ALGEBRA

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Preface

Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it's been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Algebra or needing a refresher for Algebra. I've tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you've had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Outline

Here is a listing and brief description of the material in this set of notes.

Preliminaries – In this chapter we will do a quick review of some topics that are absolutely essential to being successful in an Algebra class. We review exponents (integer and rational), radicals, polynomials, factoring polynomials, rational expressions and complex numbers.

- **Integer Exponents** – In this section we will start looking at exponents. We will give the basic properties of exponents and illustrate some of the common mistakes students make in working with exponents. Examples in this section we will be restricted to integer exponents. Rational exponents will be discussed in the next section.
- **Rational Exponents** – In this section we will define what we mean by a rational exponent and extend the properties from the previous section to rational exponents. We will also discuss how to evaluate numbers raised to a rational exponent.
- **Radicals** – In this section we will define radical notation and relate radicals to rational exponents. We will also give the properties of radicals and some of the common mistakes students often make with radicals. We will also define simplified radical form and show how to rationalize the denominator.
- **Polynomials** – In this section we will introduce the basics of polynomials a topic that will appear throughout this course. We will define the degree of a polynomial and discuss how to add, subtract and multiply polynomials.
- **Factoring Polynomials** – In this section we look at factoring polynomials a topic that will appear in pretty much every chapter in this course and so is vital that you understand it. We will discuss factoring out the greatest common factor, factoring by grouping, factoring quadratics and factoring polynomials with degree greater than 2.
- **Rational Expressions** – In this section we will define rational expressions. We will discuss how to reduce a rational expression lowest terms and how to add, subtract, multiply and divide rational expressions.
- **Complex Numbers** – In this section we give a very quick primer on complex numbers including standard form, adding, subtracting, multiplying and dividing them.

Solving Equations and Inequalities – In this chapter we will look at one of the most important topics of the class. The ability to solve equations and inequalities is vital to surviving this class and many of the later math classes you might take. We will discuss solving linear and quadratic equations as well as applications. In addition, we will discuss solving polynomial and rational inequalities as well as absolute value equations and inequalities.

- **Solutions and Solution Sets** – In this section we introduce some of the basic notation and ideas involved in solving equations and inequalities. We define solutions for equations and inequalities and solution sets.
- **Linear Equations** – In this section we give a process for solving linear equations, including equations with rational expressions, and we illustrate the process with several examples. In addition, we discuss a subtlety involved in solving equations that students often overlook.
- **Applications of Linear Equations** – In this section we discuss a process for solving applications in general although we will focus only on linear equations here. We will work applications in pricing, distance/rate problems, work rate problems and mixing problems.

Equations With More Than One Variable – In this section we will look at solving equations with more than one variable in them. These equations will have multiple variables in them and we will be asked to solve the equation for one of the variables. This is something that we will be asked to do on a fairly regular basis.
Quadratic Equations, Part I – In this section we will start looking at solving quadratic equations. Specifically, we will concentrate on solving quadratic equations by factoring and the square root property in this section.
Quadratic Equations, Part II – In this section we will continue solving quadratic equations. We will use completing the square to solve quadratic equations in this section and use that to derive the quadratic formula. The quadratic formula is a quick way that will allow us to quickly solve any quadratic equation.
Quadratic Equations : A Summary – In this section we will summarize the topics from the last two sections. We will give a procedure for determining which method to use in solving quadratic equations and we will define the discriminant which will allow us to quickly determine what kind of solutions we will get from solving a quadratic equation.
Applications of Quadratic Equations – In this section we will revisit some of the applications we saw in the linear application section, only this time they will involve solving a quadratic equation. Included are examples in distance/rate problems and work rate problems.
Equations Reducible to Quadratic Form – Not all equations are in what we generally consider quadratic equations. However, some equations, with a proper substitution can be turned into a quadratic equation. These types of equations are called quadratic in form. In this section we will solve this type of equation.
Equations with Radicals – In this section we will discuss how to solve equations with square roots in them. As we will see we will need to be very careful with the potential solutions we get as the process used in solving these equations can lead to values that are not, in fact, solutions to the equation.
Linear Inequalities – In this section we will start solving inequalities. We will concentrate on solving linear inequalities in this section (both single and double inequalities). We will also introduce interval notation.
Polynomial Inequalities – In this section we will continue solving inequalities. However, in this section we move away from linear inequalities and move on to solving inequalities that involve polynomials of degree at least 2.
Rational Inequalities – We continue solving inequalities in this section. We now will solve inequalities that involve rational expressions, although as we'll see the process here is pretty much identical to the process used when solving inequalities with polynomials.
Absolute Value Equations – In this section we will give a geometric as well as a mathematical definition of absolute value. We will then proceed to solve equations that involve an absolute value. We will also work an example that involved two absolute values.
Absolute Value Inequalities – In this final section of the Solving chapter we will solve inequalities that involve absolute value. As we will see the process for solving inequalities with a < (*i.e.* a less than) is very different from solving an inequality with a > (*i.e.* greater than).

Graphing and Functions – In this chapter we'll look at two very important topics in an Algebra class. First, we will start discussing graphing equations by introducing the Cartesian (or Rectangular) coordinates system and illustrating use of the coordinate system to graph lines and circles. We will also formally define a function and discuss graph functions and combining functions. We will also discuss inverse functions.

Graphing – In this section we will introduce the Cartesian (or Rectangular) coordinate system. We will define/introduce ordered pairs, coordinates, quadrants, and *x* and *y*-intercepts. We will illustrate these concepts with a couple of quick examples

Lines – In this section we will discuss graphing lines. We will introduce the concept of slope and discuss how to find it from two points on the line. In addition, we will introduce the standard form of the line as well as the point-slope form and slope-intercept form of the line. We will finish off the section with a discussion on parallel and perpendicular lines.

Circles – In this section we discuss graphing circles. We introduce the standard form of the circle and show how to use completing the square to put an equation of a circle into standard form.

The Definition of a Function – In this section we will formally define relations and functions. We also give a "working definition" of a function to help understand just what a function is. We introduce function notation and work several examples illustrating how it works. We also define the domain and range of a function. In addition, we introduce piecewise functions in this section.

Graphing Functions – In this section we discuss graphing functions including several examples of graphing piecewise functions.

Combining functions – In this section we will discuss how to add, subtract, multiply and divide functions. In addition, we introduce the concept of function composition.

Inverse Functions – In this section we define one-to-one and inverse functions. We also discuss a process we can use to find an inverse function and verify that the function we get from this process is, in fact, an inverse function.

Common Graphs – In this chapter we will look at graphing some of the more common functions you might be asked to graph. We graph parabolas, ellipses, hyperbolas and rational functions in this chapter. We will also look at transformations of functions and introduce the concept of symmetry.

Lines, Circles and Piecewise Functions – This section is here only to acknowledge that we've already talked about graphing these in a previous chapter.

Parabolas – In this section we will be graphing parabolas. We introduce the vertex and axis of symmetry for a parabola and give a process for graphing parabolas. We also illustrate how to use completing the square to put the parabola into the form $f(x) = a(x-h)^2 + k$.

Ellipses – In this section we will graph ellipses. We introduce the standard form of an ellipse and how to use it to quickly graph an ellipse.

Hyperbolas – In this section we will graph hyperbolas. We introduce the standard form of a hyperbola and how to use it to quickly graph a hyperbola.

Miscellaneous Functions – In this section we will graph a couple of common functions that don't really take all that much work to do but will be needed in later sections. We'll be looking at the constant function, square root, absolute value and a simple cubic function.

Transformations – In this section we will be looking at vertical and horizontal shifts of graphs as well as reflections of graphs about the *x* and *y*-axis. Collectively these are often called transformations and if we understand them they can often be used to allow us to quickly graph some fairly complicated functions.

Symmetry – In this section we introduce the idea of symmetry. We discuss symmetry about the *x*-axis, *y*-axis and the origin and we give methods for determining what, if any symmetry, a graph will have without having to actually graph the function.

[Rational Functions] – In this section we will discuss a process for graphing rational functions. We will also introduce the ideas of vertical and horizontal asymptotes as well as how to determine if the graph of a rational function will have them.

[Polynomial Functions] – In this chapter we will take a more detailed look at polynomial functions. We will discuss dividing polynomials, finding zeroes of polynomials and sketching the graph of polynomials. We will also look at partial fractions (even though this doesn't really involve polynomial functions).

[Dividing Polynomials] – In this section we'll review some of the basics of dividing polynomials. We will define the remainder and divisor used in the division process and introduce the idea of synthetic division. We will also give the Division Algorithm.

[Zeroes/Roots of Polynomials] – In this section we'll define the zero or root of a polynomial and whether or not it is a simple root or has multiplicity *k*. We will also give the Fundamental Theorem of Algebra and The Factor Theorem as well as a couple of other useful Facts.

[Graphing Polynomials] – In this section we will give a process that will allow us to get a rough sketch of the graph of some polynomials. We discuss how to determine the behavior of the graph at *x*-intercepts and the leading coefficient test to determine the behavior of the graph as we allow *x* to increase and decrease without bound.

[Finding Zeroes of Polynomials] – As we saw in the previous section in order to sketch the graph of a polynomial we need to know what it's zeroes are. However, if we are not able to factor the polynomial we are unable to do that process. So, in this section we'll look at a process using the Rational Root Theorem that will allow us to find some of the zeroes of a polynomial and in special cases all of the zeroes.

[Partial Fractions] – In this section we will take a look at the process of partial fractions and finding the partial fraction decomposition of a rational expression. What we will be asking here is what "smaller" rational expressions did we add and/or subtract to get the given rational expression. This is a process that has a lot of uses in some later math classes. It can show up in Calculus and Differential Equations for example.

[Exponential and Logarithm Functions] – In this chapter we will introduce two very important functions in many areas : the exponential and logarithm functions. We will look at their basic properties, applications and solving equations involving the two functions. If you are in a field that takes you into the sciences or engineering then you will be running into both of these functions.

[Exponential Functions] – In this section we will introduce exponential functions. We will give some of the basic properties and graphs of exponential functions. We will also discuss what many people consider to be the exponential function, $f\left(x\right)=\mathbf{e}^{x}$.

[Logarithm Functions] – In this section we will introduce logarithm functions. We give the basic properties and graphs of logarithm functions. In addition, we discuss how to evaluate some basic logarithms including the use of the change of base formula. We will also discuss the common logarithm, $\log\left(x\right)$, and the natural logarithm, $\ln\left(x\right)$.

[Solving Exponential Equations] – In this section we will discuss a couple of methods for solving equations that contain exponentials.

[Solving Logarithm Equations] – In this section we will discuss a couple of methods for solving equations that contain logarithms. Also, as we'll see, with one of the methods we will need to be careful of the results of the method as it is always possible that the method gives values that are, in fact, not solutions to the equation.

Applications – In this section we will look at a couple of applications of exponential functions and an application of logarithms. We look at compound interest, exponential growth and decay and earthquake intensity.

Systems of Equations – In this chapter we will take a look at solving systems of equations. We will solve linear as well as nonlinear systems of equations. We will also take a quick look at using augmented matrices to solve linear systems of equations.

Linear Systems with Two Variables – In this section we will solve systems of two equations and two variables. We will use the method of substitution and method of elimination to solve the systems in this section. We will also introduce the concepts of inconsistent systems of equations and dependent systems of equations.

Linear Systems with Three Variables – In this section we will work a couple of quick examples illustrating how to use the method of substitution and method of elimination introduced in the previous section as they apply to systems of three equations.

Augmented Matrices – In this section we will look at another method for solving systems. We will introduce the concept of an augmented matrix. This will allow us to use the method of Gauss-Jordan elimination to solve systems of equations. We will use the method with systems of two equations and systems of three equations.

More on the Augmented Matrix – In this section we will revisit the cases of inconsistent and dependent solutions to systems and how to identify them using the augmented matrix method.

Nonlinear Systems – In this section we will take a quick look at solving nonlinear systems of equations. A nonlinear system of equations is a system in which at least one of the equations is not linear, *i.e.* has degree of two or more. Note as well that the discussion here does not cover all the possible solution methods for nonlinear systems. Solving nonlinear systems is often a much more involved process than solving linear systems.

Chapter 1 : Preliminaries

The purpose of this chapter is to review several topics that will arise time and again throughout this material. Many of the topics here are so important to an Algebra class that if you don't have a good working grasp of them you will find it very difficult to successfully complete the course. Also, it is assumed that you've seen the topics in this chapter somewhere prior to this class and so this chapter should be mostly a review for you. However, since most of these topics are so important to an Algebra class we will make sure that you do understand them by doing a quick review of them here.

Exponents and polynomials are integral parts of any Algebra class. If you do not remember the basic exponent rules and how to work with polynomials you will find it very difficult, if not impossible, to pass an Algebra class. This is especially true with factoring polynomials. There are more than a few sections in an Algebra course where the ability to factor is absolutely essential to being able to do the work in those sections. In fact, in many of these sections factoring will be the first step taken.

It is important that you leave this chapter with a good understanding of this material! If you don't understand this material you will find it difficult to get through the remaining chapters. Here is a brief listing of the material covered in this chapter.

Integer Exponents – In this section we will start looking at exponents. We will give the basic properties of exponents and illustrate some of the common mistakes students make in working with exponents. Examples in this section we will be restricted to integer exponents. Rational exponents will be discussed in the next section.
Rational Exponents – In this section we will define what we mean by a rational exponent and extend the properties from the previous section to rational exponents. We will also discuss how to evaluate numbers raised to a rational exponent.
Radicals – In this section we will define radical notation and relate radicals to rational exponents. We will also give the properties of radicals and some of the common mistakes students often make with radicals. We will also define simplified radical form and show how to rationalize the denominator.
Polynomials – In this section we will introduce the basics of polynomials a topic that will appear throughout this course. We will define the degree of a polynomial and discuss how to add, subtract and multiply polynomials.
Factoring Polynomials – In this section we look at factoring polynomials a topic that will appear in pretty much every chapter in this course and so is vital that you understand it. We will discuss factoring out the greatest common factor, factoring by grouping, factoring quadratics and factoring polynomials with degree greater than 2.
Rational Expressions – In this section we will define rational expressions. We will discuss how to reduce a rational expression lowest terms and how to add, subtract, multiply and divide rational expressions.
Complex Numbers – In this section we give a very quick primer on complex numbers including standard form, adding, subtracting, multiplying and dividing them.

Section 1-1 : Integer Exponents

We will start off this chapter by looking at integer exponents. In fact, we will initially assume that the exponents are positive as well. We will look at zero and negative exponents in a bit.

Let's first recall the definition of exponentiation with positive integer exponents. If *a* is any number and *n* is a positive integer then,

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

So, for example,

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$$

We should also use this opportunity to remind ourselves about parenthesis and conventions that we have in regard to exponentiation and parenthesis. This will be particularly important when dealing with negative numbers. Consider the following two cases.

$$\left(-2\right)^4 \qquad \text{and} \qquad -2^4$$

These will have different values once we evaluate them. When performing exponentiation remember that it is only the quantity that is immediately to the left of the exponent that gets the power.

In the first case there is a parenthesis immediately to the left so that means that everything in the parenthesis gets the power. So, in this case we get,

$$\left(-2\right)^4 = \left(-2\right)\left(-2\right)\left(-2\right)\left(-2\right) = 16$$

In the second case however, the 2 is immediately to the left of the exponent and so it is only the 2 that gets the power. The minus sign will stay out in front and will NOT get the power. In this case we have the following,

$$-2^4 = -\left(2^4\right) = -\left(2 \cdot 2 \cdot 2 \cdot 2\right) = -\left(16\right) = -16$$

We put in some extra parenthesis to help illustrate this case. In general, they aren't included and we would write instead,

$$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$$

The point of this discussion is to make sure that you pay attention to parenthesis. They are important and ignoring parenthesis or putting in a set of parenthesis where they don't belong can completely change the answer to a problem. Be careful. Also, this warning about parenthesis is not just intended for exponents. We will need to be careful with parenthesis throughout this course.

Now, let's take care of zero exponents and negative integer exponents. In the case of zero exponents we have,

$$a^0 = 1 \qquad \text{provided } a \neq 0$$

Notice that it is required that *a* not be zero. This is important since 0^0 is not defined. Here is a quick example of this property.

$$\left(-1268\right)^0 = 1$$

We have the following definition for negative exponents. If *a* is any non-zero number and *n* is a positive integer (yes, positive) then,

$$a^{-n} = \frac{1}{a^n}$$

Can you see why we required that *a* not be zero? Remember that division by zero is not defined and if we had allowed *a* to be zero we would have gotten division by zero. Here are a couple of quick examples for this definition,

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25} \qquad\qquad (-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64} = -\frac{1}{64}$$

Here are some of the main properties of integer exponents. Accompanying each property will be a quick example to illustrate its use. We will be looking at more complicated examples after the properties.

Properties

1. $a^n a^m = a^{n+m}$ Example : $a^{-9}a^4 = a^{-9+4} = a^{-5}$

2. $\left(a^n\right)^m = a^{nm}$ Example : $\left(a^7\right)^3 = a^{(7)(3)} = a^{21}$

3. $\dfrac{a^n}{a^m} = \begin{cases} a^{n-m} \\ \dfrac{1}{a^{m-n}} \end{cases}, \quad a \neq 0$ Example : $\dfrac{a^4}{a^{11}} = a^{4-11} = a^{-7}$, $\dfrac{a^4}{a^{11}} = \dfrac{1}{a^{11-4}} = \dfrac{1}{a^7} = a^{-7}$

4. $(ab)^n = a^n b^n$ Example : $(ab)^{-4} = a^{-4}b^{-4}$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$ Example : $\left(\dfrac{a}{b}\right)^8 = \dfrac{a^8}{b^8}$

6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$ Example : $\left(\dfrac{a}{b}\right)^{-10} = \left(\dfrac{b}{a}\right)^{10} = \dfrac{b^{10}}{a^{10}}$

7. $(ab)^{-n} = \dfrac{1}{(ab)^n}$ Example : $(ab)^{-20} = \dfrac{1}{(ab)^{20}}$

8. $\dfrac{1}{a^{-n}} = a^n$ Example : $\dfrac{1}{a^{-2}} = a^2$

9. $\dfrac{a^{-n}}{b^{-m}}=\dfrac{b^m}{a^n}$	Example : $\dfrac{a^{-6}}{b^{-17}}=\dfrac{b^{17}}{a^6}$
10. $\left(a^n b^m\right)^k = a^{nk}b^{mk}$	Example : $\left(a^4 b^{-9}\right)^3 = a^{(4)(3)}b^{(-9)(3)} = a^{12}b^{-27}$
11. $\left(\dfrac{a^n}{b^m}\right)^k=\dfrac{a^{nk}}{b^{mk}}$	Example : $\left(\dfrac{a^6}{b^5}\right)^2=\dfrac{a^{(6)(2)}}{b^{(5)(2)}}=\dfrac{a^{12}}{b^{10}}$

Notice that there are two possible forms for the third property. Which form you use is usually dependent upon the form you want the answer to be in.

Note as well that many of these properties were given with only two terms/factors but they can be extended out to as many terms/factors as we need. For example, property 4 can be extended as follows.

$$\left(abcd\right)^n = a^n b^n c^n d^n$$

We only used four factors here, but hopefully you get the point. Property 4 (and most of the other properties) can be extended out to meet the number of factors that we have in a given problem.

There are several common mistakes that students make with these properties the first time they see them. Let's take a look at a couple of them.

Consider the following case.

$$\begin{aligned} \text{Correct} &: \quad ab^{-2} = a\frac{1}{b^2}=\frac{a}{b^2} \\ \text{Incorrect} &: \quad ab^{-2} \neq \frac{1}{ab^2} \end{aligned}$$

In this case only the *b* gets the exponent since it is immediately off to the left of the exponent and so only this term moves to the denominator. Do NOT carry the *a* down to the denominator with the *b*. Contrast this with the following case.

$$\left(ab\right)^{-2} = \frac{1}{\left(ab\right)^2}$$

In this case the exponent is on the set of parenthesis and so we can just use property 7 on it and so both the *a* and the *b* move down to the denominator. Again, note the importance of parenthesis and how they can change an answer!

Here is another common mistake.

$$\begin{aligned} \text{Correct} &: \quad \frac{1}{3a^{-5}}=\frac{1}{3}\frac{1}{a^{-5}}=\frac{1}{3}a^5 \\ \text{Incorrect} &: \quad \frac{1}{3a^{-5}} \neq 3a^5 \end{aligned}$$

In this case the exponent is only on the *a* and so to use property 8 on this we would have to break up the fraction as shown and then use property 8 only on the second term. To bring the 3 up with the *a* we would have needed the following.

$$\frac{1}{(3a)^{-5}} = (3a)^5$$

Once again, notice this common mistake comes down to being careful with parenthesis. This will be a constant refrain throughout these notes. We must always be careful with parenthesis. Misusing them can lead to incorrect answers.

Let's take a look at some more complicated examples now.

Example 1 Simplify each of the following and write the answers with only positive exponents.

(a) $\left(4x^{-4}y^5\right)^3$

(b) $\left(-10z^2y^{-4}\right)^2\left(z^3y\right)^{-5}$

(c) $\dfrac{n^{-2}m}{7m^{-4}n^{-3}}$

(d) $\dfrac{5x^{-1}y^{-4}}{\left(3y^5\right)^{-2}x^9}$

(e) $\left(\dfrac{z^{-5}}{z^{-2}x^{-1}}\right)^6$

(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$

Solution

Note that when we say "simplify" in the problem statement we mean that we will need to use all the properties that we can to get the answer into the required form. Also, a "simplified" answer will have as few terms as possible and each term should have no more than a single exponent on it.

There are many different paths that we can take to get to the final answer for each of these. In the end the answer will be the same regardless of the path that you used to get the answer. All that this means for you is that as long as you used the properties you can take the path that you find the easiest. The path that others find to be the easiest may not be the path that you find to be the easiest. That is okay.

Also, we won't put quite as much detail in using some of these properties as we did in the examples given with each property. For instance, we won't show the actual multiplications anymore, we will just give the result of the multiplication.

(a) $\left(4x^{-4}y^{5}\right)^{3}$

For this one we will use property 10 first.

$$\left(4x^{-4}y^{5}\right)^{3} = 4^{3}x^{-12}y^{15}$$

Don’t forget to put the exponent on the constant in this problem. That is one of the more common mistakes that students make with these simplification problems.

At this point we need to evaluate the first term and eliminate the negative exponent on the second term. The evaluation of the first term isn’t too bad and all we need to do to eliminate the negative exponent on the second term is use the definition we gave for negative exponents.

$$\left(4x^{-4}y^{5}\right)^{3} = 64\left(\frac{1}{x^{12}}\right)y^{15} = \frac{64y^{15}}{x^{12}}$$

We further simplified our answer by combining everything up into a single fraction. This should always be done.

The middle step in this part is usually skipped. All the definition of negative exponents tells us to do is move the term to the denominator and drop the minus sign in the exponent. So, from this point on, that is what we will do without writing in the middle step.

(b) $\left(-10z^{2}y^{-4}\right)^{2}\left(z^{3}y\right)^{-5}$

In this case we will first use property 10 on both terms and then we will combine the terms using property 1. Finally, we will eliminate the negative exponents using the definition of negative exponents.

$$\left(-10z^{2}y^{-4}\right)^{2}\left(z^{3}y\right)^{-5} = \left(-10\right)^{2}z^{4}y^{-8}z^{-15}y^{-5} = 100z^{-11}y^{-13} = \frac{100}{z^{11}y^{13}}$$

There are a couple of things to be careful with in this problem. First, when using the property 10 on the first term, make sure that you square the “-10” and not just the 10 (*i.e.* don’t forget the minus sign…). Second, in the final step, the 100 stays in the numerator since there is no negative exponent on it. The exponent of “-11” is only on the *z* and so only the *z* moves to the denominator.

(c) $\dfrac{n^{-2}m}{7m^{-4}n^{-3}}$

This one isn’t too bad. We will use the definition of negative exponents to move all terms with negative exponents in them to the denominator. Also, property 8 simply says that if there is a term with a negative exponent in the denominator then we will just move it to the numerator and drop the minus sign.

So, let’s take care of the negative exponents first.

$$\frac{n^{-2}m}{7m^{-4}n^{-3}} = \frac{m^{4}n^{3}m}{7n^{2}}$$

Now simplify. We will use property 1 to combine the *m*'s in the numerator. We will use property 3 to combine the *n*'s and since we are looking for positive exponents we will use the first form of this property since that will put a positive exponent up in the numerator.

$$\frac{n^{-2}m}{7m^{-4}n^{-3}} = \frac{m^5 n}{7}$$

Again, the 7 will stay in the denominator since there isn't a negative exponent on it. It will NOT move up to the numerator with the *m*. Do not get excited if all the terms move up to the numerator or if all the terms move down to the denominator. That will happen on occasion.

(d) $\dfrac{5x^{-1}y^{-4}}{\left(3y^5\right)^{-2}x^9}$

This example is similar to the previous one except there is a little more going on with this one. The first step will be to again, get rid of the negative exponents as we did in the previous example. Any terms in the numerator with negative exponents will get moved to the denominator and we'll drop the minus sign in the exponent. Likewise, any terms in the denominator with negative exponents will move to the numerator and we'll drop the minus sign in the exponent. Notice this time, unlike the previous part, there is a term with a set of parenthesis in the denominator. Because of the parenthesis that whole term, including the 3, will move to the numerator.

Here is the work for this part.

$$\frac{5x^{-1}y^{-4}}{\left(3y^5\right)^{-2}x^9} = \frac{5\left(3y^5\right)^2}{xy^4x^9} = \frac{5(9)y^{10}}{xy^4x^9} = \frac{45y^6}{x^{10}}$$

(e) $\left(\dfrac{z^{-5}}{z^{-2}x^{-1}}\right)^6$

There are several first steps that we can take with this one. The first step that we're pretty much always going to take with these kinds of problems is to first simplify the fraction inside the parenthesis as much as possible. After we do that we will use property 5 to deal with the exponent that is on the parenthesis.

$$\left(\frac{z^{-5}}{z^{-2}x^{-1}}\right)^6 = \left(\frac{z^2x^1}{z^5}\right)^6 = \left(\frac{x}{z^3}\right)^6 = \frac{x^6}{z^{18}}$$

In this case we used the second form of property 3 to simplify the *z*'s since this put a positive exponent in the denominator. Also note that we almost never write an exponent of "1". When we have exponents of 1 we will drop them.

(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$

This one is very similar to the previous part. The main difference is negative on the outer exponent. We will deal with that once we've simplified the fraction inside the parenthesis.

$$\left(\frac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2} = \left(\frac{4a^3a^5}{b^8b}\right)^{-2} = \left(\frac{4a^8}{b^9}\right)^{-2}$$

Now at this point we can use property 6 to deal with the exponent on the parenthesis. Doing this gives us,

$$\left(\frac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2} = \left(\frac{b^9}{4a^8}\right)^{2} = \frac{b^{18}}{16a^{16}}$$

Before leaving this section we need to talk briefly about the requirement of positive only exponents in the above set of examples. This was done only so there would be a consistent final answer. In many cases negative exponents are okay and in some cases they are required. In fact, if you are on a track that will take you into calculus there are a fair number of problems in a calculus class in which negative exponents are the preferred, if not required, form.

Section 1-2 : Rational Exponents

Now that we have looked at integer exponents we need to start looking at more complicated exponents. In this section we are going to be looking at rational exponents. That is exponents in the form

$$b^{\frac{m}{n}}$$

where both *m* and *n* are integers.

We will start simple by looking at the following special case,

$$b^{\frac{1}{n}}$$

where *n* is an integer. Once we have this figured out the more general case given above will actually be pretty easy to deal with.

Let's first define just what we mean by exponents of this form.

$$a = b^{\frac{1}{n}} \qquad \text{is equivalent to} \qquad a^n = b$$

In other words, when evaluating $b^{\frac{1}{n}}$ we are really asking what number (in this case *a*) did we raise to the *n* to get *b*. Often $b^{\frac{1}{n}}$ is called the **n^th^ root of b**.

Let's do a couple of evaluations.

Example 1 Evaluate each of the following.

(a) $25^{\frac{1}{2}}$

(b) $32^{\frac{1}{5}}$

(c) $81^{\frac{1}{4}}$

(d) $\left(-8\right)^{\frac{1}{3}}$

(e) $\left(-16\right)^{\frac{1}{4}}$

(f) $-16^{\frac{1}{4}}$

Solution

When doing these evaluations, we will not actually do them directly. When first confronted with these kinds of evaluations doing them directly is often very difficult. In order to evaluate these we will remember the equivalence given in the definition and use that instead.

We will work the first one in detail and then not put as much detail into the rest of the problems.

(a) $25^{\frac{1}{2}}$

So, here is what we are asking in this problem.

$$25^{\frac{1}{2}} = ?$$

Using the equivalence from the definition we can rewrite this as,

$$?^2 = 25$$

So, all that we are really asking here is what number did we square to get 25. In this case that is (hopefully) easy to get. We square 5 to get 25. Therefore,

$$25^{\frac{1}{2}} = 5$$

(b) $32^{\frac{1}{5}}$

So what we are asking here is what number did we raise to the 5th power to get 32?

$$32^{\frac{1}{5}} = 2 \qquad \text{because} \qquad 2^5 = 32$$

(c) $81^{\frac{1}{4}}$

What number did we raise to the 4th power to get 81?

$$81^{\frac{1}{4}} = 3 \qquad \text{because} \qquad 3^4 = 81$$

(d) $(-8)^{\frac{1}{3}}$

We need to be a little careful with minus signs here, but other than that it works the same way as the previous parts. What number did we raise to the 3rd power (*i.e.* cube) to get -8?

$$(-8)^{\frac{1}{3}} = -2 \qquad \text{because} \qquad (-2)^3 = -8$$

(e) $(-16)^{\frac{1}{4}}$

This part does not have an answer. It is here to make a point. In this case we are asking what number do we raise to the 4th power to get -16. However, we also know that raising any number (positive or negative) to an even power will be positive. In other words, there is no real number that we can raise to the 4th power to get -16.

Note that this is different from the previous part. If we raise a negative number to an odd power we will get a negative number so we could do the evaluation in the previous part.

As this part has shown, we can't always do these evaluations.

(f) $-16^{\frac{1}{4}}$

Again, this part is here to make a point more than anything. Unlike the previous part this one has an answer. Recall from the previous section that if there aren't any parentheses then only the part immediately to the left of the exponent gets the exponent. So, this part is really asking us to evaluate the following term.

$$-16^{\frac{1}{4}} = -\left(16^{\frac{1}{4}}\right)$$

So, we need to determine what number raised to the 4th power will give us 16. This is 2 and so in this case the answer is,

$$-16^{\frac{1}{4}} = -\left(16^{\frac{1}{4}}\right) = -(2) = -2$$

As the last two parts of the previous example has once again shown, we really need to be careful with parenthesis. In this case parenthesis makes the difference between being able to get an answer or not.

Also, don't be worried if you didn't know some of these powers off the top of your head. They are usually fairly simple to determine if you don't know them right away. For instance, in the part b we needed to determine what number raised to the 5 will give 32. If you can't see the power right off the top of your head simply start taking powers until you find the correct one. In other words compute 2^5, 3^5, 4^5 until you reach the correct value. Of course, in this case we wouldn't need to go past the first computation.

The next thing that we should acknowledge is that all of the properties for exponents that we gave in the previous section are still valid for all rational exponents. This includes the more general rational exponent that we haven't looked at yet.

Now that we know that the properties are still valid we can see how to deal with the more general rational exponent. There are in fact two different ways of dealing with them as we'll see. Both methods involve using property 2 from the previous section. For reference purposes this property is,

$$\left(a^n\right)^m = a^{nm}$$

So, let's see how to deal with a general rational exponent. We will first rewrite the exponent as follows.

$$b^{\frac{m}{n}} = b^{\left(\frac{1}{n}\right)(m)}$$

In other words, we can think of the exponent as a product of two numbers. Now we will use the exponent property shown above. However, we will be using it in the opposite direction than what we did in the previous section. Also, there are two ways to do it. Here they are,

$$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m \qquad \text{OR} \qquad b^{\frac{m}{n}} = \left(b^m\right)^{\frac{1}{n}}$$

Using either of these forms we can now evaluate some more complicated expressions

Example 2 Evaluate each of the following.

(a) $8^{\frac{2}{3}}$

(b) $625^{\frac{3}{4}}$

(c) $\left(\dfrac{243}{32}\right)^{\frac{4}{5}}$

Solution

We can use either form to do the evaluations. However, it is usually more convenient to use the first form as we will see.

(a) $8^{\frac{2}{3}}$

Let's use both forms here since neither one is too bad in this case. Let's take a look at the first form.

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = (2)^2 = 4 \qquad\qquad 8^{\frac{1}{3}} = 2 \text{ because } 2^3 = 8$$

Now, let's take a look at the second form.

$$8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4 \qquad\qquad 64^{\frac{1}{3}} = 4 \text{ because } 4^3 = 64$$

So, we get the same answer regardless of the form. Notice however that when we used the second form we ended up taking the 3rd root of a much larger number which can cause problems on occasion.

(b) $625^{\frac{3}{4}}$

Again, let's use both forms to compute this one.

$$625^{\frac{3}{4}} = \left(625^{\frac{1}{4}}\right)^3 = (5)^3 = 125 \qquad\qquad 625^{\frac{1}{4}} = 5 \text{ because } 5^4 = 625$$

$$625^{\frac{3}{4}} = \left(625^3\right)^{\frac{1}{4}} = (244140625)^{\frac{1}{4}} = 125 \qquad\qquad \text{because } 125^4 = 244140625$$

As this part has shown the second form can be quite difficult to use in computations. The root in this case was not an obvious root and not particularly easy to get if you didn't know it right off the top of your head.

(c) $\left(\dfrac{243}{32}\right)^{\frac{4}{5}}$

In this case we'll only use the first form. However, before doing that we'll need to first use property 5 of our exponent properties to get the exponent onto the numerator and denominator.

$$\left(\frac{243}{32}\right)^{\frac{4}{5}} = \frac{243^{\frac{4}{5}}}{32^{\frac{4}{5}}} = \frac{\left(243^{\frac{1}{5}}\right)^4}{\left(32^{\frac{1}{5}}\right)^4} = \frac{(3)^4}{(2)^4} = \frac{81}{16}$$

We can also do some of the simplification type problems with rational exponents that we saw in the previous section.

Example 3 Simplify each of the following and write the answers with only positive exponents.

(a) $\left(\dfrac{w^{-2}}{16v^{\frac{1}{2}}}\right)^{\frac{1}{4}}$

(b) $\left(\dfrac{x^2 y^{-\frac{2}{3}}}{x^{-\frac{1}{2}} y^{-3}}\right)^{-\frac{1}{7}}$

Solution

(a) For this problem we will first move the exponent into the parenthesis then we will eliminate the negative exponent as we did in the previous section. We will then move the term to the denominator and drop the minus sign.

$$\frac{w^{-2\left(\frac{1}{4}\right)}}{16^{\frac{1}{4}} v^{\frac{1}{2}\left(\frac{1}{4}\right)}} = \frac{w^{-\frac{1}{2}}}{2v^{\frac{1}{8}}} = \frac{1}{2v^{\frac{1}{8}} w^{\frac{1}{2}}}$$

(b) In this case we will first simplify the expression inside the parenthesis.

$$\left(\frac{x^2 y^{-\frac{2}{3}}}{x^{-\frac{1}{2}} y^{-3}}\right)^{-\frac{1}{7}} = \left(\frac{x^2 x^{\frac{1}{2}} y^3}{y^{\frac{2}{3}}}\right)^{-\frac{1}{7}} = \left(\frac{x^{2+\frac{1}{2}} y^{3-\frac{2}{3}}}{1}\right)^{-\frac{1}{7}} = \left(x^{\frac{5}{2}} y^{\frac{7}{3}}\right)^{-\frac{1}{7}}$$

Don't worry if, after simplification, we don't have a fraction anymore. That will happen on occasion. Now we will eliminate the negative in the exponent using property 7 and then we'll use property 4 to finish the problem up.

$$\left(\frac{x^2 y^{-\frac{2}{3}}}{x^{-\frac{1}{2}} y^{-3}}\right)^{-\frac{1}{7}} = \frac{1}{\left(x^{\frac{5}{2}} y^{\frac{7}{3}}\right)^{\frac{1}{7}}} = \frac{1}{x^{\frac{5}{14}} y^{\frac{1}{3}}}$$

We will leave this section with a warning about a common mistake that students make in regard to negative exponents and rational exponents. Be careful not to confuse the two as they are totally separate topics.

In other words,

$$b^{-n} = \frac{1}{b^n}$$

and NOT

$$b^{-n} \neq b^{\frac{1}{n}}$$

This is a very common mistake when students first learn exponent rules.

Section 1-3 : Radicals

We'll open this section with the definition of the radical. If *n* is a positive integer that is greater than 1 and *a* is a real number then,

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

where *n* is called the **index**, *a* is called the **radicand**, and the symbol $\sqrt{\ }$ is called the **radical**. The left side of this equation is often called the radical form and the right side is often called the exponent form.

From this definition we can see that a radical is simply another notation for the first rational exponent that we looked at in the rational exponents section.

Note as well that the index is required in these to make sure that we correctly evaluate the radical. There is one exception to this rule and that is square root. For square roots we have,

$$\sqrt[2]{a} = \sqrt{a}$$

In other words, for square roots we typically drop the index.

Let's do a couple of examples to familiarize us with this new notation.

Example 1 Write each of the following radicals in exponent form.

(a) $\sqrt[4]{16}$

(b) $\sqrt[10]{8x}$

(c) $\sqrt{x^2 + y^2}$

Solution

(a) $\sqrt[4]{16} = 16^{\frac{1}{4}}$

(b) $\sqrt[10]{8x} = \left(8x\right)^{\frac{1}{10}}$

(c) $\sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{\frac{1}{2}}$

As seen in the last two parts of this example we need to be careful with parenthesis. When we convert to exponent form and the radicand consists of more than one term then we need to enclose the whole radicand in parenthesis as we did with these two parts. To see why this is consider the following,

$$8x^{\frac{1}{10}}$$

From our discussion of exponents in the previous sections we know that only the term immediately to the left of the exponent actually gets the exponent. Therefore, the radical form of this is,

$$8x^{\frac{1}{10}} = 8\sqrt[10]{x} \neq \sqrt[10]{8x}$$

So, we once again see that parenthesis are very important in this class. Be careful with them.

Since we know how to evaluate rational exponents we also know how to evaluate radicals as the following set of examples shows.

Example 2 Evaluate each of the following.

(a) $\sqrt{16}$ and $\sqrt[4]{16}$

(b) $\sqrt[5]{243}$

(c) $\sqrt[4]{1296}$

(d) $\sqrt[3]{-125}$

(e) $\sqrt[4]{-16}$

Solution

To evaluate these we will first convert them to exponent form and then evaluate that since we already know how to do that.

(a) These are together to make a point about the importance of the index in this notation. Let's take a look at both of these.

$$\sqrt{16} = 16^{\frac{1}{2}} = 4 \qquad \text{because } 4^2 = 16$$
$$\sqrt[4]{16} = 16^{\frac{1}{4}} = 2 \qquad \text{because } 2^4 = 16$$

So, the index is important. Different indexes will give different evaluations so make sure that you don't drop the index unless it is a 2 (and hence we're using square roots).

(b) $\sqrt[5]{243} = 243^{\frac{1}{5}} = 3 \qquad \text{because } 3^5 = 243$

(c) $\sqrt[4]{1296} = 1296^{\frac{1}{4}} = 6 \qquad \text{because } 6^4 = 1296$

(d) $\sqrt[3]{-125} = \left(-125\right)^{\frac{1}{3}} = -5 \qquad \text{because } \left(-5\right)^3 = -125$

(e) $\sqrt[4]{-16} = \left(-16\right)^{\frac{1}{4}}$

As we saw in the integer exponent section this does not have a real answer and so we can't evaluate the radical of a negative number if the index is even. Note however that we can evaluate the radical of a negative number if the index is odd as the previous part shows.

Let's briefly discuss the answer to the first part in the above example. In this part we made the claim that $\sqrt{16} = 4$ because $4^2 = 16$. However, 4 isn't the only number that we can square to get 16. We also have $\left(-4\right)^2 = 16$. So, why didn't we use -4 instead? There is a general rule about evaluating square roots (or more generally radicals with even indexes). When evaluating square roots we ALWAYS take the positive answer. If we want the negative answer we will do the following.

$$-\sqrt{16} = -4$$

This may not seem to be all that important, but in later topics this can be very important. Following this convention means that we will always get predictable values when evaluating roots.

Note that we don't have a similar rule for radicals with odd indexes such as the cube root in part (d) above. This is because there will never be more than one possible answer for a radical with an odd index.

We can also write the general rational exponent in terms of radicals as follows.

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \qquad \text{OR} \qquad a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

We now need to talk about some properties of radicals.

Properties

If n is a positive integer greater than 1 and both a and b are positive real numbers then,

1. $\sqrt[n]{a^n} = a$
2. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
3. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Note that on occasion we can allow a or b to be negative and still have these properties work. When we run across those situations we will acknowledge them. However, for the remainder of this section we will assume that a and b must be positive.

Also note that while we can "break up" products and quotients under a radical we can't do the same thing for sums or differences. In other words,

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b} \qquad \text{AND} \qquad \sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$$

If you aren't sure that you believe this consider the following quick number example.

$$5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

If we "break up" the root into the sum of the two pieces we clearly get different answers! So, be careful to not make this very common mistake!

We are going to be simplifying radicals shortly so we should next define **simplified radical form**. A radical is said to be in simplified radical form (or just simplified form) if each of the following are true.

Simplified Radial Form

1. All exponents in the radicand must be less than the index.
2. Any exponents in the radicand can have no factors in common with the index.
3. No fractions appear under a radical.
4. No radicals appear in the denominator of a fraction.

In our first set of simplification examples we will only look at the first two. We will need to do a little more work before we can deal with the last two.

Example 3 Simplify each of the following. Assume that *x*, *y*, and *z* are positive.

(a) $\sqrt{y^7}$

(b) $\sqrt[9]{x^6}$

(c) $\sqrt{18x^6y^{11}}$

(d) $\sqrt[4]{32x^9y^5z^{12}}$

(e) $\sqrt[5]{x^{12}y^4z^{24}}$

(f) $\sqrt[3]{9x^2}\,\sqrt[3]{6x^2}$

Solution

(a) $\sqrt{y^7}$

In this case the exponent (7) is larger than the index (2) and so the first rule for simplification is violated. To fix this we will use the first and second properties of radicals above. So, let's note that we can write the radicand as follows.

$$y^7 = y^6 y = \left(y^3\right)^2 y$$

So, we've got the radicand written as a perfect square times a term whose exponent is smaller than the index. The radical then becomes,

$$\sqrt{y^7} = \sqrt{\left(y^3\right)^2 y}$$

Now use the second property of radicals to break up the radical and then use the first property of radicals on the first term.

$$\sqrt{y^7} = \sqrt{\left(y^3\right)^2}\,\sqrt{y} = y^3\sqrt{y}$$

This now satisfies the rules for simplification and so we are done.

Before moving on let's briefly discuss how we figured out how to break up the exponent as we did. To do this we noted that the index was 2. We then determined the largest multiple of 2 that is less than 7, the exponent on the radicand. This is 6. Next, we noticed that 7=6+1.

Finally, remembering several rules of exponents we can rewrite the radicand as,

$$y^7 = y^6 y = y^{(3)(2)} y = \left(y^3\right)^2 y$$

In the remaining examples we will typically jump straight to the final form of this and leave the details to you to check.

(b) $\sqrt[9]{x^6}$

This radical violates the second simplification rule since both the index and the exponent have a common factor of 3. To fix this all we need to do is convert the radical to exponent form do some simplification and then convert back to radical form.

$$\sqrt[9]{x^6} = \left(x^6\right)^{\frac{1}{9}} = x^{\frac{6}{9}} = x^{\frac{2}{3}} = \left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$$

(c) $\sqrt{18x^6y^{11}}$

Now that we've got a couple of basic problems out of the way let's work some harder ones. Although, with that said, this one is really nothing more than an extension of the first example.

There is more than one term here but everything works in exactly the same fashion. We will break the radicand up into perfect squares times terms whose exponents are less than 2 (*i.e.* 1).

$$18x^6y^{11} = 9x^6y^{10}\left(2y\right) = 9\left(x^3\right)^2\left(y^5\right)^2\left(2y\right)$$

Don't forget to look for perfect squares in the number as well.

Now, go back to the radical and then use the second and first property of radicals as we did in the first example.

$$\sqrt{18x^6y^{11}} = \sqrt{9\left(x^3\right)^2\left(y^5\right)^2\left(2y\right)} = \sqrt{9}\sqrt{\left(x^3\right)^2}\sqrt{\left(y^5\right)^2}\sqrt{2y} = 3x^3y^5\sqrt{2y}$$

Note that we used the fact that the second property can be expanded out to as many terms as we have in the product under the radical. Also, don't get excited that there are no x's under the radical in the final answer. This will happen on occasion.

(d) $\sqrt[4]{32x^9y^5z^{12}}$

This one is similar to the previous part except the index is now a 4. So, instead of get perfect squares we want powers of 4. This time we will combine the work in the previous part into one step.

$$\sqrt[4]{32x^9y^5z^{12}} = \sqrt[4]{16x^8y^4z^{12}\left(2xy\right)} = \sqrt[4]{16}\sqrt[4]{\left(x^2\right)^4}\sqrt[4]{y^4}\sqrt[4]{\left(z^3\right)^4}\sqrt[4]{2xy} = 2x^2y\,z^3\sqrt[4]{2xy}$$

(e) $\sqrt[5]{x^{12}y^4z^{24}}$

Again this one is similar to the previous two parts.

$$\sqrt[5]{x^{12}y^4z^{24}} = \sqrt[5]{x^{10}z^{20}\left(x^2y^4z^4\right)} = \sqrt[5]{\left(x^2\right)^5}\sqrt[5]{\left(z^4\right)^5}\sqrt[5]{x^2y^4z^4} = x^2z^4\sqrt[5]{x^2y^4z^4}$$

In this case don't get excited about the fact that all the y's stayed under the radical. That will happen on occasion.

(f) $\sqrt[3]{9x^2}\,\sqrt[3]{6x^2}$

This last part seems a little tricky. Individually both of the radicals are in simplified form. However, there is often an unspoken rule for simplification. The unspoken rule is that we should have as few radicals in the problem as possible. In this case that means that we can use the second property of radicals to combine the two radicals into one radical and then we'll see if there is any simplification that needs to be done.

$$\sqrt[3]{9x^2}\,\sqrt[3]{6x^2} = \sqrt[3]{\left(9x^2\right)\left(6x^2\right)} = \sqrt[3]{54x^4}$$

Now that it's in this form we can do some simplification.

$$\sqrt[3]{9x^2}\,\sqrt[3]{6x^2} = \sqrt[3]{27x^3\left(2x\right)} = \sqrt[3]{27x^3}\,\sqrt[3]{2x} = 3x\sqrt[3]{2x}$$

Before moving into a set of examples illustrating the last two simplification rules we need to talk briefly about adding/subtracting/multiplying radicals. Performing these operations with radicals is much the same as performing these operations with polynomials. If you don't remember how to add/subtract/multiply polynomials we will give a quick reminder here and then give a more in depth set of examples the next section.

Recall that to add/subtract terms with *x* in them all we need to do is add/subtract the coefficients of the *x*. For example,

$$4x + 9x = \left(4+9\right)x = 13x \qquad\qquad 3x - 11x = \left(3-11\right)x = -8x$$

Adding/subtracting radicals works in exactly the same manner. For instance,

$$4\sqrt{x} + 9\sqrt{x} = \left(4+9\right)\sqrt{x} = 13\sqrt{x} \qquad 3\,\sqrt[10]{5} - 11\,\sqrt[10]{5} = \left(3-11\right)\sqrt[10]{5} = -8\,\sqrt[10]{5}$$

We've already seen some multiplication of radicals in the last part of the previous example. If we are looking at the product of two radicals with the same index then all we need to do is use the second property of radicals to combine them then simplify. What we need to look at now are problems like the following set of examples.

Example 4 Multiply each of the following. Assume that *x* is positive.

(a) $\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right)$

(b) $\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right)$

(c) $\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)$

Solution

In all of these problems all we need to do is recall how to FOIL binomials. Recall,

$$\left(3x-5\right)\left(x+2\right) = 3x\left(x\right) + 3x\left(2\right) - 5\left(x\right) - 5\left(2\right) = 3x^2 + 6x - 5x - 10 = 3x^2 + x - 10$$

With radicals we multiply in exactly the same manner. The main difference is that on occasion we'll need to do some simplification after doing the multiplication

(a) $\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right)$

$$\begin{aligned}\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right) &= \sqrt{x}\left(\sqrt{x}\right)-5\sqrt{x}+2\sqrt{x}-10\\ &= \sqrt{x^2}-3\sqrt{x}-10\\ &= x-3\sqrt{x}-10\end{aligned}$$

As noted above we did need to do a little simplification on the first term after doing the multiplication.

(b) $\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right)$

Don't get excited about the fact that there are two variables here. It works the same way!

$$\begin{aligned}\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right) &= 6\sqrt{x^2}-15\sqrt{x}\sqrt{y}-2\sqrt{x}\sqrt{y}+5\sqrt{y^2}\\ &= 6x-15\sqrt{xy}-2\sqrt{xy}+5y\\ &= 6x-17\sqrt{xy}+5y\end{aligned}$$

Again, notice that we combined up the terms with two radicals in them.

(c) $\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)$

Not much to do with this one.

$$\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)=25\sqrt{x^2}-10\sqrt{x}+10\sqrt{x}-4=25x-4$$

Notice that, in this case, the answer has no radicals. That will happen on occasion so don't get excited about it when it happens.

The last part of the previous example really used the fact that

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

If you don't recall this formula we will look at it in a little more detail in the next section.

Okay, we are now ready to take a look at some simplification examples illustrating the final two rules. Note as well that the fourth rule says that we shouldn't have any radicals in the denominator. To get rid of them we will use some of the multiplication ideas that we looked at above and the process of getting rid of the radicals in the denominator is called **rationalizing the denominator**. In fact, that is really what this next set of examples is about. They are really more examples of rationalizing the denominator rather than simplification examples.

Example 5 Rationalize the denominator for each of the following. Assume that *x* is positive.

(a) $\dfrac{4}{\sqrt{x}}$

(b) $\sqrt[5]{\dfrac{2}{x^3}}$

(c) $\dfrac{1}{3-\sqrt{x}}$

(d) $\dfrac{5}{4\sqrt{x}+\sqrt{3}}$

Solution

There are really two different types of problems that we'll be seeing here. The first two parts illustrate the first type of problem and the final two parts illustrate the second type of problem. Both types are worked differently.

(a) $\dfrac{4}{\sqrt{x}}$

In this case we are going to make use of the fact that $\sqrt[n]{a^n} = a$. We need to determine what to multiply the denominator by so that this will show up in the denominator. Once we figure this out we will multiply the numerator and denominator by this term.

Here is the work for this part.

$$\frac{4}{\sqrt{x}} = \frac{4}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{4\sqrt{x}}{\sqrt{x^2}} = \frac{4\sqrt{x}}{x}$$

Remember that if we multiply the denominator by a term we must also multiply the numerator by the same term. In this way we are really multiplying the term by 1 (since $\dfrac{a}{a}=1$) and so aren't changing its value in any way.

(b) $\sqrt[5]{\dfrac{2}{x^3}}$

We'll need to start this one off with first using the third property of radicals to eliminate the fraction from underneath the radical as is required for simplification.

$$\sqrt[5]{\frac{2}{x^3}} = \frac{\sqrt[5]{2}}{\sqrt[5]{x^3}}$$

Now, in order to get rid of the radical in the denominator we need the exponent on the *x* to be a 5. This means that we need to multiply by $\sqrt[5]{x^2}$ so let's do that.

$$\sqrt[5]{\frac{2}{x^3}} = \frac{\sqrt[5]{2}}{\sqrt[5]{x^3}}\frac{\sqrt[5]{x^2}}{\sqrt[5]{x^2}} = \frac{\sqrt[5]{2x^2}}{\sqrt[5]{x^5}} = \frac{\sqrt[5]{2x^2}}{x}$$

(c) $\dfrac{1}{3-\sqrt{x}}$

In this case we can't do the same thing that we did in the previous two parts. To do this one we will need to instead to make use of the fact that

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

When the denominator consists of two terms with at least one of the terms involving a radical we will do the following to get rid of the radical.

$$\frac{1}{3-\sqrt{x}}=\frac{1}{\left(3-\sqrt{x}\right)}\frac{3+\sqrt{x}}{\left(3+\sqrt{x}\right)}=\frac{3+\sqrt{x}}{\left(3-\sqrt{x}\right)\left(3+\sqrt{x}\right)}=\frac{3+\sqrt{x}}{9-x}$$

So, we took the original denominator and changed the sign on the second term and multiplied the numerator and denominator by this new term. By doing this we were able to eliminate the radical in the denominator when we then multiplied out.

(d) $\dfrac{5}{4\sqrt{x}+\sqrt{3}}$

This one works exactly the same as the previous example. The only difference is that both terms in the denominator now have radicals. The process is the same however.

$$\frac{5}{4\sqrt{x}+\sqrt{3}}=\frac{5}{\left(4\sqrt{x}+\sqrt{3}\right)}\frac{\left(4\sqrt{x}-\sqrt{3}\right)}{\left(4\sqrt{x}-\sqrt{3}\right)}=\frac{5\left(4\sqrt{x}-\sqrt{3}\right)}{\left(4\sqrt{x}+\sqrt{3}\right)\left(4\sqrt{x}-\sqrt{3}\right)}=\frac{5\left(4\sqrt{x}-\sqrt{3}\right)}{16x-3}$$

Rationalizing the denominator may seem to have no real uses and to be honest we won't see many uses in an Algebra class. However, if you are on a track that will take you into a Calculus class you will find that rationalizing is useful on occasion at that level.

We will close out this section with a more general version of the first property of radicals. Recall that when we first wrote down the properties of radicals we required that *a* be a positive number. This was done to make the work in this section a little easier. However, with the first property that doesn't necessarily need to be the case.

Here is the property for a general *a* (*i.e.* positive or negative)

$$\sqrt[n]{a^n}=\begin{cases}\left|a\right| & \text{if } n \text{ is even}\\ a & \text{if } n \text{ is odd}\end{cases}$$

where $\left|a\right|$ is the absolute value of *a*. If you don't recall absolute value we will cover that in detail in a section in the next chapter. All that you need to do is know at this point is that absolute value always makes *a* a positive number.

So, as a quick example this means that,

$$\sqrt[8]{x^8} = |x| \qquad \text{AND} \qquad \sqrt[11]{x^{11}} = x$$

For square roots this is,

$$\sqrt{x^2} = |x|$$

This will not be something we need to worry all that much about here, but again there are topics in courses after an Algebra course for which this is an important idea so we needed to at least acknowledge it.

Section 1-4 : Polynomials

In this section we will start looking at polynomials. Polynomials will show up in pretty much every section of every chapter in the remainder of this material and so it is important that you understand them.

We will start off with **polynomials in one variable**. Polynomials in one variable are algebraic expressions that consist of terms in the form ax^n where *n* is a non-negative (*i.e.* positive or zero) integer and *a* is a real number and is called the **coefficient** of the term. The **degree** of a polynomial in one variable is the largest exponent in the polynomial.

Note that we will often drop the "in one variable" part and just say polynomial.

Here are examples of polynomials and their degrees.

$$\begin{array}{ll} 5x^{12} - 2x^6 + x^5 - 198x + 1 & \text{degree : } 12 \\ x^4 - x^3 + x^2 - x + 1 & \text{degree : } 4 \\ 56x^{23} & \text{degree : } 23 \\ 5x - 7 & \text{degree : } 1 \\ -8 & \text{degree : } 0 \end{array}$$

So, a polynomial doesn't have to contain all powers of *x* as we see in the first example. Also, polynomials can consist of a single term as we see in the third and fifth example.

We should probably discuss the final example a little more. This really is a polynomial even it may not look like one. Remember that a polynomial is any algebraic expression that consists of terms in the form ax^n. Another way to write the last example is

$$-8x^0$$

Written in this way makes it clear that the exponent on the *x* is a zero (this also explains the degree…) and so we can see that it really is a polynomial in one variable.

Here are some examples of things that aren't polynomials.

$$4x^6 + 15x^{-8} + 1$$

$$5\sqrt{x} - x + x^2$$

$$\frac{2}{x} + x^3 - 2$$

The first one isn't a polynomial because it has a negative exponent and all exponents in a polynomial must be positive.

To see why the second one isn't a polynomial let's rewrite it a little.

$$5\sqrt{x} - x + x^2 = 5x^{\frac{1}{2}} - x + x^2$$

By converting the root to exponent form we see that there is a rational root in the algebraic expression. All the exponents in the algebraic expression must be non-negative integers in order for the algebraic expression to be a polynomial. As a general rule of thumb if an algebraic expression has a radical in it then it isn't a polynomial.

Let's also rewrite the third one to see why it isn't a polynomial.

$$\frac{2}{x}+x^3-2=2x^{-1}+x^3-2$$

So, this algebraic expression really has a negative exponent in it and we know that isn't allowed. Another rule of thumb is if there are any variables in the denominator of a fraction then the algebraic expression isn't a polynomial.

Note that this doesn't mean that radicals and fractions aren't allowed in polynomials. They just can't involve the variables. For instance, the following is a polynomial

$$\sqrt[3]{5}\,x^4-\frac{7}{12}x^2+\frac{1}{\sqrt{8}}x-5\sqrt[14]{113}$$

There are lots of radicals and fractions in this algebraic expression, but the denominators of the fractions are only numbers and the radicands of each radical are only a numbers. Each *x* in the algebraic expression appears in the numerator and the exponent is a positive (or zero) integer. Therefore this is a polynomial.

Next, let's take a quick look at **polynomials in two variables**. Polynomials in two variables are algebraic expressions consisting of terms in the form $ax^n y^m$. The degree of each term in a polynomial in two variables is the sum of the exponents in each term and the **degree** of the polynomial is the largest such sum.

Here are some examples of polynomials in two variables and their degrees.

$$x^2y-6x^3y^{12}+10x^2-7y+1 \qquad \text{degree : } 15$$
$$6x^4+8y^4-xy^2 \qquad \text{degree : } 4$$
$$x^4y^2-x^3y^3-xy+x^4 \qquad \text{degree : } 6$$
$$6x^{14}-10y^3+3x-11y \qquad \text{degree : } 14$$

In these kinds of polynomials not every term needs to have both *x*'s and *y*'s in them, in fact as we see in the last example they don't need to have any terms that contain both *x*'s and *y*'s. Also, the degree of the polynomial may come from terms involving only one variable. Note as well that multiple terms may have the same degree.

We can also talk about polynomials in three variables, or four variables or as many variables as we need. The vast majority of the polynomials that we'll see in this course are polynomials in one variable and so most of the examples in the remainder of this section will be polynomials in one variable.

Next, we need to get some terminology out of the way. A **monomial** is a polynomial that consists of exactly one term. A **binomial** is a polynomial that consists of exactly two terms. Finally, a **trinomial** is a polynomial that consists of exactly three terms. We will use these terms off and on so you should probably be at least somewhat familiar with them.

Now we need to talk about adding, subtracting and multiplying polynomials. You'll note that we left out division of polynomials. That will be discussed in a later section where we will use division of polynomials quite often.

Before actually starting this discussion we need to recall the distributive law. This will be used repeatedly in the remainder of this section. Here is the distributive law.

$$a\left(b+c\right)=ab+ac$$

We will start with adding and subtracting polynomials. This is probably best done with a couple of examples.

Example 1 Perform the indicated operation for each of the following.

(a) Add $6x^5-10x^2+x-45$ to $13x^2-9x+4$.

(b) Subtract $5x^3-9x^2+x-3$ from x^2+x+1.

Solution

(a) Add $6x^5-10x^2+x-45$ to $13x^2-9x+4$.

The first thing that we should do is actually write down the operation that we are being asked to do.

$$\left(6x^5-10x^2+x-45\right)+\left(13x^2-9x+4\right)$$

In this case the parenthesis are not required since we are adding the two polynomials. They are there simply to make clear the operation that we are performing. To add two polynomials all that we do is **combine like terms**. This means that for each term with the same exponent we will add or subtract the coefficient of that term.

In this case this is,

$$\begin{aligned}\left(6x^5-10x^2+x-45\right)+\left(13x^2-9x+4\right)&=6x^5+\left(-10+13\right)x^2+\left(1-9\right)x-45+4\\&=6x^5+3x^2-8x-41\end{aligned}$$

(b) Subtract $5x^3-9x^2+x-3$ from x^2+x+1.

Again, let's write down the operation we are doing here. We will also need to be very careful with the order that we write things down in. Here is the operation

$$x^2+x+1-\left(5x^3-9x^2+x-3\right)$$

This time the parentheses around the second term are absolutely required. We are subtracting the whole polynomial and the parenthesis must be there to make sure we are in fact subtracting the whole polynomial.

In doing the subtraction the first thing that we'll do is **distribute the minus sign** through the parenthesis. This means that we will change the sign on every term in the second polynomial. Note that all we are really doing here is multiplying a "-1" through the second polynomial using the distributive law. After distributing the minus through the parenthesis we again combine like terms.

Here is the work for this problem.

$$\begin{aligned} x^2 + x + 1 - \left(5x^3 - 9x^2 + x - 3\right) &= x^2 + x + 1 - 5x^3 + 9x^2 - x + 3 \\ &= -5x^3 + 10x^2 + 4 \end{aligned}$$

Note that sometimes a term will completely drop out after combing like terms as the *x* did here. This will happen on occasion so don't get excited about it when it does happen.

Now let's move onto multiplying polynomials. Again, it's best to do these in an example.

Example 2 Multiply each of the following.

(a) $4x^2\left(x^2 - 6x + 2\right)$

(b) $\left(3x+5\right)\left(x-10\right)$

(c) $\left(4x^2 - x\right)\left(6-3x\right)$

(d) $\left(3x+7y\right)\left(x-2y\right)$

(e) $\left(2x+3\right)\left(x^2 - x + 1\right)$

Solution

(a) $4x^2\left(x^2 - 6x + 2\right)$

This one is nothing more than a quick application of the distributive law.

$$4x^2\left(x^2 - 6x + 2\right) = 4x^4 - 24x^3 + 8x^2$$

(b)

$\left(3x+5\right)\left(x-10\right)$ This one will use the FOIL method for multiplying these two binomials.

$$\left(3x+5\right)\left(x-10\right) = \underbrace{3x^2}_{\text{First Terms}} - \underbrace{30x}_{\text{Outer Terms}} + \underbrace{5x}_{\text{Inner Terms}} - \underbrace{50}_{\text{Last Terms}} = 3x^2 - 25x - 50$$

Recall that the FOIL method will only work when multiplying two binomials. If either of the polynomials isn't a binomial then the FOIL method won't work.

Also note that all we are really doing here is multiplying every term in the second polynomial by every term in the first polynomial. The FOIL acronym is simply a convenient way to remember this.

(c) $\left(4x^2 - x\right)\left(6-3x\right)$

Again, we will just FOIL this one out.

$$\left(4x^2 - x\right)\left(6-3x\right) = 24x^2 - 12x^3 - 6x + 3x^2 = -12x^3 + 27x^2 - 6x$$

(d) $\left(3x+7y\right)\left(x-2y\right)$

We can still FOIL binomials that involve more than one variable so don't get excited about these kinds of problems when they arise.

$$\left(3x+7y\right)\left(x-2y\right) = 3x^2 - 6xy + 7xy - 14y^2 = 3x^2 + xy - 14y^2$$

(e) $\left(2x+3\right)\left(x^2-x+1\right)$

In this case the FOIL method won't work since the second polynomial isn't a binomial. Recall however that the FOIL acronym was just a way to remember that we multiply every term in the second polynomial by every term in the first polynomial.

That is all that we need to do here.

$$\left(2x+3\right)\left(x^2-x+1\right)=2x^3-2x^2+2x+3x^2-3x+3=2x^3+x^2-x+3$$

Let's work another set of examples that will illustrate some nice formulas for some special products. We will give the formulas after the example.

Example 3 Multiply each of the following.

(a) $\left(3x+5\right)\left(3x-5\right)$

(b) $\left(2x+6\right)^2$

(c) $\left(1-7x\right)^2$

(d) $4\left(x+3\right)^2$

Solution

(a) $\left(3x+5\right)\left(3x-5\right)$

We can use FOIL on this one so let's do that.

$$\left(3x+5\right)\left(3x-5\right)=9x^2-15x+15x-25=9x^2-25$$

In this case the middle terms drop out.

(b) $\left(2x+6\right)^2$

Now recall that $4^2=\left(4\right)\left(4\right)=16$. Squaring with polynomials works the same way. So in this case we have,

$$\left(2x+6\right)^2=\left(2x+6\right)\left(2x+6\right)=4x^2+12x+12x+36=4x^2+24x+36$$

(c) $\left(1-7x\right)^2$

This one is nearly identical to the previous part.

$$\left(1-7x\right)^2=\left(1-7x\right)\left(1-7x\right)=1-7x-7x+49x^2=1-14x+49x^2$$

(d) $4\left(x+3\right)^2$

This part is here to remind us that we need to be careful with coefficients. When we've got a coefficient we MUST do the exponentiation first and then multiply the coefficient.

$$4\left(x+3\right)^2=4\left(x+3\right)\left(x+3\right)=4\left(x^2+6x+9\right)=4x^2+24x+36$$

You can only multiply a coefficient through a set of parenthesis if there is an exponent of "1" on the parenthesis. If there is any other exponent then you CAN'T multiply the coefficient through the parenthesis.

