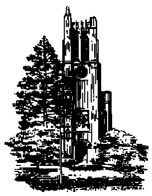
Tuesday June 14, 2016
Lecture 17

Number Theory



Notables

Homework 10; Due Thursday June 16

1. Using the method discussed in class, convert (long hand)
   a. Decimal (that is, base 10) 8888 to binary
   b. Decimal 2555 to base 6
   c. Decimal 8990 to base 19; use symbols 0-9 and A-Z
   You must show ALL your steps to receive full credit.
2. Convert $(260.260)_9$ to base 10.
3. Carry out the following additions in the given base:
   a. $(5665)_7 + (4664)_7 = (\quad)_7$
   b. $(AB56)_{16} + (9868)_{16} = (\quad)_{16}$
4. Page 255, Problem 26
5. Page 256, Problem 30(d), and Problem 48(d)
6. Page 272, Problem 4(f), 21(c), 24(b), and 40(e)

Base Conversion, Recap

- From decimal to any base:
  - For the whole (integer) part
    - Repeated divisions, collecting the remainder
  - For the fraction part
    - Repeated multiplications, collecting the whole part
- From another base to decimal
  - Expand according to positional system
- A good site for binary conversion is:
  - http://en.wikipedia.org/wiki/Binary_numeral_system

Base Conversion

Number in Base 10 xxxxxxxxxxxxxxx.yyyyyyyyyyyyyyy

/, % expand *b

Number in Base b qqqqqqqqqqqqqq.wwwwwwwwwww

Answer to the pop quiz

- Represent $(5.875)_{10}$ in binary
- Answer:
  - First the whole part
    - 5 = 2 * 2 + 1
    - 2 = 1 * 2 + 0
    - 1 = 0 * 2 + 1
  - Now, the fraction part
    - 0.875 * 2 = 1.75
    - 0.75 * 2 = 1.5
    - 0.5 * 2 = 1.0
  - Number 5.875 in decimal has 101.111 representation in base 2

Exercise

- Represent $(123.213)_4$ in base 10
  - $(123)_4 = 3 \times 4^0 + 2 \times 4^1 + 1 \times 4^2 = 3 + 8 + 16 = 27$
  - $(0.213)_4 = 2 \times 4^{-1} + 1 \times 4^{-2} + 3 \times 4^{-3} = 0.609375$
  - Thus $(123.213)_4$ is (27.609375) in base 10

Bases used in Computer Science

- The base b expansion of integer n is written as $n = (a_k \dots a_1 \, a_0)_b$
- We omit b and () for base 10
- Base 2 (Binary),
  - bits in computer, has two symbols {0,1}
  - The Presence/Absence (PandA) representation
- Base 8 (Octal),
  - has eight symbols {0,1,2,3,4,5,6,7}
- Base 16 (Hexadecimal)
  - has sixteen symbols {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}
  - Note that A=10, B=11, C=12, D=13, E=14, F=15.

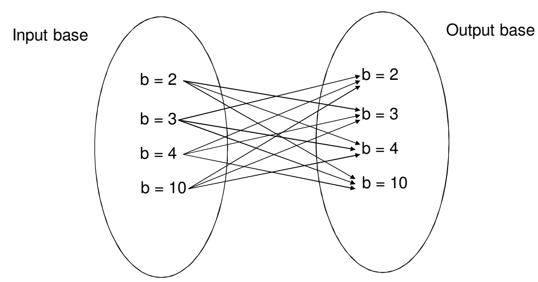
Bases used in Computer Science

- Base 2 (Binary),
  - bits in computer, has two symbols {0,1}
- Base 8 (Octal),
  - has eight symbols {0,1,2,3,4,5,6,7}
- Base 16 (Hexadecimal)
  - has sixteen symbols {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}.
  - Note that A=10, B=11, C=12, D=13, E=14, F=15.

Exercise

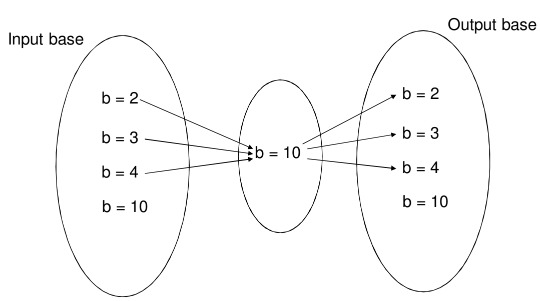
- Covert binary 110000111010
  - To octal (base 8)
    - 6072
  - To hex
    - C3A
- Convert Hex (base 16) ABC
  - To binary
    - 101010111100
  - To octal
    - 5274
- 4 bits in binary is equal to one digit in Hex. One binary byte can be represented by a two-digit Hex

