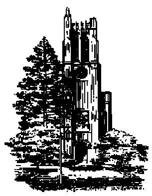
Monday June 6, 2016
Lecture 12

Countable and Uncountable Sets



Notables

- Homework#7
  - Page 168, Problem 12
  - Page 169, Problems 26((a), (c), and (g), 32, and 34
  - Page 176, Problem 4
  - Due Tuesday June 7, 2016
- Tentative Schedule for the week

Week	M	T	W	R	Topic	Section
1	6-6				Countable and uncountable sets	2.5
		6-7			Algorithms	3.1
			6-8		More on Algorithms	3.1
				6-9	Growth functions	3.2

Homework#8, Due Thursday (6/10)

- Sort the list [64, 13, 29, 32, 21, 46, 60] using
  - Exchange sort; show each pass
  - Bubble sort; show each pass
  - Merge sort; show each sublist
  - For each case, give the number of 2-number comparisons required.
- Use Newton's method to find the square root of 27 up to 2 decimal places. Use 7 as your initial guess.
- Page 204, Problem 52
- Page 216, Problem 2, 8, and 14
- Page 217, 34(a)
- Page 218, 74(a), and 74(b)

Lectures from web (Rob Shone)

- https://www.youtube.com/watch?v=sT9hAmaot8U
- https://www.youtube.com/watch?v=fRhdpyaOhEo

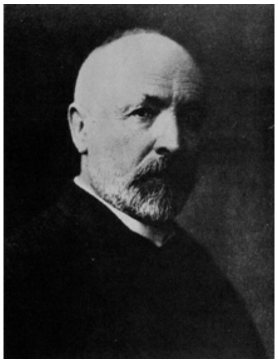
Cardinality - Countable/Uncountable Sets

- Review:
  - What's the cardinality of the set A where $A = \{2, 3, 5, 7, 11\}$?

Correspondence Principle

- If two finite sets can be placed into 1-1 onto correspondence (that is, a *bijection*) then they have the same *size*. In other words, set A and set B are of the same size if there is a bijection from set A to set B

Georg Cantor (1845-1918)



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Cantor's Definition (1874)

- Cantor extended the Correspondence definition to the infinite sets.
- Two sets *A* and *B* are defined to have the same *size* if there is a bijection from A to B.
- A set is *countable* if it has the same *size* as the set of positive integers, Z^+
  - Aside: Many call Z^+ the set of natural numbers
- In this discussion, we will use the words *size* and *cardinality* interchangeably.

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Do Z^+ and E have the same size?

- $Z^+ = \{1, 2, 3, 4, 5, 6, 7, \ldots.\}$
- $E = \{0, 2, 4, 6, 8, 10, 12, \ldots.\}$
- $f: Z^+ \to E$, $f(n) = 2n - 2$

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Do Z^+ and O have the same size?

- $Z^+ = \{1, 2, 3, 4, 5, 6, 7, \ldots.\}$
- $O = \{1, 3, 5, 7, 9, 11, 13, \ldots.\}$
- $f: Z^+ \to O$, $f(n) = 2n - 1$

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Do Z^+ and Z have the same size?

- $Z^+ = \{1, 2, 3, 4, 5, 6, 7, \ldots.\}$
- $Z = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots.\}$
- Let's "line" them up this way:
- $\{1, 2, 3, 4, 5, 6, 7, \ldots.\}$
- $\{0, 1, -1, 2, -2, 3, -3, \ldots.\}$

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Do Z^+ and Z have the same size?...

- $\{1, 2, 3, 4, 5, 6, 7, \ldots.\}$
- $\{0, 1, -1, 2, -2, 3, -3, \ldots.\}$
- $f(n) = \begin{cases} 0, & n = 1 \\ \left\lfloor \frac{n}{2} \right\rfloor, & n \text{ even} \\ -\left\lfloor \frac{n}{2} \right\rfloor, & n \text{ odd} \end{cases}$

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Transitivity Lemma

- If $f: A \rightarrow B$ is a bijection, and $g: B \rightarrow C$ is a bijection, then $h(x) = g(f(x)): A \rightarrow C$ is a bijection.
- Hence, Z^+, E, O, and Z all have the same cardinality.

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Do Z^+ and Q have the same cardinality?

- $Z^+ = \{1, 2, 3, 4, 5, 6, 7, \dots\}$
- Q = The Rational Numbers
- Let's "line" them up this way

- $Z^+ = \{1, 2, 3, 4, 5, 6, 7, \dots\}$
- $Q = \{\frac{-1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{-2}{1}, \frac{3}{1}, \frac{3}{2}, \frac{1}{3}, \frac{-1}{3}, \dots\}$

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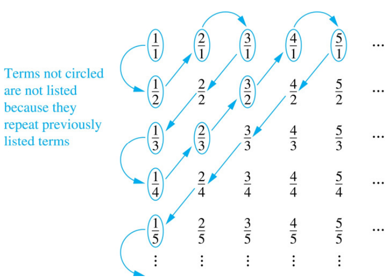
The set of positive Rational Numbers is countable