Just to illustrate the point.

$$4\left(x+3\right)^2 \neq \left(4x+12\right)^2 = \left(4x+12\right)\left(4x+12\right) = 16x^2 + 96x + 144$$

This is clearly not the same as the correct answer so be careful!

The parts of this example all use one of the following special products.

$$\left(a+b\right)\left(a-b\right) = a^2 - b^2$$

$$\left(a+b\right)^2 = a^2 + 2ab + b^2$$

$$\left(a-b\right)^2 = a^2 - 2ab + b^2$$

Be careful to not make the following mistakes!

$$\left(a+b\right)^2 \neq a^2 + b^2$$

$$\left(a-b\right)^2 \neq a^2 - b^2$$

These are very common mistakes that students often make when they first start learning how to multiply polynomials.

Section 1-5 : Factoring Polynomials

Of all the topics covered in this chapter factoring polynomials is probably the most important topic. There are many sections in later chapters where the first step will be to factor a polynomial. So, if you can't factor the polynomial then you won't be able to even start the problem let alone finish it.

Let's start out by talking a little bit about just what factoring is. Factoring is the process by which we go about determining what we multiplied to get the given quantity. We do this all the time with numbers. For instance, here are a variety of ways to factor 12.

$$12 = (2)(6) \qquad 12 = (3)(4) \qquad 12 = (2)(2)(3)$$
$$12 = \left(\frac{1}{2}\right)(24) \qquad 12 = (-2)(-6) \qquad 12 = (-2)(2)(-3)$$

There are many more possible ways to factor 12, but these are representative of many of them.

A common method of factoring numbers is to **completely factor** the number into positive prime factors. A **prime** number is a number whose only positive factors are 1 and itself. For example, 2, 3, 5, and 7 are all examples of prime numbers. Examples of numbers that aren't prime are 4, 6, and 12 to pick a few.

If we completely factor a number into positive prime factors there will only be one way of doing it. That is the reason for factoring things in this way. For our example above with 12 the complete factorization is,

$$12 = (2)(2)(3)$$

Factoring polynomials is done in pretty much the same manner. We determine all the terms that were multiplied together to get the given polynomial. We then try to factor each of the terms we found in the first step. This continues until we simply can't factor anymore. When we can't do any more factoring we will say that the polynomial is **completely factored.**

Here are a couple of examples.

$$x^2 - 16 = (x+4)(x-4)$$

This is completely factored since neither of the two factors on the right can be further factored.

Likewise,

$$x^4 - 16 = (x^2+4)(x^2-4)$$

is not completely factored because the second factor can be further factored. Note that the first factor is completely factored however. Here is the complete factorization of this polynomial.

$$x^4 - 16 = (x^2+4)(x+2)(x-2)$$

The purpose of this section is to familiarize ourselves with many of the techniques for factoring polynomials.

Greatest Common Factor

The first method for factoring polynomials will be factoring out the **greatest common factor**. When factoring in general this will also be the first thing that we should try as it will often simplify the problem.

To use this method all that we do is look at all the terms and determine if there is a factor that is in common to all the terms. If there is, we will factor it out of the polynomial. Also note that in this case we are really only using the distributive law in reverse. Remember that the distributive law states that

$$a\left(b+c\right)=ab+ac$$

In factoring out the greatest common factor we do this in reverse. We notice that each term has an *a* in it and so we "factor" it out using the distributive law in reverse as follows,

$$ab+ac=a\left(b+c\right)$$

Let's take a look at some examples.

Example 1 Factor out the greatest common factor from each of the following polynomials.

(a) $8x^4-4x^3+10x^2$

(b) $x^3y^2+3x^4y+5x^5y^3$

(c) $3x^6-9x^2+3x$

(d) $9x^2\left(2x+7\right)-12x\left(2x+7\right)$

Solution

(a) $8x^4-4x^3+10x^2$

First, we will notice that we can factor a 2 out of every term. Also note that we can factor an x^2 out of every term. Here then is the factoring for this problem.

$$8x^4-4x^3+10x^2=2x^2\left(4x^2-2x+5\right)$$

Note that we can always check our factoring by multiplying the terms back out to make sure we get the original polynomial.

(b) $x^3y^2+3x^4y+5x^5y^3$

In this case we have both *x*'s and *y*'s in the terms but that doesn't change how the process works. Each term contains and x^3 and a *y* so we can factor both of those out. Doing this gives,

$$x^3y^2+3x^4y+5x^5y^3=x^3y\left(y+3x+5x^2y^2\right)$$

(c) $3x^6-9x^2+3x$

In this case we can factor a 3*x* out of every term. Here is the work for this one.

$$3x^6-9x^2+3x=3x\left(x^5-3x+1\right)$$

Notice the "+1" where the 3*x* originally was in the final term, since the final term was the term we factored out we needed to remind ourselves that there was a term there originally. To do this we need the "+1" and notice that it is "+1" instead of "-1" because the term was originally a positive term. If it had been a negative term originally we would have had to use "-1".

One of the more common mistakes with these types of factoring problems is to forget this "1". Remember that we can always check by multiplying the two back out to make sure we get the original. To check that the "+1" is required, let's drop it and then multiply out to see what we get.

$$3x\left(x^5 - 3x\right) = 3x^6 - 9x^2 \neq 3x^6 - 9x^2 + 3x$$

So, without the "+1" we don't get the original polynomial! Be careful with this. It is easy to get in a hurry and forget to add a "+1" or "-1" as required when factoring out a complete term.

(d) $9x^2\left(2x+7\right) - 12x\left(2x+7\right)$

This one looks a little odd in comparison to the others. However, it works the same way. There is a 3*x* in each term and there is also a $2x+7$ in each term and so that can also be factored out. Doing the factoring for this problem gives,

$$9x^2\left(2x+7\right) - 12x\left(2x+7\right) = 3x\left(2x+7\right)\left(3x-4\right)$$

Factoring By Grouping

This is a method that isn't used all that often, but when it can be used it can be somewhat useful. This method is best illustrated with an example or two.

Example 2 Factor by grouping each of the following.

(a) $3x^2 - 2x + 12x - 8$

(b) $x^5 + x - 2x^4 - 2$

(c) $x^5 - 3x^3 - 2x^2 + 6$

Solution

(a) $3x^2 - 2x + 12x - 8$

In this case we *group* the first two terms and the final two terms as shown here,

$$\left(3x^2 - 2x\right) + \left(12x - 8\right)$$

Now, notice that we can factor an *x* out of the first grouping and a *4* out of the second grouping. Doing this gives,

$$3x^2 - 2x + 12x - 8 = x\left(3x-2\right) + 4\left(3x-2\right)$$

We can now see that we can factor out a common factor of $3x-2$ so let's do that to the final factored form.

$$3x^2 - 2x + 12x - 8 = \left(3x-2\right)\left(x+4\right)$$

And we're done. That's all that there is to factoring by grouping. Note again that this will not always work and sometimes the only way to know if it will work or not is to try it and see what you get.

(b) $x^5 + x - 2x^4 - 2$

In this case we will do the same initial step, but this time notice that both of the final two terms are negative so we'll factor out a "-" as well when we group them. Doing this gives,

$$\left(x^5 + x\right) - \left(2x^4 + 2\right)$$

Again, we can always distribute the "-" back through the parenthesis to make sure we get the original polynomial.

At this point we can see that we can factor an *x* out of the first term and a 2 out of the second term. This gives,

$$x^5 + x - 2x^4 - 2 = x\left(x^4 + 1\right) - 2\left(x^4 + 1\right)$$

We now have a common factor that we can factor out to complete the problem.

$$x^5 + x - 2x^4 - 2 = \left(x^4 + 1\right)\left(x - 2\right)$$

(c) $x^5 - 3x^3 - 2x^2 + 6$

This one also has a "-" in front of the third term as we saw in the last part. However, this time the fourth term has a "+" in front of it unlike the last part. We will still factor a "-" out when we group however to make sure that we don't lose track of it. When we factor the "-" out notice that we needed to change the "+" on the fourth term to a "-". Again, you can always check that this was done correctly by multiplying the "-" back through the parenthesis.

$$\left(x^5 - 3x^3\right) - \left(2x^2 - 6\right)$$

Now that we've done a couple of these we won't put the remaining details in and we'll go straight to the final factoring.

$$x^5 - 3x^3 - 2x^2 + 6 = x^3\left(x^2 - 3\right) - 2\left(x^2 - 3\right) = \left(x^2 - 3\right)\left(x^3 - 2\right)$$

Factoring by grouping can be nice, but it doesn't work all that often. Notice that as we saw in the last two parts of this example if there is a "-" in front of the third term we will often also factor that out of the third and fourth terms when we group them.

Factoring Quadratic Polynomials

First, let's note that quadratic is another term for second degree polynomial. So we know that the largest exponent in a quadratic polynomial will be a 2. In these problems we will be attempting to factor quadratic polynomials into two first degree (hence forth linear) polynomials. Until you become good at these, we usually end up doing these by trial and error although there are a couple of processes that can make them somewhat easier.

Let's take a look at some examples.

Example 3 Factor each of the following polynomials.

(a) $x^2 + 2x - 15$
(b) $x^2 - 10x + 24$
(c) $x^2 + 6x + 9$
(d) $x^2 + 5x + 1$
(e) $3x^2 + 2x - 8$
(f) $5x^2 - 17x + 6$
(g) $4x^2 + 10x - 6$

Solution

(a) $x^2+2x-15$

Okay since the first term is x^2 we know that the factoring must take the form.

$$x^2+2x-15=\left(x+\_\right)\left(x+\_\right)$$

We know that it will take this form because when we multiply the two linear terms the first term must be x^2 and the only way to get that to show up is to multiply *x* by *x*. Therefore, the first term in each factor must be an *x*. To finish this we just need to determine the two numbers that need to go in the blank spots.

We can narrow down the possibilities considerably. Upon multiplying the two factors out these two numbers will need to multiply out to get -15. In other words, these two numbers must be factors of -15. Here are all the possible ways to factor -15 using only integers.

$$(-1)(15) \qquad (1)(-15) \qquad (-3)(5) \qquad (3)(-5)$$

Now, we can just plug these in one after another and multiply out until we get the correct pair. However, there is another trick that we can use here to help us out. The correct pair of numbers must add to get the coefficient of the *x* term. So, in this case the third pair of factors will add to "+2" and so that is the pair we are after.

Here is the factored form of the polynomial.

$$x^2+2x-15=(x-3)(x+5)$$

Again, we can always check that we got the correct answer by doing a quick multiplication.

Note that the method we used here will only work if the coefficient of the x^2 term is one. If it is anything else this won't work and we really will be back to trial and error to get the correct factoring form.

(b) $x^2-10x+24$

Let's write down the initial form again,

$$x^2-10x+24=\left(x+\_\right)\left(x+\_\right)$$

Now, we need two numbers that multiply to get 24 and add to get -10. It looks like -6 and -4 will do the trick and so the factored form of this polynomial is,

$$x^2-10x+24=(x-4)(x-6)$$

(c) x^2+6x+9

Again, let's start with the initial form,

$$x^2+6x+9=\left(x+\_\right)\left(x+\_\right)$$

This time we need two numbers that multiply to get 9 and add to get 6. In this case 3 and 3 will be the correct pair of numbers. Don't forget that the two numbers can be the same number on occasion as they are here.

Here is the factored form for this polynomial.

$$x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$$

Note as well that we further simplified the factoring to acknowledge that it is a perfect square. You should always do this when it happens.

(d) $x^2 + 5x + 1$
Once again, here is the initial form,

$$x^2 + 5x + 1 = (x + \_\_)(x + \_\_)$$

Okay, this time we need two numbers that multiply to get 1 and add to get 5. There aren't two integers that will do this and so this quadratic doesn't factor.

This will happen on occasion so don't get excited about it when it does.

(e) $3x^2 + 2x - 8$
Okay, we no longer have a coefficient of 1 on the x^2 term. However, we can still make a guess as to the initial form of the factoring. Since the coefficient of the x^2 term is a 3 and there are only two positive factors of 3 there is really only one possibility for the initial form of the factoring.

$$3x^2 + 2x - 8 = (3x + \_\_)(x + \_\_)$$

Since the only way to get a $3x^2$ is to multiply a $3x$ and an x these must be the first two terms. However, finding the numbers for the two blanks will not be as easy as the previous examples. We will need to start off with all the factors of -8.

$$(-1)(8) \qquad (1)(-8) \qquad (-2)(4) \qquad (2)(-4)$$

At this point the only option is to pick a pair plug them in and see what happens when we multiply the terms out. Let's start with the fourth pair. Let's plug the numbers in and see what we get.

$$(3x+2)(x-4) = 3x^2 - 10x - 8$$

Well the first and last terms are correct, but then they should be since we've picked numbers to make sure those work out correctly. However, since the middle term isn't correct this isn't the correct factoring of the polynomial.

That doesn't mean that we guessed wrong however. With the previous parts of this example it didn't matter which blank got which number. This time it does. Let's flip the order and see what we get.

$$(3x-4)(x+2) = 3x^2 + 2x - 8$$

So, we got it. We did guess correctly the first time we just put them into the wrong spot.

So, in these problems don't forget to check both places for each pair to see if either will work.

(f) $5x^2 - 17x + 6$
Again, the coefficient of the x^2 term has only two positive factors so we've only got one possible initial form.

$$5x^2 - 17x + 6 = (5x + \_\_)(x + \_\_)$$

Next, we need all the factors of 6. Here they are.

$$(1)(6) \qquad (-1)(-6) \qquad (2)(3) \qquad (-2)(-3)$$

Don't forget the negative factors. They are often the ones that we want. In fact, upon noticing that the coefficient of the *x* is negative we can be assured that we will need one of the two pairs of negative factors since that will be the only way we will get negative coefficient there. With some trial and error we can get that the factoring of this polynomial is,

$$5x^2 - 17x + 6 = (5x - 2)(x - 3)$$

(g) $4x^2 + 10x - 6$

In this final step we've got a harder problem here. The coefficient of the x^2 term now has more than one pair of positive factors. This means that the initial form must be one of the following possibilities.

$$4x^2 + 10x - 6 = (4x + \_\_)(x + \_\_)$$

$$4x^2 + 10x - 6 = (2x + \_\_)(2x + \_\_)$$

To fill in the blanks we will need all the factors of -6. Here they are,

$$(-1)(6) \qquad (1)(-6) \qquad (-2)(3) \qquad (2)(-3)$$

With some trial and error we can find that the correct factoring of this polynomial is,

$$4x^2 + 10x - 6 = (2x - 1)(2x + 6)$$

Note as well that in the trial and error phase we need to make sure and plug each pair into both possible forms and in both possible orderings to correctly determine if it is the correct pair of factors or not.

We can actually go one more step here and factor a 2 out of the second term if we'd like to. This gives,

$$4x^2 + 10x - 6 = 2(2x - 1)(x + 3)$$

This is important because we could also have factored this as,

$$4x^2 + 10x - 6 = (4x - 2)(x + 3)$$

which, on the surface, appears to be different from the first form given above. However, in this case we can factor a 2 out of the first term to get,

$$4x^2 + 10x - 6 = 2(2x - 1)(x + 3)$$

This is exactly what we got the first time and so we really do have the same factored form of this polynomial.

Special Forms

There are some nice special forms of some polynomials that can make factoring easier for us on occasion. Here are the special forms.

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Let's work some examples with these.

Example 4 Factor each of the following.

(a) $x^2 - 20x + 100$

(b) $25x^2 - 9$

(c) $8x^3 + 1$

Solution

(a) $x^2 - 20x + 100$

In this case we've got three terms and it's a quadratic polynomial. Notice as well that the constant is a perfect square and its square root is 10. Notice as well that 2(10)=20 and this is the coefficient of the *x* term. So, it looks like we've got the second special form above. The correct factoring of this polynomial is,

$$x^2 - 20x + 100 = (x-10)^2$$

To be honest, it might have been easier to just use the general process for factoring quadratic polynomials in this case rather than checking that it was one of the special forms, but we did need to see one of them worked.

(b) $25x^2 - 9$

In this case all that we need to notice is that we've got a difference of perfect squares,

$$25x^2 - 9 = (5x)^2 - (3)^2$$

So, this must be the third special form above. Here is the correct factoring for this polynomial.

$$25x^2 - 9 = (5x+3)(5x-3)$$

(c) $8x^3 + 1$

This problem is the sum of two perfect cubes,

$$8x^3 + 1 = (2x)^3 + (1)^3$$

and so we know that it is the fourth special form from above. Here is the factoring for this polynomial.

$$8x^3 + 1 = (2x+1)(4x^2 - 2x + 1)$$

Do not make the following factoring mistake!

$$a^2 + b^2 \neq (a+b)^2$$

This just simply isn't true for the vast majority of sums of squares, so be careful not to make this very common mistake. There are rare cases where this can be done, but none of those special cases will be seen here.

Factoring Polynomials with Degree Greater than 2

There is no one method for doing these in general. However, there are some that we can do so let's take a look at a couple of examples.

Example 5 Factor each of the following.

(a) $3x^4 - 3x^3 - 36x^2$

(b) $x^4 - 25$

(c) $x^4 + x^2 - 20$

Solution

(a) $3x^4 - 3x^3 - 36x^2$

In this case let's notice that we can factor out a common factor of $3x^2$ from all the terms so let's do that first.

$$3x^4 - 3x^3 - 36x^2 = 3x^2\left(x^2 - x - 12\right)$$

What is left is a quadratic that we can use the techniques from above to factor. Doing this gives us,

$$3x^4 - 3x^3 - 36x^2 = 3x^2\left(x-4\right)\left(x+3\right)$$

Don't forget that the **FIRST** step to factoring should always be to factor out the greatest common factor. This can only help the process.

(b) $x^4 - 25$

There is no greatest common factor here. However, notice that this is the difference of two perfect squares.

$$x^4 - 25 = \left(x^2\right)^2 - \left(5\right)^2$$

So, we can use the third special form from above.

$$x^4 - 25 = \left(x^2+5\right)\left(x^2-5\right)$$

Neither of these can be further factored and so we are done. Note however, that often we will need to do some further factoring at this stage.

(c) $x^4 + x^2 - 20$

Let's start this off by working a factoring a different polynomial.

$$u^2 + u - 20 = \left(u-4\right)\left(u+5\right)$$

We used a different variable here since we'd already used *x*'s for the original polynomial.

So, why did we work this? Well notice that if we let $u = x^2$ then $u^2 = \left(x^2\right)^2 = x^4$. We can then rewrite the original polynomial in terms of *u*'s as follows,

$$x^4 + x^2 - 20 = u^2 + u - 20$$

and we know how to factor this! So factor the polynomial in *u*'s then back substitute using the fact that we know $u = x^2$.

$$\begin{aligned} x^4 + x^2 - 20 &= u^2 + u - 20 \\ &= \left(u-4\right)\left(u+5\right) \\ &= \left(x^2-4\right)\left(x^2+5\right) \end{aligned}$$

Finally, notice that the first term will also factor since it is the difference of two perfect squares. The correct factoring of this polynomial is then,

$$x^4 + x^2 - 20 = \left(x-2\right)\left(x+2\right)\left(x^2+5\right)$$

Note that this converting to *u* first can be useful on occasion, however once you get used to these this is usually done in our heads.

We did not do a lot of problems here and we didn't cover all the possibilities. However, we did cover some of the most common techniques that we are liable to run into in the other chapters of this work.

Section 1-6 : Rational Expressions

We now need to look at rational expressions. A **rational expression** is nothing more than a fraction in which the numerator and/or the denominator are polynomials. Here are some examples of rational expressions.

$$\frac{6}{x-1} \qquad \frac{z^2-1}{z^2+5} \qquad \frac{m^4+18m+1}{m^2-m-6} \qquad \frac{4x^2+6x-10}{1}$$

The last one may look a little strange since it is more commonly written $4x^2+6x-10$. However, it's important to note that polynomials can be thought of as rational expressions if we need to, although they rarely are.

There is an unspoken rule when dealing with rational expressions that we now need to address. When dealing with numbers we know that division by zero is not allowed. Well the same is true for rational expressions. So, when dealing with rational expressions we will always assume that whatever *x* is it won't give division by zero. We rarely write these restrictions down, but we will always need to keep them in mind.

For the first one listed we need to avoid $x=1$. The second rational expression is never zero in the denominator and so we don't need to worry about any restrictions. Note as well that the numerator of the second rational expression will be zero. That is okay, we just need to avoid division by zero. For the third rational expression we will need to avoid $m=3$ and $m=-2$. The final rational expression listed above will never be zero in the denominator so again we don't need to have any restrictions.

The first topic that we need to discuss here is reducing a rational expression to lowest terms. A rational expression has been **reduced to lowest terms** if all common factors from the numerator and denominator have been canceled. We already know how to do this with number fractions so let's take a quick look at an example.

$$\text{not reduced to lowest terms} \;\Rightarrow\; \frac{12}{8}=\frac{\cancel{(4)}(3)}{\cancel{(4)}(2)}=\frac{3}{2} \;\Leftarrow\; \text{reduced to lowest terms}$$

With rational expression it works exactly the same way.

$$\text{not reduced to lowest terms} \;\Rightarrow\; \frac{\cancel{(x+3)}(x-1)}{x\cancel{(x+3)}}=\frac{x-1}{x} \;\Leftarrow\; \text{reduced to lowest terms}$$

We do have to be careful with canceling however. There are some common mistakes that students often make with these problems. Recall that in order to cancel a factor it must multiply the whole numerator and the whole denominator. So, the *x*+3 above could cancel since it multiplied the whole numerator and the whole denominator. However, the *x*'s in the reduced form can't cancel since the *x* in the numerator is not times the whole numerator.

To see why the *x*'s don't cancel in the reduced form above put a number in and see what happens. Let's plug in $x = 4$.

$$\frac{4-1}{4} = \frac{3}{4} \qquad\qquad \frac{\cancel{4}-1}{\cancel{4}} = -1$$

Clearly the two aren't the same number!

So, be careful with canceling. As a general rule of thumb remember that you can't cancel something if it's got a "+" or a "-" on one side of it. There is one exception to this rule of thumb with "-" that we'll deal with in an example later on down the road.

Let's take a look at a couple of examples.

Example 1 Reduce the following rational expression to lowest terms.

(a) $\dfrac{x^2 - 2x - 8}{x^2 - 9x + 20}$

(b) $\dfrac{x^2 - 25}{5x - x^2}$

(c) $\dfrac{x^7 + 2x^6 + x^5}{x^3\left(x+1\right)^8}$

Solution

When reducing a rational expression to lowest terms the first thing that we will do is factor both the numerator and denominator as much as possible. That should always be the first step in these problems.

Also, the factoring in this section, and all successive section for that matter, will be done without explanation. It will be assumed that you are capable of doing and/or checking the factoring on your own. In other words, make sure that you can factor!

(a) $\dfrac{x^2 - 2x - 8}{x^2 - 9x + 20}$

We'll first factor things out as completely as possible. Remember that we can't cancel anything at this point in time since every term has a "+" or a "-" on one side of it! We've got to factor first!

$$\frac{x^2 - 2x - 8}{x^2 - 9x + 20} = \frac{\left(x-4\right)\left(x+2\right)}{\left(x-5\right)\left(x-4\right)}$$

At this point we can see that we've got a common factor in both the numerator and the denominator and so we can cancel the *x*-4 from both. Doing this gives,

$$\frac{x^2 - 2x - 8}{x^2 - 9x + 20} = \frac{x+2}{x-5}$$

This is also all the farther that we can go. Nothing else will cancel and so we have reduced this expression to lowest terms.

(b) $\dfrac{x^2-25}{5x-x^2}$

Again, the first thing that we'll do here is factor the numerator and denominator.

$$\frac{x^2-25}{5x-x^2}=\frac{(x-5)(x+5)}{x(5-x)}$$

At first glance it looks there is nothing that will cancel. Notice however that there is a term in the denominator that is almost the same as a term in the numerator except all the signs are the opposite.

We can use the following fact on the second term in the denominator.

$$a-b=-(b-a) \qquad \text{OR} \qquad -a+b=-(a-b)$$

This is commonly referred to as **factoring a minus sign out** because that is exactly what we've done. There are two forms here that cover both possibilities that we are liable to run into. In our case however we need the first form.

Because of some notation issues let's just work with the denominator for a while.

$$\begin{aligned} x(5-x) &= x\left[-(x-5)\right] \\ &= x\left[(-1)(x-5)\right] \\ &= x(-1)(x-5) \\ &= (-1)(x)(x-5) \\ &= -x(x-5) \end{aligned}$$

Notice the steps used here. In the first step we factored out the minus sign, but we are still multiplying the terms and so we put in an added set of brackets to make sure that we didn't forget that. In the second step we acknowledged that a minus sign in front is the same as multiplication by "-1". Once we did that we didn't really need the extra set of brackets anymore so we dropped them in the third step. Next, we recalled that we change the order of a multiplication if we need to so we flipped the *x* and the "-1". Finally, we dropped the "-1" and just went back to a negative sign in the front.

Typically, when we factor out minus signs we skip all the intermediate steps and go straight to the final step.

Let's now get back to the problem. The rational expression becomes,

$$\frac{x^2-25}{5x-x^2}=\frac{(x-5)(x+5)}{-x(x-5)}$$

At this point we can see that we do have a common factor and so we can cancel the *x-5*.

$$\frac{x^2-25}{5x-x^2}=\frac{x+5}{-x}=-\frac{x+5}{x}$$

(c) $\dfrac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}$

In this case the denominator is already factored for us to make our life easier. All we need to do is factor the numerator.

$$\frac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}=\frac{x^5\left(x^2+2x+1\right)}{x^3\left(x+1\right)^8}=\frac{x^5\left(x+1\right)^2}{x^3\left(x+1\right)^8}$$

Now we reach the point of this part of the example. There are 5 *x*'s in the numerator and 3 in the denominator so when we cancel there will be 2 left in the numerator. Likewise, there are 2 $\left(x+1\right)$'s in the numerator and 8 in the denominator so when we cancel there will be 6 left in the denominator. Here is the rational expression reduced to lowest terms.

$$\frac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}=\frac{x^2}{\left(x+1\right)^6}$$

Before moving on let's briefly discuss the answer in the second part of this example. Notice that we moved the minus sign from the denominator to the front of the rational expression in the final form. This can always be done when we need to. Recall that the following are all equivalent.

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

In other words, a minus sign in front of a rational expression can be moved onto the whole numerator or whole denominator if it is convenient to do that. We do have to be careful with this however. Consider the following rational expression.

$$\frac{-x+3}{x+1}$$

In this case the "-" on the *x* can't be moved to the front of the rational expression since it is only on the *x*. In order to move a minus sign to the front of a rational expression it needs to be times the whole numerator or denominator. So, if we factor a minus out of the numerator we could then move it into the front of the rational expression as follows,

$$\frac{-x+3}{x+1}=\frac{-\left(x-3\right)}{x+1}=-\frac{x-3}{x+1}$$

The moral here is that we need to be careful with moving minus signs around in rational expressions.

We now need to move into adding, subtracting, multiplying and dividing rational expressions.

Let's start with multiplying and dividing rational expressions. The general formulas are as follows,

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Note the two different forms for denoting division. We will use either as needed so make sure you are familiar with both. Note as well that to do division of rational expressions all that we need to do is multiply the numerator by the reciprocal of the denominator (*i.e.* the fraction with the numerator and denominator switched).

Before doing a couple of examples there are a couple of *special* cases of division that we should look at. In the general case above both the numerator and the denominator of the rational expression are fractions, however, what if one of them isn't a fraction. So let's look at the following cases.

$$\frac{a}{\frac{c}{d}} \qquad\qquad \frac{\frac{a}{b}}{c}$$

Students often make mistakes with these initially. To correctly deal with these we will turn the numerator (first case) or denominator (second case) into a fraction and then do the general division on them.

$$\frac{a}{\frac{c}{d}} = \frac{\frac{a}{1}}{\frac{c}{d}} = \frac{a}{1} \cdot \frac{d}{c} = \frac{ad}{c}$$

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$$

Be careful with these cases. It is easy to make a mistake with these and incorrectly do the division.

Now let's take a look at a couple of examples.

Example 2 Perform the indicated operation and reduce the answer to lowest terms.

(a) $\dfrac{x^2-5x-14}{x^2-3x+2} \bullet \dfrac{x^2-4}{x^2-14x+49}$

(b) $\dfrac{m^2-9}{m^2+5m+6} \div \dfrac{3-m}{m+2}$

(c) $\dfrac{y^2+5y+4}{\frac{y^2-1}{y+5}}$

Solution

Notice that with this problem we have started to move away from *x* as the main variable in the examples. Do not get so used to seeing *x*'s that you always expect them. The problems will work the same way regardless of the letter we use for the variable so don't get excited about the different letters here.

(a) $\dfrac{x^2-5x-14}{x^2-3x+2}\bullet\dfrac{x^2-4}{x^2-14x+49}$

Okay, this is a multiplication. The first thing that we should always do in the multiplication is to factor everything in sight as much as possible.

$$\frac{x^2-5x-14}{x^2-3x+2}\bullet\frac{x^2-4}{x^2-14x+49}=\frac{(x-7)(x+2)}{(x-2)(x-1)}\bullet\frac{(x-2)(x+2)}{(x-7)^2}$$

Now, recall that we can cancel things across a multiplication as follows.

$$\frac{a}{b\cancel{k}}\cdot\frac{c\cancel{k}}{d}=\frac{a}{b}\cdot\frac{c}{d}$$

Note that this ONLY works for multiplication and NOT for division!

In this case we do have multiplication so cancel as much as we can and then do the multiplication to get the answer.

$$\frac{x^2-5x-14}{x^2-3x+2}\bullet\frac{x^2-4}{x^2-14x+49}=\frac{(x+2)}{(x-1)}\bullet\frac{(x+2)}{(x-7)}=\frac{(x+2)^2}{(x-1)(x-7)}$$

(b) $\dfrac{m^2-9}{m^2+5m+6}\div\dfrac{3-m}{m+2}$

With division problems it is very easy to mistakenly cancel something that shouldn't be canceled and so the first thing we do here (before factoring!!!!) is do the division. Once we've done the division we have a multiplication problem and we factor as much as possible, cancel everything that can be canceled and finally do the multiplication.

So, let's get started on this problem.

$$\begin{aligned}\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}&=\frac{m^2-9}{m^2+5m+6}\cdot\frac{m+2}{3-m}\\&=\frac{(m-3)(m+3)}{(m+3)(m+2)}\cdot\frac{(m+2)}{(3-m)}\end{aligned}$$

Now, notice that there will be a lot of canceling here. Also notice that if we factor a minus sign out of the denominator of the second rational expression. Let's do some of the canceling and then do the multiplication.

$$\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}=\frac{(m-3)}{1}\cdot\frac{1}{-(m-3)}=\frac{(m-3)}{-(m-3)}$$

Remember that when we cancel all the terms out of a numerator or denominator there is actually a "1" left over! Now, we didn't finish the canceling to make a point. Recall that at the start of this discussion we said that as a rule of thumb we can only cancel terms if there isn't a "+" or a "-" on either side of it with one exception for the "-". We are now at that exception. If there is a "-" if front of the whole numerator or denominator, as we've got here, then we can still cancel the term. In this case the "-" acts as a "-1" that is multiplied by the whole denominator and so is a factor instead of an addition or subtraction. Here is the final answer for this part.

$$\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}=\frac{1}{-1}=-1$$

In this case all the terms canceled out and we were left with a number. This doesn't happen all that often, but as this example has shown it clearly can happen every once in a while so don't get excited about it when it does happen.

(c) $\dfrac{y^2+5y+4}{\dfrac{y^2-1}{y+5}}$

This is one of the special cases for division. So, as with the previous part, we will first do the division and then we will factor and cancel as much as we can.

Here is the work for this part.

$$\begin{aligned}\frac{y^2+5y+4}{\dfrac{y^2-1}{y+5}}&=\frac{\left(y^2+5y+4\right)\left(y+5\right)}{y^2-1}\\&=\frac{\left(y+1\right)\left(y+4\right)\left(y+5\right)}{\left(y+1\right)\left(y-1\right)}=\frac{\left(y+4\right)\left(y+5\right)}{y-1}\end{aligned}$$

Okay, it's time to move on to addition and subtraction of rational expressions. Here are the general formulas.

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}\qquad\qquad\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

As these have shown we've got to remember that in order to add or subtract rational expression or fractions we MUST have common denominators. If we don't have common denominators then we need to first get common denominators.

Let's remember how do to do this with a quick number example.

$$\frac{5}{6}-\frac{3}{4}$$

In this case we need a common denominator and recall that it's usually best to use the **least common denominator**, often denoted **lcd**. In this case the least common denominator is 12. So we need to get the denominators of these two fractions to a 12. This is easy to do. In the first case we need to multiply the denominator by 2 to get 12 so we will multiply the numerator and denominator of the first fraction by 2. Remember that we've got to multiply both the numerator and denominator by the same number since we aren't allowed to actually change the problem and this is equivalent to multiplying the fraction by 1 since $\frac{a}{a}=1$. For the second term we'll need to multiply the numerator and denominator by a 3.

$$\frac{5}{6}-\frac{3}{4}=\frac{5(2)}{6(2)}-\frac{3(3)}{4(3)}=\frac{10}{12}-\frac{9}{12}=\frac{10-9}{12}=\frac{1}{12}$$

Now, the process for rational expressions is identical. The main difficulty is in finding the least common denominator. However, there is a really simple process for finding the least common denominator for rational expressions. Here is it.

1. Factor all the denominators.
2. Write down each factor that appears at least once in any of the denominators. Do NOT write down the power that is on each factor, only write down the factor
3. Now, for each factor written down in the previous step write down the largest power that occurs in all the denominators containing that factor.
4. The product all the factors from the previous step is the least common denominator.

Let's work some examples.

Example 3 Perform the indicated operation.

(a) $\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}$

(b) $\frac{2}{z+1}-\frac{z-1}{z+2}$

(c) $\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}$

(d) $\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}$

(e) $\frac{4}{y+2}-\frac{1}{y}+1$

Solution

(a) $\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}$

For this problem there are coefficients on each term in the denominator so we'll first need the least common denominator for the coefficients. This is 6. Now, *x* (by itself with a power of 1) is the only factor that occurs in any of the denominators. So, the least common denominator for this part is *x* with the largest power that occurs on all the *x*'s in the problem, which is 5. So, the least common denominator for this set of rational expression is

$$\text{lcd} : 6x^5$$

So, we simply need to multiply each term by an appropriate quantity to get this in the denominator and then do the addition and subtraction. Let's do that.

$$\begin{aligned}\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}&=\frac{4\left(x^3\right)}{6x^2\left(x^3\right)}-\frac{1\left(2\right)}{3x^5\left(2\right)}+\frac{5\left(3x^2\right)}{2x^3\left(3x^2\right)}\\&=\frac{4x^3}{6x^5}-\frac{2}{6x^5}+\frac{15x^2}{6x^5}\\&=\frac{4x^3-2+15x^2}{6x^5}\end{aligned}$$

(b) $\dfrac{2}{z+1}-\dfrac{z-1}{z+2}$

In this case there are only two factors and they both occur to the first power and so the least common denominator is.

$$\text{lcd}:\ \left(z+1\right)\left(z+2\right)$$

Now, in determining what to multiply each part by simply compare the current denominator to the least common denominator and multiply top and bottom by whatever is "missing". In the first term we're "missing" a $z+2$ and so that's what we multiply the numerator and denominator by. In the second term we're "missing" a $z+1$ and so that's what we'll multiply in that term.

Here is the work for this problem.

$$\frac{2}{z+1}-\frac{z-1}{z+2}=\frac{2\left(z+2\right)}{\left(z+1\right)\left(z+2\right)}-\frac{\left(z-1\right)\left(z+1\right)}{\left(z+2\right)\left(z+1\right)}=\frac{2\left(z+2\right)-\left(z-1\right)\left(z+1\right)}{\left(z+1\right)\left(z+2\right)}$$

The final step is to do any multiplication in the numerator and simplify that up as much as possible.

$$\frac{2}{z+1}-\frac{z-1}{z+2}=\frac{2z+4-\left(z^2-1\right)}{\left(z+1\right)\left(z+2\right)}=\frac{2z+4-z^2+1}{\left(z+1\right)\left(z+2\right)}=\frac{-z^2+2z+5}{\left(z+1\right)\left(z+2\right)}$$

Be careful with minus signs and parenthesis when doing the subtraction.

(c) $\dfrac{y}{y^2-2y+1}-\dfrac{2}{y-1}+\dfrac{3}{y+2}$

Let's first factor the denominators and determine the least common denominator.

$$\frac{y}{\left(y-1\right)^2}-\frac{2}{y-1}+\frac{3}{y+2}$$

So, there are two factors in the denominators a *y-1* and a *y+2*. So we will write both of those down and then take the highest power for each. That means a 2 for the *y-1* and a 1 for the *y+2*. Here is the least common denominator for this rational expression.

$$\text{lcd}:\ \left(y+2\right)\left(y-1\right)^2$$

Now determine what's missing in the denominator for each term, multiply the numerator and denominator by that and then finally do the subtraction and addition.

$$\begin{aligned}\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}&=\frac{y\left(y+2\right)}{\left(y-1\right)^2\left(y+2\right)}-\frac{2\left(y-1\right)\left(y+2\right)}{\left(y-1\right)\left(y-1\right)\left(y+2\right)}+\frac{3\left(y-1\right)^2}{\left(y-1\right)^2\left(y+2\right)}\\&=\frac{y\left(y+2\right)-2\left(y-1\right)\left(y+2\right)+3\left(y-1\right)^2}{\left(y-1\right)^2\left(y+2\right)}\end{aligned}$$

Okay now let's multiply the numerator out and simplify. In the last term recall that we need to do the multiplication prior to distributing the 3 through the parenthesis. Here is the simplification work for this part.

$$\begin{aligned}\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}&=\frac{y^2+2y-2\left(y^2+y-2\right)+3\left(y^2-2y+1\right)}{\left(y-1\right)^2\left(y+2\right)}\\&=\frac{y^2+2y-2y^2-2y+4+3y^2-6y+3}{\left(y-1\right)^2\left(y+2\right)}\\&=\frac{2y^2-6y+7}{\left(y-1\right)^2\left(y+2\right)}\end{aligned}$$

(d) $\dfrac{2x}{x^2-9}-\dfrac{1}{x+3}-\dfrac{2}{x-3}$

Again, factor the denominators and get the least common denominator.

$$\frac{2x}{\left(x-3\right)\left(x+3\right)}-\frac{1}{x+3}-\frac{2}{x-3}$$

The least common denominator is,

$$\text{lcd}: \left(x-3\right)\left(x+3\right)$$

Notice that the first rational expression already contains this in its denominator, but that is okay. In fact, because of that the work will be slightly easier in this case. Here is the subtraction for this problem.

$$\begin{aligned}\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}&=\frac{2x}{\left(x-3\right)\left(x+3\right)}-\frac{1\left(x-3\right)}{\left(x+3\right)\left(x-3\right)}-\frac{2\left(x+3\right)}{\left(x-3\right)\left(x+3\right)}\\&=\frac{2x-\left(x-3\right)-2\left(x+3\right)}{\left(x-3\right)\left(x+3\right)}\\&=\frac{2x-x+3-2x-6}{\left(x-3\right)\left(x+3\right)}\\&=\frac{-x-3}{\left(x-3\right)\left(x+3\right)}\end{aligned}$$

Notice that we can actually go one step further here. If we factor a minus out of the numerator we can do some canceling.

$$\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}=\frac{-\left(x+3\right)}{\left(x-3\right)\left(x+3\right)}=\frac{-1}{x-3}$$

Sometimes this kind of canceling will happen after the addition/subtraction so be on the lookout for it.

(e) $\dfrac{4}{y+2}-\dfrac{1}{y}+1$

The point of this problem is that "1" sitting out behind everything. That isn't really the problem that it appears to be. Let's first rewrite things a little here.

$$\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}$$

In this way we see that we really have three fractions here. One of them simply has a denominator of one. The least common denominator for this part is,

$$\text{lcd} : y\left(y+2\right)$$

Here is the addition and subtraction for this problem.

$$\begin{aligned}\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}&=\frac{4y}{\left(y+2\right)\left(y\right)}-\frac{y+2}{y\left(y+2\right)}+\frac{y\left(y+2\right)}{y\left(y+2\right)}\\&=\frac{4y-\left(y+2\right)+y\left(y+2\right)}{y\left(y+2\right)}\end{aligned}$$

Notice the set of parenthesis we added onto the second numerator as we did the subtraction. We are subtracting off the whole numerator and so we need the parenthesis there to make sure we don't make any mistakes with the subtraction.

Here is the simplification for this rational expression.

$$\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}=\frac{4y-y-2+y^2+2y}{y\left(y+2\right)}=\frac{y^2+5y-2}{y\left(y+2\right)}$$

Section 1-7 : Complex Numbers

The last topic in this section is not really related to most of what we've done in this chapter, although it is somewhat related to the radicals section as we will see. We also won't need the material here all that often in the remainder of this course, but there are a couple of sections in which we will need this and so it's best to get it out of the way at this point.

In the radicals section we noted that we won't get a real number out of a square root of a negative number. For instance, $\sqrt{-9}$ isn't a real number since there is no real number that we can square and get a NEGATIVE 9.

Now we also saw that if *a* and *b* were both positive then $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. For a second let's forget that restriction and do the following.

$$\sqrt{-9} = \sqrt{(9)(-1)} = \sqrt{9}\sqrt{-1} = 3\sqrt{-1}$$

Now, $\sqrt{-1}$ is not a real number, but if you think about it we can do this for any square root of a negative number. For instance,

$$\begin{aligned}&\sqrt{-100} = \sqrt{100}\sqrt{-1} = 10\sqrt{-1}\\&\sqrt{-5} = \sqrt{5}\ \sqrt{-1}\\&\sqrt{-290} = \sqrt{290}\ \sqrt{-1} \qquad etc.\end{aligned}$$

So, even if the number isn't a perfect square we can still always reduce the square root of a negative number down to the square root of a positive number (which we or a calculator can deal with) times $\sqrt{-1}$.

So, if we just had a way to deal with $\sqrt{-1}$ we could actually deal with square roots of negative numbers. Well the reality is that, at this level, there just isn't any way to deal with $\sqrt{-1}$ so instead of dealing with it we will "make it go away" so to speak by using the following definition.

$$\boxed{i = \sqrt{-1}}$$

Note that if we square both sides of this we get,

$$\boxed{i^2 = -1}$$

It will be important to remember this later on. This shows that, in some way, *i* is the only "number" that we can square and get a negative value.

Using this definition all the square roots above become,

$$\sqrt{-9} = 3\,i \qquad\qquad \sqrt{-100} = 10\,i$$

$$\sqrt{-5} = \sqrt{5}\,i \qquad\qquad \sqrt{-290} = \sqrt{290}\,i$$

These are all examples of **complex numbers**.

The natural question at this point is probably just why do we care about this? The answer is that, as we will see in the next chapter, sometimes we will run across the square roots of negative numbers and we're going to need a way to deal with them. So, to deal with them we will need to discuss complex numbers.

So, let's start out with some of the basic definitions and terminology for complex numbers. The **standard form** of a complex number is

$$a + bi$$

where *a* and *b* are real numbers and they can be anything, positive, negative, zero, integers, fractions, decimals, it doesn't matter. When in the standard form *a* is called the **real part** of the complex number and *b* is called the **imaginary part** of the complex number.

Here are some examples of complex numbers.

$$3 + 5i \qquad \sqrt{6} - 10i \qquad \frac{4}{5} + i \qquad 16i \qquad 113$$

The last two probably need a little more explanation. It is completely possible that *a* or *b* could be zero and so in 16*i* the real part is zero. When the real part is zero we often will call the complex number a **purely imaginary number**. In the last example (113) the imaginary part is zero and we actually have a real number. So, thinking of numbers in this light we can see that the real numbers are simply a subset of the complex numbers.

The **conjugate** of the complex number $a + bi$ is the complex number $a - bi$. In other words, it is the original complex number with the sign on the imaginary part changed. Here are some examples of complex numbers and their conjugates.

complex number	conjugate
$3 + \frac{1}{2}i$	$3 - \frac{1}{2}i$
$12 - 5i$	$12 + 5i$
$1 - i$	$1 + i$
$45i$	$-45i$
101	101

Notice that the conjugate of a real number is just itself with no changes.

Now we need to discuss the basic operations for complex numbers. We'll start with addition and subtraction. The easiest way to think of adding and/or subtracting complex numbers is to think of each complex number as a polynomial and do the addition and subtraction in the same way that we add or subtract polynomials.

Example 1 Perform the indicated operation and write the answers in standard form.

(a) $(-4+7i)+(5-10i)$

(b) $(4+12i)-(3-15i)$

(c) $5i-(-9+i)$

Solution

There really isn't much to do here other than add or subtract. Note that the parentheses on the first terms are only there to indicate that we're thinking of that term as a complex number and in general aren't used.

(a) $(-4+7i)+(5-10i)=1-3i$

(b) $(4+12i)-(3-15i)=4+12i-3+15i=1+27i$

(c) $5i-(-9+i)=5i+9-i=9+4i$

Next let's take a look at multiplication. Again, with one small difference, it's probably easiest to just think of the complex numbers as polynomials so multiply them out as you would polynomials. The one difference will come in the final step as we'll see.

Example 2 Multiply each of the following and write the answers in standard form.

(a) $7i(-5+2i)$

(b) $(1-5i)(-9+2i)$

(c) $(4+i)(2+3i)$

(d) $(1-8i)(1+8i)$

Solution

(a) So all that we need to do is distribute the 7*i* through the parenthesis.

$$7i(-5+2i)=-35i+14i^2$$

Now, this is where the small difference mentioned earlier comes into play. This number is NOT in standard form. The standard form for complex numbers does not have an i^2 in it. This however is not a problem provided we recall that

$$i^2=-1$$

Using this we get,

$$7i(-5+2i)=-35i+14(-1)=-14-35i$$

We also rearranged the order so that the real part is listed first.

(b) In this case we will FOIL the two numbers and we'll need to also remember to get rid of the i^2.

$$(1-5i)(-9+2i)=-9+2i+45i-10i^2=-9+47i-10(-1)=1+47i$$

(c) Same thing with this one.

$$\left(4+i\right)\left(2+3i\right)=8+12i+2i+3i^2=8+14i+3\left(-1\right)=5+14i$$

(d) Here's one final multiplication that will lead us into the next topic.

$$\left(1-8i\right)\left(1+8i\right)=1+8i-8i-64i^2=1+64=65$$

Don't get excited about it when the product of two complex numbers is a real number. That can and will happen on occasion.

In the final part of the previous example we multiplied a number by its conjugate. There is a nice general formula for this that will be convenient when it comes to discussing division of complex numbers.

$$\left(a+bi\right)\left(a-bi\right)=a^2-abi+abi-b^2i^2=a^2+b^2$$

So, when we multiply a complex number by its conjugate we get a real number given by,

$$\boxed{\left(a+bi\right)\left(a-bi\right)=a^2+b^2}$$

Now, we gave this formula with the comment that it will be convenient when it came to dividing complex numbers so let's look at a couple of examples.

Example 3 Write each of the following in standard form.

(a) $\dfrac{3-i}{2+7i}$

(b) $\dfrac{3}{9-i}$

(c) $\dfrac{8i}{1+2i}$

(d) $\dfrac{6-9i}{2i}$

Solution

So, in each case we are really looking at the division of two complex numbers. The main idea here however is that we want to write them in standard form. Standard form does not allow for any *i*'s to be in the denominator. So, we need to get the *i*'s out of the denominator.

This is actually fairly simple if we recall that a complex number times its conjugate is a real number. So, if we multiply the numerator and denominator by the conjugate of the denominator we will be able to eliminate the *i* from the denominator.

Now that we've figured out how to do these let's go ahead and work the problems.

(a) $\dfrac{3-i}{2+7i}=\dfrac{(3-i)}{(2+7i)}\dfrac{(2-7i)}{(2-7i)}=\dfrac{6-23i+7i^2}{2^2+7^2}=\dfrac{-1-23i}{53}=-\dfrac{1}{53}-\dfrac{23}{53}i$

Notice that to officially put the answer in standard form we broke up the fraction into the real and imaginary parts.

(b) $\dfrac{3}{9-i}=\dfrac{3}{(9-i)}\dfrac{(9+i)}{(9+i)}=\dfrac{27+3i}{9^2+1^2}=\dfrac{27}{82}+\dfrac{3}{82}i$

(c) $\dfrac{8i}{1+2i}=\dfrac{8i}{(1+2i)}\dfrac{(1-2i)}{(1-2i)}=\dfrac{8i-16i^2}{1^2+2^2}=\dfrac{16+8i}{5}=\dfrac{16}{5}+\dfrac{8}{5}i$

(d) This one is a little different from the previous ones since the denominator is a pure imaginary number. It can be done in the same manner as the previous ones, but there is a slightly easier way to do the problem.

First, break up the fraction as follows.

$$\frac{6-9i}{2i}=\frac{6}{2i}-\frac{9i}{2i}=\frac{3}{i}-\frac{9}{2}$$

Now, we want the *i* out of the denominator and since there is only an *i* in the denominator of the first term we will simply multiply the numerator and denominator of the first term by an *i*.

$$\frac{6-9i}{2i}=\frac{3(i)}{i(i)}-\frac{9}{2}=\frac{3i}{i^2}-\frac{9}{2}=\frac{3i}{-1}-\frac{9}{2}=-\frac{9}{2}-3i$$

The next topic that we want to discuss here is powers of *i*. Let's just take a look at what happens when we start looking at various powers of *i*.

$$\begin{aligned}
&i^1=i && i^1=i\\
&i^2=-1 && i^2=-1\\
&i^3=i\cdot i^2=-i && i^3=-i\\
&i^4=\left(i^2\right)^2=(-1)^2=1 && i^4=1\\
&i^5=i\cdot i^4=i && i^5=i\\
&i^6=i^2\cdot i^4=(-1)(1)=-1 && i^6=-1\\
&i^7=i\cdot i^6=-i && i^7=-i\\
&i^8=\left(i^4\right)^2=(1)^2=1 && i^8=1
\end{aligned}$$

Can you see the pattern? All powers if *i* can be reduced down to one of four possible answers and they repeat every four powers. This can be a convenient fact to remember.

We next need to address an issue on dealing with square roots of negative numbers. From the section on radicals we know that we can do the following.

$$6 = \sqrt{36} = \sqrt{(4)(9)} = \sqrt{4}\,\sqrt{9} = (2)(3) = 6$$

In other words, we can break up products under a square root into a product of square roots provided both numbers are positive.

It turns out that we can actually do the same thing if **one** of the numbers is negative. For instance,

$$6i = \sqrt{-36} = \sqrt{(-4)(9)} = \sqrt{-4}\,\sqrt{9} = (2i)(3) = 6i$$

However, if BOTH numbers are negative this won't work anymore as the following shows.

$$6 = \sqrt{36} = \sqrt{(-4)(-9)} \neq \sqrt{-4}\,\sqrt{-9} = (2i)(3i) = 6i^2 = -6$$

We can summarize this up as a set of rules. If *a* and *b* are both positive numbers then,

$$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$$
$$\sqrt{-a}\,\sqrt{b} = \sqrt{-ab}$$
$$\sqrt{a}\,\sqrt{-b} = \sqrt{-ab}$$

$$\sqrt{-a}\,\sqrt{-b} \neq \sqrt{(-a)(-b)}$$

Why is this important enough to worry about? Consider the following example.

Example 4 Multiply the following and write the answer in standard form.

$$\left(2 - \sqrt{-100}\right)\left(1 + \sqrt{-36}\right)$$

Solution

If we were to multiply this out in its present form we would get,

$$\left(2 - \sqrt{-100}\right)\left(1 + \sqrt{-36}\right) = 2 + 2\sqrt{-36} - \sqrt{-100} - \sqrt{-36}\,\sqrt{-100}$$

Now, if we were not being careful we would probably combine the two roots in the final term into one which can't be done!

So, there is a general rule of thumb in dealing with square roots of negative numbers. When faced with them the first thing that you should always do is convert them to complex number. If we follow this rule we will always get the correct answer.

So, let's work this problem the way it should be worked.

$$\left(2 - \sqrt{-100}\right)\left(1 + \sqrt{-36}\right) = (2 - 10i)(1 + 6i) = 2 + 2i - 60i^2 = 62 + 2i$$

The rule of thumb given in the previous example is important enough to make again. When faced with square roots of negative numbers the first thing that you should do is convert them to complex numbers.

There is one final topic that we need to touch on before leaving this section. As we noted back in the section on radicals even though $\sqrt{9} = 3$ there are in fact two numbers that we can square to get 9. We can square both 3 and -3.

The same will hold for square roots of negative numbers. As we saw earlier $\sqrt{-9} = 3i$. As with square roots of positive numbers in this case we are really asking what did we square to get -9? Well it's easy enough to check that 3*i* is correct.

$$(3i)^2 = 9i^2 = -9$$

However, that is not the only possibility. Consider the following,

$$(-3i)^2 = (-3)^2 i^2 = 9i^2 = -9$$

and so if we square -3*i* we will also get -9. So, when taking the square root of a negative number there are really two numbers that we can square to get the number under the radical. However, we will ALWAYS take the positive number for the value of the square root just as we do with the square root of positive numbers.

Chapter 2 : Solving Equations and Inequalities

In this chapter we will look at one of the standard topics in any Algebra class. The ability to solve equations and/or inequalities is very important and is used time and again both in this class and in later classes. We will cover a wide variety of solving topics in this chapter that should cover most of the basic equations/inequalities/techniques that are involved in solving.

Here is a brief listing of the material covered in this chapter.

Polynomial Inequalities – In this section we will continue solving inequalities. However, in this section we move away from linear inequalities and move on to solving inequalities that involve polynomials of degree at least 2.
Rational Inequalities – We continue solving inequalities in this section. We now will solve inequalities that involve rational expressions, although as we'll see the process here is pretty much identical to the process used when solving inequalities with polynomials.
Absolute Value Equations – In this section we will give a geometric as well as a mathematical definition of absolute value. We will then proceed to solve equations that involve an absolute value. We will also work an example that involved two absolute values.
Absolute Value Inequalities – In this final section of the Solving chapter we will solve inequalities that involve absolute value. As we will see the process for solving inequalities with a < (*i.e.* a less than) is very different from solving an inequality with a > (*i.e.* greater than).

Section 2-1 : Solutions and Solution Sets

We will start off this chapter with a fairly short section with some basic terminology that we use on a fairly regular basis in solving equations and inequalities.

First, a **solution** to an equation or inequality is any number that, when plugged into the equation/inequality, will satisfy the equation/inequality. So, just what do we mean by satisfy? Let's work an example or two to illustrate this.

Example 1 Show that each of the following numbers are solutions to the given equation or inequality.

(a) $x = 3$ in $x^2 - 9 = 0$

(b) $y = 8$ in $3(y+1) = 4y - 5$

(c) $z = 1$ in $2(z-5) \le 4z$

(d) $z = -5$ in $2(z-5) \le 4z$

Solution

(a) We first plug the proposed solution into the equation.

$$\begin{aligned} 3^2 - 9 &\stackrel{?}{=} 0 \\ 9 - 9 &= 0 \\ 0 &= 0 \qquad \text{OK} \end{aligned}$$

So, what we are asking here is does the right side equal the left side after we plug in the proposed solution. That is the meaning of the "?" above the equal sign in the first line.

Since the right side and the left side are the same we say that $x = 3$ **satisfies** the equation.

(b) So, we want to see if $y = 8$ satisfies the equation. First plug the value into the equation.

$$\begin{aligned} 3(8+1) &\stackrel{?}{=} 4(8) - 5 \\ 27 &= 27 \qquad \text{OK} \end{aligned}$$

So, $y = 8$ satisfies the equation and so is a solution.

(c) In this case we've got an inequality and in this case "satisfy" means something slightly different. In this case we will say that a number will satisfy the inequality if, after plugging it in, we get a true inequality as a result.

Let's check $z = 1$.

$$\begin{aligned} 2(1-5) &\stackrel{?}{\le} 4(1) \\ -8 &\le 4 \qquad \text{OK} \end{aligned}$$

So, -8 is less than or equal to 4 (in fact it's less than) and so we have a true inequality. Therefore $z = 1$ will satisfy the inequality and hence is a solution

(d) This is the same inequality with a different value so let's check that.

$$2(-5-5) \overset{?}{\le} 4(-5)$$
$$-20 \le -20 \qquad \text{OK}$$

In this case -20 is less than or equal to -20 (in this case it's equal) and so again we get a true inequality and so $z = -5$ satisfies the inequality and so will be a solution.

We should also do a quick example of numbers that aren't solution so we can see how these will work as well.

Example 2 Show that the following numbers aren't solutions to the given equation or inequality.

(a) $y = -2$ in $3(y+1) = 4y - 5$

(b) $z = -12$ in $2(z-5) \le 4z$

Solution

(a) In this case we do essentially the same thing that we did in the previous example. Plug the number in and show that this time it doesn't satisfy the equation. For equations that will mean that the right side of the equation will not equal the left side of the equation.

$$3(-2+1) \overset{?}{=} 4(-2) - 5$$
$$-3 \ne -13 \qquad \text{NOT OK}$$

So, -3 is not the same as -13 and so the equation isn't satisfied. Therefore $y = -2$ isn't a solution to the equation.

(b) This time we've got an inequality. A number will not satisfy an inequality if we get an inequality that isn't true after plugging the number in.

$$2(-12-5) \overset{?}{\le} 4(-12)$$
$$-34 \not\le -48 \qquad \text{NOT OK}$$

In this case -34 is NOT less than or equal to -48 and so the inequality isn't satisfied. Therefore $z = -12$ is not a solution to the inequality.

Now, there is no reason to think that a given equation or inequality will only have a single solution. In fact, as the first example showed the inequality $2(z-5) \le 4z$ has at least two solutions. Also, you might have noticed that $x = 3$ is not the only solution to $x^2 - 9 = 0$. In this case $x = -3$ is also a solution.

We call the complete set of all solutions the **solution set** for the equation or inequality. There is also some formal notation for solution sets although we won't be using it all that often in this course. Regardless of that fact we should still acknowledge it.

For equations we denote the solution set by enclosing all the solutions in a set of braces, $\{\ \}$. For the two equations we looked at above here are the solution sets.

$$3(y+1) = 4y - 5 \qquad \text{Solution Set : } \{8\}$$
$$x^2 - 9 = 0 \qquad \text{Solution Set : } \{-3, 3\}$$

For inequalities we have a similar notation. Depending on the complexity of the inequality the solution set may be a single number or it may be a range of numbers. If it is a single number then we use the same notation as we used for equations. If the solution set is a range of numbers, as the one we looked at above is, we will use something called **set builder notation**. Here is the solution set for the inequality we looked at above.

$$\{z \mid z \geq -5\}$$

This is read as : "The set of all *z* such that *z* is greater than or equal to -5".

Most of the inequalities that we will be looking at will have simple enough solution sets that we often just shorthand this as,

$$z \geq -5$$

There is one final topic that we need to address as far as solution sets go before leaving this section. Consider the following equation and inequality.

$$x^2 + 1 = 0$$
$$x^2 < 0$$

If we restrict ourselves to only real solutions (which we won't always do) then there is no solution to the equation. Squaring *x* makes *x* greater than equal to zero, then adding 1 onto that means that the left side is guaranteed to be at least 1. In other words, there is no real solution to this equation. For the same basic reason there is no solution to the inequality. Squaring any real *x* makes it positive or zero and so will never be negative.

We need a way to denote the fact that there are no solutions here. In solution set notation we say that the solution set is **empty** and denote it with the symbol : $\varnothing$. This symbol is often called the **empty set**.

We now need to make a couple of final comments before leaving this section.

In the above discussion of empty sets we assumed that we were only looking for real solutions. While that is what we will be doing for inequalities, we won't be restricting ourselves to real solutions with equations. Once we get around to solving quadratic equations (which $x^2 + 1 = 0$ is) we will allow solutions to be complex numbers and in the case looked at above there are complex solutions to $x^2 + 1 = 0$. If you don't know how to find these at this point that is fine we will be covering that material in a couple of sections. At this point just accept that $x^2 + 1 = 0$ does have complex solutions.

Finally, as noted above we won't be using the solution set notation much in this course. It is a nice notation and does have some use on occasion especially for complicated solutions. However, for the vast majority of the equations and inequalities that we will be looking at will have simple enough solution sets that it's just easier to write down the solutions and let it go at that. Therefore, that is what we will not be using the notation for our solution sets. However, you should be aware of the notation and know what it means.

Section 2-2 : Linear Equations

We'll start off the solving portion of this chapter by solving linear equations. A **linear equation** is any equation that can be written in the form

$$ax + b = 0$$

where *a* and *b* are real numbers and *x* is a variable. This form is sometimes called the **standard form** of a linear equation. Note that most linear equations will not start off in this form. Also, the variable may or may not be an *x* so don't get too locked into always seeing an *x* there.

To solve linear equations we will make heavy use of the following facts.

1. If $a = b$ then $a + c = b + c$ for any *c*. All this is saying is that we can add a number, *c*, to both sides of the equation and not change the equation.

2. If $a = b$ then $a - c = b - c$ for any *c*. As with the last property we can subtract a number, *c*, from both sides of an equation.

3. If $a = b$ then $ac = bc$ for any *c*. Like addition and subtraction, we can multiply both sides of an equation by a number, *c*, without changing the equation.

4. If $a = b$ then $\frac{a}{c} = \frac{b}{c}$ for any non-zero *c*. We can divide both sides of an equation by a non-zero number, *c*, without changing the equation.

These facts form the basis of almost all the solving techniques that we'll be looking at in this chapter so it's very important that you know them and don't forget about them. One way to think of these rules is the following. What we do to one side of an equation we have to do to the other side of the equation. If you remember that then you will always get these facts correct.

In this section we will be solving linear equations and there is a nice simple process for solving linear equations. Let's first summarize the process and then we will work some examples.

Process for Solving Linear Equations

1. If the equation contains any fractions use the least common denominator to clear the fractions. We will do this by multiplying both sides of the equation by the LCD.

   Also, if there are variables in the denominators of the fractions identify values of the variable which will give division by zero as we will need to avoid these values in our solution.

2. Simplify both sides of the equation. This means clearing out any parenthesis and combining like terms.

3. Use the first two facts above to get all terms with the variable in them on one side of the equations (combining into a single term of course) and all constants on the other side.

4. If the coefficient of the variable is not a one use the third or fourth fact above (this will depend on just what the number is) to make the coefficient a one.

   Note that we usually just divide both sides of the equation by the coefficient if it is an integer or multiply both sides of the equation by the reciprocal of the coefficient if it is a fraction.

5. **VERIFY YOUR ANSWER!** This is the final step and the most often skipped step, yet it is probably the most important step in the process. With this step you can know whether or not you got the correct answer long before your instructor ever looks at it. We verify the answer by plugging the results from the previous steps into the **original** equation. It is very important to plug into the original equation since you may have made a mistake in the very first step that led you to an incorrect answer.

   Also, if there were fractions in the problem and there were values of the variable that give division by zero (recall the first step…) it is important to make sure that one of these values didn't end up in the solution set. It is possible, as we'll see in an example, to have these values show up in the solution set.

Let's take a look at some examples.

Example 1 Solve each of the following equations.

(a) $3\left(x+5\right)=2\left(-6-x\right)-2x$

(b) $\dfrac{m-2}{3}+1=\dfrac{2m}{7}$

(c) $\dfrac{5}{2y-6}=\dfrac{10-y}{y^2-6y+9}$

(d) $\dfrac{2z}{z+3}=\dfrac{3}{z-10}+2$

Solution

In the following problems we will describe in detail the first problem and the leave most of the explanation out of the following problems.

(a) $3\left(x+5\right)=2\left(-6-x\right)-2x$

For this problem there are no fractions so we don't need to worry about the first step in the process. The next step tells to simplify both sides. So, we will clear out any parenthesis by multiplying the numbers through and then combine like terms.

$$\begin{aligned}3\left(x+5\right)&=2\left(-6-x\right)-2x\\3x+15&=-12-2x-2x\\3x+15&=-12-4x\end{aligned}$$

The next step is to get all the *x*'s on one side and all the numbers on the other side. Which side the *x*'s go on is up to you and will probably vary with the problem. As a rule of thumb, we will usually put the variables on the side that will give a positive coefficient. This is done simply because it is often easy to lose track of the minus sign on the coefficient and so if we make sure it is positive we won't need to worry about it.

So, for our case this will mean adding 4*x* to both sides and subtracting 15 from both sides. Note as well that while we will actually put those operations in this time we normally do these operations in our head.

$$\begin{aligned} 3x+15 &= -12-4x \\ 3x+15-15+4x &= -12-4x+4x-15 \\ 7x &= -27 \end{aligned}$$

The next step says to get a coefficient of 1 in front of the *x*. In this case we can do this by dividing both sides by a 7.

$$\begin{aligned} \frac{7x}{7} &= \frac{-27}{7} \\ x &= -\frac{27}{7} \end{aligned}$$

Now, if we've done all of our work correct $x = -\frac{27}{7}$ is the solution to the equation.

