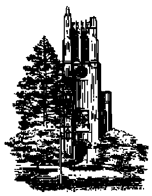
Thursday May 26, 2016
Lecture 08

Set Theory

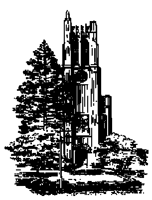


Notables

- Homework#5
  - Page 125, Problems 4, 10, 22, 32((a) and (c) only)
  - Page 136, Problems 4, 16, and 26
  - Due Tuesday May 31, 2016
- Read Chapter 2
- Tentative Schedule for the week

Week	M	T	W	R	Topic	Section
1	5-23				Nested Quantifiers, Rules of Inference	1.5, 1.6
		5-23			Proof	1.7
			5-25		Proofs, Sets	1.8
				5-26	Sets	2.1

Set Operations

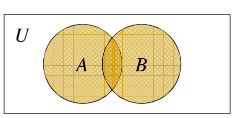


Outline

- Introduction
  - Union, Intersection, Complement, Difference,
  - Generalized Unions and Intersections
- Set Identities
  - Fundamental set identities
  - Proofs involving sets, using:
    - Membership tables
    - Set builder notation
    - Substitution

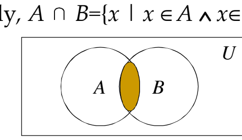
Set Union

- Definition: Let A and B be sets. The *union* of A and B, denoted $A \cup B$, is the set containing those elements that are either in A or in B, or in both.
- Formally, $A \cup B = \{x \mid x \in A \vee x \in B\}$
- **Venn Diagram**



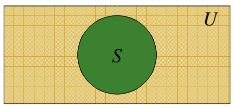
Set Intersection

- Definition: Let A and B be sets. The *intersection* of A and B, denoted by $A \cap B$, is the set containing those elements which are in both A and B.
- Formally, $A \cap B = \{x \mid x \in A \wedge x \in B\}$



Set Complement

- Definition: Let U be the universal set. The *complement* of a set S, denoted $\overline{S}$, is the set containing elements of U which are not in S.
- Formally, $\overline{S} = \{x \mid x \notin S \land x \in U\}$
- $\overline{S} = U - S$



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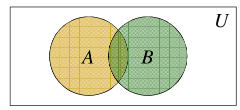
Set Difference

- Definition: Let A and B be sets. The *difference* of A and B, denoted $A - B$, is the set containing those elements that are in A but are not in B.
  - $A - B$ is also called the *complement* of B *with respect* to A.
  - $A - B = \{x \mid x \in A \land x \notin B\}$



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Disjoint Sets

- Definition: Two sets are called *disjoint* if their intersection is the empty set $\emptyset$.
- Principle of inclusion-exclusion for finite sets:
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

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Set Identities

- Sets together with operators $\{\overline{}, \cap, \cup\}$ define a special class of Boolean algebraic structures.

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Set Identities

$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

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Set Identities…

$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

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Set Identities – Example.

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- To show this identity we can use one of the following ways:
  1) Using the fact that $A = B \equiv (A \subseteq B) \wedge (B \subseteq A)$
  2) Using Set Builder notation,
  3) Using membership tables

Example: Using Set Builder…

$\overline{A \cup B}$	$= \{x \mid x \notin (A \cup B)\}$	Complement
	$= \{x \mid \neg(x \in (A \cup B))\}$	Not an element
	$= \{x \mid \neg[(x \in A) \vee (x \in B)]\}$	Union
	$= \{x \mid (x \notin A) \wedge (x \notin B)\}$	De Morgan's
	$= \{x \mid (x \in \overline{A}) \wedge (x \in \overline{B})\}$	Complement
	$= \{x \mid x \in \overline{A} \cap \overline{B}\}$	Intersection
	$= \overline{A} \cap \overline{B}$	

Example – Using Membership…

A	B	$\overline{A}$	$\overline{B}$	$A \cup B$	$\overline{A \cup B}$	$\overline{A} \cap \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

1= an element in the set 0 = an element is not in the set

More identities

$$\begin{aligned} \overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} \\ &= \overline{A} \cap \left(\overline{B} \cup \overline{C}\right) \\ &= (\overline{A} \cap \overline{B}) \cup \left(\overline{A} \cap \overline{C}\right) \end{aligned}$$

Proof of Set Identities…

- We can use Venn diagrams to get some ideas about the correctness of the identity.
- This method can't be used to prove the identities.

Generalized Union

- Definition: The *union* of a collection of sets is the set containing those elements that are members of at least one set in the collection.

$$A_1 \bigcup A_2 \bigcup \ldots \bigcup A_n = \bigcup_{i=1}^{n} A_i$$

Generalized Intersection

- Definition: The *intersection* of a collection of sets is the set containing those elements that are members of **all** the sets in the collection.

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Aside: Russell's Paradox

- Let S be the set of all sets which do not contain themselves.
  - Does S contain itself.

