Artificial Intelligence

10. Bayesian Networks

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Reminder: elementary probability

- Basic laws: $0 \leq P(\omega) \leq 1$ $\quad \sum_{\omega \in \Omega} P(\omega) = 1$
- Events: subsets of Ω: $P(A) = \sum_{\omega \in A} P(\omega)$
- Random variable $X(\omega)$ has a value in each ω
  - Distribution $P(X)$ gives probability for each possible value x
  - Joint distribution $P(X,Y)$ gives total probability for each combination x,y
- Summing out/marginalization: $P(X=x) = \sum_y P(X=x,Y=y)$
- Conditional probability: $P(X|Y) = P(X,Y)/P(Y)$
- Product rule: $P(X|Y)P(Y) = P(X,Y) = P(Y|X)P(X)$
  - Generalize to chain rule: $P(X_1,..,X_n) = \prod_i P(X_i \mid X_1,..,X_{i-1})$

Probabilistic Inference

- Probabilistic inference: compute a desired probability from a probability model
  - Typically for a query variable given evidence
  - E.g., P(airport on time | no accidents) = 0.90
  - These represent the agent's beliefs given the evidence

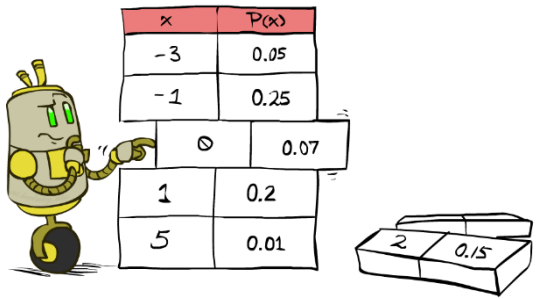
- Probabilities change with new evidence
  - P(airport on time | no accidents, 5 a.m.) = 0.95
  - P(airport on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes beliefs to be updated

Inference by Enumeration

- Probability model $P(X_1, \ldots, X_n)$ is given
- Partition the variables $X_1, \ldots, X_n$ into sets as follows:
  - Evidence variables: $E = e$
  - Query variables: Q
  - Hidden variables: H

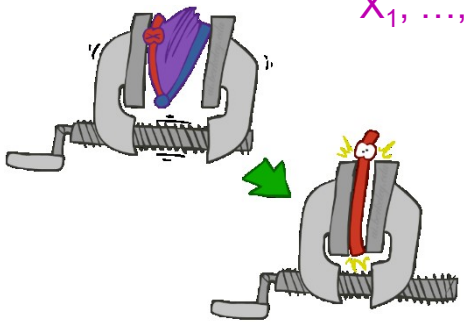
- We want:
  $P(\mathbf{Q} \mid \mathbf{e})$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out **H** from model to get joint of query and evidence

$$P(\mathbf{Q}, \mathbf{e}) = \sum_{\mathbf{h}} \underbrace{P(\mathbf{Q}, \mathbf{h}, \mathbf{e})}_{X_1, \ldots, X_n}$$



- Step 3: Normalize

$$P(\mathbf{Q} \mid \mathbf{e}) = \alpha P(\mathbf{Q}, \mathbf{e})$$

Inference by Enumeration

- P(W)?
- P(W | winter)?
- P(W | winter, cold)?

Season	Temp	Weather	P
summer	hot	sun	0.35
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.01
summer	cold	rain	0.05
summer	cold	fog	0.10
summer	cold	meteor	0.00
winter	hot	sun	0.10
winter	hot	rain	0.01
winter	hot	fog	0.01
winter	hot	meteor	0.00
winter	cold	sun	0.10
winter	cold	rain	0.10
winter	cold	fog	0.15
winter	cold	meteor	0.00

Issues with Inference by Enumeration

- Worst-case time complexity $O(d^n)$
  - exponential in the number of hidden variables
- Space complexity $O(d^n)$ to store the joint distribution
- $O(d^n)$ data points to estimate the entries in the joint distribution

Bayes' Rule

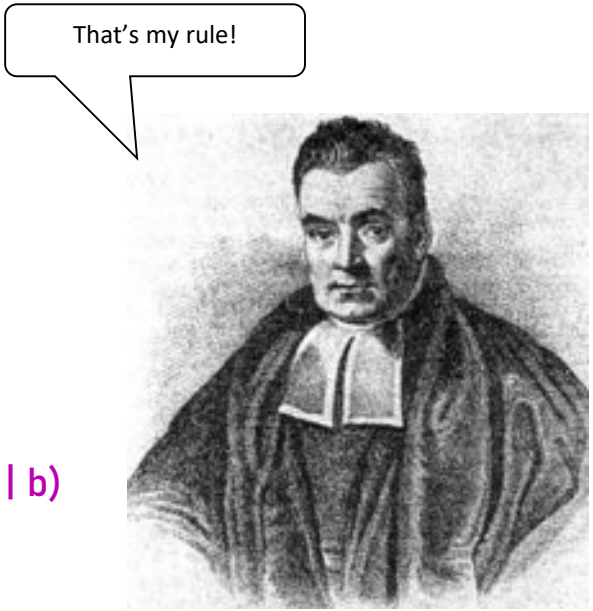
- Write the product rule both ways:

$$P(a \mid b)\, P(b) = P(a, b) = P(b \mid a)\, P(a)$$

- Dividing left and right expressions, we get:

$$P(a \mid b) = \frac{P(b \mid a)\, P(a)}{P(b)}$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Describes an "update" step from prior P(a) to posterior P(a | b)
  - Foundation of many systems we'll see later (e.g. ASR, MT)
- In the running for most important AI equation!



Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(cause \mid effect) = \frac{P(effect \mid cause)\, P(cause)}{P(effect)}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{array}{l} P(s \mid m) = 0.8 \\ P(s \mid \neg m) = 0.01 \\ P(m) = 0.0001 \end{array}\right\} \text{ Example givens}$$

$$P(m \mid s) = \frac{P(s \mid m)\, P(m)}{P(s)} \simeq \frac{0.8 \times 0.0001}{0.01}$$

  - Note: posterior probability of meningitis still very small: 0.008 (80x bigger – why?)
  - Note: you should still get stiff necks checked out! Why?