The last and final step is to then check the solution. As pointed out in the process outline we need to check the solution in the **original** equation. This is important, because we may have made a mistake in the very first step and if we did and then checked the answer in the results from that step it may seem to indicate that the solution is correct when the reality will be that we don't have the correct answer because of the mistake that we originally made.

The problem of course is that, with this solution, that checking might be a little messy. Let's do it anyway.

$$\begin{aligned} 3\left(-\frac{27}{7}+5\right) &\stackrel{?}{=} 2\left(-6-\left(-\frac{27}{7}\right)\right)-2\left(-\frac{27}{7}\right) \\ 3\left(\frac{8}{7}\right) &\stackrel{?}{=} 2\left(-\frac{15}{7}\right)+\frac{54}{7} \\ \frac{24}{7} &= \frac{24}{7} \qquad \text{OK} \end{aligned}$$

So, we did our work correctly and the solution to the equation is,

$$x = -\frac{27}{7}$$

Note that we didn't use the solution set notation here. For single solutions we will rarely do that in this class. However, if we had wanted to the solution set notation for this problem would be,

$$\left\{-\frac{27}{7}\right\}$$

Before proceeding to the next problem let's first make a quick comment about the "messiness' of this answer. Do NOT expect all answers to be nice simple integers. While we do try to keep most answer simple often they won't be so do NOT get so locked into the idea that an answer must be a simple integer that you immediately assume that you've made a mistake because of the "messiness" of the answer.

(b) $\dfrac{m-2}{3}+1=\dfrac{2m}{7}$

Okay, with this one we won't be putting quite as much explanation into the problem.

In this case we have fractions so to make our life easier we will multiply both sides by the LCD, which is 21 in this case. After doing that the problem will be very similar to the previous problem. Note as well that the denominators are only numbers and so we won't need to worry about division by zero issues.

Let's first multiply both sides by the LCD.

$$\begin{aligned} 21\left(\frac{m-2}{3}+1\right) &= \left(\frac{2m}{7}\right)21 \\ 21\left(\frac{m-2}{3}\right)+21(1) &= \left(\frac{2m}{7}\right)21 \\ 7(m-2)+21 &= (2m)(3) \end{aligned}$$

Be careful to correctly distribute the 21 through the parenthesis on the left side. Everything inside the parenthesis needs to be multiplied by the 21 before we simplify. At this point we've got a problem that is similar the previous problem and we won't bother with all the explanation this time.

$$\begin{aligned} 7(m-2)+21 &= (2m)(3) \\ 7m-14+21 &= 6m \\ 7m+7 &= 6m \\ m &= -7 \end{aligned}$$

So, it looks like $m=-7$ is the solution. Let's verify it to make sure.

$$\begin{aligned} \frac{-7-2}{3}+1 &\stackrel{?}{=} \frac{2(-7)}{7} \\ \frac{-9}{3}+1 &\stackrel{?}{=} -\frac{14}{7} \\ -3+1 &\stackrel{?}{=} -2 \\ -2 &= -2 \qquad \text{OK} \end{aligned}$$

So, it is the solution.

(c) $\dfrac{5}{2y-6}=\dfrac{10-y}{y^2-6y+9}$

This one is similar to the previous one except now we've got variables in the denominator. So, to get the LCD we'll first need to completely factor the denominators of each rational expression.

$$\frac{5}{2(y-3)}=\frac{10-y}{(y-3)^2}$$

So, it looks like the LCD is $2(y-3)^2$. Also note that we will need to avoid $y=3$ since if we plugged that into the equation we would get division by zero.

Now, outside of the *y*'s in the denominator this problem works identical to the previous one so let's do the work.

$$\begin{aligned}(2)(y-3)^2\left(\frac{5}{2(y-3)}\right)&=\left(\frac{10-y}{(y-3)^2}\right)(2)(y-3)^2\\5(y-3)&=2(10-y)\\5y-15&=20-2y\\7y&=35\\y&=5\end{aligned}$$

Now the solution is not $y=3$ so we won't get division by zero with the solution which is a good thing. Finally, let's do a quick verification.

$$\begin{aligned}\frac{5}{2(5)-6}&\stackrel{?}{=}\frac{10-5}{5^2-6(5)+9}\\\frac{5}{4}&=\frac{5}{4}\qquad\qquad\text{OK}\end{aligned}$$

So we did the work correctly.

(d) $\dfrac{2z}{z+3}=\dfrac{3}{z-10}+2$

In this case it looks like the LCD is $(z+3)(z-10)$ and it also looks like we will need to avoid $z=-3$ and $z=10$ to make sure that we don't get division by zero.

Let's get started on the work for this problem.

$$\begin{aligned}(z+3)(z-10)\left(\frac{2z}{z+3}\right)&=\left(\frac{3}{z-10}+2\right)(z+3)(z-10)\\2z(z-10)&=3(z+3)+2(z+3)(z-10)\\2z^2-20z&=3z+9+2\left(z^2-7z-30\right)\end{aligned}$$

At this point let's pause and acknowledge that we've got a z^2 in the work here. Do not get excited about that. Sometimes these will show up temporarily in these problems. You should only worry about it if it is still there after we finish the simplification work.

So, let's finish the problem.

$$\begin{aligned}\cancel{2z^2} - 20z &= 3z + 9 + \cancel{2z^2} - 14z - 60\\ -20z &= -11z - 51\\ 51 &= 9z\\ \frac{51}{9} &= z\\ \frac{17}{3} &= z\end{aligned}$$

Notice that the z^2 did in fact cancel out. Now, if we did our work correctly $z = \frac{17}{3}$ should be the solution since it is not either of the two values that will give division by zero. Let's verify this.

$$\begin{aligned}\frac{2\left(\frac{17}{3}\right)}{\frac{17}{3}+3} &\stackrel{?}{=} \frac{3}{\frac{17}{3}-10}+2\\ \frac{\frac{34}{3}}{\frac{26}{3}} &\stackrel{?}{=} \frac{3}{-\frac{13}{3}}+2\\ \frac{34}{3}\left(\frac{3}{26}\right) &\stackrel{?}{=} 3\left(-\frac{3}{13}\right)+2\\ \frac{17}{13} &= \frac{17}{13} \qquad \text{OK}\end{aligned}$$

The checking can be a little messy at times, but it does mean that we KNOW the solution is correct.

Okay, in the last couple of parts of the previous example we kept going on about watching out for division by zero problems and yet we never did get a solution where that was an issue. So, we should now do a couple of those problems to see how they work.

Example 2 Solve each of the following equations.

(a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

(b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

Solution

(a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

The first step is to factor the denominators to get the LCD.

$$\frac{2}{x+2} = \frac{-x}{(x+2)(x+3)}$$

So, the LCD is $(x+2)(x+3)$ and we will need to avoid $x=-2$ and $x=-3$ so we don't get division by zero.

Here is the work for this problem.

$$\begin{aligned}(x+2)(x+3)\left(\frac{2}{x+2}\right) &= \left(\frac{-x}{(x+2)(x+3)}\right)(x+2)(x+3)\\ 2(x+3) &= -x\\ 2x+6 &= -x\\ 3x &= -6\\ x &= -2\end{aligned}$$

So, we get a "solution" that is in the list of numbers that we need to avoid so we don't get division by zero and so we can't use it as a solution. However, this is also the only possible solution. That is okay. This just means that this equation has **no solution**.

(b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

The LCD for this equation is $x+1$ and we will need to avoid $x=-1$ so we don't get division by zero. Here is the work for this equation.

$$\begin{aligned}\left(\frac{2}{x+1}\right)(x+1) &= \left(4-\frac{2x}{x+1}\right)(x+1)\\ 2 &= 4(x+1)-2x\\ 2 &= 4x+4-2x\\ 2 &= 2x+4\\ -2 &= 2x\\ -1 &= x\end{aligned}$$

So, we once again arrive at the single value of *x* that we needed to avoid so we didn't get division by zero. Therefore, this equation has **no solution**.

So, as we've seen we do need to be careful with division by zero issues when we start off with equations that contain rational expressions.

At this point we should probably also acknowledge that provided we don't have any division by zero issues (such as those in the last set of examples) linear equations will have exactly one solution. We will never get more than one solution and the only time that we won't get any solutions is if we run across a division by zero problems with the "solution".

Before leaving this section we should note that many of the techniques for solving linear equations will show up time and again as we cover different kinds of equations so it very important that you understand this process.

Section 2- 3 : Applications of Linear Equations

We now need to discuss the section that most students hate. We need to talk about applications to linear equations. Or, put in other words, we will now start looking at story problems or word problems. Throughout history students have hated these. It is my belief however that the main reason for this is that students really don't know how to work them. Once you understand how to work them, you'll probably find that they aren't as bad as they may seem on occasion. So, we'll start this section off with a process for working applications.

Process for Working Story/Word Problems

1. **READ THE PROBLEM.**

2. **READ THE PROBLEM AGAIN.** Okay, this may be a little bit of overkill here. However, the point of these first two steps is that you must read the problem. This step is the MOST important step, but it is also the step that most people don't do properly.

   You need to read the problem very carefully and as many times as it takes. You are only done with this step when you have completely understood what the problem is asking you to do. This includes identifying all the given information and identifying what you being asked to find.

   Again, it can't be stressed enough that you've got to carefully read the problem. Sometimes a single word can completely change how the problem is worked. If you just skim the problem you may well miss that very important word.

3. Represent one of the unknown quantities with a variable and try to relate all the other unknown quantities (if there are any of course) to this variable.

4. If applicable, sketch a figure illustrating the situation. This may seem like a silly step, but it can be incredibly helpful with the next step on occasion.

5. Form an equation that will relate known quantities to the unknown quantities. To do this make use of known formulas and often the figure sketched in the previous step can be used to determine the equation.

6. Solve the equation formed in the previous step and write down the answer to all the questions. It is important to answer all the questions that you were asked. Often you will be asked for several quantities in the answer and the equation will only give one of them.

7. Check your answer. Do this by plugging into the equation, but also use intuition to make sure that the answer makes sense. Mistakes can often be identified by acknowledging that the answer just doesn't make sense.

Let's start things off with a couple of fairly basic examples to illustrate the process. Note as well that at this point it is assumed that you are capable of solving fairly simple linear equations and so not a lot of detail will be given for the actual solution stage. The point of this section is more on the set up of the equation than the solving of the equation.

Example 1 In a certain Algebra class there is a total of 350 possible points. These points come from 5 homework sets that are worth 10 points each and 3 hour exams that are worth 100 points each. A student has received homework scores of 4, 8, 7, 7, and 9 and the first two exam scores are 78 and 83. Assuming that grades are assigned according to the standard scale and there are no weights assigned to any of the grades is it possible for the student to receive an A in the class and if so what is the minimum score on the third exam that will give an A? What about a B?

Solution
Okay, let's start off by defining *p* to be the minimum required score on the third exam.

Now, let's recall how grades are set. Since there are no weights or anything on the grades, the grade will be set by first computing the following percentage.

$$\frac{\text{actual points}}{\text{total possible points}} = \text{grade percentage}$$

Since we are using the standard scale if the grade percentage is 0.9 or higher the student will get an A. Likewise, if the grade percentage is between 0.8 and 0.9 the student will get a B.

We know that the total possible points is 350 and the student has a total points (including the third exam) of,

$$4+8+7+7+9+78+83+p=196+p$$

The smallest possible percentage for an A is 0.9 and so if *p* is the minimum required score on the third exam for an A we will have the following equation.

$$\frac{196+p}{350}=0.9$$

This is a linear equation that we will need to solve for *p*.

$$196+p=0.9\left(350\right)=315 \qquad \Rightarrow \qquad p=315-196=119$$

So, the minimum required score on the third exam is 119. This is a problem since the exam is worth only 100 points. In other words, the student will not be getting an A in the Algebra class.

Now let's check if the student will get a B. In this case the minimum percentage is 0.8. So, to find the minimum required score on the third exam for a B we will need to solve,

$$\frac{196+p}{350}=0.8$$

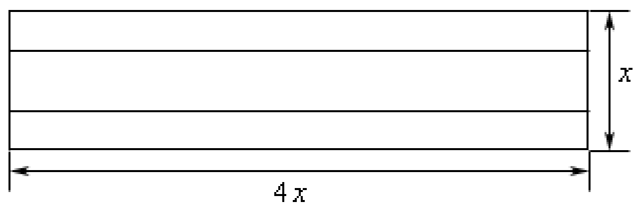
Solving this for *p* gives,

$$196+p=0.8\left(350\right)=280 \qquad \Rightarrow \qquad p=280-196=84$$

So, it is possible for the student to get a B in the class. All that the student will need to do is get at least an 84 on the third exam.

Example 2 We want to build a set of shelves. The width of the set of shelves needs to be 4 times the height of the set of shelves and the set of shelves must have three shelves in it. If there are 72 feet of wood to use to build the set of shelves what should the dimensions of the set of shelves be?

Solution
We will first define *x* to be the height of the set of shelves. This means that 4*x* is width of the set of shelves. In this case we definitely need to sketch a figure so we can correctly set up the equation. Here it is,



Now we know that there are 72 feet of wood to be used and we will assume that all of it will be used. So, we can set up the following word equation.

$$\left(\begin{array}{c}\text{length of}\\\text{vertical pieces}\end{array}\right)+\left(\begin{array}{c}\text{length of}\\\text{horizontal pieces}\end{array}\right)=72$$

It is often a good idea to first put the equation in words before actually writing down the equation as we did here. At this point, we can see from the figure there are two vertical pieces; each one has a length of *x*. Also, there are 4 horizontal pieces, each with a length of 4*x*. So, the equation is then,

$$\begin{aligned}4(4x)+2(x)&=72\\16x+2x&=72\\18x&=72\\x&=4\end{aligned}$$

So, it looks like the height of the set of shelves should be 4 feet. Note however that we haven't actually answered the question however. The problem asked us to find the dimensions. This means that we also need the width of the set of shelves. The width is 4(4)=16 feet. So the dimensions will need to be 4x16 feet.

Pricing Problems
The next couple of problems deal with some basic principles of pricing.

Example 3 A calculator has been marked up 15% and is being sold for \$78.50. How much did the store pay the manufacturer of the calculator?

Solution
First, let's define *p* to be the cost that the store paid for the calculator. The stores markup on the calculator is 15%. This means that 0.15*p* has been added on to the original price (*p*) to get the amount the calculator is being sold for. In other words, we have the following equation

$$p + 0.15p = 78.50$$

that we need to solve for *p*. Doing this gives,

$$1.15p = 78.50 \qquad \Rightarrow \qquad p = \frac{78.50}{1.15} = 68.26087$$

The store paid $68.26 for the calculator. Note that since we are dealing with money we rounded the answer down to two decimal places.

Example 4 A shirt is on sale for $15.00 and has been marked down 35%. How much was the shirt being sold for before the sale?

Solution

This problem is pretty much the opposite of the previous example. Let's start with defining *p* to be the price of the shirt before the sale. It has been marked down by 35%. This means that 0.35*p* has been subtracted off from the original price. Therefore, the equation (and solution) is,

$$\begin{aligned} p - 0.35p &= 15.00 \\ 0.65p &= 15.00 \\ p &= \frac{15.00}{0.65} = 23.0769 \end{aligned}$$

So, with rounding it looks like the shirt was originally sold for $23.08.

Distance/Rate Problems

These are some of the standard problems that most people think about when they think about Algebra word problems. The standard formula that we will be using here is

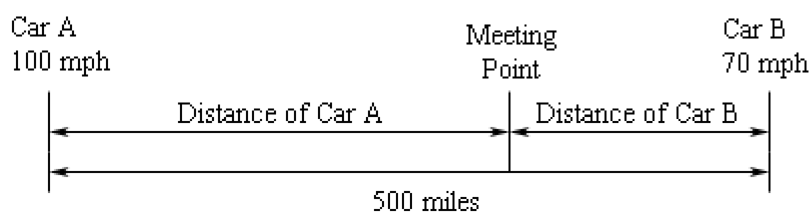
$$\text{Distance} = \text{Rate} \times \text{Time}$$

All of the problems that we'll be doing in this set of examples will use this to one degree or another and often more than once as we will see.

Example 5 Two cars are 500 miles apart and moving directly towards each other. One car is moving at a speed of 100 mph and the other is moving at 70 mph. Assuming that the cars start moving at the same time how long does it take for the two cars to meet?

Solution

Let's let *t* represent the amount of time that the cars are traveling before they meet. Now, we need to sketch a figure for this one. This figure will help us to write down the equation that we'll need to solve.



From this figure we can see that the Distance Car A travels plus the Distance Car B travels must equal the total distance separating the two cars, 500 miles.

Here is the word equation for this problem in two separate forms.

$$\begin{pmatrix}\text{Distance}\\ \text{of Car A}\end{pmatrix}+\begin{pmatrix}\text{Distance}\\ \text{of Car B}\end{pmatrix}=500$$

$$\begin{pmatrix}\text{Rate of}\\ \text{Car A}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{Car A}\end{pmatrix}+\begin{pmatrix}\text{Rate of}\\ \text{Car B}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{Car B}\end{pmatrix}=500$$

We used the standard formula here twice, once for each car. We know that the distance a car travels is the rate of the car times the time traveled by the car. In this case we know that Car A travels at 100 mph for *t* hours and that Car B travels at 70 mph for *t* hours as well. Plugging these into the word equation and solving gives us,

$$\begin{aligned} 100t+70t&=500\\ 170t&=500\\ t&=\frac{500}{170}=2.941176\text{ hrs}\end{aligned}$$

So, they will travel for approximately 2.94 hours before meeting.

Example 6 Repeat the previous example except this time assume that the faster car will start 1 hour after slower car starts.

Solution

For this problem we are going to need to be careful with the time traveled by each car. Let's let *t* be the amount of time that the slower travel car travels. Now, since the faster car starts out 1 hour after the slower car it will only travel for $t-1$ hours.

Now, since we are repeating the problem from above the figure and word equation will remain identical and so we won't bother repeating them here. The only difference is what we substitute for the time traveled for the faster car. Instead of *t* as we used in the previous example we will use $t-1$ since it travels for one hour less that the slower car.

Here is the equation and solution for this example.

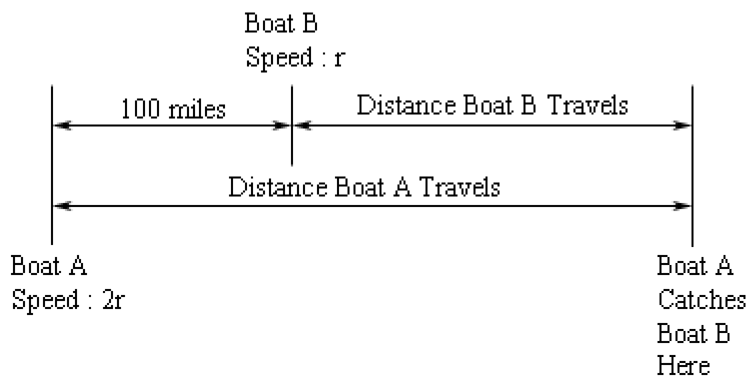
$$\begin{aligned} 100\left(t-1\right)+70t&=500\\ 100t-100+70t&=500\\ 170t&=600\\ t&=\frac{600}{170}=3.529412\text{ hrs}\end{aligned}$$

In this case the slower car will travel for 3.53 hours before meeting while the faster car will travel for 2.53 hrs (1 hour less than the slower car…).

Example 7 Two boats start out 100 miles apart and start moving to the right at the same time. The boat on the left is moving at twice the speed as the boat on the right. Five hours after starting the boat on the left catches up with the boat on the right. How fast was each boat moving?

Solution

Let's start off by letting *r* be the speed of the boat on the right (the slower boat). This means that the boat to the left (the faster boat) is moving at a speed of 2*r*. Here is the figure for this situation.



From the figure it looks like we've got the following word equation.

$$100+\begin{pmatrix}\text{Distance}\\ \text{of Boat B}\end{pmatrix}=\begin{pmatrix}\text{Distance}\\ \text{of Boat A}\end{pmatrix}$$

Upon plugging in the standard formula for the distance gives,

$$100+\begin{pmatrix}\text{Rate of}\\ \text{Boat B}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{Boat B}\end{pmatrix}=\begin{pmatrix}\text{Rate of}\\ \text{Boat A}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{Boat A}\end{pmatrix}$$

For this problem we know that the time each is 5 hours and we know that the rate of Boat A is 2*r* and the rate of Boat B is *r*. Plugging these into the work equation and solving gives,

$$\begin{aligned}100+(r)(5)&=(2r)(5)\\100+5r&=10r\\100&=5r\\20&=r\end{aligned}$$

So, the slower boat is moving at 20 mph and the faster boat is moving at 40 mph (twice as fast).

Work/Rate Problems

These problems are actually variants of the Distance/Rate problems that we just got done working. The standard equation that will be needed for these problems is,

$$\begin{pmatrix}\text{Portion of job}\\ \text{done in given time}\end{pmatrix} = \begin{pmatrix}\text{Work}\\ \text{Rate}\end{pmatrix} \times \begin{pmatrix}\text{Time Spent}\\ \text{Working}\end{pmatrix}$$

As you can see this formula is very similar to the formula we used above.

Example 8 An office has two envelope stuffing machines. Machine A can stuff a batch of envelopes in 5 hours, while Machine B can stuff a batch of envelopes in 3 hours. How long would it take the two machines working together to stuff a batch of envelopes?

Solution

Let t be the time that it takes both machines, working together, to stuff a batch of envelopes. The word equation for this problem is,

$$\begin{pmatrix}\text{Portion of job}\\ \text{done by Machine A}\end{pmatrix} + \begin{pmatrix}\text{Portion of job}\\ \text{done by Machine B}\end{pmatrix} = 1 \text{ Job}$$

$$\begin{pmatrix}\text{Work Rate}\\ \text{of Machine A}\end{pmatrix}\begin{pmatrix}\text{Time Spent}\\ \text{Working}\end{pmatrix} + \begin{pmatrix}\text{Work Rate}\\ \text{of Machine B}\end{pmatrix}\begin{pmatrix}\text{Time Spent}\\ \text{Working}\end{pmatrix} = 1$$

We know that the time spent working is t however we don't know the work rate of each machine. To get these we'll need to use the initial information given about how long it takes each machine to do the job individually. We can use the following equation to get these rates.

$$1 \text{ Job} = \begin{pmatrix}\text{Work}\\ \text{Rate}\end{pmatrix} \times \begin{pmatrix}\text{Time Spent}\\ \text{Working}\end{pmatrix}$$

Let's start with Machine A.

$$1 \text{ Job} = \left(\text{Work Rate of A}\right) \times (5) \qquad \Rightarrow \qquad \text{Work Rate of A} = \frac{1}{5}$$

Now, Machine B.

$$1 \text{ Job} = \left(\text{Work Rate of B}\right) \times (3) \qquad \Rightarrow \qquad \text{Work Rate of B} = \frac{1}{3}$$

Plugging these quantities into the main equation above gives the following equation that we need to solve.

$$\begin{aligned} \frac{1}{5}t + \frac{1}{3}t &= 1 \qquad\qquad \text{Multiplying both sides by 15}\\ 3t + 5t &= 15\\ 8t &= 15\\ t &= \frac{15}{8} = 1.875 \text{ hours} \end{aligned}$$

So, it looks like it will take the two machines, working together, 1.875 hours to stuff a batch of envelopes.

Example 9 Mary can clean an office complex in 5 hours. Working together John and Mary can clean the office complex in 3.5 hours. How long would it take John to clean the office complex by himself?

Solution

Let *t* be the amount of time it would take John to clean the office complex by himself. The basic word equation for this problem is,

$$\left(\begin{array}{c}\text{Portion of job}\\ \text{done by Mary}\end{array}\right)+\left(\begin{array}{c}\text{Portion of job}\\ \text{done by John}\end{array}\right)=1\text{ Job}$$

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of Mary}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right)+\left(\begin{array}{c}\text{Work Rate}\\ \text{of John}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right)=1$$

This time we know that the time spent working together is 3.5 hours. We now need to find the work rates for each person. We'll start with Mary.

$$1\text{ Job}=\left(\text{Work Rate of Mary}\right)\times\left(5\right)\qquad\Rightarrow\qquad\text{Work Rate of Mary}=\frac{1}{5}$$

Now we'll find the work rate of John. Notice however, that since we don't know how long it will take him to do the job by himself we aren't going to be able to get a single number for this. That is not a problem as we'll see in a second.

$$1\text{ Job}=\left(\text{Work Rate of John}\right)\times\left(t\right)\qquad\Rightarrow\qquad\text{Work Rate of John}=\frac{1}{t}$$

Notice that we've managed to get the work rate of John in terms of the time it would take him to do the job himself. This means that once we solve the equation above we'll have the answer that we want. So, let's plug into the work equation and solve for the time it would take John to do the job by himself.

$$\begin{aligned}\frac{1}{5}\left(3.5\right)+\frac{1}{t}\left(3.5\right)&=1 &&\text{Multiplying both sides by }5t\\ 3.5t+\left(3.5\right)\left(5\right)&=5t\\ 17.5&=1.5t\\ \frac{17.5}{1.5}&=t &&\Rightarrow\qquad t=11.67\text{ hrs}\end{aligned}$$

So, it looks like it would take John 11.67 hours to clean the complex by himself.

Mixing Problems

This is the final type of problems that we'll be looking at in this section. We are going to be looking at mixing solutions of different percentages to get a new percentage. The solution will consist of a secondary liquid mixed in with water. The secondary liquid can be alcohol or acid for instance.

The standard equation that we'll use here will be the following.

$$\left(\begin{array}{c}\text{Amount of secondary}\\ \text{liquid in the water}\end{array}\right)=\left(\begin{array}{c}\text{Percentage of}\\ \text{Solution}\end{array}\right)\times\left(\begin{array}{c}\text{Volume of}\\ \text{Solution}\end{array}\right)$$

Note as well that the percentage needs to be a decimal. So if we have an 80% solution we will need to use 0.80.

Example 10 How much of a 50% alcohol solution should we mix with 10 gallons of a 35% solution to get a 40% solution?

Solution

Okay, let *x* be the amount of 50% solution that we need. This means that there will be $x+10$ gallons of the 40% solution once we're done mixing the two.

Here is the basic work equation for this problem.

$$\begin{pmatrix}\text{Amount of alcohol}\\ \text{in 50\% Solution}\end{pmatrix}+\begin{pmatrix}\text{Amount of alcohol}\\ \text{in 35\% Solution}\end{pmatrix}=\begin{pmatrix}\text{Amount of alcohol}\\ \text{in 40\% Solution}\end{pmatrix}$$

$$\left(0.5\right)\begin{pmatrix}\text{Volume of}\\ \text{50\% Solution}\end{pmatrix}+\left(0.35\right)\begin{pmatrix}\text{Volume of}\\ \text{35\% Solution}\end{pmatrix}=\left(0.4\right)\begin{pmatrix}\text{Volume of}\\ \text{40\% Solution}\end{pmatrix}$$

Now, plug in the volumes and solve for *x*.

$$\begin{aligned}0.5x+0.35\left(10\right)&=0.4\left(x+10\right)\\ 0.5x+3.5&=0.4x+4\\ 0.1x&=0.5\\ x&=\frac{0.5}{0.1}=5\text{ gallons}\end{aligned}$$

So, we need 5 gallons of the 50% solution to get a 40% solution.

Example 11 We have a 40% acid solution and we want 75 liters of a 15% acid solution. How much water should we put into the 40% solution to do this?

Solution

Let *x* be the amount of water we need to add to the 40% solution. Now, we also don't how much of the 40% solution we'll need. However, since we know the final volume (75 liters) we will know that we will need $75-x$ liters of the 40% solution.

Here is the word equation for this problem.

$$\begin{pmatrix}\text{Amount of acid}\\ \text{in the water}\end{pmatrix}+\begin{pmatrix}\text{Amount of acid}\\ \text{in 40\% Solution}\end{pmatrix}=\begin{pmatrix}\text{Amount of acid}\\ \text{in 15\% Solution}\end{pmatrix}$$

Notice that in the first term we used the "Amount of acid in the water". This might look a little weird to you because there shouldn't be any acid in the water. However, this is exactly what we want. The basic equation tells us to look at how much of the secondary liquid is in the water. So, this is the correct wording. When we plug in the percentages and volumes we will think of the water as a 0% percent solution since that is in fact what it is. So, the new word equation is,

$$\left(0\right)\left(\begin{array}{c}\text{Volume}\\ \text{of Water}\end{array}\right)+\left(0.4\right)\left(\begin{array}{c}\text{Volume of}\\ \text{40\% Solution}\end{array}\right)=\left(0.15\right)\left(\begin{array}{c}\text{Volume of}\\ \text{15\% Solution}\end{array}\right)$$

Do not get excited about the zero in the first term. This is okay and will not be a problem. Let's now plug in the volumes and solve for *x*.

$$\begin{aligned}\left(0\right)\left(x\right)+\left(0.4\right)\left(75-x\right)&=\left(0.15\right)\left(75\right)\\ 30-0.4x&=11.25\\ 18.75&=0.4x\\ x&=\frac{18.75}{0.4}=46.875\text{ liters}\end{aligned}$$

So, we need to add in 46.875 liters of water to 28.125 liters of a 40% solution to get 75 liters of a 15% solution.

Section 2-4 : Equations With More Than One Variable

In this section we are going to take a look at a topic that often doesn't get the coverage that it deserves in an Algebra class. This is probably because it isn't used in more than a couple of sections in an Algebra class. However, this is a topic that can, and often is, used extensively in other classes.

What we'll be doing here is solving equations that have more than one variable in them. The process that we'll be going through here is very similar to solving linear equations, which is one of the reasons why this is being introduced at this point. There is however one exception to that. Sometimes, as we will see, the ordering of the process will be different for some problems. Here is the process in the standard order.

1. Multiply both sides by the LCD to clear out any fractions.
2. Simplify both sides as much as possible. This will often mean clearing out parenthesis and the like.
3. Move all terms containing the variable we're solving for to one side and all terms that don't contain the variable to the opposite side.
4. Get a single instance of the variable we're solving for in the equation. For the types of problems that we'll be looking at here this will almost always be accomplished by simply factoring the variable out of each of the terms.
5. Divide by the coefficient of the variable. This step will make sense as we work problems. Note as well that in these problems the "coefficient" will probably contain things other than numbers.

It is usually easiest to see just what we're going to be working with and just how they work with an example. We will also give the basic process for solving these inside the first example.

Example 1 Solve $A = P\left(1 + rt\right)$ for *r*.

Solution
What we're looking for here is an expression in the form,

$$r = \underline{\text{Equation involving numbers, } A, P, \text{and } t}$$

In other words, the only place that we want to see an *r* is on the left side of the equal sign all by itself. There should be no other *r*'s anywhere in the equation. The process given above should do that for us.

Okay, let's do this problem. We don't have any fractions so we don't need to worry about that. To simplify we will multiply the *P* through the parenthesis. Doing this gives,

$$A = P + Prt$$

Now, we need to get all the terms with an *r* on them on one side. This equation already has that set up for us which is nice. Next, we need to get all terms that don't have an *r* in them to the other side. This means subtracting a *P* from both sides.

$$A - P = Prt$$

As a final step we will divide both sides by the coefficient of *r*. Also, as noted in the process listed above the "coefficient" is not a number. In this case it is *Pt*. At this stage the coefficient of a variable is simply all the stuff that multiplies the variable.

$$\frac{A-P}{Pt}=r \qquad \Rightarrow \qquad r=\frac{A-P}{Pt}$$

To get a final answer we went ahead and flipped the order to get the answer into a more "standard" form.

We will work more examples in a bit. However, let's note a couple things first. These problems tend to seem fairly difficult at first, but if you think about it all we really did was use exactly the same process we used to solve linear equations. The main difference of course, is that there is more "mess" in this process. That brings us to the second point. Do not get excited about the mess in these problems. The problems will, on occasion, be a little messy, but the steps involved are steps that you can do! Finally, the answer will not be a simple number, but again it will be a little messy, often messier than the original equation. That is okay and expected.

Let's work some more examples.

Example 2 Solve $V=m\left(\frac{1}{b}-\frac{5aR}{m}\right)$ for *R*.

Solution

This one is fairly similar to the first example. However, it does work a little differently. Recall from the first example that we made the comment that sometimes the ordering of the steps in the process needs to be changed? Well, that's what we're going to do here.

The first step in the process tells us to clear fractions. However, since the fraction is inside a set of parentheses let's first multiply the *m* through the parenthesis. Notice as well that if we multiply the *m* through first we will in fact clear one of the fractions out automatically. This will make our work a little easier when we do clear the fractions out.

$$V=\frac{m}{b}-5aR$$

Now, clear fractions by multiplying both sides by *b*. We'll also go ahead move all terms that don't have an *R* in them to the other side.

$$Vb=m-5abR$$
$$Vb-m=-5abR$$

Be careful to not lose the minus sign in front of the 5! It's very easy to lose track of that. The final step is to then divide both sides by the coefficient of the *R*, in this case *-5ab*.

$$R=\frac{Vb-m}{-5ab}=-\frac{Vb-m}{5ab}=\frac{-\left(Vb-m\right)}{5ab}=\frac{-Vb+m}{5ab}=\frac{m-Vb}{5ab}$$

Notice as well that we did some manipulation of the minus sign that was in the denominator so that we could simplify the answer somewhat.

In the previous example we solved for *R*, but there is no reason for not solving for one of the other variables in the problems. For instance, consider the following example.

Example 3 Solve $V = m\left(\frac{1}{b} - \frac{5aR}{m}\right)$ for *b*.

Solution

The first couple of steps are identical to the previous example. First, we will multiply the *m* through the parenthesis and then we will multiply both sides by *b* to clear the fractions. We've already done this work so from the previous example we have,

$$Vb - m = -5abR$$

In this case we've got *b*'s on both sides of the equal sign and we need all terms with *b*'s in them on one side of the equation and all other terms on the other side of the equation. In this case we can eliminate the minus signs if we collect the *b*'s on the left side and the other terms on the right side. Doing this gives,

$$Vb + 5abR = m$$

Now, both terms on the right side have a *b* in them so if we factor that out of both terms we arrive at,

$$b\left(V + 5aR\right) = m$$

Finally, divide by the coefficient of *b*. Recall as well that the "coefficient" is all the stuff that multiplies the *b*. Doing this gives,

$$b = \frac{m}{V + 5aR}$$

Example 4 Solve $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ for *c*.

Solution

First, multiply by the LCD, which is abc for this problem.

$$\frac{1}{a}\left(abc\right) = \left(\frac{1}{b} + \frac{1}{c}\right)\left(abc\right)$$
$$bc = ac + ab$$

Next, collect all the *c*'s on one side (the left will probably be easiest here), factor a *c* out of the terms and divide by the coefficient.

$$bc - ac = ab$$
$$c\left(b - a\right) = ab$$
$$c = \frac{ab}{b - a}$$

Example 5 Solve $y = \dfrac{4}{5x-9}$ for *x*.

Solution

First, we'll need to clear the denominator. To do this we will multiply both sides by $5x-9$. We'll also clear out any parenthesis in the problem after we do the multiplication.

$$y\left(5x-9\right) = 4$$
$$5xy - 9y = 4$$

Now, we want to solve for *x* so that means that we need to get all terms without a *y* in them to the other side. So, add 9*y* to both sides and the divide by the coefficient of *x*.

$$5xy = 9y + 4$$
$$x = \frac{9y+4}{5y}$$

Example 6 Solve $y = \dfrac{4-3x}{1+8x}$ for *x*.

Solution

This one is very similar to the previous example. Here is the work for this problem.

$$y\left(1+8x\right) = 4-3x$$
$$y + 8xy = 4-3x$$
$$8xy + 3x = 4-y$$
$$x\left(8y+3\right) = 4-y$$
$$x = \frac{4-y}{8y+3}$$

As mentioned at the start of this section we won't be seeing this kind of problem all that often in this class. However, outside of this class (a Calculus class for example) this kind of problem shows up with surprising regularity.

Section 2-5 : Quadratic Equations - Part I

Before proceeding with this section we should note that the topic of solving quadratic equations will be covered in two sections. This is done for the benefit of those viewing the material on the web. This is a long topic and to keep page load times down to a minimum the material was split into two sections.

So, we are now going to solve quadratic equations. First, the **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad\qquad a \neq 0$$

The only requirement here is that we have an x^2 in the equation. We guarantee that this term will be present in the equation by requiring $a \neq 0$. Note however, that it is okay if *b* and/or *c* are zero.

There are many ways to solve quadratic equations. We will look at four of them over the course of the next two sections. The first two methods won't always work yet are probably a little simpler to use when they work. This section will cover these two methods. The last two methods will always work, but often require a little more work or attention to get correct. We will cover these methods in the next section.

So, let's get started.

Solving by Factoring

As the heading suggests we will be solving quadratic equations here by factoring them. To do this we will need the following fact.

$$\text{If } \ ab = 0 \ \text{ then either } \ a = 0 \ \text{ and/or } \ b = 0$$

This fact is called the **zero factor property** or **zero factor principle**. All the fact says is that if a product of two terms is zero then at least one of the terms had to be zero to start off with.

Notice that this fact will ONLY work if the product is equal to zero. Consider the following product.

$$ab = 6$$

In this case there is no reason to believe that either *a* or *b* will be 6. We could have $a = 2$ and $b = 3$ for instance. So, do not misuse this fact!

To solve a quadratic equation by factoring we first must move all the terms over to one side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form that can be factored. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don't have a zero on one side of the equation we won't be able to use the zero factor property.

Let's take a look at a couple of examples. Note that it is assumed that you can do the factoring at this point and so we won't be giving any details on the factoring. If you need a review of factoring you should go back and take a look at the Factoring section of the previous chapter.

Example 1 Solve each of the following equations by factoring.

(a) $x^2 - x = 12$
(b) $x^2 + 40 = -14x$
(c) $y^2 + 12y + 36 = 0$
(d) $4m^2 - 1 = 0$
(e) $3x^2 = 2x + 8$
(f) $10z^2 + 19z + 6 = 0$
(g) $5x^2 = 2x$

Solution

Now, as noted earlier, we won't be putting any detail into the factoring process, so make sure that you can do the factoring here.

(a) $x^2 - x = 12$

First, get everything on side of the equation and then factor.

$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$

Now at this point we've got a product of two terms that is equal to zero. This means that at least one of the following must be true.

$$x - 4 = 0 \qquad \text{OR} \qquad x + 3 = 0$$
$$x = 4 \qquad \text{OR} \qquad x = -3$$

Note that each of these is a linear equation that is easy enough to solve. What this tell us is that we have two solutions to the equation, $x = 4$ and $x = -3$. As with linear equations we can always check our solutions by plugging the solution back into the equation. We will check $x = -3$ and leave the other to you to check.

$$(-3)^2 - (-3) \stackrel{?}{=} 12$$
$$9 + 3 \stackrel{?}{=} 12$$
$$12 = 12 \qquad \text{OK}$$

So, this was in fact a solution.

(b) $x^2 + 40 = -14x$

As with the first one we first get everything on side of the equal sign and then factor.

$$x^2 + 40 + 14x = 0$$
$$(x+4)(x+10) = 0$$

Now, we once again have a product of two terms that equals zero so we know that one or both of them have to be zero. So, technically we need to set each one equal to zero and solve. However, this is usually easy enough to do in our heads and so from now on we will be doing this solving in our head.

The solutions to this equation are,

$$x = -4 \qquad \text{AND} \qquad x = -10$$

To save space we won't be checking any more of the solutions here, but you should do so to make sure we didn't make any mistakes.

(c) $y^2 + 12y + 36 = 0$

In this case we already have zero on one side and so we don't need to do any manipulation to the equation all that we need to do is factor. Also, don't get excited about the fact that we now have *y*'s in the equation. We won't always be dealing with *x*'s so don't expect to always see them.

So, let's factor this equation.

$$\begin{aligned} y^2 + 12y + 36 &= 0 \\ (y+6)^2 &= 0 \\ (y+6)(y+6) &= 0 \end{aligned}$$

In this case we've got a perfect square. We broke up the square to denote that we really do have an application of the zero factor property. However, we usually don't do that. We usually will go straight to the answer from the squared part.

The solution to the equation in this case is,

$$y = -6$$

We only have a single value here as opposed to the two solutions we've been getting to this point. We will often call this solution a **double root** or say that it has **multiplicity of 2** because it came from a term that was squared.

(d) $4m^2 - 1 = 0$

As always let's first factor the equation.

$$\begin{aligned} 4m^2 - 1 &= 0 \\ (2m-1)(2m+1) &= 0 \end{aligned}$$

Now apply the zero factor property. The zero factor property tells us that,

$$\begin{array}{ccc} 2m - 1 = 0 & \text{OR} & 2m + 1 = 0 \\ 2m = 1 & \text{OR} & 2m = -1 \\ m = \dfrac{1}{2} & \text{OR} & m = -\dfrac{1}{2} \end{array}$$

Again, we will typically solve these in our head, but we needed to do at least one in complete detail. So, we have two solutions to the equation.

$$m = \frac{1}{2} \qquad \text{AND} \qquad m = -\frac{1}{2}$$

(e) $3x^2 = 2x + 8$
Now that we've done quite a few of these, we won't be putting in as much detail for the next two problems. Here is the work for this equation.

$$3x^2 - 2x - 8 = 0$$

$$(3x+4)(x-2) = 0 \qquad \Rightarrow \qquad x = -\frac{4}{3} \quad \text{and} \quad x = 2$$

(f) $10z^2 + 19z + 6 = 0$
Again, factor and use the zero factor property for this one.

$$10z^2 + 19z + 6 = 0$$

$$(5z+2)(2z+3) = 0 \qquad \Rightarrow \qquad z = -\frac{2}{5} \quad \text{and} \quad z = -\frac{3}{2}$$

(g) $5x^2 = 2x$
This one always seems to cause trouble for students even though it's really not too bad.

First off. DO NOT CANCEL AN *x* FROM BOTH SIDES!!!! Do you get the idea that might be bad? It is. If you cancel an *x* from both sides, you WILL miss a solution so don't do it. Remember we are solving by factoring here so let's first get everything on one side of the equal sign.

$$5x^2 - 2x = 0$$

Now, notice that all we can do for factoring is to factor an *x* out of everything. Doing this gives,

$$x(5x-2) = 0$$

From the first factor we get that $x = 0$ and from the second we get that $x = \frac{2}{5}$. These are the two solutions to this equation. Note that is we'd canceled an *x* in the first step we would NOT have gotten $x = 0$ as an answer!

Let's work another type of problem here. We saw some of these back in the Solving Linear Equations section and since they can also occur with quadratic equations we should go ahead and work on to make sure that we can do them here as well.

Example 2 Solve each of the following equations.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

(b) $x + 3 + \dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

Solution
Okay, just like with the linear equations the first thing that we're going to need to do here is to clear the denominators out by multiplying by the LCD. Recall that we will also need to note value(s) of *x* that will give division by zero so that we can make sure that these aren't included in the solution.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

The LCD for this problem is $(x+1)(2x-4)$ and we will need to avoid $x=-1$ and $x=2$ to make sure we don't get division by zero. Here is the work for this equation.

$$\begin{aligned}
(x+1)(2x-4)\left(\frac{1}{x+1}\right) &= (x+1)(2x-4)\left(1-\frac{5}{2x-4}\right)\\
2x-4 &= (x+1)(2x-4) - 5(x+1)\\
2x-4 &= 2x^2 - 2x - 4 - 5x - 5\\
0 &= 2x^2 - 9x - 5\\
0 &= (2x+1)(x-5)
\end{aligned}$$

So, it looks like the two solutions to this equation are,

$$x = -\frac{1}{2} \qquad \text{and} \qquad x = 5$$

Notice as well that neither of these are the values of *x* that we needed to avoid and so both are solutions.

(b) $x + 3 + \dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

In this case the LCD is $x-1$ and we will need to avoid $x=1$ so we don't get division by zero. Here is the work for this problem.

$$\begin{aligned}
(x-1)\left(x+3+\frac{3}{x-1}\right) &= \left(\frac{4-x}{x-1}\right)(x-1)\\
(x-1)(x+3)+3 &= 4-x\\
x^2+2x-3+3 &= 4-x\\
x^2+3x-4 &= 0\\
(x-1)(x+4) &= 0
\end{aligned}$$

So, the quadratic that we factored and solved has two solutions, $x=1$ and $x=-4$. However, when we found the LCD we also saw that we needed to avoid $x=1$ so we didn't get division by zero. Therefore, this equation has a single solution,

$$x = -4$$

Before proceeding to the next topic we should address that this idea of factoring can be used to solve equations with degree larger than two as well. Consider the following example.

Example 3 Solve $5x^3 - 5x^2 - 10x = 0$.

Solution
The first thing to do is factor this equation as much as possible. In this case that means factoring out the greatest common factor first. Here is the factored form of this equation.

$$5x\left(x^2 - x - 2\right) = 0$$
$$5x\left(x-2\right)\left(x+1\right) = 0$$

Now, the zero factor property will still hold here. In this case we have a product of three terms that is zero. The only way this product can be zero is if one of the terms is zero. This means that,

$$\begin{aligned} 5x &= 0 & \Rightarrow \quad & x = 0 \\ x - 2 &= 0 & \Rightarrow \quad & x = 2 \\ x + 1 &= 0 & \Rightarrow \quad & x = -1 \end{aligned}$$

So, we have three solutions to this equation.

So, provided we can factor a polynomial we can always use this as a solution technique. The problem is, of course, that it is sometimes not easy to do the factoring.

Square Root Property
The second method of solving quadratics we'll be looking at uses the **square root property**,

$$\text{If } p^2 = d \quad \text{then} \quad p = \pm\sqrt{d}$$

There is a (potentially) new symbol here that we should define first in case you haven't seen it yet. The symbol "$\pm$" is read as : "plus or minus" and that is exactly what it tells us. This symbol is shorthand that tells us that we really have two numbers here. One is $p = \sqrt{d}$ and the other is $p = -\sqrt{d}$. Get used to this notation as it will be used frequently in the next couple of sections as we discuss the remaining solution techniques. It will also arise in other sections of this chapter and even in other chapters.

This is a fairly simple property to use, however it can only be used on a small portion of the equations that we're ever likely to encounter. Let's see some examples of this property.

Example 4 Solve each of the following equations.

(a) $x^2 - 100 = 0$
(b) $25y^2 - 3 = 0$
(c) $4z^2 + 49 = 0$
(d) $\left(2t - 9\right)^2 = 5$
(e) $\left(3x + 10\right)^2 + 81 = 0$

Solution
There really isn't all that much to these problems. In order to use the square root property all that we need to do is get the squared quantity on the left side by itself with a coefficient of 1 and the number on the other side. Once this is done we can use the square root property.

(a) $x^2 - 100 = 0$
This is a fairly simple problem so here is the work for this equation.

$$x^2 = 100 \qquad\qquad x = \pm\sqrt{100} = \pm 10$$

So, there are two solutions to this equation, $x = \pm 10$. Remember this means that there are really two solutions here, $x = -10$ and $x = 10$.

(b) $25y^2 - 3 = 0$
Okay, the main difference between this one and the previous one is the 25 in front of the squared term. The square root property wants a coefficient of one there. That's easy enough to deal with however; we'll just divide both sides by 25. Here is the work for this equation.

$$25y^2 = 3$$

$$y^2 = \frac{3}{25} \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{3}{25}} = \pm\frac{\sqrt{3}}{5}$$

In this case the solutions are a little messy, but many of these will do so don't worry about that. Also note that since we knew what the square root of 25 was we went ahead and split the square root of the fraction up as shown. Again, remember that there are really two solutions here, one positive and one negative.

(c) $4z^2 + 49 = 0$
This one is nearly identical to the previous part with one difference that we'll see at the end of the example. Here is the work for this equation.

$$4z^2 = -49$$

$$z^2 = -\frac{49}{4} \qquad \Rightarrow \qquad z = \pm\sqrt{-\frac{49}{4}} = \pm i\sqrt{\frac{49}{4}} = \pm\frac{7}{2}i$$

So, there are two solutions to this equation : $z = \pm\frac{7}{2}i$. Notice as well that they are complex solutions. This will happen with the solution to many quadratic equations so make sure that you can deal with them.

(d) $(2t - 9)^2 = 5$
This one looks different from the previous parts, however it works the same way. The square root property can be used anytime we have *something* squared equals a number. That is what we have here. The main difference of course is that the something that is squared isn't a single variable it is something else. So, here is the application of the square root property for this equation.

$$2t - 9 = \pm\sqrt{5}$$

Now, we just need to solve for *t* and despite the "plus or minus" in the equation it works the same way we would solve any linear equation. We will add 9 to both sides and then divide by a 2.

$$2t = 9 \pm \sqrt{5}$$

$$t = \frac{1}{2}\left(9 \pm \sqrt{5}\right) = \frac{9}{2} \pm \frac{\sqrt{5}}{2}$$

Note that we multiplied the fraction through the parenthesis for the final answer. We will usually do this in these problems. Also, do NOT convert these to decimals unless you are asked to. This is the standard form for these answers. With that being said we should convert them to decimals just to make sure that you can. Here are the decimal values of the two solutions.

$$t = \frac{9}{2} + \frac{\sqrt{5}}{2} = 5.61803 \qquad \text{and} \qquad t = \frac{9}{2} - \frac{\sqrt{5}}{2} = 3.38197$$

(e) $(3x+10)^2 + 81 = 0$

In this final part we'll not put much in the way of details into the work.

$$\begin{aligned} (3x+10)^2 &= -81 \\ 3x+10 &= \pm 9i \\ 3x &= -10 \pm 9i \\ x &= -\frac{10}{3} \pm 3i \end{aligned}$$

So we got two complex solutions again and notice as well that with both of the previous part we put the "plus or minus" part last. This is usually the way these are written.

As mentioned at the start of this section we are going to break this topic up into two sections for the benefit of those viewing this on the web. The next two methods of solving quadratic equations, completing the square and quadratic formula, are given in the next section.

Section 2-6 : Quadratic Equations - Part II

The topic of solving quadratic equations has been broken into two sections for the benefit of those viewing this on the web. As a single section the load time for the page would have been quite long. This is the second section on solving quadratic equations.

In the previous section we looked at using factoring and the square root property to solve quadratic equations. The problem is that both of these solution methods will not always work. Not every quadratic is factorable and not every quadratic is in the form required for the square root property.

It is now time to start looking into methods that will work for all quadratic equations. So, in this section we will look at completing the square and the quadratic formula for solving the quadratic equation,

$$ax^2 + bx + c = 0 \qquad\qquad a \neq 0$$

Completing the Square

The first method we'll look at in this section is completing the square. It is called this because it uses a process called completing the square in the solution process. So, we should first define just what completing the square is.

Let's start with

$$x^2 + bx$$

and notice that the x^2 has a coefficient of one. That is required in order to do this. Now, to this lets add $\left(\frac{b}{2}\right)^2$. Doing this gives the following **factorable** quadratic equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

This process is called **completing the square** and if we do all the arithmetic correctly we can guarantee that the quadratic will factor as a perfect square.

Let's do a couple of examples for just completing the square before looking at how we use this to solve quadratic equations.

Example 1 Complete the square on each of the following.

(a) $x^2 - 16x$

(b) $y^2 + 7y$

Solution

(a) $x^2 - 16x$

Here's the number that we'll add to the equation.

$$\left(\frac{-16}{2}\right)^2 = (-8)^2 = 64$$

Notice that we kept the minus sign here even though it will always drop out after we square things. The reason for this will be apparent in a second. Let's now complete the square.

$$x^2 - 16x + 64 = (x - 8)^2$$

Now, this is a quadratic that hopefully you can factor fairly quickly. However notice that it will always factor as *x* plus the blue number we computed above that is in the parenthesis (in our case that is -8). This is the reason for leaving the minus sign. It makes sure that we don't make any mistakes in the factoring process.

(b) $y^2 + 7y$

Here's the number we'll need this time.

$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

It's a fraction and that will happen fairly often with these so don't get excited about it. Also, leave it as a fraction. Don't convert to a decimal. Now complete the square.

$$y^2 + 7y + \frac{49}{4} = \left(y + \frac{7}{2}\right)^2$$

This one is not so easy to factor. However, if you again recall that this will ALWAYS factor as *y* plus the blue number above we don't have to worry about the factoring process.

It's now time to see how we use completing the square to solve a quadratic equation. The process is best seen as we work an example so let's do that.

Example 2 Use completing the square to solve each of the following quadratic equations.

(a) $x^2 - 6x + 1 = 0$

(b) $2x^2 + 6x + 7 = 0$

(c) $3x^2 - 2x - 1 = 0$

Solution

We will do the first problem in detail explicitly giving each step. In the remaining problems we will just do the work without as much explanation.

(a) $x^2 - 6x + 1 = 0$

So, let's get started.

Step 1 : Divide the equation by the coefficient of the x^2 term. Recall that completing the square required a coefficient of one on this term and this will guarantee that we will get that. We don't need to do that for this equation however.

Step 2 : Set the equation up so that the *x*'s are on the left side and the constant is on the right side.

$$x^2 - 6x = -1$$

Step 3 : Complete the square on the left side. However, this time we will need to add the number to both sides of the equal sign instead of just the left side. This is because we have to remember the rule that what we do to one side of an equation we need to do to the other side of the equation.

First, here is the number we add to both sides.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now, complete the square.

$$x^2 - 6x + 9 = -1 + 9$$
$$(x-3)^2 = 8$$

Step 4 : Now, at this point notice that we can use the square root property on this equation. That was the purpose of the first three steps. Doing this will give us the solution to the equation.

$$x - 3 = \pm\sqrt{8} \qquad \Rightarrow \qquad x = 3 \pm \sqrt{8}$$

And that is the process. Let's do the remaining parts now.

(b) $2x^2 + 6x + 7 = 0$

We will not explicitly put in the steps this time nor will we put in a lot of explanation for this equation. This that being said, notice that we will have to do the first step this time. We don't have a coefficient of one on the x^2 term and so we will need to divide the equation by that first.

Here is the work for this equation.

$$x^2 + 3x + \frac{7}{2} = 0$$
$$x^2 + 3x = -\frac{7}{2} \qquad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$
$$x^2 + 3x + \frac{9}{4} = -\frac{7}{2} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = -\frac{5}{4}$$
$$x + \frac{3}{2} = \pm\sqrt{-\frac{5}{4}} \qquad \Rightarrow \qquad x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$$

Don't forget to convert square roots of negative numbers to complex numbers!

(c) $3x^2 - 2x - 1 = 0$

Again, we won't put a lot of explanation for this problem.

$$x^2 - \frac{2}{3}x - \frac{1}{3} = 0$$
$$x^2 - \frac{2}{3}x = \frac{1}{3}$$

At this point we should be careful about computing the number for completing the square since *b* is now a fraction for the first time.

$$\left(\frac{-\frac{2}{3}}{2}\right)^2 = \left(-\frac{2}{3}\cdot\frac{1}{2}\right)^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$$

Now finish the problem.

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{3} + \frac{1}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{4}{9}$$

$$x - \frac{1}{3} = \pm\sqrt{\frac{4}{9}} \qquad \Rightarrow \qquad x = \frac{1}{3} \pm \frac{2}{3}$$

In this case notice that we can actually do the arithmetic here to get two integer and/or fractional solutions. We should always do this when there are only integers and/or fractions in our solution. Here are the two solutions.

$$x = \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1 \qquad \text{and} \qquad x = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

A quick comment about the last equation that we solved in the previous example is in order. Since we received integer and fractions as solutions, we could have just factored this equation from the start rather than used completing the square. In cases like this we could use either method and we will get the same result.

Now, the reality is that completing the square is a fairly long process and it's easy to make mistakes. So, we rarely actually use it to solve equations. That doesn't mean that it isn't important to know the process however. We will be using it in several sections in later chapters and is often used in other classes.

Quadratic Formula

This is the final method for solving quadratic equations and will always work. Not only that, but if you can remember the formula it's a fairly simple process as well.

We can derive the quadratic formula by completing the square on the general quadratic formula in standard form. Let's do that and we'll take it kind of slow to make sure all the steps are clear.

First, we MUST have the quadratic equation in standard form as already noted. Next, we need to divide both sides by a to get a coefficient of one on the x^2 term.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Next, move the constant to the right side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, we need to compute the number we'll need to complete the square. Again, this is one-half the coefficient of *x*, squared.

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Now, add this to both sides, complete the square and get common denominators on the right side to simplify things up a little.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now we can use the square root property on this.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Solve for *x* and we'll also simplify the square root a little.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

As a last step we will notice that we've got common denominators on the two terms and so we'll add them up. Doing this gives,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, summarizing up, provided that we start off in standard form,

$$ax^2 + bx + c = 0$$

and that's very important, then the solution to any quadratic equation is,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's work a couple of examples.

Example 3 Use the quadratic formula to solve each of the following equations.

(a) $x^2 + 2x = 7$

(b) $3q^2 + 11 = 5q$

(c) $7t^2 = 6 - 19t$

(d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$

(e) $16x - x^2 = 0$

Solution

The important part here is to make sure that before we start using the quadratic formula that we have the equation in standard form first.

(a) $x^2 + 2x = 7$

So, the first thing that we need to do here is to put the equation in standard form.

$$x^2 + 2x - 7 = 0$$

At this point we can identify the values for use in the quadratic formula. For this equation we have.

$$a = 1 \qquad b = 2 \qquad c = -7$$

Notice the "-" with *c*. It is important to make sure that we carry any minus signs along with the constants.

At this point there really isn't anything more to do other than plug into the formula.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{32}}{2} \end{aligned}$$

There are the two solutions for this equation. There is also some simplification that we can do. We need to be careful however. One of the larger mistakes at this point is to "cancel" two 2's in the numerator and denominator. Remember that in order to cancel anything from the numerator or denominator then it must be multiplied by the whole numerator or denominator. Since the 2 in the numerator isn't multiplied by the whole denominator it can't be canceled.

In order to do any simplification here we will first need to reduce the square root. At that point we can do some canceling.

$$x = \frac{-2 \pm \sqrt{(16)2}}{2} = \frac{-2 \pm 4\sqrt{2}}{2} = \frac{2\left(-1 \pm 2\sqrt{2}\right)}{2} = -1 \pm 2\sqrt{2}$$

That's a much nicer answer to deal with and so we will almost always do this kind of simplification when it can be done.

(b) $3q^2 + 11 = 5q$

Now, in this case don't get excited about the fact that the variable isn't an *x*. Everything works the same regardless of the letter used for the variable. So, let's first get the equation into standard form.

$$3q^2 + 11 - 5q = 0$$

Now, this isn't quite in the typical standard form. However, we need to make a point here so that we don't make a very common mistake that many student make when first learning the quadratic formula.

Many students will just get everything on one side as we've done here and then get the values of *a*, *b*, and *c* based upon position. In other words, often students will just let *a* be the first number listed, *b* be the second number listed and then *c* be the final number listed. This is not correct however. For the quadratic formula *a* is the coefficient of the squared term, *b* is the coefficient of the term with just the variable in it (not squared) and *c* is the constant term. So, to avoid making this mistake we should always put the quadratic equation into the official standard form.

$$3q^2 - 5q + 11 = 0$$

Now we can identify the value of *a*, *b*, and *c*.

$$a = 3 \qquad b = -5 \qquad c = 11$$

Again, be careful with minus signs. They need to get carried along with the values.

Finally, plug into the quadratic formula to get the solution.

$$\begin{aligned} q &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(11)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25 - 132}}{6} \\ &= \frac{5 \pm \sqrt{-107}}{6} \\ &= \frac{5 \pm \sqrt{107}\, i}{6} \end{aligned}$$

As with all the other methods we've looked at for solving quadratic equations, don't forget to convert square roots of negative numbers into complex numbers. Also, when *b* is negative be very careful with the substitution. This is particularly true for the squared portion under the radical. Remember that when you square a negative number it will become positive. One of the more common mistakes here is to get in a hurry and forget to drop the minus sign after you square *b*, so be careful.

(c) $7t^2 = 6 - 19t$

We won't put in quite the detail with this one that we've done for the first two. Here is the standard form of this equation.

$$7t^2 + 19t - 6 = 0$$

Here are the values for the quadratic formula as well as the quadratic formula itself.

$$a = 7 \qquad b = 19 \qquad c = -6$$

$$\begin{aligned} t &= \frac{-19 \pm \sqrt{(19)^2 - 4(7)(-6)}}{2(7)} \\ &= \frac{-19 \pm \sqrt{361 + 168}}{14} \\ &= \frac{-19 \pm \sqrt{529}}{14} \\ &= \frac{-19 \pm 23}{14} \end{aligned}$$

Now, recall that when we get solutions like this we need to go the extra step and actually determine the integer and/or fractional solutions. In this case they are,

$$t = \frac{-19 + 23}{14} = \frac{2}{7} \qquad\qquad t = \frac{-19 - 23}{14} = -3$$

Now, as with completing the square, the fact that we got integer and/or fractional solutions means that we could have factored this quadratic equation as well.

(d) $\frac{3}{y-2} = \frac{1}{y} + 1$

So, an equation with fractions in it. The first step then is to identify the LCD.

$$\text{LCD} : y(y-2)$$

So, it looks like we'll need to make sure that neither $y = 0$ or $y = 2$ is in our answers so that we don't get division by zero.

Multiply both sides by the LCD and then put the result in standard form.

$$\begin{aligned} (y)(y-2)\left(\frac{3}{y-2}\right) &= \left(\frac{1}{y} + 1\right)(y)(y-2) \\ 3y &= y - 2 + y(y-2) \\ 3y &= y - 2 + y^2 - 2y \\ 0 &= y^2 - 4y - 2 \end{aligned}$$

Okay, it looks like we've got the following values for the quadratic formula.

$$a = 1 \qquad b = -4 \qquad c = -2$$

Plugging into the quadratic formula gives,

$$\begin{aligned} y &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{24}}{2} \\ &= \frac{4 \pm 2\sqrt{6}}{2} \\ &= 2 \pm \sqrt{6} \end{aligned}$$

Note that both of these are going to be solutions since neither of them are the values that we need to avoid.

(e) $16x - x^2 = 0$

We saw an equation similar to this in the previous section when we were looking at factoring equations and it would definitely be easier to solve this by factoring. However, we are going to use the quadratic formula anyway to make a couple of points.

First, let's rearrange the order a little bit just to make it look more like the standard form.

$$-x^2 + 16x = 0$$

Here are the constants for use in the quadratic formula.

$$a = -1 \qquad\qquad b = 16 \qquad\qquad c = 0$$

There are two things to note about these values. First, we've got a negative *a* for the first time. Not a big deal, but it is the first time we've seen one. Secondly, and more importantly, one of the values is zero. This is fine. It will happen on occasion and in fact, having one of the values zero will make the work much simpler.

Here is the quadratic formula for this equation.

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{(16)^2 - 4(-1)(0)}}{2(-1)} \\ &= \frac{-16 \pm \sqrt{256}}{-2} \\ &= \frac{-16 \pm 16}{-2} \end{aligned}$$

Reducing these to integers/fractions gives,

$$x = \frac{-16+16}{-2} = \frac{0}{2} = 0 \qquad\qquad x = \frac{-16-16}{-2} = \frac{-32}{-2} = 16$$

So we get the two solutions, $x = 0$ and $x = 16$. These are exactly the solutions we would have gotten by factoring the equation.

To this point in both this section and the previous section we have only looked at equations with integer coefficients. However, this doesn't have to be the case. We could have coefficient that are fractions or decimals. So, let's work a couple of examples so we can say that we've seen something like that as well.

Example 4 Solve each of the following equations.

(a) $\frac{1}{2}x^2 + x - \frac{1}{10} = 0$

(b) $0.04x^2 - 0.23x + 0.09 = 0$

Solution

(a) There are two ways to work this one. We can either leave the fractions in or multiply by the LCD (10 in this case) and solve that equation. Either way will give the same answer. We will only do the fractional case here since that is the point of this problem. You should try the other way to verify that you get the same solution.

In this case here are the values for the quadratic formula as well as the quadratic formula work for this equation.

$$a = \frac{1}{2} \qquad b = 1 \qquad c = -\frac{1}{10}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{10}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{1 + \frac{1}{5}}}{1} = -1 \pm \sqrt{\frac{6}{5}}$$

In these cases we usually go the extra step of eliminating the square root from the denominator so let's also do that,

$$x = -1 \pm \frac{\sqrt{6}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = -1 \pm \frac{\sqrt{(6)(5)}}{5} = -1 \pm \frac{\sqrt{30}}{5}$$

If you do clear the fractions out and run through the quadratic formula then you should get exactly the same result. For the practice you really should try that.

