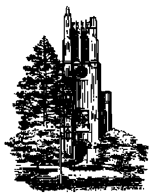
Monday May 23, 2016
Lecture 05

Predicate Logic, Proof Techniques



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Notables

- Homework#4
  - Page 78, Problem 4
  - Page 80, Problem 20, and 24
  - Page 91, Problem 6, and 8
  - Due Thursday May 26, 2016
- Tentative Schedule for the week

Week	M	T	W	R	Topic	Section
1	5-23				Nested Quantifiers, Rules of Inference	1.5, 1.6
		5-23			Proof	1.7
			5-25		Proofs	1.8
				5-26	Sets	2.1

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Example

- Consider the following predicates:
  - $E(x)$ = "x is an even integer"
  - $P(x)$ = "x is prime"
  - $Q(x,y)$ = "integer x equals integer y"
  - $L(x,y)$ = "integer x is less than integer y"
- Using the above predicates and appropriate quantifiers, symbolize the following. Assume the universe of discourse is all positive integers.
  - $F(x)$ = "x is between 1 and 9, x is even, and x is prime."
  - There is *exactly* one integer between 1 and 9, which is both even and prime.

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Solution

- Using
  - $E(x)$ = "x is an even integer"
  - $P(x)$ = "x is prime"
  - $Q(x,y)$ = "integer x equals integer y"
  - $L(x,y)$ = "integer x is less than integer y"
- $F(x)$ = "x is between 1 and 9, x is even, and x is prime."
  - $F(x) = L(1,x) \wedge L(x,9) \wedge E(x) \wedge P(x)$
- There is *exactly* one integer between 1 and 9, which is both even and prime.
  - $\exists\, x\, [\, F(x) \wedge \forall\, y\, (F(y) \rightarrow Q(y,x))]$

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Nested Quantifiers

- When we have more than one variable in a predicate, one way to make a proposition is to use nested quantifiers.
- When using nested quantifiers, we must pay attention to the order in which they are used.

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Quantification of Two Variables

proposition	*When True?*	*When False?*
$\forall x\, \forall y\, P(x, y)$ $\forall y\, \forall x\, P(x, y)$		
$\forall x\, \exists y\, P(x, y)$		
$\exists x\, \forall y\, P(x, y)$		
$\exists x\, \exists y\, P(x, y)$ $\exists y\, \exists x\, P(x, y)$		

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Quantification of Two Variables

proposition	*When True?*	*When False?*
$\forall x\, \forall y\, P(x, y)$ $\forall y\, \forall x\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x\, \exists y\, P(x, y)$		
$\exists x\, \forall y\, P(x, y)$		
$\exists x\, \exists y\, P(x, y)$ $\exists y\, \exists x\, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Quantification of Two Variables

proposition	*When True?*	*When False?*
$\forall x\, \forall y\, P(x, y)$ $\forall y\, \forall x\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x\, \exists y\, P(x, y)$	For every x there is a y for which $P(x, y)$ is true. y may depend on x	There is an x such that $P(x,y)$ is false for every y.
$\exists x\, \forall y\, P(x, y)$	There is a fixed x such that $P(x,y)$ is true for every y. Once x is decided, it cannot change.	For every x there is a y for which $P(x, y)$ is false.
$\exists x\, \exists y\, P(x, y)$ $\exists y\, \exists x\, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Examples

- Let $P(x, y)$ denote "$x + y = 5$."
  - The universe of discourse for both x and y is {1, 2, 3, 4}
- Find the truth value for $\forall x\, \forall y\, P(x, y)$
  - *Solution:*
- Find the truth value for $\exists y\, \exists x\, P(x, y)$
  - *Solution:*

Examples

- Let $P(x, y)$ denote "$x + y = 5$."
  - The universe of discourse for both x and y is {1, 2, 3, 4}
- Find the truth value for $\forall x\, \forall y\, P(x, y)$
  - *Solution:* False; x =1 and y = 3, $x + y = 4$
- Find the truth value for $\exists y\, \exists x\, P(x, y)$
  - *Solution:* True; x = 3 and y = 2, $x + y = 5$

Examples

- Let $P(x, y)$ denote "$x + y = 5$."
  - The universe of discourse for both x and y is {1, 2, 3, 4}
- Find the truth value for $\forall x\, \exists y\, P(x, y)$
  - *Solution:*
- Find the truth value for $\exists x\, \forall y\, P(x, y)$
  - *Solution:*

Examples

- Let $P(x, y)$ denote "$x + y = 5$."
  - The universe of discourse for both x and y is {1, 2, 3, 4}
- Find the truth value for $\forall x\, \exists y\, P(x, y)$
  - *Solution:* True; for any x, there exists a y such that $x + y=5$; x=1, y=4; x=2, y=3; x=3, y=2; x=4, y=1;…
- Find the truth value for $\exists x\, \forall y\, P(x, y)$
  - *Solution:* False; there is no such (fixed) x that for each y, we would have $x + y = 5$;

x=1, y=1; x=2, y=2; x=3, y=3; x=4, y=4;