Independence

- Two variables X and Y are (absolutely) independent if

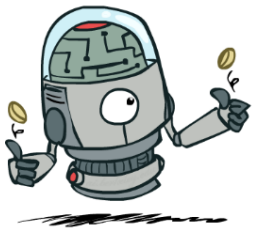
$$\forall x,y \qquad P(x, y) = P(x)\, P(y)$$

  - I.e., the joint distribution factors into a product of two simpler distributions

- Equivalently, via the product rule $P(x,y) = P(x|y)\, P(y)$,

$$P(x \mid y) = P(x) \quad \text{or} \quad P(y \mid x) = P(y)$$

- Example: two dice rolls $Roll_1$ and $Roll_2$
  - $P(Roll_1=5, Roll_2=3) = P(Roll_1=5)\, P(Roll_2=3) = 1/6 \times 1/6 = 1/36$
  - $P(Roll_2=3 \mid Roll_1=5) = P(Roll_2=3)$



Conditional Independence

- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

- X is conditionally independent of Y given Z if and only if:

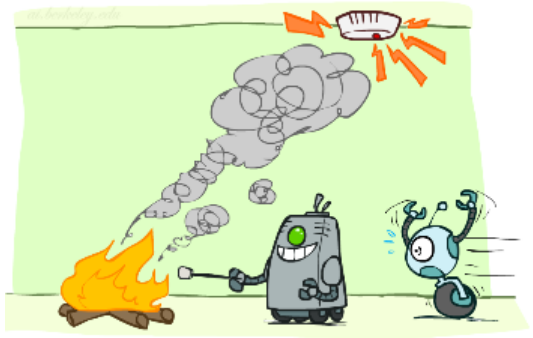
$$\forall x,y,z \quad P(x \mid y, z) = P(x \mid z)$$

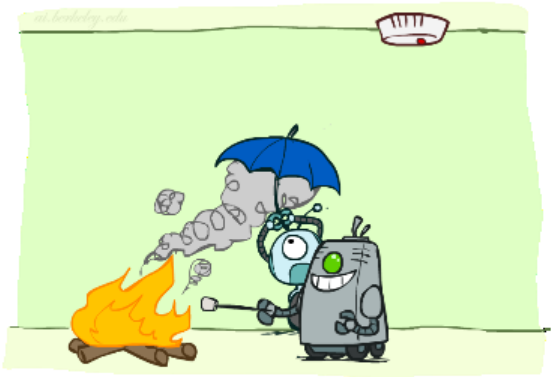
or, equivalently, if and only if

$$\forall x,y,z \quad P(x, y \mid z) = P(x \mid z)\, P(y \mid z)$$

Conditional Independence

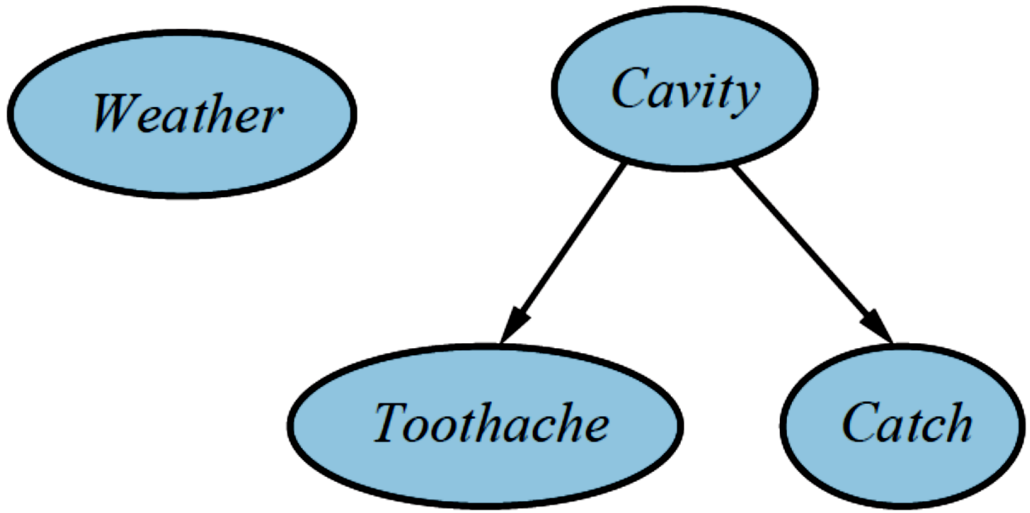
- What about this domain:
  - Fire
  - Smoke
  - Alarm





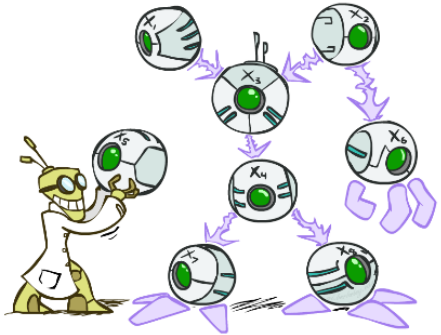
Conditional Independence

- What about this domain:
  - Cavity
  - Toothache
  - Catch
  - Weather



Bayes Nets: Big Picture

- Bayes nets: a technique for describing complex joint distributions (models) using simple, conditional distributions
  - A subset of the general class of graphical models
- Use local causality/conditional independence:
  - the world is composed of many variables,
  - each interacting locally with a few others



Bayes Nets

- Part I: Representation
- Part II: Exact inference
  - Enumeration (always exponential complexity)
  - Variable elimination (worst-case exponential complexity, often better)
  - Inference is NP-hard in general
- Part III: Approximate Inference
- Later: Learning Bayes nets from data

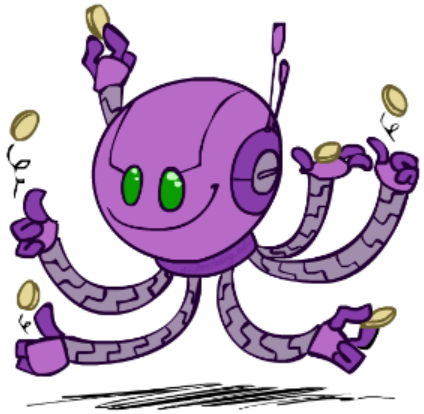
Graphical Model Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)

- Arcs: interactions
  - Indicate “direct influence” between variables
  - Formally: absence of arc encodes conditional independence (more later)

