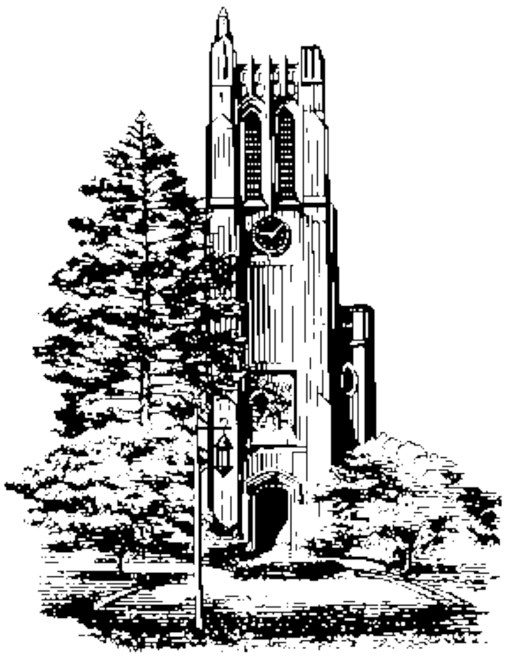
Lecture 03

Tuesday January 19, 2016



Notables

- Homework #1 is posted
  - Due Thursday September 15th, 2016
  - Convert your document to PDF before submitting it.
- Read: Top 10 Simple Things Every Computer User Should Know How to Do
  - http://lifehacker.com/5941496/top-10-simple-things-every-computer-user-should-know-how-to-do
- Udacity lecture by Steve Huffman (heads up for HW#2)
  - https://www.udacity.com/course/viewer#!/c-cs253/l-48737165/m-48723400
- Activate your Piazza account once notified
- Forthcoming topics:
  - Math review
  - System review

Summation

- Consider the following expression:
  - $S = 3 + 5 + 7 + 9 + 11 + 13 + 15 + \ldots + 35$
  - $S = (2\times1 + 1) + (2\times2 + 1) + (2\times3 + 1) + \ldots + (2\times17 + 1)$
- A more compact way of expressing S is to use the *summation notation*:

$$s = \sum_{i=1}^{17} (2i + 1)$$

Summation…

$$\sum_{n=0}^{5} \sin 2\pi n t = \sin 2\pi(0)t + \sin 2\pi(1)t + \sin 2\pi(2)t +$$

$$\sin 2\pi(3)t + \sin 2\pi(4)t + \sin 2\pi(5)t$$

$$= \sin 0 + \sin 2\pi t + \sin 4\pi t + \sin 6\pi t +$$

$$\sin 8\pi t + \sin 10\pi t$$

Exercise

- Compute the following summation:

$$\sum_{k=0}^{5} \left\lceil \frac{2k+1}{3} \right\rceil$$

Exercise

$$\sum_{k=0}^{5}\left\lceil \frac{2k+1}{3} \right\rceil = \left\lceil \frac{2\times 0+1}{3} \right\rceil + \left\lceil \frac{2\times 1+1}{3} \right\rceil + \left\lceil \frac{2\times 2+1}{3} \right\rceil + \left\lceil \frac{2\times 3+1}{3} \right\rceil + \left\lceil \frac{2\times 4+1}{3} \right\rceil + \left\lceil \frac{2\times 5+1}{3} \right\rceil$$

$$= \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{2+1}{3} \right\rceil + \left\lceil \frac{4+1}{3} \right\rceil + \left\lceil \frac{6+1}{3} \right\rceil + \left\lceil \frac{8+1}{3} \right\rceil + \left\lceil \frac{10+1}{3} \right\rceil =$$

$$= 1+1+2+3+3+4=14$$

Euler's number

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Arithmetic series

Next we will look various series

Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+\cdots+n=\sum_{k=0}^{n} k=\frac{n(n+1)}{2}$$

Arithmetic series

Proof 1: write out the series twice and add each column

$$
\begin{array}{cccccccccccccc}
 & 1 & + & 2 & + & 3 & + \cdots + & n-2 & + & n-1 & + & n \\
+ & n & + & n-1 & + & n-2 & + \cdots + & 3 & + & 2 & + & 1 \\
\hline
\end{array}
$$

$$(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$= n\,(n+1)$$

Since we added the series twice, we must divide the result by 2

Other polynomial series

We could repeat this process, after all:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

however, it is easier to see the pattern:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

