

UNIT - 1TAYLOR'S SERIES

A Taylor's Series is a series expansion of a function about a point.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

↓
(special case) $\begin{bmatrix} x=0 \\ h=x \end{bmatrix}$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

for instance : Taylor expansion of $f(x) = e^x$.

$$e^x = e^0 + xe^0 + \frac{x^2}{2!} e^0 + \frac{x^3}{3!} e^0 + \dots$$

→ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

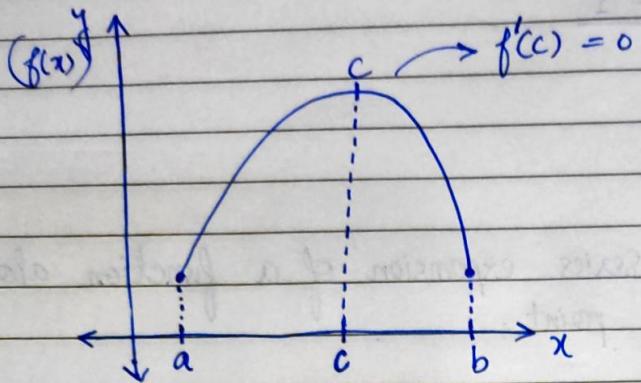
ROLLE'S THEOREM

1) If $f(x)$ is continuous on $[a, b]$.

2) If $f(x)$ is differentiable on (a, b) .

3) $f(a) = f(b)$ holds,

then, there at least one $c \in (a, b)$ such that $f'(c) = 0$.

Graphical Representation
of Rolle's Theorem.

(Ans.) $f(x) = 3 + (x-2)^{2/3}$ on $[1, 3]$ existence of Rolle's Theorem.

Ans) $f'(x) = \frac{2}{3} \frac{1}{(x-2)^{1/3}}$

so,

$$x-2 = 0$$

$$(x=2)$$

so, derivative doesn't exist on $x=2$ and $2 \in (1, 3)$.
 \therefore Rolle's Theorem doesn't hold.

Note : Polynomial, exponential, logarithmic, trigonometric, Inverse Trigonometric functions are always continuous & differentiable on their domain.

* Rolle's Theorem for POLYNOMIAL

- 1) If $f(x)$ continuous on $[a, b]$.
- 2) If $f(x)$ is differentiable on (a, b) .
- 3) $f(a) = f(b) = 0$ i.e. a & b are roots..

then,] at least one root of $f'(c)$ with 2 roots of $f(x)$ ($a \neq b$) included.

MEAN - VALUE THEOREM

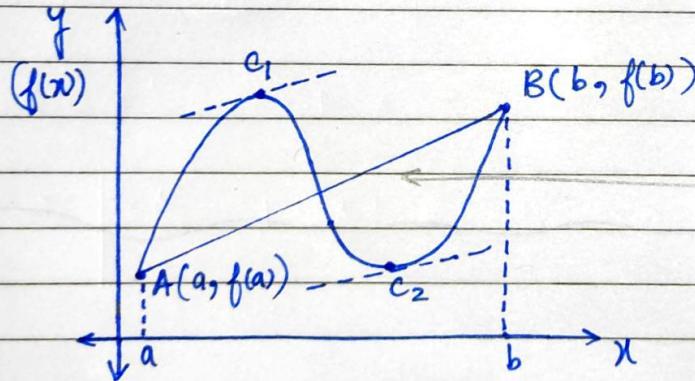
let $f(x)$ be a function which is :

- ① continuous on $[a, b]$.
- ② differentiable on (a, b) .

then,] at least one point $c \in (a, b)$ such that :-

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

• Graphical Representation .



Secant made by end points ($A \neq B$) of the curve

→ Tangents made on points c_1 & c_2 are parallel to the secant.

SIGNIFICANT FIGURES

Rules

- Non-zero digits are significant.
- Zeroes b/w non-zero digits are significant.
- Final zeroes in the decimal portion are significant.
- Final zeroes are non-significant if there's no decimal point.

