Geodesics on 2- Torus

Shashank Nag, EE19B118

PH3500, July-November 2020

1 Aim

To find the equations of the geodesics on the surface of a 2- Torus, and to numerically solve the same.

2 Theory

2.1 Geodesics

A geodesic is defined as the curve which is a shortest path on a surface between two points. It need not be the shortest path, but is actually the locally-length minimising path. When moving on a plane surface, the shortest path connecting two points is essentially a straight line. However things are not so evident when we try to find the shortest path on curved surfaces.

2.1.1 Closed Geodesics

Geodesics are shortest paths connecting two given points. It may be even possible to find paths which start and end at the same point, and have the shortest length (local minima) of this loop path (In other words, the path length function has a minima along this path). Such paths are called as Closed Geodesics. It turns out that there are only a few paths which satisfy this property.

2.2 Finding Geodesics using Calculus of Variation

In this course, we have studied how to arrive at the equations for geodesics using Calculus of Variations. The equation of geodesic in terms of the Christoffel Symbol is given as:

$$\ddot{q_l} + \frac{1}{2}\Gamma^l_{ij}\dot{q_i}\dot{q_j} = 0$$

Parametrising in terms of S,

$$\frac{d^2x_l}{dS^2} + \frac{1}{2}\Gamma^l_{ij}\frac{dq_i}{dS}\frac{dq_j}{dS} = 0$$

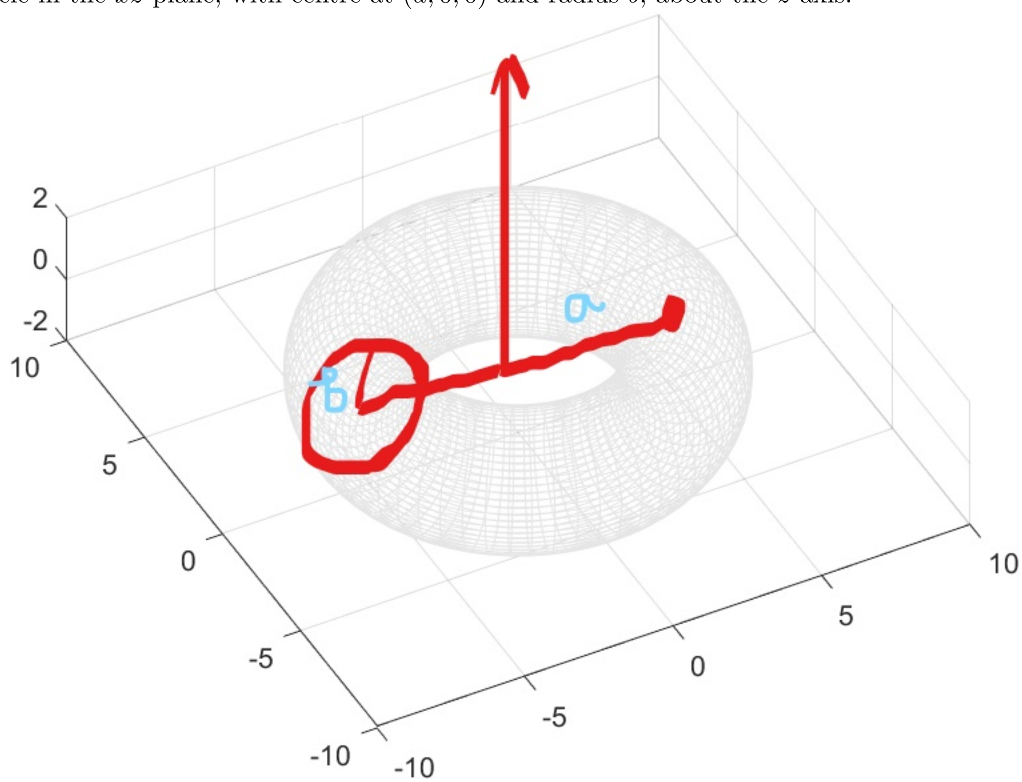
Expanding the Christoffel Symbol in terms of the g metric, we get

$$\frac{d^2x_l}{dS^2} + \frac{1}{2}g^{lk}(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k})\frac{dq_i}{dS}\frac{dq_j}{dS} = 0$$

For different values of l, we get the corresponding second order differential equations. Upon solving these equations with appropriate initial conditions, we get the required geodesics on the surface.

2.3 Parameterising the 2 - Torus

The 2 – Torus can be parameterised using the generalized coordinates, θ and χ. For this parameterization, we visualize the torus as a surface of revolution of a circle in the xz plane, with centre at $(a, 0, 0)$ and radius b, about the z axis.



Upon doing so, we obtain the cartesian coordinates of the surface, as a function of the generalized coordinates as:

$$x = (a + bcos\chi)cos\theta$$

$$y = (a + bcos\chi)sin\theta$$

$$z = bsin\chi$$

We can now proceed to find the components of the g matrix.