(b) In this case do not get excited about the decimals. The quadratic formula works in exactly the same manner. Here are the values and the quadratic formula work for this problem.

$$a = 0.04 \qquad b = -0.23 \qquad c = 0.09$$

$$\begin{aligned} x &= \frac{-(-0.23) \pm \sqrt{(-0.23)^2 - 4(0.04)(0.09)}}{2(0.04)} \\ &= \frac{0.23 \pm \sqrt{0.0529 - 0.0144}}{0.08} \\ &= \frac{0.23 \pm \sqrt{0.0385}}{0.08} \end{aligned}$$

Now, to this will be the one difference between these problems and those with integer or fractional coefficients. When we have decimal coefficients we usually go ahead and figure the two individual numbers. So, let's do that,

$$x = \frac{0.23 \pm \sqrt{0.0385}}{0.08} = \frac{0.23 \pm 0.19621}{0.08}$$

$$\begin{aligned} x &= \frac{0.23 + 0.19621}{0.08} & \text{and} \qquad & x = \frac{0.23 - 0.19621}{0.08} \\ &= 5.327625 & \text{and} \qquad & \;\;= 0.422375 \end{aligned}$$

Notice that we did use some rounding on the square root.

Over the course of the last two sections we've done quite a bit of solving. It is important that you understand most, if not all, of what we did in these sections as you will be asked to do this kind of work in some later sections.

Section 2-7 : Quadratic Equations : A Summary

In the previous two sections we've talked quite a bit about solving quadratic equations. A logical question to ask at this point is which method should we use to solve a given quadratic equation? Unfortunately, the answer is, it depends.

If your instructor has specified the method to use then that, of course, is the method you should use. However, if your instructor had NOT specified the method to use then we will have to make the decision ourselves. Here is a general set of guidelines that *may* be helpful in determining which method to use.

1. Is it clearly a square root property problem? In other words, does the equation consist ONLY of something squared and a constant. If this is true then the square root property is probably the easiest method for use.
2. Does it factor? If so, that is probably the way to go. Note that you shouldn't spend a lot of time trying to determine if the quadratic equation factors. Look at the equation and if you can quickly determine that it factors then go with that. If you can't quickly determine that it factors then don't worry about it.
3. If you've reached this point then you've determined that the equation is not in the correct form for the square root property and that it doesn't factor (or that you can't quickly see that it factors). So, at this point you're only real option is the quadratic formula.

Once you've solved enough quadratic equations the above set of guidelines will become almost second nature to you and you will find yourself going through them almost without thinking.

Notice as well that nowhere in the set of guidelines was completing the square mentioned. The reason for this is simply that it's a long method that is prone to mistakes when you get in a hurry. The quadratic formula will also always work and is much shorter of a method to use. In general, you should only use completing the square if your instructor has required you to use it.

As a solving technique completing the square should always be your last choice. This doesn't mean however that it isn't an important method. We will see the completing the square process arise in several sections in later chapters. Interestingly enough when we do see this process in later sections we won't be solving equations! This process is very useful in many situations of which solving is only one.

Before leaving this section we have one more topic to discuss. In the previous couple of sections we saw that solving a quadratic equation in standard form,

$$ax^2 + bx + c = 0$$

we will get one of the following three possible solution sets.

1. Two real distinct (*i.e.* not equal) solutions.
2. A double root. Recall this arises when we can factor the equation into a perfect square.
3. Two complex solutions.

These are the ONLY possibilities for solving quadratic equations in standard form. Note however, that if we start with rational expression in the equation we may get different solution sets because we may need avoid one of the possible solutions so we don't get division by zero errors.

Now, it turns out that all we need to do is look at the quadratic equation (in standard form of course) to determine which of the three cases that we'll get. To see how this works let's start off by recalling the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ in the quadratic formula is called the **discriminant**. It is the value of the discriminant that will determine which solution set we will get. Let's go through the cases one at a time.

1. Two real distinct solutions. We will get this solution set if $b^2 - 4ac > 0$. In this case we will be taking the square root of a positive number and so the square root will be a real number. Therefore, the numerator in the quadratic formula will be *–b* plus or minus a real number. This means that the numerator will be two different real numbers. Dividing either one by 2*a* won't change the fact that they are real, nor will it change the fact that they are different.

2. A double root. We will get this solution set if $b^2 - 4ac = 0$. Here we will be taking the square root of zero, which is zero. However, this means that the "plus or minus" part of the numerator will be zero and so the numerator in the quadratic formula will be *–b*. In other words, we will get a single real number out of the quadratic formula, which is what we get when we get a double root.

3. Two complex solutions. We will get this solution set if $b^2 - 4ac < 0$. If the discriminant is negative we will be taking the square root of negative numbers in the quadratic formula which means that we will get complex solutions. Also, we will get two since they have a "plus or minus" in front of the square root.

So, let's summarize up the results here.

1. If $b^2 - 4ac > 0$ then we will get two real solutions to the quadratic equation.
2. If $b^2 - 4ac = 0$ then we will get a double root to the quadratic equation.
3. If $b^2 - 4ac < 0$ then we will get two complex solutions to the quadratic equation.

Example 1 Using the discriminant determine which solution set we get for each of the following quadratic equations.

(a) $13x^2 + 1 = 5x$

(b) $6q^2 + 20q = 3$

(c) $49t^2 + 126t + 81 = 0$

Solution

All we need to do here is make sure the equation is in standard form, determine the value of *a*, *b*, and *c*, then plug them into the discriminant.

(a) $13x^2 + 1 = 5x$
First get the equation in standard form.

$$13x^2 - 5x + 1 = 0$$

We then have,

$$a = 13 \qquad b = -5 \qquad c = 1$$

Plugging into the discriminant gives,

$$b^2 - 4ac = \left(-5\right)^2 - 4\left(13\right)\left(1\right) = -27$$

The discriminant is negative and so we will have two complex solutions. For reference purposes the actual solutions are,

$$x = \frac{5 \pm 3\sqrt{3}\,i}{26}$$

(b) $6q^2 + 20q = 3$
Again, we first need to get the equation in standard form.

$$6q^2 + 20q - 3 = 0$$

This gives,

$$a = 6 \qquad b = 20 \qquad c = -3$$

The discriminant is then,

$$b^2 - 4ac = \left(20\right)^2 - 4\left(6\right)\left(-3\right) = 472$$

The discriminant is positive we will get two real distinct solutions. Here they are,

$$x = \frac{-20 \pm \sqrt{472}}{12} = \frac{-10 \pm \sqrt{118}}{6}$$

(c) $49t^2 + 126t + 81 = 0$
This equation is already in standard form so let's jump straight in.

$$a = 49 \qquad b = 126 \qquad c = 81$$

The discriminant is then,

$$b^2 - 4ac = \left(126\right)^2 - 4\left(49\right)\left(81\right) = 0$$

In this case we'll get a double root since the discriminant is zero. Here it is,

$$x = -\frac{9}{7}$$

For practice you should verify the solutions in each of these examples.

Section 2-8 : Applications of Quadratic Equations

In this section we're going to go back and revisit some of the applications that we saw in the Linear Applications section and see some examples that will require us to solve a quadratic equation to get the answer.

Note that the solutions in these cases will almost always require the quadratic formula so expect to use it and don't get excited about it. Also, we are going to assume that you can do the quadratic formula work and so we won't be showing that work. We will give the results of the quadratic formula, we just won't be showing the work.

Also, as we will see, we will need to get decimal answer to these and so as a general rule here we will round all answers to 4 decimal places.

Example 1 We are going to fence in a rectangular field and we know that for some reason we want the field to have an enclosed area of 75 ft^2. We also know that we want the width of the field to be 3 feet longer than the length of the field. What are the dimensions of the field?

Solution

So, we'll let *x* be the length of the field and so we know that $x+3$ will be the width of the field. Now, we also know that area of a rectangle is length times width and so we know that,

$$x\left(x+3\right)=75$$

Now, this is a quadratic equation so let's first write it in standard form.

$$\begin{aligned} x^2+3x&=75\\ x^2+3x-75&=0 \end{aligned}$$

Using the quadratic formula gives,

$$x=\frac{-3\pm\sqrt{309}}{2}$$

Now, at this point, we've got to deal with the fact that there are two solutions here and we only want a single answer. So, let's convert to decimals and see what the solutions actually are.

$$x=\frac{-3+\sqrt{309}}{2}=7.2892 \qquad \text{and} \qquad x=\frac{-3-\sqrt{309}}{2}=-10.2892$$

So, we have one positive and one negative. From the stand point of needing the dimensions of a field the negative solution doesn't make any sense so we will ignore it.

Therefore, the length of the field is 7.2892 feet. The width is 3 feet longer than this and so is 10.2892 feet.

Notice that the width is almost the second solution to the quadratic equation. The only difference is the minus sign. Do NOT expect this to always happen. In this case this is more of a function of the problem. For a more complicated set up this will NOT happen.

Now, from a physical standpoint we can see that we should expect to NOT get complex solutions to these problems. Upon solving the quadratic equation we should get either two real distinct solutions or a double root. Also, as the previous example has shown, when we get two real distinct solutions we will be able to eliminate one of them for physical reasons.

Let's work another example or two.

Example 2 Two cars start out at the same point. One car starts out driving north at 25 mph. Two hours later the second car starts driving east at 20 mph. How long after the first car starts traveling does it take for the two cars to be 300 miles apart?

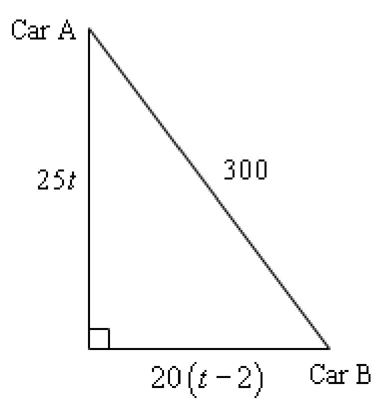
Solution

We'll start off by letting *t* be the amount of time that the first car, let's call it car A, travels. Since the second car, let's call that car B, starts out two hours later then we know that it will travel for $t-2$ hours.

Now, we know that the distance traveled by an object (or car since that's what we're dealing with here) is its speed times time traveled. So we have the following distances traveled for each car.

$$\begin{aligned}&\text{distance of car A} : 25t\\&\text{distance of car B} : 20\left(t-2\right)\end{aligned}$$

At this point a quick sketch of the situation is probably in order so we can see just what is going on. In the sketch we will assume that the two cars have traveled long enough so that they are 300 miles apart.



So, we have a right triangle here. That means that we can use the Pythagorean Theorem to say,

$$\left(25t\right)^2+\left(20\left(t-2\right)\right)^2=\left(300\right)^2$$

This is a quadratic equation, but it is going to need some fairly heavy simplification before we can solve it so let's do that.

$$\begin{aligned} 625t^2 + \left(20t - 40\right)^2 &= 90000 \\ 625t^2 + 400t^2 - 1600t + 1600 &= 90000 \\ 1025t^2 - 1600t - 88400 &= 0 \end{aligned}$$

Now, the coefficients here are quite large, but that is just something that will happen fairly often with these problems so don't worry about that. Using the quadratic formula (and simplifying that answer) gives,

$$t = \frac{1600 \pm \sqrt{365000000}}{2050} = \frac{1600 \pm 1000\sqrt{365}}{2050} = \frac{32 \pm 20\sqrt{365}}{41}$$

Again, we have two solutions and we're going to need to determine which one is the correct one, so let's convert them to decimals.

$$t = \frac{32 + 20\sqrt{365}}{41} = 10.09998 \qquad \text{and} \qquad t = \frac{32 - 20\sqrt{365}}{41} = -8.539011$$

As with the previous example the negative answer just doesn't make any sense. So, it looks like the car A traveled for 10.09998 hours when they were finally 300 miles apart.

Also, even though the problem didn't ask for it, the second car will have traveled for 8.09998 hours before they are 300 miles apart. Notice as well that this is NOT the second solution without the negative this time, unlike the first example.

Example 3 An office has two envelope stuffing machines. Working together they can stuff a batch of envelopes in 2 hours. Working separately, it will take the second machine 1 hour longer than the first machine to stuff a batch of envelopes. How long would it take each machine to stuff a batch of envelopes by themselves?

Solution

Let *t* be the amount of time it takes the first machine (Machine A) to stuff a batch of envelopes by itself. That means that it will take the second machine (Machine B) $t+1$ hours to stuff a batch of envelopes by itself.

The word equation for this problem is then,

$$\left(\begin{array}{c}\text{Portion of job}\\ \text{done by Machine A}\end{array}\right) + \left(\begin{array}{c}\text{Portion of job}\\ \text{done by Machine B}\end{array}\right) = 1 \text{ Job}$$

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of Machine A}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right) + \left(\begin{array}{c}\text{Work Rate}\\ \text{of Machine B}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right) = 1$$

We know the time spent working together (2 hours) so we need to work rates of each machine. Here are those computations.

$$1\text{ Job} = \left(\text{Work Rate of Machine A}\right)\times\left(t\right) \qquad \Rightarrow \qquad \text{Machine A} = \frac{1}{t}$$

$$1\text{ Job} = \left(\text{Work Rate of Machine B}\right)\times\left(t+1\right) \qquad \Rightarrow \qquad \text{Machine B} = \frac{1}{t+1}$$

Note that it's okay that the work rates contain *t*. In fact, they will need to so we can solve for it! Plugging into the word equation gives,

$$\begin{aligned} \left(\frac{1}{t}\right)\left(2\right) + \left(\frac{1}{t+1}\right)\left(2\right) &= 1 \\ \frac{2}{t} + \frac{2}{t+1} &= 1 \end{aligned}$$

So, to solve we'll first need to clear denominators and get the equation in standard form.

$$\begin{aligned} \left(\frac{2}{t} + \frac{2}{t+1}\right)\left(t\right)\left(t+1\right) &= \left(1\right)\left(t\right)\left(t+1\right) \\ 2\left(t+1\right) + 2t &= t^2 + t \\ 4t + 2 &= t^2 + t \\ 0 &= t^2 - 3t - 2 \end{aligned}$$

Using the quadratic formula gives,

$$t = \frac{3 \pm \sqrt{17}}{2}$$

Converting to decimals gives,

$$t = \frac{3+\sqrt{17}}{2} = 3.5616 \qquad \text{and} \qquad t = \frac{3-\sqrt{17}}{2} = -0.5616$$

Again, the negative doesn't make any sense and so Machine A will work for 3.5616 hours to stuff a batch of envelopes by itself. Machine B will need 4.5616 hours to stuff a batch of envelopes by itself. Again, unlike the first example, note that the time for Machine B was NOT the second solution from the quadratic without the minus sign.

Section 2-9 : Equations Reducible to Quadratic in Form

In this section we are going to look at equations that are called **quadratic in form** or **reducible to quadratic in form**. What this means is that we will be looking at equations that if we look at them in the correct light we can make them look like quadratic equations. At that point we can use the techniques we developed for quadratic equations to help us with the solution of the actual equation.

It is usually best with these to show the process with an example so let's do that.

Example 1 Solve $x^4 - 7x^2 + 12 = 0$

Solution

Now, let's start off here by noticing that

$$x^4 = \left(x^2\right)^2$$

In other words, we can notice here that the variable portion of the first term (*i.e.* ignore the coefficient) is nothing more than the variable portion of the second term squared. Note as well that all we really needed to notice here is that the exponent on the first term was twice the exponent on the second term.

This, along with the fact that third term is a constant, means that this equation is reducible to quadratic in form. We will solve this by first defining,

$$u = x^2$$

Now, this means that

$$u^2 = \left(x^2\right)^2 = x^4$$

Therefore, we can write the equation in terms of *u*'s instead of *x*'s as follows,

$$x^4 - 7x^2 + 12 = 0 \qquad \Rightarrow \qquad u^2 - 7u + 12 = 0$$

The new equation (the one with the *u*'s) is a quadratic equation and we can solve that. In fact, this equation is factorable, so the solution is,

$$u^2 - 7u + 12 = (u-4)(u-3) = 0 \qquad \Rightarrow \qquad u = 3,\ u = 4$$

So, we get the two solutions shown above. These aren't the solutions that we're looking for. We want values of *x*, not values of *u*. That isn't really a problem once we recall that we've defined

$$u = x^2$$

To get values of *x* for the solution all we need to do is plug in *u* into this equation and solve that for *x*. Let's do that.

$$\begin{array}{llll} u = 3: & 3 = x^2 & \Rightarrow & x = \pm\sqrt{3} \\ u = 4: & 4 = x^2 & \Rightarrow & x = \pm\sqrt{4} = \pm 2 \end{array}$$

So, we have four solutions to the original equation, $x = \pm 2$ and $x = \pm\sqrt{3}$.

So, the basic process is to check that the equation is reducible to quadratic in form then make a quick substitution to turn it into a quadratic equation. We solve the new equation for *u*, the variable from the substitution, and then use these solutions and the substitution definition to get the solutions to the equation that we really want.

In most cases to make the check that it's reducible to quadratic in form all that we really need to do is to check that one of the exponents is twice the other. There is one exception to this that we'll see here once we get into a set of examples.

Also, once you get "good" at these you often don't really need to do the substitution either. We will do them to make sure that the work is clear. However, these problems can be done without the substitution in many cases.

Example 2 Solve each of the following equations.

(a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$
(b) $y^{-6} - 9y^{-3} + 8 = 0$
(c) $z - 9\sqrt{z} + 14 = 0$
(d) $t^4 - 4 = 0$

Solution

(a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

Okay, in this case we can see that,

$$\frac{2}{3} = 2\left(\frac{1}{3}\right)$$

and so one of the exponents is twice the other so it looks like we've got an equation that is reducible to quadratic in form. The substitution will then be,

$$u = x^{\frac{1}{3}} \qquad\qquad u^2 = \left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$$

Substituting this into the equation gives,

$$\begin{aligned} u^2 - 2u - 15 &= 0 \\ (u-5)(u+3) &= 0 \qquad \Rightarrow \qquad u = -3,\ u = 5 \end{aligned}$$

Now that we've gotten the solutions for *u* we can find values of *x*.

$$\begin{aligned} u = -3: &\qquad x^{\frac{1}{3}} = -3 \qquad \Rightarrow \qquad x = (-3)^3 = -27 \\ u = 5: &\qquad x^{\frac{1}{3}} = 5 \qquad \Rightarrow \qquad x = (5)^3 = 125 \end{aligned}$$

So, we have two solutions here $x = -27$ and $x = 125$.

(b) $y^{-6} - 9y^{-3} + 8 = 0$

For this part notice that,

$$-6 = 2(-3)$$

and so we do have an equation that is reducible to quadratic form. The substitution is,

$$u = y^{-3} \qquad\qquad u^2 = \left(y^{-3}\right)^2 = y^{-6}$$

The equation becomes,

$$\begin{aligned} u^2 - 9u + 8 &= 0 \\ (u-8)(u-1) &= 0 \qquad\qquad u = 1,\ u = 8 \end{aligned}$$

Now, going back to *y*'s is going to take a little more work here, but shouldn't be too bad.

$$u = 1: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 1 \quad \Rightarrow \quad y^3 = \frac{1}{1} = 1 \quad \Rightarrow \quad y = (1)^{\frac{1}{3}} = 1$$

$$u = 8: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 8 \quad \Rightarrow \quad y^3 = \frac{1}{8} \quad \Rightarrow \quad y = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$$

The two solutions to this equation are $y = 1$ and $y = \frac{1}{2}$.

(c) $z - 9\sqrt{z} + 14 = 0$

This one is a little trickier to see that it's quadratic in form, yet it is. To see this recall that the exponent on the square root is one-half, then we can notice that the exponent on the first term is twice the exponent on the second term. So, this equation is in fact reducible to quadratic in form.

Here is the substitution.

$$u = \sqrt{z} \qquad\qquad u^2 = \left(\sqrt{z}\right)^2 = z$$

The equation then becomes,

$$\begin{aligned} u^2 - 9u + 14 &= 0 \\ (u-7)(u-2) &= 0 \qquad\qquad u = 2,\ u = 7 \end{aligned}$$

Now go back to *z*'s.

$$u = 2: \quad \Rightarrow \quad \sqrt{z} = 2 \quad \Rightarrow \quad z = (2)^2 = 4$$

$$u = 7: \quad \Rightarrow \quad \sqrt{z} = 7 \quad \Rightarrow \quad z = (7)^2 = 49$$

The two solutions for this equation are $z = 4$ and $z = 49$

(d) $t^4 - 4 = 0$

Now, this part is the exception to the rule that we've been using to identify equations that are reducible to quadratic in form. There is only one term with a *t* in it. However, notice that we can write the equation as,

$$\left(t^2\right)^2 - 4 = 0$$

So, if we use the substitution,

$$u = t^2 \qquad\qquad u^2 = \left(t^2\right)^2 = t^4$$

the equation becomes,

$$u^2 - 4 = 0$$

and so it is reducible to quadratic in form.

Now, we can solve this using the square root property. Doing that gives,

$$u = \pm\sqrt{4} = \pm 2$$

Now, going back to *t*'s gives us,

$$\begin{aligned} u &= 2: & &\Rightarrow & t^2 &= 2 & &\Rightarrow & t &= \pm\sqrt{2} \\ u &= -2: & &\Rightarrow & t^2 &= -2 & &\Rightarrow & t &= \pm\sqrt{-2} = \pm\sqrt{2}\, i \end{aligned}$$

In this case we get four solutions and two of them are complex solutions. Getting complex solutions out of these are actually more common that this set of examples might suggest. The problem is that to get some of the complex solutions requires knowledge that we haven't (and won't) cover in this course. So, they don't show up all that often.

All of the examples to this point gave quadratic equations that were factorable or in the case of the last part of the previous example was an equation that we could use the square root property on. That need not always be the case however. It is more than possible that we would need the quadratic formula to do some of these. We should do an example of one of these just to make the point.

Example 3 Solve $2x^{10} - x^5 - 4 = 0$.

Solution

In this case we can reduce this to quadratic in form by using the substitution,

$$u = x^5 \qquad\qquad u^2 = x^{10}$$

Using this substitution the equation becomes,

$$2u^2 - u - 4 = 0$$

This doesn't factor and so we'll need to use the quadratic formula on it. From the quadratic formula the solutions are,

$$u = \frac{1 \pm \sqrt{33}}{4}$$

Now, in order to get back to *x*'s we are going to need decimals values for these so,

$$u = \frac{1+\sqrt{33}}{4} = 1.68614 \qquad\qquad u = \frac{1-\sqrt{33}}{4} = -1.18614$$

Now, using the substitution to get back to *x*'s gives the following,

$$u = 1.68614 \qquad x^5 = 1.68614 \qquad x = \left(1.68614\right)^{\frac{1}{5}} = 1.11014$$

$$u = -1.18614 \qquad x^5 = -1.18614 \qquad x = \left(-1.18614\right)^{\frac{1}{5}} = -1.03473$$

We had to use a calculator to get the final answer for these. This is one of the reasons that you don't tend to see too many of these done in an Algebra class. The work and/or answers tend to be a little messy.

Section 2-10 : Equations with Radicals

The title of this section is maybe a little misleading. The title seems to imply that we're going to look at equations that involve any radicals. However, we are going to restrict ourselves to equations involving square roots. The techniques we are going to apply here can be used to solve equations with other radicals, however the work is usually significantly messier than when dealing with square roots. Therefore, we will work only with square roots in this section.

Before proceeding it should be mentioned as well that in some Algebra textbooks you will find this section in with the equations reducible to quadratic form material. The reason is that we will in fact end up solving a quadratic equation in most cases. However, the approach is significantly different and so we're going to separate the two topics into different sections in this course.

It is usually best to see how these work with an example.

Example 1 Solve $x = \sqrt{x+6}$.

Solution

In this equation the basic problem is the square root. If that weren't there we could do the problem. The whole process that we're going to go through here is set up to eliminate the square root. However, as we will see, the steps that we're going to take can actually cause problems for us. So, let's see how this all works.

Let's notice that if we just square both sides we can make the square root go away. Let's do that and see what happens.

$$\begin{aligned} \left(x\right)^2 &= \left(\sqrt{x+6}\right)^2 \\ x^2 &= x+6 \\ x^2 - x - 6 &= 0 \\ \left(x-3\right)\left(x+2\right) &= 0 \qquad \Rightarrow \qquad x = 3,\ x = -2 \end{aligned}$$

Upon squaring both sides we see that we get a factorable quadratic equation that gives us two solutions $x = 3$ and $x = -2$.

Now, for no apparent reason, let's do something that we haven't actually done since the section on solving linear equations. Let's check our answers. Remember as well that we need to check the answers in the original equation! That is very important.

Let's first check $x = 3$

$$\begin{aligned} 3 &\stackrel{?}{=} \sqrt{3+6} \\ 3 &= \sqrt{9} \qquad \text{OK} \end{aligned}$$

So $x = 3$ is a solution. Now let's check $x = -2$.

$$-2 \stackrel{?}{=} \sqrt{-2+6}$$
$$-2 \neq \sqrt{4} = 2 \qquad \text{NOT OK}$$

We have a problem. Recall that square roots are ALWAYS positive and so $x = -2$ does not work in the original equation. One possibility here is that we made a mistake somewhere. We can go back and look however, and we'll quickly see that we haven't made a mistake.

So, what is the deal? Remember that our first step in the solution process was to square both sides. Notice that if we plug $x = -2$ into the quadratic we solved it would in fact be a solution to that. When we squared both sides of the equation we actually changed the equation and, in the process, introduced a solution that is not a solution to the original equation.

With these problems it is vitally important that you check your solutions as this will often happen. When this does we only take the values that are actual solutions to the original equation.

So, the original equation had a single solution $x = 3$.

Now, as this example has shown us, we have to be very careful in solving these equations. When we solve the quadratic we will get two solutions and it is possible both of these, one of these, or none of these values to be solutions to the original equation. The only way to know is to check your solutions!

Let's work a couple more examples that are a little more difficult.

Example 2 Solve each of the following equations.

(a) $y + \sqrt{y-4} = 4$

(b) $1 = t + \sqrt{2t-3}$

(c) $\sqrt{5z+6} - 2 = z$

Solution

(a) $y + \sqrt{y-4} = 4$

In this case let's notice that if we just square both sides we're going to have problems.

$$\left(y + \sqrt{y-4}\right)^2 = \left(4\right)^2$$
$$y^2 + 2y\sqrt{y-4} + y - 4 = 16$$

Before discussing the problem we've got here let's make sure you can do the squaring that we did above since it will show up on occasion. All that we did here was use the formula

$$\left(a+b\right)^2 = a^2 + 2ab + b^2$$

with $a = y$ and $b = \sqrt{y-4}$. You will need to be able to do these because while this may not have worked here we will need to this kind of work in the next set of problems.

Now, just what is the problem with this? Well recall that the point behind squaring both sides in the first problem was to eliminate the square root. We haven't done that. There is still a square root in the problem and we've made the remainder of the problem messier as well.

So, what we're going to need to do here is make sure that we've got a square root all by itself on one side of the equation before squaring. Once that is done we can square both sides and the square root really will disappear.

Here is the correct way to do this problem.

$$\begin{aligned} \sqrt{y-4} &= 4-y & \text{now square both sides} \\ \left(\sqrt{y-4}\right)^2 &= (4-y)^2 \\ y-4 &= 16-8y+y^2 \\ 0 &= y^2-9y+20 \\ 0 &= (y-5)(y-4) \qquad \Rightarrow \qquad & y=4,\; y=5 \end{aligned}$$

As with the first example we will need to make sure and check both of these solutions. Again, make sure that you check in the original equation. Once we've square both sides we've changed the problem and so checking there won't do us any good. In fact, checking there could well lead us into trouble.

First $y=4$.

$$\begin{aligned} 4+\sqrt{4-4} &\stackrel{?}{=} 4 \\ 4 &= 4 \qquad \text{OK} \end{aligned}$$

So, that is a solution. Now $y=5$.

$$\begin{aligned} 5+\sqrt{5-4} &\stackrel{?}{=} 4 \\ 5+\sqrt{1} &\stackrel{?}{=} 4 \\ 6 &\neq 4 \qquad \text{NOT OK} \end{aligned}$$

So, as with the first example we worked there is in fact a single solution to the original equation, $y=4$.

(b) $1=t+\sqrt{2t-3}$

Okay, so we will again need to get the square root on one side by itself before squaring both sides.

$$\begin{aligned} 1-t &= \sqrt{2t-3} \\ (1-t)^2 &= \left(\sqrt{2t-3}\right)^2 \\ 1-2t+t^2 &= 2t-3 \\ t^2-4t+4 &= 0 \\ (t-2)^2 &= 0 \qquad \Rightarrow \qquad t=2 \end{aligned}$$

So, we have a double root this time. Let's check it to see if it really is a solution to the original equation.

$$1 \stackrel{?}{=} 2 + \sqrt{2(2)-3}$$
$$1 \stackrel{?}{=} 2 + \sqrt{1}$$
$$1 \neq 3$$

So, $t = 2$ isn't a solution to the original equation. Since this was the only possible solution, this means that there are **no solutions** to the original equation. This doesn't happen too often, but it does happen so don't be surprised by it when it does.

(c) $\sqrt{5z+6} - 2 = z$

This one will work the same as the previous two.

$$\begin{aligned} \sqrt{5z+6} &= z+2 \\ \left(\sqrt{5z+6}\right)^2 &= (z+2)^2 \\ 5z+6 &= z^2+4z+4 \\ 0 &= z^2 - z - 2 \\ 0 &= (z-2)(z+1) \qquad \Rightarrow \qquad z=-1, \ \ z=2 \end{aligned}$$

Let's check these possible solutions start with $z = -1$.

$$\begin{aligned} \sqrt{5(-1)+6} - 2 &\stackrel{?}{=} -1 \\ \sqrt{1} - 2 &\stackrel{?}{=} -1 \\ -1 &= -1 \qquad \text{OK} \end{aligned}$$

So, that's was a solution. Now let's check $z = 2$.

$$\begin{aligned} \sqrt{5(2)+6} - 2 &\stackrel{?}{=} 2 \\ \sqrt{16} - 2 &\stackrel{?}{=} 2 \\ 4 - 2 &= 2 \qquad \text{OK} \end{aligned}$$

This was also a solution.

So, in this case we've now seen an example where both possible solutions are in fact solutions to the original equation as well.

So, as we've seen in the previous set of examples once we get our list of possible solutions anywhere from none to all of them can be solutions to the original equation. Always remember to check your answers!

Okay, let's work one more set of examples that have an added complexity to them. To this point all the equations that we've looked at have had a single square root in them. However, there can be more than one square root in these equations. The next set of examples is designed to show us how to deal with these kinds of problems.

Example 3 Solve each of the following equations.

(a) $\sqrt{2x-1}-\sqrt{x-4}=2$

(b) $\sqrt{t+7}+2=\sqrt{3-t}$

Solution

In both of these there are two square roots in the problem. We will work these in basically the same manner however. The first step is to get one of the square roots by itself on one side of the equation then square both sides. At this point the process is different so we'll see how to proceed from this point once we reach it in the first example.

(a) $\sqrt{2x-1}-\sqrt{x-4}=2$

So, the first thing to do is get one of the square roots by itself. It doesn't matter which one we get by itself. We'll end up the same solution(s) in the end.

$$\begin{aligned}\sqrt{2x-1}&=2+\sqrt{x-4}\\ \left(\sqrt{2x-1}\right)^2&=\left(2+\sqrt{x-4}\right)^2\\ 2x-1&=4+4\sqrt{x-4}+x-4\\ 2x-1&=4\sqrt{x-4}+x\end{aligned}$$

Now, we still have a square root in the problem, but we have managed to eliminate one of them. Not only that, but what we've got left here is identical to the examples we worked in the first part of this section. Therefore, we will continue now work this problem as we did in the previous sets of examples.

$$\begin{aligned}\left(x-1\right)^2&=\left(4\sqrt{x-4}\right)^2\\ x^2-2x+1&=16\left(x-4\right)\\ x^2-2x+1&=16x-64\\ x^2-18x+65&=0\\ \left(x-13\right)\left(x-5\right)&=0 \qquad\Rightarrow\qquad x=13,\ x=5\end{aligned}$$

Now, let's check both possible solutions in the original equation. We'll start with $x=13$

$$\begin{aligned}\sqrt{2\left(13\right)-1}-\sqrt{13-4}&\stackrel{?}{=}2\\ \sqrt{25}-\sqrt{9}&\stackrel{?}{=}2\\ 5-3&=2 \qquad \text{OK}\end{aligned}$$

So, the one is a solution. Now let's check $x=5$.

$$\sqrt{2(5)-1}-\sqrt{5-4}\stackrel{?}{=}2$$
$$\sqrt{9}-\sqrt{1}\stackrel{?}{=}2$$
$$3-1=2 \qquad \text{OK}$$

So, they are both solutions to the original equation.

(b) $\sqrt{t+7}+2=\sqrt{3-t}$

In this case we've already got a square root on one side by itself so we can go straight to squaring both sides.

$$\left(\sqrt{t+7}+2\right)^2=\left(\sqrt{3-t}\right)^2$$
$$t+7+4\sqrt{t+7}+4=3-t$$
$$t+11+4\sqrt{t+7}=3-t$$

Next, get the remaining square root back on one side by itself and square both sides again.

$$4\sqrt{t+7}=-8-2t$$
$$\left(4\sqrt{t+7}\right)^2=\left(-8-2t\right)^2$$
$$16\left(t+7\right)=64+32t+4t^2$$
$$16t+112=64+32t+4t^2$$
$$0=4t^2+16t-48$$
$$0=4\left(t^2+4t-12\right)$$
$$0=4\left(t+6\right)\left(t-2\right) \qquad \Rightarrow \qquad t=-6,\ \ t=2$$

Now check both possible solutions starting with $t=2$.

$$\sqrt{2+7}+2\stackrel{?}{=}\sqrt{3-2}$$
$$\sqrt{9}+2\stackrel{?}{=}\sqrt{1}$$
$$3+2\neq 1 \qquad \text{NOT OK}$$

So, that wasn't a solution. Now let's check $t=-6$.

$$\sqrt{-6+7}+2\stackrel{?}{=}\sqrt{3-(-6)}$$
$$\sqrt{1}+2\stackrel{?}{=}\sqrt{9}$$
$$1+2=3 \qquad \text{OK}$$

It looks like in this case we've got a single solution, $t=-6$.

So, when there is more than one square root in the problem we are again faced with the task of checking our possible solutions. It is possible that anywhere from none to all of the possible solutions will in fact be solutions and the only way to know for sure is to check them in the original equation.

Section 2-11 : Linear Inequalities

To this point in this chapter we've concentrated on solving equations. It is now time to switch gears a little and start thinking about solving inequalities. Before we get into solving inequalities we should go over a couple of the basics first.

At this stage of your mathematical career it is assumed that you know that

$$a < b$$

means that *a* is some number that is strictly less that *b*. It is also assumed that you know that

$$a \ge b$$

means that *a* is some number that is either strictly bigger than *b* or is exactly equal to *b*. Likewise, it is assumed that you know how to deal with the remaining two inequalities. > (greater than) and $\le$ (less than or equal to).

What we want to discuss is some notational issues and some subtleties that sometimes get students when they really start working with inequalities.

First, remember that when we say that *a* is less than *b* we mean that *a* is to the left of *b* on a number line. So,

$$-1000 < 0$$

is a true inequality.

Next, don't forget how to correctly interpret $\le$ and $\ge$. Both of the following are true inequalities.

$$4 \le 4 \qquad\qquad -6 \le 4$$

In the first case 4 is equal to 4 and so it is "less than or equal" to 4. In the second case -6 is strictly less than 4 and so it is "less than or equal" to 4. The most common mistake is to decide that the first inequality is not a true inequality. Also be careful to not take this interpretation and translate it to < and/or >. For instance,

$$4 < 4$$

is not a true inequality since 4 is equal to 4 and not strictly less than 4.

Finally, we will be seeing many **double inequalities** throughout this section and later sections so we can't forget about those. The following is a double inequality.

$$-9 < 5 \le 6$$

In a double inequality we are saying that both inequalities must be simultaneously true. In this case 5 is definitely greater than -9 and at the same time is less than or equal to 6. Therefore, this double inequality is a true inequality.

On the other hand,

$$10 \le 5 < 20$$

is not a true inequality. While it is true that 5 is less than 20 (so the second inequality is true) it is not true that 5 is greater than or equal to 10 (so the first inequality is not true). If even one of the inequalities in a double inequality is not true then the whole inequality is not true. This point is more important than you might realize at this point. In a later section we will run across situations where many students try to combine two inequalities into a double inequality that simply can't be combined, so be careful.

The next topic that we need to discuss is the idea of **interval notation**. Interval notation is some very nice shorthand for inequalities and will be used extensively in the next few sections of this chapter.

The best way to define interval notation is the following table. There are three columns to the table. Each row contains an inequality, a graph representing the inequality and finally the interval notation for the given inequality.

Inequality	Graph	Interval Notation
$a \le x \le b$		$\left[a,b\right]$
$a < x < b$		$\left(a,b\right)$
$a \le x < b$		$\left[a,b\right)$
$a < x \le b$		$\left(a,b\right]$
$x > a$		$\left(a,\infty\right)$
$x \ge a$		$\left[a,\infty\right)$
$x < b$		$\left(-\infty,b\right)$
$x \le b$		$\left(-\infty,b\right]$

Remember that a bracket, "[" or "]", means that we include the endpoint while a parenthesis, "(" or ")", means we don't include the endpoint.

Now, with the first four inequalities in the table the interval notation is really nothing more than the graph without the number line on it. With the final four inequalities the interval notation is almost the graph, except we need to add in an appropriate infinity to make sure we get the correct portion of the number line. Also note that infinities NEVER get a bracket. They only get a parenthesis.

We need to give one final note on interval notation before moving on to solving inequalities. Always remember that when we are writing down an interval notation for an inequality that the number on the left must be the smaller of the two.

It's now time to start thinking about solving linear inequalities. We will use the following set of facts in our solving of inequalities. Note that the facts are given for <. We can however, write down an equivalent set of facts for the remaining three inequalities.

1. If $a < b$ then $a + c < b + c$ and $a - c < b - c$ for any number *c*. In other words, we can add or subtract a number to both sides of the inequality and we don't change the inequality itself.

2. If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$. So, provided *c* is a positive number we can multiply or divide both sides of an inequality by the number without changing the inequality.

3. If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. In this case, unlike the previous fact, if *c* is negative we need to flip the direction of the inequality when we multiply or divide both sides by the inequality by *c*.

These are nearly the same facts that we used to solve linear equations. The only real exception is the third fact. This is the important fact as it is often the most misused and/or forgotten fact in solving inequalities.

If you aren't sure that you believe that the sign of *c* matters for the second and third fact consider the following number example.

$$-3 < 5$$

I hope that we would all agree that this is a true inequality. Now multiply both sides by 2 and by -2.

$$\begin{aligned} -3 &< 5 \\ -3(2) &< 5(2) \\ -6 &< 10 \end{aligned} \qquad\qquad \begin{aligned} -3 &< 5 \\ -3(-2) &> 5(-2) \\ 6 &> -10 \end{aligned}$$

Sure enough, when multiplying by a positive number the direction of the inequality remains the same, however when multiplying by a negative number the direction of the inequality does change.

Okay, let's solve some inequalities. We will start off with inequalities that only have a single inequality in them. In other words, we'll hold off on solving double inequalities for the next set of examples.

The thing that we've got to remember here is that we're asking to determine all the values of the variable that we can substitute into the inequality and get a true inequality. This means that our solutions will, in most cases, be inequalities themselves.

Example 1 Solving the following inequalities. Give both inequality and interval notation forms of the solution.

(a) $-2(m-3) < 5(m+1) - 12$

(b) $2(1-x) + 5 \le 3(2x-1)$

Solution

Solving single linear inequalities follow pretty much the same process for solving linear equations. We will simplify both sides, get all the terms with the variable on one side and the numbers on the other side, and then multiply/divide both sides by the coefficient of the variable to get the solution. The one thing that you've got to remember is that if you multiply/divide by a negative number then switch the direction of the inequality.

(a) $-2(m-3) < 5(m+1) - 12$

There really isn't much to do here other than follow the process outlined above.

$$\begin{aligned} -2(m-3) &< 5(m+1) - 12 \\ -2m + 6 &< 5m + 5 - 12 \\ -7m &< -13 \\ m &> \frac{13}{7} \end{aligned}$$

You did catch the fact that the direction of the inequality changed here didn't you? We divided by a "-7" and so we had to change the direction. The inequality form of the solution is $m > \frac{13}{7}$. The interval notation for this solution is, $\left(\frac{13}{7}, \infty\right)$.

(b) $2(1-x) + 5 \le 3(2x-1)$

Again, not much to do here.

$$\begin{aligned} 2(1-x) + 5 &\le 3(2x-1) \\ 2 - 2x + 5 &\le 6x - 3 \\ 10 &\le 8x \\ \frac{10}{8} &\le x \\ \frac{5}{4} &\le x \end{aligned}$$

Now, with this inequality we ended up with the variable on the right side when it more traditionally on the left side. So, let's switch things around to get the variable onto the left side. Note however, that we're going to need also switch the direction of the inequality to make sure that we don't change the answer. So, here is the inequality notation for the inequality.

$$x \ge \frac{5}{4}$$

The interval notation for the solution is $\left[\frac{5}{4}, \infty\right)$.

Now, let's solve some double inequalities. The process here is similar in some ways to solving single inequalities and yet very different in other ways. Since there are two inequalities there isn't any way to get the variables on "one side" of the inequality and the numbers on the other. It is easier to see how these work if we do an example or two so let's do that.

Example 2 Solve each of the following inequalities. Give both inequality and interval notation forms for the solution.

(a) $-6 \le 2\left(x-5\right) < 7$

(b) $-3 < \frac{3}{2}\left(2-x\right) \le 5$

(c) $-14 < -7\left(3x+2\right) < 1$

Solution

(a) $-6 \le 2\left(x-5\right) < 7$

The process here is fairly similar to the process for single inequalities, but we will first need to be careful in a couple of places. Our first step in this case will be to clear any parenthesis in the middle term.

$$-6 \le 2x-10 < 7$$

Now, we want the *x* all by itself in the middle term and only numbers in the two outer terms. To do this we will add/subtract/multiply/divide as needed. The only thing that we need to remember here is that if we do something to middle term we need to do the same thing to BOTH of the out terms. One of the more common mistakes at this point is to add something, for example, to the middle and only add it to one of the two sides.

Okay, we'll add 10 to all three parts and then divide all three parts by two.

$$4 \le 2x < 17$$

$$2 \le x < \frac{17}{2}$$

That is the inequality form of the answer. The interval notation form of the answer is $\left[2, \frac{17}{2}\right)$.

(b) $-3 < \frac{3}{2}\left(2-x\right) \le 5$

In this case the first thing that we need to do is clear fractions out by multiplying all three parts by 2. We will then proceed as we did in the first part.

$$-6 < 3\left(2-x\right) \le 10$$

$$-6 < 6-3x \le 10$$

$$-12 < -3x \le 4$$

Now, we're not quite done here, but we need to be very careful with the next step. In this step we need to divide all three parts by -3. However, recall that whenever we divide both sides of an inequality by a negative number we need to switch the direction of the inequality. For us, this means that both of the inequalities will need to switch direction here.

$$4 > x \ge -\frac{4}{3}$$

So, there is the inequality form of the solution. We will need to be careful with the interval notation for the solution. First, the interval notation is NOT $\left(4,-\frac{4}{3}\right]$. Remember that in interval notation the smaller number must always go on the left side! Therefore, the correct interval notation for the solution is $\left[-\frac{4}{3},4\right)$.

Note as well that this does match up with the inequality form of the solution as well. The inequality is telling us that *x* is any number between 4 and $-\frac{4}{3}$ or possibly $-\frac{4}{3}$ itself and this is exactly what the interval notation is telling us.

Also, the inequality could be flipped around to get the smaller number on the left if we'd like to. Here is that form,

$$-\frac{4}{3} \le x < 4$$

When doing this make sure to correctly deal with the inequalities as well.

(c) $-14 < -7\left(3x+2\right) < 1$

Not much to this one. We'll proceed as we've done the previous two.

$$\begin{gathered} -14 < -21x - 14 < 1 \\ 0 < -21x < 15 \end{gathered}$$

Don't get excited about the fact that one of the sides is now zero. This isn't a problem. Again, as with the last part, we'll be dividing by a negative number and so don't forget to switch the direction of the inequalities.

$$\begin{aligned} &0 > x > -\frac{15}{21} \\ &0 > x > -\frac{5}{7} \qquad \text{OR} \qquad -\frac{5}{7} < x < 0 \end{aligned}$$

Either of the inequalities in the second row will work for the solution. The interval notation of the solution is $\left(-\frac{5}{7},0\right)$.

When solving double inequalities make sure to pay attention to the inequalities that are in the original problem. One of the more common mistakes here is to start with a problem in which one of the inequalities is < or > and the other is $\le$ or $\ge$, as we had in the first two parts of the previous example, and then by the final answer they are both < or > or they are both $\le$ or $\ge$. In other words, it is easy to all of a sudden make both of the inequalities the same. Be careful with this.

There is one final example that we want to work here.

Example 3 If $-1 < x < 4$ then determine *a* and *b* in $a < 2x + 3 < b$.

Solution

This is easier than it may appear at first. All we are really going to do is start with the given inequality and then manipulate the middle term to look like the second inequality. Again, we'll need to remember that whatever we do to the middle term we'll also need to do to the two outer terms.

So, first we'll multiply everything by 2.

$$-2 < 2x < 8$$

Now add 3 to everything.

$$1 < 2x + 3 < 11$$

We've now got the middle term identical to the second inequality in the problems statement and so all we need to do is pick off *a* and *b*. From this inequality we can see that $a = 1$ and $b = 11$.

Section 2-12 : Polynomial Inequalities

It is now time to look at solving some more difficult inequalities. In this section we will be solving (single) inequalities that involve polynomials of degree at least two. Or, to put it in other words, the polynomials won't be linear any more. Just as we saw when solving equations the process that we have for solving linear inequalities just won't work here.

Since it's easier to see the process as we work an example let's do that. As with the linear inequalities, we are looking for all the values of the variable that will make the inequality true. This means that our solution will almost certainly involve inequalities as well. The process that we're going to go through will give the answers in that form.

Example 1 Solve $x^2 - 10 < 3x$.

Solution

There is a fairly simple process to solving these. If you can remember it you'll always be able to solve these kinds of inequalities.

Step 1 : Get a zero on one side of the inequality. It doesn't matter which side has the zero, however, we're going to be factoring in the next step so keep that in mind as you do this step. Make sure that you've got something that's going to be easy to factor.

$$x^2 - 3x - 10 < 0$$

Step 2 : If possible, factor the polynomial. Note that it won't always be possible to factor this, but that won't change things. This step is really here to simplify the process more than anything. Almost all of the problems that we're going to look at will be factorable.

$$(x-5)(x+2) < 0$$

Step 3 : Determine where the polynomial is zero. Notice that these points won't make the inequality true (in this case) because $0 < 0$ is NOT a true inequality. That isn't a problem. These points are going to allow us to find the actual solution.

In our case the polynomial will be zero at $x = -2$ and $x = 5$.

Now, before moving on to the next step let's address why we want these points.

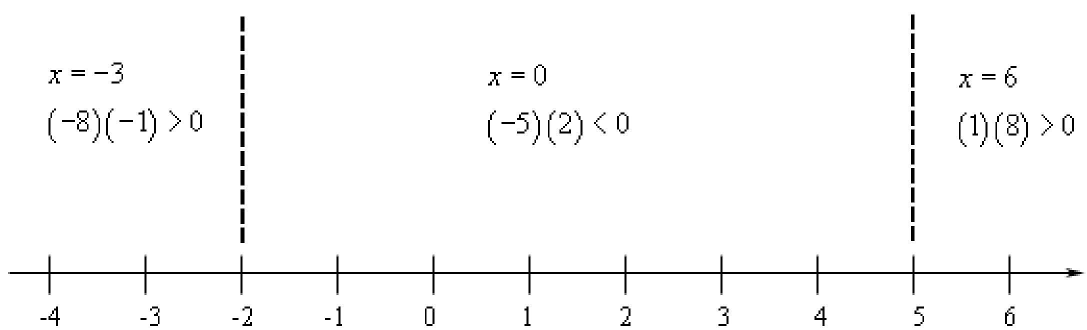
We haven't discussed graphing polynomials yet, however, the graphs of polynomials are nice smooth functions that have no breaks in them. This means that as we are moving across the number line (in any direction) if the value of the polynomial changes sign (say from positive to negative) then it MUST go through zero!

So, that means that these two numbers ($x = 5$ and $x = -2$) are the ONLY places where the polynomial can change sign. The number line is then divided into three regions. In each region if the inequality is satisfied by one point from that region then it is satisfied for ALL points in that region. If this wasn't true (*i.e* it was positive at one point in the region and negative at another) then it must also be zero somewhere in that region, but that can't happen as we've already determined all the places where the polynomial can be zero! Likewise, if the inequality isn't satisfied for some point in that region then it isn't satisfied for ANY point in that region.

This leads us into the next step.

Step 4 : Graph the points where the polynomial is zero (*i.e.* the points from the previous step) on a number line and pick a **test point** from each of the regions. Plug each of these test points into the polynomial and determine the sign of the polynomial at that point.

This is the step in the process that has all the work, although it isn't too bad. Here is the number line for this problem.



Now, let's talk about this a little. When we pick test points make sure that you pick easy numbers to work with. So, don't choose large numbers or fractions unless you are forced to by the problem.

Also, note that we plugged the test points into the factored from of the polynomial and all we're really after here is whether or not the polynomial is positive or negative. Therefore, we didn't actually bother with values of the polynomial just the sign and we can get that from the product shown. The product of two negatives is a positive, *etc*.

We are now ready for the final step in the process.

Step 5 : Write down the answer. Recall that we discussed earlier that if any point from a region satisfied the inequality then ALL points in that region satisfied the inequality and likewise if any point from a region did not satisfy the inequality then NONE of the points in that region would satisfy the inequality.

This means that all we need to do is look up at the number line above. If the test point from a region satisfies the inequality then that region is part of the solution. If the test point doesn't satisfy the inequality then that region isn't part of the solution.

Now, also notice that any value of x that will satisfy the original inequality will also satisfy the inequality from Step 2 and likewise, if an x satisfies the inequality from Step 2 then it will satisfy the original inequality.

So, that means that all we need to do is determine the regions in which the polynomial from Step 2 is negative. For this problem that is only the middle region. The inequality and interval notation for the solution to this inequality are,

$$-2 < x < 5 \qquad\qquad\qquad (-2,5)$$

Notice that we do need to exclude the endpoints since we have a strict inequality (< in this case) in the inequality.

Okay, that seems like a long process, however, it really isn't. There was lots of explanation in the previous example. The remaining examples won't be as long because we won't need quite as much explanation in them.

Example 2 Solve $x^2 - 5x \geq -6$.

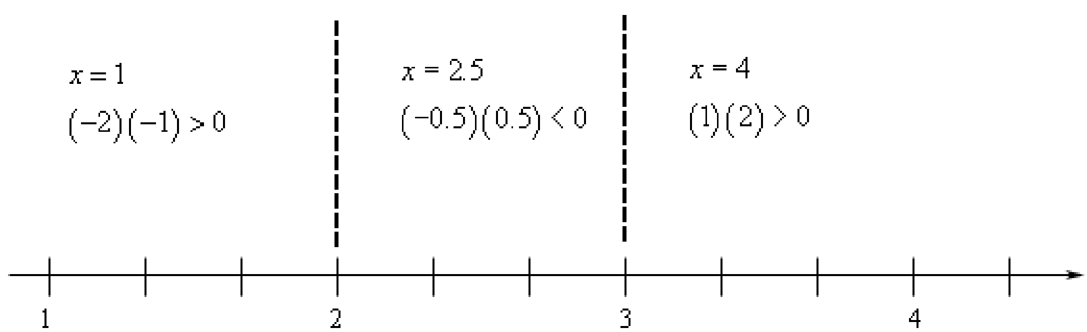
Solution

Okay, this time we'll just go through the process without all the explanations and steps. The first thing to do is get a zero on one side and factor the polynomial if possible.

$$x^2 - 5x + 6 \geq 0$$
$$(x-3)(x-2) \geq 0$$

So, the polynomial will be zero at $x = 2$ and $x = 3$. Notice as well that unlike the previous example, these will be solutions to the inequality since we've got a "greater than or equal to" in the inequality.

Here is the number line for this example.



Notice that in this case we were forced to choose a decimal for one of the test points.

Now, we want regions where the polynomial will be positive. So, the first and last regions will be part of the solution. Also, in this case, we've got an "or equal to" in the inequality and so we'll need to include the endpoints in our solution since at this points we get zero for the inequality and $0 \geq 0$ is a true inequality.

Here is the solution in both inequality and interval notation form.

$$-\infty < x \leq 2 \qquad \text{and} \qquad 3 \leq x < \infty$$
$$(-\infty, 2] \qquad \text{and} \qquad [3, \infty)$$

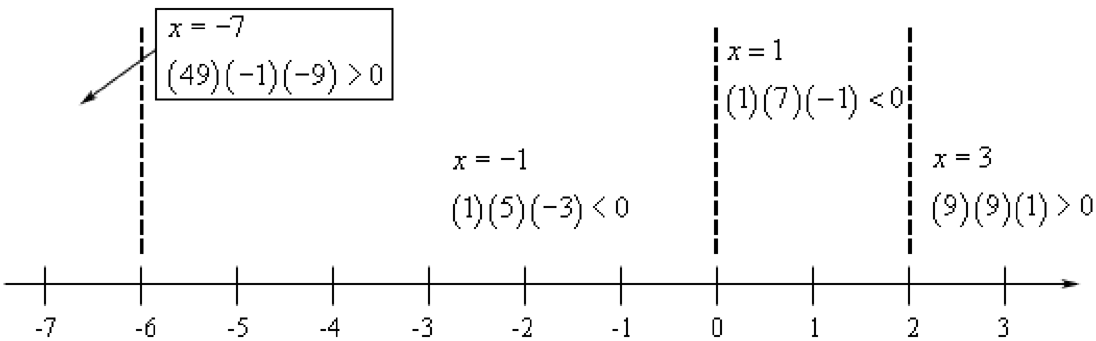
Example 3 Solve $x^4 + 4x^3 - 12x^2 \le 0$.

Solution

Again, we'll just jump right into the problem. We've already got zero on one side so we can go straight to factoring.

$$x^4 + 4x^3 - 12x^2 \le 0$$
$$x^2\left(x^2 + 4x - 12\right) \le 0$$
$$x^2\left(x+6\right)\left(x-2\right) \le 0$$

So, this polynomial is zero at $x = -6$, $x = 0$ and $x = 2$. Here is the number line for this problem.



First, notice that unlike the first two examples these regions do NOT alternate between positive and negative. This is a common mistake that students make. You really do need to plug in test points from each region. Don't ever just plug in for the first region and then assume that the other regions will alternate from that point.

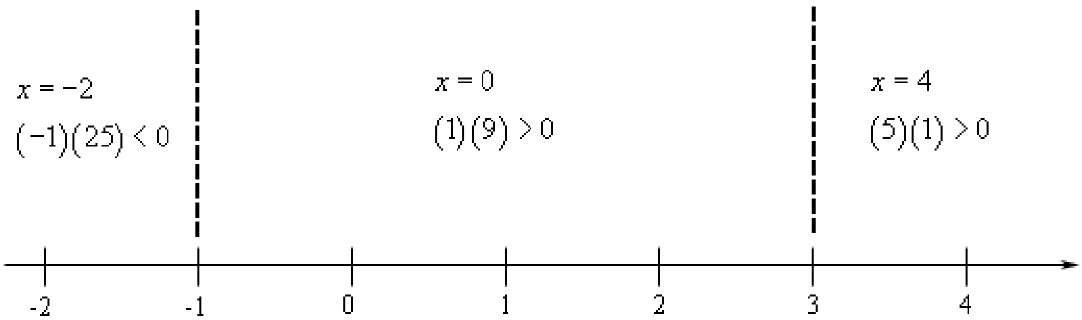
Now, for our solution we want regions where the polynomial will be negative (that's the middle two here) or zero (that's all three points that divide the regions). So, we can combine up the middle two regions and the three points into a single inequality in this case. The solution, in both inequality and interval notation form, is.

$$-6 \le x \le 2 \qquad\qquad \left[-6, 2\right]$$

Example 4 Solve $\left(x+1\right)\left(x-3\right)^2 > 0$.

Solution

The first couple of steps have already been done for us here. So, we can just straight into the work. This polynomial will be zero at $x = -1$ and $x = 3$. Here is the number line for this problem.



Again, note that the regions don't alternate in sign!

For our solution to this inequality we are looking for regions where the polynomial is positive (that's the last two in this case), however we don't want values where the polynomial is zero this time since we've got a strict inequality (>) in this problem. This means that we want the last two regions, but not $x = 3$.

So, unlike the previous example we can't just combine up the two regions into a single inequality since that would include a point that isn't part of the solution. Here is the solution for this problem.

$$-1 < x < 3 \qquad \text{and} \qquad 3 < x < \infty$$
$$\left(-1,3\right) \qquad \text{and} \qquad \left(3,\infty\right)$$

Now, all of the examples that we've worked to this point involved factorable polynomials. However, that doesn't have to be the case. We can work these inequalities even if the polynomial doesn't factor. We should work one of these just to show you how they work.

Example 5 Solve $3x^2 - 2x - 11 > 0$.

Solution

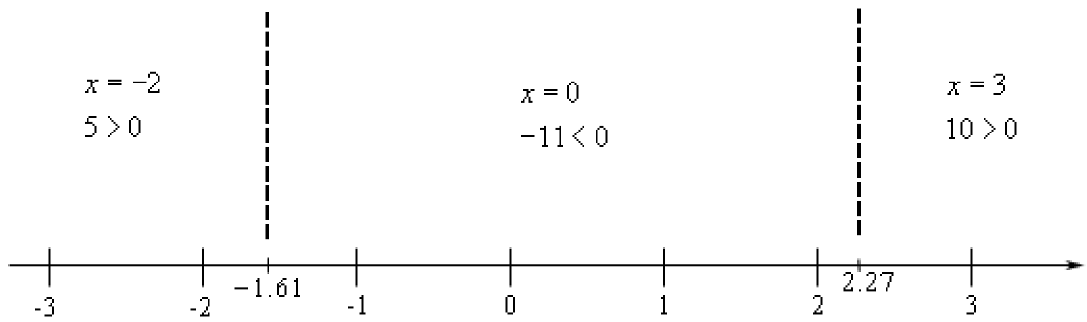
In this case the polynomial doesn't factor so we can't do that step. However, we do still need to know where the polynomial is zero. We will have to use the quadratic formula for that. Here is what the quadratic formula gives us.

$$x = \frac{1 \pm \sqrt{34}}{3}$$

In order to work the problem we'll need to reduce this to decimals.

$$x = \frac{1 + \sqrt{34}}{3} = 2.27698 \qquad\qquad x = \frac{1 - \sqrt{34}}{3} = -1.61032$$

From this point on the process is identical to the previous examples. In the number line below the dashed lines are at the approximate values of the two decimals above and the inequalities show the value of the quadratic evaluated at the test points shown.



So, it looks like we need the two outer regions for the solution. Here is the inequality and interval notation for the solution.

$$-\infty < x < \frac{1-\sqrt{34}}{3} \qquad \text{and} \qquad \frac{1+\sqrt{34}}{3} < x < \infty$$

$$\left(-\infty, \frac{1-\sqrt{34}}{3}\right) \qquad \text{and} \qquad \left(\frac{1+\sqrt{34}}{3}, \infty\right)$$

Section 2-13 : Rational Inequalities

In this section we will solve inequalities that involve rational expressions. The process for solving rational inequalities is nearly identical to the process for solving polynomial inequalities with a few minor differences.

Let's just jump straight into some examples.

Example 1 Solve $\frac{x+1}{x-5} \le 0$.

Solution

Before we get into solving these we need to point out that these DON'T solve in the same way that we've solve equations that contained rational expressions. With equations the first thing that we always did was clear out the denominators by multiplying by the least common denominator. That won't work with these however.

Since we don't know the value of *x* we can't multiply both sides by anything that contains an *x*. Recall that if we multiply both sides of an inequality by a negative number we will need to switch the direction of the inequality. However, since we don't know the value of *x* we don't know if the denominator is positive or negative and so we won't know if we need to switch the direction of the inequality or not. In fact, to make matters worse, the denominator will be both positive and negative for values of *x* in the solution and so that will create real problems.

So, we need to leave the rational expression in the inequality.

Now, the basic process here is the same as with polynomial inequalities. The first step is to get a zero on one side and write the other side as a single rational inequality. This has already been done for us here.

The next step is to factor the numerator and denominator as much as possible. Again, this has already been done for us in this case.

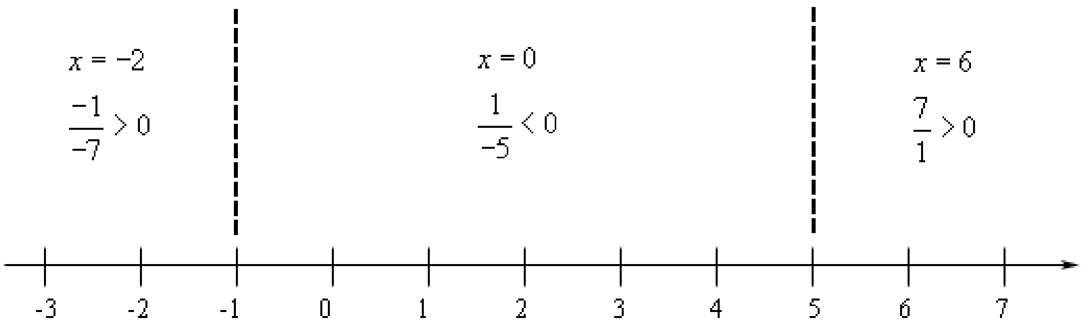
The next step is to determine where both the numerator and the denominator are zero. In this case these values are.

$$\text{numerator}: x = -1 \qquad\qquad \text{denominator}: x = 5$$

Now, we need these numbers for a couple of reasons. First, just like with polynomial inequalities these are the only numbers where the rational expression *may* change sign. So, we'll build a number line using these points to define ranges out of which to pick test points just like we did with polynomial inequalities.

There is another reason for needing the value of *x* that make the denominator zero however. No matter what else is going on here we do have a rational expression and that means we need to avoid division by zero and so knowing where the denominator is zero will give us the values of *x* to avoid for this.

Here is the number line for this inequality.



So, we need regions that make the rational expression negative. That means the middle region. Also, since we've got an "or equal to" part in the inequality we also need to include where the inequality is zero, so this means we include $x = -1$. Notice that we will also need to avoid $x = 5$ since that gives division by zero.

The solution for this inequality is,

$$-1 \le x < 5 \qquad\qquad [-1,5)$$

Example 2 Solve $\dfrac{x^2+4x+3}{x-1} > 0$.

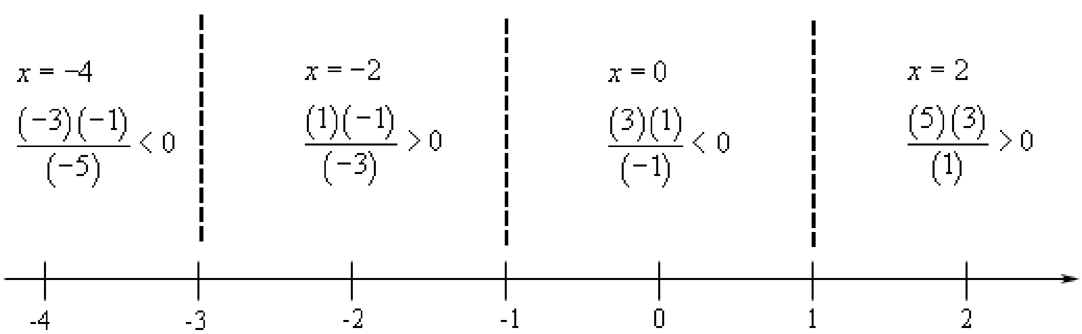
Solution

We've got zero on one side so let's first factor the numerator and determine where the numerator and denominator are both zero.

$$\frac{(x+1)(x+3)}{x-1} > 0$$

$$\text{numerator} : x = -1, \quad x = -3 \qquad\qquad \text{denominator} : x = 1$$

Here is the number line for this one.



In the problem we are after values of *x* that make the inequality strictly positive and so that looks like the second and fourth region and we won't include any of the endpoints here. The solution is then,

$$-3 < x < -1 \qquad \text{and} \qquad 1 < x < \infty$$

$$(-3,-1) \qquad \text{and} \qquad (1,\infty)$$

Example 3 Solve $\dfrac{x^2-16}{(x-1)^2}<0$.

