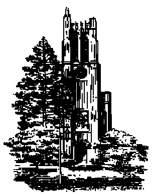
Wednesday June 22, 2016
Lecture 22

Basics of Counting



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Notables

- Homework #12
  - Page 525, Problem 12
  - Page 525, Problem 14
  - Find the solution to $a_n = 2a_{n-1} + 2n^2$ with $a_1 = 4$.
  - Page 396, Problems 2, 4, 6, 12, 36
  - Page 413, Problem 6
  - Page 414, Problem 22
  - Due Thursday June 23
- Read Chapter 6

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Combinations & Permutations

- Definition: An *r-combination* of n objects is an *unordered* selection of r of these objects where replacement is not allowed. If the objects are unique, then the r-combination is just an r-element subset of the set of objects.
  - Example: From the set $S = \{1,2,3,4\}$ we can choose two elements in six different ways, namely, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$
- The notation $C(n,r)$ denotes the number of r-combinations of n *distinct* objects.
  - Alternative notation is $\binom{n}{r}$

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r-Combination

- Note that:
  - $C(n,n) = 1$ since there is just one way to select n objects out of n objects
  - $C(n,1) = n$ since there are n ways to select one object out of n objects.

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r-Permutations

- Definition: An r-permutation is an *ordered* arrangement of r objects from n objects where replacement is not allowed.
  - Example: From the set $S = \{1,2,3,4\}$ we can arrange two elements in twelve different ways, namely, (1,2) , (2,1), (1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)
- The notation $P(n,r)$ denotes the number of r-permutations of n *distinct* objects without replacement.

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Combination, Permutation: recap

- Combination:
  - Selection when order does not matter
  - With or without replacement
  - The notation $C(n,r)$ denotes the number of r-combinations of n *distinct* objects when replacement is not allowed
- Permutation
  - Arrangement when order does matter
  - With or without replacement
  - The notation $P(n,r)$ denotes the number of r-permutations of n *distinct* objects when replacement is not allowed.

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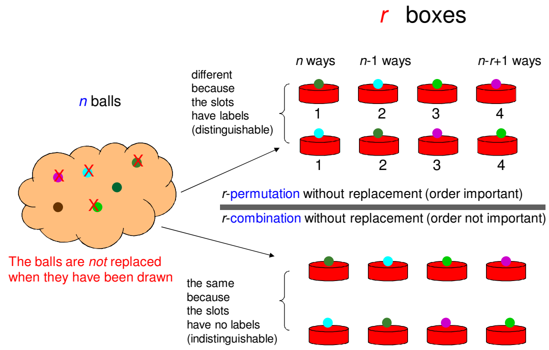
Permutation, Combination



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Relation between $C(n,r)$ and $P(n,r)$

- Claim 1: $P(n,r) = C(n,r) \times P(r,r)$
  - Proof: We can make an ordered arrangement of r out of n distinct objects by first selecting r objects and then arranging these r objects in order. This is an application of the Product Rule.
- Claim 2: $C(n,r) = C(n-1,r-1) + C(n-1,r)$
  - Proof: Suppose one of the distinct objects is marked as red. Then the r-element subsets consist of two types of subsets, namely, those which include the red object (corresponding to the first term), and those that exclude the red object (corresponding to the second term). This is an application of the Sum Rule.

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r-Permutations

- Claim: $P(n,n) = n!$
  - Proof: By induction on n
    - Basis: $n = 1$; $P(1,1) = 1 = 1!$
    - Hypothesis: Assume $P(n-1, n-1) = (n-1)!$
    - Step: Want to show $P(n,n) = n!$. To arrange n objects in order, we first single out one of the objects and arrange the remaining n-1 objects in order. By induction hypothesis, there are $P(n-1, n-1) = (n-1)!$. Now, for each of the arrangements, we can place the special object in n different ways, namely, n-2 places between the objects and two at the end. Thus, according to the product rule, we have $P(n,n) = n\times P(n-1, n-1) = n\times (n-1)! = n!$

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r-Permutations

- Claim: $P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)) = \dfrac{n!}{(n-r)!}$
  - Proof: In arranging r objects out of n in order, the first object can be chosen in n ways, the second object can be chosen in $(n - 1)$ way, …., and the r-th object can be chosen in $(n - (r - 1))$ ways. By the product rule we have
    $P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-(r-1))$
    $= n!/(n-r)!$

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$C(n,r)$

Recall that we have $P(n,r) = C(n,r) \times P(r,r)$

Thus $C(n,r) = \dfrac{P(n,r)}{P(r,r)} = \dfrac{n!}{r!(n-r)!}$

Note that $C(n,r) = C(n, n-r)$

Recall that another notation for $C(n,r)$ is $\binom{n}{r}$ read as n-choose-r.