2.4 Finding elements of g

$$g_{\theta\theta} = \frac{\partial X_i}{\partial \theta}\frac{\partial X_i}{\partial \theta} = \frac{\partial X}{\partial \theta}\frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial \theta}\frac{\partial Y}{\partial \theta} + \frac{\partial Z}{\partial \theta}\frac{\partial Z}{\partial \theta}$$

$$= (a + bcos\chi)^2 cos^2\theta + (a + bcos\chi)^2 sin^2\theta + 0$$

$$= (a + bcos\chi)^2$$

$$g_{\theta\chi} = \frac{\partial X_i}{\partial \theta}\frac{\partial X_i}{\partial \chi} = \frac{\partial X}{\partial \theta}\frac{\partial X}{\partial \chi} + \frac{\partial Y}{\partial \theta}\frac{\partial Y}{\partial \chi} + \frac{\partial Z}{\partial \theta}\frac{\partial Z}{\partial \chi}$$

$$= (a + bcos\chi)bsin\theta cos\theta sin\chi - (a + bcos\chi)bsin\theta cos\theta sin\chi + 0$$

$$= 0$$

$$g_{\chi\theta} = \frac{\partial X_i}{\partial \chi}\frac{\partial X_i}{\partial \theta} = \frac{\partial X}{\partial \chi}\frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial \chi}\frac{\partial Y}{\partial \theta} + \frac{\partial Z}{\partial \chi}\frac{\partial Z}{\partial \theta}$$

$$= (a + bcos\chi)bsin\theta cos\theta sin\chi - (a + bcos\chi)bsin\theta cos\theta sin\chi + 0$$

$$= 0$$

$$g_{\chi\chi} = \frac{\partial X_i}{\partial \chi}\frac{\partial X_i}{\partial \chi} = \frac{\partial X}{\partial \chi}\frac{\partial X}{\partial \chi} + \frac{\partial Y}{\partial \chi}\frac{\partial Y}{\partial \chi} + \frac{\partial Z}{\partial \chi}\frac{\partial Z}{\partial \chi}$$

$$= b^2 sin^2\chi cos^2\theta + b^2 sin^2\chi sin^2\theta + b^2 cos^2\chi$$

$$= b^2$$

We can now express the g matrix as:

$$g = \begin{pmatrix} (a + bcos\chi)^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

And its inverse as

$$g^{-1} = \begin{pmatrix} \frac{1}{(a+bcos\chi)^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}$$

2.5 Finding the Solution to the Geodesic Equations

We now get back to the geodesic equations

$$\frac{d^2x_l}{dS^2} + \frac{1}{2}g^{lk}(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k})\frac{dq_i}{dS}\frac{dq_j}{dS} = 0$$

For $x_l = \theta$,

$$\frac{d^2\theta}{dS^2} + \frac{1}{2}g^{\theta k}(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k})\frac{dq_i}{dS}\frac{dq_j}{dS} = 0$$

Summing up over appropriate indices, we get the differential equation as

$$\frac{d^2\theta}{dS^2} - \frac{2bsin\chi}{(a + bcos\chi)}\frac{d\theta}{dS}\frac{d\chi}{dS} = 0......(1)$$

Similarly, for $x_l = \chi$,

$$\frac{d^2\chi}{dS^2} + \frac{1}{2}g^{\chi k}(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k})\frac{dq_i}{dS}\frac{dq_j}{dS} = 0$$

Summing up over appropriate indices, we get the differential equation as

$$\frac{d^2\chi}{dS^2} + \frac{(a + bcos\chi)}{b}sin\chi(\frac{d\theta}{dS})^2 = 0......(2)$$

Equation (1) appears variable separable. Solving the same, we get

$$\frac{d\theta}{dS} = \frac{k}{(a + bcos\chi)^2}......(3)$$

where k is an arbitrary constant

Using (3), and (2), we can get an expression for $\frac{d\chi}{dS}$

$$\frac{d\chi}{dS} = \pm\sqrt{l - \frac{k^2}{b^2(a + bcos\chi)^2}}......(4)$$

where l is another arbitrary constant

We now have the expression for $\frac{d\theta}{dS}$ and $\frac{d\chi}{dS}$. Taking their ratio, we get $\frac{d\theta}{d\chi}$, and then by numerically integrating it using initial conditions, we should be able to find the geodesic through a given point as $\chi = \chi(\theta)$.

$$\frac{d\theta}{d\chi} = \pm\frac{bk}{(a + bcos\chi)\sqrt{lb^2(a + bcos\chi)^2 - k^2}}......(5)$$

However, issues arise when we try to approach using this method. The reason for this is, we have two arbitrary constants k and l, in the expression for $\frac{d\theta}{dS}$ and $\frac{d\chi}{dS}$, with no physical meaning or connection attached to them. It so appears, that choosing any arbitrary pair of values for these constants yields no solution to the geodesic equations. So, we need to look at some other alternative to arrive at the geodesics or which could relate these constants. Clairaut relation appears to be an effective tool in this regard.