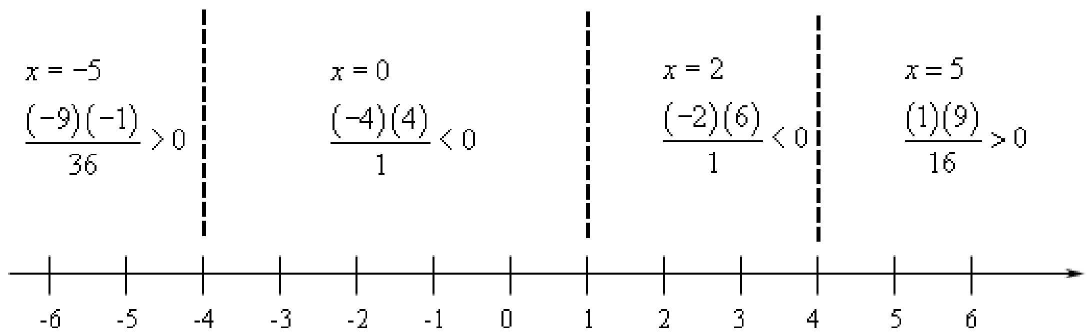
Solution

There really isn't too much to this example. We'll first need to factor the numerator and then determine where the numerator and denominator are zero.

$$\frac{(x-4)(x+4)}{(x-1)^2}<0$$

$$\text{numerator} : x=-4, \quad x=4 \qquad\qquad \text{denominator} : x=1$$

The number line for this problem is,



So, as with the polynomial inequalities we can not just assume that the regions will always alternate in sign. Also, note that while the middle two regions do give negative values in the rational expression we need to avoid $x=1$ to make sure we don't get division by zero. This means that we will have to write the answer as two inequalities and/or intervals.

$$-4<x<1 \qquad \text{and} \qquad 1<x<4$$

$$(-4,1) \qquad \text{and} \qquad (1,4)$$

Once again, it's important to note that we really do need to test each region and not just assume that the regions will alternate in sign.

Next, we need to take a look at some examples that don't already have a zero on one side of the inequality.

Example 4 Solve $\dfrac{3x+1}{x+4} \ge 1$.

Solution

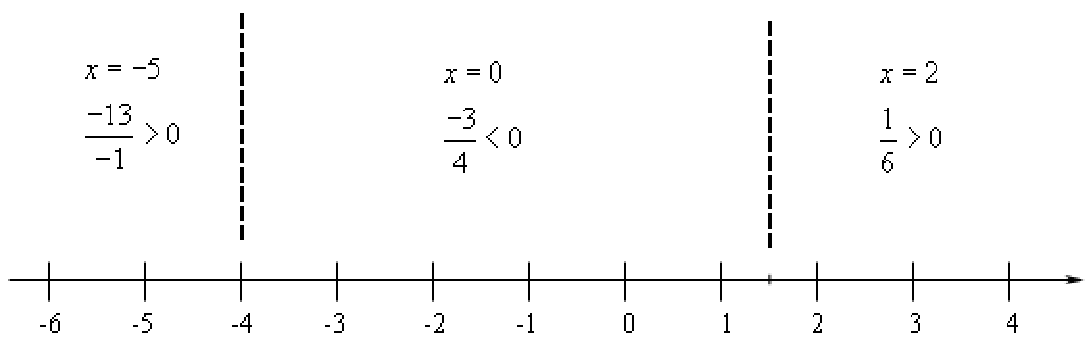
The first thing that we need to do here is subtract 1 from both sides and then get everything into a single rational expression.

$$\begin{aligned} \frac{3x+1}{x+4} - 1 &\ge 0 \\ \frac{3x+1}{x+4} - \frac{x+4}{x+4} &\ge 0 \\ \frac{3x+1-(x+4)}{x+4} &\ge 0 \\ \frac{2x-3}{x+4} &\ge 0 \end{aligned}$$

In this case there is no factoring to do so we can go straight to identifying where the numerator and denominator are zero.

$$\text{numerator} : x = \frac{3}{2} \qquad\qquad \text{denominator} : x = -4$$

Here is the number line for this problem.



Okay, we want values of *x* that give positive and/or zero in the rational expression. This looks like the outer two regions as well as $x = \frac{3}{2}$. As with the first example we will need to avoid $x = -4$ since that will give a division by zero error.

The solution for this problem is then,

$$\begin{array}{ccc} -\infty < x < -4 & \text{and} & \frac{3}{2} \le x < \infty \\ (-\infty, -4) & \text{and} & \left[\frac{3}{2}, \infty\right) \end{array}$$

Example 5 Solve $\frac{x-8}{x} \le 3-x$.

Solution

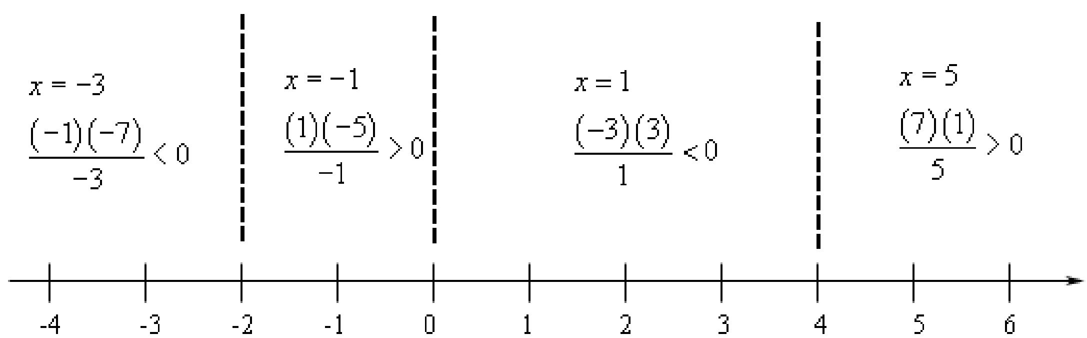
So, again, the first thing to do is to get a zero on one side and then get everything into a single rational expression.

$$\begin{aligned} \frac{x-8}{x}+x-3 &\le 0 \\ \frac{x-8}{x}+\frac{x(x-3)}{x} &\le 0 \\ \frac{x-8+x^2-3x}{x} &\le 0 \\ \frac{x^2-2x-8}{x} &\le 0 \\ \frac{(x-4)(x+2)}{x} &\le 0 \end{aligned}$$

We also factored the numerator above so we can now determine where the numerator and denominator are zero.

$$\text{numerator}: x=-2, \quad x=4 \qquad\qquad \text{denominator}: x=0$$

Here is the number line for this problem.



The solution for this inequality is then,

$$-\infty < x \le -2 \qquad \text{and} \qquad 0 < x \le 4$$

$$(-\infty,-2] \qquad \text{and} \qquad (0,4]$$

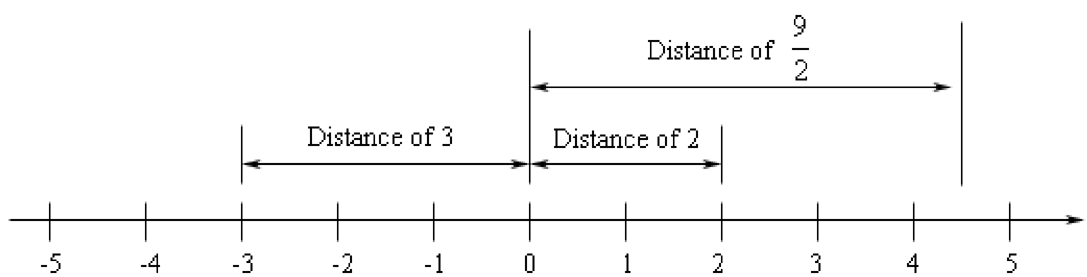
Section 2-14 : Absolute Value Equations

In the final two sections of this chapter we want to discuss solving equations and inequalities that contain absolute values. We will look at equations with absolute value in them in this section and we'll look at inequalities in the next section.

Before solving however, we should first have a brief discussion of just what absolute value is. The notation for the absolute value of *p* is $\left| p \right|$. Note as well that the absolute value bars are NOT parentheses and, in many cases, don't behave as parentheses so be careful with them.

There are two ways to define absolute value. There is a geometric definition and a mathematical definition. We will look at both.

Geometric Definition
In this definition we are going to think of $\left| p \right|$ as the distance of *p* from the origin on a number line. Also, we will always use a positive value for distance. Consider the following number line.



From this we can get the following values of absolute value.

$$\left| 2 \right| = 2 \qquad \left| -3 \right| = 3 \qquad \left| \frac{9}{2} \right| = \frac{9}{2}$$

All that we need to do is identify the point on the number line and determine its distance from the origin. Note as well that we also have $\left| 0 \right| = 0$.

Mathematical Definition
We can also give a strict mathematical/formula definition for absolute value. It is,

$$\left| p \right| = \begin{cases} p & \text{if } p \ge 0 \\ -p & \text{if } p < 0 \end{cases}$$

This tells us to look at the sign of *p* and if it's positive we just drop the absolute value bar. If *p* is negative we drop the absolute value bars and then put in a negative in front of it.

So, let's see a couple of quick examples.

$$\begin{aligned} |4| &= 4 && \text{because } 4 \geq 0 \\ |-8| &= -(-8) = 8 && \text{because } -8 < 0 \\ |0| &= 0 && \text{because } 0 \geq 0 \end{aligned}$$

Note that these give exactly the same value as if we'd used the geometric interpretation.

One way to think of absolute value is that it takes a number and makes it positive. In fact, we can guarantee that,

$$|p| \geq 0$$

regardless of the value of *p*.

We do need to be careful however to not misuse either of these definitions. For example, we can't use the definition on

$$|-x|$$

because we don't know the value of *x*.

Also, don't make the mistake of assuming that absolute value just makes all minus signs into plus signs. In other words, don't make the following mistake,

$$|4x-3| \neq 4x+3$$

This just isn't true! If you aren't sure that you believe that then plug in a number for *x*. For example, if $x = -1$ we would get,

$$7 = |-7| = |4(-1)-3| \neq 4(-1)+3 = -1$$

There are a couple of problems with this. First, the numbers are clearly not the same and so that's all we really need to prove that the two expressions aren't the same. There is also the fact however that the right number is negative and we will never get a negative value out of an absolute value! That also will guarantee that these two expressions aren't the same.

The definitions above are easy to apply if all we've got are numbers inside the absolute value bars. However, once we put variables inside them we've got to start being very careful.

It's now time to start thinking about how to solve equations that contain absolute values. Let's start off fairly simple and look at the following equation.

$$|p| = 4$$

Now, if we think of this from a geometric point of view this means that whatever *p* is it must have a distance of 4 from the origin. Well there are only two numbers that have a distance of 4 from the origin, namely 4 and -4. So, there are two solutions to this equation,

$$p = -4 \qquad \text{or} \qquad p = 4$$

Now, if you think about it we can do this for any positive number, not just 4. So, this leads to the following general formula for equations involving absolute value.

$$\boxed{\text{If} \quad |p| = b,\ b > 0 \quad \text{then} \quad p = -b \ \text{ or } \ p = b}$$

Notice that this does **require** the *b* be a positive number. We will deal with what happens if *b* is zero or negative in a bit.

Let's take a look at some examples.

Example 1 Solve each of the following.

(a) $|2x-5| = 9$

(b) $|1-3t| = 20$

(c) $|5y-8| = 1$

Solution

Now, remember that absolute value does not just make all minus signs into plus signs! To solve these, we've got to use the formula above since in all cases the number on the right side of the equal sign is positive.

(a) $|2x-5| = 9$

There really isn't much to do here other than using the formula from above as noted above. All we need to note is that in the formula above *p* represents whatever is on the inside of the absolute value bars and so in this case we have,

$$2x - 5 = -9 \qquad \text{or} \qquad 2x - 5 = 9$$

At this point we've got two linear equations that are easy to solve.

$$\begin{aligned} 2x &= -4 & \qquad \text{or} \qquad & 2x = 14 \\ x &= -2 & \qquad \text{or} \qquad & x = 7 \end{aligned}$$

So, we've got two solutions to the equation $x = -2$ and $x = 7$.

(b) $|1-3t| = 20$

This one is pretty much the same as the previous part so we won't put as much detail into this one.

$$\begin{aligned} 1 - 3t &= -20 & \qquad \text{or} \qquad & 1 - 3t = 20 \\ -3t &= -21 & \qquad \text{or} \qquad & -3t = 19 \\ t &= 7 & \qquad \text{or} \qquad & t = -\frac{19}{3} \end{aligned}$$

The two solutions to this equation are $t=-\frac{19}{3}$ and $t=7$.

(c) $\left|5y-8\right|=1$

Again, not much more to this one.

$$\begin{aligned} 5y-8&=-1 &\qquad \text{or} \qquad && 5y-8&=1\\ 5y&=7 &\qquad \text{or} \qquad && 5y&=9\\ y&=\frac{7}{5} &\qquad \text{or} \qquad && y&=\frac{9}{5}\end{aligned}$$

In this case the two solutions are $y=\frac{7}{5}$ and $y=\frac{9}{5}$.

Now, let's take a look at how to deal with equations for which *b* is zero or negative. We'll do this with an example.

Example 2 Solve each of the following.

(a) $\left|10x-3\right|=0$

(b) $\left|5x+9\right|=-3$

Solution

(a) Let's approach this one from a geometric standpoint. This is saying that the quantity in the absolute value bars has a distance of zero from the origin. There is only one number that has the property and that is zero itself. So, we must have,

$$10x-3=0 \qquad\Rightarrow\qquad x=\frac{3}{10}$$

In this case we get a single solution.

(b) Now, in this case let's recall that we noted at the start of this section that $\left|p\right|\geq 0$. In other words, we can't get a negative value out of the absolute value. That is exactly what this equation is saying however. Since this isn't possible that means there is **no solution** to this equation.

So, summarizing we can see that if *b* is zero then we can just drop the absolute value bars and solve the equation. Likewise, if *b* is negative then there will be no solution to the equation.

To this point we've only looked at equations that involve an absolute value being equal to a number, but there is no reason to think that there has to only be a number on the other side of the equal sign. Likewise, there is no reason to think that we can only have one absolute value in the problem. So, we need to take a look at a couple of these kinds of equations.

Example 3 Solve each of the following.

(a) $\left|x-2\right|=3x+1$

(b) $\left|4x+3\right|=3-x$

(c) $\left|2x-1\right|=\left|4x+9\right|$

Solution

At first glance the formula we used above will do us no good here. It requires the right side of the equation to be a positive number. It turns out that we can still use it here, but we're going to have to be careful with the answers as using this formula will, on occasion introduce an incorrect answer. So, while we can use the formula we'll need to make sure we check our solutions to see if they really work.

(a) $\left|x-2\right|=3x+1$

So, we'll start off using the formula above as we have in the previous problems and solving the two linear equations.

$$\begin{array}{lcl} x-2=-\left(3x+1\right)=-3x-1 & \text{or} & x-2=3x+1 \\ 4x=1 & \text{or} & -2x=3 \\ x=\dfrac{1}{4} & \text{or} & x=-\dfrac{3}{2} \end{array}$$

Okay, we've got two potential answers here. There is a problem with the second one however. If we plug this one into the equation we get,

$$\begin{aligned} \left|-\frac{3}{2}-2\right| &\stackrel{?}{=} 3\left(-\frac{3}{2}\right)+1 \\ \left|-\frac{7}{2}\right| &\stackrel{?}{=} -\frac{7}{2} \\ \frac{7}{2} &\neq -\frac{7}{2} \qquad \text{NOT OK} \end{aligned}$$

We get the same number on each side but with opposite signs. This will happen on occasion when we solve this kind of equation with absolute values. Note that we really didn't need to plug the solution into the whole equation here. All we needed to do was check the portion without the absolute value and if it was negative then the potential solution will NOT in fact be a solution and if it's positive or zero it will be solution.

We'll leave it to you to verify that the first potential solution does in fact work and so there is a single solution to this equation : $x=\frac{1}{4}$ and notice that this is less than 2 (as our assumption required) and so is a solution to the equation with the absolute value in it.

So, all together there is a single solution to this equation : $x=\frac{1}{4}$.

(b) $\left|4x+3\right| = 3-x$

This one will work in pretty much the same way so we won't put in quite as much explanation.

$$\begin{aligned} 4x+3 &= -\left(3-x\right) = -3+x & \text{or} \qquad 4x+3 &= 3-x \\ 3x &= -6 & \text{or} \qquad 5x &= 0 \\ x &= -2 & \text{or} \qquad x &= 0 \end{aligned}$$

Now, before we check each of these we should give a quick warning. Do not make the assumption that because the first potential solution is negative it won't be a solution. We only exclude a potential solution if it makes the portion without absolute value bars negative. In this case both potential solutions will make the portion without absolute value bars positive and so both are in fact solutions.

So in this case, unlike the first example, we get two solutions : $x=-2$ and $x=0$.

(c) $\left|2x-1\right| = \left|4x+9\right|$

This case looks very different from any of the previous problems we've worked to this point and in this case the formula we've been using doesn't really work at all. However, if we think about this a little we can see that we'll still do something similar here to get a solution.

Both sides of the equation contain absolute values and so the only way the two sides are equal will be if the two quantities inside the absolute value bars are equal or equal but with opposite signs. Or in other words, we must have,

$$\begin{aligned} 2x-1 &= -\left(4x+9\right) = -4x-9 & \text{or} \qquad 2x-1 &= 4x+9 \\ 6x &= -8 & \text{or} \qquad -2x &= 10 \\ x &= -\frac{8}{6} = -\frac{4}{3} & \text{or} \qquad x &= -5 \end{aligned}$$

Now, we won't need to verify our solutions here as we did in the previous two parts of this problem. Both with be solutions provided we solved the two equations correctly. However, it will probably be a good idea to verify them anyway just to show that the solution technique we used here really did work properly.

Let's first check $x = -\frac{4}{3}$.

$$\begin{aligned} \left|2\left(-\frac{4}{3}\right)-1\right| &\stackrel{?}{=} \left|4\left(-\frac{4}{3}\right)+9\right| \\ \left|-\frac{11}{3}\right| &\stackrel{?}{=} \left|\frac{11}{3}\right| \\ \frac{11}{3} &= \frac{11}{3} \qquad \text{OK} \end{aligned}$$

In the case the quantities inside the absolute value were the same number but opposite signs. However, upon taking the absolute value we got the same number and so $x = -\frac{4}{3}$ is a solution.

Now, let's check $x = -5$.

$$\begin{aligned} \left|2(-5)-1\right| &\stackrel{?}{=} \left|4(-5)+9\right| \\ \left|-11\right| &\stackrel{?}{=} \left|-11\right| \\ 11 &= 11 \qquad \text{OK} \end{aligned}$$

In the case we got the same value inside the absolute value bars.

So, as suggested above both answers did in fact work and both are solutions to the equation.

So, as we've seen in the previous set of examples we need to be a little careful if there are variables on both sides of the equal sign. If one side does not contain an absolute value then we need to look at the two potential answers and make sure that each is in fact a solution.

Section 2-15 : Absolute Value Inequalities

In the previous section we solved equations that contained absolute values. In this section we want to look at inequalities that contain absolute values. We will need to examine two separate cases.

Inequalities Involving < and $\le$

As we did with equations let's start off by looking at a fairly simple case.

$$\left| p \right| \le 4$$

This says that no matter what *p* is it must have a distance of no more than 4 from the origin. This means that *p* must be somewhere in the range,

$$-4 \le p \le 4$$

We could have a similar inequality with the < and get a similar result.

In general, we have the following formulas to use here,

If	$\left\| p \right\| \le b,\ \ b > 0$	then	$-b \le p \le b$
If	$\left\| p \right\| < b,\ \ b > 0$	then	$-b < p < b$

Notice that this does **require** *b* to be positive just as we did with equations.

Let's take a look at a couple of examples.

Example 1 Solve each of the following.

(a) $\left| 2x - 4 \right| < 10$

(b) $\left| 9m + 2 \right| \le 1$

(c) $\left| 3 - 2z \right| \le 5$

Solution

(a) $\left| 2x - 4 \right| < 10$

There really isn't much to do other than plug into the formula. As with equations *p* simply represents whatever is inside the absolute value bars. So, with this first one we have,

$$-10 < 2x - 4 < 10$$

Now, this is nothing more than a fairly simple double inequality to solve so let's do that.

$$-6 < 2x < 14$$

$$-3 < x < 7$$

The interval notation for this solution is $\left(-3, 7 \right)$.

(b) $\left|9m+2\right| \le 1$

Not much to do here.

$$\begin{aligned} -1 \le 9m+2 \le 1 \\ -3 \le 9m \le -1 \\ -\frac{1}{3} \le m \le -\frac{1}{9} \end{aligned}$$

The interval notation is $\left[-\frac{1}{3}, -\frac{1}{9}\right]$.

(c) $\left|3-2z\right| \le 5$

We'll need to be a little careful with solving the double inequality with this one, but other than that it is pretty much identical to the previous two parts.

$$\begin{aligned} -5 \le 3-2z \le 5 \\ -8 \le -2z \le 2 \\ 4 \ge z \ge -1 \end{aligned}$$

In the final step don't forget to switch the direction of the inequalities since we divided everything by a negative number. The interval notation for this solution is $\left[-1, 4\right]$.

Inequalities Involving > and $\ge$

Once again let's start off with a simple number example.

$$\left|p\right| \ge 4$$

This says that whatever *p* is it must be at least a distance of 4 from the origin and so *p* must be in one of the following two ranges,

$$p \le -4 \qquad \text{or} \qquad p \ge 4$$

Before giving the general solution we need to address a common mistake that students make with these types of problems. Many students try to combine these into a single double inequality as follows,

$$-4 \ge p \ge 4$$

While this may seem to make sense we can't stress enough that THIS IS NOT CORRECT!! Recall what a double inequality says. In a double inequality we require that both of the inequalities be satisfied simultaneously. The double inequality above would then mean that *p* is a number that is simultaneously smaller than -4 and larger than 4. This just doesn't make sense. There is no number that satisfies this.

These solutions must be written as two inequalities.

Here is the general formula for these.

If	$\lvert p\rvert \ge b,\ b>0$	then	$p \le -b$	or	$p \ge b$
If	$\lvert p\rvert > b,\ b>0$	then	$p < -b$	or	$p > b$

Again, we will **require** that *b* be a positive number here. Let's work a couple of examples.

Example 2 Solve each of the following.

(a) $\lvert 2x-3\rvert > 7$

(b) $\lvert 6t+10\rvert \ge 3$

(c) $\lvert 2-6y\rvert > 10$

Solution

(a) $\lvert 2x-3\rvert > 7$

Again, *p* represents the quantity inside the absolute value bars so all we need to do here is plug into the formula and then solve the two linear inequalities.

$$\begin{array}{ccc} 2x-3<-7 & \text{or} & 2x-3>7 \\ 2x<-4 & \text{or} & 2x>10 \\ x<-2 & \text{or} & x>5 \end{array}$$

The interval notation for these are $\left(-\infty,-2\right)$ or $\left(5,\infty\right)$.

(b) $\lvert 6t+10\rvert \ge 3$

Let's just plug into the formulas and go here,

$$\begin{array}{ccc} 6t+10\le -3 & \text{or} & 6t+10\ge 3 \\ 6t\le -13 & \text{or} & 6t\ge -7 \\ t\le -\dfrac{13}{6} & \text{or} & t\ge -\dfrac{7}{6} \end{array}$$

The interval notation for these are $\left(-\infty,-\dfrac{13}{6}\right]$ or $\left[-\dfrac{7}{6},\infty\right)$.

(c) $\lvert 2-6y\rvert > 10$

Again, not much to do here.

$$\begin{array}{ccc} 2-6y<-10 & \text{or} & 2-6y>10 \\ -6y<-12 & \text{or} & -6y>8 \\ y>2 & \text{or} & y<-\dfrac{4}{3} \end{array}$$

Notice that we had to switch the direction of the inequalities when we divided by the negative number! The interval notation for these solutions is $\left(2,\infty\right)$ or $\left(-\infty,-\dfrac{4}{3}\right)$.

Okay, we next need to take a quick look at what happens if *b* is zero or negative. We'll do these with a set of examples and let's start with zero.

Example 3 Solve each of the following.

(a) $\left|3x+2\right|<0$

(b) $\left|x-9\right|\le 0$

(c) $\left|2x-4\right|\ge 0$

(d) $\left|3x-9\right|>0$

Solution

These four examples seem to cover all our bases.

(a) Now we know that $\left|p\right|\ge 0$ and so can't ever be less than zero. Therefore, in this case there is no solution since it is impossible for an absolute value to be strictly less than zero (*i.e.* negative).

(b) This is almost the same as the previous part. We still can't have absolute value be less than zero, however it can be equal to zero. So, this will have a solution only if

$$\left|x-9\right|=0$$

and we know how to solve this from the previous section.

$$x-9=0 \qquad\Rightarrow\qquad x=9$$

(c) In this case let's again recall that no matter what *p* is we are guaranteed to have $\left|p\right|\ge 0$. This means that no matter what *x* is we can be assured that $\left|2x-4\right|\ge 0$ will be true since absolute values will always be positive or zero.

The solution in this case is all real numbers, or all possible values of *x*. In inequality notation this would be $-\infty<x<\infty$.

(d) This one is nearly identical to the previous part except this time note that we don't want the absolute value to ever be zero. So, we don't care what value the absolute value takes as long as it isn't zero. This means that we just need to avoid value(s) of *x* for which we get,

$$\left|3x-9\right|=0 \qquad\Rightarrow\qquad 3x-9=0 \qquad\Rightarrow\qquad x=3$$

The solution in this case is all real numbers except $x=3$.

Now, let's do a quick set of examples with negative numbers.

Example 4 Solve each of the following.

(a) $\left|4x+15\right|<-2$ and $\left|4x+15\right|\le -2$

(b) $\left|2x-9\right|\ge -8$ and $\left|2x-9\right|>-8$

Solution

Notice that we're working these in pairs, because this time, unlike the previous set of examples the solutions will be the same for each.

Both (all four?) of these will make use of the fact that no matter what *p* is we are guaranteed to have $|p| \geq 0$. In other words, absolute values are always positive or zero.

(a) Okay, if absolute values are always positive or zero there is no way they can be less than or equal to a negative number.

Therefore, there is no solution for either of these.

(b) In this case if the absolute value is positive or zero then it will always be greater than or equal to a negative number.

The solution for each of these is then all real numbers.

Chapter 3 : Graphing and Functions

In this chapter we will be introducing two topics that are very important in an algebra class. We will start off the chapter with a brief discussion of graphing. This is not really the main topic of this chapter, but we need the basics down before moving into the second topic of this chapter. The next chapter will contain the remainder of the graphing discussion.

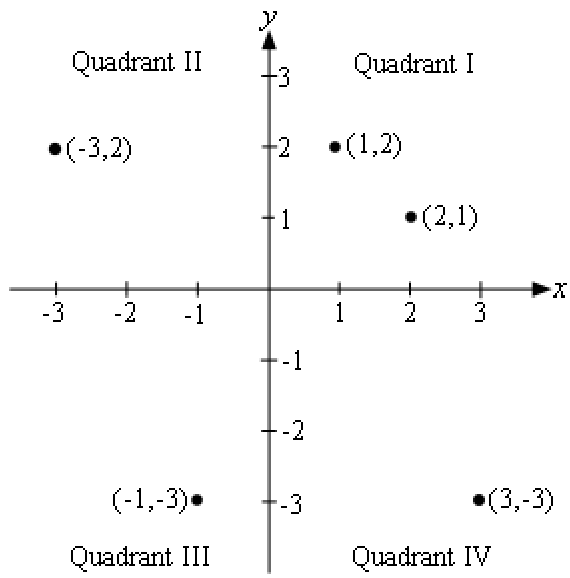
The second topic that we'll be looking at is that of functions. This is probably one of the more important ideas that will come out of an Algebra class. When first studying the concept of functions many students don't really understand the importance or usefulness of functions and function notation. The importance and/or usefulness of functions and function notation will only become apparent in later chapters and later classes. In fact, there are some topics that can only be done easily with function and function notation.

Here is a brief listing of the topics in this chapter.

Section 3-1 : Graphing

In this section we need to review some of the basic ideas in graphing. It is assumed that you've seen some graphing to this point and so we aren't going to go into great depth here. We will only be reviewing some of the basic ideas.

We will start off with the Rectangular or Cartesian coordinate system. This is just the standard axis system that we use when sketching our graphs. Here is the Cartesian coordinate system with a few points plotted.



The horizontal and vertical axes, typically called the ***x*-axis** and the ***y*-axis** respectively, divide the coordinate system up into quadrants as shown above. In each quadrant we have the following signs for *x* and *y*.

Quadrant I	$x > 0$, or *x* positive	$y > 0$, or *y* positive
Quadrant II	$x < 0$, or *x* negative	$y > 0$, or *y* positive
Quadrant III	$x < 0$, or *x* negative	$y < 0$, or *y* negative
Quadrant IV	$x > 0$, or *x* positive	$y < 0$, or *y* negative

Each point in the coordinate system is defined by an **ordered pair** of the form (x, y). The first number listed is the ***x*-coordinate** of the point and the second number listed is the ***y*-coordinate** of the point. The ordered pair for any given point, (x, y), is called the **coordinates** for the point.

The point where the two axes cross is called the **origin** and has the coordinates $(0,0)$.

Note as well that the order of the coordinates is important. For example, the point $\left(2,1\right)$ is the point that is two units to the right of the origin and then 1 unit up, while the point $\left(1,2\right)$ is the point that is 1 unit to the right of the origin and then 2 units up.

We now need to discuss graphing an equation. The first question that we should ask is what exactly is a graph of an equation? A graph is the set of all the ordered pairs whose coordinates satisfy the equation.

For instance, the point $\left(2,-3\right)$ is a point on the graph of $y=\left(x-1\right)^2-4$ while $\left(1,5\right)$ isn't on the graph. How do we tell this? All we need to do is take the coordinates of the point and plug them into the equation to see if they satisfy the equation. Let's do that for both to verify the claims made above.

$\left(2,-3\right)$:

In this case we have $x=2$ and $y=-3$ so plugging in gives,

$$\begin{aligned} -3 &\stackrel{?}{=} \left(2-1\right)^2-4 \\ -3 &\stackrel{?}{=} \left(1\right)^2-4 \\ -3 &= -3 \qquad\qquad \text{OK} \end{aligned}$$

So, the coordinates of this point satisfies the equation and so it is a point on the graph.

$\left(1,5\right)$:

Here we have $x=1$ and $y=5$. Plugging these in gives,

$$\begin{aligned} 5 &\stackrel{?}{=} \left(1-1\right)^2-4 \\ 5 &\stackrel{?}{=} \left(0\right)^2-4 \\ 5 &\neq -4 \qquad\qquad \text{NOT OK} \end{aligned}$$

The coordinates of this point do NOT satisfy the equation and so this point isn't on the graph.

Now, how do we sketch the graph of an equation? Of course, the answer to this depends on just how much you know about the equation to start off with. For instance, if you know that the equation is a line or a circle we've got simple ways to determine the graph in these cases. There are also many other kinds of equations that we can usually get the graph from the equation without a lot of work. We will see many of these in the next chapter.

However, let's suppose that we don't know ahead of time just what the equation is or any of the ways to quickly sketch the graph. In these cases we will need to recall that the graph is simply all the points that satisfy the equation. So, all we can do is plot points. We will pick values of *x*, compute *y* from the equation and then plot the ordered pair given by these two values.

How, do we determine which values of *x* to choose? Unfortunately, the answer there is we guess. We pick some values and see what we get for a graph. If it looks like we've got a pretty good sketch we stop. If not we pick some more. Knowing the values of *x* to choose is really something that we can only get with experience and some knowledge of what the graph of the equation will *probably* look like. Hopefully, by the end of this course you will have gained some of this knowledge.

Let's take a quick look at a graph.

Example 1 Sketch the graph of $y = (x-1)^2 - 4$.

Solution

Now, this is a parabola and after the next chapter you will be able to quickly graph this without much effort. However, we haven't gotten that far yet and so we will need to choose some values of *x*, plug them in and compute the *y* values.

As mentioned earlier, it helps to have an idea of what this graph is liable to look like when picking values of *x*. So, don't worry at this point why we chose the values that we did. After the next chapter you would also be able to choose these values of *x*.

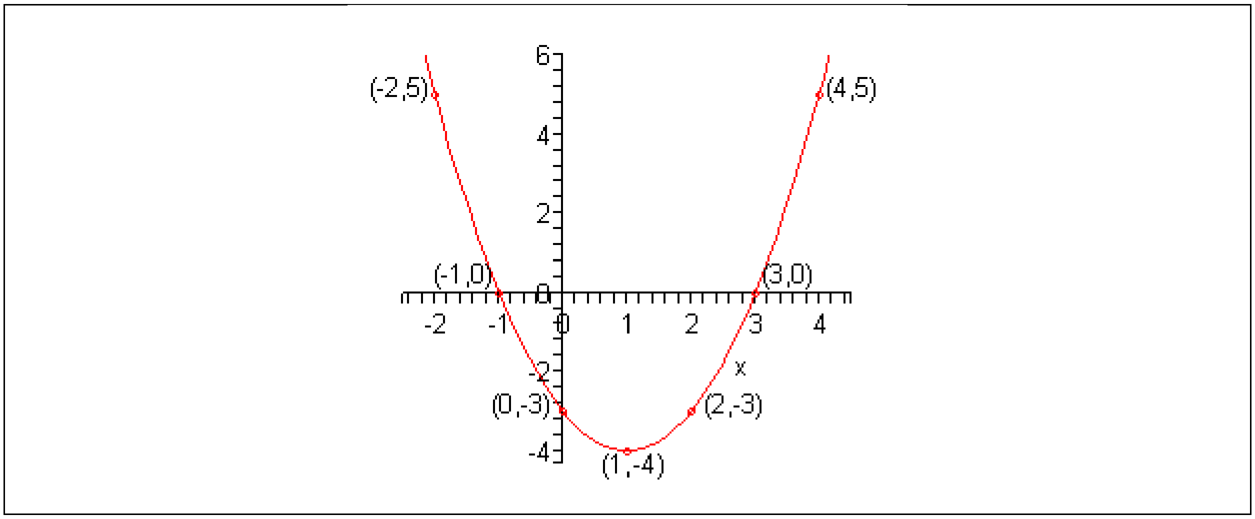
Here is a table of values for this equation.

x	*y*	(x,y)
-2	5	$(-2,5)$
-1	0	$(-1,0)$
0	-3	$(0,-3)$
1	-4	$(1,-4)$
2	-3	$(2,-3)$
3	0	$(3,0)$
4	5	$(4,5)$

Let's verify the first one and we'll leave the rest to you to verify. For the first one we simply plug $x = -2$ into the equation and compute *y*.

$$\begin{aligned} y &= (-2-1)^2 - 4 \\ &= (-3)^2 - 4 \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

Here is the graph of this equation.



Notice that when we set up the axis system in this example, we only set up as much as we needed. For example, since we didn't go past -2 with our computations we didn't go much past that with our axis system.

Also, notice that we used a different scale on each of the axes. With the horizontal axis we incremented by 1's while on the vertical axis we incremented by 2. This will often be done in order to make the sketching easier.

The final topic that we want to discuss in this section is that of **intercepts**. Notice that the graph in the above example crosses the *x*-axis in two places and the *y*-axis in one place. All three of these points are called intercepts. We can, and often will be, more specific however.

We often will want to know if an intercept crosses the *x* or *y*-axis specifically. So, if an intercept crosses the *x*-axis we will call it an ***x*-intercept**. Likewise, if an intercept crosses the *y*-axis we will call it a ***y*-intercept**.

Now, since the *x*-intercept crosses *x*-axis then the *y* coordinates of the *x*-intercept(s) will be zero. Also, the *x* coordinate of the *y*-intercept will be zero since these points cross the *y*-axis. These facts give us a way to determine the intercepts for an equation. To find the *x*-intercepts for an equation all that we need to do is set $y = 0$ and solve for *x*. Likewise, to find the *y*-intercepts for an equation we simply need to set $x = 0$ and solve for *y*.

Let's take a quick look at an example.

Example 2 Determine the *x*-intercepts and *y*-intercepts for each of the following equations.

(a) $y = x^2 + x - 6$

(b) $y = x^2 + 2$

(c) $y = (x+1)^2$

Solution

As verification for each of these we will also sketch the graph of each function. We will leave the details of the sketching to you to verify. Also, these are all parabolas and as mentioned earlier we will be looking at these in detail in the next chapter.

(a) $y = x^2 + x - 6$

Let's first find the *y*-intercept(s). Again, we do this by setting $x = 0$ and solving for *y*. This is usually the easier of the two. So, let's find the *y*-intercept(s).

$$y = (0)^2 + 0 - 6 = -6$$

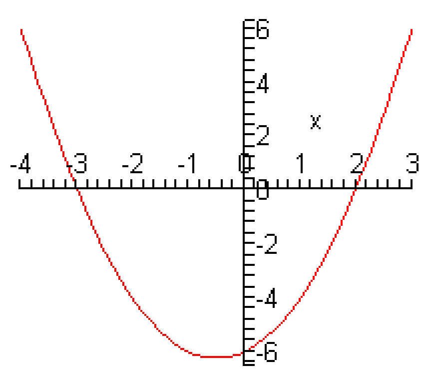
So, there is a single *y*-intercept : $(0,-6)$.

The work for the *x*-intercept(s) is almost identical except in this case we set $y = 0$ and solve for *x*. Here is that work.

$$\begin{aligned} 0 &= x^2 + x - 6 \\ 0 &= (x+3)(x-2) \qquad \Rightarrow \qquad x = -3, \ x = 2 \end{aligned}$$

For this equation there are two *x*-intercepts : $(-3,0)$ and $(2,0)$. Oh, and you do remember how to solve quadratic equations right?

For verification purposes here is sketch of the graph for this equation.



(b) $y = x^2 + 2$

First, the *y*-intercepts.

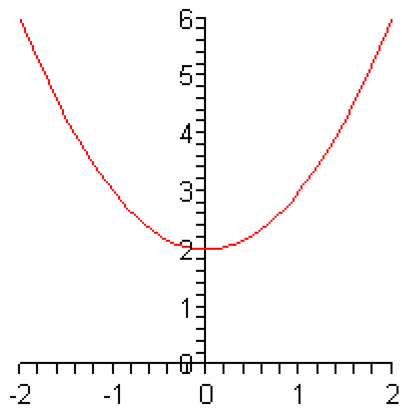
$$y = (0)^2 + 2 = 2 \qquad \Rightarrow \qquad (0,2)$$

So, we've got a single *y*-intercepts. Now, the *x*-intercept(s).

$$0 = x^2 + 2$$
$$-2 = x^2 \qquad \Rightarrow \qquad x = \pm\sqrt{2}\,i$$

Okay, we got complex solutions from this equation. What this means is that we will not have any *x*-intercepts. Note that it is perfectly acceptable for this to happen so don't worry about it when it does happen.

Here is the graph for this equation.



Sure enough, it doesn't cross the *x*-axis.

(c) $y = (x+1)^2$

Here is the *y*-intercept work for this equation.

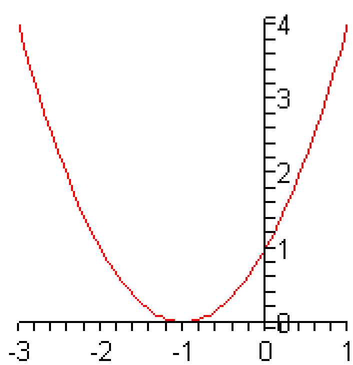
$$y = (0+1)^2 = 1 \qquad \Rightarrow \qquad (0,1)$$

Now the *x*-intercept work.

$$0 = (x+1)^2 \qquad \Rightarrow \qquad x = -1 \qquad \Rightarrow \qquad (-1,0)$$

In this case we have a single *x*-intercept.

Here is a sketch of the graph for this equation.



Now, notice that in this case the graph doesn't actually cross the *x*-axis at $x = -1$. This point is still called an *x*-intercept however.

We should make one final comment before leaving this section. In the previous set of examples all the equations were quadratic equations. This was done only because they exhibited the range of behaviors that we were looking for and we would be able to do the work as well. You should not walk away from this discussion of intercepts with the idea that they will only occur for quadratic equations. They can, and do, occur for many different equations.

Section 3-2 : Lines

Let's start this section off with a quick mathematical definition of a line. Any equation that can be written in the form,

$$Ax + By = C$$

where we can't have both *A* and *B* be zero simultaneously is a line. It is okay if one of them is zero, we just can't have both be zero. Note that this is sometimes called the **standard form** of the line.

Before we get too far into this section it would probably be helpful to recall that a line is defined by any two points that are on the line. Given two points that are on the line we can graph the line and/or write down the equation of the line. This fact will be used several times throughout this section.

One of the more important ideas that we'll be discussing in this section is that of **slope**. The slope of a line is a measure of the *steepness* of a line and it can also be used to measure whether a line is increasing or decreasing as we move from left to right. Here is the precise definition of the slope of a line.

Given any two points on the line say, $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$, the slope of the line is given by,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

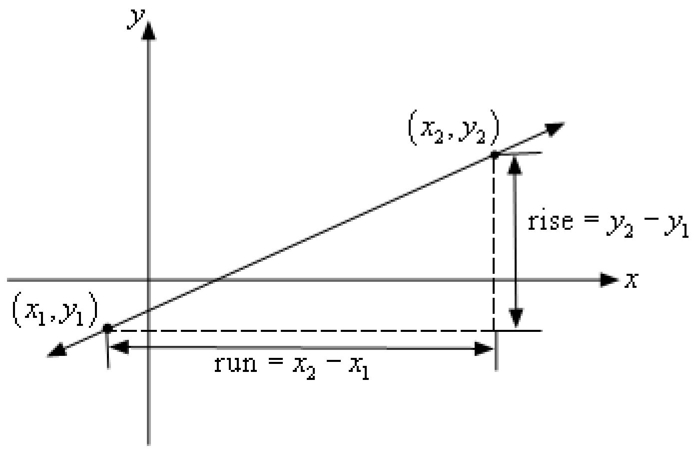
In other words, the slope is the difference in the *y* values divided by the difference in the *x* values. Also, do not get worried about the subscripts on the variables. These are used fairly regularly from this point on and are simply used to denote the fact that the variables are both *x* or *y* values but are, in all likelihood, different.

When using this definition do not worry about which point should be the first point and which point should be the second point. You can choose either to be the first and/or second and we'll get exactly the same value for the slope.

There is also a geometric "definition" of the slope of the line as well. You will often hear the slope as being defined as follows,

$$m = \frac{\text{rise}}{\text{run}}$$

The two definitions are identical as the following diagram illustrates. The numerators and denominators of both definitions are the same.



Note as well that if we have the slope (written as a fraction) and a point on the line, say (x_1, y_1), then we can easily find a second point that is also on the line. Before seeing how this can be done let's take the convention that if the slope is negative we will put the minus sign on the numerator of the slope. In other words, we will assume that the *rise* is negative if the slope is negative. Note as well that a negative *rise* is really a *fall*.

So, we have the slope, written as a fraction, and a point on the line, (x_1, y_1). To get the coordinates of the second point, (x_2, y_2) all that we need to do is start at (x_1, y_1) then move to the right by the *run* (or denominator of the slope) and then up/down by *rise* (or the numerator of the slope) depending on the sign of the *rise*. We can also write down some equations for the coordinates of the second point as follows,

$$x_2 = x_1 + \text{run}$$
$$y_2 = y_1 + \text{rise}$$

Note that if the slope is negative then the *rise* will be a negative number.

Let's compute a couple of slopes.

Example 1 Determine the slope of each of the following lines. Sketch the graph of each line.

(a) The line that contains the two points $(-2,-3)$ and $(3,1)$.

(b) The line that contains the two points $(-1,5)$ and $(0,-2)$.

(c) The line that contains the two points $(-3,2)$ and $(5,2)$.

(d) The line that contains the two points $(4,3)$ and $(4,-2)$.

Solution

Okay, for each of these all that we'll need to do is use the slope formula to find the slope and then plot the two points and connect them with a line to get the graph.

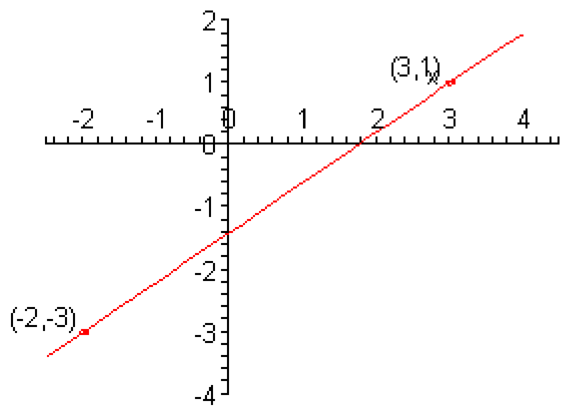
(a) The line that contains the two points $\left(-2,-3\right)$ and $\left(3,1\right)$.

Do not worry which point gets the subscript of 1 and which gets the subscript of 2. Either way will get the same answer. Typically, we'll just take them in the order listed. So, here is the slope for this part.

$$m = \frac{1-\left(-3\right)}{3-\left(-2\right)} = \frac{1+3}{3+2} = \frac{4}{5}$$

Be careful with minus signs in these computations. It is easy to lose track of them. Also, when the slope is a fraction, as it is here, leave it as a fraction. Do not convert to a decimal unless you absolutely have to.

Here is a sketch of the line.



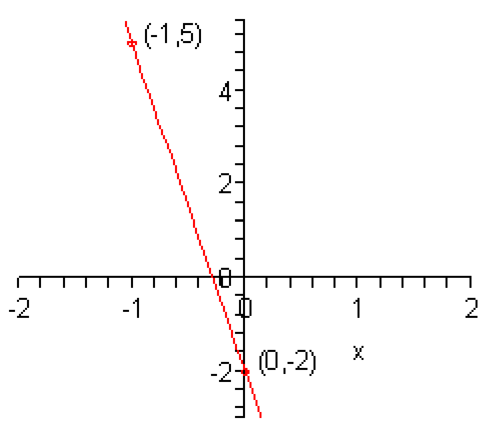
Notice that this line increases as we move from left to right.

(b) The line that contains the two points $\left(-1,5\right)$ and $\left(0,-2\right)$.

Here is the slope for this part.

$$m = \frac{-2-5}{0-\left(-1\right)} = \frac{-7}{1} = -7$$

Again, watch out for minus signs. Here is a sketch of the graph.



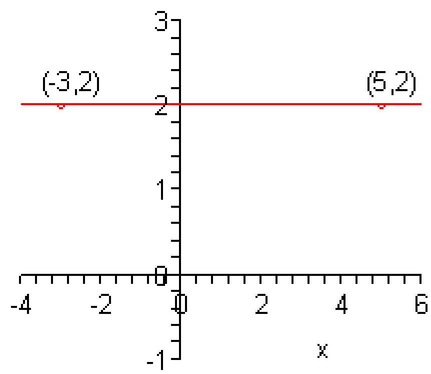
This line decreases as we move from left to right.

(c) The line that contains the two points $(-3,2)$ and $(5,2)$.

Here is the slope for this line.

$$m = \frac{2-2}{5-(-3)} = \frac{0}{8} = 0$$

We got a slope of zero here. That is okay, it will happen sometimes. Here is the sketch of the line.



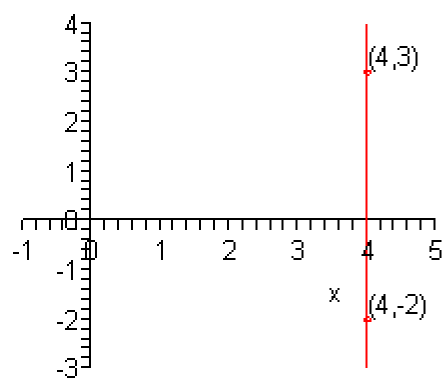
In this case we've got a horizontal line.

(d) The line that contains the two points $(4,3)$ and $(4,-2)$.

The final part. Here is the slope computation.

$$m = \frac{-2-3}{4-4} = \frac{-5}{0} = \text{undefined}$$

In this case we get division by zero which is undefined. Again, don't worry too much about this it will happen on occasion. Here is a sketch of this line.



This final line is a vertical line.

We can use this set of examples to see some general facts about lines.

First, we can see from the first two parts that the larger the number (ignoring any minus signs) the steeper the line. So, we can use the slope to tell us something about just how steep a line is.

Next, we can see that if the slope is a positive number then the line will be increasing as we move from left to right. Likewise, if the slope is a negative number then the line will decrease as we move from left to right.

We can use the final two parts to see what the slopes of horizontal and vertical lines will be. A horizontal line will always have a slope of zero and a vertical line will always have an undefined slope.

We now need to take a look at some special forms of the equation of the line.

We will start off with horizontal and vertical lines. A horizontal line with a *y*-intercept of *b* will have the equation,

$$\boxed{y = b}$$

Likewise, a vertical line with an *x*-intercept of *a* will have the equation,

$$\boxed{x = a}$$

So, if we go back and look that the last two parts of the previous example we can see that the equation of the line for the horizontal line in the third part is

$$y = 2$$

while the equation for the vertical line in the fourth part is

$$x = 4$$

The next special form of the line that we need to look at is the **point-slope form** of the line. This form is very useful for writing down the equation of a line. If we know that a line passes through the point $\left(x_1, y_1\right)$ and has a slope of *m* then the point-slope form of the equation of the line is,

$$\boxed{y - y_1 = m\left(x - x_1\right)}$$

Sometimes this is written as,

$$\boxed{y = y_1 + m\left(x - x_1\right)}$$

The form it's written in usually depends on the instructor that is teaching the class.

As stated earlier this form is particularly useful for writing down the equation of a line so let's take a look at an example of this.

Example 2 Write down the equation of the line that passes through the two points $\left(-2,4\right)$ and $\left(3,-5\right)$.

Solution

At first glance it may not appear that we'll be able to use the point-slope form of the line since this requires a single point (we've got two) and the slope (which we don't have). However, that fact that we've got two points isn't really a problem; in fact, we can use these two points to determine the missing slope of the line since we do know that we can always find that from any two points on the line.

So, let's start off my finding the slope of the line.

$$m = \frac{-5-4}{3-(-2)} = -\frac{9}{5}$$

Now, which point should we use to write down the equation of the line? We can actually use either point. To show this we will use both.

First, we'll use $(-2,4)$. Now that we've gotten the point all that we need to do is plug into the formula. We will use the second form.

$$y = 4 - \frac{9}{5}\left(x - (-2)\right) = 4 - \frac{9}{5}(x+2)$$

Now, let's use $(3,-5)$.

$$y = -5 - \frac{9}{5}(x-3)$$

Okay, we claimed that it wouldn't matter which point we used in the formula, but these sure look like different equations. It turns out however, that these really are the same equation. To see this all that we need to do is distribute the slope through the parenthesis and then simplify.

Here is the first equation.

$$\begin{aligned} y &= 4 - \frac{9}{5}(x+2) \\ &= 4 - \frac{9}{5}x - \frac{18}{5} \\ &= -\frac{9}{5}x + \frac{2}{5} \end{aligned}$$

Here is the second equation.

$$\begin{aligned} y &= -5 - \frac{9}{5}(x-3) \\ &= -5 - \frac{9}{5}x + \frac{27}{5} \\ &= -\frac{9}{5}x + \frac{2}{5} \end{aligned}$$

So, sure enough they are the same equation.

The final special form of the equation of the line is probably the one that most people are familiar with. It is the **slope-intercept form**. In this case if we know that a line has slope *m* and has a *y*-intercept of $(0,b)$ then the slope-intercept form of the equation of the line is,

$$y = mx + b$$

This form is particularly useful for graphing lines. Let's take a look at a couple of examples.

Example 3 Determine the slope of each of the following equations and sketch the graph of the line.

(a) $2y - 6x = -2$

(b) $3y + 4x = 6$

Solution

Okay, to get the slope we'll first put each of these in slope-intercept form and then the slope will simply be the coefficient of the *x* (including sign). To graph the line we know the *y*-intercept of the line, that's the number without an *x* (including sign) and as discussed above we can use the slope to find a second point on the line. At that point there isn't anything to do other than sketch the line.

(a) $2y - 6x = -2$

First solve the equation for *y*. Remember that we solved equations like this back in the previous chapter.

$$\begin{aligned} 2y &= 6x - 2 \\ y &= 3x - 1 \end{aligned}$$

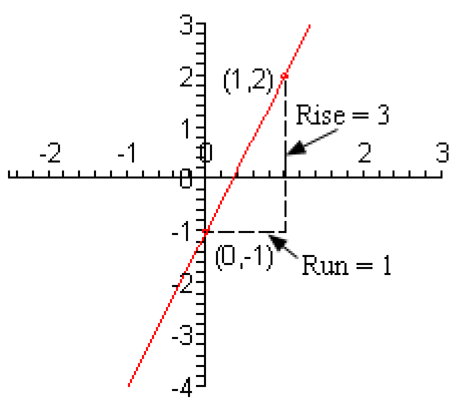
So, the slope for this line is 3 and the *y*-intercept is the point $(0, -1)$. Don't forget to take the sign when determining the *y*-intercept. Now, to find the second point we usually like the slope written as a fraction to make it clear what the *rise* and *run* are. So,

$$m = 3 = \frac{3}{1} = \frac{\text{rise}}{\text{run}} \qquad \Rightarrow \qquad \text{rise} = 3, \quad \text{run} = 1$$

The second point is then,

$$x_2 = 0 + 1 = 1 \qquad\qquad y_2 = -1 + 3 = 2 \qquad\qquad \Rightarrow \qquad\qquad (1, 2)$$

Here is a sketch of the graph of the line.



(b) $3y + 4x = 6$

Again, solve for *y*.

$$3y = -4x + 6$$
$$y = -\frac{4}{3}x + 2$$

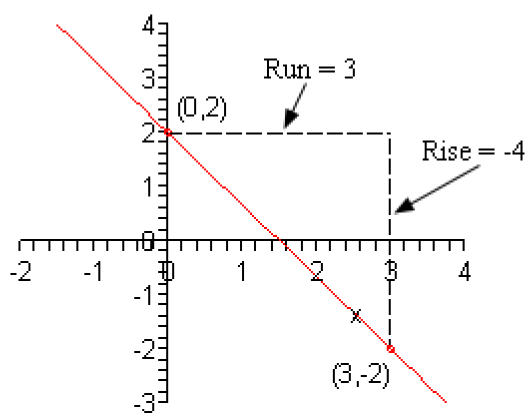
In this case the slope is $-\frac{4}{3}$ and the *y*-intercept is $(0,2)$. As with the previous part let's first determine the *rise* and the *run*.

$$m = -\frac{4}{3} = \frac{-4}{3} = \frac{\text{rise}}{\text{run}} \qquad \Rightarrow \qquad \text{rise} = -4, \ \ \text{run} = 3$$

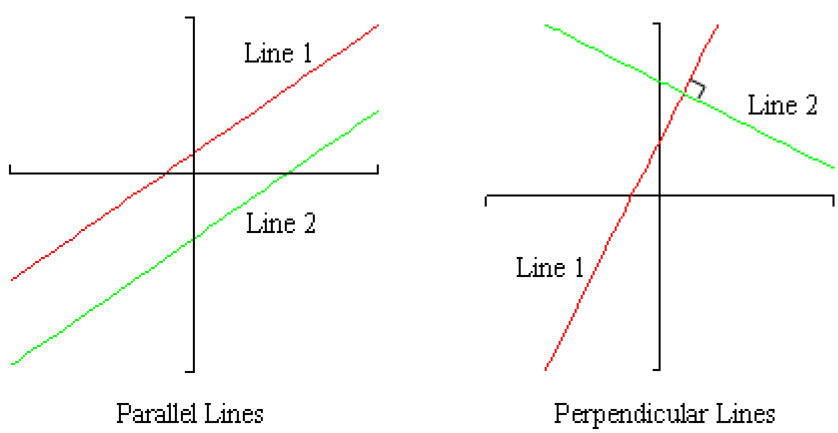
Again, remember that if the slope is negative make sure that the minus sign goes with the numerator. The second point is then,

$$x_2 = 0 + 3 = 3 \qquad y_2 = 2 + (-4) = -2 \qquad \Rightarrow \qquad (3,-2)$$

Here is the sketch of the graph for this line.



The final topic that we need to discuss in this section is that of parallel and perpendicular lines. Here is a sketch of parallel and perpendicular lines.



Suppose that the slope of Line 1 is m_1 and the slope of Line 2 is m_2. We can relate the slopes of parallel lines and we can relate slopes of perpendicular lines as follows.

$$\text{parallel}: m_1 = m_2$$
$$\text{perpendicular}: m_1 m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}$$

Note that there are two forms of the equation for perpendicular lines. The second is the more common and in this case we usually say that m_2 is the negative reciprocal of m_1.

Example 4 Determine if the line that passes through the points $(-2,-10)$ and $(6,-1)$ is parallel, perpendicular or neither to the line given by $7y - 9x = 15$.

Solution
Okay, in order to do answer this we'll need the slopes of the two lines. Since we have two points for the first line we can use the formula for the slope,

$$m_1 = \frac{-1-(-10)}{6-(-2)} = \frac{9}{8}$$

We don't actually need the equation of this line and so we won't bother with it.

Now, to get the slope of the second line all we need to do is put it into slope-intercept form.

$$\begin{aligned} 7y &= 9x + 15 \\ y &= \frac{9}{7}x + \frac{15}{7} \qquad \Rightarrow \qquad m_2 = \frac{9}{7} \end{aligned}$$

Okay, since the two slopes aren't the same (they're close, but still not the same) the two lines are not parallel. Also,

$$\left(\frac{9}{8}\right)\left(\frac{9}{7}\right) = \frac{81}{56} \neq -1$$

so the two lines aren't perpendicular either.

Therefore, the two lines are neither parallel nor perpendicular.

Example 5 Determine the equation of the line that passes through the point $(8,2)$ and is,

(a) parallel to the line given by $10y + 3x = -2$

(b) perpendicular to the line given by $10y + 3x = -2$.

Solution
In both parts we are going to need the slope of the line given by $10y + 3x = -2$ so let's actually find that before we get into the individual parts.

$$\begin{aligned} 10y &= -3x - 2 \\ y &= -\frac{3}{10}x - \frac{1}{5} \qquad \Rightarrow \qquad m_1 = -\frac{3}{10} \end{aligned}$$

Now, let's work the example.

(a) parallel to the line given by $10y+3x=-2$

In this case the new line is to be parallel to the line given by $10y+3x=-2$ and so it must have the same slope as this line. Therefore, we know that,

$$m_2=-\frac{3}{10}$$

Now, we've got a point on the new line, $(8,2)$, and we know the slope of the new line, $-\frac{3}{10}$, so we can use the point-slope form of the line to write down the equation of the new line. Here is the equation,

$$\begin{aligned} y &= 2-\frac{3}{10}(x-8) \\ &= 2-\frac{3}{10}x+\frac{24}{10} \\ &= -\frac{3}{10}x+\frac{44}{10} \\ y &= -\frac{3}{10}x+\frac{22}{5} \end{aligned}$$

(b) perpendicular to the line given by $10y+3x=-2$

For this part we want the line to be perpendicular to $10y+3x=-2$ and so we know we can find the new slope as follows,

$$m_2=-\frac{1}{-\frac{3}{10}}=\frac{10}{3}$$

Then, just as we did in the previous part we can use the point-slope form of the line to get the equation of the new line. Here it is,

$$\begin{aligned} y &= 2+\frac{10}{3}(x-8) \\ &= 2+\frac{10}{3}x-\frac{80}{3} \\ y &= \frac{10}{3}x-\frac{74}{3} \end{aligned}$$

Section 3-3 : Circles

In this section we are going to take a quick look at circles. However, before we do that we need to give a quick formula that hopefully you'll recall seeing at some point in the past.

Given two points $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ the distance between them is given by,

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

So, why did we remind you of this formula? Well, let's recall just what a circle is. A circle is all the points that are the same distance, *r* – called the radius, from a point, $\left(h, k\right)$ - called the center. In other words, if $\left(x, y\right)$ is any point that is on the circle then it has a distance of *r* from the center, $\left(h, k\right)$.

If we use the distance formula on these two points we would get,

$$r = \sqrt{\left(x - h\right)^2 + \left(y - k\right)^2}$$

Or, if we square both sides we get,

$$\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$$

This is the **standard form** of the equation of a circle with radius *r* and center $\left(h, k\right)$.

Example 1 Write down the equation of a circle with radius 8 and center $\left(-4, 7\right)$.

Solution

Okay, in this case we have $r = 8$, $h = -4$ and $k = 7$ so all we need to do is plug them into the standard form of the equation of the circle.

$$\begin{aligned} \left(x - \left(-4\right)\right)^2 + \left(y - 7\right)^2 &= 8^2 \\ \left(x + 4\right)^2 + \left(y - 7\right)^2 &= 64 \end{aligned}$$

Do not square out the two terms on the left. Leaving these terms as they are will allow us to quickly identify the equation as that of a circle and to quickly identify the radius and center of the circle.

Graphing circles is a fairly simple process once we know the radius and center. In order to graph a circle all we really need is the right most, left most, top most and bottom most points on the circle. Once we know these it's easy to sketch in the circle.

Nicely enough for us these points are easy to find. Since these are points on the circle we know that they must be a distance of *r* from the center. Therefore, the points will have the following coordinates.

$$\text{right most point : } (h+r, k)$$
$$\text{left most point : } (h-r, k)$$
$$\text{top most point : } (h, k+r)$$
$$\text{bottom most point : } (h, k-r)$$

In other words all we need to do is add r on to the x coordinate or y coordinate of the point to get the right most or top most point respectively and subtract r from the x coordinate or y coordinate to get the left most or bottom most points.

Let's graph some circles.

Example 2 Determine the center and radius of each of the following circles and sketch the graph of the circle.

(a) $x^2 + y^2 = 1$

(b) $x^2 + (y-3)^2 = 4$

(c) $(x-1)^2 + (y+4)^2 = 16$

Solution

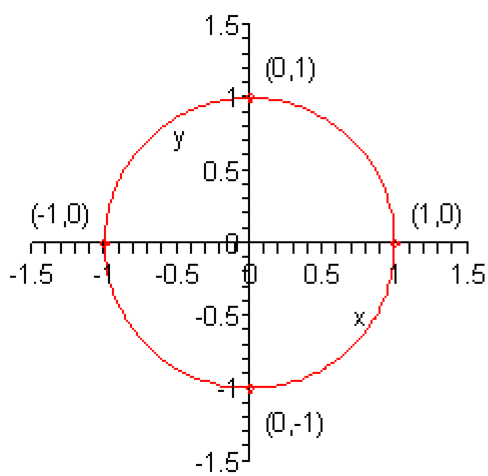
In all of these all that we really need to do is compare the equation to the standard form and identify the radius and center. Once that is done find the four points talked about above and sketch in the circle.

(a) $x^2 + y^2 = 1$

In this case it's just x and y squared by themselves. The only way that we could have this is to have both h and k be zero. So, the center and radius is,

$$\text{center} = (0,0) \qquad\qquad \text{radius} = \sqrt{1} = 1$$

Don't forget that the radius is the square root of the number on the other side of the equal sign. Here is a sketch of this circle.



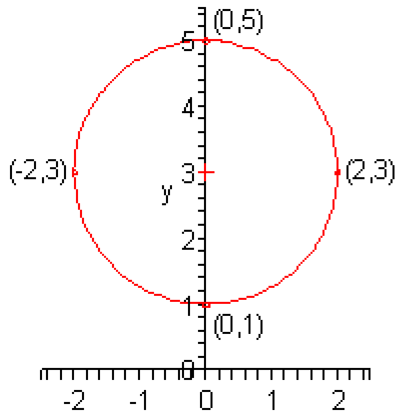
A circle centered at the origin with radius 1 (*i.e.* this circle) is called the **unit circle**. The unit circle is very useful in a Trigonometry class.

(b) $x^2+(y-3)^2=4$

In this part, it looks like the *x* coordinate of the center is zero as with the previous part. However, this time there is something more with the *y* term and so comparing this term to the standard form of the circle we can see that the *y* coordinate of the center must be 3. The center and radius of this circle is then,

$$\text{center}=(0,3) \qquad \text{radius}=\sqrt{4}=2$$

Here is a sketch of the circle. The center is marked with a red cross in this graph.



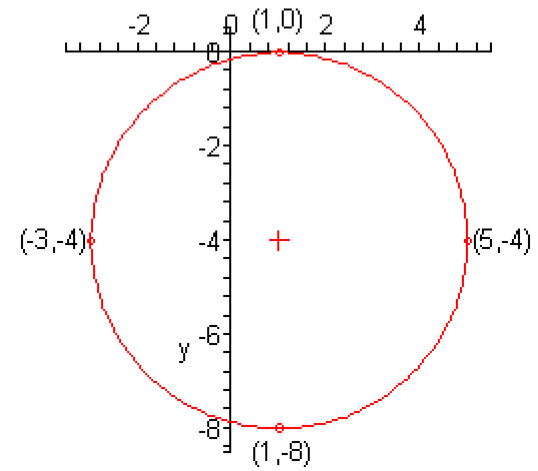
(c) $(x-1)^2+(y+4)^2=16$

For this part neither of the coordinates of the center are zero. By comparing our equation with the standard form it's fairly easy to see (hopefully…) that the *x* coordinate of the center is 1. The *y* coordinate isn't too bad either, but we do need to be a little careful. In this case the term is $(y+4)^2$ and in the standard form the term is $(y-k)^2$. Note that the signs are different. The only way that this can happen is if *k* is negative. So, the *y* coordinate of the center must be -4.

The center and radius for this circle are,

$$\text{center}=(1,-4) \qquad \text{radius}=\sqrt{16}=4$$

Here is a sketch of this circle with the center marked with a red cross.



So, we've seen how to deal with circles that are already in the standard form. However, not all circles will start out in the standard form. So, let's take a look at how to put a circle in the standard form.

Example 3 Determine the center and radius of each of the following.

(a) $x^2 + y^2 + 8x + 7 = 0$

(b) $x^2 + y^2 - 3x + 10y - 1 = 0$

Solution

Neither of these equations are in standard form and so to determine the center and radius we'll need to put it into standard form. We actually already know how to do this. Back when we were solving quadratic equations we saw a way to turn a quadratic polynomial into a perfect square. The process was called completing the square.