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Example

- How many positive integers between 100 and 999 inclusive
  - Are divisible by 7?
  - Are odd?
  - Have the same three decimal digits
  - Are not divisible by 4?
  - Are divisible by either 3 or 4?
  - Are not divisible by either 3 or 4
  - Are divisible by 3 but not by 4?
  - Are divisible by 3 and 4.

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Solution:

(a) How many are divisible by 7? Answer: $\left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{99}{7} \right\rfloor = 142 - 14 = 128.$

(b) How many are odd? Answer: we first find how many are divisible by 2, that is, $\left\lfloor \frac{999}{2} \right\rfloor - \left\lfloor \frac{99}{2} \right\rfloor = 499 - 49 = 450,$ and then substract from the total, that is, $999 - 99 - 450 = 450.$

(c) Have the same three decimal digits? Answer: 111, 222, ..., 999 for a total of 9.

(d) Are not divisible by 4? Answer: First finding the ones which are divisible by 4 $\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 249 - 24 = 225.$ So, the answer is $999 - 99 - 225 = 675$

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(e) Are divisible by either 3 or 4? Answer:

$\left\lfloor \frac{999}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor = 333 - 33 = 300$ are divisible by 3

$\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 249 - 24 = 225$ are divisible by 4

$\left\lfloor \frac{999}{3\times 4} \right\rfloor - \left\lfloor \frac{99}{3\times 4} \right\rfloor = 83 - 8 = 75$ are divisible by both 3 and 4

Thus, there are $300 + 225 - 75 = 450$ which are divisible by either 3 or 4.

(f) Are not divisible by either 3 or 4? Answer: $999 - 99 - 450 = 450.$

(g) Are divisible by 3 but not by 4? Answer: $300 - 75 = 225$

(h) Are divisible by 3 and 4? Answer: 75.

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Example

- In how many ways can three numbers be selected from the numbers 1, 2, 3,…, 300 such that their sum is divisible by 3?

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Solution

- The 300 numbers 1, 2, 3,…, 300 can be divided into the following three sets:
  - Set $S0$: Those that are divisible by 3
  - Set $S1$: Those when divided by 3 give remainder 1.
  - Set $S2$: Those when divided by 3 give remainder 2.
- Note that $|S0| = |S1| = |S2| = 100$, and the sets are, of course, mutually exclusive.

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Solution

- So, the problem reduces to:
  - Selecting 3 numbers from $S0$, or
  - Selecting 3 numbers from $S1$, or
  - Selecting 3 numbers from $S2$, or
  - Three numbers one from each set.
- Therefore, the answer is:
  - $C(100,3) + C(100,3) + C(100,3) + 100\times 100\times 100$ =1,485,100.

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Principle of Inclusion & Exclusion

- Consider a set X of n distinct objects. Let p_1, p_2, …, p_r be a set of properties that these objects may have. In general, these properties are not mutually exclusive.
- Let $N(p_i)$ denote the number of objects in set X that have property p_i. Note that an object having property p_i is included in the $N(p_i)$ regardless of other properties it may have.

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Principle of Inclusion & Exclusion

- Example:
  - Let $S = \{1, 2, \ldots, 10\}$, p_1 = even, p_2 = odd & prime, p_3 = prime, p_4 = odd
    - $N(p_1) =$
    - $N(p_2) =$
    - $N(p_3) =$
    - $N(p_4) =$
    - $N(\neg p_3) =$

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Principle of Inclusion & Exclusion

- Example:
  - Let $S = \{1, 2, \ldots, 10\}$, p_1 = even, p_2 = odd & prime, p_3 = prime, p_4 = odd
    - $N(p_1) = 5$
    - $N(p_2) = 3$
    - $N(p_3) = 4$
    - $N(p_4) = 5$
    - $N(\neg p_3) = 6$

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Principle of Inclusion & Exclusion

- The notation $N(p \wedge q)$ denotes the number of distinct objects that have both property p and property q. Let n be the total number of objects. Note these facts:

$N(\neg p) = n - N(p)$

$N(q \wedge \neg p) = N(q) - N(p \wedge q)$

$N(p \wedge \neg q) = N(p) - N(p \wedge q)$

$N(\neg q \wedge \neg p) = n - N(p \wedge \neg q) - N(\neg p \wedge q) - N(p \wedge q)$

$= n - N(p) - N(q) + N(p \wedge q)$

$N(p \vee q) = N(p) + N(q) - N(p \wedge q)$

We can extend this to three properties, four properties, etc.