2.5.1 Clairaut's Relation

Clairaut Relation is very effective for finding geodesics on a surface of revolution. Since a $2-$Torus is a surface of revolution of a circle about the $z-$axis, we find Clairaut relation particularly useful in this case. The Clairaut Relation states that: If γ is a geodesic on a surface of revolution S, and ρ is the distance of a point of S from the axis of rotation, and ψ is the angle between γ and the meridians of S, then $\rho sin\psi$ is constant along γ. For our case of interest, (2-Torus), the axis of revolution is the $z-$axis, and the distance ρ is given by

$$\rho = \sqrt{x^2 + y^2}$$

$$\rho = a + bcos\chi$$

We define the constant mentioned by the Clairaut Relation as h. So,

$$h = \rho sin\psi = (a + bcos\chi)sin\psi = const$$

From the Clairaut parameterisation, we obtain the relation for $\frac{d\theta}{d\chi}$ as:

$$\frac{d\theta}{d\chi} = \pm\frac{bh}{(a + bcos\chi)\sqrt{(a + bcos\chi)^2 - k^2}} ......(6)$$

Comparing this with (5), we can clearly relate the constants:

$$k = h$$

$$l = \frac{1}{b^2}$$

So, now all we have to do is vary h for a given initial point, and thereby get different geodesics from a given initial point. Substituting these constants back in (3) and (4), we get:

$$\frac{d\theta}{dS} = \frac{h}{(a + bcos\chi)^2}$$

$$\frac{d\chi}{dS} = \pm\sqrt{\frac{1}{b^2} - \frac{h^2}{b^2(a + bcos\chi)^2}}$$

3 Numerically Solving the Geodesic Equations

The problem at hand is to numerically solve the differential equations, given the initial condition (initial point) and value of constants. To numerically solve the differential equation, we use the ode45 function of MATLAB. ode45 computes the value of the integral given a differential equation using the RK45 method of approximation of the integral. For the purpose of simplicity, let us assume the values of the constants as :

$$a = 5$$

$$b = 2$$

As the differential equation is largely dependent on h, we so find that varying the value of h generates different kinds of geodesics. Also, since the term inside the square root, i.e., $(a + bcos\chi)^2 – h^2$ should be positive, it imposes restrictions on the values taken by h.

However, upon using this approach, we run into some errors for certain values of h. We observe that the ode45 solver of MATLAB generates complex solutions to the differential equation. This issue arises due to the two possible signs ($\pm$) of the $\frac{d\chi}{dS}$. To avoid this complication, we turn to the original pair of second order differential equations, and solve them using the ode45 solver itself.

$$\frac{d^2\theta}{dS^2} - \frac{2bsin\chi}{(a + bcos\chi)}\frac{d\theta}{dS}\frac{d\chi}{dS} = 0$$

$$\frac{d^2\chi}{dS^2} + \frac{(a + bcos\chi)}{b}sin\chi(\frac{d\theta}{dS})^2 = 0$$

For the ode45 solver, we define the y- matrix as :

$$y = \begin{pmatrix} \theta & \chi & \frac{d\theta}{dS} & \frac{d\chi}{dS} \end{pmatrix}$$

We define the derivative as derivative w.r.t. S, S being the parameter.

So, the $\frac{dy}{dt}$ matrix becomes

$$\frac{dy}{dt} = \begin{pmatrix} \frac{d\theta}{dS} & \frac{d\chi}{dS} & \frac{d^2\theta}{dS^2} & \frac{d^2\chi}{dS^2} \end{pmatrix}$$

The terms in the $\frac{dy}{dt}$ matrix are written in terms of the terms of the y matrix, wherein $\frac{d\theta}{dS} and \frac{d\chi}{dS}$ are straightaway written as y(3) and y(4) respectively, while the double derivatives are written using the second order equations.

We now need to specify the initial conditions for the y row vector, denoted by the y0 row vector, i.e., we need to fix the initial point, and the initial gradient of the geodesic.

Upon fixing the initial point on the geodesic, we change the value of h to change the initial slope. The initial conditions matrix is given by -

$$y_0 = \begin{pmatrix} \theta_0 & \chi_0 & (\frac{d\theta}{dS})_0 = \frac{h}{(a+bcos\chi_0)^2} & (\frac{d\chi}{dS})_0 = \pm\sqrt{\frac{1}{b^2} - \frac{h^2}{b^2(a+bcos\chi_0)^2}} \end{pmatrix}$$

As it is just the initial gradient that gets affected, we can retain only the positive sign for the $(\frac{d\chi}{dS})_0$ term in the y_0 row vector.

Upon fixing these values, we are ready to numerically simulate the geodesics. The MATLAB code written for the same is submitted as an attachment to this report.

3.1 Geodesics for different values of h

The key parameter on which we have a control, apart from the initial point is h. So, we would generate the different geodesics corresponding to the different values of h. Note that all the constants (h, θ_0, and χ_0) only affect the initial conditions.

