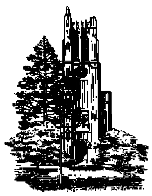
Tuesday May 31, 2016
Lecture 09

Functions



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Notables

- Homework#6
  - Page 152, Problems 2, 6
  - Page 153, Problems 8, 20, and 22
  - Due Thursday June 2, 2016
- Tentative Schedule for the week

Week	M	T	W	R	Topic	Section
1						
		5-31			functions	2.3
			6-1		Sequences and summations	2.4
				6-2	Cardinality of sets	2.5

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Monotonic Functions

- Definition: Let A and B be subsets of $\boldsymbol{R}$, the set of real numbers. A function f: $A \to B$ is *strictly increasing* if

  $$\forall x \in A\ \forall y \in A \quad x < y \ \to\ f(x) < f(y).$$

- f: $A \to B$ is *strictly decreasing* if

  $$\forall x \in A\ \forall y \in A \quad x < y \ \to\ f(x) > f(y).$$

- Note that strictly increasing, or strictly decreasing (*strictly monotone*) functions have to be one-to-one.

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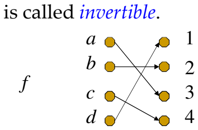
Inverse Function

- Definition: Let function f: $A \to B$ be a bijection. The *inverse function* of f, denoted f^{-1}, is the function, $f^{-1}: B \to A$, that assigns to each element b of B the element a of A such that $f(a) = b$.
  - $\forall a \in A\ \forall b \in B \quad f(a) = b \ \to\ f^{-1}(b) = a.$
  - f is called *invertible*.



$f^{-1}(1) = d$
$f^{-1}(2) = b$
$f^{-1}(3) = a$
$f^{-1}(4) = c$

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Inverse Function…

- Example: Let $f: \mathbf{Z} \to \mathbf{Z}$, where $f(x) = x + 1$.
  - f is a bijection; what is f^{-1}?
  - Suppose $f(x) = y$; then $x + 1 = y$; so $x = y - 1 = f^{-1}(y)$

    $f^{-1}(x) = x - 1$.

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Identity Function

- Let A be a set. The *identity function* on A is the function $\iota_A : A \to A$, where

  $$\forall x \in A \ \ \iota_A(x) = x.$$

- Notes:
  - ι_A assigns each element of A to itself.
  - ι_A is a bijection.

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Characteristic and Constant Functions

- Let A be a set. The *characteristic function* $f : A \to \{0, 1\}$, maps each element of A to either 0 or 1.
- Let A be a set. The *constant function* $f : A \to \{t\}$ maps each element of A to the same value t.

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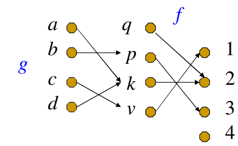
Compositions of Functions

- Definition: Let g be a function from A to B and f a function from B to C, that is,
  $g: A \to B$ $\qquad$ $f: B \to C$
- The *composition* of f and g, denoted $f \circ g$, is function from A to C, defined as follows
  $\forall x \in A \ \ (f \circ g)(x) = f(g(x))$.

$A=\{a, b, c, d\}$

$B=\{q, p, k, v\}$

$C=\{1, 2, 3, 4\}$



$(f \circ g)(a) = 2$
$(f \circ g)(b) = 3$
$(f \circ g)(c) = 1$
$(f \circ g)(d) = 2$

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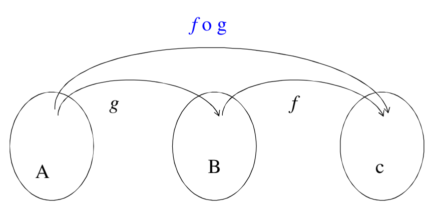
Composition of Functions



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Example

- Consider the two functions
  $g: \mathbf{Z} \to \mathbf{Z}$, where $g(x) = 3x + 2$.
  $f: \mathbf{Z} \to \mathbf{Z}$, where $f(x) = 2x + 3$
  - What are $f \circ g$, and $g \circ f$?
  - $f \circ g: \mathbf{Z} \to \mathbf{Z}$, where
    $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3$
    $= 6x + 7$
  - $g \circ f: \mathbf{Z} \to \mathbf{Z}$, where
    $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2$
    $= 6x + 11$

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Example

Consider the following $R \to R$ functions:

$$f(x) = x^3 - 4x \qquad g(x) = \frac{1}{x^2+1} \qquad h(x) = x^4$$

Compute the following:

1. $f \circ f$
2. $f \circ g$
3. $g \circ f$
4. $h \circ g$
5. $f \circ h$
6. $f \circ (h \circ g)$
7. $(f \circ h) \circ g$

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Example

Consider t he following $R \to R$ functions :

$$f(x) = x^3 - 4x \qquad g(x) = \frac{1}{x^2+1} \qquad h(x) = x^4$$

Compute the following :

1. $f \circ f = f(x^3 - 4x) = (x^3 - 4x)^3 - 4(x^3 - 4x)$
2. $f \circ g = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right) = \frac{1 - 4(x^2+1)^2}{(x^2+1)^3}$
3. $g \circ f = g(f(x)) = g(x^3 - 4x) = \frac{1}{(x^3 - 4x)^2 + 1}$

Note that $f \circ g \neq g \circ f$

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Example….