This is exactly what we want to do here, although in this case we aren't solving anything and we're going to have to deal with the fact that we've got both *x* and *y* in the equation. Let's step through the process with the first part.

(a) $x^2 + y^2 + 8x + 7 = 0$

We'll go through the process in a step by step fashion with this one.

Step 1 : First get the constant on one side by itself and at the same time group the *x* terms together and the *y* terms together.

$$x^2 + 8x + y^2 = -7$$

In this case there was only one term with a *y* in it and two with *x*'s in them.

Step 2 : For each variable with two terms complete the square on those terms.

So, in this case that means that we only need to complete the square on the *x* terms. Recall how this is done. We first take half the coefficient of the *x* and square it.

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

We then add this to both sides of the equation.

$$x^2+8x+16+y^2=-7+16=9$$

Now, the first three terms will factor as a perfect square.

$$\left(x+4\right)^2+y^2=9$$

<u>Step 3 :</u> This is now the standard form of the equation of a circle and so we can pick the center and radius right off this. They are,

$$\text{center}=\left(-4,0\right) \qquad\qquad \text{radius}=\sqrt{9}=3$$

(b) $x^2+y^2-3x+10y-1=0$

In this part we'll go through the process a little quicker. First get terms properly grouped and placed.

$$\underbrace{x^2-3x}_{\text{complete the square}}+\underbrace{y^2+10y}_{\text{complete the square}}=1$$

Now, as noted above we'll need to complete the square twice here, once for the *x* terms and once for the *y* terms. Let's first get the numbers that we'll need to add to both sides.

$$\left(-\frac{3}{2}\right)^2=\frac{9}{4} \qquad\qquad \left(\frac{10}{2}\right)^2=\left(5\right)^2=25$$

Now, add these to both sides of the equation.

$$\underbrace{x^2-3x+\frac{9}{4}}_{\text{factor this}}+\underbrace{y^2+10y+25}_{\text{factor this}}=1+\frac{9}{4}+25=\frac{113}{4}$$

When adding the numbers to both sides make sure and place them properly. This means that we need to put the number from the coefficient of the *x* with the *x* terms and the number from the coefficient of the *y* with the *y* terms. This placement is important since this will be the only way that the quadratics will factor as we need them to factor.

Now, factor the quadratics as show above. This will give the standard form of the equation of the circle.

$$\left(x-\frac{3}{2}\right)^2+\left(y+5\right)^2=\frac{113}{4}$$

This looks a little messier than the equations that we've seen to this point. However, this is something that will happen on occasion so don't get excited about it. Here is the center and radius for this circle.

$$\text{center}=\left(\frac{3}{2},-5\right) \qquad\qquad \text{radius}=\sqrt{\frac{113}{4}}=\frac{\sqrt{113}}{2}$$

Do not get excited about the messy radius or fractions in the center coordinates.

Section 3-4 : The Definition of a Function

We now need to move into the second topic of this chapter. Before we do that however we need a quick definition taken care of.

Definition of Relation

A **relation** is a set of ordered pairs.

This seems like an odd definition but we'll need it for the definition of a function (which is the main topic of this section). However, before we actually give the definition of a function let's see if we can get a handle on just what a relation is.

Think back to Example 1 in the Graphing section of this chapter. In that example we constructed a set of ordered pairs we used to sketch the graph of $y = (x-1)^2 - 4$. Here are the ordered pairs that we used.

$$(-2,5) \;\; (-1,0) \;\; (0,-3) \;\; (1,-4) \;\; (2,-3) \;\; (3,0) \;\; (4,5)$$

Any of the following are then relations because they consist of a set of ordered pairs.

$$\begin{aligned}
&\{(-2,5) \;\; (-1,0) \;\; (2,-3)\} \\
&\{(-1,0) \;\; (0,-3) \;\; (2,-3) \;\; (3,0) \;\; (4,5)\} \\
&\{(3,0) \;\; (4,5)\} \\
&\{(-2,5) \;\; (-1,0) \;\; (0,-3) \;\; (1,-4) \;\; (2,-3) \;\; (3,0) \;\; (4,5)\}
\end{aligned}$$

There are of course many more relations that we could form from the list of ordered pairs above, but we just wanted to list a few possible relations to give some examples. Note as well that we could also get other ordered pairs from the equation and add those into any of the relations above if we wanted to.

Now, at this point you are probably asking just why we care about relations and that is a good question. Some relations are very special and are used at almost all levels of mathematics. The following definition tells us just which relations are these special relations.

Definition of a Function

A **function** is a relation for which each value from the set the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.

Okay, that is a mouth full. Let's see if we can figure out just what it means. Let's take a look at the following example that will hopefully help us figure all this out.

Example 1 The following relation is a function.

$$\{(-1,0)\ \ (0,-3)\ \ (2,-3)\ \ (3,0)\ \ (4,5)\}$$

Solution

From these ordered pairs we have the following sets of first components (*i.e.* the first number from each ordered pair) and second components (*i.e.* the second number from each ordered pair).

$$1^{\text{st}} \text{ components : } \{-1,0,2,3,4\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{0,-3,0,5\}$$

For the set of second components notice that the "-3" occurred in two ordered pairs but we only listed it once.

To see why this relation is a function simply pick any value from the set of first components. Now, go back up to the relation and find every ordered pair in which this number is the first component and list all the second components from those ordered pairs. The list of second components will consist of exactly one value.

For example, let's choose 2 from the set of first components. From the relation we see that there is exactly one ordered pair with 2 as a first component, $(2,-3)$. Therefore, the list of second components (*i.e.* the list of values from the set of second components) associated with 2 is exactly one number, -3.

Note that we don't care that -3 is the second component of a second ordered par in the relation. That is perfectly acceptable. We just don't want there to be any more than one ordered pair with 2 as a first component.

We looked at a single value from the set of first components for our quick example here but the result will be the same for all the other choices. Regardless of the choice of first components there will be exactly one second component associated with it.

Therefore, this relation is a function.

In order to really get a feel for what the definition of a function is telling us we should probably also check out an example of a relation that is not a function.

Example 2 The following relation is not a function.

$$\{(6,10)\ \ (-7,3)\ \ (0,4)\ \ (6,-4)\}$$

Solution

Don't worry about where this relation came from. It is just one that we made up for this example.

Here is the list of first and second components

$$1^{\text{st}} \text{ components : } \{6,-7,0\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{10,3,4,-4\}$$

From the set of first components let's choose 6. Now, if we go up to the relation we see that there are two ordered pairs with 6 as a first component : $\left(6,10\right)$ and $\left(6,-4\right)$. The list of second components associated with 6 is then : 10, -4.

The list of second components associated with 6 has two values and so this relation is not a function.

Note that the fact that if we'd chosen -7 or 0 from the set of first components there is only one number in the list of second components associated with each. This doesn't matter. The fact that we found even a single value in the set of first components with more than one second component associated with it is enough to say that this relation is not a function.

As a final comment about this example let's note that if we removed the first and/or the fourth ordered pair from the relation we would have a function!

So, hopefully you have at least a feeling for what the definition of a function is telling us.

Now that we've forced you to go through the actual definition of a function let's give another "working" definition of a function that will be much more useful to what we are doing here.

The actual definition works on a relation. However, as we saw with the four relations we gave prior to the definition of a function and the relation we used in Example 1 we often get the relations from some equation.

It is important to note that not all relations come from equations! The relation from the second example for instance was just a set of ordered pairs we wrote down for the example and didn't come from any equation. This can also be true with relations that are functions. They do not have to come from equations.

However, having said that, the functions that we are going to be using in this course do all come from equations. Therefore, let's write down a definition of a function that acknowledges this fact.

Before we give the "working" definition of a function we need to point out that this is NOT the actual definition of a function, that is given above. This is simply a good "working definition" of a function that ties things to the kinds of functions that we will be working with in this course.

"Working Definition" of Function

A **function** is an equation for which any *x* that can be plugged into the equation will yield exactly one *y* out of the equation.

There it is. That is the definition of functions that we're going to use and will probably be easier to decipher just what it means.

Before we examine this a little more note that we used the phrase "*x* that can be plugged into" in the definition. This tends to imply that not all *x*'s can be plugged into an equation and this is in fact correct. We will come back and discuss this in more detail towards the end of this section, however at this point just remember that we can't divide by zero and if we want real numbers out of the equation we can't take the square root of a negative number. So, with these two examples it is clear that we will not always be able to plug in every *x* into any equation.

Further, when dealing with functions we are always going to assume that both *x* and *y* will be real numbers. In other words, we are going to forget that we know anything about complex numbers for a little bit while we deal with this section.

Okay, with that out of the way let's get back to the definition of a function and let's look at some examples of equations that are functions and equations that aren't functions.

Example 3 Determine which of the following equations are functions and which are not functions.

(a) $y = 5x + 1$

(b) $y = x^2 + 1$

(c) $y^2 = x + 1$

(d) $x^2 + y^2 = 4$

Solution

The "working" definition of function is saying is that if we take all possible values of *x* and plug them into the equation and solve for *y* we will get exactly one value for each value of *x*. At this stage of the game it can be pretty difficult to actually show that an equation is a function so we'll mostly talk our way through it. On the other hand, it's often quite easy to show that an equation isn't a function.

(a) $y = 5x + 1$

So, we need to show that no matter what *x* we plug into the equation and solve for *y* we will only get a single value of *y*. Note as well that the value of *y* will probably be different for each value of *x*, although it doesn't have to be.

Let's start this off by plugging in some values of *x* and see what happens.

$$\begin{aligned} x = -4: &\qquad y = 5\left(-4\right) + 1 = -20 + 1 = -19 \\ x = 0: &\qquad y = 5\left(0\right) + 1 = 0 + 1 = 1 \\ x = 10: &\qquad y = 5\left(10\right) + 1 = 50 + 1 = 51 \end{aligned}$$

So, for each of these values of *x* we got a single value of *y* out of the equation. Now, this isn't sufficient to claim that this is a function. In order to officially prove that this is a function we need to show that this will work no matter which value of *x* we plug into the equation.

Of course, we can't plug all possible value of *x* into the equation. That just isn't physically possible. However, let's go back and look at the ones that we did plug in. For each *x*, upon plugging in, we first multiplied the *x* by 5 and then added 1 onto it. Now, if we multiply a number by 5 we will get a single value from the multiplication. Likewise, we will only get a single value if we add 1 onto a number. Therefore, it seems plausible that based on the operations involved with plugging *x* into the equation that we will only get a single value of *y* out of the equation.

So, this equation is a function.

(b) $y = x^2 + 1$

Again, let's plug in a couple of values of *x* and solve for *y* to see what happens.

$$x = -1: \qquad y = (-1)^2 + 1 = 1 + 1 = 2$$
$$x = 3: \qquad y = (3)^2 + 1 = 9 + 1 = 10$$

Now, let's think a little bit about what we were doing with the evaluations. First, we squared the value of *x* that we plugged in. When we square a number there will only be one possible value. We then add 1 onto this, but again, this will yield a single value.

So, it seems like this equation is also a function.

Note that it is okay to get the same *y* value for different *x*'s. For example,

$$x = -3: \qquad y = (-3)^2 + 1 = 9 + 1 = 10$$

We just can't get more than one *y* out of the equation after we plug in the *x*.

(c) $y^2 = x + 1$

As we've done with the previous two equations let's plug in a couple of value of *x*, solve for *y* and see what we get.

$$x = 3: \qquad y^2 = 3 + 1 = 4 \qquad \Rightarrow \qquad y = \pm 2$$
$$x = -1: \qquad y^2 = -1 + 1 = 0 \qquad \Rightarrow \qquad y = 0$$
$$x = 10: \qquad y^2 = 10 + 1 = 11 \qquad \Rightarrow \qquad y = \pm\sqrt{11}$$

Now, remember that we're solving for *y* and so that means that in the first and last case above we will actually get two different *y* values out of the *x* and so this equation is NOT a function.

Note that we can have values of *x* that will yield a single *y* as we've seen above, but that doesn't matter. If even one value of *x* yields more than one value of *y* upon solving the equation will not be a function.

What this really means is that we didn't need to go any farther than the first evaluation, since that gave multiple values of *y*.

(d) $x^2 + y^2 = 4$

With this case we'll use the lesson learned in the previous part and see if we can find a value of *x* that will give more than one value of *y* upon solving. Because we've got a y^2 in the problem this shouldn't be too hard to do since solving will eventually mean using the square root property which will give more than one value of *y*.

$$x = 0: \qquad 0^2 + y^2 = 4 \qquad \Rightarrow \qquad y^2 = 4 \qquad \Rightarrow \qquad y = \pm 2$$

So, this equation is not a function. Recall, that from the previous section this is the equation of a circle. Circles are never functions.

Hopefully these examples have given you a better feel for what a function actually is.

We now need to move onto something called **function notation**. Function notation will be used heavily throughout most of the remaining chapters in this course and so it is important to understand it.

Let's start off with the following quadratic equation.

$$y = x^2 - 5x + 3$$

We can use a process similar to what we used in the previous set of examples to convince ourselves that this is a function. Since this is a function we will denote it as follows,

$$f\left(x\right) = x^2 - 5x + 3$$

So, we replaced the *y* with the notation $f\left(x\right)$. This is read as "f of *x*". Note that there is nothing special about the *f* we used here. We could just have easily used any of the following,

$$g\left(x\right) = x^2 - 5x + 3 \qquad h\left(x\right) = x^2 - 5x + 3 \qquad R\left(x\right) = x^2 - 5x + 3$$

The letter we use does not matter. What is important is the "$\left(x\right)$" part. The letter in the parenthesis must match the variable used on the right side of the equal sign.

It is very important to note that $f\left(x\right)$ is really nothing more than a really fancy way of writing *y*. If you keep that in mind you may find that dealing with function notation becomes a little easier.

Also, this is **NOT** a multiplication of *f* by *x*! This is one of the more common mistakes people make when they first deal with functions. This is just a notation used to denote functions.

Next we need to talk about **evaluating functions**. Evaluating a function is really nothing more than asking what its value is for specific values of *x*. Another way of looking at it is that we are asking what the *y* value for a given *x* is.

Evaluation is really quite simple. Let's take the function we were looking at above

$$f\left(x\right) = x^2 - 5x + 3$$

and ask what its value is for $x = 4$. In terms of function notation we will "ask" this using the notation $f\left(4\right)$. So, when there is something other than the variable inside the parenthesis we are really asking what the value of the function is for that particular quantity.

Now, when we say the value of the function we are really asking what the value of the equation is for that particular value of *x*. Here is $f\left(4\right)$.

$$f\left(4\right) = \left(4\right)^2 - 5\left(4\right) + 3 = 16 - 20 + 3 = -1$$

Notice that evaluating a function is done in exactly the same way in which we evaluate equations. All we do is plug in for *x* whatever is on the inside of the parenthesis on the left. Here's another evaluation for this function.

$$f(-6) = (-6)^2 - 5(-6) + 3 = 36 + 30 + 3 = 69$$

So, again, whatever is on the inside of the parenthesis on the left is plugged in for *x* in the equation on the right. Let's take a look at some more examples.

Example 4 Given $f(x) = x^2 - 2x + 8$ and $g(x) = \sqrt{x+6}$ evaluate each of the following.

(a) $f(3)$ and $g(3)$
(b) $f(-10)$ and $g(-10)$
(c) $f(0)$
(d) $f(t)$
(e) $f(t+1)$ and $f(x+1)$
(f) $f(x^3)$
(g) $g(x^2 - 5)$

Solution

(a) $f(3)$ **and** $g(3)$

Okay we've got two function evaluations to do here and we've also got two functions so we're going to need to decide which function to use for the evaluations. The key here is to notice the letter that is in front of the parenthesis. For $f(3)$ we will use the function $f(x)$ and for $g(3)$ we will use $g(x)$. In other words, we just need to make sure that the variables match up.

Here are the evaluations for this part.

$$f(3) = (3)^2 - 2(3) + 8 = 9 - 6 + 8 = 11$$
$$g(3) = \sqrt{3+6} = \sqrt{9} = 3$$

(b) $f(-10)$ **and** $g(-10)$

This one is pretty much the same as the previous part with one exception that we'll touch on when we reach that point. Here are the evaluations.

$$f(-10) = (-10)^2 - 2(-10) + 8 = 100 + 20 + 8 = 128$$

Make sure that you deal with the negative signs properly here. Now the second one.

$$g(-10) = \sqrt{-10+6} = \sqrt{-4}$$

We've now reached the difference. Recall that when we first started talking about the definition of functions we stated that we were only going to deal with real numbers. In other words, we only plug in real numbers and we only want real numbers back out as answers. So, since we would get a complex number out of this we can't plug -10 into this function.

(c) $f\left(0\right)$

Not much to this one.

$$f\left(0\right) = \left(0\right)^2 - 2\left(0\right) + 8 = 8$$

Again, don't forget that this isn't multiplication! For some reason students like to think of this one as multiplication and get an answer of zero. Be careful.

(d) $f\left(t\right)$

The rest of these evaluations are now going to be a little different. As this one shows we don't need to just have numbers in the parenthesis. However, evaluation works in exactly the same way. We plug into the *x*'s on the right side of the equal sign whatever is in the parenthesis. In this case that means that we plug in *t* for all the *x*'s.

Here is this evaluation.

$$f\left(t\right) = t^2 - 2t + 8$$

Note that in this case this is pretty much the same thing as our original function, except this time we're using *t* as a variable.

(e) $f\left(t+1\right)$ **and** $f\left(x+1\right)$

Now, let's get a little more complicated, or at least they appear to be more complicated. Things aren't as bad as they may appear however. We'll evaluate $f\left(t+1\right)$ first. This one works exactly the same as the previous part did. All the *x*'s on the left will get replaced with $t+1$. We will have some simplification to do as well after the substitution.

$$\begin{aligned} f\left(t+1\right) &= \left(t+1\right)^2 - 2\left(t+1\right) + 8 \\ &= t^2 + 2t + 1 - 2t - 2 + 8 \\ &= t^2 + 7 \end{aligned}$$

Be careful with parenthesis in these kinds of evaluations. It is easy to mess up with them.

Now, let's take a look at $f\left(x+1\right)$. With the exception of the *x* this is identical to $f\left(t+1\right)$ and so it works exactly the same way.

$$\begin{aligned} f\left(x+1\right) &= \left(x+1\right)^2 - 2\left(x+1\right) + 8 \\ &= x^2 + 2x + 1 - 2x - 2 + 8 \\ &= x^2 + 7 \end{aligned}$$

Do not get excited about the fact that we reused *x*'s in the evaluation here. In many places where we will be doing this in later sections there will be *x*'s here and so you will need to get used to seeing that.

(f) $f\left(x^3\right)$

Again, don't get excited about the *x*'s in the parenthesis here. Just evaluate it as if it were a number.

$$\begin{aligned} f\left(x^3\right) &= \left(x^3\right)^2 - 2\left(x^3\right) + 8 \\ &= x^6 - 2x^3 + 8 \end{aligned}$$

(g) $g\left(x^2 - 5\right)$

One more evaluation and this time we'll use the other function.

$$\begin{aligned} g\left(x^2 - 5\right) &= \sqrt{x^2 - 5 + 6} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

Function evaluation is something that we'll be doing a lot of in later sections and chapters so make sure that you can do it. You will find several later sections very difficult to understand and/or do the work in if you do not have a good grasp on how function evaluation works.

While we are on the subject of function evaluation we should now talk about **piecewise functions**. We've actually already seen an example of a piecewise function even if we didn't call it a function (or a piecewise function) at the time. Recall the mathematical definition of absolute value.

$$\left|x\right| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is a function and if we use function notation we can write it as follows,

$$f\left(x\right) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is also an example of a piecewise function. A piecewise function is nothing more than a function that is broken into pieces and which piece you use depends upon value of *x*. So, in the absolute value example we will use the top piece if *x* is positive or zero and we will use the bottom piece if *x* is negative.

Let's take a look at evaluating a more complicated piecewise function.

Example 5 Given,

$$g(t) = \begin{cases} 3t^2 + 4 & \text{if } t \le -4 \\ 10 & \text{if } -4 < t \le 15 \\ 1 - 6t & \text{if } t > 15 \end{cases}$$

evaluate each of the following.

(a) $g(-6)$

(b) $g(-4)$

(c) $g(1)$

(d) $g(15)$

(e) $g(21)$

Solution

Before starting the evaluations here let's notice that we're using different letters for the function and variable than the ones that we've used to this point. That won't change how the evaluation works. Do not get so locked into seeing *f* for the function and *x* for the variable that you can't do any problem that doesn't have those letters.

Now, to do each of these evaluations the first thing that we need to do is determine which inequality the number satisfies, and it will only satisfy a single inequality. When we determine which inequality the number satisfies we use the equation associated with that inequality.

So, let's do some evaluations.

(a) $g(-6)$

In this case -6 satisfies the top inequality and so we'll use the top equation for this evaluation.

$$g(-6) = 3(-6)^2 + 4 = 112$$

(b) $g(-4)$

Now we'll need to be a little careful with this one since -4 shows up in two of the inequalities. However, it only satisfies the top inequality and so we will once again use the top function for the evaluation.

$$g(-4) = 3(-4)^2 + 4 = 52$$

(c) $g(1)$

In this case the number, 1, satisfies the middle inequality and so we'll use the middle equation for the evaluation. This evaluation often causes problems for students despite the fact that it's actually one of the easiest evaluations we'll ever do. We know that we evaluate functions/equations by plugging in the number for the variable. In this case there are no variables. That isn't a problem. Since there aren't any variables it just means that we don't actually plug in anything and we get the following,

$$g(1) = 10$$

(d) $g\left(15\right)$

Again, like with the second part we need to be a little careful with this one. In this case the number satisfies the middle inequality since that is the one with the equal sign in it. Then like the previous part we just get,

$$g\left(15\right)=10$$

Don't get excited about the fact that the previous two evaluations were the same value. This will happen on occasion.

(e) $g\left(21\right)$

For the final evaluation in this example the number satisfies the bottom inequality and so we'll use the bottom equation for the evaluation.

$$g\left(21\right)=1-6\left(21\right)=-125$$

Piecewise functions do not arise all that often in an Algebra class however, they do arise in several places in later classes and so it is important for you to understand them if you are going to be moving on to more math classes.

As a final topic we need to come back and touch on the fact that we can't always plug every *x* into every function. We talked briefly about this when we gave the definition of the function and we saw an example of this when we were evaluating functions. We now need to look at this in a little more detail.

First, we need to get a couple of definitions out of the way.

Domain and Range

The **domain** of an equation is the set of all *x*'s that we can plug into the equation and get back a real number for *y*. The **range** of an equation is the set of all *y*'s that we can ever get out of the equation.

Note that we did mean to use equation in the definitions above instead of functions. These are really definitions for equations. However, since functions are also equations we can use the definitions for functions as well.

Determining the range of an equation/function can be pretty difficult to do for many functions and so we aren't going to really get into that. We are much more interested here in determining the domains of functions. From the definition the domain is the set of all *x*'s that we can plug into a function and get back a real number. At this point, that means that we need to avoid division by zero and taking square roots of negative numbers.

Let's do a couple of quick examples of finding domains.

Example 6 Determine the domain of each of the following functions.

(a) $g\left(x\right)=\dfrac{x+3}{x^2+3x-10}$

(b) $f\left(x\right)=\sqrt{5-3x}$

(c) $h\left(x\right)=\dfrac{\sqrt{7x+8}}{x^2+4}$

(d) $R\left(x\right)=\dfrac{\sqrt{10x-5}}{x^2-16}$

Solution

The domains for these functions are all the values of *x* for which we don't have division by zero or the square root of a negative number. If we remember these two ideas finding the domains will be pretty easy.

(a) $g\left(x\right)=\dfrac{x+3}{x^2+3x-10}$

So, in this case there are no square roots so we don't need to worry about the square root of a negative number. There is however a possibility that we'll have a division by zero error. To determine if we will we'll need to set the denominator equal to zero and solve.

$$x^2+3x-10=\left(x+5\right)\left(x-2\right)=0 \qquad x=-5,\ x=2$$

So, we will get division by zero if we plug in $x=-5$ or $x=2$. That means that we'll need to avoid those two numbers. However, all the other values of *x* will work since they don't give division by zero. The domain is then,

$$\text{Domain : All real numbers except } x=-5 \text{ and } x=2$$

(b) $f\left(x\right)=\sqrt{5-3x}$

In this case we won't have division by zero problems since we don't have any fractions. We do have a square root in the problem and so we'll need to worry about taking the square root of a negative numbers.

This one is going to work a little differently from the previous part. In that part we determined the value(s) of *x* to avoid. In this case it will be just as easy to directly get the domain. To avoid square roots of negative numbers all that we need to do is require that

$$5-3x\geq 0$$

This is a fairly simple linear inequality that we should be able to solve at this point.

$$5\geq 3x \qquad \Rightarrow \qquad x\leq\frac{5}{3}$$

The domain of this function is then,

$$\text{Domain : } x\leq\frac{5}{3}$$

(c) $h\left(x\right)=\dfrac{\sqrt{7x+8}}{x^2+4}$

In this case we've got a fraction, but notice that the denominator will never be zero for any real number since x^2 is guaranteed to be positive or zero and adding 4 onto this will mean that the denominator is always at least 4. In other words, the denominator won't ever be zero. So, all we need to do then is worry about the square root in the numerator.

To do this we'll require,

$$\begin{aligned}7x+8&\geq 0\\7x&\geq -8\\x&\geq -\frac{8}{7}\end{aligned}$$

Now, we can actually plug in any value of *x* into the denominator, however, since we've got the square root in the numerator we'll have to make sure that all *x*'s satisfy the inequality above to avoid problems. Therefore, the domain of this function is

$$\text{Domain} : x\geq -\frac{8}{7}$$

(d) $R\left(x\right)=\dfrac{\sqrt{10x-5}}{x^2-16}$

In this final part we've got both a square root and division by zero to worry about. Let's take care of the square root first since this will probably put the largest restriction on the values of *x*. So, to keep the square root happy (*i.e.* no square root of negative numbers) we'll need to require that,

$$\begin{aligned}10x-5&\geq 0\\10x&\geq 5\\x&\geq \frac{1}{2}\end{aligned}$$

So, at the least we'll need to require that $x\geq\frac{1}{2}$ in order to avoid problems with the square root.

Now, let's see if we have any division by zero problems. Again, to do this simply set the denominator equal to zero and solve.

$$x^2-16=\left(x-4\right)\left(x+4\right)=0 \qquad \Rightarrow \qquad x=-4,\ x=4$$

Now, notice that $x=-4$ doesn't satisfy the inequality we need for the square root and so that value of *x* has already been excluded by the square root. On the other hand, $x=4$ does satisfy the inequality. This means that it is okay to plug $x=4$ into the square root, however, since it would give division by zero we will need to avoid it.

The domain for this function is then,

$$\text{Domain} : x\geq\frac{1}{2} \quad \text{except} \quad x=4$$

Section 3-5 : Graphing Functions

Now we need to discuss graphing functions. If we recall from the previous section we said that $f\left(x\right)$ is nothing more than a fancy way of writing *y*. This means that we already know how to graph functions. We graph functions in exactly the same way that we graph equations. If we know ahead of time what the function is a graph of we can use that information to help us with the graph and if we don't know what the function is ahead of time then all we need to do is plug in some *x*'s compute the value of the function (which is really a *y* value) and then plot the points.

Example 1 Sketch the graph of $f\left(x\right)=\left(x-1\right)^3+1$.

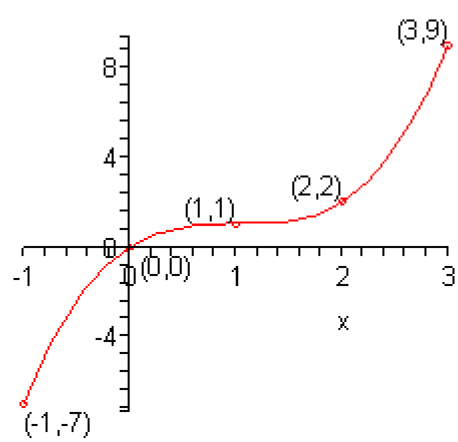
Solution

Now, as we talked about when we first looked at graphing earlier in this chapter we'll need to pick values of *x* to plug in and knowing the values to pick really only comes with experience. Therefore, don't worry so much about the values of *x* that we're using here. By the end of this chapter you'll also be able to correctly pick these values.

Here are the function evaluations.

x	*f(x)*	$\left(x,y\right)$
-1	-7	$\left(-1,-7\right)$
0	0	$\left(0,0\right)$
1	1	$\left(1,1\right)$
2	2	$\left(2,2\right)$
3	9	$\left(3,9\right)$

Here is the sketch of the graph.



So, graphing functions is pretty much the same as graphing equations.

There is one function that we've seen to this point that we didn't really see anything like when we were graphing equations in the first part of this chapter. That is piecewise functions. So, we should graph a couple of these to make sure that we can graph them as well.

Example 2 Sketch the graph of the following piecewise function.

$$g(x) = \begin{cases} -x^2 + 4 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$

Solution

Okay, now when we are graphing piecewise functions we are really graphing several functions at once, except we are only going to graph them on very specific intervals. In this case we will be graphing the following two functions,

$$\begin{array}{lcl} -x^2 + 4 & \text{on} & x < 1 \\ 2x - 1 & \text{on} & x \geq 1 \end{array}$$

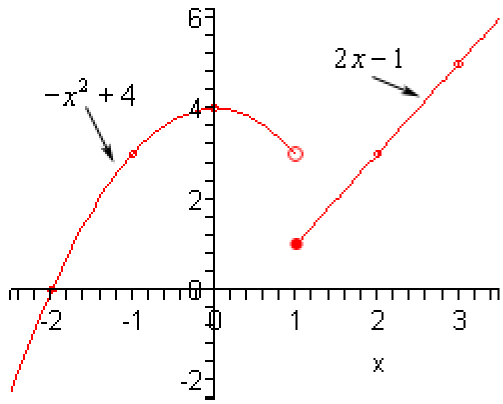
We'll need to be a little careful with what is going on right at $x = 1$ since technically that will only be valid for the bottom function. However, we'll deal with that at the very end when we actually do the graph. For now, we will use $x = 1$ in both functions.

The first thing to do here is to get a table of values for each function on the specified range and again we will use $x = 1$ in both even though technically it only should be used with the bottom function.

x	$-x^2+4$	(x,y)
-2	0	$(-2,0)$
-1	3	$(-1,3)$
0	4	$(0,4)$
1	3	$(1,3)$

x	$2x-1$	(x,y)
1	1	$(1,1)$
2	3	$(2,3)$
3	5	$(3,5)$

Here is a sketch of the graph and notice how we denoted the points at $x = 1$. For the top function we used an open dot for the point at $x = 1$ and for the bottom function we used a closed dot at $x = 1$. In this way we make it clear on the graph that only the bottom function really has a point at $x = 1$.



Notice that since the two graphs didn't meet at $x=1$ we left a blank space in the graph. Do NOT connect these two points with a line. There really does need to be a break there to signify that the two portions do not meet at $x=1$.

Sometimes the two portions will meet at these points and at other times they won't. We shouldn't ever expect them to meet or not to meet until we've actually sketched the graph.

Let's take a look at another example of a piecewise function.

Example 3 Sketch the graph of the following piecewise function.

$$h(x)=\begin{cases} x+3 & \text{if } x<-2 \\ x^2 & \text{if } -2\le x<1 \\ -x+2 & \text{if } x\ge 1 \end{cases}$$

Solution

In this case we will be graphing three functions on the ranges given above. So, as with the previous example we will get function values for each function in its specified range and we will include the endpoints of each range in each computation. When we graph we will acknowledge which function the endpoint actually belongs with by using a closed dot as we did previously. Also, the top and bottom functions are lines and so we don't really need more than two points for these two. We'll get a couple more points for the middle function.

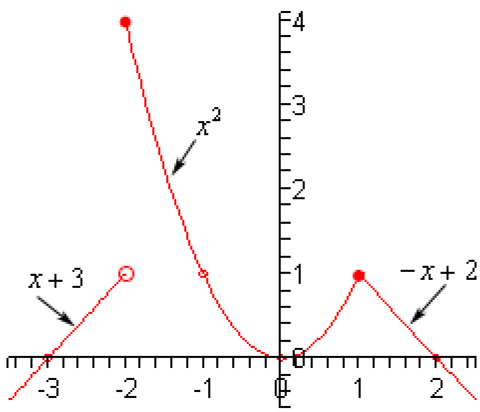
x	$x+3$	(x,y)
-3	0	$(-3,0)$
-2	1	$(-2,1)$

x	x^2	(x,y)
-2	4	$(-2,4)$

-1	1	$(-1,1)$
0	0	$(0,0)$
1	1	$(1,1)$

x	$-x+2$	(x,y)
1	1	$(1,1)$
2	0	$(2,0)$

Here is the sketch of the graph.



Note that in this case two of the portions met at the breaking point $x=1$ and at the other breaking point, $x=-2$, they didn't meet up. As noted in the previous example sometimes they meet up and sometimes they won't.

Section 3-6 : Combining Functions

The topic with functions that we need to deal with is combining functions. For the most part this means performing basic arithmetic (addition, subtraction, multiplication, and division) with functions. There is one new way of combing functions that we'll need to look at as well.

Let's start with basic arithmetic of functions. Given two functions $f\left(x\right)$ and $g\left(x\right)$ we have the following notation and operations.

$$\left(f+g\right)\left(x\right)=f\left(x\right)+g\left(x\right) \qquad \left(f-g\right)\left(x\right)=f\left(x\right)-g\left(x\right)$$

$$\left(fg\right)\left(x\right)=f\left(x\right)g\left(x\right) \qquad \left(\frac{f}{g}\right)\left(x\right)=\frac{f\left(x\right)}{g\left(x\right)}$$

Sometimes we will drop the $\left(x\right)$ part and just write the following,

$$f+g=f\left(x\right)+g\left(x\right) \qquad f-g=f\left(x\right)-g\left(x\right)$$

$$fg=f\left(x\right)g\left(x\right) \qquad \frac{f}{g}=\frac{f\left(x\right)}{g\left(x\right)}$$

Note as well that we put *x*'s in the parenthesis, but we will often put in numbers as well. Let's take a quick look at an example.

Example 1 Given $f\left(x\right)=2+3x-x^2$ and $g\left(x\right)=2x-1$ evaluate each of the following.

(a) $\left(f+g\right)\left(4\right)$
(b) $g-f$
(c) $\left(fg\right)\left(x\right)$
(d) $\left(\dfrac{f}{g}\right)\left(0\right)$

Solution

By evaluate we mean one of two things depending on what is in the parenthesis. If there is a number in the parenthesis then we want a number. If there is an *x* (or no parenthesis, since that implies and *x*) then we will perform the operation and simplify as much as possible.

(a) $\left(f+g\right)\left(4\right)$

In this case we've got a number so we need to do some function evaluation.

$$\begin{aligned}\left(f+g\right)\left(4\right)&=f\left(4\right)+g\left(4\right)\\&=\left(2+3\left(4\right)-4^2\right)+\left(2\left(4\right)-1\right)\\&=-2+7\\&=5\end{aligned}$$

(b) $g - f$

Here we don't have an *x* or a number so this implies the same thing as if there were an *x* in parenthesis. Therefore, we'll subtract the two functions and simplify. Note as well that this is written in the opposite order from the definitions above, but it works the same way.

$$\begin{aligned} g - f &= g(x) - f(x) \\ &= 2x - 1 - \left(2 + 3x - x^2\right) \\ &= 2x - 1 - 2 - 3x + x^2 \\ &= x^2 - x - 3 \end{aligned}$$

(c) $(fg)(x)$

As with the last part this has an *x* in the parenthesis so we'll multiply and then simplify.

$$\begin{aligned} (fg)(x) &= f(x)g(x) \\ &= \left(2 + 3x - x^2\right)(2x - 1) \\ &= 4x + 6x^2 - 2x^3 - 2 - 3x + x^2 \\ &= -2x^3 + 7x^2 + x - 2 \end{aligned}$$

(d) $\left(\dfrac{f}{g}\right)(0)$

In this final part we've got a number so we'll once again be doing function evaluation.

$$\begin{aligned} \left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} \\ &= \frac{2 + 3(0) - (0)^2}{2(0) - 1} \\ &= \frac{2}{-1} \\ &= -2 \end{aligned}$$

Now we need to discuss the new method of combining functions. The new method of combining functions is called **function composition**. Here is the definition.

Given two functions $f(x)$ and $g(x)$ we have the following two definitions.

1. The **composition** of $f(x)$ and $g(x)$ (note the order here) is,

   $$(f \circ g)(x) = f\left[g(x)\right]$$

2. The **composition of** $g(x)$ and $f(x)$ (again, note the order) is,

   $$(g \circ f)(x) = g\left[f(x)\right]$$

We need to note a couple of things here about function composition. First this is **NOT** multiplication. Regardless of what the notation may suggest to you this is simply not multiplication.

Second, the order we've listed the two functions is very important since more often than not we will get different answers depending on the order we've listed them.

Finally, function composition is really nothing more than function evaluation. All we're really doing is plugging the second function listed into the first function listed. In the definitions we used $[\ \ ]$ for the function evaluation instead of the standard $(\ \ )$ to avoid confusion with too many sets of parenthesis, but the evaluation will work the same.

Let's take a look at a couple of examples.

Example 2 Given $f(x) = 2 + 3x - x^2$ and $g(x) = 2x - 1$ evaluate each of the following.

(a) $(fg)(x)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

Solution

(a) These are the same functions that we used in the first set of examples and we've already done this part there so we won't redo all the work here. It is here only here to prove the point that function composition is NOT function multiplication.

Here is the multiplication of these two functions.

$$(fg)(x) = -2x^3 + 7x^2 + x - 2$$

(b) Now, for function composition all you need to remember is that we are going to plug the second function listed into the first function listed. If you can remember that you should always be able to write down the basic formula for the composition.

Here is this function composition.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f[2x-1]\end{aligned}$$

Now, notice that since we've got a formula for $g(x)$ we went ahead and plugged that in first. Also, we did this kind of function evaluation in the first section we looked at for functions. At the time it probably didn't seem all that useful to be looking at that kind of function evaluation, yet here it is.

Let's finish this problem out.

$$
\begin{aligned}
(f \circ g)(x) &= f\left[g(x)\right] \\
&= f\left[2x-1\right] \\
&= 2+3(2x-1)-(2x-1)^2 \\
&= 2+6x-3-\left(4x^2-4x+1\right) \\
&= -1+6x-4x^2+4x-1 \\
&= -4x^2+10x-2
\end{aligned}
$$

Notice that this is very different from the multiplication! Remember that function composition is NOT function multiplication.

(c) We'll not put in the detail in this part as it works essentially the same as the previous part.

$$
\begin{aligned}
(g \circ f)(x) &= g\left[f(x)\right] \\
&= g\left[2+3x-x^2\right] \\
&= 2\left(2+3x-x^2\right)-1 \\
&= 4+6x-2x^2-1 \\
&= -2x^2+6x+3
\end{aligned}
$$

Notice that this is NOT the same answer as that from the second part. In most cases the order in which we do the function composition will give different answers.

The ideas from the previous example are important enough to make again. First, function composition is NOT function multiplication. Second, the order in which we do function composition is important. In most case we will get different answers with a different order. Note however, that there are times when we will get the same answer regardless of the order.

Let's work a couple more examples.

Example 3 Given $f(x)=x^2-3$ and $h(x)=\sqrt{x+1}$ evaluate each of the following.

(a) $(f \circ h)(x)$

(b) $(h \circ f)(x)$

(c) $(f \circ f)(x)$

(d) $(h \circ h)(8)$

(e) $(f \circ h)(4)$

Solution

(a) $(f \circ h)(x)$

Not much to do here other than run through the formula.

$$\begin{aligned}(f \circ h)(x) &= f\left[h(x)\right] \\ &= f\left[\sqrt{x+1}\right] \\ &= \left(\sqrt{x+1}\right)^2 - 3 \\ &= x+1-3 \\ &= x-2\end{aligned}$$

(b) $(h \circ f)(x)$

Again, not much to do here.

$$\begin{aligned}(h \circ f)(x) &= h\left[f(x)\right] \\ &= h\left[x^2-3\right] \\ &= \sqrt{x^2-3+1} \\ &= \sqrt{x^2-2}\end{aligned}$$

(c) $(f \circ f)(x)$

Now in this case do not get excited about the fact that the two functions here are the same. Composition works the same way.

$$\begin{aligned}(f \circ f)(x) &= f\left[f(x)\right] \\ &= f\left[x^2-3\right] \\ &= \left(x^2-3\right)^2 - 3 \\ &= x^4-6x^2+9-3 \\ &= x^4-6x^2+6\end{aligned}$$

(d) $(h \circ h)(8)$

In this case, unlike all the previous examples, we've got a number in the parenthesis instead of an *x*, but it works in exactly the same manner.

$$\begin{aligned}(h \circ h)(8) &= h\left[h(8)\right] \\ &= h\left[\sqrt{8+1}\right] \\ &= h\left[\sqrt{9}\right] \\ &= h[3] \\ &= \sqrt{3+1} \\ &= 2\end{aligned}$$

(e) $\left(f\circ h\right)\left(4\right)$

Again, we've got a number here. This time there are actually two ways to do this evaluation. The first is to simply use the results from the first part since that is a formula for the general function composition.

If we do the problem that way we get,

$$\left(f\circ h\right)\left(4\right)=4-2=2$$

We could also do the evaluation directly as we did in the previous part. The answers should be the same regardless of how we get them. So, to get another example down of this kind of evaluation let's also do the evaluation directly.

$$\begin{aligned}\left(f\circ h\right)\left(4\right)&=f\left[h\left(4\right)\right]\\&=f\left[\sqrt{4+1}\right]\\&=f\left[\sqrt{5}\right]\\&=\left(\sqrt{5}\right)^{2}-3\\&=5-3\\&=2\end{aligned}$$

So, sure enough we got the same answer, although it did take more work to get it.

Example 4 Given $f\left(x\right)=3x-2$ and $g\left(x\right)=\frac{x}{3}+\frac{2}{3}$ evaluate each of the following.

(a) $\left(f\circ g\right)\left(x\right)$

(b) $\left(g\circ f\right)\left(x\right)$

Solution

(a) Hopefully, by this point these aren't too bad.

$$\begin{aligned}\left(f\circ g\right)\left(x\right)&=f\left[g\left(x\right)\right]\\&=f\left[\frac{x}{3}+\frac{2}{3}\right]\\&=3\left(\frac{x}{3}+\frac{2}{3}\right)-2\\&=x+2-2\\&=x\end{aligned}$$

Looks like things simplified down considerable here.

(b) All we need to do here is use the formula so let's do that.

$$\begin{aligned}(g\circ f)(x) &= g\left[f(x)\right]\\ &= g\left[3x-2\right]\\ &= \frac{1}{3}(3x-2)+\frac{2}{3}\\ &= x-\frac{2}{3}+\frac{2}{3}\\ &= x\end{aligned}$$

So, in this case we get the same answer regardless of the order we did the composition in.

So, as we've seen from this last example it is possible to get the same answer from both compositions on occasion. In fact when the answer from both composition is *x*, as it is in this case, we know that these two functions are very special functions. In fact, they are so special that we're going to devote the whole next section to these kinds of functions. So, let's move onto the next section.

Section 3-7 : Inverse Functions

In the last example from the previous section we looked at the two functions $f\left(x\right)=3x-2$ and $g\left(x\right)=\frac{x}{3}+\frac{2}{3}$ and saw that

$$\left(f\circ g\right)\left(x\right)=\left(g\circ f\right)\left(x\right)=x$$

and as noted in that section this means that these are very special functions. Let's see just what makes them so special. Consider the following evaluations.

$$f\left(-1\right)=3\left(-1\right)-2=-5 \qquad \Rightarrow \qquad g\left(-5\right)=\frac{-5}{3}+\frac{2}{3}=\frac{-3}{3}=-1$$

$$g\left(2\right)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right)=3\left(\frac{4}{3}\right)-2=4-2=2$$

In the first case we plugged $x=-1$ into $f\left(x\right)$ and got a value of -5. We then turned around and plugged $x=-5$ into $g\left(x\right)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x=2$ into $g\left(x\right)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f\left(x\right)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$\left(g\circ f\right)\left(-1\right)=g\left[f\left(-1\right)\right]=g\left[-5\right]=-1$$

and the second case is really,

$$\left(f\circ g\right)\left(2\right)=f\left[g\left(2\right)\right]=f\left[\frac{4}{3}\right]=2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x=-1$ into $f\left(x\right)$ and then plugged the result from this function evaluation back into $g\left(x\right)$ and in some way $g\left(x\right)$ undid what $f\left(x\right)$ had done to $x=-1$ and gave us back the original *x* that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. This is a fairly simple definition of one-to-one but it takes an example of a function that isn't one-to-one to show just what it means. Before doing that however we should note that this definition of one-to-one is not really the mathematically correct definition of one-to-one. It is identical to the mathematically correct definition it just doesn't use all the notation from the formal definition.

Now, let's see an example of a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words, there are two different values of *x* that produce the same value of *y*. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \le x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often a tedious and difficult process. For the most part we are going to assume that the functions that we're going to be dealing with in this section are one-to-one. We did need to talk about one-to-one functions however since only one-to-one functions can be inverse functions.

Now, let's formally define just what inverse functions are.

Inverse Functions

Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise, we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there is a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this section if one is true then the other will also be true. However, there are functions (they are far beyond the scope of this course however) for which it is possible for only of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with *y*.

$$y = 3x - 2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x = 3y - 2$$

Now, solve for *y*.

$$x + 2 = 3y$$
$$\frac{1}{3}(x+2) = y$$
$$\frac{x}{3} + \frac{2}{3} = y$$

Finally replace *y* with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $\left(f \circ f^{-1}\right)(x) = x$ is true.

$$\begin{aligned}\left(f \circ f^{-1}\right)(x) &= f\left[f^{-1}(x)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

Example 2 Given $g(x) = \sqrt{x-3}$ find $g^{-1}(x),\ x \ge 0$.

Solution

Now the fact that we're now using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$y = \sqrt{x-3}$$
$$x = \sqrt{y-3}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$x = \sqrt{y-3}$$
$$x^2 = y - 3$$
$$x^2 + 3 = y$$

This inverse is then,

$$g^{-1}(x) = x^2 + 3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}\left(g^{-1}\circ g\right)(x) &= g^{-1}\left[g(x)\right]\\ &= g^{-1}\left(\sqrt{x-3}\right)\\ &= \left(\sqrt{x-3}\right)^2+3\\ &= x-3+3\\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

Before we move on we should also acknowledge the restrictions of $x \geq 0$ that we gave in the problem statement but never apparently did anything with. Note that this restriction is required to make sure that the inverse, $g^{-1}(x)$ given above is in fact one-to-one.

Without this restriction the inverse would not be one-to-one as is easily seen by a couple of quick evaluations.

$$g^{-1}(1)=(1)^2+3=4 \qquad\qquad g^{-1}(-1)=(-1)^2+3=4$$

Therefore, the restriction is required in order to make sure the inverse is one-to-one.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x)=\dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$\begin{aligned}y&=\frac{x+4}{2x-5}\\ x&=\frac{y+4}{2y-5}\end{aligned}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned}x(2y-5)&=y+4\\ 2xy-5x&=y+4\\ 2xy-y&=4+5x\\ (2x-1)y&=4+5x\\ y&=\frac{4+5x}{2x-1}\end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally, we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned}
\left(h \circ h^{-1}\right)(x) &= h\left[h^{-1}(x)\right] \\
&= h\left[\frac{4+5x}{2x-1}\right] \\
&= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}
\end{aligned}$$

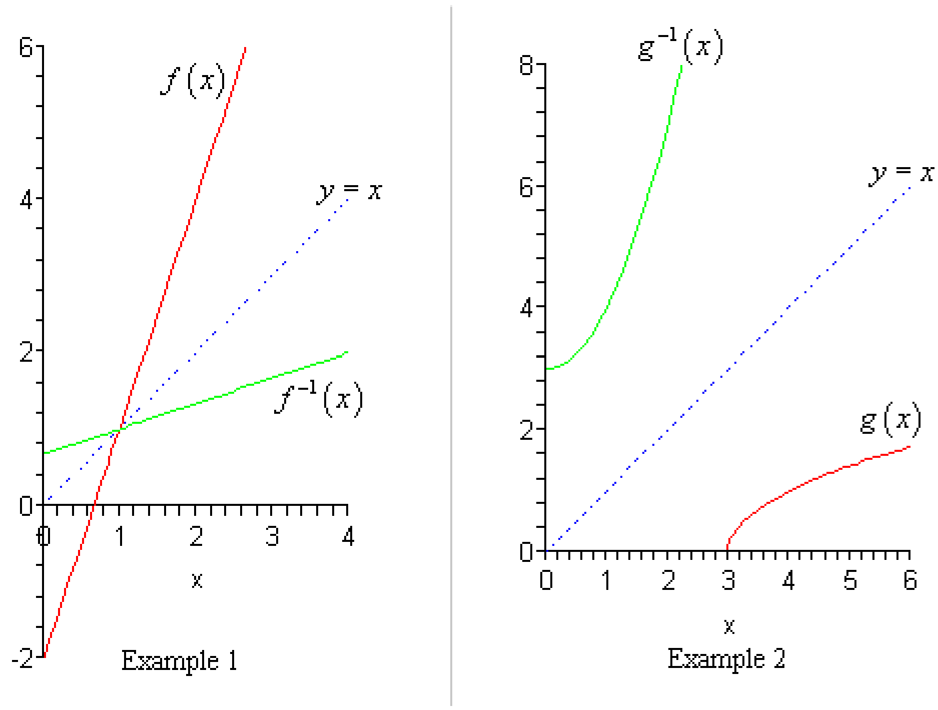
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$\begin{aligned}
\left(h \circ h^{-1}\right)(x) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\
&= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\
&= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)} \\
&= \frac{4+5x+8x-4}{8+10x-10x+5} \\
&= \frac{13x}{13} \\
&= x
\end{aligned}$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and its inverse.

Here is the graph of the function and inverse from the first two examples. We'll not deal with the final example since that is a function that we haven't really talked about graphing yet.



In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

Chapter 4 : Common Graphs

We started the process of graphing in the previous chapter. However, since the main focus of that chapter was functions we didn't graph all that many equations or functions. In this chapter we will now look at graphing a wide variety of equations and functions.

Here is a listing of the topics that we'll be looking at in this chapter.

Lines, Circles and Piecewise Functions – This section is here only to acknowledge that we've already talked about graphing these in a previous chapter.
Parabolas – In this section we will be graphing parabolas. We introduce the vertex and axis of symmetry for a parabola and give a process for graphing parabolas. We also illustrate how to use completing the square to put the parabola into the form $f\left(x\right)=a\left(x-h\right)^2+k$.
Ellipses – In this section we will graph ellipses. We introduce the standard form of an ellipse and how to use it to quickly graph an ellipse.
Hyperbolas – In this section we will graph hyperbolas. We introduce the standard form of a hyperbola and how to use it to quickly graph a hyperbola.
Miscellaneous Functions – In this section we will graph a couple of common functions that don't really take all that much work to do but will be needed in later sections. We'll be looking at the constant function, square root, absolute value and a simple cubic function.
Transformations – In this section we will be looking at vertical and horizontal shifts of graphs as well as reflections of graphs about the *x* and *y*-axis. Collectively these are often called transformations and if we understand them they can often be used to allow us to quickly graph some fairly complicated functions.
Symmetry – In this section we introduce the idea of symmetry. We discuss symmetry about the *x*-axis, *y*-axis and the origin and we give methods for determining what, if any symmetry, a graph will have without having to actually graph the function.
Rational Functions – In this section we will discuss a process for graphing rational functions. We will also introduce the ideas of vertical and horizontal asymptotes as well as how to determine if the graph of a rational function will have them.

Section 4-1 : Lines, Circles and Piecewise Functions

We're not really going to do any graphing in this section. In fact, this section is here only to acknowledge that we've already looked at these equations and functions in the previous chapter.

Here are the appropriate sections to see for these.

Lines : Graphing and Functions – **Lines**

Circles : Graphing and Functions – **Circles**

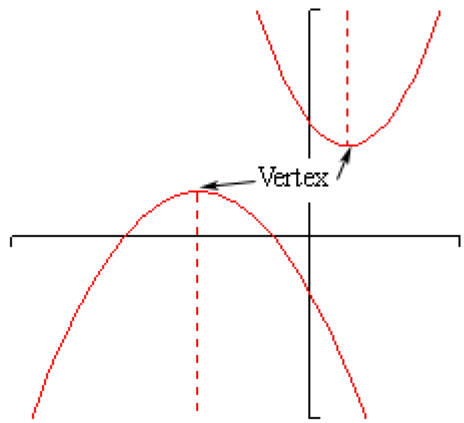
Piecewise Functions : Graphing and Functions – **Graphing Functions**

Section 4-1 : Parabolas

In this section we want to look at the graph of a quadratic function. The most general form of a quadratic function is,

$$f\left(x\right) = ax^2 + bx + c$$

The graphs of quadratic functions are called **parabolas**. Here are some examples of parabolas.



All parabolas are vaguely "U" shaped and they will have a highest or lowest point that is called the **vertex**. Parabolas may open up or down and may or may not have *x*-intercepts and they will always have a single *y*-intercept.

Note as well that a parabola that opens down will always open down and a parabola that opens up will always open up. In other words, a parabola will not all of a sudden turn around and start opening up if it has already started opening down. Similarly, if it has already started opening up it will not turn around and start opening down all of a sudden.

The dashed line with each of these parabolas is called the **axis of symmetry**. Every parabola has an axis of symmetry and, as the graph shows, the graph to either side of the axis of symmetry is a mirror image of the other side. This means that if we know a point on one side of the parabola we will also know a point on the other side based on the axis of symmetry. We will see how to find this point once we get into some examples.

We should probably do a quick review of intercepts before going much farther. Intercepts are the points where the graph will cross the *x* or *y*-axis. We also saw a graph in the section where we introduced intercepts where an intercept just touched the axis without actually crossing it.

Finding intercepts is a fairly simple process. To find the *y*-intercept of a function $y = f\left(x\right)$ all we need to do is set $x = 0$ and evaluate to find the *y* coordinate. In other words, the *y*-intercept is the point $\left(0, f\left(0\right)\right)$. We find *x*-intercepts in pretty much the same way. We set $y = 0$ and solve the resulting equation for the *x* coordinates. So, we will need to solve the equation,

$$f\left(x\right) = 0$$

Now, let's get back to parabolas. There is a basic process we can always use to get a pretty good sketch of a parabola. Here it is.

Sketching Parabolas

1. Find the vertex. We'll discuss how to find this shortly. It's fairly simple, but there are several methods for finding it and so will be discussed separately.

2. Find the *y*-intercept, $\left(0, f\left(0\right)\right)$.

3. Solve $f\left(x\right)=0$ to find the *x* coordinates of the *x*-intercepts if they exist. As we will see in our examples we can have 0, 1, or 2 *x*-intercepts.

4. Make sure that you've got at least one point to either side of the vertex. This is to make sure we get a somewhat accurate sketch. If the parabola has two *x*-intercepts then we'll already have these points. If it has 0 or 1 *x*-intercept we can either just plug in another *x* value or use the *y*-intercept and the axis of symmetry to get the second point.

5. Sketch the graph. At this point we've gotten enough points to get a fairly decent idea of what the parabola will look like.

Now, there are two forms of the parabola that we will be looking at. This first form will make graphing parabolas very easy. Unfortunately, most parabolas are not in this form. The second form is the more common form and will require slightly (and only slightly) more work to sketch the graph of the parabola.

Let's take a look at the first form of the parabola.

$$f\left(x\right)=a\left(x-h\right)^2+k$$

There are two pieces of information about the parabola that we can instantly get from this function. First, if *a* is positive then the parabola will open up and if *a* is negative then the parabola will open down. Secondly, the vertex of the parabola is the point $\left(h,k\right)$. Be very careful with signs when getting the vertex here.

So, when we are lucky enough to have this form of the parabola we are given the vertex for free.

Let's see a couple of examples here.

Example 1 Sketch the graph of each of the following parabolas.

(a) $f(x) = 2(x+3)^2 - 8$

(b) $g(x) = -(x-2)^2 - 1$

(c) $h(x) = x^2 + 4$

Solution

Okay, in all of these we will simply go through the process given above to find the needed points and the graph.

(a) $f(x) = 2(x+3)^2 - 8$

First, we need to find the vertex. We will need to be careful with the signs however. Comparing our equation to the form above we see that we must have $h = -3$ and $k = -8$ since that is the only way to get the correct signs in our function. Therefore, the vertex of this parabola is,

$$(-3,-8)$$

Now let's find the *y*-intercept. This is nothing more than a quick function evaluation.

$$f(0) = 2(0+3)^2 - 8 = 2(9) - 8 = 10 \qquad\qquad y-\text{intercept}: \ (0,10)$$

Next, we need to find the *x*-intercepts. This means we'll need to solve an equation. However, before we do that we can actually tell whether or not we'll have any before we even start to solve the equation.

In this case we have $a = 2$ which is positive and so we know that the parabola opens up. Also the vertex is a point below the *x*-axis. So, we know that the parabola will have at least a few points below the *x*-axis and it will open up. Therefore, since once a parabola starts to open up it will continue to open up eventually we will have to cross the *x*-axis. In other words, there are *x*-intercepts for this parabola.

To find them we need to solve the following equation.

$$0 = 2(x+3)^2 - 8$$

We solve equations like this back when we were solving quadratic equations so hopefully you remember how to do them.

$$\begin{aligned} 2(x+3)^2 &= 8 \\ (x+3)^2 &= 4 \\ x+3 &= \pm\sqrt{4} = \pm 2 \\ x &= -3 \pm 2 \qquad\qquad \Rightarrow \qquad\qquad x = -1, \ x = -5 \end{aligned}$$

The two *x*-intercepts are then,

$$(-5,0) \quad \text{and} \quad (-1,0)$$

Now, at this point we've got points on either side of the vertex so we are officially done with finding the points. However, let's talk a little bit about how to find a second point using the *y*-intercept and the axis of symmetry since we will need to do that eventually.

First, notice that the *y*-intercept has an *x* coordinate of 0 while the vertex has an *x* coordinate of -3. This means that the *y*-intercept is a distance of 3 to the right of the axis of symmetry since that will move straight up from the vertex.

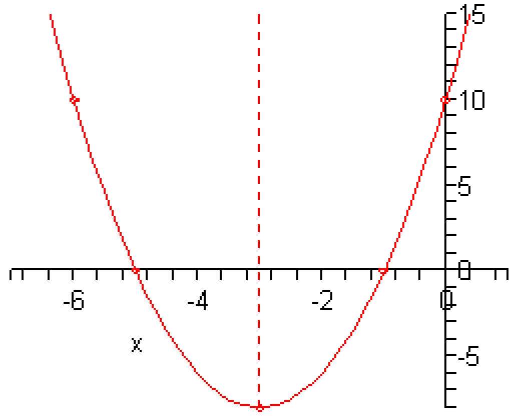
Now, the left part of the graph will be a mirror image of the right part of the graph. So, since there is a point at $y = 10$ that is a distance of 3 to the right of the axis of symmetry there must also be a point at $y = 10$ that is a distance of 3 to the left of the axis of symmetry.

So, since the *x* coordinate of the vertex is -3 and this new point is a distance of 3 to the left its *x* coordinate must be -6. The coordinates of this new point are then $(-6, 10)$. We can verify this by evaluating the function at $x = -6$. If we are correct we should get a value of 10. Let's verify this.

$$f(-6) = 2(-6+3)^2 - 8 = 2(-3)^2 - 8 = 2(9) - 8 = 10$$

So, we were correct. Note that we usually don't bother with the verification of this point.

Okay, it's time to sketch the graph of the parabola. Here it is.



Note that we included the axis of symmetry in this graph and typically we won't. It was just included here since we were discussing it earlier.

(b) $g(x) = -(x-2)^2 - 1$

Okay with this one we won't put in quite a much detail. First let's notice that $a = -1$ which is negative and so we know that this parabola will open downward.

Next, by comparing our function to the general form we see that the vertex of this parabola is $(2, -1)$. Again, be careful to get the signs correct here!

Now let's get the *y*-intercept.

$$g(0) = -(0-2)^2 - 1 = -(-2)^2 - 1 = -4 - 1 = -5$$

The *y*-intercept is then $(0,-5)$.

Now, we know that the vertex starts out below the *x*-axis and the parabola opens down. This means that there can't possibly be *x*-intercepts since the *x* axis is above the vertex and the parabola will always open down. This means that there is no reason, in general, to go through the solving process to find what won't exist.

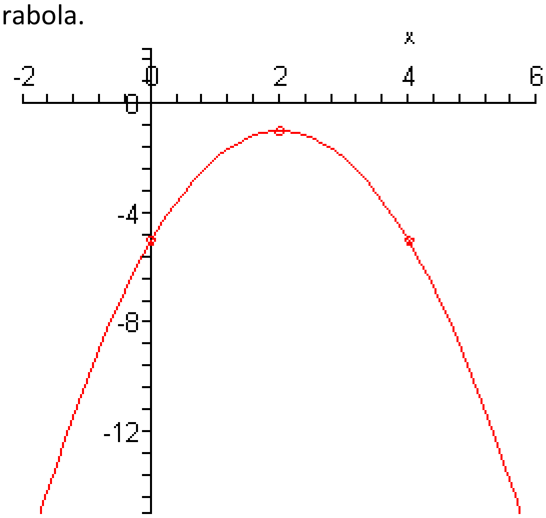
However, let's do it anyway. This will show us what to look for if we don't catch right away that they won't exist from the vertex and direction the parabola opens. We'll need to solve,

$$\begin{aligned} 0 &= -(x-2)^2 - 1 \\ (x-2)^2 &= -1 \\ x-2 &= \pm i \\ x &= 2 \pm i \end{aligned}$$

So, we got complex solutions. Complex solutions will always indicate no *x*-intercepts.

Now, we do want points on either side of the vertex so we'll use the *y*-intercept and the axis of symmetry to get a second point. The *y*-intercept is a distance of two to the left of the axis of symmetry and is at $y = -5$ and so there must be a second point at the same *y* value only a distance of 2 to the right of the axis of symmetry. The coordinates of this point must then be $(4,-5)$.

Here is the sketch of this parabola.



(c) $h(x) = x^2 + 4$

This one is actually a fairly simple one to graph. We'll first notice that it will open upwards.

Now, the vertex is probably the point where most students run into trouble here. Since we have x^2 by itself this means that we must have $h = 0$ and so the vertex is $(0,4)$.

Note that this means there will not be any *x*-intercepts with this parabola since the vertex is above the *x*-axis and the parabola opens upwards.

Next, the *y*-intercept is,

$$h(0) = (0)^2 + 4 = 4 \qquad\qquad y - \text{intercept}: \ (0,4)$$

The *y*-intercept is exactly the same as the vertex. This will happen on occasion so we shouldn't get too worried about it when that happens. Although this will mean that we aren't going to be able to use the *y*-intercept to find a second point on the other side of the vertex this time. In fact, we don't even have a point yet that isn't the vertex!

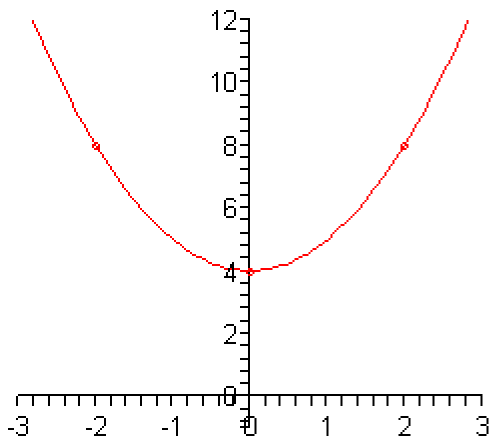
So, we'll need to find a point on either side of the vertex. In this case since the function isn't too bad we'll just plug in a couple of points.

$$h(-2) = (-2)^2 + 4 = 8 \qquad \Rightarrow \qquad (-2,8)$$

$$h(2) = (2)^2 + 4 = 8 \qquad \Rightarrow \qquad (2,8)$$

Note that we could have gotten the second point here using the axis of symmetry if we'd wanted to.

Here is a sketch of the graph.



Okay, we've seen some examples now of this form of the parabola. However, as noted earlier most parabolas are not given in that form. So, we need to take a look at how to graph a parabola that is in the general form.

$$f(x) = ax^2 + bx + c$$

In this form the sign of *a* will determine whether or not the parabola will open upwards or downwards just as it did in the previous set of examples. Unlike the previous form we will not get the vertex for free this time. However, it is will easy to find. Here is the vertex for a parabola in the general form.

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

To get the vertex all we do is compute the *x* coordinate from *a* and *b* and then plug this into the function to get the *y* coordinate. Not quite as simple as the previous form, but still not all that difficult.

Note as well that we will get the *y*-intercept for free from this form. The *y*-intercept is,

$$f\left(0\right) = a\left(0\right)^2 + b\left(0\right) + c = c \qquad \Rightarrow \qquad \left(0, c\right)$$

so we won't need to do any computations for this one.

Let's graph some parabolas.

