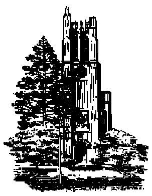
Thursday June 23, 2016
Lecture 23

Basics of Counting



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Notables

- Homework #13
  - Page 405, Problems 2 and 6
  - Page 432, Problem 16
  - Page 581, Problems 2, 6, and 8
  - Page 606, Problem 3
  - Page 615, Problem 2
  - Page 616, Problems 24, and 36
  - Due Wednesday June 29
- Read Chapter 9

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Permutation & Combination

Type	Replacement Allowed?	Formula for r-p/c of n
r-permutation	No	$n!/(n-r)!$
r-combination	No	$C(n, r) = n!/r!(n-r)!$
r-permutation	Yes	n^r
r-combination	Yes	$C(n+r-1, r) = (n+r-1)!/r!(n-1)!$

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Classification of Occupancy Problems

Distinguished balls? r	Distinguished Cells? n	Can Cells be empty	No. of ways to place r balls in n cells
Yes	Yes	Yes	n^r
Yes	Yes	No	$n!S(r,n)$
Yes	No	Yes	$S(r,1)+S(r,2)+S(r,3)+\ldots+S(r,n)$
Yes	No	No	$S(r,n)$
No	Yes	Yes	$C(n+r-1, r)$
No	Yes	No	$C(r-1, n-1)$

Stirling Number of the second kind

$$S(r,n)=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$$

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Examples

- Q1: How many bit strings contain exactly $n-1$ 1's and r zeros?
  - Answer: $C(n+r-1, r) = C(n+r-1, n-1)$
  - Example of such a string with $n = 4$ and $r = 7$
    - 0010001001
- Q2: In how many ways can we fill n distinguished cells with r undistinguished balls, when empty cells are allowed?
- Transformation from Q1 to Q2:
  - The 1's partition the bit strings into n "regions" as follows:
    region 1 region 1…. region 1 region
    Note that there are n regions. Let each regions correspond to a unique cell. The content of each region corresponds to the balls in the corresponding cell.

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Example

- Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ and $T = \{a, b, c\}$. How many onto functions $f: S \to T$ are there?
  - Solution: This is the same as placing $r = 7$ distinguished balls into $n = 3$ distinguished cells, where a cell cannot be empty. So, the answer is:
    $3!S(7,3) = 1806$

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Example

How many nonnegative integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$?

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Example

How many nonnegative integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$

Solution: This corresponds to placing $r = 17$ indistinguished balls into $n = 4$ distinguished cells where a cell could be empty. So, it is

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!} = \frac{(4+17-1)!}{17!(4-1)!} = \frac{20!}{17!3!} = 1140$$

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Example

How many positive integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$?

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Example

How many positive integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$

Solution: This corresponds to placing $r = 17$ indistinguished balls into $n = 4$ distinguished cells where a cell cannot be empty. So, it is

$$C(r-1, n-1) = \frac{(r-1)!}{(n-1)!(r-n)!} = \frac{(17-1)!}{3!(17-4)!} = \frac{16!}{3!13!} = 560$$

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Example

How many integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$ with the constraints $x_1 \ge 1,\ x_2 \ge 2,\ x_3 \ge 3,\ x_4 \ge 5$?

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Example

How many integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 17$ with the constraints $x_1 \ge 1,\ x_2 \ge 2,\ x_3 \ge 3,\ x_4 \ge 5$?

Solution: This corresponds to placing

$r = 17 - (1+2+3+5) = 6$

indistinguished balls into $n = 4$ distinguished cells where a cell could be empty. So, it is

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!} = \frac{(4+6-1)!}{6!(4-1)!} = \frac{9!}{6!3!} = 84$$

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Example

- How many nonnegative integers less than 100 have the sum of their digits equal to 15?
  - Solution:
    Want to find integer solutions to $x+y = 15$ where $0 \le x \le 9$ and $0 \le y \le 9$.
    Note that
    Solution $(x \le 9 \land y \le 9)$
    = Unrestricted Solution – Solution($\neg(x \le 9 \land y \le 9)$)
    = Unrestricted Solution – Solution($\neg(x \le 9) \lor \neg(y \le 9)$)
    = Unrestricted Solution – Solution($(x > 9) \lor y > 9$))
    = Unrestricted Solution – Solution($x > 9$) – Solution($y > 9$)
  - $C(2 + 15 – 1,15) – 2*C(2 + 5 – 1,5) = 4$.

Example

How many integer solutions are there to the equation:

$x_1 + x_2 + x_3 + x_4 + x_5 = 21$

with the constraints

$0 \le x_1 \le 3,\ 1 \le x_2 < 4,\ x_3 \ge 15,\ x_4 \ge 0,\ x_5 \ge 0$?

Solution: First let's impose the restrictions that $x_2 \ge 1$ and $x_3 \ge 15$.
Then the problem reduces to counting the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$ with the constraints $0 \le x_1 \le 3,\ 0 \le x_2' \le 2,\ x'_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0$.
Note that the restrictions $x_1 \le 3$ and $x_2' \le 2$ cannot be violated simultaneously.
If we count the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, substract the number of its solutions in which $x_1 \ge 4$, and substract the number of its solutions in which $x_2' \ge 3$, then we will have the answer.

There are $C(5+5-1,5) = C(9,5) = 126$ solutions of the unrestricted equation.
Applying the first restriction reduces the equation to $x_1' + x_2' + x_3' + x_4 + x_5 = 1$, which has $C(5+1-1,1) = C(5,1) = 5$ solutions.
Applying the second restriction reduces the equation to $x_1 + x_2'' + x_3' + x_4 + x_5 = 2$, which has $C(5+2-1,2) = C(6,2) = 15$ solutions.
Therefore, the answer is $126 - 5 - 15 = 106$.

Generalized Permutations

- Question: How many different strings can be made by reordering the letters in the word MISSISSIPPI
  - The answer is not the number of permutations of 11 objects as the objects are not all distinct.
- The following two problems are equivalent
  - The number of different permutations of n objects, where there are n_i objects of type i, $i = 1,2,\ldots, k$. Note that $n = n_1 + n_2 + \ldots + n_k$
  - The number of ways to distribute n distinguishable objects into k distinguishable cells so that n_i objects are placed into box i, $i = 1,2,\ldots, k$

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