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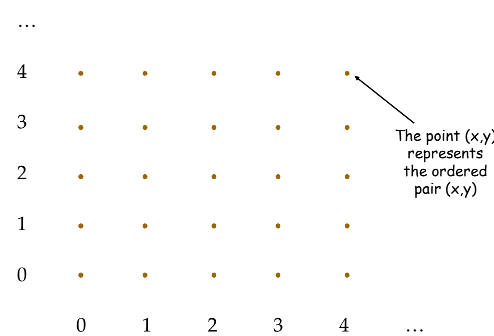
Theorem: N and $N \times N$ have the same cardinality



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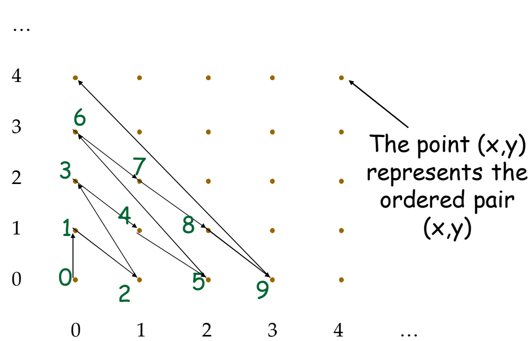
Theorem: ℕ and ℕxℕ have the same cardinality



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Do Z^+ and R have the same cardinality?

$Z^+ = \{1, 2, 3, 4, 5, 6, 7, \dots\}$

R = The Real Numbers

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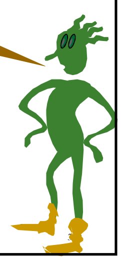
Theorem: The set ***W*** of reals between 0 and 1 is not countable

- Proof by contradiction:
- Suppose ***W*** is countable. Let f be a 1-1 onto function from ***Z***$^+$ to ***W***. Make a list L as follows:
- 1: decimal expansion of $f(1)$
  2: decimal expansion of $f(2)$
- …
- k: decimal expansion of $f(k)$
- …

Theorem: The set W of reals between 0 and 1 is not countable.

- So the list may look like this
  - 1: 0.33333333333333333333333…
  - 2: 0.3141592656578395938594982…
  - 3: 0.5678567778573846592385933….
  - 4: 0.99999878654666666666663…..
  - ….
  - k: 0.3453222142435553452211 23235…
- …
- Form a new real number $r = 0.r_1r_2r_3r_4 \ldots.$ where $r_i = 3$ if $d_{ii} \neq 3$ *and* $r_i = 4$ if $d_{ii} = 3$
- $r = 0.4333\ldots$
- r is not any of the numbers in the list.

The set of reals is uncountable!



Countable/Uncountable Sets

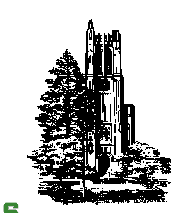
- Facts:
  - If A and B are countable sets, then $A \cup B$ is also *countable*.
  - A subset of a countable set is *countable*.
  - If A is uncountable and $A \subseteq B$ then B is *uncountable*.
  - There are functions which are *uncomputable*.
  - The power set of ***Z***$^+$ is uncountable.
- There are *uncomputable* functions; a function is said to be *computable* if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.

Uncomputable functions

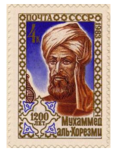
- Consider the following function:
- $f: Z^+ \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- If x is a real number whose decimal representation is $0.d_1d_2d_3$ ... then we associate to x the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Two different real numbers must have different decimal representations, so the corresponding functions are different. Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable

Chapter 3

Algorithms



What is an Algorithm



Persian textbook author, Abu Abdullah Muhammad bin Musa al-Khwarizmi (780-850)

- An *Algorithm* is a finite set of *instructions* for solving a class of problems on a computer, conforming to the following properties
  - *Definiteness*: Each instruction must be "precisely" defined.
  - *Effectiveness*: Each instruction must be sufficiently "basic".
  - *Finiteness*: An algorithm must always terminate.

Basic computer operations on numbers

- Operations on numbers
  - Addition
  - Subtraction
  - Multiplication
  - Division
  - Shift
  - Compare/testing
- How are other functions/operations, such as the nth root, sine, log,…, done?
  - These functions are either expressed in terms of "basic" operations, or in terms of procedures that only require "basic" operations.

Algorithm example

- *Take* two positive integers n and $b > 1$.
- *Divide* n by b, and let q and r be the quotient and the remainder, respectively.
- *Print* r
- *If* q is zero, *stop*.
- *Treat* q as n and *go to*

Algorithm example

- Step 1: *Take* two positive integers n and $b > 1$.
- Step 2: *Divide* n by b, and let q and r be the quotient and the remainder, respectively.
- Step 3: *Print* r
- Step 4: *If* q is zero, *stop*.
- Step 5: *Treat* q as n and *go to* Step 2.

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Let's try it for $n = 27$ and $b = 3$

Algorithm examples

- Carry out the following calculation
  - 3 + 2*4
- Find the max value in the following list
  - [3, 2, 14, 6, 5]

Examples of algorithms

- Searching
  - Linear Search
  - Binary search
- Sorting
  - Exchange sort
  - Bubble sort
  - Merge Sort
- Computing SQRT