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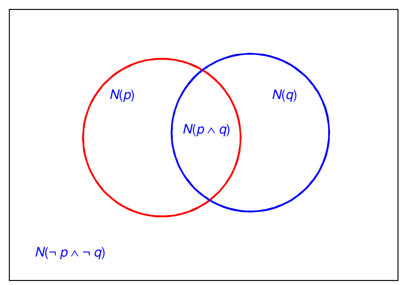
$N(\neg q \wedge \neg p) = n - N(p \wedge \neg q) - N(\neg p \wedge q) - N(p \wedge q)$

$N(\neg q \wedge \neg p) = n - N(p) - N(q) + N(p \wedge q)$



Set S, |S| = n

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Principle of Inclusion & Exclusion

- We can easily extent these to more that two properties. For example, for three properties
  - $N(\neg p \wedge \neg q \wedge \neg r) = n - N(p) - N(q) - N(r) + N(p \wedge q) + N(p \wedge r) + N(q \wedge r) - N(p \wedge q \wedge r)$

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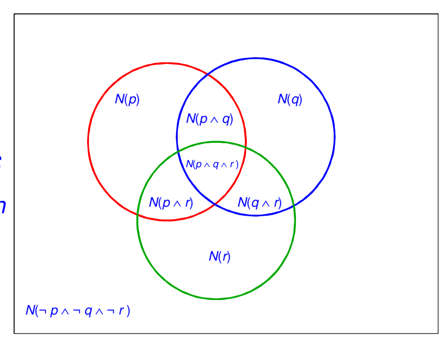
$N(\neg p \wedge \neg q \wedge \neg r) = n - [N(p) + N(q) + N(r) - N(p \wedge q) - N(p \wedge r) - N(q \wedge r) + N(p \wedge q \wedge r)]$

Set S

|S| = n



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Summary: Principle of Inclusion & Exclusion

- $N(\neg p) = n - N(p)$
- $N(\neg p \wedge q) = N(q) - N(p \wedge q)$
- $N(\neg p \wedge \neg q) = n - N(p) - N(q) + N(p \wedge q)$
- $N(p \vee q) = N(p) + N(q) - N(p \wedge q)$

We can easily extent these to more that two properties. For example, for three properties

- $N(\neg p \wedge \neg q \wedge \neg r) = n - N(p) - N(q) - N(r) + N(p \wedge q) + N(p \wedge r) + N(q \wedge r) - N(p \wedge q \wedge r)$
- $N(\neg p \wedge \neg q \wedge \neg r) = n - [N(p) + N(q) + N(r) - N(p \wedge q) - N(p \wedge r) - N(q \wedge r) + N(p \wedge q \wedge r)]$

Note that n is the total number of objects, N(property) is the number of objects having "property"

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Example

- How many integers between 1 and 250, inclusive, are not divisible by any of the integers 3, 5, and 7?

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Answer:

Let t, f, and s respectively denote the properties that a number is divisible by 3, 5, and 7. So,

$$N(t) = \left\lfloor \frac{250}{3} \right\rfloor = 83,\ N(f) = \left\lfloor \frac{250}{5} \right\rfloor = 50,\ N(s) = \left\lfloor \frac{250}{7} \right\rfloor = 35$$

$$N(t \wedge f) = \left\lfloor \frac{250}{3\times 5} \right\rfloor = 16,\ N(t \wedge s) = \left\lfloor \frac{250}{3\times 7} \right\rfloor = 11,\ N(f \wedge s) = \left\lfloor \frac{250}{5\times 7} \right\rfloor = 7$$

$$N(t \wedge f \wedge s) = \left\lfloor \frac{250}{3\times 5\times 7} \right\rfloor = 2.$$

$$N(\neg t \wedge \neg f \wedge \neg s) = n - N(t) - N(f) - N(s) + N(t \wedge f) + N(t \wedge s) + N(f \wedge s) - N(t \wedge f \wedge s)$$

$$= 250 - 83 - 50 - 35 + 16 + 11 + 7 - 2 = 114.$$

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Example

- Let S ={1, 2, 3, 4, 5, 6, 7} and T = {a, b ,c}. How many onto functions $f: S \to T$ are there?

