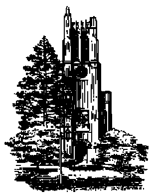
Wednesday May 18, 2016
Lecture 03

Logic



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Notables

- Homework#2
  - Page 35, problems 10(b), 22, 34(c)
  - Page 53, Problems 2, and 16
  - Does
    $(p \to q) \to (r \to s)$ logically imply
    $(p \to r) \to (q \to s)$
    justify your answer.
  - Due tomorrow Thursday May 19, 2016
- Exam#1 on This Thursday

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Some Important Equivalences

$p \wedge \mathrm{T} \equiv p$ $p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}$ $p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

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Some Important Equivalences

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee \neg p \equiv \mathrm{T}$ $p \wedge \neg p \equiv \mathrm{F}$ $p \to q \equiv \neg p \vee q$	Tautology Contradiction Implication

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How to show logical equivalences?

- By means of a Truth Table.
  - Already did few examples
- By derivation, using known logical equivalences.

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Example: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p\vee(q\wedge r)$	$p \vee q$	$p \vee r$	$(p\vee q)\wedge(p\vee r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

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Example: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p\vee(q\wedge r)$	$p \vee q$	$p \vee r$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

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Show that $\neg (p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg (p \vee (\neg p \wedge q))$	$\equiv \neg p \wedge \neg (\neg p \wedge q)$	De Morgan's law
	$\equiv \neg p \wedge (\neg (\neg p) \vee \neg q)$	De Morgan's law
	$\equiv \neg p \wedge (p \vee \neg q)$	Double negation
	$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law
	$\equiv F \vee (\neg p \wedge \neg q)$	Contradiction
	$\equiv (\neg p \wedge \neg q) \vee F$	Commutative
	$\equiv \neg p \wedge \neg q$	Identity law

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Exercise

- Show that $[(p\wedge q) \rightarrow (p\vee q)]$ is a tautology
  - (a) using a truth table
  - (b) using derivations

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Is $[(p\wedge q) \rightarrow (p\vee q)] \equiv T$?

$(p\wedge q) \rightarrow (p\vee q)$	$\equiv \neg(p\wedge q) \vee (p\vee q)$	Implication
	$\equiv (\neg p \vee \neg q) \vee (p\vee q)$	De Morgan's law
	$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	Assoc. & Comm.
	$\equiv T \vee T$	Tautology
	$\equiv T$	Domination

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Substitution Rules

- Let H be a wff. Further let w be a proposition variable in H. If H is a tautology, then the wff Q obtained from H by replacing every occurrence of w by another proposition, say r, is also a tautology.
  - Example:

    $H = (p \wedge (p \rightarrow q)\ ) \rightarrow q$ is a tautology.

    If we replace p with $q \rightarrow s$ to get

    $Q = ((q \rightarrow s) \wedge ((q \rightarrow s) \rightarrow q)) \rightarrow q$

    it remains a tautology

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Substitution Rules…

- Let H be a wff. Further let w be a proposition variable in H. Let Q be the wff obtained from H by replacing some occurrence of w by another proposition u which is logically equivalent to w. Then Q is logically equivalent to H.
  - Example:

    $H = \neg [(p \rightarrow q) \wedge (p \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

    $Q = \neg [(\neg p \vee q) \wedge (p \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

    Note that in this case, H does not have to be a tautology.

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Duality Law

- Let H be a wff. The wff Q obtained from H by replacing in H each $\wedge$ with $\vee$, each $\vee$ with $\wedge$, each T with F and each F with T, is called the *dual* of H. Example:
  - Let $H = (p \vee q) \wedge r$. Its dual is $Q = (p \wedge q) \vee r$.
- Suppose $H(p_1, p_2, \ldots, p_n)$ and Q are dual. Then $\neg H \equiv Q(\neg p_1, \neg p_2, \ldots, \neg p_n)$.
- Example:
  - $\neg((p \vee q) \wedge r) \equiv (\neg p \wedge (\neg q)) \vee (\neg r)$

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From Truth Table to wff

- Given a Truth Table, how would we find the corresponding wff?

p	q	r	*wff?*
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

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From Truth Table to wff

- Given a Truth Table, how would we find the corresponding wff?

p	q	r	W	X	Y	Z	$wff = W \vee X \vee Y \vee Z$
T	T	T	T	F	F	F	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	T	F	F	T
F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	T	T

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Aside: Quantifying problems' hardness