Example: Coin Flips

- N independent coin flips

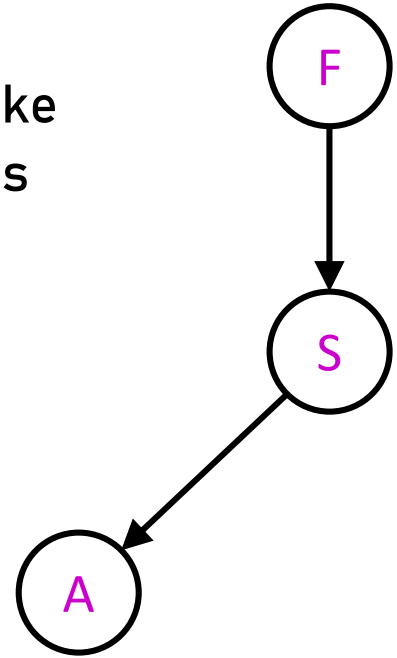


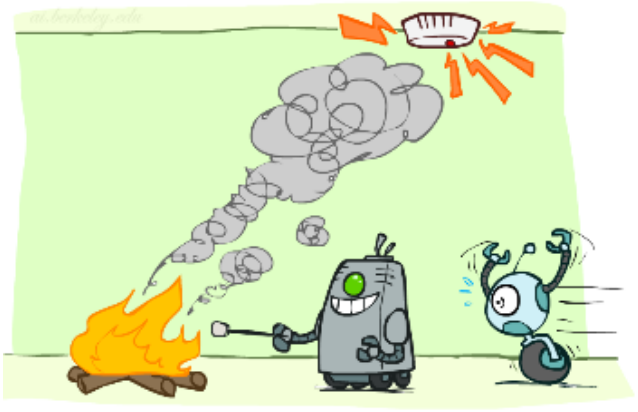
X_1 X_2 . . . X_n

- No interactions between variables: absolute independence

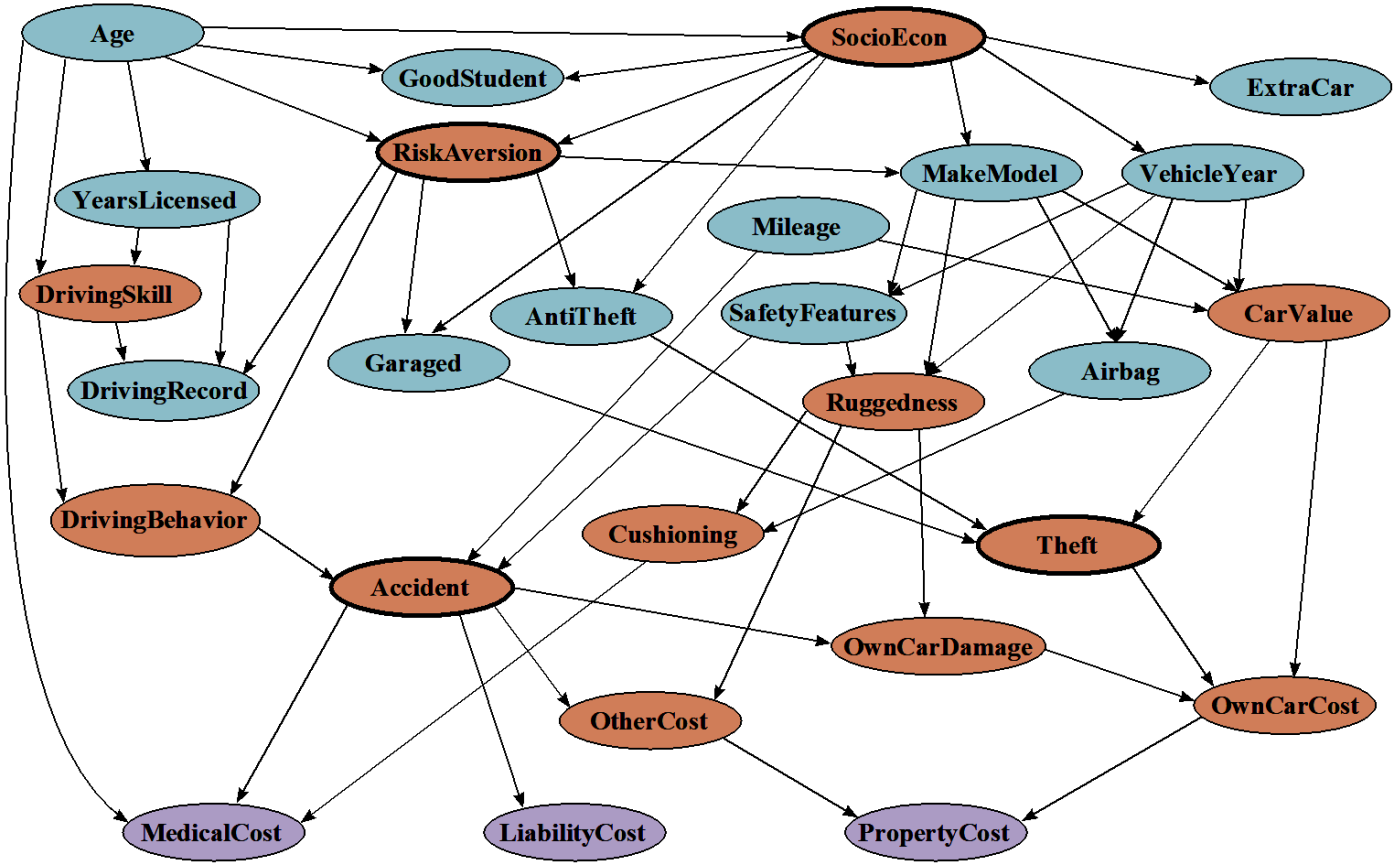
Example: Smoke alarm

- Variables:
  - F: There is fire
  - S: There is smoke
  - A: Alarm sounds

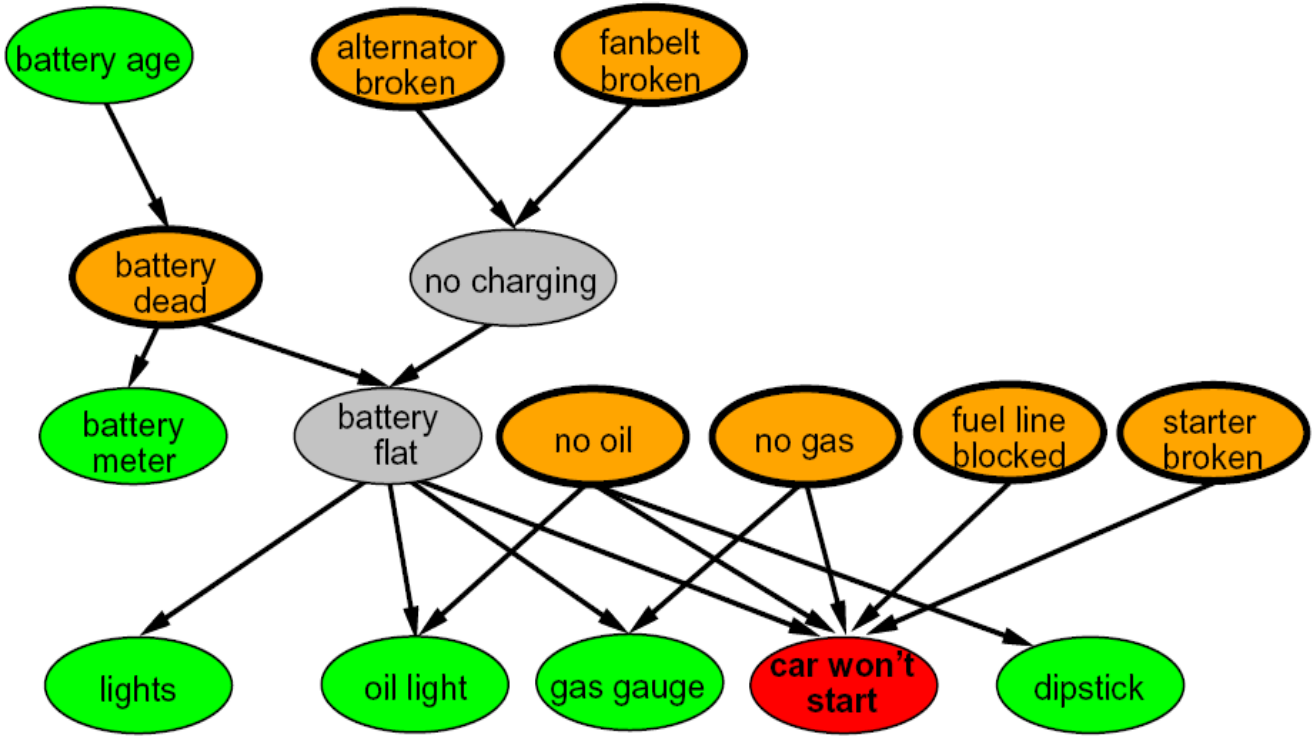




Example Bayes' Net: Car Insurance



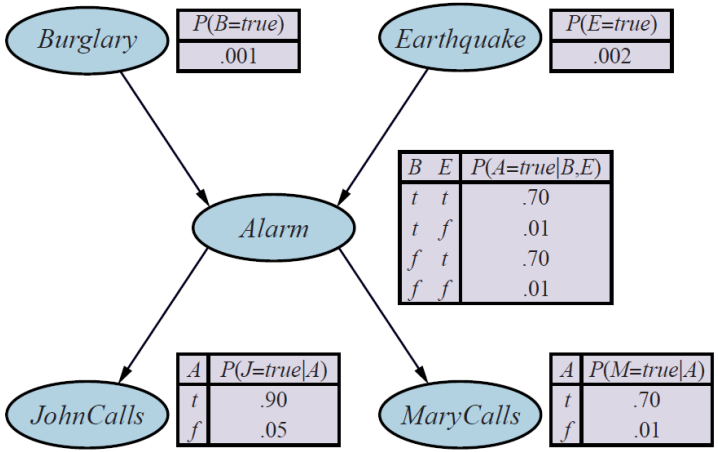
Example Bayes' Net: Car Won't Start



Bayes Net Syntax and Semantics

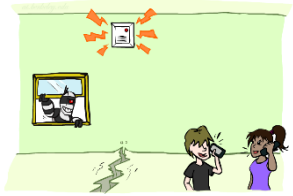
Bayes Net Syntax

- A set of nodes, one per variable X_i
- A directed, acyclic graph
- A conditional distribution for each node given its parent variables in the graph
  - CPT (conditional probability table); each row is a distribution for child given values of its parents



Bayes net = Topology (graph) + Local Conditional Probabilities

Example: Alarm Network

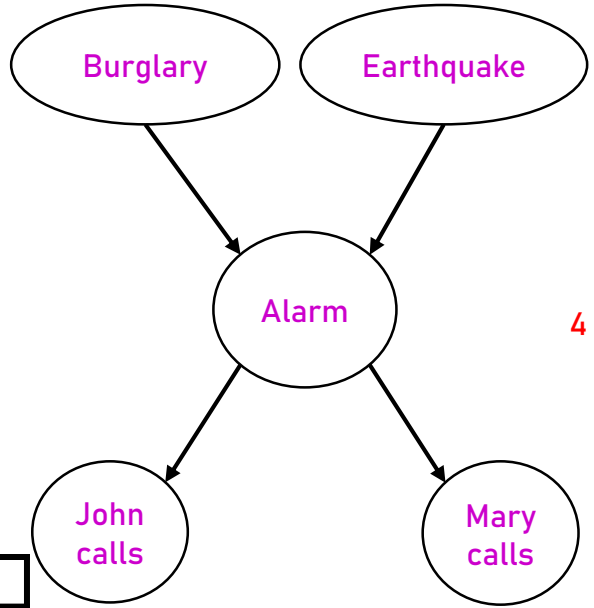


P(B)	
true	false
0.001	0.999

1

P(E)	
true	false