Example

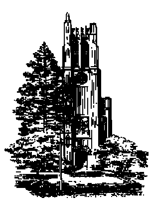
- Consider the following predicates:
  - $E(x)$ = "x is an even integer"
  - $P(x)$ = "x is prime"
  - $Q(x,y)$ = "integer x equals integer y"
  - $L(x,y)$ = "integer x is less than integer y"
- Using the above predicates and appropriate quantifiers, symbolize the following proposition. Assume the universe of discourse is all positive integers.
  - There are two distinct prime integers whose sum is even.

Solution

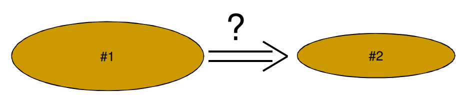
- Using
  - $E(x)$ = "x is an even integer"
  - $P(x)$ = "x is prime"
  - $Q(x,y)$ = "integer x equals integer y"
  - $L(x,y)$ = "integer x is less than integer y"
- There are two distinct prime integers whose sum is even.
  - Solution: $\exists x \exists y\, [\neg Q(x,y) \wedge E(x+y) \wedge P(x) \wedge P(y)]$

Rules of Inference

Section 1.6



Review: Logical Implications



#1	#2	Answer?
T	T	Yes
T	F	No
F	T	Yes
F	F	Yes

Once again: How are these questions related?

1. Does p *logically imply* c ?
2. Is the proposition $(p \rightarrow c)$ a tautology?
3. Is the proposition $(\neg p \vee c)$ is a tautology?
4. Is the proposition $(\neg c \rightarrow \neg p)$ is a tautology?
5. Is the proposition $(p \wedge \neg c)$ is a contradiction?

Formal Proofs

- A proof is equivalent to establishing a logical implication chain
- Given premises (hypotheses) $h_1, h_2, \ldots, h_n$ and conclusion c, to give a *formal proof* for establishing that the hypotheses logically imply the conclusion, entails establishing the logical implication

$$h_1 \wedge h_2 \wedge \ldots \wedge h_n \Rightarrow c$$

- To do so, we need to give a finite chain

$$p_1, p_2, \ldots, c$$

of wffs such that each wff p_r is:
  - Rule I: one of the premises or a tautology, or
  - Rule II: p_r can *logically be implied* by one or more propositions p_k where $1 \le k < r$.

The Textbook Notation

$$\begin{array}{l} h_1 \\ \vdots \\ h_2 \\ \vdots \\ h_n \\ \vdots \\ \hline \therefore\ c \end{array}$$

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Proof Methods
$h_1 \wedge h_2 \wedge \ldots \wedge h_n \Rightarrow c$?

- Let $p = h_1 \wedge h_2 \wedge \ldots \wedge h_n$. The following propositions are equivalent:
  1. $p \Rightarrow c$ — Direct
  2. $(p \to c)$ is a tautology. — Direct
  3. $(\neg p \vee c)$ is a tautology. — Direct
  4. $(\neg c \to \neg p)$ is a tautology. — Contrapositive
  5. $(p \wedge \neg c)$ is a contradiction. — Contradiction

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Proof Methods

		Direct	Direct	Contrapositive	Contradiction
p	c	$p \to c$	$\neg p \vee c$	$\neg c \to \neg p$	$p \wedge \neg c$
T	T	T	T	T	F
T	F	F	F	F	T
F	T	T	T	T	F
F	F	T	T	T	F

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Terminology

- Axiom or Postulate: An underlying assumptions/postulates to begin the logical argument with.
- Rules of inference: Logical facts and implications used to draw conclusions from the given postulates.
- Proof: A sequence of propositions that forms a valid argument.
- Fallacies: Incorrect reasoning

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Terminology…

- Theorem: A proposition that can be shown to be true.
- Lemma: A simple theorem used in the proof of other theorems.
- Corollary: A fact that can be immediately deduced from a Theorem/Lemma.
- Conjecture: A proposition whose correctness is unknown.