- The Satisfiability problem:
  - Is there a truth assignment that would make an input wff to have truth value True? Example:
    - $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$
- Distinction between:
  - Finding a solution
  - Verifying a solution

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Logical Implications (Sec 1.5)

- Given two wffs p and q, we say that p *logically implies* q, denoted $p \Rightarrow q$, if whenever p is true q is also true.
- Note that if $p \Rightarrow q$, then the proposition $p \to q$ is definitely true. Also, when the proposition $p \to q$ is true, we definitely have $p \Rightarrow q$. Thus, to check if $p \Rightarrow q$ one can check to see if the proposition $p \to q$ is a tautology.
- The left hand side of $\Rightarrow$ is called the set of *premises* and the right hand side is called the *conclusion*.
- A *proof* is nothing but establishing a *chain of logical implications*!

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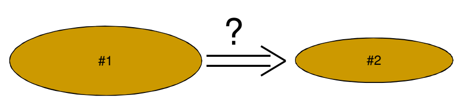
Logical Implications



#1	#2	Answer?
T	T	Yes
T	F	No
F	T	Yes
F	F	Yes

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Does $((p \to q) \wedge p) \Rightarrow q$?

p	q	$p \to q$	$(p \to q) \wedge p$	q	*Is the conclusion true whenever the premise is true*
T	T	T	T	T	Yes
T	F	F	F	F	Yes
F	T	T	F	T	Yes
F	F	T	F	F	Yes

Relation between $w \Rightarrow c$ and $w \to c$

w	c	Does $w \Rightarrow c$?	$w \to c$
T	T	Yes	T
T	F	No	F
F	T	Yes	T
F	F	Yes	T

Does $((p \to q) \wedge p) \Rightarrow q$?

p	q	$p \to q$	$(p \to q) \wedge p$	q	*Is the conclusion true whenever the premise is true*	In other words, is $((p \to q) \wedge p) \to q$ is a tautology
T	T	T	T	T	Yes	T
T	F	F	F	F	Yes	T
F	T	T	F	T	Yes	T
F	F	T	F	F	Yes	T

Does $((p \to q) \wedge q) \Rightarrow p$?

p	q	$p \to q$	$(p \to q) \wedge q$	p	*Is the conclusion true whenever the premise is true*
T	T	T	T		
T	F	F	F		
F	T	T	T		
F	F	T	F		

Does $((p \to q) \wedge q) \Rightarrow p$?

p	q	$p \to q$	$(p \to q) \wedge q$	p	*Is the conclusion true whenever the premise is true*
T	T	T	T	T	YES
T	F	F	F	T	YES
F	T	T	T	F	NO
F	F	T	F	F	YES

Example

- p = "You are born in U.S."
- q = "You are a U.S. citizen."
- Based on the current laws, clearly $p \Rightarrow q$. Note that you may still be a U.S. citizen and not being born in U.S. That is, in $p \Rightarrow q$, the logical implication still holds when the conclusion is true but the premise is not. *In fact, when the premise is false anything can be concluded* !

Example

- Consider the following reasoning
  - If Canada is a country then New York is a city. We know that New York is a city. Therefore, Canada must be a country.

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Example

- Let p = "Canada is a country."
- Let q = "New York is a city."
- $p \rightarrow q$ = "If Canada is a country then New York is a city."
- $(p \rightarrow q) \wedge q$ = "If Canada is a country then New York is a city. And New York is a city."
- Given above, can we conclude that Canada is a country?
- Does $((p \rightarrow q) \wedge q) \Rightarrow p$?
  - The conclusion that "Canada is a Country" does not follow logically from the above "argument", even though the conclusion is true in itself. Once again, the system will determine the validity of the implication once the implication is stated.

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Understanding Logical Implication

- The point being that in the logical implications we are not so much concerned with the conclusion being true or false, rather, we are concerned with determining whether the conclusion follows logically from the premise; that is, whether the "*argument*" is valid.

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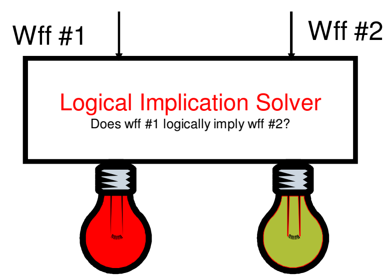
The Light Machine



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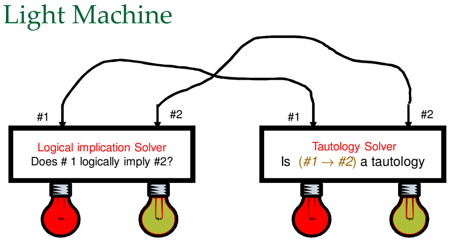
The Light Machine



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How are these questions related?