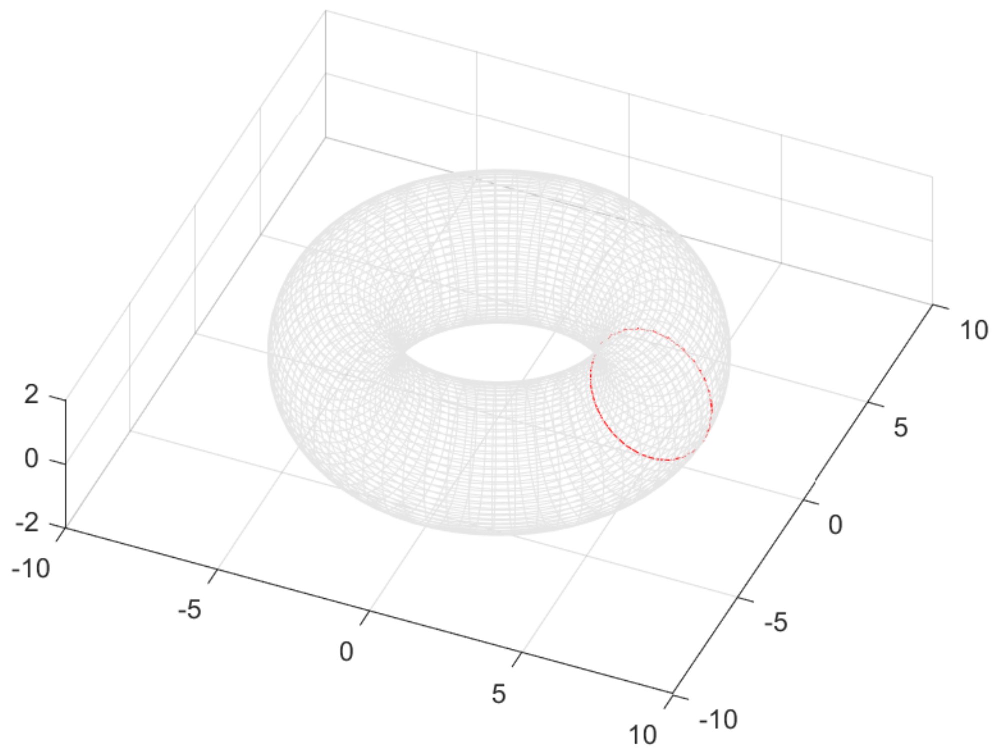
3.1.1 h=0

For h=0, $(\frac{d\theta}{dS})_0 = \frac{h}{(a+bcos\chi_0)^2} = 0$. So for a given θ_0, the geodesic would just be a meridian about that θ_0.

Choosing $\theta_0 = 0$ and $\chi_0 = 0$, we get the y_0 matrix as:

$$y_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{b} \end{pmatrix}$$

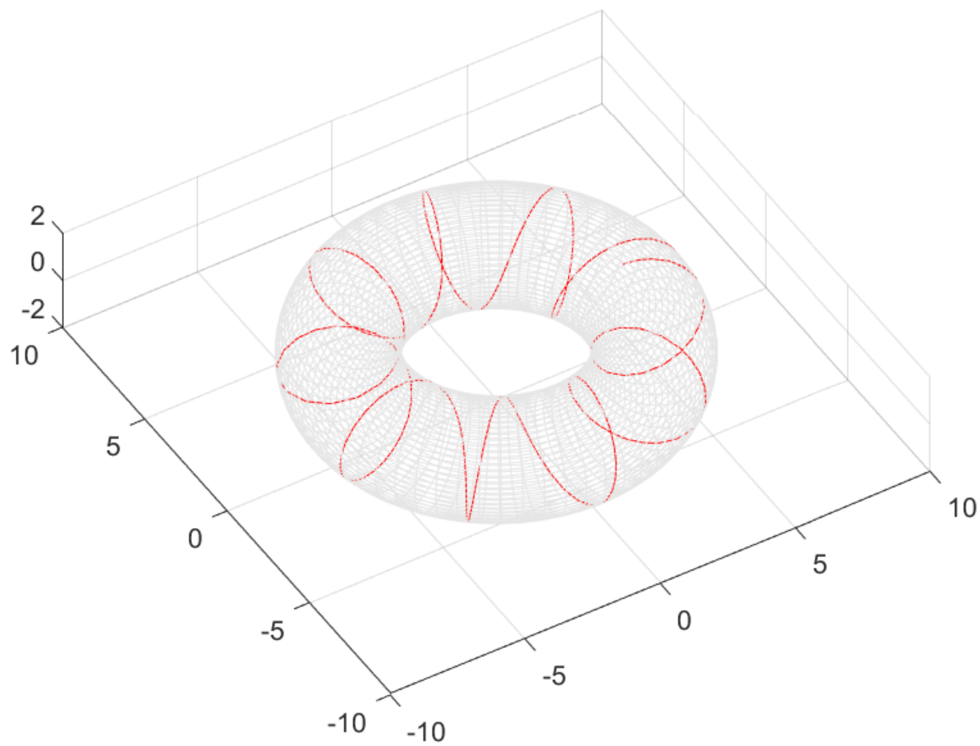
Solving the second order equations and plotting the obtained values, we get the following plot:



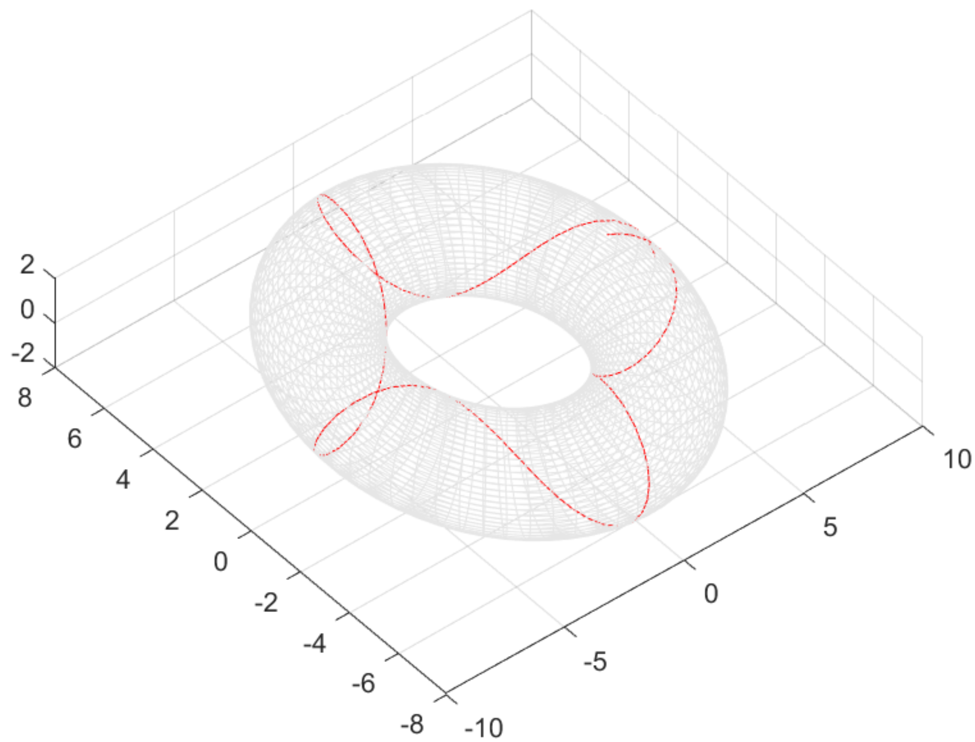
Similarly, for different values of θ_0, we get different meridians on the 2 - Torus. Note that these geodesics are closed geodesics.

3.1.2 $0 < h < a - b$

For our case of interest, $a - b = 3$. So we choose $h = 1$, and keep the initial point as $\theta_0 = 0$ and $\chi_0 = 0$ and substitute the corresponding initial conditions column vector y_0.



Similarly, we can solve the equations with the initial value of h as 2.

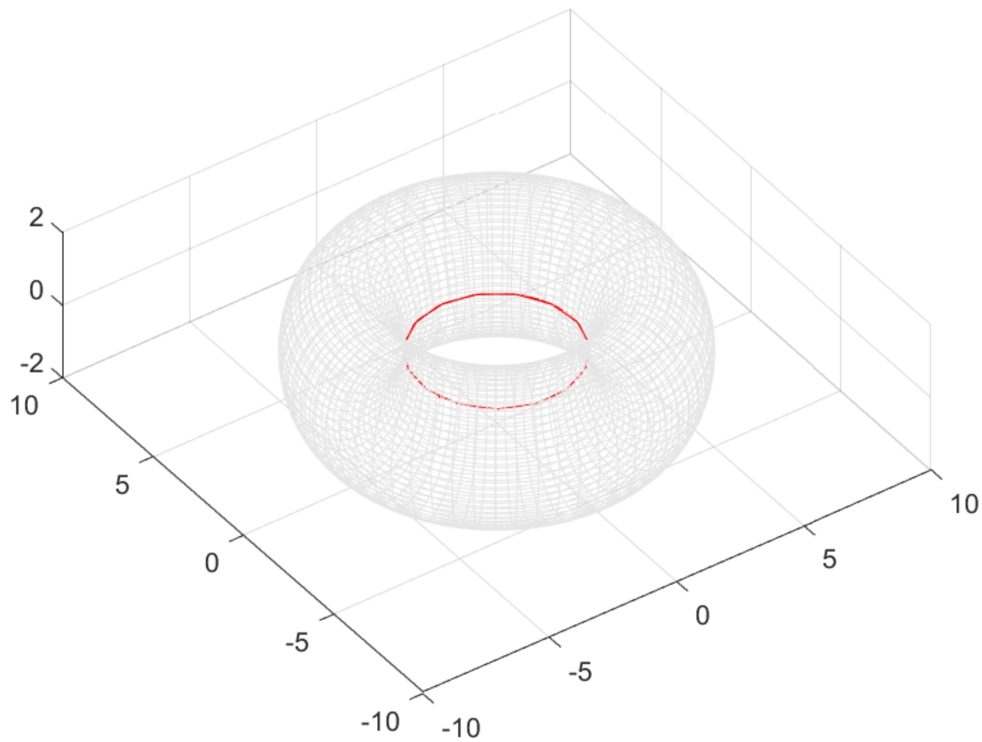


Note that these geodesics are not closed geodesics. If we wish to find a geodesic passing through two points, we need to identify the slope at which the geodesic has to be launched from the initial point, such that it passes through the second point. This path between the two points will have a local extrema of the path length.