Recall that $f(x) = x^3 - 4x$ $\quad g(x) = \dfrac{1}{x^2+1}$ $\quad h(x) = x^4$

4. $h \circ g = h(g(x)) = \left(\dfrac{1}{x^2+1}\right)^4$ $\quad$ 5. $f \circ h = f(x^4) = \left((x^4)\right)^3 - 4(x^4)$

6. $f \circ (h \circ g) = f\left(\left(\dfrac{1}{x^2+1}\right)^4\right) = \left(\left(\dfrac{1}{x^2+1}\right)^4\right)^3 - 4\left(\left(\dfrac{1}{x^2+1}\right)^4\right)$

7. $(f \circ h) \circ g = \left(\left(\dfrac{1}{x^2+1}\right)^4\right)^3 - 4\left(\left(\dfrac{1}{x^2+1}\right)^4\right)$

Note that at least for this example we have

$$f \circ (h \circ g) = (f \circ h) \circ g$$

In fact, the above is true always.

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Graph of a Function

- Definition: Let $f: A \rightarrow B$. The *graph* of f is the set of ordered pairs

$$G_f = \{(x, f(x)) \mid x \in A\}.$$

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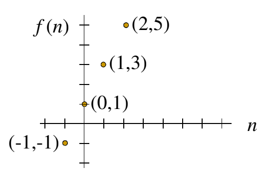
Graph of a Function….

- The graph of the function $f: \mathbf{Z} \rightarrow \mathbf{Z}$, where $f(n) = 2n + 1$, is $G_f = \{(n, 2n + 1) \mid n \in \mathbf{Z}\}$



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Important Integer Functions

- Whole numbers constitute the backbone of discrete mathematics. We often need to convert fractions or arbitrary real numbers to integers. These integer functions will help us do that.
- Besides the identity function, some important functions are:
  - The *floor* function,
  - The *ceiling* function,
  - The *mod* function.

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Floor Function

- Definition: The *floor function* from $\boldsymbol{R}$ to $\boldsymbol{Z}$ assigns to the real number x, the largest integer $\leq x$.

  The value of the floor function at x is denoted by $\lfloor x \rfloor$.
- Examples:
  - $\lfloor 18 \rfloor = 18$
  - $\lfloor 3.75 \rfloor = 3$
  - $\lfloor -4.5 \rfloor = -5$

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Ceiling Function

- The *ceiling function* from $\boldsymbol{R}$ to $\boldsymbol{Z}$ assigns to the real number x the smallest integer $\geq x$. The value of the ceiling function at x is denoted by $\lceil x \rceil$.
- Examples:
  - $\lceil 18 \rceil = 18$
  - $\lceil 3.75 \rceil = 4$
  - $\lceil -4.5 \rceil = -4$

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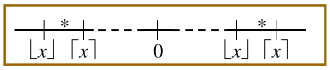
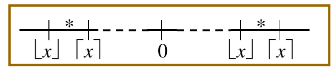
Floor and Ceiling Functions, recap



Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

- $\forall x \in \boldsymbol{R}\ \forall n \in \boldsymbol{Z}\quad \lfloor x \rfloor = n \;\leftrightarrow\; n \le x < n+1.$
- $\forall x \in \boldsymbol{R}\ \forall n \in \boldsymbol{Z}\quad \lfloor x \rfloor = n \;\leftrightarrow\; x-1 < n \le x.$
- $\forall x \in \boldsymbol{R}\ \forall n \in \boldsymbol{Z}\quad \lceil x \rceil = n \;\leftrightarrow\; n-1 < x \le n.$
- $\forall x \in \boldsymbol{R}\ \forall n \in \boldsymbol{Z}\quad \lceil x \rceil = n \;\leftrightarrow\; x \le n < x+1.$
- $\forall x \in \boldsymbol{R}\quad x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1.$
- $\forall x \in \boldsymbol{R}\quad \lfloor -x \rfloor = -\lceil x \rceil$
- $\forall x \in \boldsymbol{R}\quad \lceil -x \rceil = -\lfloor x \rfloor$
- $\forall x \in \boldsymbol{R}\ \forall m \in \boldsymbol{Z}\quad \lfloor x+m \rfloor = \lfloor x \rfloor + m$
- $\forall x \in \boldsymbol{R}\ \forall m \in \boldsymbol{Z}\quad \lceil x+m \rceil = \lceil x \rceil + m$