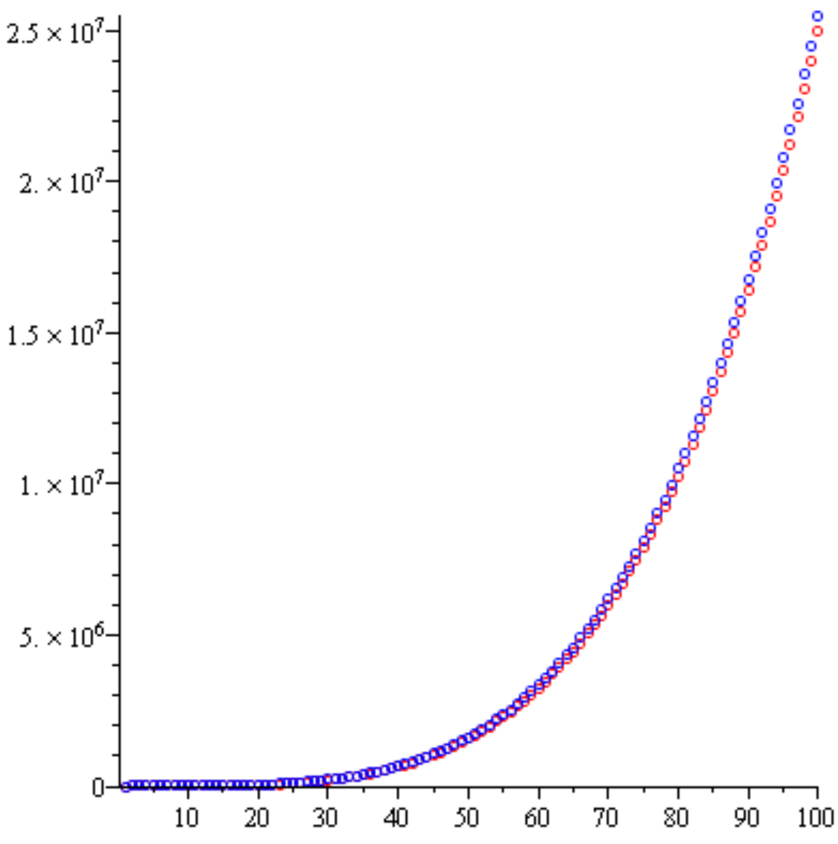
$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \approx \frac{n^4}{4}$$

