RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN
COMPUTER SCIENCE VI AUTONOMOUS INTELLIGENT SYSTEMS
Prof. Dr. Sven Behnke
Friedrich-Hirzebruch-Allee 8

Cognitive Robotics

Assignment 1

Due Tuesday, October 24th, before class.

1.1) Camilla receives a positive outcome on a first-stage test for a serious but rare disease. The test reports false positives with a probability of 0.06. With a probability of 0.2, the test will report a negative result, even though the tested person has the disease. Compute the probability that Camilla has the disease, given the positive test result and the fact that one out of 150,000 in the population suffers from this disease!

4 points

1.2) A robot is equipped with an unreliable person detector that outputs either "Person" or "No person".
If there is a person in front of the robot, it indicates "Person" with probability 0.6.
However, if there is no person in front of the robot, the detector also indicates "Person" with probability 0.2.
Before observing the detector, the prior belief of the robot about a person being in front of it is 0.3.
What is the posterior probability of a person being in front of the robot when the detector outputs "No person"?

4 points

1.3) Consider a two-dimensional state $\boldsymbol{x}=(x_1, x_2)$, where x_1 is the position of a cart on a horizontal linear rail (in m) and x_2 is its horizontal velocity (in m/s).
The distance between two time steps t and $t+1$ is 0.1 seconds.
Describe the matrix **A** that maps $\boldsymbol{x}_t$ to $\boldsymbol{x}_{t+1}$ in the noiseless case: $\boldsymbol{x}_{t+1} = \mathbf{A}\,\boldsymbol{x}_t$.

2 points

1.4) Consider now control actions u_t (in Newton) that accelerate the cart with m = 10 kg constantly during a time step.
How should matrix **B** look like that maps control actions to state changes: $\boldsymbol{x}_{t+1}=\mathbf{A}\,\boldsymbol{x}_t + \mathbf{B}\,u_t$?

2 points

1.5) Suppose you can only measure the cart position. How should matrix **C** look like that maps state to measurements z_t, $z_t = \mathbf{C}\,x_t$?

2 points

1.6) Start with a state $x_0 = (2, -1)$ that has a covariance $\Sigma_0 = \begin{pmatrix} 0.4, & 0, \\ 0, & 0.6 \end{pmatrix}$.

Assume that the motion has noise covariance $R = \begin{pmatrix} 0.2, & 0, \\ 0, & 0.4 \end{pmatrix}$.

What is the prediction of a Kalman filter for t=1 (=0.1s) when $u_1 = 20$ N ?
Compute mean and the covariance of the state!

3 points

1.7) Now, we make a position measurement of $z_1 = 1.9$ m with standard deviation 0.3.
What are the mean and the covariance of the corrected state?

3 points

1.1) Camilla receives a positive outcome on a first-stage test for a serious but rare disease. The test reports false positives with a probability of 0.06. With a probability of 0.2, the test will report a negative result, even though the tested person has the disease. Compute the probability that Camilla has the disease, given the positive test result and the fact that one out of 150,000 in the population suffers from this disease!

4 points

Q1)

$$z_{obs} = \text{positive}$$

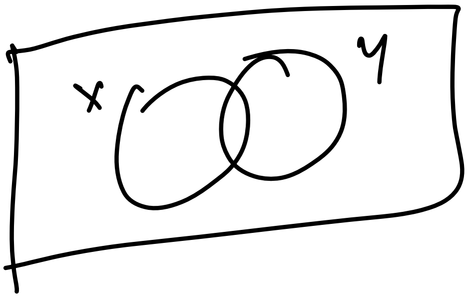
$$p(z \mid \text{negative}) = 0.06$$

$$p(-z \mid \text{positive}) = 0.2 \Rightarrow p(z \mid \text{positive}) = 0.8$$

$$p(\text{positive} \mid z) = ?$$

Given, $p(\text{positive}) = 1/150{,}000$

$$p(\text{positive} \mid z) = \frac{p(z \mid \text{positive}) * p(\text{positive})}{p(z)}$$



$$P(x \mid y) + p(-x \mid y) =$$

$$\frac{P(x \cap y)}{P(y)} + \frac{p(\bar{x} \cap y)}{P(y)} = \frac{P(y)}{P(y)} = 1$$

$$p(\text{positive} \mid z) = \frac{(0.8) * \left(\frac{1}{150000}\right)}{p(z \mid \text{positive}) * p(\text{positive}) + p(z \mid \text{negative}) * p(\text{negative})}$$

$$\Rightarrow \frac{(0.8) * \frac{1}{150000}}{(0.8) * \frac{1}{150000} + 0.06 * \left(\frac{149999}{150000}\right)}$$

$$\Rightarrow \frac{0.8}{(0.8) + (0.06)(149999)}$$

$$\boxed{\Rightarrow 0.000889}$$

1.2) A robot is equipped with an unreliable person detector that outputs either "Person" or "No person".