Conversion between Bases



A total of 12 converters

Conversion between Bases…



A total of 6 converters

Other number representations

- Theorem: Every integer can be *uniquely* represented in the form:

$$n = a_k 3^k + a_{k-1} 3^{k-1} + \cdots + a_1 3^1 + a_0$$

$$\text{where} \quad a_i \in \{-1, 0, 1\}$$

- This is known as the Balanced Ternary Expansion
- Example,
  - Find BTE of 79.
  - $79 = 3^4 - 3 + 1$

Example

- Give the BTE representation of decimal number 223.
  - $223 = 2*3^4 + 2*3^3 + 2*3 + 1$
  - $223 = 2*3^4 + 2*3^3 + (3-1)*3 + 1$
  - $223= 2*3^4 + 2*3^3 + 3^2 - 3 + 1$
  - $223= 2*3^4 + (3-1)*3^3 + 3^2 - 3 + 1$
  - $223= 2*3^4 + 3^4 - 3^3 + 3^2 - 3 + 1$
  - $223= 3^5 - 3^3 + 3^2 - 3 + 1 = (1,0,-1,1,-1,1)_{BTE}$

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In-Class Exercise

- Covert decimal 238 to BTE
  - $238 = 2\times3^4 + 2\times3^3 + 2\times3^2 + 3 + 1 = (22211)_3$
  - $238 = (3-1)\times3^4 + (3-1)\times3^3 + (3-1)\times3^2 + 3 + 1$
  - $238 = 3^5 - 3^4 + 3^4 - 3^3 + 3^3 - 3^2 + 3 + 1$
  - $238 = 3^5 - 3^2 + 3 + 1 = (1,0,0,-1,1,1)_{BTE}$

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Other number representations…

- Theorem: Every integer x can be *uniquely* represented in the form:

$$x = a_n n! + a_{n-1}(n-1)! + \cdots + a_2 2! + a_1 1!$$
$$\text{where } a_i \text{ is an integer with}$$
$$0 \le a_i \le i \quad \text{for } i = 1, 2, \ldots, n.$$

- This is known as the Cantor Expansion
- Example,
  - Find Cantor expansion of 87.
  - $87 = 3*4! + 2*3! + 2! + 1!$

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In-Class Exercise

- Convert decimal 319 to Cantor Expansion
  - $319 = 159\times2 + 1$
  - $159 = 53\times3 + 0$
  - $53 = 13\times4 + 1$
  - $13 = 2\times5 + 3$
  - $2 = 0\times6 + 2$
  - $319 = 2\times5! + 3\times4! + 1\times3! + 0\times2! + 1\times1! = (23101)_{CE}$

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Algorithms for Integer Operation

- Computers use binary numbers to perform operations in integers.
- In order to handle integer operations in binary system first we should convert the decimal numbers into integers and then do the operations using binary numbers.

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Addition of Integers in Base 2

- A procedure to perform addition is based on the usual method for adding numbers with pencil and paper.
- Example:

1 1	← carry bits	
1 1 1 0	→	$(14)_{10}$
+ 1 0 1 1	→	$(11)_{10}$
= 1 1 0 0 1	→	$(25)_{10}$

- An overflow may happen when adding two numbers.

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Multiplying Integers in Base 2

- Example

$$
\begin{array}{rrcl}
 & 110 & \rightarrow & (6)_{10} \\
\times & \underline{101} & \rightarrow & (5)_{10} \\
 & 110 & & \\
+ & 000 & & \\
 & \underline{110} & & \\
= & 11110 & \rightarrow & (30)_{10}
\end{array}
$$

- Up to $2n$ bits may be needed to represent the product of two n-bit numbers.

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More Examples on Base Arithmetic

- Compute $222 + 222$ in base 3.
  - Answer:
    - $(1221)_3$
- Compute $222 + 222$ in base 4.
  - Answer:
    - $(1110)_4$
- Compute $222 + 222$ in base 5.
  - Answer:
    - $(444)_5$

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Modular Exponentiation

- In cryptography it is often required to compute
  $b^n \bmod m$
  efficiently.
- Direct computation, that is, computing b^n first and then dividing it by m to find the remainder is impractical (when dealing with very large numbers).
- Example: Compute $3^{644} \bmod 645$.
  - Answer is 36

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Modular Exponentiation…

Suppose b, n and m are positive integers, and

and that we have computed $b^n \equiv r \pmod{m}$.