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Rules of inference

- Rules of inference are used to deduce conclusions from hypotheses. These are the logical implication questions for which the answer is YES
- Consider the question:

  Does $[p \wedge (p \to q)]$ logically imply q
  - The answer is YES as $[p \wedge (p \to q)] \to q$ is a tautology
  - It is the basis of the rule of inference called *modus ponens*, which can be represented by the symbolic form

    $$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

    which means
  - Given that both p and $p \to q$ are true, we can conclude that q is true. In other words

    $[p \wedge (p \to q)]$ logically implies q

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Few Tautologies

$p \;\rightarrow\; (p \vee q)$ $(p \wedge q) \;\rightarrow\; p$
$[(p) \wedge (q)] \;\rightarrow\; (p \wedge q)$
$[p \wedge (p \rightarrow q)] \;\rightarrow\; q$ $[\neg q \wedge (p \rightarrow q)] \;\rightarrow\; \neg p$
$[(p \rightarrow q) \wedge (q \rightarrow r)] \;\rightarrow\; (p \rightarrow r)$
$[(p \vee q) \wedge \neg p] \;\rightarrow\; q$

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Rules of Inference

p $\therefore p \vee q$	p **logically implies** $(p \vee q)$	$p \;\rightarrow\; (p \vee q)$ is a tautology	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q)$ **logically implies** p	$(p \wedge q) \;\rightarrow\; p$ is a tautology	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)]$ **logically implies** $(p \wedge q)$	$[(p) \wedge (q)] \;\rightarrow\; (p \wedge q)$ is a tautology	Conjunction
p $p \rightarrow q$ $\therefore q$	$[p \wedge (p \rightarrow q)]$ **logically implies** q	$[p \wedge (p \rightarrow q)] \;\rightarrow\; q$ is a tautology	Modus ponens

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Rules of Inference…

$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)]$ **logically implies** $\neg p$	$[\neg q \wedge (p \rightarrow q)] \;\rightarrow\; \neg p$ is a tautology	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)]$ **logically implies** $(p \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \;\rightarrow\; (p \rightarrow r)$ is a tautology	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p]$ **logically implies** q	$[(p \vee q) \wedge \neg p] \;\rightarrow\; q$ is a tautology	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)]$ **logically implies** $q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \;\rightarrow q \vee r$ is a tautology	Resolution

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Rules of Inference for Quantifications

Rule of Inference	*Name*	*Comments*
$\forall x\, P(x)$ $\therefore P(c)$	Universal Specification/Instantiation (US) or (UI)	for any c in the domain, we get to choose c
$P(c)$ $\therefore \forall x\, P(x)$	Universal generalization (UG)	for an arbitrary c, not a particular one
$\exists x\, P(x)$ $\therefore P(c)$	Existential Specification/Instantiation (ES) or (EI)	for some specific c (unknown)
$P(c)$ $\therefore \exists x\, P(x)$	Existential generalization (EG)	Finding one c such that $P(c)$

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Formal Proofs…..

- Example:

  Given:

  $h_1 = p \rightarrow q, \quad h_2 = q \rightarrow r, \quad h_3 = p, \quad c = r$

  we want to prove, that is, establish the logical implication that $h_1 \wedge h_2 \wedge h_3 \Rightarrow c$.
- We can use any of the following approaches
  - Using Truth Table
  - Using derivations
  - Using a chain of logical implications
    - Note that if $A \Rightarrow B$ and $B \Rightarrow C$ then $A \Rightarrow C$

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Does $(p \rightarrow q) \wedge (q \rightarrow r) \wedge p \Rightarrow r$?
Truth Table Method

p	q	r	$p \rightarrow q$	$q \rightarrow r$	p	r
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	T	T	F	T
F	F	F	T	T	F	F

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Does $(p \to q) \wedge (q \to r) \wedge p \Rightarrow r$?
Chain Method

- $h_1 = p \to q$, $h_2 = q \to r$, $h_3 = p$, $c = r$
  We want to prove that $h_1 \wedge h_2 \wedge h_3 \Rightarrow c$
  1. p Rule I
  2. $p \to q$ Rule I
  3. q Rule II, modus ponens
  4. $q \to r$ Rule I
  5. r Rule II, 3&4, modus ponens

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Example

- Prove the theorem:
  "If integer n is odd, then n^2 is odd."
- *Solution:*
  - It is given that n is an odd integer.
  - Thus $n = 2k + 1$, for some integer k.
  - Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
  - Therefore, n^2 is odd.

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Example

- Prove that if integer 3 is even, then all integers greater than 1 are even.
  - Proof:
    - 3 is even Premise
    - 2 is even Math Fact
    - 3+2 is even Math Fact & Premise
    - 4 is even Math Fact
    - 5 is even Math Fact & Premise
    - ….

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Example: Hypothetical Syllogism

- Let
  $h_1 = p \to q$
  $h_2 = q \to s$
  $c = p \to s$
  We want to establish $h_1 \wedge h_2 \Rightarrow c$.

Use only Modus ponens rule.

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Solution

- $(p \to q) \wedge (q \to s) \Rightarrow p \to s$

1. $p \to q$ *Premise*
2. $q \to s$ *Premise*
3. p *Assumption (p = False, NTP)*
4. q *3 & 1, Modus Ponens (MP)*
5. s *4 & 2, MP*
6. $p \to s$

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