0.002	0.998

1



B	E	P(A|B,E)	
		true	false
true	true	0.95	0.05
true	false	0.94	0.06
false	true	0.29	0.71
false	false	0.001	0.999

4

A	P(J|A)	
	true	false
true	0.9	0.1
false	0.05	0.95

2

A	P(M|A)	
	true	false
true	0.7	0.3
false	0.01	0.99

2

Number of free parameters in each CPT:

1. Parent range sizes $d_1,\ldots,d_k$
2. Child range size d
3. Each row must sum to 1

$$(d-1)\ \Pi_i\ d_i$$

General formula for sparse BNs

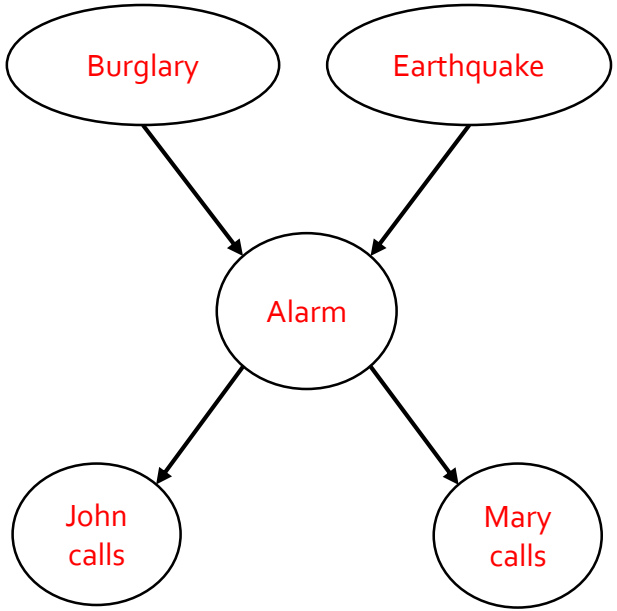
- Suppose
  - n variables
  - Maximum range size is d
  - Maximum number of parents is k
- Full joint distribution has size $O(d^n)$
- Bayes net has size $O(n \cdot d^k)$
  - Linear scaling with n as long as causal structure is local

Bayes net global semantics

- Bayes nets encode joint distributions as product of cond distributions on each variable:
  - $P(X_1,..,X_n) = \prod_i P(X_i \mid \text{Parents}(X_i))$

Example

$P(b, \neg e, a, \neg j, \neg m) =$



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A	P(J|A)	
	true	false
true	0.9	0.1
false	0.05	0.95