1. Does p logically imply c?
2. Is the proposition $(p \rightarrow c)$ a tautology?
3. Is the proposition $(\neg p \vee c)$ is a tautology?
4. Is the proposition $(\neg c \rightarrow \neg p)$ is a tautology?
5. Is the proposition $(p \wedge \neg c)$ is a contradiction?

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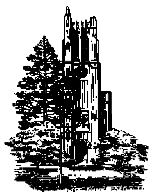
Predicate Calculus

Section 1.3 & 1.4



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Limitation of Propositional Logic

- Consider the following reasoning
  - All cats have tails
  - Gouchi is a cat

    Therefore, Gouchi has tail.

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Outline

- Introduction
  - Predicates
  - Propositional functions
- Quantifiers
  - Universal quantification
  - Existential quantification
- Translating sentences into logical expressions
- Binding variables
- Negations

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Introduction

- Remember that the sentence "x+1 = 3" is NOT a proposition. Why?
  - Can we produce a proposition from it?
- The sentence "x plus 1 equals 3" has two parts, namely, the *variable* x which is the subject of the sentence, and the property "plus one equals 3" which is the *predicate*.
- Grammatically, a predicate is the part of the sentence that says something about the subject.

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Introduction….

- If P denotes the predicate "plus 1 equals 3", then the sentence "x plus 1 equals 3" can be denoted by $P(x)$.
- $P(x)$ is called a *propositional function* with variable x.
- Once a value is assigned to x, then $P(x)$ becomes a proposition - it has a truth value.
  - What is the truth value of $P(1)$, $P(2)$, etc?

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Universe of discourse

- Let $Q(x, y)$ denote "$x = y - 4$"
  - What is truth value of $Q(1, 2)$, $Q(1, 5)$, etc?
  - Find all values of x, y which make $Q(x, y)$ true.
    - We need to specify the *Universe of Discourse* which is the *domain* of consideration.
- In general, $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at n-tuple $(x_1, x_2, \ldots, x_n)$

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Propositional Functions

- How can we turn a propositional function into a proposition?
  - Assigning values to its variables
  - Using quantifiers.
    - *Universal quantifier*
    - *Existential quantifier*

Quantifiers: Universal

- Definition: The universal quantification of $P(x)$, denoted by $\forall x\ P(x)$, is the *proposition*:

  "*every* x in the universe of discourse has property P."
- $\forall$ is called the universal quantifier, and $\forall x\ P(x)$ is read as.
  - "For all x $P(x)$," or
  - "For every x $P(x)$."

Example

- Let's consider the proposition "Every student in this class has studied calculus."
  - Let $P(x)$ denote "x has studied calculus"
  - $\forall x\ P(x)$, where the universe of discourse consists of the students in this class.
- Here is another way of stating the same thing but using a different universe of discourse
  - Let $P(x)$ denote "x has studied calculus"
  - Let $S(x)$ is the proposition "x is in this class" and the universe of discourse is the set of all students.
  - $\forall x\ (S(x) \rightarrow P(x))$

The Universal Quantifier

- Note that when the universe of discourse is finite, then

  $\forall x\ P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

  where $x_1, x_2, \ldots, x_n$ are values in the universe of discourse.
- Note again that $\forall x\ P(x)$ is a proposition.

Quantifiers: The existential

- Definition: The *existential quantification* of $P(x)$, denoted by $\exists x\ P(x)$, is the proposition:

  "*One or more* x in the universe of discourse has property P."
- $\exists$ is called the *existential quantifier*, and it is read as:
  - "There exists an x such that $P(x)$," or
  - "There is at least one x such that $P(x)$," or
  - "For some x $P(x)$."

Examples

- Let $P(x)$ denote "$x > 3$."
  - What is the truth value of $\exists x\ P(x)$, where the universe of discourse is the set of real numbers?
  - *Solution:* $\exists x\ P(x)$ is true; since "$x > 3$" is true , for instance, when $x = 4$.
  - What if we change the universe of discourse to the set of negative numbers?

The Existential Quantifier

- Note that when the universe of discourse is finite, then
  $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$
  where $x_1, x_2, \ldots, x_n$ are values in the universe of discourse.
- Note again that $\exists x\, P(x)$ is a proposition.

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