Recitation 24

1 Review of Stoke's and Gauss's Theorems

- Stoke's theorem is the generalization of Green's theorem to an orientable 2D surface in 3D space.
- Gauss's theorem (also called the divergence theorem in 3D) is the generalization of the 2D divergence theorem to closed surfaces in 3D.

Problem 1

(Section 7.3 Problem 12 from the textbook)

Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented by a normal with nonnegative k-component. Let $\mathbf{F}(x, y, z) = x^3\mathbf{i} + e^{y^2}\mathbf{j} + ze^{xy}\mathbf{k}$ Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$. (Hint: Argue that you can integrate over a different surface.)

Problem 2

Show that for any bounded region Ω and any vector field $\mathbf{F}$

$$\oiint_{\partial\Omega} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

Problem 3

Let S_r be the sphere with radius r centered at the origin, oriented with outward normal. Suppose $\mathbf{F}$ is a vector field such that

$$\oiint_{S_r} \mathbf{F} \cdot d\mathbf{S} = \frac{1}{2}r^3 + r^4$$

Consider $\Omega = \left\{(x, y, z) \mid 25 \leq x^2 + y^2 + z^2 \leq 49\right\}$. Compute

$$\iiint_\Omega \operatorname{div} \mathbf{F} dV$$

Solution, Problem 1

The boundary of S is the ellipse $4x^2+y^2=4$ in the $z=0$ plane. By Stokes's theorem

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}=\oint_{\partial S}\mathbf{F}\cdot d\mathbf{S}=\iint_{S'}\nabla\times\mathbf{F}\cdot d\mathbf{S}$$

where S' is any piecewise smooth, orientable surface with $\partial S'=\partial S$ (subject to appropriate orientation). One computes that

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^3 & e^{y^2} & ze^{xy}\end{vmatrix}=xze^{xy}\mathbf{i}-yze^{xy}\mathbf{j}$$

This has no k-component. So let us take for S' the portion of the $z=0$ plane inside the ellipse. Hence $\mathbf{n}=\mathbf{k}$ so that

$$\begin{aligned}\iint_{S'}\nabla\times\mathbf{F}\cdot d\mathbf{S}&=\iint_{S'}(xze^{xy}\mathbf{i}-yze^{xy}\mathbf{j})\cdot\mathbf{k}dS\\&=\iint_{S'}0dS=0\end{aligned}$$

Because for any vector fields $\mathbf{F}$, divcurl $\mathbf{F}=0$, from the divergence theorem we have

$$\oiint_{\partial\Omega}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iiint_{\Omega}\operatorname{divcurl}\mathbf{F}dV=0$$

Solution, Problem 3

Let B_r be the ball with radius r. From the divergence theorem we have

$$\oiint_{S_r}\mathbf{F}\cdot d\mathbf{S}=\iiint_{B_r}\operatorname{div}\mathbf{F}dV$$

Therefore

$$\iiint_{B_r}\operatorname{div}\mathbf{F}dV=\frac{1}{2}r^3+r^4$$

Thus

$$\iiint_{\Omega}\operatorname{div}\mathbf{F}dV\cdot=\iiint_{B_7}\operatorname{div}\mathbf{F}dV-\iiint_{B_5}\operatorname{div}\mathbf{F}dV=\frac{1}{2}\left(7^3-5^3\right)+\left(7^4-5^4\right)$$