Example 2 Sketch the graph of each of the following parabolas.

(a) $g\left(x\right) = 3x^2 - 6x + 5$

(b) $f\left(x\right) = -x^2 + 8x$

(c) $f\left(x\right) = x^2 + 4x + 4$

Solution

(a) For this parabola we've got $a = 3$, $b = -6$ and $c = 5$. Make sure that you're careful with signs when identifying these values. So we know that this parabola will open up since *a* is positive.

Here are the evaluations for the vertex.

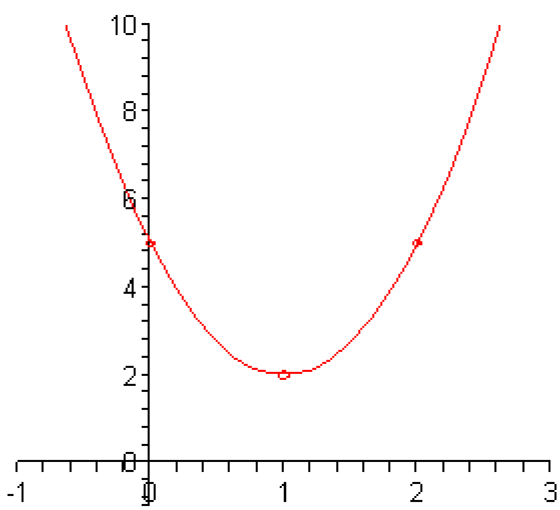
$$x = -\frac{-6}{2\left(3\right)} = -\frac{-6}{6} = 1$$

$$y = g\left(1\right) = 3\left(1\right)^2 - 6\left(1\right) + 5 = 3 - 6 + 5 = 2$$

The vertex is then $\left(1, 2\right)$. Now at this point we also know that there won't be any *x*-intercepts for this parabola since the vertex is above the *x*-axis and it opens upward.

The *y*-intercept is $\left(0, 5\right)$ and using the axis of symmetry we know that $\left(2, 5\right)$ must also be on the parabola.

Here is a sketch of the parabola.



(b) In this case $a = -1$, $b = 8$ and $c = 0$. From these we see that the parabola will open downward since *a* is negative. Here are the vertex evaluations.

$$x = -\frac{8}{2(-1)} = -\frac{8}{-2} = 4$$

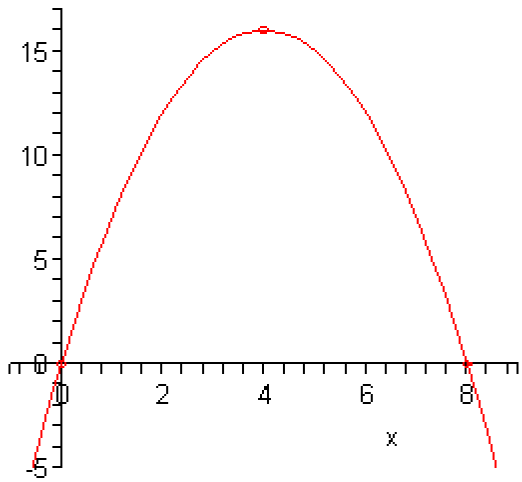
$$y = f(4) = -(4)^2 + 8(4) = 16$$

So, the vertex is $(4,16)$ and we also can see that this time there will be *x*-intercepts. In fact, let's go ahead and find them now.

$$\begin{aligned} 0 &= -x^2 + 8x \\ 0 &= x(-x+8) \qquad \Rightarrow \qquad x = 0,\ x = 8 \end{aligned}$$

So, the *x*-intercepts are $(0,0)$ and $(8,0)$. Notice that $(0,0)$ is also the *y*-intercept. This will happen on occasion so don't get excited about it when it does.

At this point we've got all the information that we need in order to sketch the graph so here it is,



(c) In this final part we have $a = 1$, $b = 4$ and $c = 4$. So, this parabola will open up.

Here are the vertex evaluations.

$$x = -\frac{4}{2(1)} = -\frac{4}{2} = -2$$

$$y = f(-2) = (-2)^2 + 4(-2) + 4 = 0$$

So, the vertex is $(-2,0)$. Note that since the *y* coordinate of this point is zero it is also an *x*-intercept. In fact it will be the only *x*-intercept for this graph. This makes sense if we consider the fact that the vertex, in this case, is the lowest point on the graph and so the graph simply can't touch the *x*-axis anywhere else.

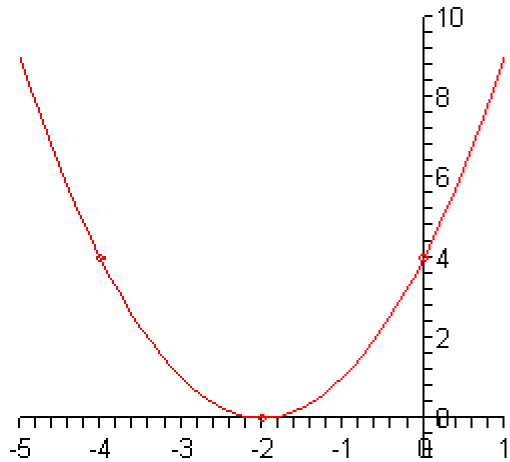
The fact that this parabola has only one *x*-intercept can be verified by solving as we've done in the other examples to this point.

$$0 = x^2 + 4x + 4$$
$$0 = (x+2)^2 \qquad \Rightarrow \qquad x = -2$$

Sure enough there is only one *x*-intercept. Note that this will mean that we're going to have to use the axis of symmetry to get a second point from the *y*-intercept in this case.

Speaking of which, the *y*-intercept in this case is $(0,4)$. This means that the second point is $(-4,4)$.

Here is a sketch of the graph.



As a final topic in this section we need to briefly talk about how to take a parabola in the general form and convert it into the form

$$f(x) = a(x-h)^2 + k$$

This will use a modified completing the square process. It's probably best to do this with an example.

Example 3 Convert each of the following into the form $f(x) = a(x-h)^2 + k$.

(a) $f(x) = 2x^2 - 12x + 3$

(b) $f(x) = -x^2 + 10x - 1$

Solution

Okay, as we pointed out above we are going to complete the square here. However, it is a slightly different process than the other times that we've seen it to this point.

(a) The thing that we've got to remember here is that we must have a coefficient of 1 for the *x*² term in order to complete the square. So, to get that we will first factor the coefficient of the *x*² term out of the whole right side as follows.

$$f(x) = 2\left(x^2 - 6x + \frac{3}{2}\right)$$

Note that this will often put fractions into the problem that is just something that we'll need to be able to deal with. Also note that if we're lucky enough to have a coefficient of 1 on the x^2 term we won't have to do this step.

Now, this is where the process really starts differing from what we've seen to this point. We still take one-half the coefficient of x and square it. However, instead of adding this to both sides we do the following with it.

$$\left(-\frac{6}{2}\right)^2 = (-3)^2 = 9$$

$$f(x) = 2\left(x^2 - 6x + 9 - 9 + \frac{3}{2}\right)$$

We add and subtract this quantity inside the parenthesis as shown. Note that all we are really doing here is adding in zero since 9-9=0! The order listed here is important. We MUST add first and then subtract.

The next step is to factor the first three terms and combine the last two as follows.

$$f(x) = 2\left((x-3)^2 - \frac{15}{2}\right)$$

As a final step we multiply the 2 back through.

$$f(x) = 2(x-3)^2 - 15$$

And there we go.

(b) Be careful here. We don't have a coefficient of 1 on the x^2 term, we've got a coefficient of -1. So, the process is identical outside of that so we won't put in as much detail this time.

$$\begin{aligned} f(x) &= -\left(x^2 - 10x + 1\right) \qquad\qquad \left(-\frac{10}{2}\right)^2 = (-5)^2 = 25 \\ &= -\left(x^2 - 10x + 25 - 25 + 1\right) \\ &= -\left((x-5)^2 - 24\right) \\ &= -(x-5)^2 + 24 \end{aligned}$$

Section 4-3 : Ellipses

In a previous section we looked at graphing circles and since circles are really special cases of ellipses we've already got most of the tools under our belts to graph ellipses. All that we really need here to get us started is then **standard form** of the ellipse and a little information on how to interpret it.

Here is the standard form of an ellipse.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Note that the right side MUST be a 1 in order to be in standard form. The point (h,k) is called the **center** of the ellipse.

To graph the ellipse all that we need are the right most, left most, top most and bottom most points. Once we have those we can sketch in the ellipse. Here are formulas for finding these points.

$$\begin{aligned}&\text{right most point}: (h+a,k)\\&\text{left most point}: (h-a,k)\\&\text{top most point}: (h,k+b)\\&\text{bottom most point}: (h,k-b)\end{aligned}$$

Note that *a* is the square root of the number under the *x* term and is the amount that we move right and left from the center. Also, *b* is the square root of the number under the *y* term and is the amount that we move up or down from the center.

Let's sketch some graphs.

Example 1 Sketch the graph of each of the following ellipses.

(a) $\frac{(x+2)^2}{9}+\frac{(y-4)^2}{25}=1$

(b) $\frac{x^2}{49}+\frac{(y-3)^2}{4}=1$

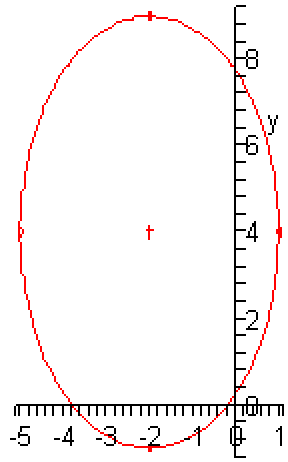
(c) $4(x+1)^2+(y+3)^2=1$

Solution

(a) So, the center of this ellipse is $(-2,4)$ and as usual be careful with signs here! Also, we have $a=3$ and $b=5$. So, the points are,

$$\begin{aligned}&\text{right most point}: (1,4)\\&\text{left most point}: (-5,4)\\&\text{top most point}: (-2,9)\\&\text{bottom most point}: (-2,-1)\end{aligned}$$

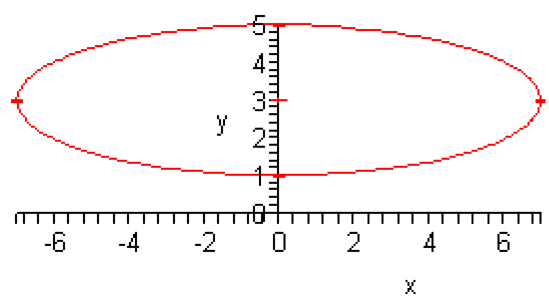
Here is a sketch of this ellipse.



(b) The center for this part is $(0,3)$ and we have $a = 7$ and $b = 2$. The points we need are,

$$\text{right most point}: (7,3)$$
$$\text{left most point}: (-7,3)$$
$$\text{top most point}: (0,5)$$
$$\text{bottom most point}: (0,1)$$

Here is the sketch of this ellipse.



(c) Now with this ellipse we're going to have to be a little careful as it isn't quite in standard form yet. Here is the standard form for this ellipse.

$$\frac{(x+1)^2}{\frac{1}{4}} + (y+3)^2 = 1$$

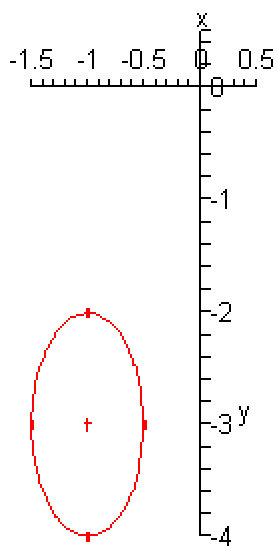
Note that in order to get the coefficient of 4 in the numerator of the first term we will need to have a $\frac{1}{4}$ in the denominator. Also, note that we don't even have a fraction for the *y* term. This implies that there is in fact a 1 in the denominator. We could put this in if it would be helpful to see what is going on here.

$$\frac{(x+1)^2}{\frac{1}{4}} + \frac{(y+3)^2}{1} = 1$$

So, in this form we can see that the center is $(-1,-3)$ and that $a = \frac{1}{2}$ and $b = 1$. The points for this ellipse are,

$$\text{right most point : } \left(-\frac{1}{2},-3\right)$$
$$\text{left most point : } \left(-\frac{3}{2},-3\right)$$
$$\text{top most point : } (-1,-2)$$
$$\text{bottom most point : } (-1,-4)$$

Here is this ellipse.



Finally, let's address a comment made at the start of this section. We said that circles are really nothing more than a special case of an ellipse. To see this let's assume that $a = b$. In this case we have,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$$

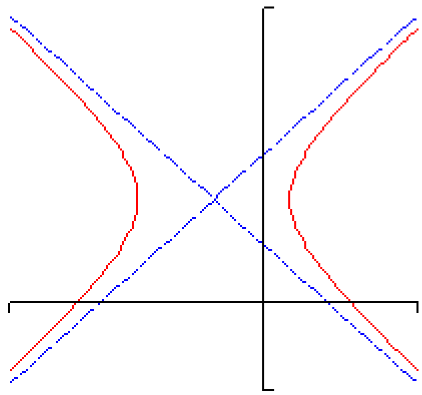
Note that we acknowledged that $a = b$ and used *a* in both cases. Now if we clear denominators we get,

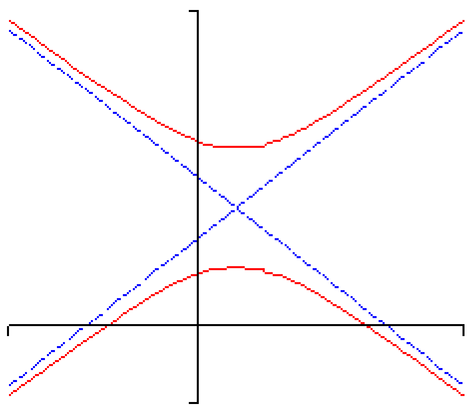
$$\left(x-h\right)^2+\left(y-k\right)^2=a^2$$

This is the standard form of a circle with center $\left(h,k\right)$ and radius *a*. So, circles really are special cases of ellipses.

Section 4-4 : Hyperbolas

The next graph that we need to look at is the hyperbola. There are two basic forms of a hyperbola. Here are examples of each.





Hyperbolas consist of two vaguely parabola shaped pieces that open either up and down or right and left. Also, just like parabolas each of the pieces has a vertex. Note that they aren't really parabolas, they just resemble parabolas.

There are also two lines on each graph. These lines are called asymptotes and as the graphs show as we make x large (in both the positive and negative sense) the graph of the hyperbola gets closer and closer to the asymptotes. The asymptotes are not officially part of the graph of the hyperbola. However, they are usually included so that we can make sure and get the sketch correct. The point where the two asymptotes cross is called the center of the hyperbola.

There are two **standard forms** of the hyperbola, one for each type shown above. Here is a table giving each form as well as the information we can get from each one.

Form	$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$	$\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$
Center	(h,k)	(h,k)
Opens	Opens left and right	Opens up and down
Vertices	$(h+a,k)$ and $(h-a,k)$	$(h,k+b)$ and $(h,k-b)$
Slope of Asymptotes	$\pm\dfrac{b}{a}$	$\pm\dfrac{b}{a}$
Equations of Asymptotes	$y=k\pm\dfrac{b}{a}(x-h)$	$y=k\pm\dfrac{b}{a}(x-h)$

Note that the difference between the two forms is which term has the minus sign. If the *y* term has the minus sign then the hyperbola will open left and right. If the *x* term has the minus sign then the hyperbola will open up and down.

We got the equations of the asymptotes by using the point-slope form of the line and the fact that we know that the asymptotes will go through the center of the hyperbola.

Let's take a look at a couple of these.

Example 1 Sketch the graph of each of the following hyperbolas.

(a) $\dfrac{(x-3)^2}{25}-\dfrac{(y+1)^2}{49}=1$

(b) $\dfrac{y^2}{9}-(x+2)^2=1$

Solution

(a) Now, notice that the *y* term has the minus sign and so we know that we're in the first column of the table above and that the hyperbola will be opening left and right.

The first thing that we should get is the center since pretty much everything else is built around that. The center in this case is $(3,-1)$ and as always watch the signs! Once we have the center we can get the vertices. These are $(8,-1)$ and $(-2,-1)$.

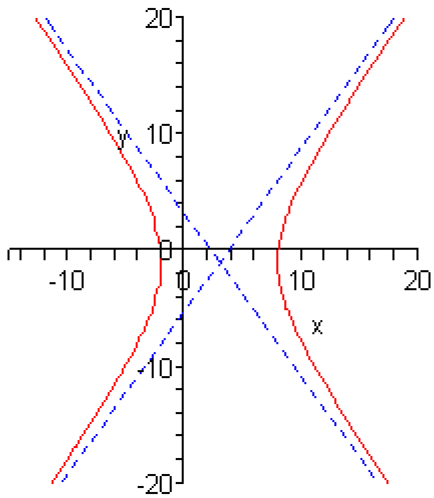
Next, we should get the slopes of the asymptotes. These are always the square root of the number under the *y* term divided by the square root of the number under the *x* term and there will always be a positive and a negative slope. The slopes are then $\pm\dfrac{7}{5}$.

Now that we've got the center and the slopes of the asymptotes we can get the equations for the asymptotes. They are,

$$y = -1 + \frac{7}{5}(x-3) \qquad \text{and} \qquad y = -1 - \frac{7}{5}(x-3)$$

We can now start the sketching. We start by sketching the asymptotes and the vertices. Once these are done we know what the basic shape should look like so we sketch it in making sure that as *x* gets large we move in closer and closer to the asymptotes.

Here is the sketch for this hyperbola.



(b) In this case the hyperbola will open up and down since the *x* term has the minus sign. Now, the center of this hyperbola is $(-2,0)$. Remember that since there is a y^2 term by itself we had to have $k=0$. At this point we also know that the vertices are $(-2,3)$ and $(-2,-3)$.

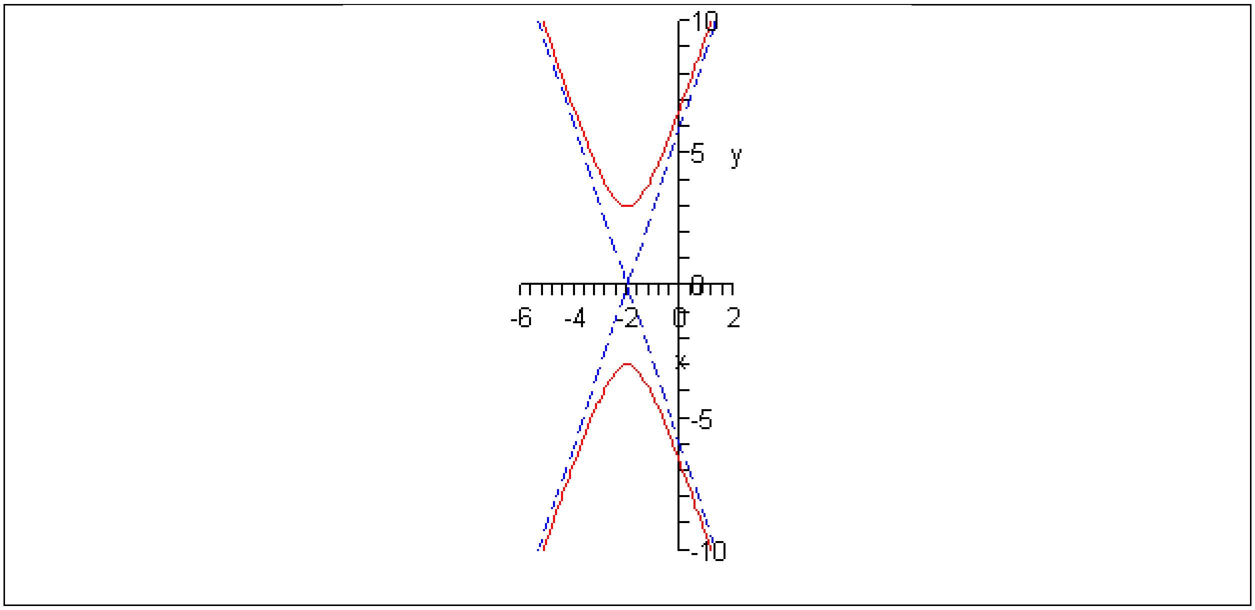
In order to see the slopes of the asymptotes let's rewrite the equation a little.

$$\frac{y^2}{9} - \frac{(x+2)^2}{1} = 1$$

So, the slopes of the asymptotes are $\pm\frac{3}{1} = \pm 3$. The equations of the asymptotes are then,

$$y = 0 + 3(x+2) = 3x+6 \qquad \text{and} \qquad y = 0 - 3(x+2) = -3x-6$$

Here is the sketch of this hyperbola.


10
5
y
0
-5
-10
-6
-4
-2
0
2
x

Section 4-5 : Miscellaneous Functions

The point of this section is to introduce you to some other functions that don't really require the work to graph that the ones that we've looked at to this point in this chapter. For most of these all that we'll need to do is evaluate the function as some *x*'s and then plot the points.

Constant Function

This is probably the easiest function that we'll ever graph and yet it is one of the functions that tend to cause problems for students.

The most general form for the constant function is,

$$f\left(x\right)=c$$

where *c* is some number.

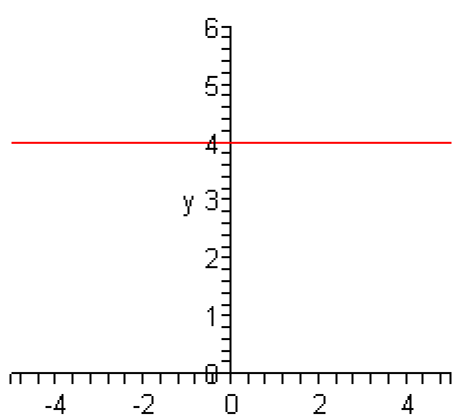
Let's take a look at $f\left(x\right)=4$ so we can see what the graph of constant functions look like. Probably the biggest problem students have with these functions is that there are no *x*'s on the right side to plug into for evaluation. However, all that means is that there is no substitution to do. In other words, no matter what *x* we plug into the function we will always get a value of 4 (or *c* in the general case) out of the function.

So, every point has a *y* coordinate of 4. This is exactly what defines a horizontal line. In fact, if we recall that $f\left(x\right)$ is nothing more than a fancy way of writing *y* we can rewrite the function as,

$$y=4$$

And this is exactly the equation of a horizontal line.

Here is the graph of this function.



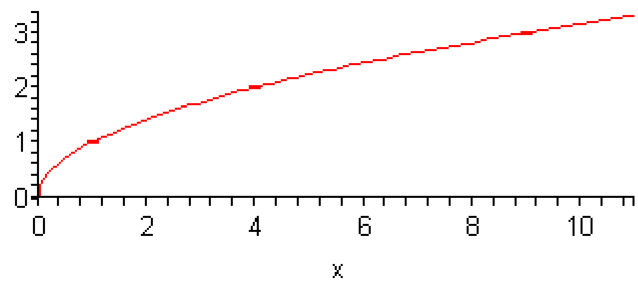
Square Root

Next, we want to take a look at $f(x) = \sqrt{x}$. First, note that since we don't want to get complex numbers out of a function evaluation we have to restrict the values of *x* that we can plug in. We can only plug in value of *x* in the range $x \geq 0$. This means that our graph will only exist in this range as well.

To get the graph we'll just plug in some values of *x* and then plot the points.

x	*f(x)*
0	0
1	1
4	2
9	3

The graph is then,



Absolute Value

We've dealt with this function several times already. It's now time to graph it. First, let's remind ourselves of the definition of the absolute value function.

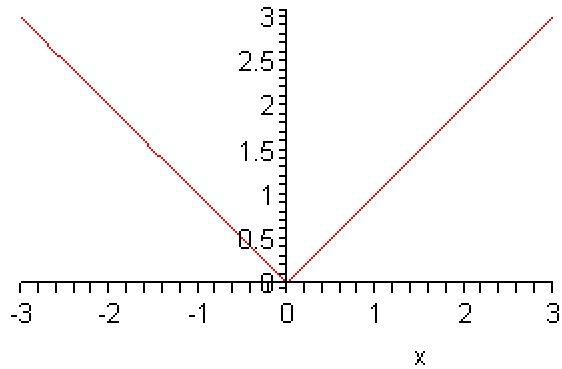
$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is a piecewise function and we've seen how to graph these already. All that we need to do is get some points in both ranges and plot them.

Here are some function evaluations.

x	*f(x)*
0	0
1	1
-1	1
2	2
-2	2

Here is the graph of this function.



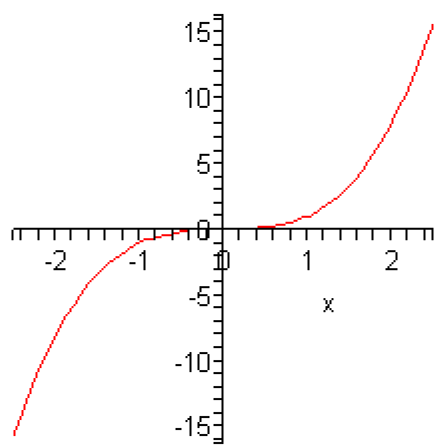
So, this is a "V" shaped graph.

Cubic Function

We're not actually going to look at a general cubic polynomial here. We'll do some of those in the next chapter. Here we are only going to look at $f(x) = x^3$. There really isn't much to do here other than just plugging in some points and plotting.

x	*f(x)*
0	0
1	1
-1	-1
2	8
-2	-8

Here is the graph of this function.



We will need some of these in the next section so make sure that you can identify these when you see them and can sketch their graphs fairly quickly.

Section 4- 6 : Transformations

In this section we are going to see how knowledge of some fairly simple graphs can help us graph some more complicated graphs. Collectively the methods we're going to be looking at in this section are called **transformations**.

Vertical Shifts

The first transformation we'll look at is a vertical shift.

Given the graph of $f(x)$ the graph of $g(x) = f(x) + c$ will be the graph of $f(x)$ shifted up by *c* units if *c* is positive and or down by *c* units if *c* is negative.

So, if we can graph $f(x)$ getting the graph of $g(x)$ is fairly easy. Let's take a look at a couple of examples.

Example 1 Using transformations sketch the graph of the following functions.

(a) $g(x) = x^2 + 3$

(b) $f(x) = \sqrt{x} - 5$

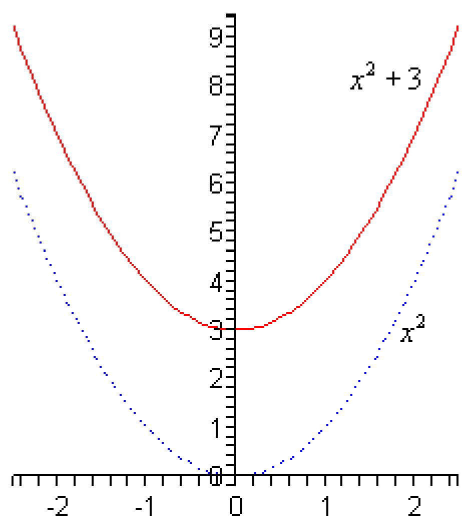
Solution

The first thing to do here is graph the function without the constant which by this point should be fairly simple for you. Then shift accordingly.

(a) $g(x) = x^2 + 3$

In this case we first need to graph x^2 (the dotted line on the graph below) and then pick this up and shift it upwards by 3. Coordinate wise this will mean adding 3 onto all the *y* coordinates of points on x^2.

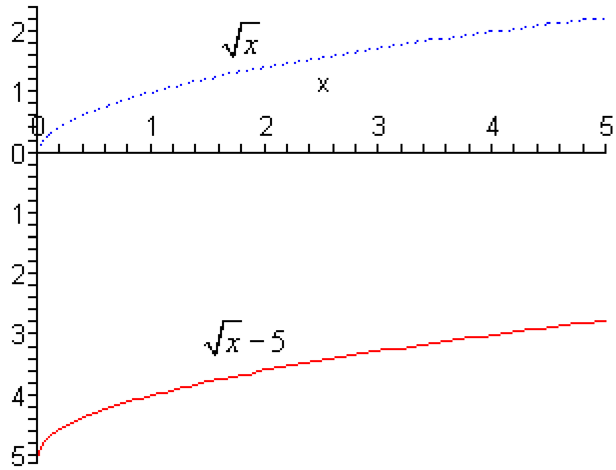
Here is the sketch for this one.



(b) $f(x) = \sqrt{x} - 5$

Okay, in this case we're going to be shifting the graph of $\sqrt{x}$ (the dotted line on the graph below) down by 5. Again, from a coordinate standpoint this means that we subtract 5 from the *y* coordinates of points on $\sqrt{x}$.

Here is this graph.



So, vertical shifts aren't all that bad if we can graph the "base" function first. Note as well that if you're not sure that you believe the graphs in the previous set of examples all you need to do is plug a couple values of *x* into the function and verify that they are in fact the correct graphs.

Horizontal Shifts

These are fairly simple as well although there is one bit where we need to be careful.

Given the graph of $f(x)$ the graph of $g(x) = f(x+c)$ will be the graph of $f(x)$ shifted left by *c* units if *c* is positive and or right by *c* units if *c* is negative.

Now, we need to be careful here. A positive *c* shifts a graph in the negative direction and a negative *c* shifts a graph in the positive direction. They are exactly opposite than vertical shifts and it's easy to flip these around and shift incorrectly if we aren't being careful.

Example 2 Using transformations sketch the graph of the following functions.

(a) $h(x) = (x+2)^3$

(b) $g(x) = \sqrt{x-4}$

Solution

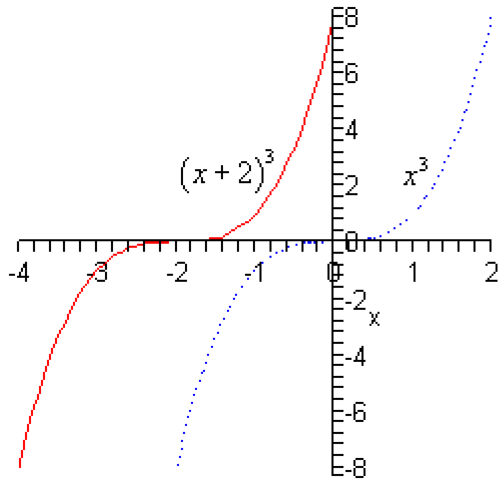
(a) $h(x) = (x+2)^3$

Okay, with these we need to first identify the "base" function. That is the function that's being shifted. In this case it looks like we are shifting $f(x) = x^3$. We can then see that,

$$h(x) = (x+2)^3 = f(x+2)$$

In this case $c = 2$ and so we're going to shift the graph of $f(x) = x^3$ (the dotted line on the graph below) and move it 2 units to the left. This will mean subtracting 2 from the *x* coordinates of all the points on $f(x) = x^3$.

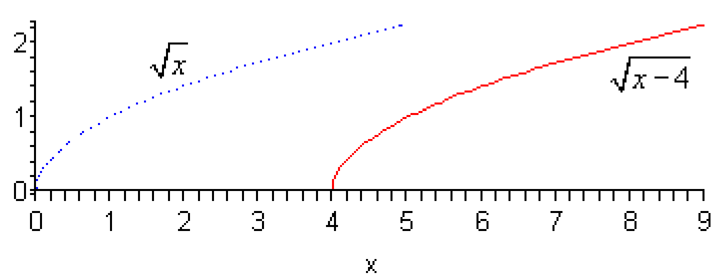
Here is the graph for this problem.



(b) $g(x) = \sqrt{x-4}$

In this case it looks like the base function is $\sqrt{x}$ and it also looks like $c = -4$ and so we will be shifting the graph of $\sqrt{x}$ (the dotted line on the graph below) to the right by 4 units. In terms of coordinates this will mean that we're going to add 4 onto the *x* coordinate of all the points on $\sqrt{x}$.

Here is the sketch for this function.



Vertical and Horizontal Shifts

Now we can also combine the two shifts we just got done looking at into a single problem. If we know the graph of $f(x)$ the graph of $g(x) = f(x+c) + k$ will be the graph of $f(x)$ shifted left or right by *c* units depending on the sign of *c* and up or down by *k* units depending on the sign of *k*.

Let's take a look at a couple of examples.

Example 3 Use transformation to sketch the graph of each of the following.

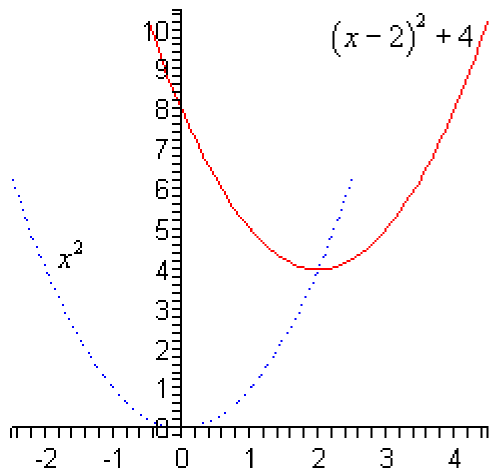
(a) $f(x) = (x-2)^2 + 4$

(b) $g(x) = |x+3| - 5$

Solution

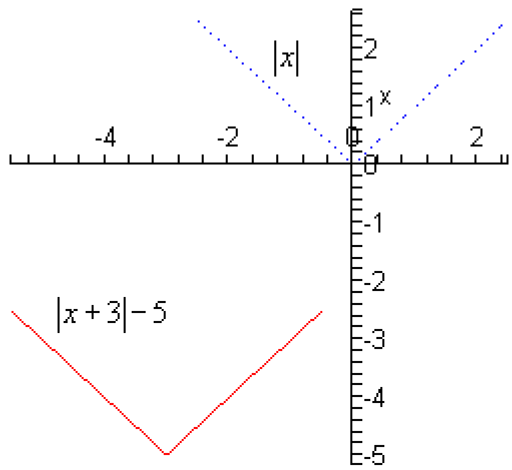
(a) $f(x) = (x-2)^2 + 4$

In this part it looks like the base function is x^2 and it looks like will be shift this to the right by 2 (since $c = -2$) and up by 4 (since $k = 4$). Here is the sketch of this function.



(b) $g(x) = |x+3| - 5$

For this part we will be shifting $|x|$ to the left by 3 (since $c = 3$) and down 5 (since $k = -5$). Here is the sketch of this function.



Reflections

The final set of transformations that we're going to be looking at in this section aren't shifts, but instead they are called reflections and there are two of them.

Reflection about the *x*-axis.

Given the graph of $f(x)$ then the graph of $g(x) = -f(x)$ is the graph of $f(x)$ *reflected* about the *x*-axis. This means that the signs on the all the *y* coordinates are changed to the opposite sign.

Reflection about the *y*-axis.

Given the graph of $f(x)$ then the graph of $g(x) = f(-x)$ is the graph of $f(x)$ *reflected* about the *y*-axis. This means that the signs on the all the *x* coordinates are changed to the opposite sign.

Here is an example of each.

Example 4 Using transformation sketch the graph of each of the following.

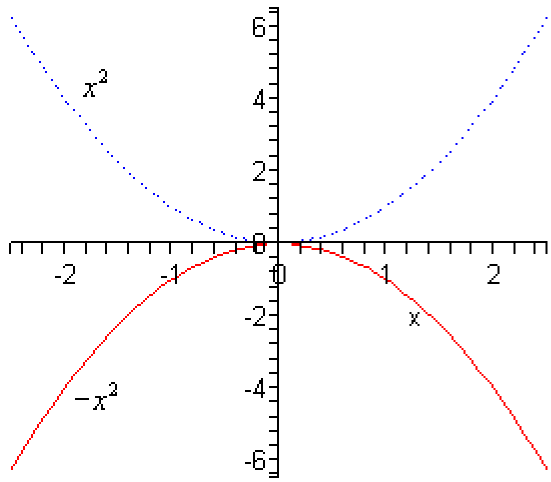
(a) $g(x) = -x^2$

(b) $h(x) = \sqrt{-x}$

Solution

(a) Based on the placement of the minus sign (*i.e.* it's outside the square and NOT inside the square, or $(-x)^2$) it looks like we will be reflecting x^2 about the *x*-axis. So, again, the means that all we do is change the sign on all the *y* coordinates.

Here is the sketch of this graph.



(b) Now with this one let's first address the minus sign under the square root in more general terms. We know that we can't take the square roots of negative numbers, however the presence of that minus sign doesn't necessarily cause problems. We won't be able to plug positive values of *x* into the function since that would give square roots of negative numbers. However, if *x* were negative, then the negative of a negative number is positive and that is okay. For instance,

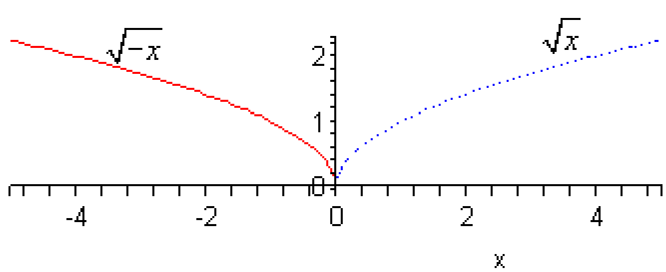
$$h\left(-4\right)=\sqrt{-\left(-4\right)}=\sqrt{4}=2$$

So, don't get all worried about that minus sign.

Now, let's address the reflection here. Since the minus sign is under the square root as opposed to in front of it we are doing a reflection about the *y*-axis. This means that we'll need to change all the signs of points on $\sqrt{x}$.

Note as well that this syncs up with our discussion on this minus sign at the start of this part.

Here is the graph for this function.



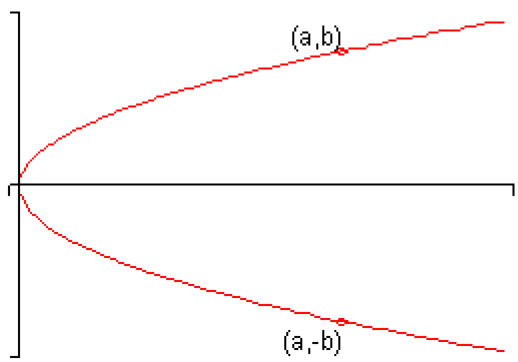
Section 4-7 : Symmetry

In this section we are going to take a look at something that we used back when we were graphing parabolas. However, we're going to take a more general view of it this section. Many graphs have **symmetry** to them.

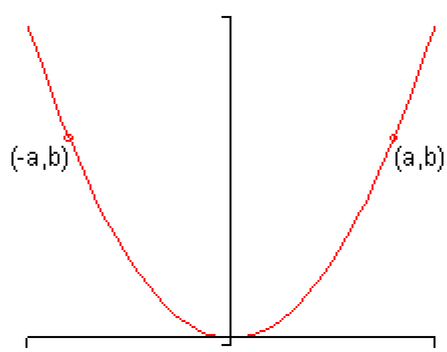
Symmetry can be useful in graphing an equation since it says that if we know one portion of the graph then we will also know the remaining (and symmetric) portion of the graph as well. We used this fact when we were graphing parabolas to get an extra point of some of the graphs.

In this section we want to look at three types of symmetry.

1. A graph is said to be **symmetric about the *x*-axis** if whenever (a,b) is on the graph then so is $(a,-b)$. Here is a sketch of a graph that is symmetric about the *x*-axis.



2. A graph is said to be **symmetric about the *y*-axis** if whenever (a,b) is on the graph then so is $(-a,b)$. Here is a sketch of a graph that is symmetric about the *y*-axis.



3. A graph is said to be **symmetric about the origin** if whenever (a,b) is on the graph then so is $(-a,-b)$. Here is a sketch of a graph that is symmetric about the origin.

Note that most graphs don't have any kind of symmetry. Also, it is possible for a graph to have more than one kind of symmetry. For example, the graph of a circle centered at the origin exhibits all three symmetries.

Tests for Symmetry

We've some fairly simply tests for each of the different types of symmetry.

1. A graph will have symmetry about the x-axis if we get an equivalent equation when all the y's are replaced with $-y$.

2. A graph will have symmetry about the y-axis if we get an equivalent equation when all the x's are replaced with $-x$.

3. A graph will have symmetry about the origin if we get an equivalent equation when all the y's are replaced with $-y$ and all the x's are replaced with $-x$.

We will define just what we mean by an "equivalent equation" when we reach an example of that. For the majority of the examples that we're liable to run across this will mean that it is exactly the same equation.

Let's test a few equations for symmetry. Note that we aren't going to graph these since most of them would actually be fairly difficult to graph. The point of this example is only to use the tests to determine the symmetry of each equation.

Example 1 Determine the symmetry of each of the following equations.

(a) $y = x^2 - 6x^4 + 2$

(b) $y = 2x^3 - x^5$

(c) $y^4 + x^3 - 5x = 0$

(d) $y = x^3 + x^2 + x + 1$

(e) $x^2 + y^2 = 1$

Solution

(a) $y = x^2 - 6x^4 + 2$

We'll first check for symmetry about the *x*-axis. This means that we need to replace all the *y*'s with –*y*. That's easy enough to do in this case since there is only one *y*.

$$-y = x^2 - 6x^4 + 2$$

Now, this is not an equivalent equation since the terms on the right are identical to the original equation and the term on the left is the opposite sign. So, this equation doesn't have symmetry about the *x*-axis.

Next, let's check symmetry about the *y*-axis. Here we'll replace all *x*'s with –*x*.

$$y = (-x)^2 - 6(-x)^4 + 2$$
$$y = x^2 - 6x^4 + 2$$

After simplifying we got exactly the same equation back out which means that the two are equivalent. Therefore, this equation does have symmetry about the *y*-axis.

Finally, we need to check for symmetry about the origin. Here we replace both variables.

$$-y = (-x)^2 - 6(-x)^4 + 2$$
$$-y = x^2 - 6x^4 + 2$$

So, as with the first test, the left side is different from the original equation and the right side is identical to the original equation. Therefore, this isn't equivalent to the original equation and we don't have symmetry about the origin.

(b) $y = 2x^3 - x^5$

We'll not put in quite as much detail here. First, we'll check for symmetry about the *x*-axis.

$$-y = 2x^3 - x^5$$

We don't have symmetry here since the one side is identical to the original equation and the other isn't. So, we don't have symmetry about the *x*-axis.

Next, check for symmetry about the *y*-axis.

$$y = 2(-x)^3 - (-x)^5$$
$$y = -2x^3 + x^5$$

Remember that if we take a negative to an odd power the minus sign can come out in front. So, upon simplifying we get the left side to be identical to the original equation, but the right side is now the opposite sign from the original equation and so this isn't equivalent to the original equation and so we don't have symmetry about the *y*-axis.

Finally, let's check symmetry about the origin.

$$\begin{aligned} -y &= 2\left(-x\right)^3 - \left(-x\right)^5 \\ -y &= -2x^3 + x^5 \end{aligned}$$

Now, this time notice that all the signs in this equation are exactly the opposite form the original equation. This means that it IS equivalent to the original equation since all we would need to do is multiply the whole thing by "-1" to get back to the original equation.

Therefore, in this case we have symmetry about the origin.

(c) $y^4 + x^3 - 5x = 0$
First, check for symmetry about the *x*-axis.

$$\begin{aligned} \left(-y\right)^4 + x^3 - 5x &= 0 \\ y^4 + x^3 - 5x &= 0 \end{aligned}$$

This is identical to the original equation and so we have symmetry about the *x*-axis.

Now, check for symmetry about the *y*-axis.

$$\begin{aligned} y^4 + \left(-x\right)^3 - 5\left(-x\right) &= 0 \\ y^4 - x^3 + 5x &= 0 \end{aligned}$$

So, some terms have the same sign as the original equation and other don't so there isn't symmetry about the *y*-axis.

Finally, check for symmetry about the origin.

$$\begin{aligned} \left(-y\right)^4 + \left(-x\right)^3 - 5\left(-x\right) &= 0 \\ y^4 - x^3 + 5x &= 0 \end{aligned}$$

Again, this is not the same as the original equation and isn't exactly the opposite sign from the original equation and so isn't symmetric about the origin.

(d) $y = x^3 + x^2 + x + 1$
First, symmetry about the *x*-axis.

$$-y = x^3 + x^2 + x + 1$$

It looks like no symmetry about the *x*-axis

Next, symmetry about the *y*-axis.

$$y = (-x)^3 + (-x)^2 + (-x) + 1$$
$$y = -x^3 + x^2 - x + 1$$

So, no symmetry here either.

Finally, symmetry about the origin.

$$-y = (-x)^3 + (-x)^2 + (-x) + 1$$
$$-y = -x^3 + x^2 - x + 1$$

And again, no symmetry here either.

This function has no symmetry of any kind. That's not unusual as most functions don't have any of these symmetries.

(e) $x^2 + y^2 = 1$

Check *x*-axis symmetry first.

$$x^2 + (-y)^2 = 1$$
$$x^2 + y^2 = 1$$

So, it's got symmetry about the *x*-axis symmetry.

Next, check for *y*-axis symmetry.

$$(-x)^2 + y^2 = 1$$
$$x^2 + y^2 = 1$$

Looks like it's also got *y*-axis symmetry.

Finally, symmetry about the origin.

$$(-x)^2 + (-y)^2 = 1$$
$$x^2 + y^2 = 1$$

So, it's also got symmetry about the origin.

Note that this is a circle centered at the origin and as noted when we first started talking about symmetry it does have all three symmetries.

Section 4-8 : Rational Functions

In this final section we need to discuss graphing rational functions. It's is probably best to start off with a fairly simple one that we can do without all that much knowledge on how these work.

Let's sketch the graph of $f\left(x\right)=\frac{1}{x}$. First, since this is a rational function we are going to have to be careful with division by zero issues. So, we can see from this equation that we'll have to avoid $x=0$ since that will give division by zero.

Now, let's just plug in some values of *x* and see what we get.

x	*f(x)*
-4	-0.25
-2	-0.5
-1	-1
-0.1	-10
-0.01	-100
0.01	100
0.1	10
1	1
2	0.5
4	0.25

So, as *x* get large (positively and negatively) the function keeps the sign of *x* and gets smaller and smaller. Likewise, as we approach $x=0$ the function again keeps the same sign as *x* but starts getting quite large. Here is a sketch of this graph.

First, notice that the graph is in two pieces. Almost all rational functions will have graphs in multiple pieces like this.

Next, notice that this graph does not have any intercepts of any kind. That's easy enough to check for ourselves.

Recall that a graph will have a *y*-intercept at the point $\left(0, f\left(0\right)\right)$. However, in this case we have to avoid $x = 0$ and so this graph will never cross the *y*-axis. It does get very close to the *y*-axis, but it will never cross or touch it and so no *y*-intercept.

Next, recall that we can determine where a graph will have *x*-intercepts by solving $f\left(x\right) = 0$. For rational functions this may seem like a mess to deal with. However, there is a nice fact about rational functions that we can use here. A rational function will be zero at a particular value of *x* only if the numerator is zero at that *x* and the denominator isn't zero at that *x*. In other words, to determine if a rational function is ever zero all that we need to do is set the numerator equal to zero and solve. Once we have these solutions we just need to check that none of them make the denominator zero as well.

In our case the numerator is one and will never be zero and so this function will have no *x*-intercepts. Again, the graph will get very close to the *x*-axis but it will never touch or cross it.

Finally, we need to address the fact that graph gets very close to the *x* and *y*-axis but never crosses. Since there isn't anything special about the axis themselves we'll use the fact that the *x*-axis is really the line given by $y = 0$ and the *y*-axis is really the line given by $x = 0$.

In our graph as the value of *x* approaches $x = 0$ the graph starts gets very large on both sides of the line given by $x = 0$. This line is called a **vertical asymptote.**

Also, as *x* get very large, both positive and negative, the graph approaches the line given by $y = 0$. This line is called a **horizontal asymptote**.

Here are the general definitions of the two asymptotes.

1. The line $x = a$ is a **vertical asymptote** if the graph increases or decreases without bound on one or both sides of the line as *x* moves in closer and closer to $x = a$.
2. The line $y = b$ is a **horizontal asymptote** if the graph approaches $y = b$ as *x* increases or decreases without bound. Note that it doesn't have to approach $y = b$ as *x* BOTH increases and decreases. It only needs to approach it on one side in order for it to be a horizontal asymptote.

Determining asymptotes is actually a fairly simple process. First, let's start with the rational function,

$$f\left(x\right) = \frac{ax^n + \cdots}{bx^m + \cdots}$$

where *n* is the largest exponent in the numerator and *m* is the largest exponent in the denominator.

We then have the following facts about asymptotes.

1. The graph will have a vertical asymptote at $x = a$ if the denominator is zero at $x = a$ and the numerator isn't zero at $x = a$.

2. If $n < m$ then the *x*-axis is the horizontal asymptote.
3. If $n = m$ then the line $y = \dfrac{a}{b}$ is the horizontal asymptote.
4. If $n > m$ there will be no horizontal asymptotes.

The process for graphing a rational function is fairly simple. Here it is.

Process for Graphing a Rational Function

1. Find the intercepts, if there are any. Remember that the *y*-intercept is given by $\left(0, f\left(0\right)\right)$ and we find the *x*-intercepts by setting the numerator equal to zero and solving.

2. Find the vertical asymptotes by setting the denominator equal to zero and solving.

3. Find the horizontal asymptote, if it exits, using the fact above.

4. The vertical asymptotes will divide the number line into regions. In each region graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if we need to we should get several points to determine the general shape of the graph.

5. Sketch the graph.

Note that the sketch that we'll get from the process is going to be a fairly rough sketch but that is okay. That's all that we're really after is a basic idea of what the graph will look at.

Let's take a look at a couple of examples.

Example 1 Sketch the graph of the following function.

$$f\left(x\right) = \frac{3x+6}{x-1}$$

Solution

So, we'll start off with the intercepts. The *y*-intercept is,

$$f\left(0\right) = \frac{6}{-1} = -6 \qquad \Rightarrow \qquad \left(0,-6\right)$$

The *x*-intercepts will be,

$$\begin{aligned} 3x + 6 &= 0 \\ x &= -2 \qquad \Rightarrow \qquad \left(-2,0\right) \end{aligned}$$

Now, we need to determine the asymptotes. Let's first find the vertical asymptotes.

$$x-1=0 \qquad \Rightarrow \qquad x=1$$

So, we've got one vertical asymptote. This means that there are now two regions of *x*'s. They are $x<1$ and $x>1$.

Now, the largest exponent in the numerator and denominator is 1 and so by the fact there will be a horizontal asymptote at the line.

$$y=\frac{3}{1}=3$$

Now, we just need points in each region of *x*'s. Since the *y*-intercept and *x*-intercept are already in the left region we won't need to get any points there. That means that we'll just need to get a point in the right region. It doesn't really matter what value of *x* we pick here we just need to keep it fairly small so it will fit onto our graph.

$$f\left(2\right)=\frac{3\left(2\right)+6}{2-1}=\frac{12}{1}=12 \qquad \Rightarrow \qquad \left(2,12\right)$$

Okay, putting all this together gives the following graph.

Note that the asymptotes are shown as dotted lines.

Example 2 Sketch the graph of the following function.

$$f\left(x\right)=\frac{9}{x^2-9}$$

Solution

Okay, we'll start with the intercepts. The *y*-intercept is,

$$f\left(0\right)=\frac{9}{-9}=-1 \qquad \Rightarrow \qquad \left(0,-1\right)$$

The numerator is a constant and so there won't be any *x*-intercepts since the function can never be zero.

Next, we'll have vertical asymptotes at,

$$x^2 - 9 = 0 \qquad \Rightarrow \qquad x = \pm 3$$

So, in this case we'll have three regions to our graph : $x < -3$, $-3 < x < 3$, $x > 3$.

Also, the largest exponent in the denominator is 2 and since there are no *x*'s in the numerator the largest exponent is 0, so by the fact the *x*-axis will be the horizontal asymptote.

Finally, we need some points. We'll use the following points here.

$$f(-4) = \frac{9}{7} \qquad \left(-4, \frac{9}{7}\right)$$

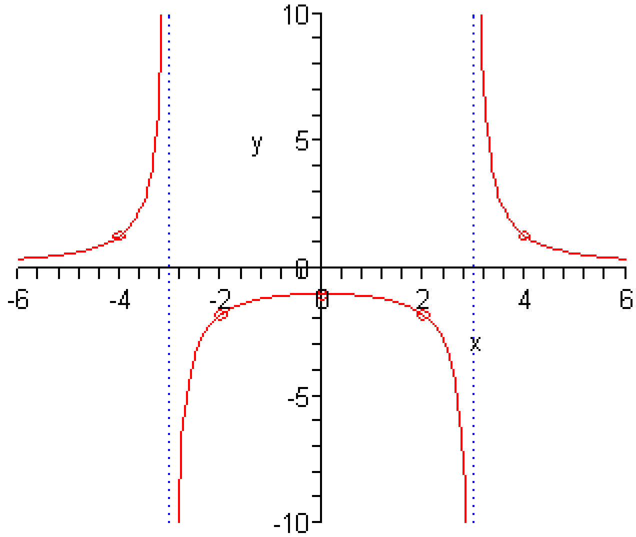
$$f(-2) = -\frac{9}{5} \qquad \left(-2, -\frac{9}{5}\right)$$

$$f(2) = -\frac{9}{5} \qquad \left(2, -\frac{9}{5}\right)$$

$$f(4) = \frac{9}{7} \qquad \left(4, \frac{9}{7}\right)$$

Notice that along with the *y*-intercept we actually have three points in the middle region. This is because there are a couple of possible behaviors in this region and we'll need to determine the actual behavior. We'll see the other main behaviors in the next examples and so this will make more sense at that point.

Here is the sketch of the graph.



Example 3 Sketch the graph of the following function.

$$f(x) = \frac{x^2 - 4}{x^2 - 4x}$$

Solution

This time notice that if we were to plug in $x = 0$ into the denominator we would get division by zero. This means there will not be a *y*-intercept for this graph. We have however, managed to find a vertical asymptote already.

Now, let's see if we've got *x*-intercepts.

$$x^2 - 4 = 0 \qquad \Rightarrow \qquad x = \pm 2$$

So, we've got two of them.

We've got one vertical asymptote, but there may be more so let's go through the process and see.

$$x^2 - 4x = x\left(x-4\right) = 0 \qquad \Rightarrow \qquad x = 0,\ x = 4$$

So, we've got two again and the three regions that we've got are $x < 0$, $0 < x < 4$ and $x > 4$.

Next, the largest exponent in both the numerator and denominator is 2 so by the fact there will be a horizontal asymptote at the line,

$$y = \frac{1}{1} = 1$$

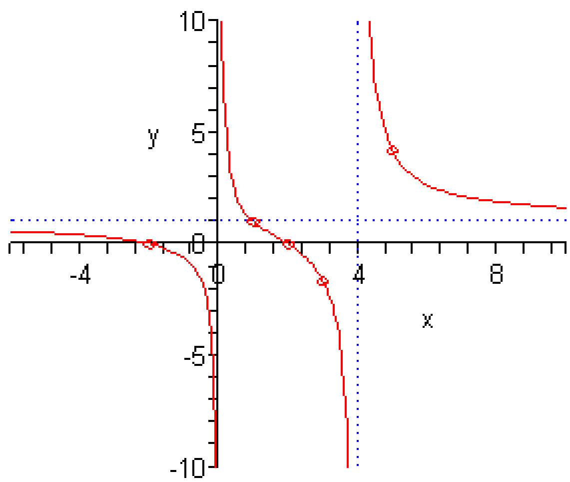
Now, one of the *x*-intercepts is in the far left region so we don't need any points there. The other *x*-intercept is in the middle region. So, we'll need a point in the far right region and as noted in the previous example we will want to get a couple more points in the middle region to completely determine its behavior.

$$f\left(1\right) = 1 \qquad \left(1,1\right)$$

$$f\left(3\right) = -\frac{5}{3} \qquad \left(3,-\frac{5}{3}\right)$$

$$f\left(5\right) = \frac{21}{5} \qquad \left(5,\frac{21}{5}\right)$$

Here is the sketch for this function.



Notice that this time the middle region doesn't have the same behavior at the asymptotes as we saw in the previous example. This can and will happen fairly often. Sometimes the behavior at the two asymptotes will be the same as in the previous example and sometimes it will have the opposite behavior at each asymptote as we see in this example. Because of this we will always need to get a couple of points in these types of regions to determine just what the behavior will be.

Chapter 5 : Polynomial Functions

In this chapter we are going to take a more in depth look at polynomials. We've already solved and graphed second degree polynomials (*i.e.* quadratic equations/functions) and we now want to extend things out to more general polynomials. We will take a look at finding solutions to higher degree polynomials and how to get a rough sketch for a higher degree polynomial.

We will also be looking at Partial Fractions in this chapter. It doesn't really have anything to do with graphing polynomials but needed to be put somewhere and this chapter seemed like as good a place as any.

Here is a brief listing of the material in this chapter.

Dividing Polynomials – In this section we'll review some of the basics of dividing polynomials. We will define the remainder and divisor used in the division process and introduce the idea of synthetic division. We will also give the Division Algorithm.
Zeroes/Roots of Polynomials – In this section we'll define the zero or root of a polynomial and whether or not it is a simple root or has multiplicity k. We will also give the Fundamental Theorem of Algebra and The Factor Theorem as well as a couple of other useful Facts.
Graphing Polynomials – In this section we will give a process that will allow us to get a rough sketch of the graph of some polynomials. We discuss how to determine the behavior of the graph at x-intercepts and the leading coefficient test to determine the behavior of the graph as we allow x to increase and decrease without bound.
Finding Zeroes of Polynomials – As we saw in the previous section in order to sketch the graph of a polynomial we need to know what it's zeroes are. However, if we are not able to factor the polynomial we are unable to do that process. So, in this section we'll look at a process using the Rational Root Theorem that will allow us to find some of the zeroes of a polynomial and in special cases all of the zeroes.
Partial Fractions – In this section we will take a look at the process of partial fractions and finding the partial fraction decomposition of a rational expression. What we will be asking here is what "smaller" rational expressions did we add and/or subtract to get the given rational expression. This is a process that has a lot of uses in some later math classes. It can show up in Calculus and Differential Equations for example.

Section 5-1 : Dividing Polynomials

In this section we're going to take a brief look at dividing polynomials. This is something that we'll be doing off and on throughout the rest of this chapter and so we'll need to be able to do this.

Let's do a quick example to remind us how long division of polynomials works.

Example 1 Divide $5x^3 - x^2 + 6$ by $x - 4$.

Solution

Let's first get the problem set up.

$$x-4\overline{)5x^3 - x^2 + 0x + 6}$$

Recall that we need to have the terms written down with the exponents in decreasing order and to make sure we don't make any mistakes we add in any missing terms with a zero coefficient.

Now we ask ourselves what we need to multiply $x-4$ to get the first term in first polynomial. In this case that is $5x^2$. So multiply $x-4$ by $5x^2$ and subtract the results from the first polynomial.

$$\begin{array}{r l} & 5x^2 \\ x-4 & \overline{)5x^3 - x^2 + 0x + 6} \\ & \underline{-\left(5x^3 - 20x^2\right)} \\ & \qquad\qquad 19x^2 + 0x + 6 \end{array}$$

The new polynomial is called the **remainder**. We continue the process until the degree of the remainder is less than the degree of the **divisor**, which is $x-4$ in this case. So, we need to continue until the degree of the remainder is less than 1.

Recall that the **degree** of a polynomial is the highest exponent in the polynomial. Also, recall that a constant is thought of as a polynomial of degree zero. Therefore, we'll need to continue until we get a constant in this case.

Here is the rest of the work for this example.

$$\begin{array}{r} 5x^2+19x+76 \\ x-4\overline{\smash{\big)}5x^3-x^2+0x+6} \\ -\underline{\left(5x^3-20x^2\right)} \\ 19x^2+0x+6 \\ -\underline{\left(19x^2-76x\right)} \\ 76x+6 \\ -\underline{\left(76x-304\right)} \\ 310 \end{array}$$

Okay, now that we've gotten this done, let's remember how we write the actual answer down. The answer is,

$$\frac{5x^3-x^2+6}{x-4}=5x^2+19x+76+\frac{310}{x-4}$$

There is actually another way to write the answer from the previous example that we're going to find much more useful, if for no other reason that it's easier to write down. If we multiply both sides of the answer by $x-4$ we get,

$$5x^3-x^2+6=\left(x-4\right)\left(5x^2+19x+76\right)+310$$

In this example we divided the polynomial by a linear polynomial in the form of $x-r$ and we will be restricting ourselves to only these kinds of problems. Long division works for much more general division, but these are the kinds of problems we are going to seeing the later sections.

In fact, we will be seeing these kinds of divisions so often that we'd like a quicker and more efficient way of doing them. Luckily there is something out there called **synthetic division** that works wonderfully for these kinds of problems. In order to use synthetic division we must be dividing a polynomial by a linear term in the form $x-r$. If we aren't then it won't work.

Let's redo the previous problem with synthetic division to see how it works.

Example 2 Use synthetic division to divide $5x^3-x^2+6$ by $x-4$.

Solution
Okay with synthetic division we pretty much ignore all the *x*'s and just work with the numbers in the polynomials.

First, let's notice that in this case *r=4*.

Now we need to set up the process. There are many different notations for doing this. We'll be using the following notation.

$$4 \rfloor\ 5 \quad -1 \quad 0 \quad 6$$

The numbers to the right of the vertical bar are the coefficients of the terms in the polynomial written in order of decreasing exponent. Also notice that any missing terms are acknowledged with a coefficient of zero.

Now, it will probably be easier to write down the process and then explain it so here it is.

$$\begin{array}{r|rrrr} 4 & 5 & -1 & 0 & 6 \\ & & 20 & 76 & 304 \\ \hline & 5 & 19 & 76 & 310 \end{array}$$

The first thing we do is drop the first number in the top line straight down as shown. Then along each diagonal we multiply the starting number by *r* (which is 4 in this case) and put this number in the second row. Finally, add the numbers in the first and second row putting the results in the third row. We continue this until we get reach the final number in the first row.

Now, notice that the numbers in the bottom row are the coefficients of the quadratic polynomial from our answer written in order of decreasing exponent and the final number in the third row is the remainder.

The answer is then the same as the first example.

$$5x^3 - x^2 + 6 = (x-4)(5x^2 + 19x + 76) + 310$$

We'll do some more examples of synthetic division in a bit. However, we really should generalize things out a little first with the following fact.

Division Algorithm

Given a polynomial *P(x)* with degree at least 1 and any number *r* there is another polynomial *Q(x)*, called the **quotient**, with degree one less than the degree of *P(x)* and a number *R*, called the **remainder**, such that,

$$P(x) = (x-r)Q(x) + R$$

Note as well that *Q(x)* and *R* are unique, or in other words, there is only one *Q(x)* and *R* that will work for a given *P(x)* and *r*.

So, with the one example we've done to this point we can see that,

$$Q(x) = 5x^2 + 19x + 76 \qquad \text{and} \qquad R = 310$$

Now, let's work a couple more synthetic division problems.

Example 3 Use synthetic division to do each of the following divisions.

(a) Divide $2x^3 - 3x - 5$ by $x + 2$

(b) Divide $4x^4 - 10x^2 + 1$ by $x - 6$

Solution

(a) Divide $2x^3 - 3x - 5$ by $x + 2$

Okay in this case we need to be a little careful here. We MUST divide by a term in the form $x - r$ in order for this to work and that minus sign is absolutely required. So, we're first going to need to write $x + 2$ as,

$$x + 2 = x - \left(-2\right)$$

and in doing so we can see that $r = -2$.

We can now do synthetic division and this time we'll just put up the results and leave it to you to check all the actual numbers.

$$\begin{array}{r|rrrr} \underline{-2} & 2 & 0 & -3 & -5 \\ & & -4 & 8 & -10 \\ \hline & 2 & -4 & 5 & -15 \end{array}$$

So, in this case we have,

$$2x^3 - 3x - 5 = \left(x + 2\right)\left(2x^2 - 4x + 5\right) - 15$$

(b) Divide $4x^4 - 10x^2 + 1$ by $x - 6$

In this case we've got *r*=6. Here is the work.

$$\begin{array}{r|rrrrr} \underline{6} & 4 & 0 & -10 & 0 & 1 \\ & 0 & 24 & 144 & 804 & 4824 \\ \hline & 4 & 24 & 134 & 804 & 4825 \end{array}$$

In this case we then have.

$$4x^4 - 10x^2 + 1 = \left(x - 6\right)\left(4x^3 + 24x^2 + 134x + 804\right) + 4825$$

So, just why are we doing this? That's a natural question at this point. One answer is that, down the road in a later section, we are going to want to get our hands on the *Q(x)*. Just why we might want to do that will have to wait for an explanation until we get to that point.

There is also another reason for this that we are going to make heavy usage of later on. Let's first start out with the division algorithm.

$$P\left(x\right) = \left(x - r\right)Q\left(x\right) + R$$

Now, let's evaluate the polynomial *P(x)* at *r*. If we had an actual polynomial here we could evaluate *P(x)* directly of course, but let's use the division algorithm and see what we get,

$$\begin{aligned} P(r) &= (r-r)Q(r)+R \\ &= (0)Q(r)+R \\ &= R \end{aligned}$$

Now, that's convenient. The remainder of the division algorithm is also the value of the polynomial evaluated at *r*. So, from our previous examples we now know the following function evaluations.

$$\begin{aligned} &\text{If } P(x)=5x^3-x^2+6 \text{ then } P(4)=310 \\ &\text{If } P(x)=2x^3-3x-5 \text{ then } P(-2)=-15 \\ &\text{If } P(x)=4x^4-10x^2+1 \text{ then } P(6)=4825 \end{aligned}$$

This is a very quick method for evaluating polynomials. For polynomials with only a few terms and/or polynomials with "small" degree this may not be much quicker that evaluating them directly. However, if there are many terms in the polynomial and they have large degrees this can be much quicker and much less prone to mistakes than computing them directly.