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Example

- Let S ={1, 2, 3, 4, 5, 6, 7} and T = {a, b ,c}. How many onto functions $f: S \to T$ are there?
- Answer: Define the following properties
  - p: functions NOT having a as an image
  - q: functions NOT having b as an image
  - r: functions NOT having c as an image

  we are looking for $N(\neg p \wedge \neg q \wedge \neg r)$. For that, we need to find n, $N(p)$, $N(q)$, $N(r)$, $N(p \wedge q)$, $N(p \wedge r)$, $N(q \wedge r)$, and $N(p \wedge q \wedge r)$

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Solution

- First, we'll compute the total number of functions as follows. Each element of the domain can be mapped to any one of the elements of the range. Thus, the total no. of functions is $n = 3^7 = 2187$
- $N(p) = 2^7$, $N(q) = 2^7$, $N(r) = 2^7 = 128$
- $N(p \wedge q) = N(p \wedge r) = N(q \wedge r) = 1$
- $N(p \wedge q \wedge r) = 0$
- $N(\neg p \wedge \neg q \wedge \neg r) = n - N(p) - N(q) - N(r) + N(p \wedge q) + N(p \wedge r) + N(q \wedge r) - N(p \wedge q \wedge r)$
  $= 2187 - 128 - 128 - 128 + 1 + 1 + 1 - 0 = 1806.$

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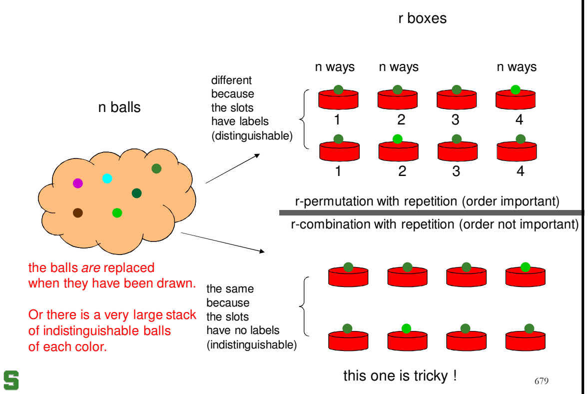
Example: Selecting three from {1,2,3,4}

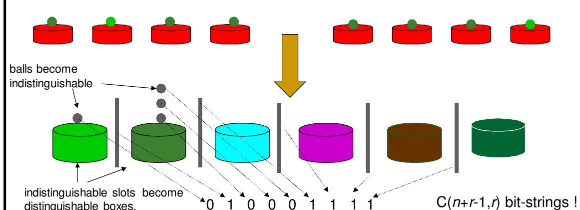
- Without replacement
  - 1,2,3
  - 1,2,4
  - 1,3,4
  - 2,3,4
- With replacement
  - 1,2,3
  - 1,2,4
  - 1,3,4
  - 2,3,4
  - 1,1,2
  - 1,1,3
  - 1,1,4
  - 2,2,1
  - 2,2,3
  - 2,2,4
  - 3,3,1
  - 3,3,2
  - 3,3,4
  - 4,4,1
  - 4,4,2
  - 4,4,3
  - 1,1,1
  - 2,2,2
  - 3,3,3
  - 4,4,4

Permutation, Combination with replacement



Combination with replacement…

- How to count the *r*-combination with replacement?
  - One strategy could be to start from drawing where the order matters and try to count the number of ways we over-counted (last example). However, there is a much smarter way!
- Map the problem to a bit-string as follows



Permutation & Combination

Type	Replacement Allowed?	Formula for *r*-p/c of *n*
r-permutation	No	$n!/(n-r)!$
r-combination	No	$n!/r!(n-r)!$
r-permutation	Yes	n^r
r-combination	Yes	$(n+r-1)!/r!(n-1)!$

Classification of Occupancy Problems

Distinguished balls? r	Distinguished Cells? n	Can Cells be empty	No. of ways to place r balls in n cells
Yes	Yes	Yes	n^r
Yes	Yes	No	$n!S(r,n)$
Yes	No	Yes	$S(r,1)+S(r,2)+S(r,3)+ \ldots + S(r,n)$
Yes	No	No	$S(r,n)$
No	Yes	Yes	$C(n+r-1, r)$
No	Yes	No	$C(r-1, n-1)$

Stirling Number of the second kind

$$S(r,n) = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$$

Classification of Occupancy Problems

Distinguished balls? r	Distinguished Cells? n	Can Cells be empty	No. of ways to place r balls in n cells	Comments
Yes	Yes	Yes	n^r	The same as r-permutation of n distinct objects when replacement is allowed
Yes	No	No	$S(r,n)$	$r \geq n$ The number of ways of partitioning a set of r elements into n nonempty subsets
Yes	Yes	No	$n!S(r,n)$	
No	Yes	Yes	$C(n+r-1, r)$	The same as r-combination of n distinct objects when replacement is allowed
No	Yes	No	$C(r-1, n-1)$	$r \geq n$ The same as $(r-n)$-combination of n distinct objects when replacement is allowed