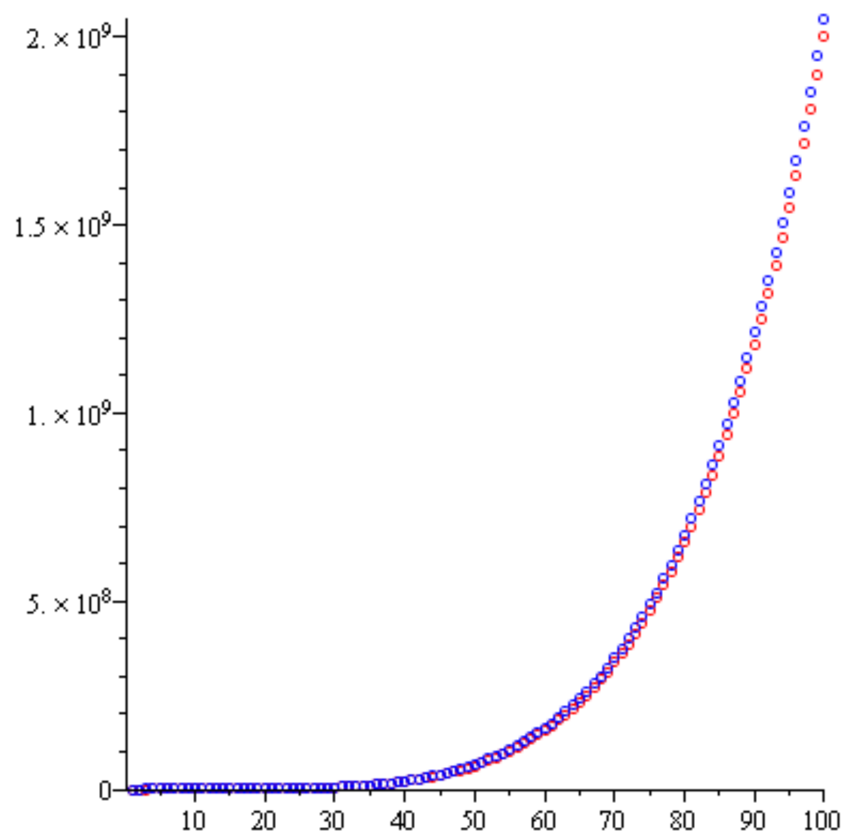
Other polynomial series

We can generalize this formula

$$\sum_{k=0}^{n} k^d \approx \frac{n^{d+1}}{d+1}$$

Demonstrating with $d = 3$ and $d = 4$:





Geometric series

The next series we will look at is the geometric series with common ratio r:

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

and if $|r| < 1$ then it is also true that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Geometric series

Note that we can use a change-of-index with summations like we do with integration:

$$\sum_{i=1}^{n} r^i = \sum_{i=1}^{n} r r^{i-1} = r \sum_{i=1}^{n} r^{i-1}$$

Letting $j = i - 1$:

$$= r \sum_{j=0}^{n-1} r^j = r \frac{1 - r^n}{1 - r}$$

Geometric series

A common geometric series will involve the ratios $r = ½$ and $r = 2$:

$$\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2 - 2^{-n} \qquad \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = 2$$

$$\sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

Series Example

The series $3 - 12 + 48$ can be written using sigma notation (also called summation notation):

$$\sum_{k=0}^{m} a_k$$

Find m.

$m =$ ☐

Write an expression for a_k in terms of k.

$a_k =$ ☐

Series Example

The series $3 - 12 + 48$ can be written using sigma notation (also called summation notation):

$$\sum_{k=0}^{m} a_k$$

Find m.

$m =$

Write an expression for a_k in terms of k.

$a_k =$

1 / 4

Thinking about the problem

Key observation: The sequence $3, -12, 48$ is geometric because each term is -4 times the term before it.

Our goal: Find m and write an expression for a_k in terms of k.

2 / 4

Step 1: Find m

m is the value of k that gives us our last term, 48. The lower limit of the sigma notation is 0, so k starts at 0:

k	0	1	2
a_k	4	-12	48

So, $m = 2$.

Series Example

The series $3 - 12 + 48$ can be written using sigma notation (also called summation notation):

$$\sum_{k=0}^{m} a_k$$

Find m.

$m =$

Write an expression for a_k in terms of k.

$a_k =$

3 / 4

Step 2: Find a_k

The key to solving this problem is realizing that when $a_k = (-4)^k$ we get a series where each term is -4 times the term before it.

$$\sum_{k=0}^{2} (-4)^k = 1 - 4 + 16$$

Multiplying each term by 3 gives us the terms we want.

$$\sum_{k=0}^{2} 3(-4)^k = 3 - 12 + 48$$

So $a_k = 3(-4)^k$.

[Is there another way to find the formula?]