That is, $b^n = md + r$. How can we compute $(b^n)^2 \bmod m$?

Observe that $(b^n)^2 = (md + r)^2 = (md)^2 + 2mdr + r^2$.

Thus, the remainder of $(b^n)^2$ divided by m comes from

the remainder of r^2 divided by m. That is,

$(b^n)^2 \bmod m = (b^n \bmod m)^2 \bmod m$.

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Modular Exponentiation…

- To compute $b^n \bmod m$, first represent n in powers of 2.
  - Example: Compute $3^{644} \bmod 645$,

$$n = 644 = (1010000100)_2 = 2^9 + 2^7 + 2^2$$

$$3^{644} = 3^{(2^9+2^7+2^2)} = 3^{2^9} \times 3^{2^7} \times 3^{2^2}$$

$$3^{644} \bmod 645$$

$$= (3^{2^9} \bmod 645) \times (3^{2^7} \bmod 645) \times (3^{2^2} \bmod 645)$$

$$= 111 \times 396 \times 81 \bmod 645 = 36$$

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Example

- Computer
  $23^{450} \bmod 987$
  using the fact that:
  - $23^{200} \equiv 739 \bmod 987$
  - $23^{150} \equiv 967 \bmod 987$
  - $23^{100} \equiv 667 \bmod 987$
    - Answer: $23^{450} \bmod 987 \equiv 739*967*667 \bmod 987$
    - $739*967*667 \equiv 883 \bmod 987$
    - $23^{450} \equiv 883 \bmod 987$

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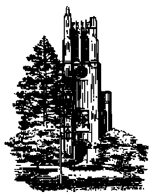
Primes

Section 4.3



Primes

- Definition: A positive integer $p > 1$ is called *prime* if the only positive factors of p are 1 and p.
  - A positive integer that is greater than 1 and is not prime is called *composite*.
  - Note that number 1 is neither prime nor composite.
  - Some primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47…
  - How many digits in the largest prime found to-date (2016)?
    - 22,338,618 digits
    - A Mersenne prime = $2^{74,207,281} - 1$

Prime Theorems

- Theorem 1: Every positive integer $n > 1$ has a prime factor $\leq n$. Why?
- Theorem 2: Every positive integer $n > 1$ can be written *uniquely* as a prime or as the product of primes, in non-decreasing order. (proof later)
- Theorem 3: If n is a composite integer, then n has a prime factor $\leq \sqrt{n}$. Why?
- Corollary: If n does not have a prime factor $\leq \sqrt{n}$ then it is prime. Why?
- Conjecture: Every even integer > 2 can be written as the sum of two primes.

Prime Factorization Algorithm

1. Try to find a prime factor of n beginning with 2, 3, 5, … up to $\sqrt{n}$
2. If no prime factor is found, n is prime.
3. If a prime factor p is found, continue by factoring n/p in the same way. Note that n/p doesn't have any prime factor less than p so start with p again.

- Question: For this method, don't we need to know the list of primes?

Example

- Find Prime factors of 3960
  - Dividing by 2: 3960 = 2 x 1960
  - Dividing by 2: 1960 = 2 x 990
  - Dividing by 2: 990 = 2 x 495
  - Dividing by 2: not possible
  - Dividing by 3: 495 = 3 x 165
  - Dividing by 3: 165 = 3 x 55
  - Dividing by 3: not possible
  - Dividing by 5: 55 = 5 x 11
  - So, the answer is : 2x2x2x3x3x5x11= 2^3x3^2x5x11

Example

- Find prime factors of 143
  - Dividing by 2: not possible
  - Dividing by 3: not possible
  - Dividing by 5: not possible
  - Dividing by 7: not possible
  - Dividing by 11: 143 = 11x13

How to find all primes ≤ 100?

- First note that if a number ≤ 100 is composite it has a factor ≤ 10.
- Find all the primes ≤ 10; these are 2, 3, 5, and 7.
- Then eliminate all multiples (other than itself) of 2, all multiples of 3, all multiples of 5, and finally, all multiples of 7.
- All the remaining numbers are primes ≤ 100.
- These are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.
- We can now use this list and find all the primes ≤ 10000

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Question

- Let p be a prime number. Further, let p_1, p_2, …, p_k be all the primes < p. Moreover, let $N = p_1 \times p_2 \times \ldots \times p_k \times p + 1$
  - For example, when $p = 5$, then $N = 2 \times 3 \times 5 + 1 = 31$
- Prove or disprove that N as defined above is always a prime number.
- Disprove: p = 13, N = 2*3*5*7*11*13+1 = 30031= 59*509