Example

Prove or disprove the following statements about real numbers:

a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$

b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

c) $\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil \in \{0,1\}$

d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$

e) $\left\lceil \frac{x}{2} \right\rceil = \left\lceil \frac{x+1}{2} \right\rceil$

Example: Solution

Prove or disprove the following statements about real numbers:

a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ Answer: True

b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ Answer: False, try $x = \frac{1}{2}$

c) $\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil \in \{0,1\}$ Answer: True

d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ Answer: Fasle, try $x = \frac{1}{4}, y = 3$

e) $\left\lceil \frac{x}{2} \right\rceil = \left\lceil \frac{x+1}{2} \right\rceil$ Answer: Fasle, try $x = 4$

Division

- Which relation defines *dividing* 101 by 11?
  - 101 = 11 × 8 + 13
  - 101 = 11 × 11 - 20
  - 101 = 11 × 9 + 2

Division…

- Let n be an integer and m a positive integer. Then there are *unique* integers q and r, with $0 \le r < m$, such that $n = mq + r$
  - n is called the *dividend*
  - m is called the *divisor*
  - q is called the *quotient*
  - r is called the *remainder*
  - Examples: 101 = 11 × 9 + 2
    - How about: 101 = 11 × 8 + 13
  - Examples: -11 = 3(-4) + 1
    - How about: -11 = 3(-3) - 2
  - Remainder cannot be a negative number

The mod Function

- When dividing an integer n by a number m, the quotient of the division is $\lfloor n/m \rfloor$. What about a simple notation for the remainder of this division?
- That's what the mod function is about:
- n mod m
- m is called *modulus*
- $n = m \times \underbrace{\lfloor n/m \rfloor}_{quotient} + \underbrace{n \bmod m}_{remainder}$

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Example

- Formally, the *mod* function is a mapping: $mod : \mathbf{Z} \times \mathbf{Z}^+ \rightarrow \mathbf{N}$ where
  $n \bmod m = n - m \times \lfloor n/m \rfloor$
- Examples:

  $5 \bmod 3 = 5 - (3 \times \lfloor 5/3 \rfloor) = 5 - (3 \times \lfloor 1.6 \rfloor) = 5 - (3 \times 1) = 2$

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Example

- $n \bmod m = n - (m \times \lfloor n/m \rfloor)$
- Examples:

  $-5 \bmod 3 = -5 - (3 \times \lfloor -5/3 \rfloor)$
  $= -5 - (3 \times \lfloor -1.6 \rfloor) = -5 - (3 \times (-2)) = 1.$

- We also write:
  $5 \equiv 2 \bmod 3$,
  $9 \equiv 0 \bmod 3$,
  $-5 \equiv 1 \bmod 3$.

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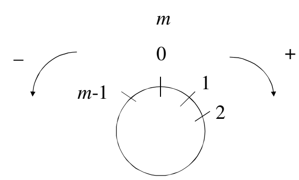
mod Function



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List Search Methods

- Problem: Given a *list* of elements, how fast can we decide whether or not a given input element belongs to the list?
  - Linear search
  - Binary search; need to sort the list first
  - Hash table

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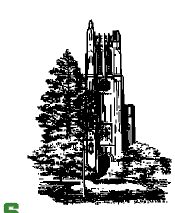
Hash Functions



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Hash Functions

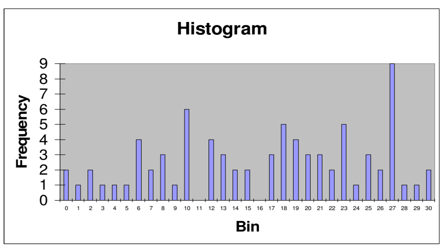
- A *hash function* *h*: *keys* → *integers* maps "keys" to "small" integers (buckets)
- Ideal features:
  - The function should be easy to compute
  - The range values should be "evenly" distributed
  - Given *an image*, it should not be "easy" to find its *pre-image*
- Applications
  - Searching/indexing
  - Information hiding
  - File signature

Hashing for Indexing

- A *hash function* *h*: *keys* → *integers* maps "keys" to "small" integers (buckets)
- Ideally this mapping is done in a "random" manner so that the *bucket* values are evenly distributed despite irregularities in the keys.
- For simplicity, we will assume that the keys are also integers, denoted by k, and the number of buckets is demoted by m. Note that the buckets are indexed 0 through $m - 1$.

Example

- Storing CSE 260, both sections, PIDs \ A
  - Using Hash Function $h(PID) = k \bmod 31$



Simple Hash Functions

- $h(k) = k \bmod m$
  - Suggestion: Choose m to be a prime number that isn't close to a power of 2.
- $h(k) = k(k + 3) \bmod m$