A	P(M|A)	
	true	false
true	0.7	0.3
false	0.01	0.99

$P(b)\ P(\neg e)\ P(a|b,\neg e)\ P(\neg j|a)\ P(\neg m|a)$

$=.001x.998x.94x.1x.3=.000028$

Conditional independence in BNs

- Compare the Bayes net global semantics

$$P(X_1,..,X_n) = \prod_i P(X_i \mid \text{Parents}(X_i))$$

with the chain rule identity

$$P(X_1,..,X_n) = \prod_i P(X_i \mid X_1,\ldots,X_{i-1})$$

Conditional independence in BNs

- Let $X_1,..,X_n$ be sorted in topological order according graph, i.e., parents before children, so

$$\text{Parents}(X_i) \subseteq X_1,\ldots,X_{i-1}$$

- So the Bayes net asserts conditional independences

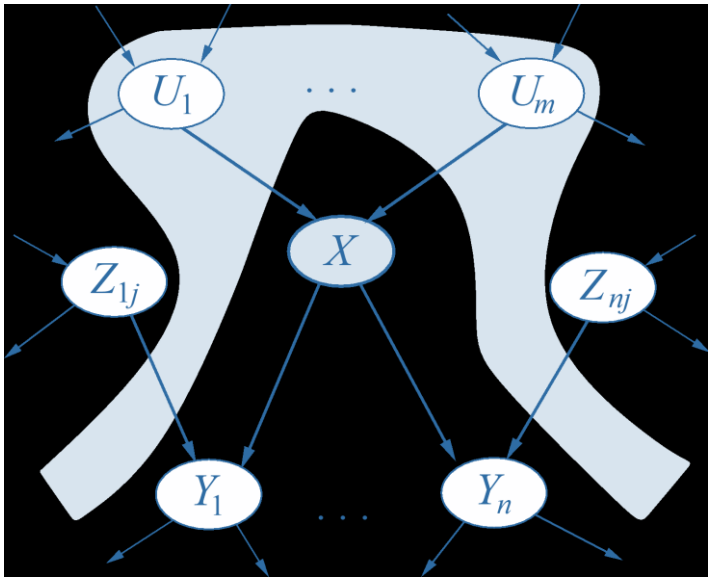
$$P(X_i \mid X_1,\ldots,X_{i-1}) = P(X_i \mid \text{Parents}(X_i))$$

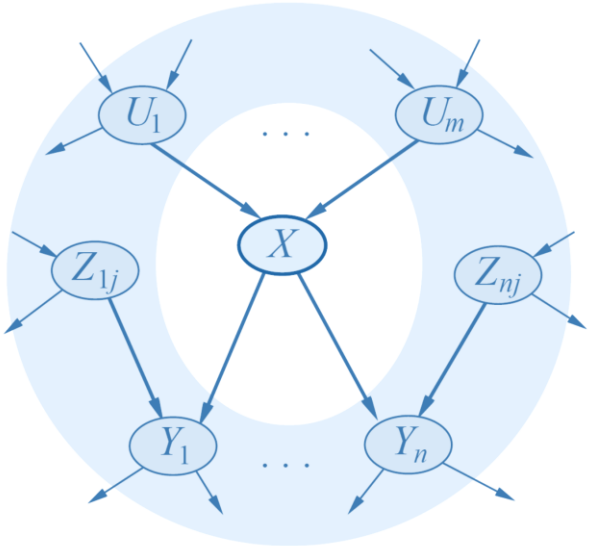
  - To ensure these are valid, choose parents for node X_i that "shield" it from other predecessors

- $P(M \mid J, A, E, B) = P(M \mid A)$

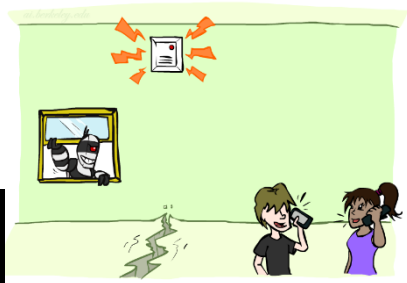
Conditional independence semantics

- Every variable is conditionally independent of its non-descendants given its parents
  - Markov blanket: parents, children and children's parents
- Conditional independence semantics <=> global semantics





Example: Burglary



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P(E)	
true	false
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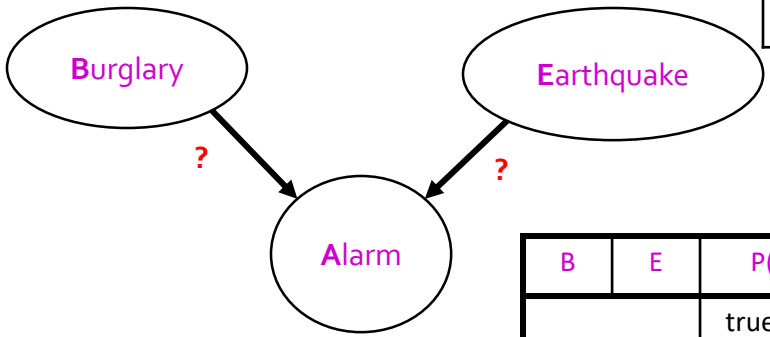
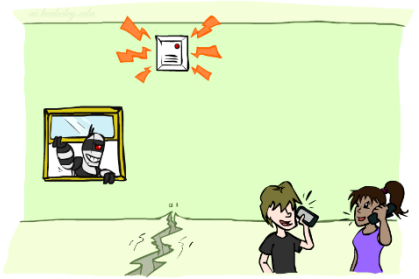


- Burglary
- Earthquake
- Alarm

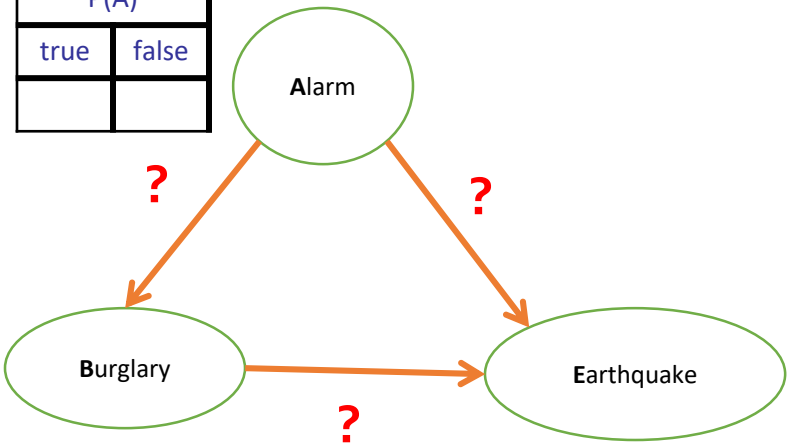
B	E	P(A|B,E)	
		true	false
true	true	0.95	0.05
true	false	0.94	0.06
false	true	0.29	0.71
false	false	0.001	0.999

Example: Burglary

- Alarm
- Burglary
- Earthquake



P(A)	
true	false



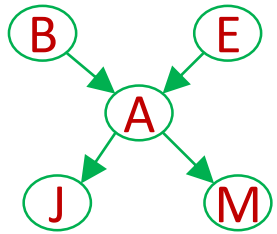
A	P(B|A)	
	true	false
true	?	
false		

A	B	P(E|A,B)	
		true	false
true	true	?	
true	false		
false	true		
false	false		

Inference by Enumeration in Bayes Net

- Reminder of inference by enumeration
  - Any probability of interest can be computed by summing entries from the joint distribution: $P(Q \mid e) = \alpha \sum_h P(Q, h, e)$
  - Entries from the joint distribution can be obtained from a BN by multiplying the corresponding conditional probabilities
- $P(B \mid j, m) = \alpha \sum_e \sum_a P(B, e, a, j, m)$

