UNIVERSITY OF WATERLOO
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
FINAL EXAMINATION
SPRING 2011

Course Abbreviation and Number: ECE-207
Course Title: Signals and Systems
Section: 001
Instructor: A.J. Heunis

Date of Examination: Tuesday 2 August, 2011
Time Period: Start time 9:00am, End time 11.30am
Duration of exam: 2 hours 30 minutes
Number of exam pages (including cover sheet): 8 pages
Exam type: Closed book
Materials allowed: None
Special aids permitted: Hand calculators only
Exam Venue: MC-2034 and MC-2035

Answer all five questions
Total marks = 100
All questions carry 20 marks
Mark allocation within each question is shown in square brackets.

1. (a) [13] A right-sided discrete-time signal $x[k]$ satisfies the identity

$$\sum_{k=0}^{n} 4^{-k} x[n-k] = 3^{-n} u[n], \quad n = 0, 1, 2, \ldots$$

Determine the z-transform of $x[k]$.
Hint: use the z-transform of the convolution of two signals.

(b) [7] Determine the z-transform of the right-sided signal

$$x[k] = k4^k.$$

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2. A periodic signal $x(t)$ satisfies the Dirichlet conditions, has period $T = 2\pi$, and is the input to the following system:

$$(D+1)y(t) = x(t).$$

The Fourier coefficients of the corresponding periodic response $y(t)$ are

$$c_1 = \frac{-1}{4j(1+j)}, \qquad c_{-1} = \frac{1}{4j(1-j)}, \qquad c_5 = \frac{1}{4j(1+5j)}, \qquad c_{-5} = \frac{-1}{4j(1-5j)},$$

and $c_k = 0$ for all other values of k. Determine

(a) [15] the signal $x(t)$.
(b) [5] the average power of the signal $x(t)$.

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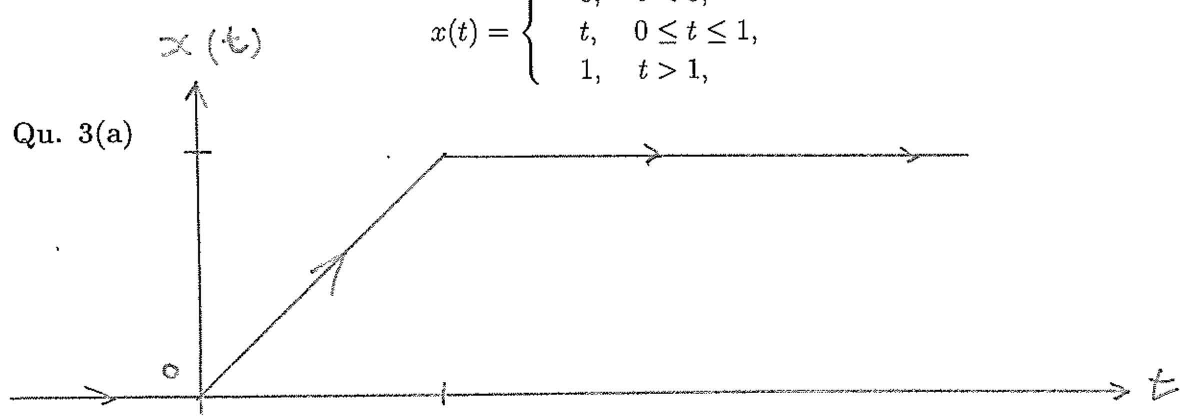
3. (a) [13] Determine the Fourier transform of the signal $x(t)$ given by (see Fig. Qu. 3(a)):

$$x(t) = \begin{cases} 0, & t < 0, \\ t, & 0 \le t \le 1, \\ 1, & t > 1, \end{cases}$$

Fig. Qu. 3(a)



Hint: use the Fourier transform of the derivatives of the signal.

Qu. 3(b) continues on the next page

3(b) [7] Determine the Fourier transform of the signal

$$x(t) = \frac{1}{(1+jt)^5}.$$

Hint: Use the duality property of Fourier transforms and the table of Fourier transforms.

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4. (a) [7] Show that the signal

$$x[k] = 3^k u[k-1]$$

has the z-transform

$$X(z) = \frac{3}{z-3}.$$

(b) [6] A discrete-time signal $y[k]$ has z-transform given by

$$Y(z) = \log(1 - 3z^{-1}).$$

Show that

$$\mathcal{Z}\{ky[k]\}(z) = \frac{-3}{z-3}.$$

(c) [7] Use (a) and (b) to determine the signal $y[k]$ in (b).

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5. (a) [6] Given the signals

$$x(t) = e^t u(-t) \qquad y(t) = e^{-t}u(t),$$

use Fourier transforms to calculate the convolution

$$z(t) = (x * y)(t).$$

(b) [14] Determine the signal $y(t)$ whose Fourier transform is

$$Y(\omega) = \frac{4}{\omega^2}\sin^2(\omega).$$

Hint: The Fourier transform of the standard rectangular pulse $z(t)$ defined by

$$z(t) = \begin{cases} 1, & -1 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

is

$$Z(\omega) = \frac{2}{\omega}\sin(\omega).$$

USEFUL FACTS:

Trig. formulae: For α in radians we have

$$\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j}.$$

Energy in signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt, \qquad E = \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Average power in signals:

$$P = \lim_{a\to\infty} \frac{1}{2a}\int_{-a}^{a} |x(t)|^2 dt, \qquad P = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} |x[k]|^2.$$

Even and odd parts of a signal:

$$x_e(t) = \frac{1}{2}[x(t)+x(-t)], \qquad x_o(t) = \frac{1}{2}[x(t)-x(-t)],$$

Sifting formulae: For a continuous-time signal $x(t)$ or discrete-time signal $x[k]$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \text{(continuous)}, \qquad x[k] = \sum_{n=-\infty}^{\infty} x[n]\delta[k-n] \quad \text{(discrete)}.$$

For a continuous-time signal $x(t)$, for which $x(t)$, $x^{(1)}(t)$, $x^{(2)}(t), \ldots, x^{(n)}(t)$, are continuous functions of t, we have

$$x^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)d\tau, \qquad n = 1, 2, \ldots$$

Convolution of two signals:

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau, \quad \text{(continuous)} \qquad (x_1 * x_2)[k] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[k-n], \quad \text{(discrete)}.$$

Output of a linear system: A continuous-time linear time-invariant system with impulse response $h(t)$ [a discrete-time linear time-invariant system with impulse response $h[k]$] and initially at rest, has response to a continuous-time input $x(t)$ [discrete-time input $x[k]$] given by

$$y(t) = (h * x)(t) \quad \text{in continuous-time}, \qquad y[k] = (h * x)[k] \quad \text{in discrete-time}.$$

Zero-input response: For the linear system

$$(*)\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

with

$$Q(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_r)^{m_r},$$

the zero-input response is given by

$$y_{zi}(t) \triangleq \sum_{i=1}^{r} \sum_{j=1}^{m_i} c_{i,j} t^{j-1} e^{\lambda_i t},$$

where constants $c_{i,j}$ are determined from the initial conditions.