Ques.) find no. of significant figures :-

1)	58	-	2
2)	603	-	3
3)	1008	-	4
4)	95080	-	4
5)	70	-	1
6)	0.80	-	2
7)	18.00	-	4
8)	0.0003	-	1
9)	0.0030	-	2

ROUNDING OFF

1)	39.72431	(3D)	-	39.724
2)	79.87893	(3D)	-	79.879
3)	2.589951	(2D)	-	2.59
4)	59.4865	(3D)	-	59.486
5)	47.6975	(3D)	-	47.698

UNIT - 1BISECTION METHOD

→ This method is based on the repeated application of the intermediate value theorem to obtain an approximate root of the equation $f(x) = 0$.

Choose 'a' & 'b' such that : $f(a) \rightarrow +ve$ $f(b) \rightarrow -ve$

↓ i.e.

$$f(a) \cdot f(b) < 0$$

then, a root lies b/w a and b. The mean value is taken as the first approximate value of the root.

$$x_0 = \frac{a+b}{2}$$

- If, $f(x_0) = 0$, then obviously $x = x_0$ is a root of the equation $f(x) = 0$.
- But if, $f(a) \cdot f(x_0) < 0$ i.e. $f(a) \rightarrow +ve$ & $f(x_0) \rightarrow -ve$

then, the next approx. root lies in the interval (a, x_0) . 0.

OR if, $f(b) \cdot f(x_0) < 0$ i.e. $f(b) \rightarrow -ve$ & $f(x_0) \rightarrow +ve$.
then, the next approx. root lies in the interval (x_0, b) .

Note : The new interval is of the length $\left(\frac{b-a}{2}\right)$.

Now,

- If the root lie in the interval (a, x_0) , then the new approximate root will be :-

$$x_1 = \frac{a+x_0}{2}$$

- If the root lie in the interval (x_0, b) , then the new approx. root will be :-

$$x_1 = \frac{x_0+b}{2}$$

③ If $f(x_1) \cdot f(a) < 0$ i.e. $f(a) \rightarrow +ve$ & $f(x_1) \rightarrow -ve$, then the next approx. root (x_2) will lie in the interval (a, x_1) .

Otherwise, the interval will be (x_1, x_0) .

Note : The length of this interval will be $\left(\frac{b-a}{2^2}\right)$.

→ This process is repeated until the interval (which contains the root) is as small as desired, say numerically less than some given small number $\epsilon > 0$.

In General :-

The length of the interval at the n^{th} step will be $\frac{b-a}{2^n}$

- Number of iteration required to find desired solution by BISECTION METHOD.

$$\frac{b-a}{2^n} < \epsilon$$

$$\Rightarrow \frac{2^n}{b-a} > \frac{1}{\epsilon}$$

$$\Rightarrow 2^n > \frac{b-a}{\epsilon}$$

$$\Rightarrow n > \log_2 \left(\frac{b-a}{\epsilon} \right)$$

Ques. 1) find real root of $f(x) = x^3 - 2x - 5$

Ans. 1.) (Hit & Trial)

$$f(0) \rightarrow -\text{ve.}$$

$$f(1) \rightarrow -\text{ve.}$$

$$f(2) \rightarrow -\text{ve.}$$

$$f(3) \rightarrow +\text{ve.}$$

Say,

$$a = 2$$

$$b = 3$$

$$\longrightarrow [a, b] = [2, 3]$$

• 1st Iteration

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2}$$

$$\Rightarrow x_0 = 2.5$$

• 2nd Iteration

$$x_1 = \frac{a+x_0}{2} = \frac{2+2.5}{2}$$

$$\Rightarrow x_1 = 2.25$$

Now,

$$f(x_0) = 5.625 > 0$$

(+ve)

$$f(x_1) = 1.89062 > 0$$

(+ve)

• 3rd Iteration

$$x_2 = \frac{a+x_1}{2} = \frac{2+2.25}{2}$$

$$\Rightarrow x_2 = 2.125$$

Now,

$$f(x_2) \rightarrow +\text{ve}$$

Say,

Next approx. root will lie in interval $[2, 2.25]$.