4 / 4

The answer

$$a_k = 3(-4)^k$$

$$m = 2$$

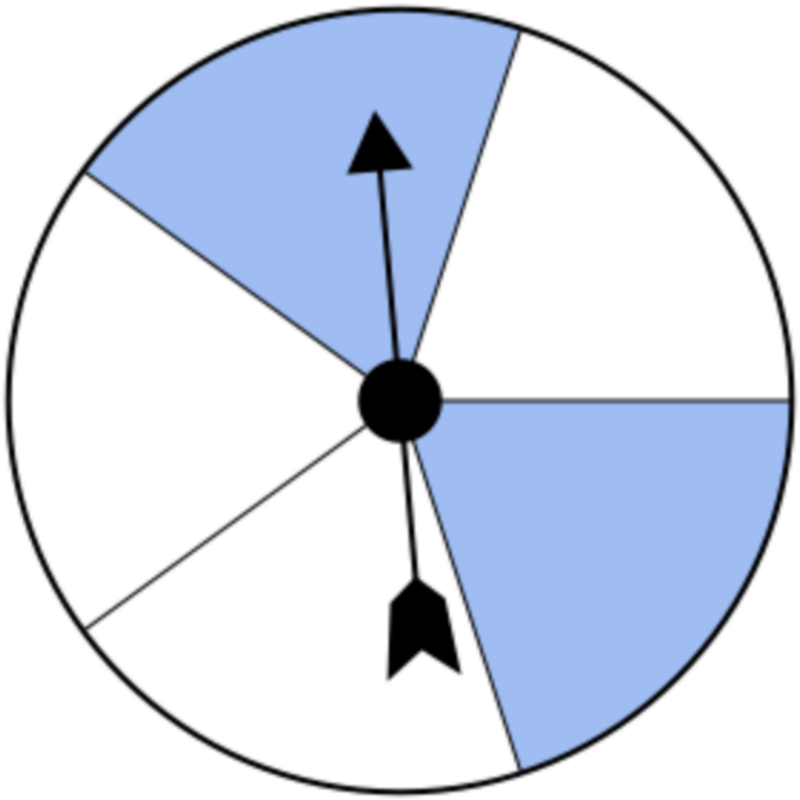
Probability= Number of Favorable Outcomes/Number of Possible Outcomes

SIMPLE PROBABILITY

Example 1

What is $\mathrm{P}(\text{shaded sector})$**?**

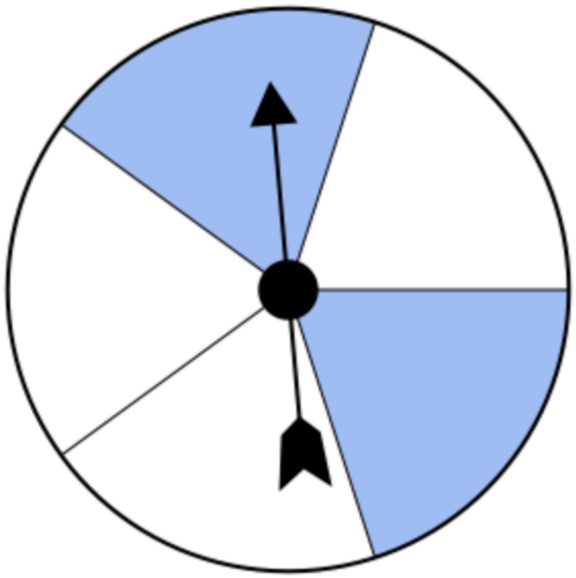
If necessary, round your answer to 2 *decimal places.*



Example 1 Answer

What is P(shaded sector)?

If necessary, round your answer to 2 decimal places.



1 / 3

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

2 / 3

There are 2 favorable outcomes (the 2 shaded sectors).

There are 5 possible outcomes since there are 5 equal sectors.

3 / 3

$$\text{P(shaded sector)} = \frac{2}{5} = 0.4$$

Probability Theory

- Random Experiment
  - An experiment whose outcome is not known before hand
    - Tossing a coin
    - Rolling a die
    - Number of students showing up for class
  - The set of all the outcomes from a random experiment is called its *sample space*.
    - {Head, Tail}; sample space of tossing a coin
  - *Probability space* refers to assigning a number (called *probability*) between 0 and 1 to each outcome in the sample space with the proviso that the numbers sum up to 1.

Examples

- Random Experiment#1
  - Will Mike show up for class
    - Sample space

      {Null, Mike}
    - Probability space

      {0.2, 0.8}
- Random Experiment#2
  - Will Barb show up for class
    - Sample space

      {Null, Barb}
    - Probability space

      {0.4, 0.6}

Examples….

- Random Experiment#3
  - Who is going to show up for class
    - Sample space

      {Null, Barb, Mike, Barb & Mike}
    - What is the probability space

      {0.08, 0.12, 0.32, 0.48}

Rolling Two Dice

- Let (die #1, die#2) denote a possible outcome of rolling two dice. Here is a listing of all possible outcomes

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- What events can we define here?