Heaviside expansion theorem: Let $X(s) \triangleq N(s)/D(s)$ be a coprime rational function such that $\deg(N) < \deg(D)$, and factorize the denominator polynomial $D(s)$ as follows

$$D(s) = a_n(s-p_1)^{m_1}(s-p_2)^{m_2}\dots(s-p_l)^{m_l}.$$

Then $X(s)$ can be expanded as

$$X(s) = \sum_{i=1}^{l} \left\{ \sum_{j=1}^{m_i} \frac{r_{ij}}{(s-p_i)^j} \right\}, \quad \text{where} \quad r_{ij} = \frac{1}{(m_i-j)!} \frac{d^{m_i-j}}{ds^{m_i-j}} \left[X(s)(s-p_i)^{m_i}\right]\big|_{s=p_i},$$

for $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m_i$.

Table of Laplace Transforms: For a continuous-time signal $x(t)$, $-\infty < t < \infty$ define

$$\mathcal{L}\{x(t)\}(s) \equiv X(s) \triangleq \int_{0-}^{\infty} x(t)e^{-st}\,dt.$$

Signal $x(t),\ t \geq 0$	Laplace Transform $X(s)$
$\delta(t)$	1
$\delta^{(n)}(t)$, $n = 1, 2, \dots$	s^n
$u(t)$	$1/s$
t^n, $n = 0, 1, 2, \dots$	$n!/s^{n+1}$
$e^{\alpha t}$, α complex	$1/(s-\alpha)$
$t^n e^{\alpha t}$, α complex $n = 0, 1, 2, \dots$	$n!/(s-\alpha)^{n+1}$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
$\cos \omega t$	$s/(s^2+\omega^2)$
$e^{\alpha t}\sin \omega t$	$\omega/[(s-\alpha)^2+\omega^2]$
$e^{\alpha t}\cos \omega t$	$(s-\alpha)/[(s-\alpha)^2+\omega^2]$

Main Properties of Laplace Transforms: Suppose signals $x(t)$ and $y(t)$ have Laplace transforms $X(s)$ and $Y(s)$ respectively, and let α be a complex constant. Then:

$$\begin{aligned}
\mathcal{L}\{x^{(n)}(t)\}(s) &= s^n X(s) - s^{n-1}x(0-) - \dots - sx^{(n-2)}(0-) - x^{(n-1)}(0-). \\
\mathcal{L}\left\{\int_{0-}^{t} x(\tau)\,d\tau\right\}(s) &= \frac{X(s)}{s}. \\
\mathcal{L}\{e^{\alpha t}x(t)\}(s) &= X(s-\alpha). \\
\mathcal{L}\{(x*y)(t)\}(s) &= X(s)Y(s).
\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s).\\ x(0) &= \lim_{s\to\infty} sX(s).\end{aligned}$$

Frequency Response: Suppose the system (*) is BIBO stable with transfer function $H(s) \triangleq P(s)/Q(s)$. If the input signal $x(t)$ is given by $x(t) = A\cos(\omega t+\theta),\ \ t \geq 0,\ x(t) = 0,\ \ t < 0$, with corresponding output $y(t)$ then

$$\lim_{t\to\infty} y(t) = y_{ss}(t) \quad \text{where} \quad y_{ss}(t) = A|H(j\omega)|\cos(\omega t+\theta+\angle H(j\omega)).$$

Table of z-Transforms: For a discrete-time signal $x[k],\ -\infty < k < +\infty$ define

$$\mathcal{Z}\{x[k]\}(z) \equiv X(z) \triangleq \sum_{k=0}^{\infty} x[k]z^{-k}.$$

Signal $x[k],\ \ k \geq 0$	z-transform $X(z)$
$\delta[k]$	1
$\delta_n[k]$, $n = 1, 2, \ldots$	z^{-N}
$u[k]$	$z/(z-1)$
α^k, α complex	$z/(z-\alpha)$
$(k)_{n-1}\alpha^{k-n+1}/(n-1)!$ α complex, $n = 0, 1, 2, \ldots$	$z/(z-\alpha)^n$
$\sin(k\omega T)$	$[z\sin(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\cos(k\omega T)$	$[z^2 - z\cos(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\alpha^k\sin(k\omega T)$	$[\alpha z\sin(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$
$\alpha^k\cos(k\omega T)$	$[z^2 - \alpha z\cos(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$

Main Properties of z-Transforms: Let signals $x[k]$ and $y[k]$ have z-transforms $X(z)$ and $Y(z)$ respectively, let α be a complex constant, and let N be a positive integer. Then:

$$\begin{aligned}\mathcal{Z}\{x[k+N]\}(z) &= z^N X(z) - z^N x[0] - z^{N-1}x[1] - \ldots - z^2 x[N-2] - zx[N-1].\\ \mathcal{Z}\{x[k-N]\}(z) &= z^{-N}X(z) + \left\{x[-N] + z^{-1}x[1-N] + z^{-2}x[2-N] + \ldots + z^{1-N}x[-1]\right\}.\\ \mathcal{Z}\{\alpha^k x[k]\}(z) &= X\left(\frac{z}{\alpha}\right).\\ \mathcal{Z}\{x * y\}(z) &= X(z)Y(z).\\ \lim_{k\to\infty} x[k] &= \lim_{z\to 1}(z-1)X(z).\\ x[0] &= \lim_{z\to\infty} X(z).\\ \mathcal{Z}\{kx[k]\}(z) &= -z\frac{dX(z)}{dz}.\end{aligned}$$

Fourier Series: Let $x(t)$ be a periodic signal with a period $T > 0$. The exponential Fourier series expansion of $x(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}, \quad \text{and} \quad a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\, dt.$$

Parseval Theorem for Fourier Series: Suppose that $x(t)$ is a periodic signal satisfying the Dirichlet conditions. Then the average power in $x(t)$ is given by

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Fourier Transform Relations: Suppose $x(t)$ is a signal and define

$$X(\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt. \qquad \text{Then} \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega,$$

provided the integrals make sense.

