science applications:
  - Evaluation of recursive database queries.
  - Analysis of reachability (connectivity) of transition graphs in communication networks.
  - Construction of parsing automata in compilers.