LAB 11 : Hidden Markov Model

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

Please refer to the following article to understand Hidden Markov Model

Here we will be dealing with 3 major problems :

1. Evaluation Problem
2. Learning Problem
3. Decoding Problem

1. Evaluation Problem : Implementation of Forward and Backward Algorithm

Forward

Forward

Backward

Backward

In [ ]:
```
data = pd.read_csv('data_python.csv') ## Read the data, change the path accordingly

V = data['Visible'].values

# Transition Probabilities
a = np.array(((0.54, 0.46), (0.49, 0.51)))

# Emission Probabilities
b = np.array(((0.16, 0.26, 0.58), (0.25, 0.28, 0.47)))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5)) ## Write your code here


def forward(V, a, b, initial_distribution):
    alpha = np.zeros((V.shape[0], a.shape[0]))
    ## Write your code here

    alpha[0, :] = initial_distribution * b[:, V[0]]
    for t in range(1, V.shape[0]):
        alpha[t, :] = (alpha[t-1] @ a) * b[:, V[t]]

    return alpha


alpha = forward(V, a, b, initial_distribution)
print("alpha:")
print(alpha)
```

```
def backward(V, a, b):
    beta = np.zeros((V.shape[0], a.shape[0]))

    ## Write your code here
    beta[V.shape[0]-1] = np.ones(a.shape[0])
    for t in range(V.shape[0]-2, -1, -1):
        for j in range(a.shape[0]):
            beta[t, j] = np.sum(a[j, :] * beta[t+1] * b[:, V[t+1]])
    return beta


beta = backward(V, a, b)
print("beta")
print(beta)
```

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```

1. Learning Problem : Implementation of Baum Welch Algorithm

Baum-Welch

Baum-Welch

In [ ]:

```
def baum_welch(V, a, b, initial_distribution, n_iter=100):

  M = a.shape[0]
  T = len(V)
  K = b.shape[1]

  ## Write your code here
  for n in range(n_iter):
    alpha = forward(V, a, b, initial_distribution)
    beta = backward(V, a, b)
    Xi = np.zeros((M, M, T - 1))
    for t in range(T-1):
```

```
      den = np.dot(np.dot(alpha[t, :].T, a) * b[:, V[t + 1]].T, beta[t + 1, :])
      for i in range(M):
        num = alpha[t, i] * a[i, :] * b[:, V[t+1]] * beta[t+1, :].T
        Xi[i, :, t] = num
      Xi[:, :, t] /= den

    gamma = np.sum(Xi, axis=1)
    a = np.sum(Xi, axis=2)/np.sum(gamma, axis=1).reshape((-1, 1))

    gamma = np.hstack((gamma, np.sum(Xi[:, :, T - 2], axis=0).reshape((-1, 1))))

    den = np.sum(gamma, axis=1).reshape(-1, 1)
    for k in range(K):
      b[:, k] = np.sum(gamma[:, V==k], axis=1)

    b = np.divide(b, den)

  return (a,b)

data = pd.read_csv('data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))

a, b = baum_welch(V, a, b, initial_distribution, n_iter=100)
print({"a": a, "b": b})
```

```
{'a': array([[0.53816345, 0.46183655],
       [0.48664443, 0.51335557]]), 'b': array([[0.16277513, 0.26258073, 0.57464414],
       [0.2514996 , 0.27780971, 0.47069069]])}
```

1. Decoding Problem : Implementation of Viterbi Algorithm

Viterbi Algorithm

Viterbi

In [ ]:

```
def viterbi(V, a, b, initial_distribution):

  ## Write your code here
  T = V.shape[0]
  M = a.shape[0]

  omega = np.zeros((T, M))
  omega[0, :] = np.log(initial_distribution * b[:, V[0]])

  prev = np.zeros((T - 1, M))
  for t in range(1, T):
      for j in range(M):
          probability = omega[t - 1] + np.log(a[:, j]) + np.log(b[j, V[t]])
          prev[t - 1, j] = np.argmax(probability)
```

```
            omega[t, j] = np.max(probability)

    S = np.zeros(T)
    last_state = np.argmax(omega[T - 1, :])
    S[0] = last_state
    backtrack_index = 1
    for i in range(T - 2, -1, -1):
        S[backtrack_index] = prev[i, int(last_state)]
        last_state = prev[i, int(last_state)]
        backtrack_index += 1

    S = np.flip(S, axis=0)

    result = []
    for s in S: result.append('B' if s else 'A')

    return result


data = pd.read_csv('data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))

a, b = baum_welch(V, a, b, initial_distribution, n_iter=100)

result = viterbi(V, a, b, initial_distribution)
print(result)
```

```
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'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A']
```