Main Properties of Fourier Transforms: Suppose that $x(t)$ and $y(t)$ are signals with Fourier transforms $X(\omega)$ and $Y(\omega)$ respectively and let t_0, ω_0 and a be real constants. Then:

$$\begin{aligned}
X(-\omega) &= X^*(\omega), \qquad \text{when } x(t) \text{ is real-valued.} \\
\mathcal{F}\{x(t-t_0)\}(\omega) &= e^{-j\omega t_0}X(\omega). \\
\mathcal{F}\{e^{j\omega_0 t}x(t)\} &= X(\omega-\omega_0). \\
\mathcal{F}\{x^{(n)}(t)\}(\omega) &= (j\omega)^n X(\omega), \qquad\qquad n = 1,2,3,\ldots \\
\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\}(\omega) &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \\
\mathcal{F}\{x(at)\}(\omega) &= \frac{1}{|a|}X\left(\frac{\omega}{a}\right). \\
\mathcal{F}\{X(t)\}(\omega) &= 2\pi x(-\omega). \\
\mathcal{F}\{x(t)*y(t)\}(\omega) &= X(\omega)Y(\omega). \\
\mathcal{F}\{x(t)y(t)\}(\omega) &= \frac{1}{2\pi}(X*Y)(\omega). \\
\mathcal{F}\{x(-t)\}(\omega) &= X(-\omega).
\end{aligned}$$

Table of Fourier Transforms:

See next page

Signal $x(t)$	Fourier Transform $X(\omega)$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$\delta^{(n)}(t)$, $n = 1, 2, \ldots$	$(j\omega)^n$
$u(t)$	$1/(j\omega) + \pi\delta(\omega)$
$e^{-\alpha t}u(t)$, α complex, $\mathrm{re}(\alpha) > 0$,	$1/(\alpha + j\omega)$
$t^n e^{-\alpha t}u(t)$, α complex, $\mathrm{re}(\alpha) > 0$, $n = 0, 1, 2, \ldots$	$n!/(\alpha + j\omega)^{n+1}$
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin\omega_0 t$	$(\pi/j)[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$

Parseval Theorem for Fourier Transforms: Suppose that signal $x(t)$ has Fourier transform $X(\omega)$ and is such that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty.$$

Then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

1. (a) Given

$$\sum_{k=0}^{n} 4^{-k} x[n-k] = 3^{-n} u[n] \quad \text{—— (1)}$$

$$n = 0, 1, 2, 3 \ldots$$

But

$$x[k] = 0 \quad \text{all } k < 0 \quad \text{—— (2)}$$

(since $x[k]$ is <u>right-sided</u>). Then

$$x[n-k] \overset{(2)}{=} 0 \quad \text{when } k > n \quad \text{—— (3)}$$

Then

$$\sum_{k=0}^{n} 4^{-k} x[n-k] \overset{(3)}{=} \sum_{k=0}^{\infty} 4^{-k} x[n-k]$$

clear!

$$\overset{\downarrow}{=} \sum_{k=0}^{\infty} \underbrace{4^{-k} u[k]}_{y[k]} \, x[n-k]$$

$$= \sum_{k=0}^{\infty} y[k] \, x[\underset{\substack{\uparrow \\ \text{fixed!}}}{n}-k] \quad \text{— (4)}$$

where

$$y[k] \triangleq 4^{-k} u[k] \quad \text{— (5)}$$

From (4)

$$\sum_{k=0}^{n} 4^{-k} x[k-n] = (x*y)[n] \quad \text{— (6)}$$

thus

$$(x*y)[n] \overset{(6)\,(1)}{\overset{\downarrow}{=}} 3^{-n} u[n], \quad n = 0, 1, 2 \ldots$$

or

$$(x * y)[k] = 3^{-k} u[k], \quad k = 0, 1, 2 \ldots \quad \text{—— (7)}$$

From (7)

$$\mathcal{Z}\{(x * y)[k]\}(z) \overset{(7)}{=} \mathcal{Z}\{3^{-k} u[k]\}(z) \quad \text{—— (8)}$$

Now

$$\mathcal{Z}\{(x * y)[k]\}(z) \underset{\substack{\uparrow \\ \text{Properties of z transforms}}}{=} X(z)\,Y(z) \quad \text{—— (9)}$$

and

$$\mathcal{Z}\{3^{-k} u[k]\}(z) \overset{\substack{\text{from tables} \\ \downarrow}}{=} \frac{z}{z - (3^{-1})}$$

$$= \frac{3z}{3z - 1} \quad \text{—— (10)}$$

Combine (8) - (10)

$$X(z)Y(z) = \frac{3z}{3z-1} \quad \text{—— (11)}$$

Now

$$Y(z) \overset{(5)}{=} \mathcal{Z}\{4^{-k}u[k]\}(z)$$

$$= \frac{z}{z-(4^{-1})} = \frac{4z}{4z-1} \quad \text{—— (12)}$$

Put (12) in (11):

$$X(z)\left[\frac{4z}{4z-1}\right] = \frac{3z}{3z-1}$$

or

$$X(z) = \frac{3}{4}\cdot\left[\frac{4z-1}{5z-1}\right]$$

1. (b) $$x[k] \triangleq k\,4^k, \quad k = 0,1,2,\ldots \quad \text{—— (13)}$$

Put $$y[k] \triangleq 4^k \quad \text{"} \quad \text{"} \quad \text{—— (14)}$$

Then $$x[k] \overset{(13)(14)}{=} k\,y[k] \quad \text{"} \quad \text{"} \quad \text{—— (15)}$$

$$X(z) \overset{(15)}{=} Z\{k\,y[k]\}(z)$$

$$\underset{\substack{\uparrow \\ \text{property of } Z\text{-transform}}}{=} -z\,\frac{dY(z)}{dz} \quad \text{—— (16)}$$