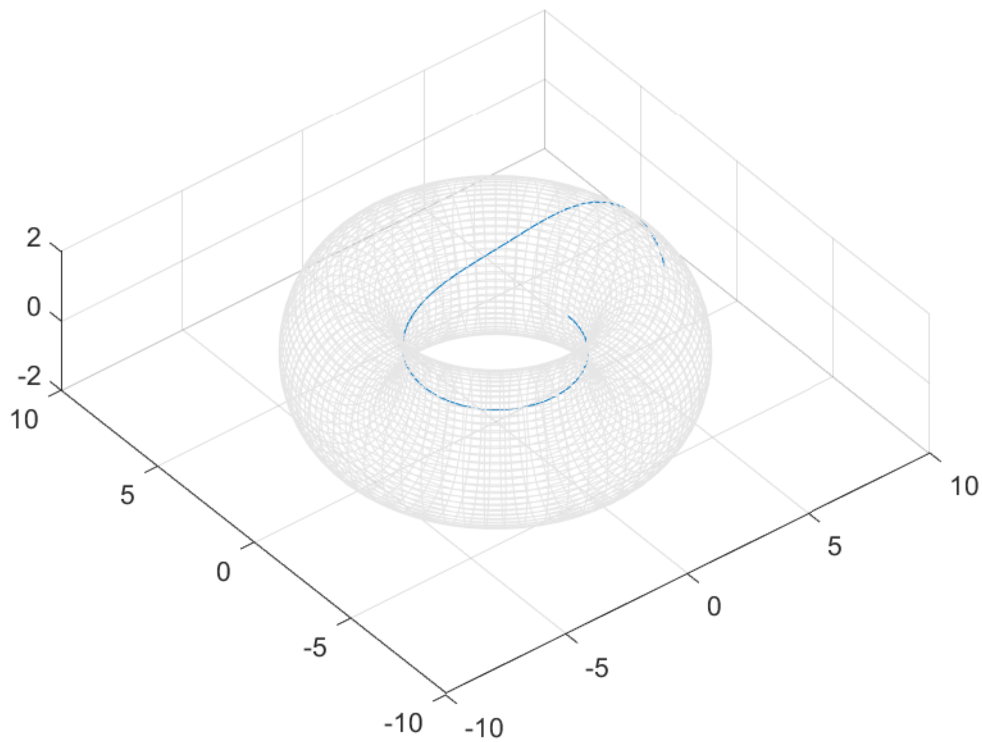
3.1.3 $h = a - b$

For $h = a - b = 3$, an interesting phenomenon takes place. The expression $(\frac{d\chi}{dS})_0 = \pm\sqrt{\frac{1}{b^2} - \frac{h^2}{b^2(a+bcos\chi_0)^2}}$ becomes 0, for $\chi_0 = \pi$, i.e., if the initial point lies on the inner equator.

So, if $\chi_0 = \pi$, then the geodesic is just the inner equator.



Incase the initial point does not correspond to $\chi_0 = \pi$, it is observed that the geodesic asymptotically approaches the inner equator.