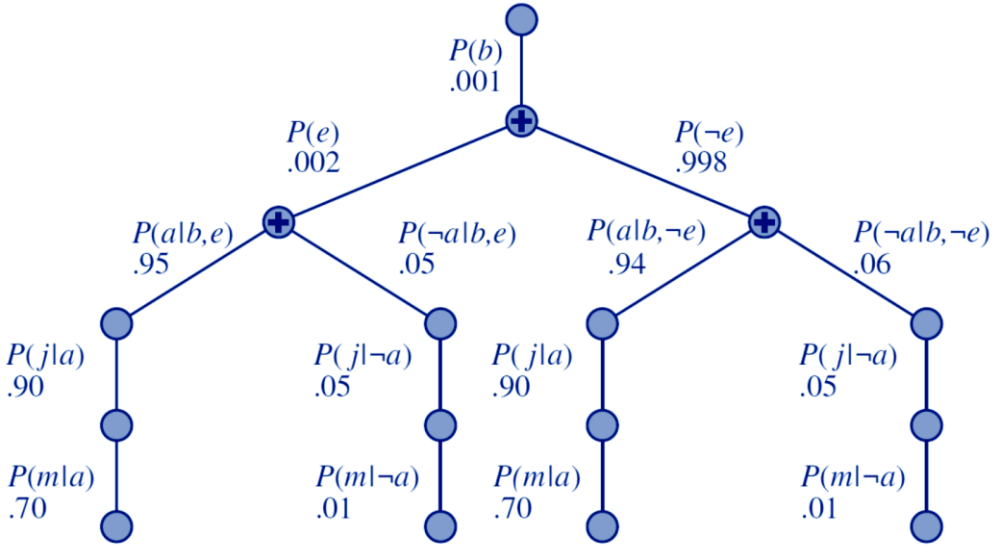
  $= \alpha \sum_e \sum_a P(B)\, P(e)\, P(a \mid B,e)\, P(j \mid a)\, P(m \mid a)$
- So inference in Bayes nets means computing sums of products of numbers: sounds easy!!
- Problem: sums of exponentially many products!

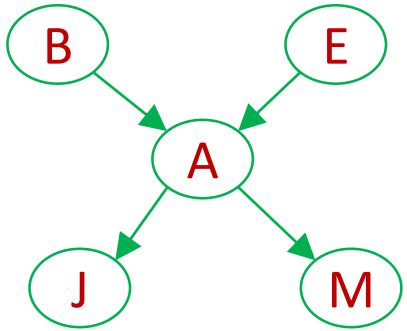


Inference by Enumeration in Bayes Net

- $P(B \mid j, m) = \alpha \sum_e \sum_a P(B, e, a, j, m)$

  $= \alpha \sum_{e,a} P(B)\, P(e)\, P(a \mid B, e)\, P(j \mid a)\, P(m \mid a)$
- $P(b \mid j, m) = \alpha * 0.00059224$, $P(\neg b \mid j, m) = \alpha * 0.0014919$
  - $P(B \mid j, m) = \langle 0.284, 0.716 \rangle$





Can we do better?

$\sum_{e,a} P(B)\ P(e)\ P(a|B,e)\ P(j|a)\ P(m|a)$

$= P(B)P(e)P(a|B,e)P(j|a)P(m|a) + P(B)P(\neg e)P(a|B,\neg e)P(j|a)P(m|a)$

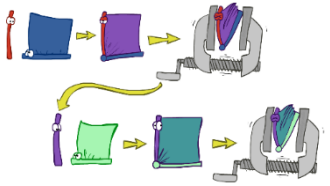
$+ P(B)P(e)P(\neg a|B,e)P(j|\neg a)P(m|\neg a) + P(B)P(\neg e)P(\neg a|B,\neg e)P(j|\neg a)P(m|\neg a)$

Lots of repeated subexpressions!

Can we do better?

- Consider uwy + uwz + uxy + uxz + vwy + vwz + vxy +vxz
  - 16 multiplies, 7 adds
  - Lots of repeated subexpressions!
- Rewrite as (u+v)(w+x)(y+z)
  - 2 multiplies, 3 adds

Variable elimination: The basic ideas

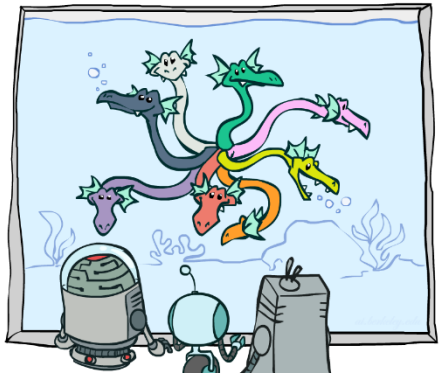


- Eliminate repeated calculations
- Move summations inwards as far as possible

$$P(B \mid j, m) = \alpha \sum_{e,a} P(B)\, P(e)\, P(a \mid B, e)\, P(j \mid a)\, P(m \mid a)$$
$$= \alpha\, P(B) \sum_{e} P(e) \sum_{a} P(a \mid B, e)\, P(j \mid a)\, P(m \mid a)$$

- Do the calculation from the inside out
  - i.e., sum over a first, then sum over e
  - Problem: P(a | B, e) isn't a single number, it's a bunch of different numbers depending on the values of B and e
  - Solution: use arrays of numbers (of various dimensions) with appropriate operations on them; these are called factors

Factor Zoo I



- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - |X| x |Y| matrix
  - Sums to 1

P(A,J)

A \ J	true	false
true	0.09	0.01
false	0.045	0.855

- Projected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for one x, all y
  - |Y|-element vector
  - Sums to P(x)

$P(a,J) = P_a(J)$

A \ J	true	false
true	0.09	0.01

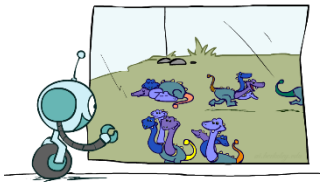
Number of random variables (Capitals) = table's dimensionality

Factor Zoo II

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1

P(J|a)

A \ J	true	false
true	0.9	0.1



- Family of conditionals:
- P(X | Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

P(J|A)

A \ J	true	false
true	0.9	0.1
false	0.05	0.95

} - P(J|a)

} - P(J|¬a)

Operation 1: Pointwise product

- First basic operation: pointwise product of factors (similar to a database join, not matrix multiply!)
  - New factor has union of variables of the two original factors
  - Each entry is the product of the corresponding entries from the original factors

$$P(J|A) \times P(A) = P(A,J)$$

- Example:

P(A)

true	0.1
false	0.9

x

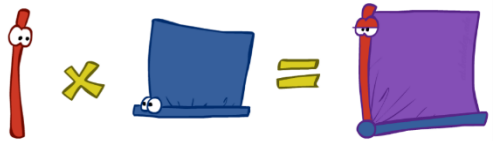
P(J|A)