Now $$Y(z) \overset{(14)}{=} \mathcal{Z}\{4^k u[k]\}(z)$$

$$\underset{\text{tables}}{=} \frac{z}{z-4} \quad \text{—— } (17)$$

then $$X(z) \overset{(17)\,(16)}{=} -z \frac{d}{dz}\left[\frac{z}{z-4}\right]$$

$$= -z\left[\frac{(1)(z-4) - z(1)}{(z-4)^2}\right]$$

$$= \frac{4z}{(z-4)^2}.$$

2.

$$(D+1)\,y(t) = x(t) \quad \text{—— (1)}$$

$x(t)$ has period $T = 2\pi$ —— (2)

(a) From (1)

$$Q(D) = D+1 \qquad P(D) = 1 \quad \text{—— (3)}$$

∴ transfer function of (1) is

$$H(s) = \frac{P(s)}{Q(s)} \overset{(3)}{=} \frac{1}{s+1} \quad \text{—— (4)}$$

Write $\{a_k\}$ for Fourier coeffts of $x(t)$. Then

$$c_k = H(jk\omega_0)\,a_k \quad \text{—— (5)}$$

2.2

Now $\omega_0 \triangleq \frac{2\pi}{T} \overset{(2)}{=} 1$ —— (6)

From (5) (6)

$$a_k \overset{(5)(6)}{=} \frac{c_k}{H(jk)}$$

$$\overset{(4)}{=} (1 + jk)\, c_k \text{ —— (7)}$$

Then

$$a_1 \overset{(7)}{=} (1+j)\, c_1$$

$$= -\frac{(1+j)}{4j\,(1+j)}$$

$$= -\frac{1}{4j} \text{ —— (8)}$$

2.3

$$a_{-1} \overset{7}{=} (1-j)\, c_{-1}$$

$$= \frac{(1-j)}{4j(1-j)} = \frac{1}{4j} \quad \text{—— } 9$$

$$a_{5} \overset{7}{=} (1+5j)\, c_{5}$$

$$= \frac{(1+5j)}{4j(1+5j)} = \frac{1}{4j} \quad \text{—— } 10$$

$$a_{-5} \overset{7}{=} (1-5j)\, c_{-5}$$

$$= \frac{-(1-5j)}{4j(1-5j)} = \frac{-1}{4j} \quad \text{—— } 11$$

also

$$a_k = 0 \quad \text{all other values of } k \quad \text{——} (12)$$

$\uparrow$ given $c_k = 0$ " " "

Then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\overset{(6)}{=} \sum_{k=-\infty}^{\infty} a_k e^{jkt} \quad \text{——} (13)$$

$$\overset{(12)}{=} a_{-5} e^{-5jt} + a_{-1} e^{-jt} + a_1 e^{jt} + a_5 e^{5jt}$$

$$\overset{(8) \text{ to } (11)}{=} -\frac{1}{4j} e^{-5jt} + \frac{1}{4j} e^{-jt} - \frac{1}{4j} e^{jt} + \frac{1}{4j} e^{5jt}$$

2.5

$$= \frac{1}{2}\left[\frac{e^{5jt} - e^{-5jt}}{2j}\right] - \frac{1}{2}\left[\frac{e^{jt} - e^{-jt}}{2j}\right]$$

$$= \frac{1}{2}\left[\sin(5t) - \sin(t)\right].$$

(b) average power of $x(t)$: From Parseval

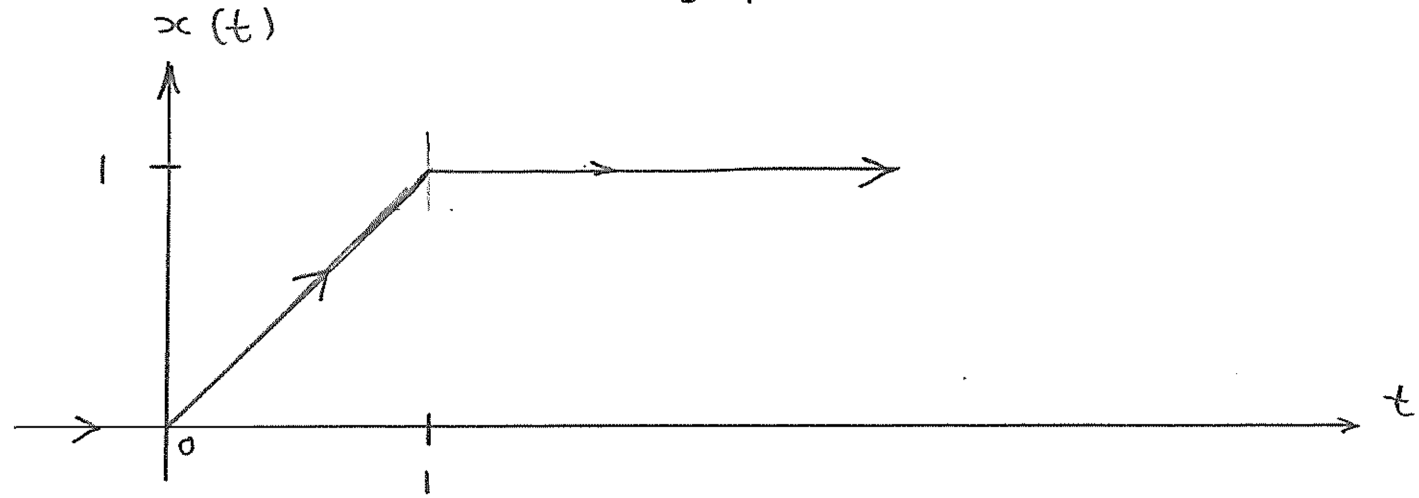
$$\underline{P} \overset{(13)}{=} \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\overset{(12)}{=} |a_{-5}|^2 + |a_{-1}|^2 + |a_1|^2 + |a_5|^2$$