Example

- What is the probability of rolling 6?
  - There are five out of 36 ways to get 6
  - 5/36
- What is the probability of rolling 7 or 11?
  - 8/36

Dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Two-Dice sum event

Example 1

If you flip three fair coins, what is the probability that the first two flips will both be heads, and the third flip will be either heads or tails?

Example 1 Solution

1 / 4

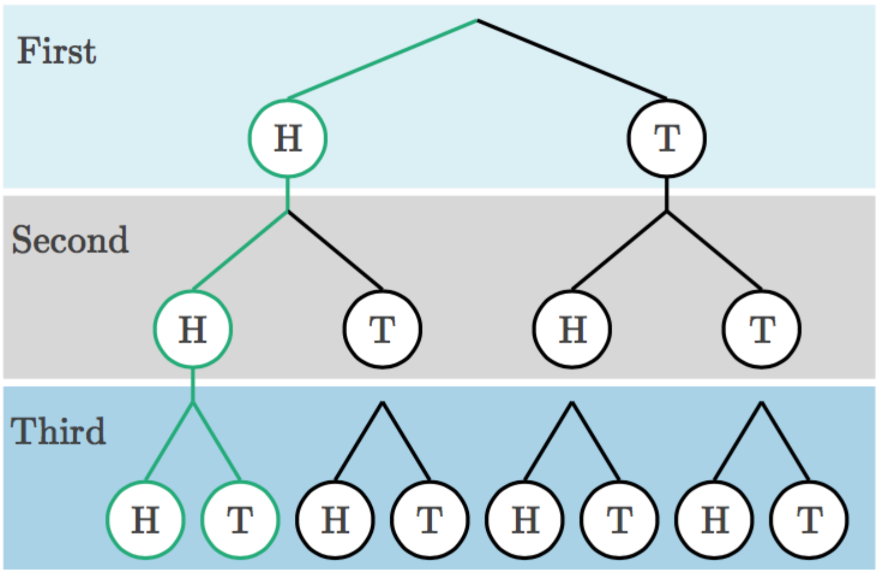
$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total possible outcomes}}$$

2 / 4

If you flip three coins, there are 2 possible outcomes for each individual flip, so there are $2 \times 2 \times 2 = 8$ total possible outcomes. Since the coin is fair, each outcome is equally likely.

3 / 4

Each path through the tree represents one outcome. The green paths show the 2 favorable outcomes.



4 / 4

The probability of getting heads on the first two flips, and either heads or tails on the third flip is $\frac{2}{8}$. We can simplify this fraction to $\frac{1}{4}$.

Random variables

- A *variable, say* X, is a placeholder for values.
  - When the values are outcomes of a random (statistical) experiment, the variable is called a random variable.
  - P(X) represents the probability of X
  - $P(X = r)$ refers to the probability that the random variable X is equal to a particular value, denoted by r. As an example, $P(X = 1)$ refers to the probability that the random variable X takes on value 1.
  - The *probability distribution* of X is a function that gives, for each value that X takes on, its probability.

Example

- Let X represent the sum of the digits in a typical PID.
  - How do we determine its probability distribution?
    - What are the values that X can take on?
    - What is the probability of each value?

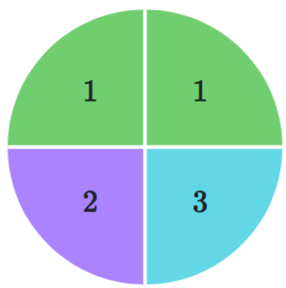
Expected Value

- Expected value uses probability to tell us what outcomes to expect in the long run.

Expected Value : Example 1

Problem 1: Board game spinner

A board game uses the spinner shown below to determine how many spaces a player will move forward on each turn. The probability is $\frac{1}{2}$ that the player moves forward 1 space, and $\frac{1}{4}$ that the player moves forward either 2 or 3 spaces.



What is the expected value for the number of spaces a player moves forward on a turn?

If necessary, round your answer to the nearest hundredth.