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Infinitude of Primes

- Theorem 4: There are infinitely many primes.
- Proof:
  - What method of proof should we use?
  - We'll do a proof by contradiction as follows:
    - Assume there is a finite number of primes, and let p be the largest (last) prime.
    - Consider the number $N = p! + 1$, that is $N = 1\times2\times3\times4\times \ldots \times(p\text{-}2)\times(p\text{-}1)\times p + 1$
    - Based on Theorem 1, N must have a prime factor (could be itself!)
    - N does not have any non-unity factor ≤ p. WHY?
    - N has a prime factor which is > p.
    - There is a prime > p.
    - A contradiction. Therefore, there infinitely many primes.

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Prime numbers distribution

- Let $\pi(x)$ be the number of primes ≤ x
- It is known that asymptotically
  $\pi(x) \approx x / \ln x$
- Asymptotically, the x^{th} prime is about $x \ln x$

x	$\pi(x)$	$x / \ln x$
1000	168	145
10000	1229	1086
100000	9592	8686
1000000	78498	72382
10000000	664579	620420
100000000	5761455	5428681

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Prime theorems…

- Theorem: If n is not prime, then $2^n - 1$ is not prime. In other words, if $2^n - 1$ is prime, so is n. (Note that n being prime does not imply $2^n - 1$ is prime. Try $n = 11$)
- Proof: Let $r > 1$ and $s > 1$ be positive integers. It is known that
  $$x^{r\cdot s} - 1 = (x^s - 1)(x^{s(r-1)} + x^{s(r-2)} + \cdots + x^s + 1)$$
- For example: $x^{3\cdot 2} - 1 = (x^2 - 1)\big(x^{2(3-1)} + x^{2(3-2)} + x^{2(3-3)}\big) = (x^2 - 1)(x^4 + x^2 + 1) = x^6 - 1$
- Therefore, using the above fact, and setting $x = 2$, when $n = r \cdot s$, that is, not prime, then $2^n - 1$ is not prime.

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More on Primes (Fermat's Little Theorem)

- Consider the following function:
  $f(n) = 2^{n-1}\ mod\ n$
  for prime integers $n > 2$
  - $f(3) = 2^{3\text{-}1}$ mod 3 = 4 mod 3 = 1
  - $f(5) = 2^{5\text{-}1}$ mod 5 = 16 mod 5 = 1
  - $f(7) = 2^{7\text{-}1}$ mod 7 = 64 mod 7 = 1
  - ….
  - $f(89) = 2^{89\text{-}1}$ mod 89 = 618970019642690137449562112 mod 89 = 1
  - …
- Any observation?
  - Holds for every prime, but the converse is not true.
  - $f(341) = 2^{341\text{-}1}$ mod 341 = 1 but 341 = 11x31 is obviously not prime.

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Greatest Common Divisors

- Definition: Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and $d \mid b$, is called the *greatest common divisor* of a and b, and is denoted by $d = \gcd(a, b)$.
- How to find gcd?
  - Euclid's Algorithm
  - Using prime factorization

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Euclid's Algorithm for GCD

1. Input: Positive integers m and n
2. Divide m by n and let r be the remainder
3. If $r = 0$, the algorithm terminates with n as the answer.
4. Set $m \leftarrow n$, followed by set $n \leftarrow r$, and then go to step 2.

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Example of Euler's GCD algorithm

- Find gcd(252,198)
  - $252 = 1 \times 198 + 54$
  - $198 = 3 \times 54 + 36$
  - $54 = 1 \times 36 + 18$
  - $36 = 2 \times 18$
  - Thus gcd(252, 198) = 18

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Fact on GCD

- Theorem: Let n and m be positive integers. Then there are integers s and t such that $\gcd(n, m) = sn + tm$
- Proof: Using the next-to-last division in GCD algorithm, and we working our way up we can obtain the desired relation. In the previous example, we would find the following :

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Example

- Find gcd(252, 198)
  - $252 = 1 \times 198 + 54$
  - $198 = 3 \times 54 + 36$
  - $54 = 1 \times 36 + 18$
  - $36 = 2 \times 18$
- $\gcd(n, m) = sn + tm$
  - $18 = 54 - 1 \times 36 = 54 - 1 \times (198 - 3 \times 54)$
    $= (252 - 1 \times 198) - 1 \times (198 - 3 \times (252 - 1 \times 198))$
    $= 4 \times 252 - 5 \times 198$