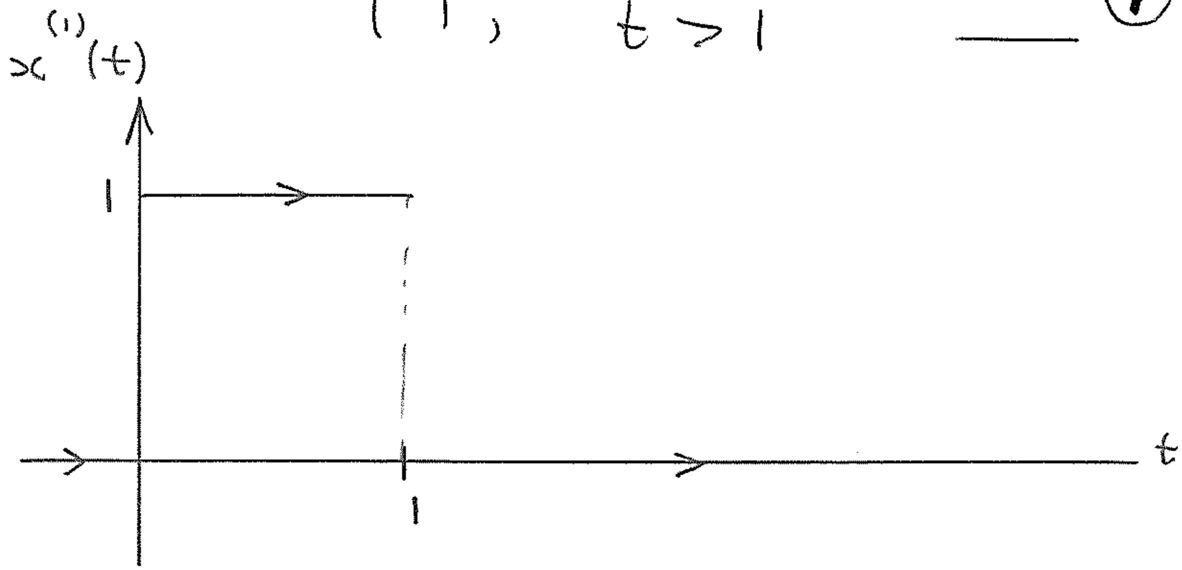
$$\overset{(8)-(11)}{=} \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{4}$$

3 (a)

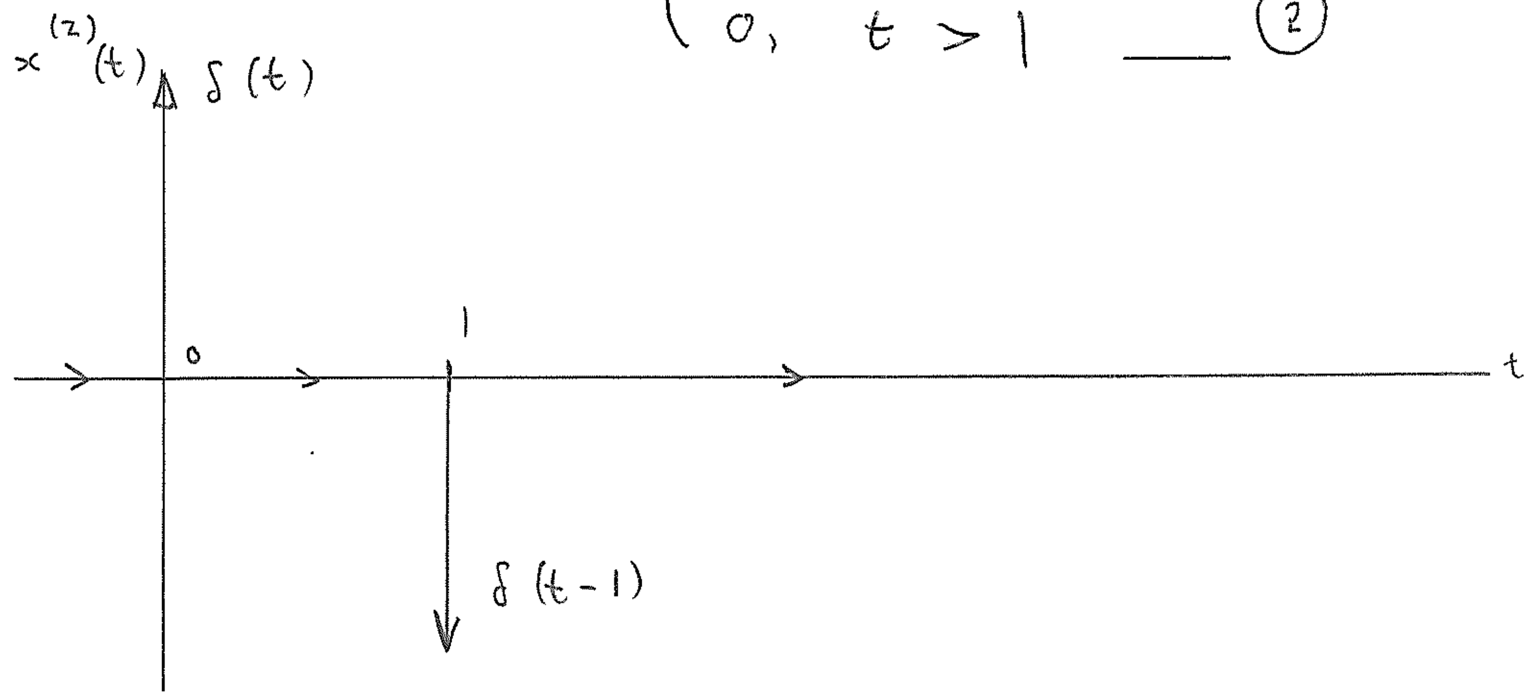


$$x(t) \triangleq \begin{cases} 0, & t < 0 \\ t, & 0 \leq t \leq 1 \\ 1, & t > 1 \end{cases} \quad \text{——} \quad (1)$$



3.2

$$x^{(1)}(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases} \quad \text{—— } (2)$$



$$x^{(2)}(t) = \delta(t) - \delta(t-1) \quad \text{—— } (3)$$

Then

$$\mathcal{F}\{x^{(2)}(t)\}(\omega) = \mathcal{F}\{\delta(t) - \delta(t-1)\}(\omega)$$

$$= \underbrace{\mathcal{F}\{\delta(t)\}(\omega)}^{=1} - \mathcal{F}\{\delta(t-1)\}(\omega)$$

$$= 1 - e^{-j\omega(1)} \underbrace{\mathcal{F}\{\delta(t)\}(\omega)}_{=1} \qquad \text{(properties of } \mathcal{F}\text{-tforms)}$$

$$= 1 - e^{-j\omega} \quad \text{——— } (4)$$

Now of course

$$x^{(1)}(t) = \int_{-\infty}^{t} x^{(2)}(\tau)\,d\tau \quad \text{——— } (5)$$

so that

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) \overset{(5)}{=} \mathcal{F}\left\{\int_{-\infty}^{t} x^{(2)}(\tau)\,d\tau\right\}(\omega)$$

property of $\mathcal{F}$-tforms

$$= \frac{1}{j\omega}\,\mathcal{F}\{x^{(2)}(t)\}(\omega) + \pi\,\mathcal{F}\{x^{(2)}(t)\}(0)\,\delta(\omega) \quad \text{——— } (6)$$

Now

$$\mathcal{F}\{x^{(2)}(t)\}(0) \overset{(6)}{=} 1 - e^{-j(0)} = 1 - 1 = 0 \quad \text{——— } (7)$$

Combine (4) (6) (7).