Next root in interval : $[2, 2.25]$

Next root in interval : $[2, 2.125]$

• 4th Iteration

$$x_3 = \frac{2+2.125}{2} = 2.0625$$

$$\text{Now, } f(x_3) = -\text{ve}$$

Say,

Next approx. root in the interval : $[2.0625, 2.125]$

• 5th Iteration

$$x_4 = \frac{2.0625 + 2.125}{2}$$

$$\Rightarrow x_4 = 2.09375$$

Now,

$$f(x_4) \rightarrow -\text{ve}$$

• 6th Iteration

$$x_5 = \frac{2.09375 + 2.125}{2}$$

$$\Rightarrow x_5 = 2.109375$$

Ques. 2.) $f(x) = x \sin x + \cos x$. Using Bisection method find root of $f(x)$.

Ans. 2.) $f(0) = 1$ (+ve) $f(2) \rightarrow 1.40244$ (+ve)
 $f(1) = 1.38177$ (+ve) $f(3) \rightarrow -0.5666$ (-ve)

So, $a = 2$ \longrightarrow interval : $[2, 3]$
 $b = 3$

• 1st Iteration

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2}$$

$$\Rightarrow x_0 = 2.5$$

Now,

$$f(x_0) \rightarrow +ve$$

So,

Next Interval of approx. root : $(2.5, 3)$

• 2nd Iteration

$$x_1 = \frac{2.5+3}{2}$$

$$\Rightarrow x_1 = 2.75$$

Now,

$$f(x_1) \Rightarrow +ve$$

So,

Next approx. root in the interval : $(2.75, 3)$

• 3rd Iteration

$$x_2 = \frac{2.75+3}{2}$$

$$\Rightarrow x_2 = 2.875$$

Now,

$$f(x_2) \Rightarrow -ve$$

So,

Next approx. root in the interval : $(2.75, 2.875)$

• 4th Iteration

$$x_3 = \frac{2.75+2.875}{2}$$

$$\Rightarrow x_3 = 2.8125$$

Now,

$$f(x_3) \rightarrow -ve$$

So,

Next Interval of approx. root : $(2.75, 2.8125)$

• 5th Iteration

$$x_4 = \frac{2.75+2.8125}{2}$$

$$\Rightarrow x_4 = 2.78125$$

Now,

$$f(x_4) \rightarrow +ve$$

So,

Next approx. root in the interval : $(2.78125, 2.8125)$

• 6th Iteration

$$x_5 = \frac{2.78125+2.8125}{2}$$

$$\Rightarrow x_5 = 2.796875$$

CONVERGENCE ANALYSIS OF BISECTION METHOD

- $f(x) = 0$ (given function)
- $x = \alpha$ is actual root of $f(x)$
- $x_n \rightarrow$ is n^{th} approx. root / successive approximation of actual root $x = \alpha$.

→ x_n is said to converge to actual root $x = \alpha$ with order $q \geq 1$ if $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^q$

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^q$$

$C \rightarrow$ Rate of convergence.
 $q \rightarrow$ Order of convergence.

Note : When $q=1$ & $0 < C < 1$, then convergence is said to be of 1st Order.

- Order of Convergence $\rightarrow q$.
- Rate of Convergence $\rightarrow C$.

BISECTION METHOD ALWAYS CONVERGES.

→ let ' α ' be actual root of $f(x)=0 \Rightarrow f(\alpha) = 0$.