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GCD

- Find the prime factorization of a and b.
  - If $a = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$,
    $b = p_1^{b_1} p_2^{b_2} \dots p_n^{b_n}$,
  then
  $\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \dots p_n^{\min(a_n,b_n)}$
- Example:
  - Find gcd(1000, 625)
    - $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^3 \times 5^3$
    - $625 = 5 \times 5 \times 5 \times 5 = 5^4 = 2^0 \times 5^4$
    - $\gcd(1000,625) = 2^{\min(0,3)} \times 5^{\min(3,4)} = 2^0 \times 5^3 = 5^3 = 125$

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Relatively Prime Integers

- Definition: Positive integers a and b are *relatively prime* if $\gcd(a, b) = 1$.
- Definition: Positive integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \le i < j \le n$.

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Least Common Multiples

- Definition: Let a and b be positive integers. The *least common multiple* of a and b is the smallest positive integer that is divisible by both a and b.
  It is denoted by $\mathrm{lcm}(a, b)$.
  - If $a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$,
    $b = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n}$, then
    $\mathrm{lcm}(a, b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \ldots p_n^{\max(a_n,b_n)}$
- Find lcm(1000,625)
  - $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^3 \times 5^3$
  - $625 = 5 \times 5 \times 5 \times 5 = 5^4 = 2^0 \times 5^4$
  - $\mathrm{lcm}(1000,625) = 2^{\max(0,3)} \times 5^{\max(3,4)} = 2^3 \times 5^4 = 5000$

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Observation

- We computed that:
  - $\gcd(1000,625) = 2^{\min(0,3)} \times 5^{\min(3,4)} = 2^0 \times 5^3 = 5^3 = 125$
  - $\mathrm{lcm}(1000,625) = 2^{\max(0,3)} \times 5^{\max(3,4)} = 2^3 \times 5^4 = 5000$
  - Note that:
    - $\gcd(1000,625) \times \mathrm{lcm}(1000,625) = 125 \times 5000 = 625000$
  - Any observation?

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gcd and lcm relationship

- Theorem: $\forall a \in Z^+ \; \forall b \in Z^+$
  $$a \times b = \gcd(a, b) \times \mathrm{lcm}(a, b)$$
- Proof:
  - Just use the expressions for gcd and lcm using prime factorizations of a and b.

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Modulo inverse

- Definition: Let n and $m > 1$ be integers. If there is an integer s such that $n \times s \equiv 1 \pmod{m}$, then we say s is an *inverse* of n.
  - Example:
    - 12 is an inverse of 3 modulo 7 because $12 \times 3 \equiv 1 \pmod{7}$
    - What is inverse of 12 modulo 6?
      - It does not exist

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Inverse ....

- Theorem: Let n and $m > 1$ be integers. An inverse of n exists *if and only if* n and m are relatively prime.
- Proof outline: Use the theorem that there are integers s and t such that
  $sn + tm = \gcd(m,n) = 1$, or
  $sn + tm \equiv 1 \bmod m$
  Since $tm \equiv 0 \bmod m$, it implies that $sn \equiv 1 \bmod m$.
  Thus n has an inverse.

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Example

- Find an inverse of 101 modulo 4620
- Solution:
  - Check that gcd(101, 4620) = 1
  - Then find s, and t such that
    - $s101 + t4620 = \gcd(101, 4620) = 1$
    - It turns out that $s = 1601$ and $t = -35$ would work
    - Thus $s = 1601$ is an invers of 101 modulo 4620
      - $1601 \times 101 = 35 \times 4620 + 1$

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Solving congruence equalities

- Find n, in terms of k, r, and m, that satisfies the following congruence equality,
- $kn \equiv r \pmod{m}$
- $kn = qm + r$
- Find inverse of k modulo m, that is $\bar{k}$, where $k \cdot \bar{k} \equiv 1 \pmod{m}$. This implies $k \cdot \bar{k} \equiv q'm + 1$
- $\bar{k}kn = \bar{k}qm + r\bar{k}$
- $(q'm + 1)n = \bar{k}qm + r\bar{k}$
- $n = \bar{k}qm - q'mn + r\bar{k} \equiv r\bar{k} \pmod{m}$

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Example

- Find n, in terms of k, r, and m, that satisfies the following congruence equality,
- $4n \equiv 5 \pmod{9}$
  - What is the *solution space* for this problem?
    - $9q + 5$
    - First find an inverse of 4 modulo 9, which is 7.
  - $7 \times 4 \times n \equiv 7 \times 5 \pmod{9}$
  - $n \equiv 8$

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