If there is a person in front of the robot, it indicates "Person" with probability 0.6.

However, if there is no person in front of the robot, the detector also indicates "Person" with probability 0.2.

Before observing the detector, the prior belief of the robot about a person being in front of it is 0.3.

What is the posterior probability of a person being in front of the robot when the detector outputs "No person"?

4 points

Q2)

$p(z \mid person) = 0.6$ | $p(-z \mid person) = 0.4$

$p(z \mid no\ person) = 0.2$ | $p(-z \mid no\ person) = 0.8$

$p(person) = 0.3$ | $p(no\ person) = 0.7$

$$p(person \mid -z) = \frac{p(-z \mid person) * p(person)}{p(-z)}$$

$$\Rightarrow \frac{(0.4) * (0.3)}{(0.4)(0.3) + (0.8)(0.7)}$$

$$\Rightarrow \frac{0.12}{0.12 + 0.56} \Rightarrow \frac{\cancel{0.12}^{\cancel{6}\,3}}{\cancel{0.68}_{\cancel{34}\,17}}$$

$$\Rightarrow 3/17 \Rightarrow \boxed{0.17647}$$

1.3) Consider a two-dimensional state $\mathbf{x}=(x_1, x_2)$, where x_1 is the position of a cart on a horizontal linear rail (in m) and x_2 is its horizontal velocity (in m/s).
The distance between two time steps t and $t+1$ is 0.1 seconds.
Describe the matrix **A** that maps $\mathbf{x}_t$ to $\mathbf{x}_{t+1}$ in the noiseless case: $\mathbf{x}_{t+1} = \mathbf{A}\,\mathbf{x}_t$.

2 points

Q3)

$$x = [x_1, x_2]$$

$$x_t = [x_{1t}, x_{2t}]$$

$$x_{t+1} = [x_{1t} + x_{2t}(0.1), x_{2t}]$$

$$= A x_t$$

$$\Rightarrow A \Rightarrow \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

1.4) Consider now control actions u_t (in Newton) that accelerate the cart with m = 10 kg constantly during a time step.
How should matrix **B** look like that maps control actions to state changes: $\mathbf{x}_{t+1}=\mathbf{A}\,\mathbf{x}_t + \mathbf{B}\,u_t$?

2 points

Q4)

$$x_{t+1} = x_t + v_t \Delta t + \frac{1}{2} a_t \Delta t^2$$

$$v_{t+1} = v_t + a_t \Delta t$$

$$X_{t+1} = [x_{t+1}, v_{t+1}]$$

$$X_{t+1} = \underset{2\times 2}{A} \underset{2\times 1}{\begin{bmatrix} x_t \\ v_t \end{bmatrix}} + \underset{2\times 1}{B}\, \underset{1\times 1}{u_t}$$

$$\Rightarrow \begin{bmatrix} x_t + v_t \Delta t + \frac{1}{2} a_t \Delta t^2 \\ v_t + a_t \Delta t \end{bmatrix} =$$

$$A \begin{bmatrix} x_t \\ v_t \end{bmatrix} + B u_t$$

$$\underset{[N]}{u_t} = \text{Force} = 10\,kg \cdot a_{t\,[m/s^2]}$$

$$\Rightarrow \begin{bmatrix} x_t + v_t \Delta t + \frac{1}{2} a_t \Delta t^2 \\ v_t + a_t \Delta t \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_t \\ v_t \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_t$$

$$\Rightarrow x_t + v_t \Delta t + \frac{1}{2} a_t \Delta t^2 = a_1 x_t + a_2 v_t + b_1 u_t \rightarrow 1$$

$$\Rightarrow v_t + a_t \Delta t = a_3 x_t + a_4 v_t + b_2 u_t \rightarrow 2$$

1

$$x_t + 0.1 v_t + \frac{u_t}{20}(0.01) = a_1 x_t + a_2 v_t + b_1 u_t$$

$a_1 = 1$

$a_2 = 0.1$

$b_1 = \frac{\cancel{10}5}{100 \times \cancel{2}00} \Rightarrow 0.0005$

2

$$v_t + a_t(0.1) = a_3 x_t + a_4 v_t + b_2 u_t$$

$a_3 = 0$

$a_4 = 1$

$b_2 = 0.01$

$A \Rightarrow \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \quad B \Rightarrow \begin{bmatrix} 0.0005 \\ 0.01 \end{bmatrix}$

1.5) Suppose you can only measure the cart position. How should matrix **C** look like that maps state to measurements z_t, $z_t = \mathbf{C}\, x_t$?