[ ] spaces

Expected Value : Example 1 Answer

To find expected value, multiply each outcome by its probability and add those results together.

$$\text{expected value} = \frac{1}{2} \cdot 1 \text{ space} + \frac{1}{4} \cdot 2 \text{ spaces} + \frac{1}{4} \cdot 3 \text{ spaces}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{4}$$

$$= 1.75$$

The expected value is 1.75 spaces.

Standard deviation

- A measure of dispersion from the mean μ
- It is represented by the symbol sigma, σ, and is given by the following formula:

$$\sigma = \sqrt{\sum_{i=1}^{N} p_i (x_i - \mu)^2}, \quad \text{where} \quad \mu = \sum_{i=1}^{N} p_i x_i .$$

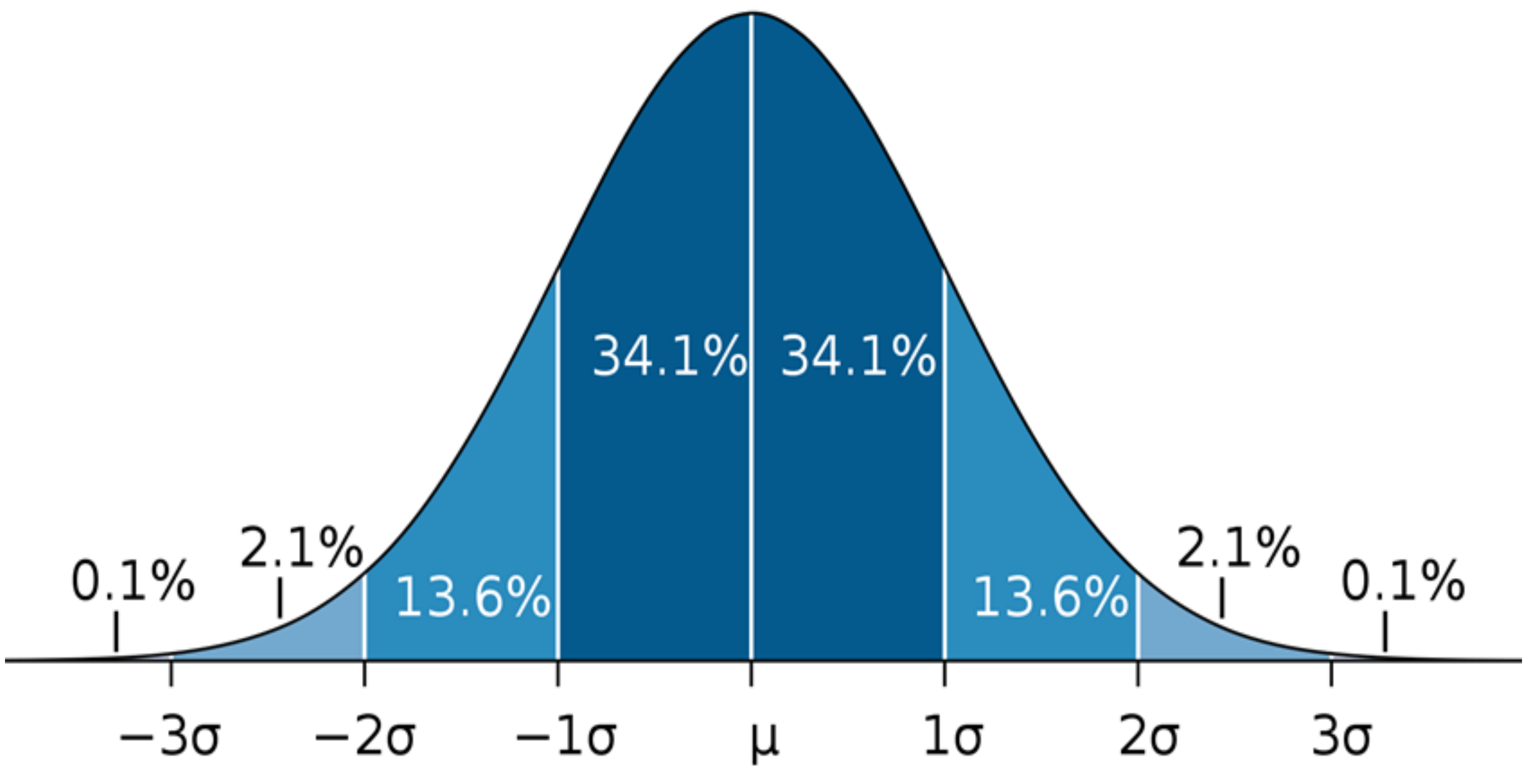
Two-Dice example

Dice Sum (value)	Prob	value * prob	prob * (value - mean)^2
2	0.03	0.06	0.69
3	0.06	0.17	0.89
4	0.08	0.33	0.75
5	0.11	0.56	0.44
6	0.14	0.83	0.14
7	0.17	1.17	0.00
8	0.14	1.11	0.14
9	0.11	1.00	0.44
10	0.08	0.83	0.75
11	0.06	0.61	0.89
12	0.03	0.33	0.69
	Mean	7.00	
	Std	2.42	