A \ J	true	false
true	0.9	0.1
false	0.05	0.95

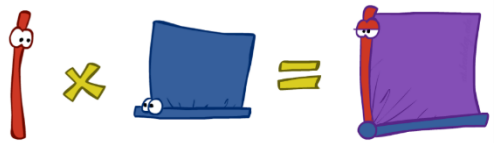
=

P(A,J)

A \ J	true	false
true	0.09	0.01
false	0.045	0.855



Operation 1: Pointwise product



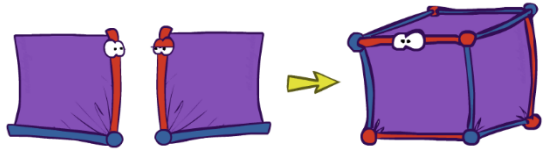
- $f(X_1, \ldots, X_j, Y_1, \ldots, Y_k) * g(Y_1, \ldots, Y_k, Z_1, \ldots, Z_l) = h(X_1, \ldots, X_j, Y_1, \ldots, Y_k, Z_1, \ldots, Z_l)$
  - If all variables are binary, the pointwise product g has 2^{j+k+l} entries

X	Y	$\mathbf{f}(X,Y)$	Y	Z	$\mathbf{g}(Y,Z)$	X	Y	Z	$\mathbf{h}(X,Y,Z)$
t	t	.3	t	t	.2	t	t	t	$.3 \times .2 = .06$
t	f	.7	t	f	.8	t	t	f	$.3 \times .8 = .24$
f	t	.9	f	t	.6	t	f	t	$.7 \times .6 = .42$
f	f	.1	f	f	.4	t	f	f	$.7 \times .4 = .28$
						f	t	t	$.9 \times .2 = .18$
						f	t	f	$.9 \times .8 = .72$
						f	f	t	$.1 \times .6 = .06$
						f	f	f	$.1 \times .4 = .04$

Figure 13.12 Illustrating pointwise multiplication: $\mathbf{f}(X,Y) \times \mathbf{g}(Y,Z) = \mathbf{h}(X,Y,Z)$.

Example: Making larger factors

- Example: P(A,J) x P(A,M) = P(A,J,M)



P(A,J)

A \ J	true	false
true	0.09	0.01
false	0.045	0.855

x

P(A,M)

A \ M	true	false
true	0.07	0.03
false	0.009	0.891

=

P(A,J,M)

J \ M	true	false
true		
false		.0003

.8 A=false

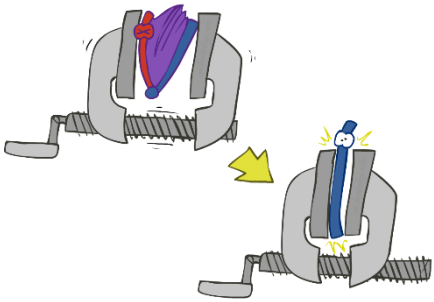
A=true

Making larger factors

- Example: P(U,V) x P(V,W) x P(W,X) = P(U,V,W,X)
- Sizes: [10,10] x [10,10] x [10,10] = [10,10,10,10]
  - i.e., 300 numbers blows up to 10,000 numbers!
  - Factor blowup can make VE very expensive

Operation 2: Summing out a variable

- Second basic operation: summing out (or eliminating) a variable from a factor
  - Shrinks a factor to a smaller one
- Example: $\sum_j P(A,J) = P(A,j) + P(A,\neg j) = P(A)$



P(A,J)

A \ J	true	false
true	0.09	0.01
false	0.045	0.855

Sum out J

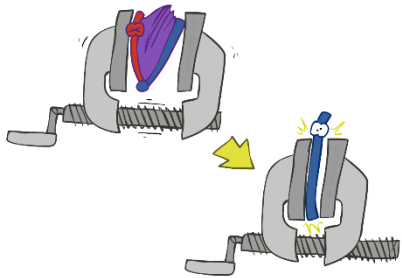
P(A)

true	0.1
false	0.9

Summing out from a product of factors

• Project the factors each way first, then sum the products

Example:



$\sum_a P(a|B,e) \times P(j|a) \times P(m|a)$

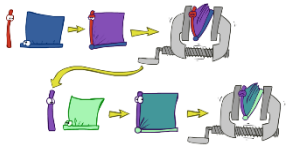
$= P(a|B,e) \times P(j|a) \times P(m|a)$

$+ P(\neg a|B,e) \times P(j|\neg a) \times P(m|\neg a)$

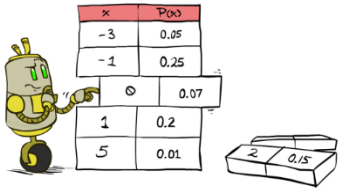
$$\mathbf{h}_2(Y,Z) = \sum_x \mathbf{h}(X,Y,Z) = \mathbf{h}(x,Y,Z) + \mathbf{h}(\neg x,Y,Z)$$

$$= \begin{pmatrix} .06 & .24 \\ .42 & .28 \end{pmatrix} + \begin{pmatrix} .18 & .72 \\ .06 & .04 \end{pmatrix} = \begin{pmatrix} .24 & .96 \\ .48 & .32 \end{pmatrix}$$

Variable Elimination



- Query: P(Q | E1=e1, …, Ek=ek)
- Start with initial factors:
  - Local CPTs (but instantiated by evidence)



- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable Hj
  - Eliminate (sum out) Hj from the product of all factors mentioning Hj

- Join all remaining factors and normalize

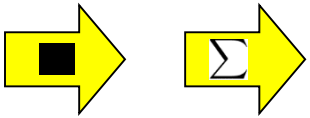
Example Query P(B | j,m)

$$\mathbf{P}(B \mid j,m) = \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$$

P(B) P(E) P(A|B,E) P(j|A) P(m|A)

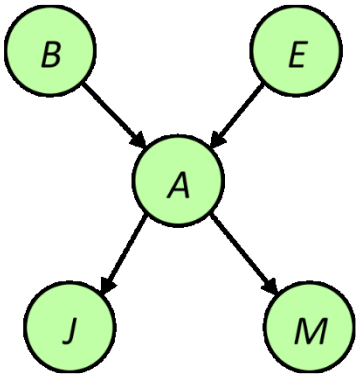
Choose A

P(A|B,E)
P(j|A)
P(m|A)

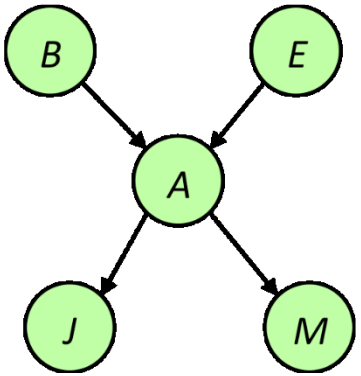


P(j,m|B,E)

P(B) P(E) P(j,m|B,E)