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) \overset{(6)(7)}{=} \frac{1}{j\omega}\,\mathcal{F}\{x^{(2)}(t)\}(\omega)$$

$$\overset{(4)}{=} \frac{1}{j\omega}\left(1 - e^{-j\omega}\right) \quad \text{—— (8)}$$

Again

$$x(t) = \int_{-\infty}^{t} x^{(1)}(\tau)\,d\tau \quad \text{—— (9)}$$

i.e.

$$\mathcal{F}\{x(t)\}(\omega) = \mathcal{F}\left\{\int_{-\infty}^{t} x^{(1)}(\tau)\,d\tau\right\}(\omega)$$

property of F-t forms

$$= \frac{1}{j\omega}\,\mathcal{F}\{x^{(1)}(t)\}(\omega) + \pi\,\mathcal{F}\{x^{(1)}(t)\}(0)\,\delta(\omega) \quad \text{—— (10)}$$

Now

$$\mathcal{F}\{x^{(1)}(t)\}(0) \overset{8}{=} \frac{1}{j(0)}\left(1 - e^{j0}\right) = \frac{0}{0}$$

(indeterminate!)

But note that

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) \overset{\text{definition!!}}{=} \int_{-\infty}^{\infty} x^{(1)}(\tau)\, e^{-j\omega\tau}\, d\tau \quad \text{— } 11$$

so that

$$\mathcal{F}\{x^{(1)}(t)\}(0) \overset{11}{=} \int_{-\infty}^{\infty} x^{(1)}(\tau)\, \underbrace{e^{-j\tau(0)}}_{=1}\, d\tau$$

$$= \int_{-\infty}^{\infty} x^{(1)}(\tau)\, d\tau \overset{2}{=} \int_{0}^{1} 1\, d\tau$$

$$= 1 \quad \text{— } 12$$

Thus

$$\mathcal{F}\{x(t)\}(\omega) \overset{10\,12}{=} \frac{1}{j\omega}\,\mathcal{F}\{x^{(1)}(t)\}(\omega) + \pi\,(1)\,\delta(\omega)$$

$$\overset{8}{=} \frac{(1 - e^{-j\omega})}{(j\omega)^2} + \pi\,\delta(\omega)$$

$$= \frac{e^{-j\omega} - 1}{\omega^2} + \pi\,\delta(\omega).$$

3 (b)

$$x(t) \triangleq \frac{1}{(2+jt)^5} \quad \text{—— (13)}$$

From tables

$$\mathcal{F}\left\{e^{-\alpha t} \cdot t^n \cdot u(t)\right\}(\omega) = \frac{n!}{(\alpha + j\omega)^{n+1}} \quad \text{—— (14)}$$

Take

$$\alpha = 2 \qquad n = 4 \quad \text{—— (15)}$$

Define

$$y(t) \triangleq e^{-\alpha t} t^n \cdot u(t)$$

$$\overset{(15)}{=} \frac{e^{-2t} t^4 u(t)}{4!} \quad \text{—— (16)}$$

then

$$\mathcal{F}\{y(t)\}(\omega) \overset{(16)}{=} \frac{1}{4!} \mathcal{F}\left\{e^{-2t} \cdot t^4 \cdot u(t)\right\}(\omega)$$

$$\overset{14}{=} \frac{1}{4!} \cdot \frac{4!}{(2+j\omega)^5}$$

$$= \frac{1}{(2+j\omega)^5} \quad \text{—— (17)}$$

ie $$Y(\omega) = \frac{1}{(2+j\omega)^5} \quad \text{—— (18)}$$

By duality

$$\underbrace{\mathcal{F}\{Y(t)\}(\omega)}_{\overset{(18)}{=}} = \underbrace{2\pi\, y(-\omega)}_{\overset{(16)}{=}}$$

$$\mathcal{F}\left\{\frac{1}{(2+jt)^5}\right\}(\omega) = \frac{2\pi \cdot e^{-2(-\omega)}\,(-\omega)^4\,u(-\omega)}{4!}$$

$$= \frac{2\pi e^{\omega} \cdot (\omega)^4 \cdot u(-\omega)}{4!}$$

4 (a) $$x[k] \triangleq 3^k u[k-1] \quad \text{—— (1)}$$

Put $$x_1[k] \triangleq 3^k u[k] \quad \text{—— (2)}$$

Then $$x[k] \overset{(1)}{=} 3 \cdot 3^{k-1} u[k-1]$$

$$\overset{(2)}{=} 3\, x_1[k-1] \quad \text{—— (3)}$$

Thus $$X(z) \triangleq \mathcal{Z}\{x[k]\}(z)$$

$$\overset{(3)}{=} 3\, \mathcal{Z}\{x_1[k-1]\}(z) \quad \text{—— (4)}$$

Now

$$\mathcal{Z}\{x_1[k-1]\}(z) \underset{\uparrow}{=} z^{-1}X_1(z) + \{x_1[-1]\} \quad \text{—— } 5$$

property of $\mathcal{Z}$ - transforms

also

$$X_1(z) \underset{\uparrow}{=} \frac{z}{z-3} \quad \text{—— } 6$$

from 2 and tables

and

$$x_1[-1] \overset{2}{=} (3)^{-1}\overbrace{u[-1]}^{=0} = 0 \quad \text{—— } 7$$

Put 6 7 in 5:

$$\mathcal{Z}\{x_1[k-1]\}(z) = \frac{1}{z-3} \quad \text{—— } 8$$

ie $\quad X(z) \overset{4\ 8}{=} \dfrac{3}{z-3}$ ___ 9

4 (b) given $\quad Y(z) \triangleq \log(1 - 3z^{-1})$ ___ 10

property of Z-transform

Then

$$\mathcal{Z}\{k\,y[k]\}(z) = -z \frac{d}{dz} Y(z)$$

$$\overset{10}{=} -z \frac{d}{dz}\left[\log(1-3z^{-1})\right]$$

$$= (-z) \cdot \frac{1}{(1-3z^{-1})} \cdot \frac{d}{dz}(1-3z^{-1})$$

$$= (-z) \cdot \frac{1}{(1-3z^{-1})} (-3) \frac{d}{dz}(z^{-1})$$

$$= \frac{3z}{(1-3z^{-1})} \cdot (-z^{-2})$$

$$= \frac{-3z^{-1}}{(1-3z^{-1})}$$

$$= -\frac{3}{z-3} \quad \text{——— (11)}$$

4 (c) From (11)

$$k\,y[k] = \mathcal{Z}^{-1}\left\{\frac{-3}{z-3}\right\}[k]$$

$$= -\mathcal{Z}^{-1}\left\{\frac{3}{z-3}\right\}[k] \quad \text{——— (12)}$$

see (1)

From (9) $$\mathcal{Z}\{x[k]\}(z) = \frac{3}{z-3}$$

ie $$\mathcal{Z}^{-1}\left\{\frac{3}{z-3}\right\}[k] = x[k]$$

$$\overset{(1)}{=} (3^{k})u[k-1] \quad \text{——— (13)}$$

Put (13) in (12):

$$k\,y[k] = -(3^{k})u[k-1]$$

or $$y[k] = -\frac{(3^{k})u[k-1]}{k}.$$

5 (a)

$$x(t) \triangleq e^{t}\, u(-t) \quad \text{—— (1)}$$

$$y(t) \triangleq e^{-t}\, u(t) \quad \text{—— (2)}$$

From (1) (2) with tables

$$X(\omega) = \frac{1}{1 - j\omega} \qquad Y(\omega) = \frac{1}{1 + j\omega} \quad \text{—— (3)}$$

Now

$$\mathcal{F}\left\{(x * y)(t)\right\}(\omega) \overset{\text{property of } \mathcal{F}\text{-transforms}}{=} X(\omega)\, Y(\omega) \quad \text{—— (4)}$$

From (4)

$$(x * y)(t) = \mathcal{F}^{-1}\left\{X(\omega) Y(\omega)\right\}(t)$$

$$= \mathcal{F}^{-1}\left\{\frac{1}{(1 - j\omega)(1 + j\omega)}\right\}(t) \quad \text{—— (5)}$$

5.2

$$\frac{1}{(1-\vartheta)(1+\vartheta)} = \frac{r_{11}}{1-\vartheta} + \frac{r_{12}}{1+\vartheta} \quad \text{—} \; (6)$$

then

$$r_{11} = \left.\frac{1}{1+\vartheta}\right|_{\vartheta=1} = \frac{1}{2} \quad \text{—} \; (7)$$

$$r_{12} = \left.\frac{1}{1-\vartheta}\right|_{\vartheta=-1} = \frac{1}{2} \quad \text{—} \; (8)$$

From (6) (7) (8)

$$\frac{1}{(1-\vartheta)(1+\vartheta)} = \frac{1}{2}\left[\frac{1}{1-\vartheta} + \frac{1}{1+\vartheta}\right] \quad \text{—} \; (9)$$

From (9)

$$\frac{1}{(1-j\omega)(1+j\omega)} = \frac{1}{2}\left[\frac{1}{1-j\omega} + \frac{1}{1+j\omega}\right] \quad \text{—} \; (10)$$

From 5 10

$$(x * y)(t) = \frac{1}{2}\mathcal{F}^{-1}\left\{\frac{1}{1-j\omega}\right\}(t) + \frac{1}{2}\mathcal{F}^{-1}\left\{\frac{1}{1+j\omega}\right\}(t) \quad \text{— (11)}$$

Now

$$\mathcal{F}^{-1}\left\{\frac{1}{1+j\omega}\right\}(t) \underset{\substack{\uparrow \\ \text{from tables}}}{=} e^{-t}u(t) \quad \text{— (12)}$$

Thus

$$\mathcal{F}\{e^{-t}u(t)\}(\omega) = \frac{1}{1+j\omega} \quad \text{— (13)}$$

From 13

$$\mathcal{F}\{e^{t}u(-t)\}(\omega) = \mathcal{F}\{e^{-(-t)}u(-t)\}(\omega)$$

$$= \frac{1}{1+j(-\omega)} = \frac{1}{1-j\omega} \quad \text{— (14)}$$

from 13 and property $\mathcal{F}\{x(-t)\}(\omega) = X(-\omega)$

From (14)

$$\mathcal{F}^{-1}\left\{\frac{1}{1-j\omega}\right\}(t) = e^{t}.u(-t) \quad \text{—— (15)}$$

From (11) (12) (15)

$$(x * y)(t) = \frac{1}{2}\left[e^{-t}u(t) + e^{t}u(-t)\right].$$

5(b) For the signal

$$z(t) = \begin{cases} 1, & -1 \leq t \leq 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{—— (16)}$$

we are given

$$Z(\omega) = \frac{2}{\omega}\sin(\omega) \quad \text{—— (17)}$$

Now

$$Y(\omega) \triangleq \frac{4}{\omega^2}\sin^2(\omega) \quad \text{—— (18)}$$

Now

$$\mathcal{F}\{(z*z)(t)\}(\omega) \overset{\text{property of } \mathcal{F}\text{-transforms}}{=} Z(\omega)\cdot Z(\omega)$$

$$\underset{(17)(18)}{=} Y(\omega) \equiv \mathcal{F}\{y(t)\}(\omega)$$

ie

$$\mathcal{F}\{y(t)\}(\omega) = \mathcal{F}\{(z*z)(t)\}(\omega) \quad \text{—— (19)}$$

or

$$y(t) = (z*z)(t)$$

$$= \int_{-\infty}^{\infty} z(\tau)\, z(t-\tau)\, d\tau \quad \text{—— (20)}$$

Now evaluate rhs of (20): From (16)

$$z(t-\tau) = \begin{cases} 1, & -1 \le t-\tau \le 1 \\ 0, & \text{o.w.} \end{cases} \quad \text{—— (21)}$$

and

$$-1 \leq t - \tau \leq 1 \iff t - 1 \leq \tau \leq t + 1 \quad \text{—} \quad (22)$$