2 points

Q5)

$$z_t = \text{position} = x_t$$

$$\underset{1\times 1}{\begin{bmatrix} x_t \end{bmatrix}} = \underset{1\times 2}{C} \underset{2\times 1}{\begin{bmatrix} x_t \\ v_t \end{bmatrix}}$$

$$\boxed{C \Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}}$$

1.6) Start with a state $x_0 = (2, -1)$ that has a covariance $\Sigma_0 = ($ 0.4, 0,
0, 0.6).

Assume that the motion has noise covariance R = (0.2, 0,
0, 0.4).

What is the prediction of a Kalman filter for t=1 (=0.1s) when $u_1 = 20$ N ?
Compute mean and the covariance of the state!

3 points

$$x_{t+1} = A x_t + B u_t$$

$$\Sigma\{x_{t+1}\} = A\, \Sigma\{x_t\}\, A^T + R$$

$$x_{t+1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0.0005 \\ 0.01 \end{bmatrix} \begin{bmatrix} 20 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & -0.1 \\ & -1 \end{pmatrix} + \begin{bmatrix} \cancel{5\times10^{-4}} \; 10^{-2} \; \cancel{\times 2 \times 10^{-3}} \\ 0.2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1.9 \\ -1 \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0.2 \end{bmatrix}$$

$$\boxed{X_{t+1} \Rightarrow \begin{bmatrix} 1.901 \\ -0.8 \end{bmatrix}}$$

$$\Sigma\{X_{t+1}\} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.4 & 0.06 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.4 + 0.006 & 0.06 \\ 0.06 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\boxed{\Sigma\{X_{t+1}\} \Rightarrow \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix}}$$

1.7) Now, we make a position measurement of z_1= 1.9 m with standard deviation 0.3. What are the mean and the covariance of the corrected state?

3 points

Q7)

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$ // apply motion model
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$ // compute Kalman gain
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$ // compare expected with observed measurement
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$
9. Return μ_t, Σ_t

$$\bar{\mu}_t = \begin{bmatrix} 1.901 \\ -0.8 \end{bmatrix} \qquad \bar{\Sigma}_t = \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix}$$

$$C_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad Q_t = 0.09 = (0.3)^2$$

$$K_t = \underset{2\times2}{\bar{\Sigma}_t} \underset{2\times1}{C_t^T} \Big(\underset{1\times2}{C_t} \underset{2\times2}{\bar{\Sigma}_t} \underset{2\times1}{C_t^T} + \underset{1\times1}{Q_t} \Big)^{-1}$$

$$K_t \Rightarrow 2\times1$$

$$K_t = \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.09 \right)^{-1}$$

$$\mu_t \Rightarrow \begin{bmatrix} 1.901 \\ -0.8 \end{bmatrix} + K_t\left(1.9 - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1.901 \\ -0.8 \end{bmatrix}\right)$$

$$K_t \Rightarrow \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0.606 + 0.09 \end{bmatrix}^{-1}$$

$$\Rightarrow \begin{bmatrix} 0.606 \\ 0.06 \end{bmatrix}\begin{bmatrix} 0.615 \end{bmatrix}$$

$$K_t \Rightarrow \begin{bmatrix} 0.37269 \\ 0.03690 \end{bmatrix}$$

$$\mu_t \Rightarrow \begin{bmatrix} 1.901 \\ -0.8 \end{bmatrix} + \begin{bmatrix} 0.37269 \\ 0.03690 \end{bmatrix}(-0.001)$$

$$\mu_t \Rightarrow \begin{bmatrix} 1.901 - 0.00037269 \\ -0.8 - 0.00003690 \end{bmatrix}$$

$$\mu_t \Rightarrow \begin{bmatrix} 1.90062731 \\ -0.8000369 \end{bmatrix}$$

$$\Sigma_t = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.37269 \\ 0.03690 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 0.606 & 0.06 \\ 0.06 & 1 \end{bmatrix}$$