Computer Representation of Sets

- Use an arbitrary ordering of the elements of the finite universal set U:
  - $(a_1, a_2, \ldots, a_n)$
- Represent a subset A of U with the bit string of length n:
  - ith bit is: 1 if $a_i \in A$
    0 if $a_i \notin A$

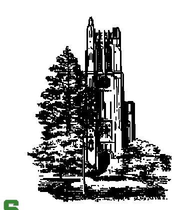
Computer Representation…

- Example: U = {1, 2, pen, guitar, car, keyboard}
  - We need a 6-bit string $(a_1, a_2, \ldots, a_6)$ and the following mapping
    (1, pen, keyboard, 2, car, guitar) = $(a_1, a_2, \ldots, a_6)$
- Now, let A = {2, pen, car, keyboard}. The string corresponding to A is (0,1,1,1,1,0)

Computer Representation…

- With the string representation of a set, and the following conventions, we can do any operation
  - Set complementation: Invert each bit (0 to 1, 1 to 0)
  - Set union: Do a bit-wise "addition"
    - $0 + 1 = 1 + 0 = 1$
    - $1 + 1 = 1, 0 + 0 = 0$
  - Set Intersection: Do a bit-wise "multiplication"
    - $0 \times 1 = 1 \times 0 = 0$
    - $1 \times 1 = 1, 0 \times 0 = 0$

Functions



Outline

- Introduction: definitions, example and terminology, image of a subset of domain, sum and product of functions
- One-to-one functions: strictly increasing, decreasing functions
- Onto functions
- Bijections: identity functions
- Graphs of Functions
- Inverse Functions
- Compositions of Functions
- Some Important Functions: Floor, Ceiling and Factorial functions

Function

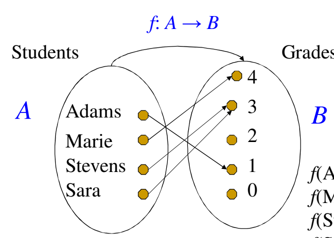
- Example
  - Consider your final grades in CSE 260. Your grades will be one of the values from the set {4, 3.5, 3, 2.5, 2, 1.5, 1, 0}
- What kind of properties does this *assignment* have?

Introduction

- Definition: Let A, B be sets. A *function* f from A to B, denoted $f: A \to B$, is an assignment where each element of A is assigned exactly one element of B.
- Notation:
  - $f: A \to B$
  - We write $f(a) = b$, if b is the element of B assigned under f to the element a of A.
  - We also say f *maps* A to B
- Formally, f is a function from A to B if and only if $\forall x \in A\ \exists! y \in B\ \ f(x) = y$.

  where $\exists!$ is the *uniqueness* quantifier.

Example



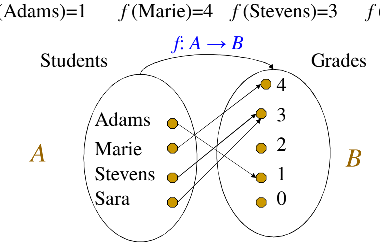
f(Adams)=1
f(Marie)=4
f(Stevens)=3
f(Sara)=3

Domain, Co-domain, …

- Definition: Let f be a function from A to B, that is, $f: A \to B$. Then A is called the *domain* of f, and B is the *codomain* of f.
- If $f(a) = b$, then
  - b is called the *image* of a, and
  - a is a *pre-image* of b.
- The *range* of f is the set of all images of elements of A.
- How are codomain and range related?
  - Range is a subset of the codomain

Example

f(Adams)=1 f(Marie)=4 f(Stevens)=3 f(Sara)=3



The image of 'Marie' is *4*; the pre-images of *3* are 'Stevens' and 'Sara'; the range of f is {*1*, *3*, *4*}.

Sum, Products of real-valued Functions

- Definition: Let f_1 and f_2 be functions from A to $\boldsymbol{R}$. Then $f_1 + f_2$ and $f_1 \cdot f_2$ are also functions from A to R, defined as:
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
  - $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$
- Note that if f_1 and f_2 do not have the same domain, the above operations do not make sense.

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Image of a subset of a domain

- Definition: Let f be a function from A to B, and let S be a subset of A.

  The *image* of S, denoted $f(S)$, is the subset of B consisting of the images of the elements of S.
  - Formally: $f(S) = \{ f(s) \mid s \in S\}$.
  - Note that $f(A)$ is the range of f.
- In the previous Example,
  - $f(\{\text{Adams, Sara}\}) = \{3, 1\}$

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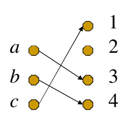
One-to-one Functions

- Definition: A function f from A to B is said to be *one-to-one*, or *injective*, if and only if distinct elements of the domain have distinct images. That is,

$$\forall x \in A\ \forall y \in A\ \ f(x) = f(y) \rightarrow x = y.$$



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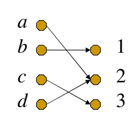
Onto Functions

- Definition: A function f from A to B is said to be *onto*, or *surjective*, if and only if its range and codomain are the same. That is,

$$\forall y \in B\ \exists x \in A\ \ f(x) = y.$$



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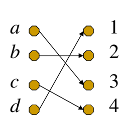
Bijections

- Definition: A function $f: A \rightarrow B$ is a *bijection*, or *one-to-one correspondence*, if it is both *one-to-one* and *onto*.
  - Note that the cardinalities (when dealing with finite sets) of domain and codomain of a bijection are equal.



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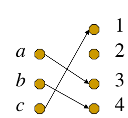
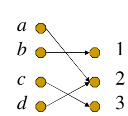
Summary of Function Types

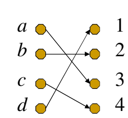
1-to1 but not onto



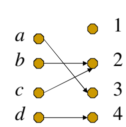
onto but not 1-to-1

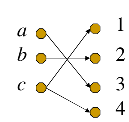


both 1-to-1 and onto; bijection



Neither 1-to-1 nor onto





Not a function

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