From (21) (22)

$$z(t - \tau) = I[t-1, t+1](\tau) \quad \text{—} \quad (23)$$

also from (16)

$$z(\tau) = I[-1, 1](\tau) \quad \text{—} \quad (24)$$

then

$$z(\tau)\, z(t - \tau) \overset{(24)\,(23)}{=} I[-1, 1](\tau)\, I[t-1, t+1](\tau)$$

$$= I[\alpha(t), \beta(t)](\tau) \quad \text{—} \quad (25)$$

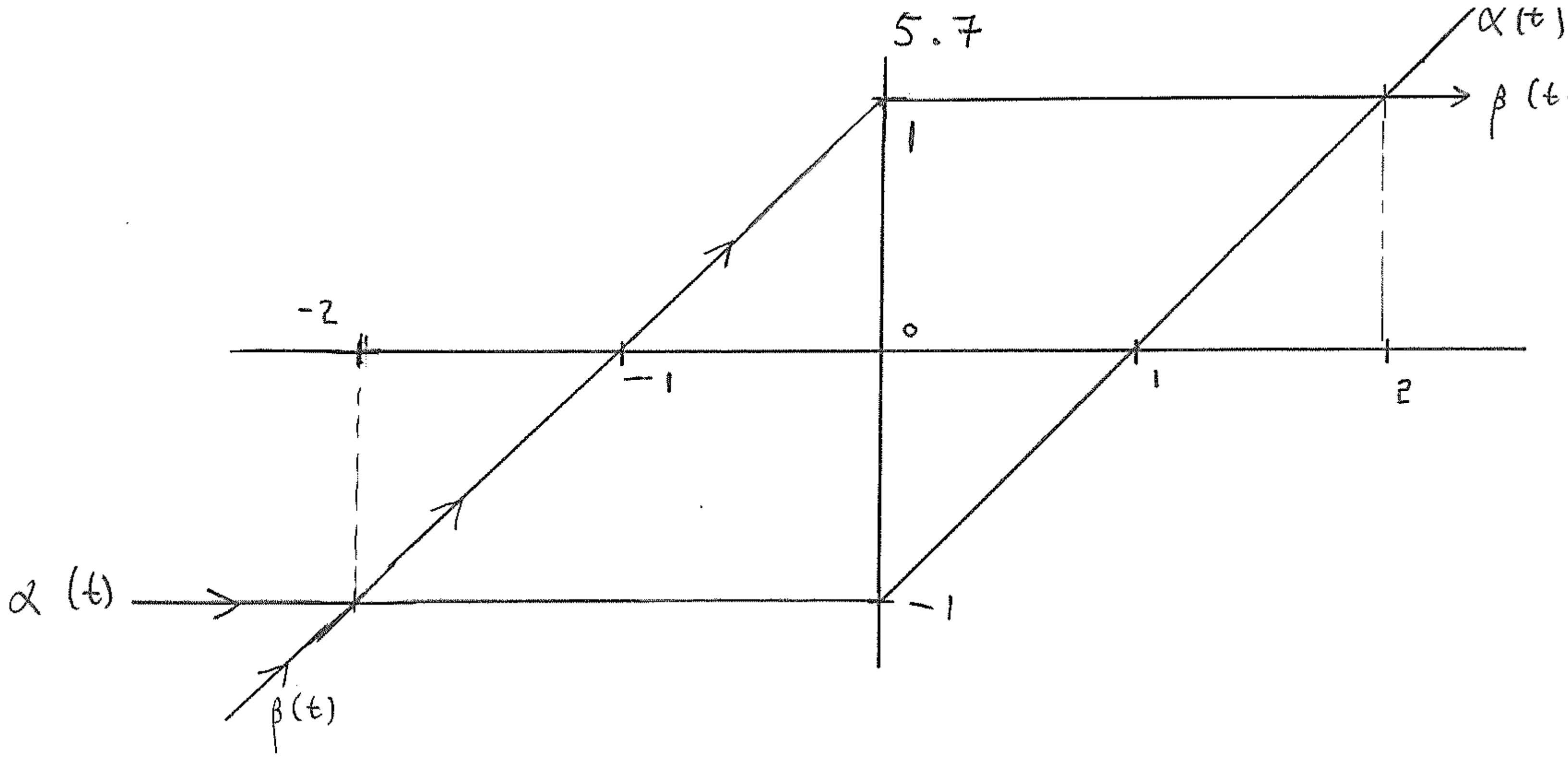
where

$$\alpha(t) \triangleq \max\{t-1, -1\} \qquad \beta(t) \triangleq \min\{t+1, 1\} \quad \text{—} \quad (26)$$

From (20) (25)

$$y(t) = \int_{-\infty}^{\infty} I[\alpha(t), \beta(t)](\tau)\, d\tau \quad \text{—} \quad (27)$$

5.7



<u>$t < -2$</u> : $\alpha(t) > \beta(t)$

$\therefore \quad y(t) \overset{27}{=} 0$ —— 28

<u>$-2 \leq t \leq 0$</u> : $\alpha(t) \leq \beta(t)$. From 27

$$y(t) = \int_{\alpha(t)}^{\beta(t)} 1 \, d\tau \quad \text{—— 29}$$

and $\alpha(t) \overset{26}{=} -1 \qquad \beta(t) \overset{26}{=} t+1$ —— 30

From 29 30

$$y(t) = \int_{-1}^{t+1} d\tau = (t+1) - (-1)$$
$$= t + 2 \quad \text{— } 31$$

<u>$0 \leq t \leq 2$</u>: $\alpha(t) \leq \beta(t)$. From 27

$$y(t) = \int_{\alpha(t)}^{\beta(t)} 1 \,.\, d\tau \quad \text{— } 32$$

and

$$\alpha(t) \overset{26}{=} t - 1 \qquad \beta(t) = 1 \quad \text{— } 33$$

ie

$$y(t) \underset{32\ 33}{=} \int_{t-1}^{1} d\tau = 1 - (t-1)$$
$$= 2 - t \quad \text{— } 34$$

<u>$t > 2$</u>: $\alpha(t) > \beta(t)$:

$$\therefore\ y(t) \overset{27}{=} 0 \quad \text{— } 35$$

From (35) (34) (31) (28)

$$y(t) = \begin{cases} 0, & t < -2 \\ t + 2, & -2 \le t \le 0 \\ 2 - t, & 0 \le t \le 2 \\ 0, & t > 2 \end{cases}$$