1. Use the built-in **hmmlearn** package to fit the data and generate the result using the decoder

In [ ]:

```
!pip install hmmlearn==0.2.7
```

```
Looking in indexes: https://pypi.org/simple, https://us-python.pkg.dev/colab-wheels/
public/simple/
Collecting hmmlearn==0.2.7
  Downloading hmmlearn-0.2.7-cp37-cp37m-manylinux_2_12_x86_64.manylinux2010_x86_64.w
hl (129 kB)
     |████████████████████████████████| 129 kB 5.1 MB/s
Requirement already satisfied: scikit-learn>=0.16 in /usr/local/lib/python3.7/dist-p
ackages (from hmmlearn==0.2.7) (1.0.2)
Requirement already satisfied: numpy>=1.10 in /usr/local/lib/python3.7/dist-packages
(from hmmlearn==0.2.7) (1.21.6)
Requirement already satisfied: scipy>=0.19 in /usr/local/lib/python3.7/dist-packages
(from hmmlearn==0.2.7) (1.7.3)
Requirement already satisfied: joblib>=0.11 in /usr/local/lib/python3.7/dist-package
s (from scikit-learn>=0.16->hmmlearn==0.2.7) (1.2.0)
Requirement already satisfied: threadpoolctl>=2.0.0 in /usr/local/lib/python3.7/dist
-packages (from scikit-learn>=0.16->hmmlearn==0.2.7) (3.1.0)
Installing collected packages: hmmlearn
Successfully installed hmmlearn-0.2.7
```

In [ ]:

```
## Write your code heredata = pd.read_csv('data_python.csv')
from hmmlearn.hmm import MultinomialHMM

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
```

```
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))
```

In [ ]:

```
model = MultinomialHMM(n_components=2)
model.startprob_ = initial_distribution
model.transmat_ = a
model.emissionprob_ = b
```

In [ ]:

```
_, seq = model.decode(np.array([V]).reshape(-1, 1))

result = []
for s in seq: result.append('B' if s else 'A')
print(result)
```

```
['B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'B', 'A', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'B', 'A', 'B', 'A', 'B', 'B', 'A',
'B', 'A', 'B', 'A', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'B', 'A',
'A', 'A', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'B', 'B', 'B', 'B', 'A', 'B', 'A', 'B',
'B', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'A', 'B', 'B', 'B', 'A', 'B',
'B', 'B', 'A', 'B', 'B', 'B', 'A', 'B', 'A', 'B', 'B', 'A', 'B', 'A', 'A', 'A', 'B',
'A', 'B', 'A', 'B', 'B', 'B', 'A', 'A', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A',
'A', 'A', 'B', 'B', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'B', 'B', 'B', 'A',
'B', 'A', 'A', 'B', 'B', 'B', 'B', 'B', 'A', 'A', 'B', 'A', 'B', 'B', 'B', 'A', 'A',
'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'B', 'A', 'B', 'A', 'B', 'A',
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'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'B', 'A', 'B', 'A', 'B',
'A', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'A', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'B', 'B', 'B',
'A', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'A', 'A',
'A', 'B', 'A', 'B', 'B', 'B', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'B',
'A', 'B', 'A', 'B', 'B', 'A', 'B', 'A', 'A', 'B', 'B', 'B', 'B', 'B', 'B', 'B', 'B',
'A', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'A', 'A', 'B', 'B', 'A', 'A', 'A',
'A', 'A', 'A', 'B', 'B', 'A', 'A', 'B', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'A', 'A',
'B', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'A', 'A', 'B', 'B', 'A', 'B', 'B', 'A', 'B',
'B', 'B', 'B', 'B', 'A', 'B', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'B', 'B', 'A', 'A',
'A', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'B', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'B',
'B', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'A', 'B', 'B', 'A', 'B', 'A', 'B', 'A',
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'A', 'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'A',
'B', 'A', 'B', 'A', 'B', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'A', 'A', 'A',
'B', 'B', 'B', 'B', 'A', 'B', 'A', 'A', 'A', 'B', 'A', 'B', 'A', 'A', 'B', 'B', 'A',
'B', 'B', 'B', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'A', 'B', 'A', 'A', 'B', 'B',
'B', 'A', 'B', 'B', 'A', 'A', 'B', 'A']
```