→ let x_n is successive approximation of actual root α .

$$x_n = \epsilon_n + \alpha$$

$\epsilon \rightarrow$ Error (gap b/w actual & approx. root)

and, $x_{n+1} = e_{n+1} + \alpha$ ①

So,

$$x_{n+1} = \frac{x_{n-1} + x_n}{2}$$

$$\Rightarrow x_{n+1} = \frac{e_{n+1} + \alpha + e_n + \alpha}{2}$$

$$\Rightarrow e_{n+1} + \alpha = \frac{e_{n+1} + e_n}{2} + \alpha$$

$$\Rightarrow e_{n+1} = e_n \left[1 + \frac{e_{n-1}}{e_n} \right]$$

$\left(\because \frac{e_{n-1}}{e_n}$ is v.v. small, we can neglect it. $\right)$

$$\Rightarrow e_{n+1} \approx \frac{e_n}{2}$$

$$\Rightarrow e_{n+1} = \frac{1}{2} (e_n)^1$$

So,

Rate of Convergence = $1/2$

Order of Convergence = 1

NEWTON - RAPHSON METHOD

→ Let x_0 be approximate root of $f^n \cdot f(x)$. And,
 $x_1 = x_0 + h$ be the correct root, then :-

$$\begin{aligned} f(x_1) &= 0 \\ \Rightarrow f(x_0 + h) &= 0 \end{aligned}$$

$$\Rightarrow f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

$$\Rightarrow f(x_0) + h f'(x_0) = 0 \quad (\text{Neglecting higher powers})$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

So,

$$x_1 = x_0 + h$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

so,

In General,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (n \rightarrow 0, 1, 2, \dots)$$

Ques. 1.) Use Newton - Raphson method to find approx. root of $(30)^{-1/5}$.

Ans.)

$$x = (30)^{-1/5}$$

$$\Rightarrow x = \left(\frac{1}{30}\right)^{1/5}$$

$$\Rightarrow x^5 = \frac{1}{30}$$

So,

$$f(x) = x^5 - \frac{1}{30}$$

$$\Rightarrow f(x) = 30x^5 - 1$$

So,

let, $x_0 = \frac{1}{2}$ (choose x_0 such that $f(x)$ is nearly equal to zero)

So,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n \rightarrow 0, 1, 2, \dots)$$

$n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = \frac{1}{2} - \frac{30x_0^5 - 1}{150x_0^4}$$

$$\Rightarrow x_1 = \frac{1}{2} - \frac{30(\frac{1}{2})^5 - 1}{150(\frac{1}{2})^4}$$

$$\Rightarrow x_1 = \frac{1}{2} + \frac{0.0625}{0.375}$$

$$\Rightarrow x_1 = 0.50667$$

$n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 0.50667 - \frac{30(0.50667)^5 - 1}{150(0.50667)^4}$$

$$\Rightarrow x_2 = 0.50667 - \frac{0.001689}{0.885}$$

$$\Rightarrow x_2 = 0.50649$$

$n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 0.50649 - \frac{30(0.50649)^5 - 1}{150(0.50649)^4}$$

$$\Rightarrow x_3 = 0.50649 + \frac{0.0005611}{9.87131}$$

$$\Rightarrow x_3 = 0.50654$$

Ques. 2) Use N-R method to find approximate root of $(24)^{\frac{1}{3}}$.

Ans.) let, $x = (24)^{\frac{1}{3}}$

$$\Rightarrow x^3 = 24$$

say, $f(x) = x^3 - 24$

let, $x_0 = 3$

We know :- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x^3 - 24}{3x^2}$$

$$\Rightarrow x_1 = 3 - \frac{(3)^3 - 24}{3(3)^2}$$

$$\Rightarrow x_1 = 2.888889$$

$n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 2.8889 - \frac{(2.8889)^3 - 24}{3(2.8889)^2}$$

$$\Rightarrow x_2 = 2.8889 - \frac{0.10974}{25.037}$$

$$\Rightarrow x_2 = 2.8845$$

 $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 2.8845 - \frac{(2.8845)^3 - 24}{3(2.8845)^2}$$

$$\Rightarrow x_3 = 2.884499$$

Ques.) Use N-R method to derive formula for $N^{\frac{1}{q}}$, $N > 0$ & hence find value of $18^{\frac{1}{13}}$ correct to 4 decimal places.