Central limit theorem

- Let N independent random variables perform *task* x, each with a fixed probability p.
  - Students *entering* a class with probability p
  - Coins tossed with heads *appearing* with probability p
- Then, the number of occurrences of task x has a *normal distribution* with mean $\mu = Np$, and standard deviation of $\sigma = \sqrt{p(1-p)N}$.
  - Note that the event μ also has the highest probability.
  - Note that $\pm\ \sigma$ around the mean gives a 68% confidence, that is 68% of the occurrences will be in the range of $\mu \pm \sigma$.

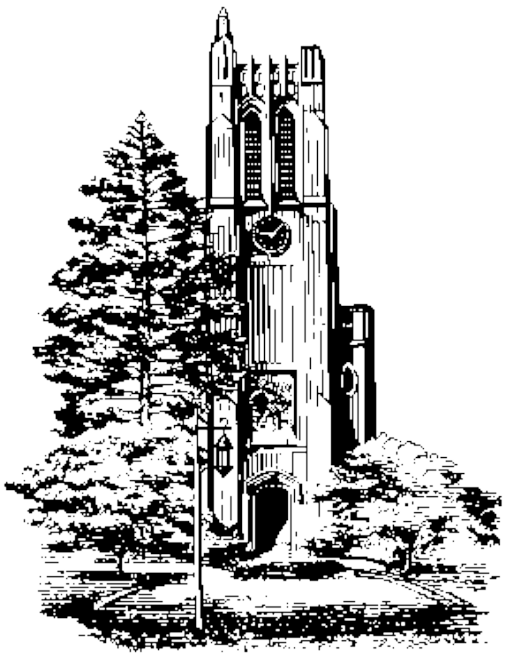
Normal Distribution



Simulating random events

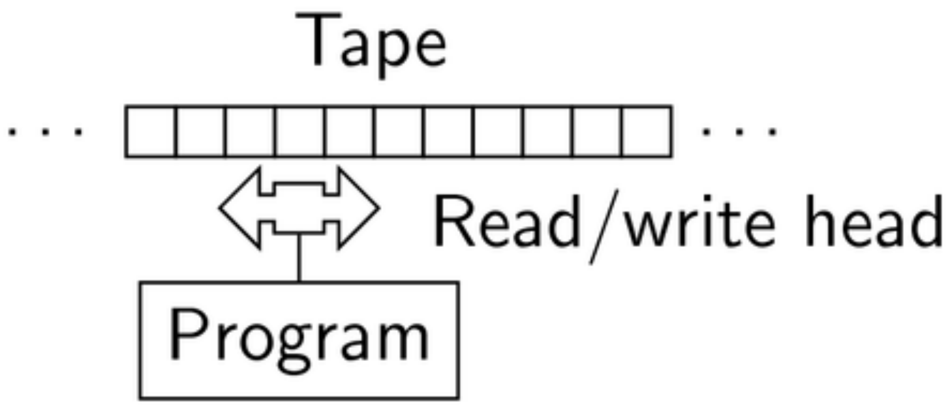
- Do we know the probability distribution?
- How to select an item from a list at random
  - Random number generators

System Review



Most powerful computer

- Alan Turing (1912-1954)
  - Turing machine



  - Simulator

    http://ironphoenix.org/tril/tm/

Von Neumann Architecture