Example

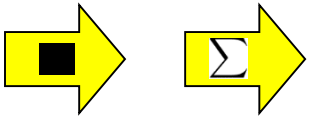


Query P(B | j,m)

P(B) P(E) P(j,m|B,E)

Choose E

P(E)
P(j,m|B,E)

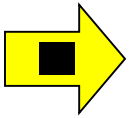


P(j,m|B)

P(B) P(j,m|B)

Finish with B

P(B)
P(j,m|B)



P(j,m,B)

P(B | j,m)

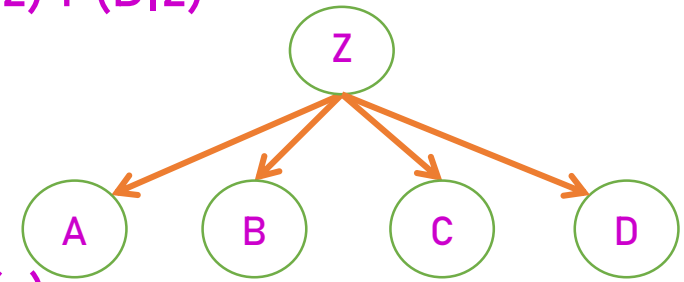
Order matters

- Order the terms Z, A, B C, D

  $P(D) = \alpha \sum_{z,a,b,c} P(z)\, P(a|z)\, P(b|z)\, P(c|z)\, P(D|z)$

  $= \alpha \sum_{z} P(z) \sum_{a} P(a|z) \sum_{b} P(b|z) \sum_{c} P(c|z)\, P(D|z)$

  - Largest factor has 2 variables (D,Z)



- Order the terms A, B C, D, Z

  $P(D) = \alpha \sum_{a,b,c,z} P(a|z)\, P(b|z)\, P(c|z)\, P(D|z)\, P(z)$

  $= \alpha \sum_{a} \sum_{b} \sum_{c} \sum_{z} P(a|z)\, P(b|z)\, P(c|z)\, P(D|z)\, P(z)$

  - Largest factor has 4 variables (A,B,C,D)
  - In general, with n leaves, factor of size 2^n

- Finding optimal ordering is intractable!

Order matters

- Exercise: P(J | b) = ?

Order matters

- Exercise: $P(J \mid b) = \alpha\, P(b) \sum_e P(e) \sum_a P(a \mid b, e)\, P(J \mid a) \sum_m P(m \mid a)$

VE: Computational and Space Complexity

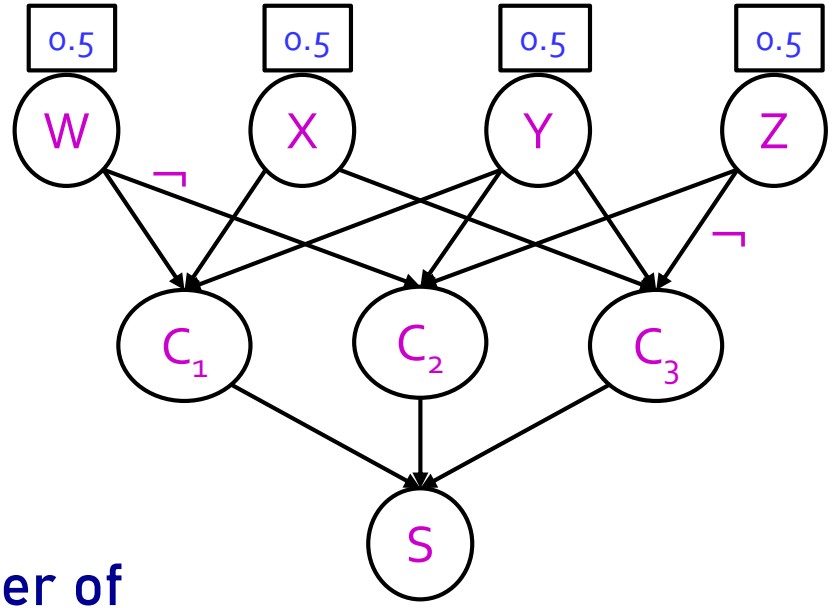
- The computational and space complexity of variable elimination is determined by the largest factor (and it's space that kills you)

- The elimination ordering can greatly affect the size of the largest factor.
  - E.g., previous slide's example 2^n vs. 2

- Does there always exist an ordering that only results in small factors?
  - No!

Worst Case Complexity? Reduction from SAT

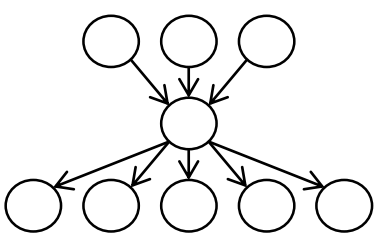
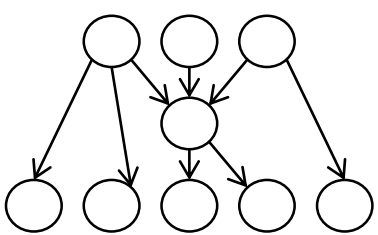
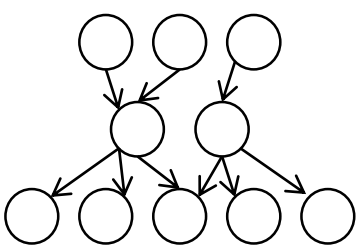
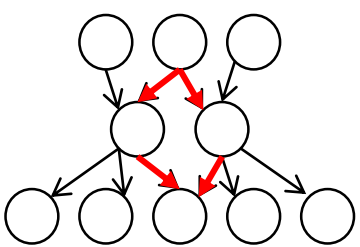
- Variables: W, X, Y, Z
- CNF clauses:
  1. $C_1 = W \vee X \vee Y$
  2. $C_2 = Y \vee Z \vee \neg W$
  3. $C_3 = X \vee Y \vee \neg Z$
- Sentence $S = C_1 \wedge C_2 \wedge C_3$
- $P(S) > 0$ iff S is satisfiable
  - => NP-hard
- $P(S) = K \times 0.5^n$ where K is the number of satisfying assignments for clauses
  - => #P-hard



Polytrees

- A polytree is a directed graph with no undirected cycles

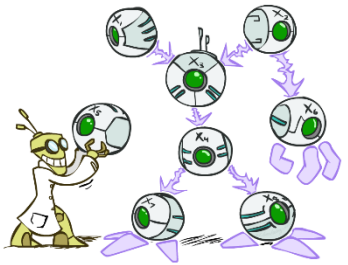
- For polytrees the complexity of variable elimination is linear in the network size if you eliminate from the leave towards the roots

Summary

- Independence and conditional independence are important forms of probabilistic knowledge
- Bayes net encode joint distributions efficiently by taking advantage of conditional independence
  - Global joint probability = product of local conditionals

- Exact inference = sums of products of conditional probabilities from the network



Next time

- Bayes nets
- Elementary inference in Bayes nets