Ans.) let, $x = N^{\frac{1}{q}}$

$$\Rightarrow x^q = N$$

$$\Rightarrow x^q - N = 0$$

S.Q

$$f(x) = x^q - N$$

$$\Rightarrow f(x_n) = x_n^q - N$$

S.Q

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{x_n^3 - 18}{3x_n^2}} \quad (\text{General formula})$$

None,

$$x = 18^{\frac{1}{3}}$$

$$\Rightarrow x^3 = 18$$

$$\Rightarrow x^3 - 18 = 0$$

So,

$$f(x) = x^3 - 18$$

$$\Rightarrow f(x_n) = x_n^3 - 18$$

So,

$$x_{n+1} = x_n - \frac{x_n^3 - 18}{3x_n^2}$$

$n=0$

$$x_1 = x_0 - \frac{x_0^3 - 18}{3x_0^2}$$

(let, $x_0 = 2.5$)

$$\Rightarrow x_1 = 2.5 - \left(\frac{(2.5)^3 - 18}{3(2.5)^2} \right)$$

$$\Rightarrow x_1 = 2.5 + 0.12667$$

$$\Rightarrow \boxed{x_1 = 2.62667}$$

$n=1$

$$x_2 = x_1 - \left(\frac{x_1^3 - 18}{3x_1^2} \right)$$

$$\Rightarrow x_2 = (2.62667) - \left(\frac{(2.62667)^3 - 18}{3(2.62667)^2} \right)$$

$$\Rightarrow \boxed{x_2 = 2.62093}$$

$n=2$

$$x_3 = x_2 - \left(\frac{(x_2)^3 - 18}{3x_2^2} \right)$$

$$\Rightarrow x_3 = 2.62093 - \left(\frac{(2.62093)^3 - 18}{3(2.62093)^2} \right)$$

$$\Rightarrow x_3 = 2.62074$$

$n=3$

$$x_4 = x_3 - \frac{x_3^3 - 18}{3x_3^2}$$

$$\Rightarrow x_4 = (2.62074) - \left(\frac{(2.62074)^3 - 18}{3(2.62074)^2} \right)$$

$$\Rightarrow x_4 = 2.62074$$

CONVERGENCE ANALYSIS OF NEWTON-RAPHSON METHOD

We know : $\epsilon_n = x_n - \alpha$

So, $\epsilon_{n+1} = x_{n+1} - \alpha$

$\Rightarrow \epsilon_{n+1} = \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) - \alpha$

$(x_n = \epsilon_n + \alpha)$
(from eqn. ①)

$\Rightarrow \epsilon_{n+1} = \left(\epsilon_n + \alpha - \frac{f(\epsilon_n + \alpha)}{f'(\epsilon_n + \alpha)} \right) - \alpha$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(x+\epsilon_n)}{f'(x+\epsilon_n)}$$

Using Taylor's expansion, we get,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \left(\frac{f(x) + \epsilon_n f'(x) + \frac{\epsilon_n^2}{2!} f''(x) + \dots}{f'(x) + \epsilon_n f''(x) + \frac{\epsilon_n^2}{2!} f'''(x) + \dots} \right)$$

(neglecting higher power as they are v.v. small)

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \left(\frac{f(x) + \epsilon_n f'(x) + \frac{\epsilon_n^2}{2!} f''(x)}{f'(x)} \right)$$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(x)}{f'(x)} - \frac{\epsilon_n f'(x)}{f'(x)} - \frac{\epsilon_n^2}{2!} \frac{f''(x)}{f'(x)}$$

$$\Rightarrow \epsilon_{n+1} = \cancel{\epsilon_n} - \cancel{\epsilon_n} - \frac{\epsilon_n^2}{2!} \frac{f''(x)}{f'(x)} \rightarrow (\text{constant} = k)$$

$$\Rightarrow \epsilon_{n+1} = K \epsilon_n^2$$

so, NR method has 2nd Order Convergence.