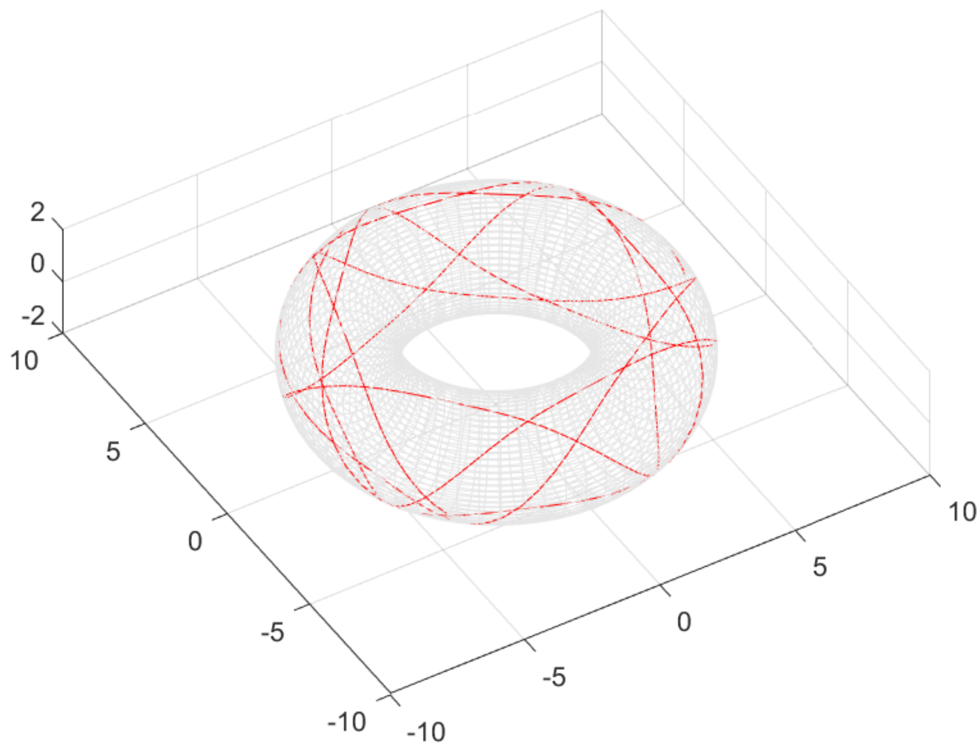


3.1.4 $a - b < h < a + b$

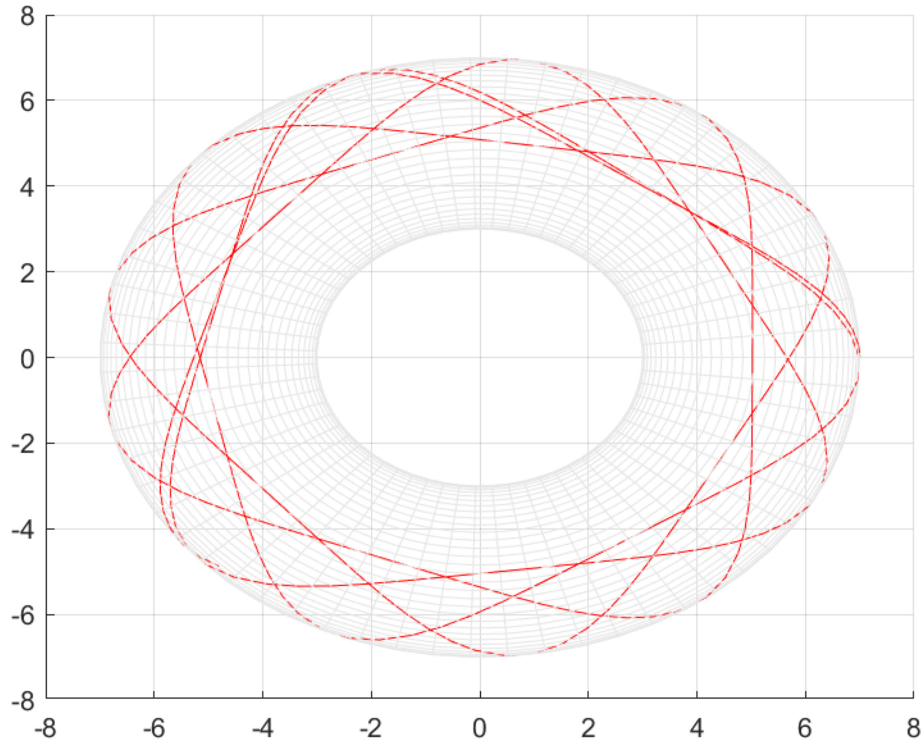
For $3 < h < 7$, we see a peculiar feature. The geodesic doesn't appear to move along a certain region about the inner equator. This arises due to the fact that the term in $\frac{d\chi}{dS} = \pm\sqrt{\frac{1}{b^2} - \frac{h^2}{b^2(a+bcos\chi)^2}}$, should be real.

This imposes the condition on h as, $(a + bcos\chi)^2 - h^2 \geq 0$. Thus, it imposes restrictions on the values χ can take. Hence, the geodesic is such that it is restricted outside a certain region about the inner equator.

For $h = 5$, with $\chi_0 = 0$ and $\theta_0 = 0$, upon numerically computing we get the following geodesic:

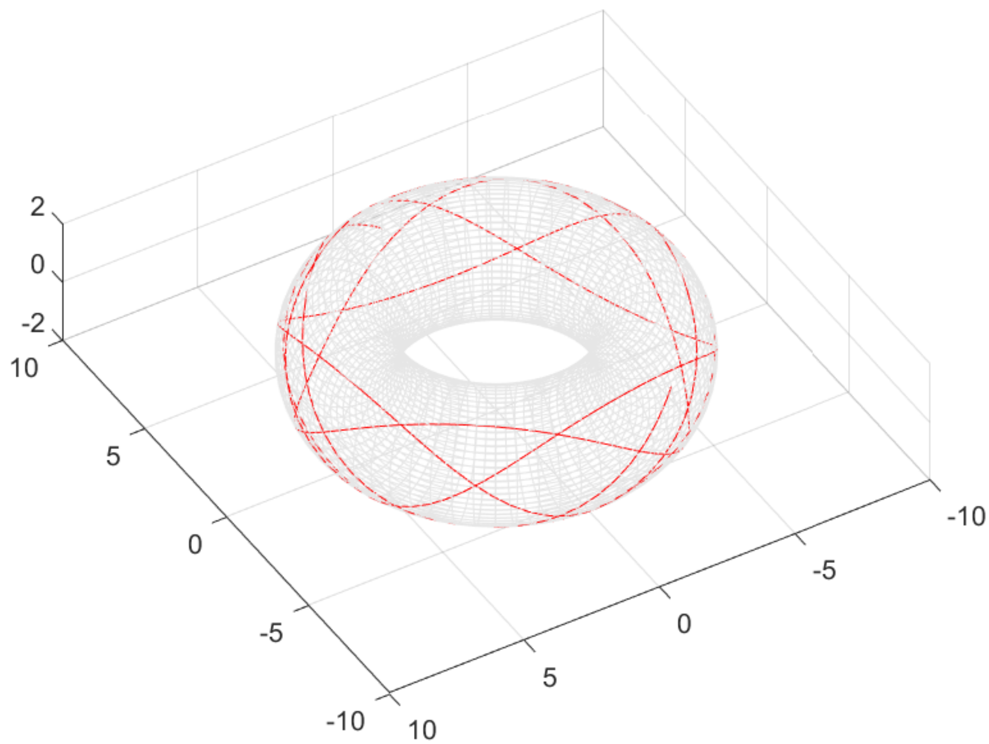


The fact that it is restricted outside a certain region is clear from a different view of the same plot.



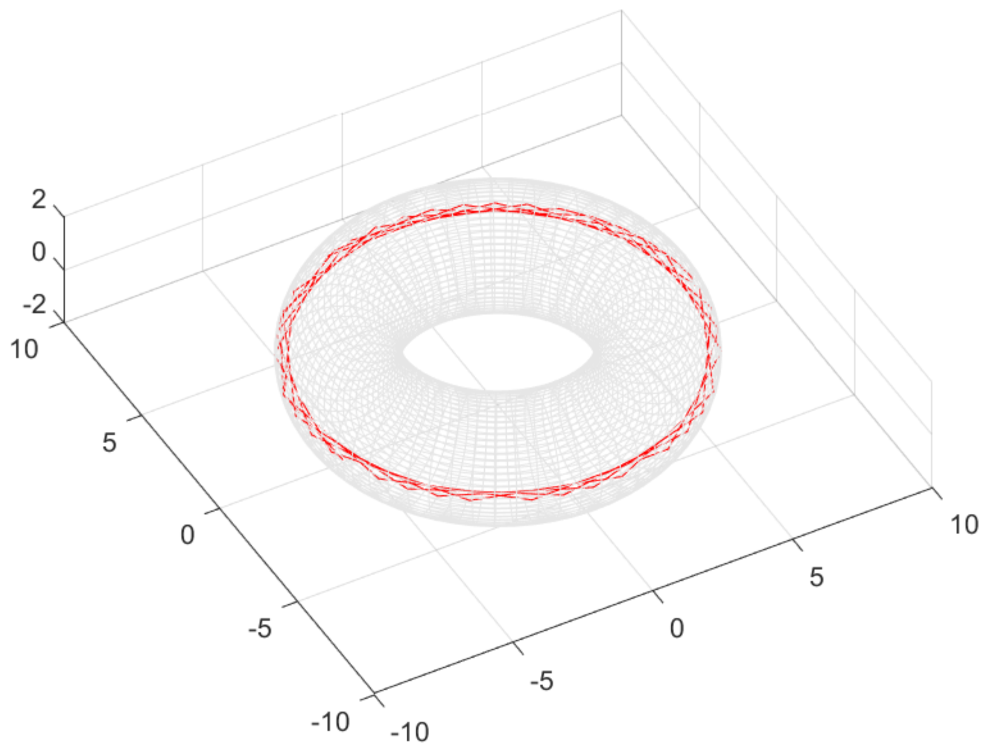
The boundary of the region outside which the geodesics could exist are marked by two curves (parallel to the xy plane) known as the barrier curves.

Similarly, we get a restricted plot for $h = 6$:



3.1.5 $h = a + b$

For $h = a + b = 7$, the only value χ can take is $\chi = 0$, i.e., the outer equator. Thus, if we set $\chi_0 = 0$, and solve the differential equations using $h = 7$, we would get the geodesic as the outer equator.



3.1.6 $h > a + b$

For $h > a + b$, we get complex valued terms as the solutions of the geodesic, which clearly shows that the geodesics do not exist corresponding to these values of h (and consequently, for those gradients).

4 Results/ Conclusion

We could thus successfully compute the geodesics on the surface of 2 - Torus numerically. Apart from the equations for Geodesics derived using Calculus of Variation, the Clairaut Relation was particularly using in relating the arbitrary constants. Geodesics are the local extremas of the path length connecting two points on a surface. In this project, rather than looking at this approach, we tried to find the geodesics given a starting point, and an initial gradient, as it is much more convenient to do so. We could extend the same to find the geodesics between two points, by finding the appropriate initial gradient, which would lead the geodesic to pass through the final point.

We also note that for different initial conditions (initial points and derivatives), the geodesics obey certain properties. For some initial derivatives, the geodesics are bound in the regions outside two barrier curves.

We also found that there are some cases where the geodesics so obtained are closed geodesics, which include the meridians(circles with constant θ) and the equators (inner and outer equators).

5 References

1. The Curvature and Geodesics of the Torus, https://en.calameo.com/read/0041379718b0adddda038

2. Geodesic Equation and Clairaut Relation, https://mathproblems123.files.wordpress.com/2010/04/geodesiceqn.pdf