As noted, we will be using this fact in a later section to greatly reduce the amount of work we'll need to do in those problems.

Section 5-2 : Zeroes/Roots of Polynomials

We'll start off this section by defining just what a **root** or **zero** of a polynomial is. We say that $x = r$ is a root or zero of a polynomial, $P(x)$, if $P(r) = 0$. In other words, $x = r$ is a root or zero of a polynomial if it is a solution to the equation $P(x) = 0$.

In the next couple of sections we will need to find all the zeroes for a given polynomial. So, before we get into that we need to get some ideas out of the way regarding zeroes of polynomials that will help us in that process.

The process of finding the zeros of $P(x)$ really amount to nothing more than solving the equation $P(x) = 0$ and we already know how to do that for second degree (quadratic) polynomials. So, to help illustrate some of the ideas were going to be looking at let's get the zeroes of a couple of second degree polynomials.

Let's first find the zeroes for $P(x) = x^2 + 2x - 15$. To do this we simply solve the following equation.

$$x^2 + 2x - 15 = (x+5)(x-3) = 0 \qquad \Rightarrow \qquad x = -5,\ x = 3$$

So, this second degree polynomial has two zeroes or roots.

Now, let's find the zeroes for $P(x) = x^2 - 14x + 49$. That will mean solving,

$$x^2 - 14x + 49 = (x-7)^2 = 0 \qquad \Rightarrow \qquad x = 7$$

So, this second degree polynomial has a single zero or root. Also, recall that when we first looked at these we called a root like this a **double root**.

We solved each of these by first factoring the polynomial and then using the zero factor property on the factored form. When we first looked at the zero factor property we saw that it said that if the product of two terms was zero then one of the terms had to be zero to start off with.

The zero factor property can be extended out to as many terms as we need. In other words, if we've got a product of *n* terms that is equal to zero, then at least one of them had to be zero to start off with. So, if we could factor higher degree polynomials we could then solve these as well.

Let's take a look at a couple of these.

Example 1 Find the zeroes of each of the following polynomials.

(a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x+4)$

Solution

In each of these the factoring has been done for us. Do not worry about factoring anything like this. You won't be asked to do any factoring of this kind anywhere in this material. There are only here to make the point that the zero factor property works here as well. We will also use these in a later example.

(a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

Okay, in this case we do have a product of 3 terms however the first is a constant and will not make the polynomial zero. So, from the final two terms it looks like the polynomial will be zero for $x = -1$ and $x = 2$. Therefore, the zeroes of this polynomial are,

$$x = -1 \text{ and } x = 2$$

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

We've also got a product of three terms in this polynomial. However, since the first is now an *x* this will introduce a third zero. The zeroes for this polynomial are,

$$x = -5, \ x = 0, \text{ and } x = 3$$

because each of these will make one of the terms, and hence the whole polynomial, zero.

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x+4)$

With this polynomial we have four terms and the zeroes here are,

$$x = -5, \ x = -1, \ x = 1, \text{ and } x = -4$$

Now, we've got some terminology to get out of the way. If *r* is a zero of a polynomial and the exponent on the term that produced the root is *k* then we say that *r* has **multiplicity *k***. Zeroes with a multiplicity of 1 are often called **simple** zeroes.

For example, the polynomial $P(x) = x^2 - 10x + 25 = (x-5)^2$ will have one zero, $x = 5$, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as,

$$P(x) = x^2 - 10x + 25 = (x-5)(x-5)$$

Written this way the term $x - 5$ shows up twice and each term gives the same zero, $x = 5$. Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

Example 2 List the multiplicities of the zeroes of each of the following polynomials.

(a) $P(x) = x^2 + 2x - 15$

(b) $P(x) = x^2 - 14x + 49$

(c) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2 (x-2)^3$

(d) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4 (x-3)^3 (x+5)$

(e) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3 (x-1)^2 (x+5)(x+4)$

Solution
We've already determined the zeroes of each of these in previous work or examples in this section so we won't redo that work. In each case we will simply write down the previously found zeroes and then go back to the factored form of the polynomial, look at the exponent on each term and give the multiplicity.

(a) In this case we've got two simple zeroes : $x = -5,\ x = 3$.

(b) Here $x = 7$ is a zero of multiplicity 2.

(c) There are two zeroes for this polynomial : $x = -1$ with multiplicity 2 and $x = 2$ with multiplicity 3.

(d) We have three zeroes in this case. : $x = -5$ which is simple, $x = 0$ with multiplicity of 4 and $x = 3$ with multiplicity 3.

(e) In the final case we've got four zeroes. $x = -5$ which is simple, $x = -1$ with multiplicity of 3, $x = 1$ with multiplicity 2 and $x = -4$ which is simple.

This example leads us to several nice facts about polynomials. Here is the first and probably the most important.

Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree *n* then $P(x)$ will have exactly *n* zeroes, some of which may repeat.

This fact says that if you list out all the zeroes and listing each one *k* times where *k* is its multiplicity you will have exactly *n* numbers in the list. Another way to say this fact is that the multiplicity of all the zeroes must add to the degree of the polynomial.

We can go back to the previous example and verify that this fact is true for the polynomials listed there.

This will be a nice fact in a couple of sections when we go into detail about finding all the zeroes of a polynomial. If we know an upper bound for the number of zeroes for a polynomial then we will know when we've found all of them and so we can stop looking.

Note as well that some of the zeroes may be complex. In this section we have worked with polynomials that only have real zeroes but do not let that lead you to the idea that this theorem will only apply to real zeroes. It is completely possible that complex zeroes will show up in the list of zeroes.

The next fact is also very useful at times.

The Factor Theorem

For the polynomial $P\left(x\right)$,

1. If *r* is a zero of $P\left(x\right)$ then $x-r$ will be a factor of $P\left(x\right)$.
2. If $x-r$ is a factor of $P\left(x\right)$ then *r* will be a zero of $P\left(x\right)$.

Again, if we go back to the previous example we can see that this is verified with the polynomials listed there.

The factor theorem leads to the following fact.

Fact 1

If $P\left(x\right)$ is a polynomial of degree *n* and *r* is a zero of $P\left(x\right)$ then $P\left(x\right)$ can be written in the following form.

$$P\left(x\right)=\left(x-r\right)Q\left(x\right)$$

where $Q\left(x\right)$ is a polynomial with degree $n-1$. $Q\left(x\right)$ can be found by dividing $P\left(x\right)$ by $x-r$.

There is one more fact that we need to get out of the way.

Fact 2

If $P\left(x\right)=\left(x-r\right)Q\left(x\right)$ and $x=t$ is a zero of $Q\left(x\right)$ then $x=t$ will also be a zero of $P\left(x\right)$.

This fact is easy enough to verify directly. First, if $x=t$ is a zero of $Q\left(x\right)$ then we know that,

$$Q\left(t\right)=0$$

since that is what it means to be a zero. So, if $x=t$ is to be a zero of $P\left(x\right)$ then all we need to do is show that $P\left(t\right)=0$ and that's actually quite simple. Here it is,

$$P\left(t\right)=\left(t-r\right)Q\left(t\right)=\left(t-r\right)\left(0\right)=0$$

and so $x=t$ is a zero of $P\left(x\right)$.

Let's work an example to see how these last few facts can be of use to us.

Example 3 Given that $x = 2$ is a zero of $P(x) = x^3 + 2x^2 - 5x - 6$ find the other two zeroes.

Solution
First, notice that we really can say the other two since we know that this is a third degree polynomial and so by The Fundamental Theorem of Algebra we will have exactly 3 zeroes, with some repeats possible.

So, since we know that $x = 2$ is a zero of $P(x) = x^3 + 2x^2 - 5x - 6$ the Fact 1 tells us that we can write $P(x)$ as,

$$P(x) = (x-2)Q(x)$$

and $Q(x)$ will be a quadratic polynomial. Then we can find the zeroes of $Q(x)$ by any of the methods that we've looked at to this point and by Fact 2 we know that the two zeroes we get from $Q(x)$ will also by zeroes of $P(x)$. At this point we'll have 3 zeroes and so we will be done.

So, let's find $Q(x)$. To do this all we need to do is a quick synthetic division as follows.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

Before writing down $Q(x)$ recall that the final number in the third row is the remainder and that we know that $P(2)$ must be equal to this number. So, in this case we have that $P(2) = 0$. If you think about it, we should already know this to be true. We were given in the problem statement the fact that $x = 2$ is a zero of $P(x)$ and that means that we must have $P(2) = 0$.

So, why go on about this? This is a great check of our synthetic division. Since we know that $x = 2$ is a zero of $P(x)$ and we get any other number than zero in that last entry we will know that we've done something wrong and we can go back and find the mistake.

Now, let's get back to the problem. From the synthetic division we have,

$$P(x) = (x-2)(x^2 + 4x + 3)$$

So, this means that,

$$Q(x) = x^2 + 4x + 3$$

and we can find the zeroes of this. Here they are,

$$Q(x) = x^2 + 4x + 3 = (x+3)(x+1) \qquad \Rightarrow \qquad x = -3,\ x = -1$$

So, the three zeroes of $P(x)$ are $x = -3$, $x = -1$ and $x = 2$.

As an aside to the previous example notice that we can also now completely factor the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$. Substituting the factored form of $Q(x)$ into $P(x)$ we get,

$$P(x) = (x-2)(x+3)(x+1)$$

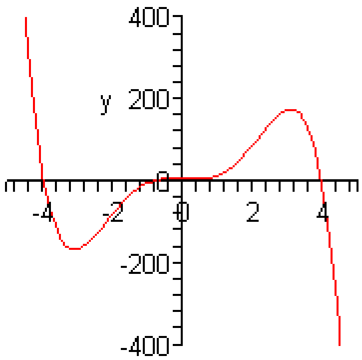
This is how the polynomials in the first set of examples were factored by the way. Those require a little more work than this, but they can be done in the same manner.

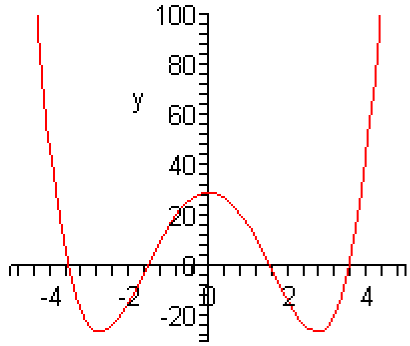
Section 5-3 : Graphing Polynomials

In this section we are going to look at a method for getting a rough sketch of a general polynomial. The only real information that we're going to need is a complete list of all the zeroes (including multiplicity) for the polynomial.

In this section we are going to either be given the list of zeroes or they will be easy to find. In the next section we will go into a method for determining a large portion of the list for most polynomials. We are graphing first since the method for finding all the zeroes of a polynomial can be a little long and we don't want to obscure the details of this section in the mess of finding the zeroes of the polynomial.

Let's start off with the graph of couple of polynomials.





Do not worry about the equations for these polynomials. We are giving these only so we can use them to illustrate some ideas about polynomials.

First, notice that the graphs are nice and smooth. There are no holes or breaks in the graph and there are no sharp corners in the graph. The graphs of polynomials will always be nice smooth curves.

Secondly, the "humps" where the graph changes direction from increasing to decreasing or decreasing to increasing are often called **turning points**. If we know that the polynomial has degree *n* then we will know that there will be at most $n-1$ turning points in the graph.

While this won't help much with the actual graphing process it will be a nice check. If we have a fourth degree polynomial with 5 turning point then we will know that we've done something wrong since a fourth degree polynomial will have no more than 3 turning points.

Next, we need to explore the relationship between the *x*-intercepts of a graph of a polynomial and the zeroes of the polynomial. Recall that to find the *x*-intercepts of a function we need to solve the equation

$$P(x)=0$$

Also, recall that $x=r$ is a zero of the polynomial, $P(x)$, provided $P(r)=0$. But this means that $x=r$ is also a solution to $P(x)=0$.

In other words, the zeroes of a polynomial are also the *x*-intercepts of the graph. Also, recall that *x*-intercepts can either cross the *x*-axis or they can just touch the *x*-axis without actually crossing the axis.

Notice as well from the graphs above that the *x*-intercepts can either flatten out as they cross the *x*-axis or they can go through the *x*-axis at an angle.

The following fact will relate all of these ideas to the multiplicity of the zero.

Fact

If $x = r$ is a zero of the polynomial $P(x)$ with multiplicity *k* then,

1. If *k* is odd then the *x*-intercept corresponding to $x = r$ will cross the *x*-axis.
2. If *k* is even then the *x*-intercept corresponding to $x = r$ will only touch the *x*-axis and not actually cross it.

Furthermore, if $k > 1$ then the graph will flatten out at $x = r$.

Finally, notice that as we let *x* get large in both the positive or negative sense (*i.e.* at either end of the graph) then the graph will either increase without bound or decrease without bound. This will always happen with every polynomial and we can use the following test to determine just what will happen at the endpoints of the graph.

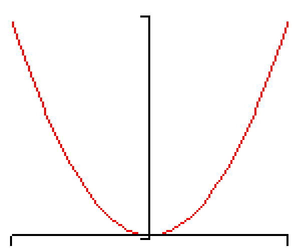
Leading Coefficient Test

Suppose that $P(x)$ is a polynomial with degree *n*. So we know that the polynomial must look like,

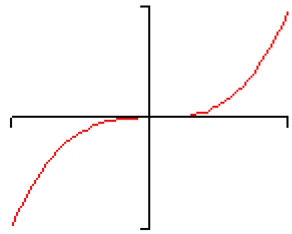
$$P(x) = ax^n + \cdots$$

We don't know if there are any other terms in the polynomial, but we do know that the first term will have to be the one listed since it has degree *n*. We now have the following facts about the graph of $P(x)$ at the ends of the graph.

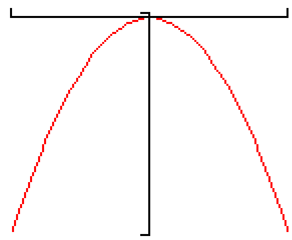
1. If $a > 0$ and *n* is even then the graph of $P(x)$ will increase without bound positively at both endpoints. A good example of this is the graph of x^2.



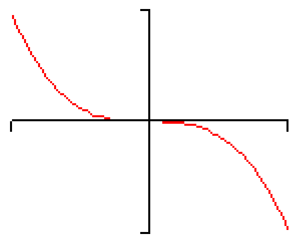
2. If $a > 0$ and *n* is odd then the graph of $P(x)$ will increase without bound positively at the right end and decrease without bound at the left end. A good example of this is the graph of x^3.



3. If $a < 0$ and *n* is even then the graph of $P(x)$ will decrease without bound positively at both endpoints. A good example of this is the graph of -x^2.



4. If $a < 0$ and *n* is odd then the graph of $P(x)$ will decrease without bound positively at the right end and increase without bound at the left end. A good example of this is the graph of -x^3.



Okay, now that we've got all that out of the way we can finally give a process for getting a rough sketch of the graph of a polynomial.

Process for Graphing a Polynomial

1. Determine all the zeroes of the polynomial and their multiplicity. Use the fact above to determine the *x*-intercept that corresponds to each zero will cross the *x*-axis or just touch it and if the *x*-intercept will flatten out or not.

2. Determine the *y*-intercept, $\left(0, P(0)\right)$.

3. Use the leading coefficient test to determine the behavior of the polynomial at the end of the graph.

4. Plot a few more points. This is left intentionally vague. The more points that you plot the better the sketch. At the least you should plot at least one at either end of the graph and at least one point between each pair of zeroes.

We should give a quick warning about this process before we actually try to use it. This process assumes that all the zeroes are real numbers. If there are any complex zeroes then this process may miss some pretty important features of the graph.

Let's sketch a couple of polynomials.

Example 1 Sketch the graph of $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40$.

Solution

We found the zeroes and multiplicities of this polynomial in the previous section so we'll just write them back down here for reference purposes.

$$\begin{aligned} x &= -1 \quad (\text{multiplicity } 2) \\ x &= 2 \quad\;\; (\text{multiplicity } 3) \end{aligned}$$

So, from the fact we know that $x = -1$ will just touch the *x*-axis and not actually cross it and that $x = 2$ will cross the *x*-axis and will be flat as it does this since the multiplicity is greater than 1.

Next, the *y*-intercept is $(0, -40)$.

The coefficient of the 5th degree term is positive and since the degree is odd we know that this polynomial will increase without bound at the right end and decrease without bound at the left end.

Finally, we just need to evaluate the polynomial at a couple of points. The points that we pick aren't really all that important. We just want to pick points according to the guidelines in the process outlined above and points that will be fairly easy to evaluate. Here are some points. We will leave it to you to verify the evaluations.

$$P(-2) = -320 \qquad P(1) = -20 \qquad P(3) = 80$$

Now, to actually sketch the graph we'll start on the left end and work our way across to the right end. First, we know that on the left end the graph decreases without bound as we make *x* more and more negative and this agrees with the point that we evaluated at $x = -2$.

So, as we move to the right the function will actually be increasing at $x = -2$ and we will continue to increase until we hit the first *x*-intercept at $x = -1$. At this point we know that the graph just touches the *x*-axis without actually crossing it. This means that at $x = 0$ the graph must be a turning point.

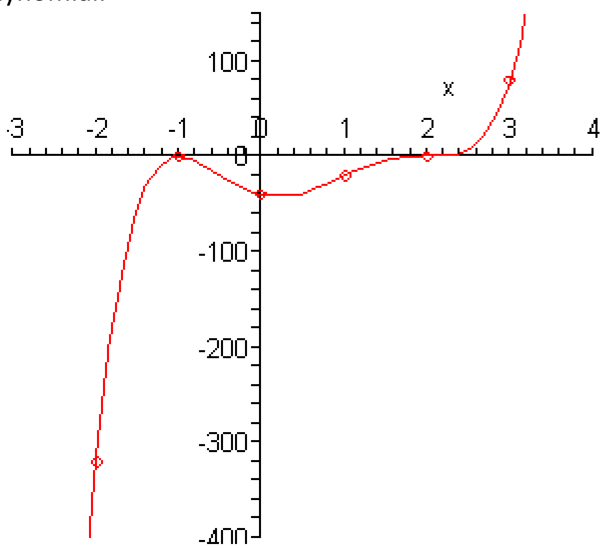
The graph is now decreasing as we move to the right. Again, this agrees with the next point that we'll run across, the *y*-intercept.

Now, according to the next point that we've got, $x = 1$, the graph must have another turning point somewhere between $x = 0$ and $x = 1$ since the graph is higher at $x = 1$ than at $x = 0$. Just where this turning point will occur is very difficult to determine at this level so we won't worry about trying to find it. In fact, determining this point usually requires some Calculus.

So, we are moving to the right and the function is increasing. The next point that we hit is the *x*-intercept at $x = 2$ and this one crosses the *x*-axis so we know that there won't be a turning point here as there was at the first *x*-intercept. Therefore, the graph will continue to increase through this point until we hit the final point that we evaluated the function at, $x = 3$.

At this point we've hit all the *x*-intercepts and we know that the graph will increase without bound at the right end and so it looks like all we need to do is sketch in an increasing curve.

Here is a sketch of the polynomial.



Note that one of the reasons for plotting points at the ends is to see just how fast the graph is increasing or decreasing. We can see from the evaluations that the graph is decreasing on the left end much faster than it's increasing on the right end.

Okay, let's take a look at another polynomial. This time we'll go all the way through the process of finding the zeroes.

Example 2 Sketch the graph of $P(x) = x^4 - x^3 - 6x^2$.

Solution
First, we'll need to factor this polynomial as much as possible so we can identify the zeroes and get their multiplicities.

$$P(x) = x^4 - x^3 - 6x^2 = x^2\left(x^2 - x - 6\right) = x^2(x-3)(x+2)$$

Here is a list of the zeroes and their multiplicities.

$$x = -2 \quad (\text{multiplicity } 1)$$
$$x = 0 \quad (\text{multiplicity } 2)$$
$$x = 3 \quad (\text{multiplicity } 1)$$

So, the zeroes at $x = -2$ and $x = 3$ will correspond to *x*-intercepts that cross the *x*-axis since their multiplicity is odd and will do so at an angle since their multiplicity is NOT at least 2. The zero at $x = 0$ will not cross the *x*-axis since its multiplicity is even.

The *y*-intercept is $(0,0)$ and notice that this is also an *x*-intercept.

The coefficient of the 4th degree term is positive and so since the degree is even we know that the polynomial will increase without bound at both ends of the graph.

Finally, here are some function evaluations.

$$P(-3) = 54 \qquad P(-1) = -4 \qquad P(1) = -6 \qquad P(4) = 96$$

Now, starting at the left end we know that as we make *x* more and more negative the function must increase without bound. That means that as we move to the right the graph will actually be decreasing.

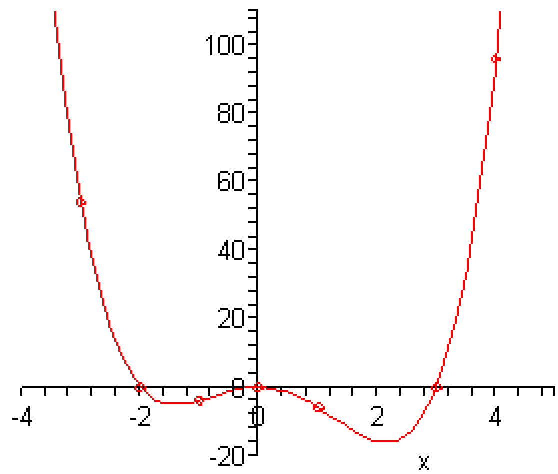
At $x = -3$ the graph will be decreasing and will continue to decrease when we hit the first *x*-intercept at $x = -2$ since we know that this *x*-intercept will cross the *x*-axis.

Next, since the next *x*-intercept is at $x = 0$ we will have to have a turning point somewhere so that the graph can increase back up to this *x*-intercept. Again, we won't worry about where this turning point actually is.

Once we hit the *x*-intercept at $x = 0$ we know that we've got to have a turning point since this *x*-intercept doesn't cross the *x*-axis. Therefore, to the right of $x = 0$ the graph will now be decreasing.

It will continue to decrease until it hits another turning point (at some unknown point) so that the graph can get back up to the *x*-axis for the next *x*-intercept at $x = 3$. This is the final *x*-intercept and since the graph is increasing at this point and must increase without bound at this end we are done.

Here is a sketch of the graph.



Example 3 Sketch the graph of $P(x) = -x^5 + 4x^3$.

Solution
As with the previous example we'll first need to factor this as much as possible.

$$P(x) = -x^5 + 4x^3 = -\left(x^5 - 4x^3\right) = -x^3\left(x^2 - 4\right) = -x^3(x-2)(x+2)$$

Notice that we first factored out a minus sign to make the rest of the factoring a little easier. Here is a list of all the zeroes and their multiplicities.

$$\begin{aligned} x &= -2 \quad (\text{multiplicity } 1) \\ x &= 0 \quad (\text{multiplicity } 3) \\ x &= 2 \quad (\text{multiplicity } 1) \end{aligned}$$

So, all three zeroes correspond to *x*-intercepts that actually cross the *x*-axis since all their multiplicities are odd, however, only the *x*-intercept at $x = 0$ will cross the *x*-axis flattened out.

The *y*-intercept is $(0,0)$ and as with the previous example this is also an *x*-intercept.

In this case the coefficient of the 5th degree term is negative and so since the degree is odd the graph will increase without bound on the left side and decrease without bound on the right side.

Here are some function evaluations.

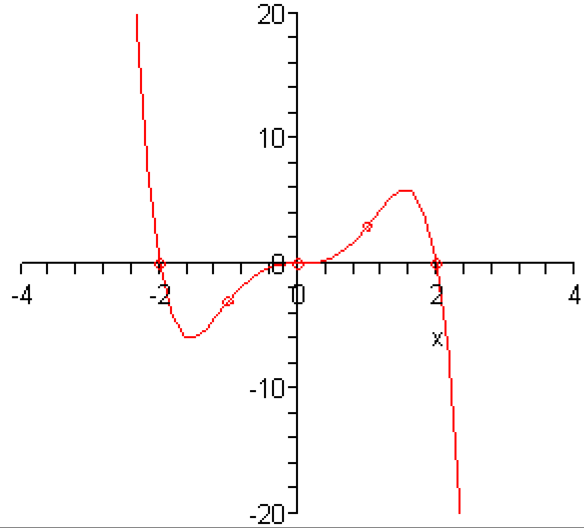
$$P(-3) = 135 \qquad P(-1) = -3 \qquad P(1) = 3 \qquad P(3) = -135$$

Alright, this graph will start out much as the previous graph did. At the left end the graph will be decreasing as we move to the right and will decrease through the first *x*-intercept at $x = -2$ since know that this *x*-intercept crosses the *x*-axis.

Now at some point we'll get a turning point so the graph can get back up to the next *x*-intercept at $x = 0$ and the graph will continue to increase through this point since it also crosses the *x*-axis. Note as well that the graph should be flat at this point as well since the multiplicity is greater than one.

Finally, the graph will reach another turning point and start decreasing so it can get back down to the final *x*-intercept at $x = 2$. Since we know that the graph will decrease without bound at this end we are done.

Here is the sketch of this polynomial.



The process that we've used in these examples can be a difficult process to learn. It takes time to learn how to correctly interpret the results.

Also, as pointed out at various spots there are several situations that we won't be able to deal with here. To find the majority of the turning points we would need some Calculus, which we clearly don't have. Also, the process does require that we have all the zeroes and that they all be real numbers.

Even with these drawbacks however, the process can at least give us an idea of what the graph of a polynomial will look like.

Section 5-4 : Finding Zeroes of Polynomials

We've been talking about zeroes of polynomial and why we need them for a couple of sections now. We haven't, however, really talked about how to actually find them for polynomials of degree greater than two. That is the topic of this section. Well, that's kind of the topic of this section. In general, finding all the zeroes of any polynomial is a fairly difficult process. In this section we will give a process that will find all rational (*i.e.* integer or fractional) zeroes of a polynomial. We will be able to use the process for finding all the zeroes of a polynomial provided all but at most two of the zeroes are rational. If more than two of the zeroes are not rational then this process will not find all of the zeroes.

We will need the following theorem to get us started on this process.

Rational Root Theorem

If the rational number $x = \frac{b}{c}$ is a zero of the n^th^ degree polynomial,

$$P(x) = sx^n + \cdots + t$$

where all the coefficients are integers then *b* will be a factor of *t* and *c* will be a factor of *s*.

Note that in order for this theorem to work then the zero must be reduced to lowest terms. In other words, it will work for $\frac{4}{3}$ but not necessarily for $\frac{20}{15}$.

Let's verify the results of this theorem with an example.

Example 1 Verify that the roots of the following polynomial satisfy the rational root theorem.

$$P(x) = 12x^3 - 41x^2 - 38x + 40 = (x-4)(3x-2)(4x+5)$$

Solution

From the factored form we can see that the zeroes are,

$$x = 4 = \frac{4}{1} \qquad x = \frac{2}{3} \qquad x = -\frac{5}{4}$$

Notice that we wrote the integer as a fraction to fit it into the theorem. Also, with the negative zero we can put the negative onto the numerator or denominator. It won't matter.

So, according to the rational root theorem the numerators of these fractions (with or without the minus sign on the third zero) must all be factors of 40 and the denominators must all be factors of 12.

Here are several ways to factor 40 and 12.

$$40 = (4)(10) \qquad 40 = (2)(20) \qquad 40 = (5)(8) \qquad 40 = (-5)(-8)$$
$$12 = (1)(12) \qquad 12 = (3)(4) \qquad 12 = (-3)(-4)$$

From these we can see that in fact the numerators are all factors of 40 and the denominators are all factors of 12. Also note that, as shown, we can put the minus sign on the third zero on either the numerator or the denominator and it will still be a factor of the appropriate number.

So, why is this theorem so useful? Well, for starters it will allow us to write down a list of *possible* rational zeroes for a polynomial and more importantly, any rational zeroes of a polynomial WILL be in this list.

In other words, we can quickly determine all the rational zeroes of a polynomial simply by checking all the numbers in our list.

Before getting into the process of finding the zeroes of a polynomial let's see how to come up with a list of possible rational zeroes for a polynomial.

Example 2 Find a list of all possible rational zeroes for each of the following polynomials.

(a) $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$

(b) $P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9$

Solution

(a) $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$

Now, just what does the rational root theorem say? It says that if $x = \frac{b}{c}$ is to be a zero of $P(x)$ then *b* must be a factor of 6 and *c* must be a factor of 1. Also, as we saw in the previous example we can't forget negative factors.

So, the first thing to do is actually to list all possible factors of 1 and 6. Here they are.

$$\begin{array}{ll} 6: & \pm 1, \pm 2, \pm 3, \pm 6 \\ 1: & \pm 1 \end{array}$$

Now, to get a list of possible rational zeroes of the polynomial all we need to do is write down all possible fractions that we can form from these numbers where the numerators must be factors of 6 and the denominators must be factors of 1. This is actually easier than it might at first appear to be.

There is a very simple shorthanded way of doing this. Let's go through the first one in detail then we'll do the rest quicker. First, take the first factor from the numerator list, including the $\pm$, and divide this by the first factor (okay, only factor in this case) from the denominator list, again including the $\pm$. Doing this gives,

$$\frac{\pm 1}{\pm 1}$$

This looks like a mess, but it isn't too bad. There are four fractions here. They are,

$$\frac{+1}{+1} = 1 \qquad \frac{+1}{-1} = -1 \qquad \frac{-1}{+1} = -1 \qquad \frac{-1}{-1} = 1$$

Notice however, that the four fractions all reduce down to two possible numbers. This will always happen with these kinds of fractions. What we'll do from now on is form the fraction, do any simplification of the numbers, ignoring the $\pm$, and then drop one of the $\pm$.

So, the list possible rational zeroes for this polynomial is,

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 6}{\pm 1} = \pm 6$$

So, it looks there are only 8 possible rational zeroes and in this case they are all integers. Note as well that any rational zeroes of this polynomial WILL be somewhere in this list, although we haven't found them yet.

(b) $P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9$

We'll not put quite as much detail into this one. First get a list of all factors of -9 and 2. Note that the minus sign on the 9 isn't really all that important since we will still get a $\pm$ on each of the factors.

$$\begin{aligned} -9: &\quad \pm 1, \pm 3, \pm 9 \\ 2: &\quad \pm 1, \pm 2 \end{aligned}$$

Now, the factors of -9 are all the possible numerators and the factors of 2 are all the possible denominators.

Here then is a list of all possible rational zeroes of this polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 9}{\pm 1} = \pm 9$$

$$\frac{\pm 1}{\pm 2} = \pm \frac{1}{2} \qquad \frac{\pm 3}{\pm 2} = \pm \frac{3}{2} \qquad \frac{\pm 9}{\pm 2} = \pm \frac{9}{2}$$

So, we've got a total of 12 possible rational zeroes, half are integers and half are fractions.

The following fact will also be useful on occasion in finding the zeroes of a polynomial.

Fact

If $P(x)$ is a polynomial and we know that $P(a) > 0$ and $P(b) < 0$ then somewhere between *a* and *b* is a zero of $P(x)$.

What this fact is telling us is that if we evaluate the polynomial at two points and one of the evaluations gives a positive value (*i.e.* the point is above the *x*-axis) and the other evaluation gives a negative value (*i.e.* the point is below the *x*-axis), then the only way to get from one point to the other is to go through the *x*-axis. Or, in other words, the polynomial must have a zero, since we know that zeroes are where a graph touches or crosses the *x*-axis.

Note that this fact doesn't tell us what the zero is, it only tells us that one will exist. Also, note that if both evaluations are positive or both evaluations are negative there may or may not be a zero between them.

Here is the process for determining all the rational zeroes of a polynomial.

Process for Finding Rational Zeroes

1. Use the rational root theorem to list all possible rational zeroes of the polynomial $P(x)$.

2. Evaluate the polynomial at the numbers from the first step until we find a zero. Let's suppose the zero is $x = r$, then we will know that it's a zero because $P(r) = 0$. Once this has been determined that it is in fact a zero write the original polynomial as
$$P(x) = (x - r)Q(x)$$

3. Repeat the process using $Q(x)$ this time instead of $P(x)$. This repeating will continue until we reach a second degree polynomial. At this point we can solve this directly for the remaining zeroes.

To simplify the second step we will use synthetic division. This will greatly simplify our life in several ways. First, recall that the last number in the final row is the polynomial evaluated at *r* and if we do get a zero the remaining numbers in the final row are the coefficients for $Q(x)$ and so we won't have to go back and find that.

Also, in the evaluation step it is usually easiest to evaluate at the possible integer zeroes first and then go back and deal with any fractions if we have to.

Let's see how this works.

Example 3 Determine all the zeroes of $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$.

Solution

We found the list of all possible rational zeroes in the previous example. Here they are.

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$$

We now need to start evaluating the polynomial at these numbers. We can start anywhere in the list and will continue until we find zero.

To do the evaluations we will build a **synthetic division table**. In a synthetic division table do the multiplications in our head and drop the middle row just writing down the third row and since we will be going through the process multiple times we put all the rows into a table.

Here is the first synthetic division table for this problem.

$$\begin{array}{r|rrrrrl} & 1 & -7 & 17 & -17 & 6 & \\ \hline -1 & 1 & -8 & 25 & -42 & 48 & = P(-1) \neq 0 \\ 1 & 1 & -6 & 11 & -6 & 0 & = P(1) = 0\,!! \end{array}$$

So, we found a zero. Before getting into that let's recap the computations here to make sure you can do them.

The top row is the coefficients from the polynomial and the first column is the numbers that we're evaluating the polynomial at.

Each row (after the first) is the third row from the synthetic division process. Let's quickly look at the first couple of numbers in the second row. The number in the second column is the first coefficient dropped down. The number in the third column is then found by multiplying the -1 by 1 and adding to the -7. This gives the -8. For the fourth number is then -1 times -8 added onto 17. This is 25, *etc.*

You can do regular synthetic division if you need to, but it's a good idea to be able to do these tables as it can help with the process.

Okay, back to the problem. We now know that $x = 1$ is a zero and so we can write the polynomial as,

$$P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 = (x-1)\left(x^3 - 6x^2 + 11x - 6\right)$$

Now we need to repeat this process with the polynomial $Q(x) = x^3 - 6x^2 + 11x - 6$. So, the first thing to do is to write down all possible rational roots of this polynomial and in this case we're lucky enough to have the first and last numbers in this polynomial be the same as the original polynomial, that usually won't happen so don't always expect it. Here is the list of all possible rational zeroes of this polynomial.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Now, before doing a new synthetic division table let's recall that we are looking for zeroes to $P(x)$ and from our first division table we determined that $x = -1$ is NOT a zero of $P(x)$ and so there is no reason to bother with that number again.

This is something that we should always do at this step. Take a look at the list of new possible rational zeros and ask are there any that can't be rational zeroes of the original polynomial. If there are some, throw them out as we will already know that they won't work. So, a reduced list of numbers to try here is,

$$1, \pm 2, \pm 3, \pm 6$$

Note that we do need to include $x = 1$ in the list since it is possible for a zero to occur more that once (*i.e.* multiplicity greater than one).

Here is the synthetic division table for this polynomial.

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ \hline 1 & 1 & -5 & 6 & 0 = P(1) = 0!! \end{array}$$

So, $x=1$ is also a zero of $Q(x)$ and we can now write $Q(x)$ as,

$$Q(x)=x^3-6x^2+11x-6=(x-1)(x^2-5x+6)$$

Now, technically we could continue the process with x^2-5x+6, but this is a quadratic equation and we know how to find zeroes of these without a complicated process like this so let's just solve this like we normally would.

$$x^2-5x+6=(x-2)(x-3)=0 \qquad \Rightarrow \qquad x=2,\ x=3$$

Note that these two numbers are in the list of possible rational zeroes.

Finishing up this problem then gives the following list of zeroes for $P(x)$.

$$\begin{aligned} x&=1 \quad (\text{multiplicity } 2) \\ x&=2 \quad (\text{multiplicity } 1) \\ x&=3 \quad (\text{multiplicity } 1) \end{aligned}$$

Note that $x=1$ has a multiplicity of 2 since it showed up twice in our work above.

Before moving onto the next example let's also note that we can now completely factor the polynomial $P(x)=x^4-7x^3+17x^2-17x+6$. We know that each zero will give a factor in the factored form and that the exponent on the factor will be the multiplicity of that zero. So, the factored form is,

$$P(x)=x^4-7x^3+17x^2-17x+6=(x-1)^2(x-2)(x-3)$$

Let's take a look at another example.

Example 4 Find all the zeroes of $P(x)=2x^4+x^3+3x^2+3x-9$.

Solution

From the second example we know that the list of all possible rational zeroes is,

$$\frac{\pm1}{\pm1}=\pm1 \qquad \frac{\pm3}{\pm1}=\pm3 \qquad \frac{\pm9}{\pm1}=\pm9$$

$$\frac{\pm1}{\pm2}=\pm\frac{1}{2} \qquad \frac{\pm3}{\pm2}=\pm\frac{3}{2} \qquad \frac{\pm9}{\pm2}=\pm\frac{9}{2}$$

The next step is to build up the synthetic division table. When we've got fractions it's usually best to start with the integers and do those first. Also, this time we'll start with doing all the negative integers first. We are doing this to make a point on how we can use the fact given above to help us identify zeroes.

$$\begin{array}{r|rrrrl} & 2 & 1 & 3 & 3 & -9 \\ \hline -9 & 2 & -17 & 156 & -1401 & 12600 = P(-9) \ne 0 \\ -3 & 2 & -5 & 18 & -51 & 144 = P(-3) \ne 0 \\ -1 & 2 & -1 & 4 & -1 & -8 = P(-1) \ne 0 \end{array}$$

Now, we haven't found a zero yet, however let's notice that $P(-3) = 144 > 0$ and $P(-1) = -8 < 0$ and so by the fact above we know that there must be a zero somewhere between $x = -3$ and $x = -1$. Now, we can also notice that $x = -\frac{3}{2} = -1.5$ is in this range and is the only number in our list that is in this range and so there is a chance that this is a zero. Let's run through synthetic division real quick to check and see if it's a zero and to get the coefficients for $Q(x)$ if it is a zero.

$$\begin{array}{r|rrrrr} & 2 & 1 & 3 & 3 & -9 \\ \hline -\frac{3}{2} & 2 & -2 & 6 & -6 & 0 \end{array}$$

So, we got a zero in the final spot which tells us that this was a zero and $Q(x)$ is,

$$Q(x) = 2x^3 - 2x^2 + 6x - 6$$

We now need to repeat the whole process with this polynomial. Also, unlike the previous example we can't just reuse the original list since the last number is different this time. So, here are the factors of -6 and 2.

$$\begin{aligned} -6: &\quad \pm 1, \pm 2, \pm 3, \pm 6 \\ 2: &\quad \pm 1, \pm 2 \end{aligned}$$

Here is a list of all possible rational zeroes for $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 6}{\pm 1} = \pm 6$$

$$\frac{\pm 1}{\pm 2} = \pm \frac{1}{2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \pm \frac{3}{2} \qquad \frac{\pm 6}{\pm 2} = \pm 3$$

Notice that some of the numbers appear in both rows and so we can shorten the list by only writing them down once. Also, remember that we are looking for zeroes of $P(x)$ and so we can exclude any number in this list that isn't also in the original list we gave for $P(x)$. So, excluding previously checked numbers that were not zeros of $P(x)$ as well as those that aren't in the original list gives the following list of possible number that we'll need to check.

$$1,\ 3,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$$

Again, we've already checked $x = -3$ and $x = -1$ and know that they aren't zeroes so there is no reason to recheck them. Let's again start with the integers and see what we get.

$$\begin{array}{r|rrrr} & 2 & -2 & 6 & -6 \\ \hline 1 & 2 & 0 & 6 & 0 \end{array} = P(1) = 0\,!\,!$$

So, $x = 1$ is a zero of $Q(x)$ and we can now write $Q(x)$ as,

$$Q(x) = 2x^3 - 2x^2 + 6x - 6 = (x-1)\left(2x^2 + 6\right)$$

and as with the previous example we can solve the quadratic by other means.

$$\begin{aligned} 2x^2 + 6 &= 0 \\ x^2 &= -3 \\ x &= \pm\sqrt{3}\,i \end{aligned}$$

So, in this case we get a couple of complex zeroes. That can happen.

Here is a complete list of all the zeroes for $P(x)$ and note that they all have multiplicity of one.

$$x = -\frac{3}{2},\ \ x = 1,\ \ x = -\sqrt{3}\ i,\ \ x = \sqrt{3}\ i$$

So, as you can see this is a fairly lengthy process and we only did the work for two 4th degree polynomials. The larger the degree the longer and more complicated the process. With that being said, however, it is sometimes a process that we've got to go through to get zeroes of a polynomial.

Section 5-5 : Partial Fractions

This section doesn't really have a lot to do with the rest of this chapter, but since the subject needs to be covered and this was a fairly short chapter it seemed like as good a place as any to put it.

So, let's start with the following. Let's suppose that we want to add the following two rational expressions.

$$\begin{aligned}\frac{8}{x+1}-\frac{5}{x-4}&=\frac{8\left(x-4\right)}{\left(x+1\right)\left(x-4\right)}-\frac{5\left(x+1\right)}{\left(x+1\right)\left(x-4\right)}\\&=\frac{8x-32-\left(5x+5\right)}{\left(x+1\right)\left(x-4\right)}\\&=\frac{3x-37}{\left(x+1\right)\left(x-4\right)}\end{aligned}$$

What we want to do in this section is to start with rational expressions and ask what simpler rational expressions did we add and/or subtract to get the original expression. The process of doing this is called **partial fractions** and the result is often called the **partial fraction decomposition**.

The process can be a little long and on occasion messy, but it is actually fairly simple. We will start by trying to determine the partial fraction decomposition of,

$$\frac{P\left(x\right)}{Q\left(x\right)}$$

where both *P(x)* and *Q(x)* are polynomials and the degree of *P(x)* is smaller than the degree of *Q(x)*. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$\left(ax+b\right)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{\left(ax+b\right)^2}+\cdots+\dfrac{A_k}{\left(ax+b\right)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$\left(ax^2+bx+c\right)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\dfrac{A_kx+B_k}{\left(ax^2+bx+c\right)^k}$

Notice that the first and third cases are really special cases of the second and fourth cases respectively if we let $k = 1$. Also, it will always be possible to factor any polynomial down into a product of linear factors ($ax + b$) and quadratic factors ($ax^2 + bx + c$) some of which may be raised to a power.

There are several methods for determining the coefficients for each term and we will go over each of those as we work the examples. Speaking of which, let's get started on some examples.

Example 1 Determine the partial fraction decomposition of each of the following.

(a) $\dfrac{8x-42}{x^2+3x-18}$

(b) $\dfrac{9-9x}{2x^2+7x-4}$

(c) $\dfrac{4x^2}{(x-1)(x-2)^2}$

(d) $\dfrac{9x+25}{(x+3)^2}$

Solution

We'll go through the first one in great detail to show the complete partial fraction process and then we'll leave most of the explanation out of the remaining parts.

(a) $\dfrac{8x-42}{x^2+3x-18}$

The first thing to do is factor the denominator as much as we can.

$$\frac{8x-42}{x^2+3x-18} = \frac{8x-42}{(x+6)(x-3)}$$

So, by comparing to the table above it looks like the partial fraction decomposition must look like,

$$\frac{8x-42}{x^2+3x-18} = \frac{A}{x+6} + \frac{B}{x-3}$$

Note that we've got different coefficients for each term since there is no reason to think that they will be the same.

Now, we need to determine the values of *A* and *B*. The first step is to actually add the two terms back up. This is usually simpler than it might appear to be. Recall that we first need the least common denominator, but we've already got that from the original rational expression. In this case it is,

$$LCD = (x+6)(x-3)$$

Now, just look at each term and compare the denominator to the LCD. Multiply the numerator and denominator by whatever is missing then add. In this case this gives,

$$\frac{8x-42}{x^2+3x-18} = \frac{A(x-3)}{(x+6)(x-3)} + \frac{B(x+6)}{(x+6)(x-3)} = \frac{A(x-3)+B(x+6)}{(x+6)(x-3)}$$

We need values of *A* and *B* so that the numerator of the expression on the left is the same as the numerator of the term on the right. Or,

$$8x - 42 = A(x-3) + B(x+6)$$

This needs to be true regardless of the *x* that we plug into this equation. As noted above there are several ways to do this. One way will always work but can be messy and will often require knowledge that we don't have yet. The other way will not always work, but when it does it will greatly reduce the amount of work required.

In this set of examples, the second (and easier) method will always work so we'll be using that here. Here we are going to make use of the fact that this equation must be true regardless of the *x* that we plug in.

So, let's pick an *x*, plug it in and see what happens. For no apparent reason let's try plugging in $x = 3$. Doing this gives,

$$\begin{aligned} 8(3) - 42 &= A(3-3) + B(3+6) \\ -18 &= 9B \\ -2 &= B \end{aligned}$$

Can you see why we choose this number? By choosing $x = 3$ we got the term involving *A* to drop out and we were left with a simple equation that we can solve for *B*.

Now, we could also choose $x = -6$ for exactly the same reason. Here is what happens if we use this value of *x*.

$$\begin{aligned} 8(-6) - 42 &= A(-6-3) + B(-6+6) \\ -90 &= -9A \\ 10 &= A \end{aligned}$$

So, by correctly picking *x* we were able to quickly and easily get the values of *A* and *B*. So, all that we need to do at this point is plug them in to finish the problem. Here is the partial fraction decomposition for this part.

$$\frac{8x-42}{x^2+3x-18} = \frac{10}{x+6} + \frac{-2}{x-3} = \frac{10}{x+6} - \frac{2}{x-3}$$

Notice that we moved the minus sign on the second term down to make the addition a subtraction. We will always do that.

(b) $\dfrac{9-9x}{2x^2+7x-4}$

Okay, in this case we won't put quite as much detail into the problem. We'll first factor the denominator and then get the form of the partial fraction decomposition.

$$\frac{9-9x}{2x^2+7x-4} = \frac{9-9x}{(2x-1)(x+4)} = \frac{A}{2x-1} + \frac{B}{x+4}$$

In this case the LCD is $(2x-1)(x+4)$ and so adding the two terms back up give,

$$\frac{9-9x}{2x^2+7x-4}=\frac{A(x+4)+B(2x-1)}{(2x-1)(x+4)}$$

Next, we need to set the two numerators equal.

$$9-9x=A(x+4)+B(2x-1)$$

Now all that we need to do is correctly pick values of *x* that will make one of the terms zero and solve for the constants. Note that in this case we will need to make one of them a fraction. This is fairly common so don't get excited about it. Here is this work.

$$\begin{array}{llll} x=-4: & 45=-9B & \Rightarrow & B=-5 \\ x=\dfrac{1}{2}: & \dfrac{9}{2}=A\left(\dfrac{9}{2}\right) & \Rightarrow & A=1 \end{array}$$

The partial fraction decomposition for this expression is,

$$\frac{9-9x}{2x^2+7x-4}=\frac{1}{2x-1}-\frac{5}{x+4}$$

(c) $\dfrac{4x^2}{(x-1)(x-2)^2}$

In this case the denominator has already been factored for us. Notice as well that we've now got a linear factor to a power. So, recall from our table that this means we will get 2 terms in the partial fraction decomposition from this factor. Here is the form of the partial fraction decomposition for this expression.

$$\frac{4x^2}{(x-1)(x-2)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

Now, remember that the LCD is just the denominator of the original expression so in this case we've got $(x-1)(x-2)^2$. Adding the three terms back up gives us,

$$\frac{4x^2}{(x-1)(x-2)^2}=\frac{A(x-2)^2+B(x-1)(x-2)+C(x-1)}{(x-1)(x-2)^2}$$

Remember that we just need to add in the factors that are missing to each term.

Now set the numerators equal.

$$4x^2=A(x-2)^2+B(x-1)(x-2)+C(x-1)$$

In this case we've got a slightly different situation from the previous two parts. Let's start by picking a couple of values of *x* and seeing what we get since there are two that should jump right out at us as being particularly useful.

$$x = 1: \qquad 4 = A(-1)^2 \qquad \Rightarrow \qquad A = 4$$
$$x = 2: \qquad 16 = C(1) \qquad \Rightarrow \qquad C = 16$$

So, we can get *A* and *C* in the same manner that we've been using to this point. However, there is no value of *x* that will allow us to eliminate the first and third term leaving only the middle term that we can use to solve for *B*. While this may appear to be a problem it actually isn't. At this point we know two of the three constants. So all we need to do is chose any other value of *x* that would be easy to work with ($x = 0$ seems particularly useful here), plug that in along with the values of *A* and *C* and we'll get a simple equation that we can solve for *B*.

Here is that work.

$$\begin{aligned} 4(0)^2 &= (4)(-2)^2 + B(-1)(-2) + 16(-1) \\ 0 &= 16 + 2B - 16 \\ 0 &= 2B \\ 0 &= B \end{aligned}$$

In this case we got $B = 0$ this will happen on occasion, but do not expect it to happen in all cases. Here is the partial fraction decomposition for this part.

$$\frac{4x^2}{(x-1)(x-2)^2} = \frac{4}{x-1} + \frac{16}{(x-2)^2}$$

(d) $\dfrac{9x+25}{(x+3)^2}$

Again, the denominator has already been factored for us. In this case the form of the partial fraction decomposition is,

$$\frac{9x+25}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

Adding the two terms together gives,

$$\frac{9x+25}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2}$$

Notice that in this case the second term already had the LCD under it and so we didn't need to add anything in that time.

Setting the numerators equal gives,

$$9x + 25 = A(x+3) + B$$

Now, again, we can get *B* for free by picking $x = -3$.

$$9(-3) + 25 = A(-3+3) + B$$
$$-2 = B$$

To find *A* we will do the same thing that we did in the previous part. We'll use $x = 0$ and the fact that we know what *B* is.

$$25 = A(3) - 2$$
$$27 = 3A$$
$$9 = A$$

In this case, notice that the constant in the numerator of the first isn't zero as it was in the previous part. Here is the partial fraction decomposition for this part.

$$\frac{9x+25}{(x+3)^2} = \frac{9}{x+3} - \frac{2}{(x+3)^2}$$

Now, we need to do a set of examples with quadratic factors. Note however, that this is where the work often gets fairly messy and in fact we haven't covered the material yet that will allow us to work many of these problems. We can work some simple examples however, so let's do that.

Example 2 Determine the partial fraction decomposition of each of the following.

(a) $\dfrac{8x^2-12}{x(x^2+2x-6)}$

(b) $\dfrac{3x^3+7x-4}{(x^2+2)^2}$

Solution

(a) $\dfrac{8x^2-12}{x(x^2+2x-6)}$

In this case the *x* that sits in the front is a linear term since we can write it as,

$$x = x + 0$$

and so the form of the partial fraction decomposition is,

$$\frac{8x^2-12}{x(x^2+2x-6)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x-6}$$

Now we'll use the fact that the LCD is $x(x^2+2x-6)$ and add the two terms together,

$$\frac{8x^2-12}{x(x^2+2x-6)} = \frac{A(x^2+2x-6)+x(Bx+C)}{x(x^2+2x-6)}$$

Next, set the numerators equal.

$$8x^2 - 12 = A\left(x^2 + 2x - 6\right) + x\left(Bx + C\right)$$

This is where the process changes from the previous set of examples. We could choose $x = 0$ to get the value of *A*, but that's the only constant that we could get using this method and so it just won't work all that well here.

What we need to do here is multiply the right side out and then collect all the like terms as follows,

$$8x^2 - 12 = Ax^2 + 2Ax - 6A + Bx^2 + Cx$$
$$8x^2 - 12 = \left(A + B\right)x^2 + \left(2A + C\right)x - 6A$$

Now, we need to choose *A*, *B*, and *C* so that these two are equal. That means that the coefficient of the x^2 term on the right side will have to be 8 since that is the coefficient of the x^2 term on the left side. Likewise, the coefficient of the *x* term on the right side must be zero since there isn't an *x* term on the left side. Finally the constant term on the right side must be -12 since that is the constant on the left side.

We generally call this **setting coefficients equal** and we'll write down the following equations.

$$\begin{aligned} A + B &= 8 \\ 2A + C &= 0 \\ -6A &= -12 \end{aligned}$$

Now, we haven't talked about how to solve systems of equations yet, but this is one that we can do without that knowledge. We can solve the third equation directly for *A* to get that $A = 2$. We can then plug this into the first two equations to get,

$$\begin{aligned} 2 + B &= 8 \qquad \Rightarrow \qquad B = 6 \\ 2\left(2\right) + C &= 0 \qquad \Rightarrow \qquad C = -4 \end{aligned}$$

So, the partial fraction decomposition for this expression is,

$$\frac{8x^2 - 12}{x\left(x^2 + 2x - 6\right)} = \frac{2}{x} + \frac{6x - 4}{x^2 + 2x - 6}$$

(b) $\dfrac{3x^3 + 7x - 4}{\left(x^2 + 2\right)^2}$

Here is the form of the partial fraction decomposition for this part.

$$\frac{3x^3 + 7x - 4}{\left(x^2 + 2\right)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{\left(x^2 + 2\right)^2}$$

Adding the two terms up gives,

$$\frac{3x^3+7x-4}{\left(x^2+2\right)^2}=\frac{\left(Ax+B\right)\left(x^2+2\right)+Cx+D}{\left(x^2+2\right)^2}$$

Now, set the numerators equal and we might as well go ahead and multiply the right side out and collect up like terms while we're at it.

$$3x^3+7x-4=\left(Ax+B\right)\left(x^2+2\right)+Cx+D$$
$$3x^3+7x-4=Ax^3+2Ax+Bx^2+2B+Cx+D$$
$$3x^3+7x-4=Ax^3+Bx^2+\left(2A+C\right)x+2B+D$$

Setting coefficients equal gives,

$$\begin{aligned} A&=3\\ B&=0\\ 2A+C&=7\\ 2B+D&=-4 \end{aligned}$$

In this case we got *A* and *B* for free and don't get excited about the fact that $B=0$. This is not a problem and in fact when this happens the remaining work is often a little easier. So, plugging the known values of *A* and *B* into the remaining two equations gives,

$$2\left(3\right)+C=7 \qquad \Rightarrow \qquad C=1$$
$$2\left(0\right)+D=-4 \qquad \Rightarrow \qquad D=-4$$

The partial fraction decomposition is then,

$$\frac{3x^3+7x-4}{\left(x^2+2\right)^2}=\frac{3x}{x^2+2}+\frac{x-4}{\left(x^2+2\right)^2}$$

Chapter 6 : Exponential and Logarithm Functions

In this chapter we are going to look at exponential and logarithm functions. Both of these functions are very important and need to be understood by anyone who is going on to later math courses. These functions also have applications in science, engineering, and business to name a few areas. In fact, these functions can show up in just about any field that uses even a small degree of mathematics.

Many students find both of these functions, especially logarithm functions, difficult to deal with. This is probably because they are so different from any of the other functions that they've looked at to this point and logarithms use a notation that will be new to almost everyone in an algebra class. However, you will find that once you get past the notation and start to understand some of their properties they really aren't too bad.

Here is a listing of the topics covered in this chapter.

Exponential Functions – In this section we will introduce exponential functions. We will give some of the basic properties and graphs of exponential functions. We will also discuss what many people consider to be the exponential function, $f\left(x\right)=\mathbf{e}^{x}$.

Logarithm Functions – In this section we will introduce logarithm functions. We give the basic properties and graphs of logarithm functions. In addition, we discuss how to evaluate some basic logarithms including the use of the change of base formula. We will also discuss the common logarithm, $\log\left(x\right)$, and the natural logarithm, $\ln\left(x\right)$.

Solving Exponential Equations – In this section we will discuss a couple of methods for solving equations that contain exponentials.

Solving Logarithm Equations – In this section we will discuss a couple of methods for solving equations that contain logarithms. Also, as we'll see, with one of the methods we will need to be careful of the results of the method as it is always possible that the method gives values that are, in fact, not solutions to the equation.

Applications – In this section we will look at a couple of applications of exponential functions and an application of logarithms. We look at compound interest, exponential growth and decay and earthquake intensity.

Section 6-1 : Exponential Functions

Let's start off this section with the definition of an exponential function.

If *b* is any number such that $b > 0$ and $b \neq 1$ then an **exponential function** is a function in the form,

$$f\left(x\right) = b^x$$

where *b* is called the **base** and *x* can be any real number.

Notice that the *x* is now in the exponent and the base is a fixed number. This is exactly the opposite from what we've seen to this point. To this point the base has been the variable, *x* in most cases, and the exponent was a fixed number. However, despite these differences these functions evaluate in exactly the same way as those that we are used to. We will see some examples of exponential functions shortly.

Before we get too far into this section we should address the restrictions on *b*. We avoid one and zero because in this case the function would be,

$$f\left(x\right) = 0^x = 0 \qquad \text{and} \qquad f\left(x\right) = 1^x = 1$$

and these are constant functions and won't have many of the same properties that general exponential functions have.

Next, we avoid negative numbers so that we don't get any complex values out of the function evaluation. For instance, if we allowed $b = -4$ the function would be,

$$f\left(x\right) = \left(-4\right)^x \qquad \Rightarrow \qquad f\left(\frac{1}{2}\right) = \left(-4\right)^{\frac{1}{2}} = \sqrt{-4}$$

and as you can see there are some function evaluations that will give complex numbers. We only want real numbers to arise from function evaluation and so to make sure of this we require that *b* not be a negative number.

Now, let's take a look at a couple of graphs. We will be able to get most of the properties of exponential functions from these graphs.

Example 1 Sketch the graph of $f\left(x\right) = 2^x$ and $g\left(x\right) = \left(\frac{1}{2}\right)^x$ on the same axis system.

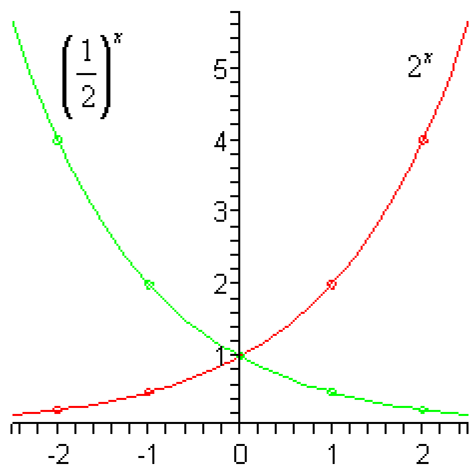
Solution

Okay, since we don't have any knowledge on what these graphs look like we're going to have to pick some values of *x* and do some function evaluations. Function evaluation with exponential functions works in exactly the same manner that all function evaluation has worked to this point. Whatever is in the parenthesis on the left we substitute into all the *x*'s on the right side.

Here are some evaluations for these two functions,

x	$f(x)=2^x$	$g(x)=\left(\frac{1}{2}\right)^x$
-2	$f(-2)=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$	$g(-2)=\left(\frac{1}{2}\right)^{-2}=\left(\frac{2}{1}\right)^{2}=4$
-1	$f(-1)=2^{-1}=\frac{1}{2^1}=\frac{1}{2}$	$g(-1)=\left(\frac{1}{2}\right)^{-1}=\left(\frac{2}{1}\right)^{1}=2$
0	$f(0)=2^0=1$	$g(0)=\left(\frac{1}{2}\right)^{0}=1$
1	$f(1)=2^1=2$	$g(1)=\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$
2	$f(2)=2^2=4$	$g(2)=\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$

Here is the sketch of the two graphs.



Note as well that we could have written $g(x)$ in the following way,

$$g(x)=\left(\frac{1}{2}\right)^x=\frac{1}{2^x}=2^{-x}$$

Sometimes we'll see this kind of exponential function and so it's important to be able to go between these two forms.

Now, let's talk about some of the properties of exponential functions.

Properties of $f\left(x\right)=b^x$

1. The graph of $f\left(x\right)$ will always contain the point $\left(0,1\right)$. Or put another way, $f\left(0\right)=1$ regardless of the value of *b*.

2. For every possible *b* we have $b^x > 0$. Note that this implies that $b^x \neq 0$.

3. If $0 < b < 1$ then the graph of b^x will decrease as we move from left to right. Check out the graph of $\left(\frac{1}{2}\right)^x$ above for verification of this property.

4. If $b > 1$ then the graph of b^x will increase as we move from left to right. Check out the graph of 2^x above for verification of this property.

5. If $b^x = b^y$ then $x = y$

All of these properties except the final one can be verified easily from the graphs in the first example. We will hold off discussing the final property for a couple of sections where we will actually be using it.

As a final topic in this section we need to discuss a special exponential function. In fact this is so special that for many people this is THE exponential function. Here it is,

$$f\left(x\right)=\mathbf{e}^x$$

where $\mathbf{e}=2.718281828\ldots$. Note the difference between $f\left(x\right)=b^x$ and $f\left(x\right)=\mathbf{e}^x$. In the first case *b* is any number that meets the restrictions given above while **e** is a very specific number. Also note that **e** is not a terminating decimal.

This special exponential function is very important and arises naturally in many areas. As noted above, this function arises so often that many people will think of this function if you talk about exponential functions. We will see some of the applications of this function in the final section of this chapter.

Let's get a quick graph of this function.

Example 2 Sketch the graph of $f\left(x\right)=\mathbf{e}^x$.

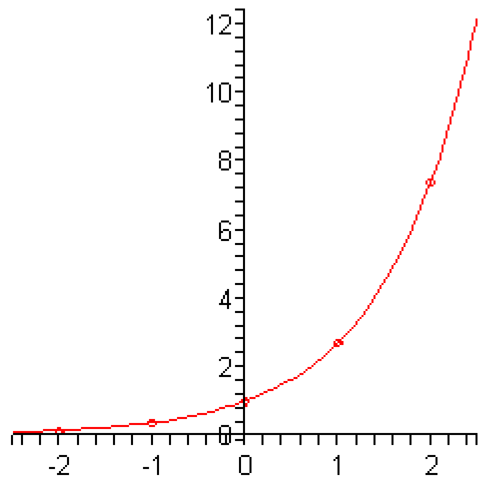
Solution

Let's first build up a table of values for this function.

x	-2	-1	0	1	2
f(x)	0.1353…	0.3679…	1	2.718…	7.389…

To get these evaluation (with the exception of $x = 0$) you will need to use a calculator. In fact, that is part of the point of this example. Make sure that you can run your calculator and verify these numbers.

Here is a sketch of this graph.



Notice that this is an increasing graph as we should expect since $\mathbf{e} = 2.718281827\ldots > 1$.

There is one final example that we need to work before moving onto the next section. This example is more about the evaluation process for exponential functions than the graphing process. We need to be very careful with the evaluation of exponential functions.

Example 3 Sketch the graph of $g(x) = 5\mathbf{e}^{1-x} - 4$.

Solution

Here is a quick table of values for this function.

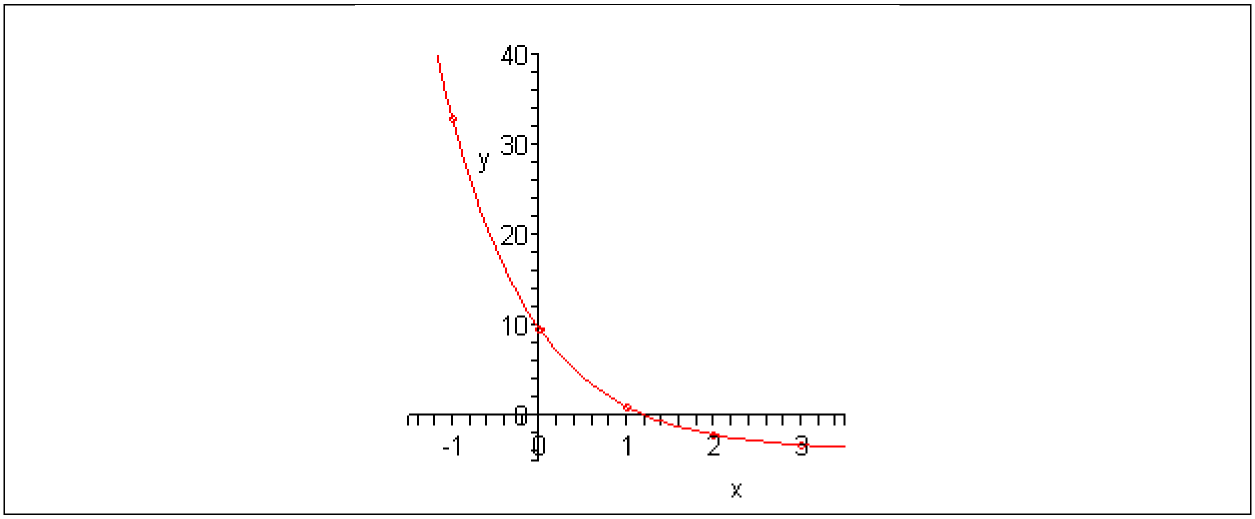
x	-1	0	1	2	3
g(x)	32.945…	9.591…	1	-2.161…	-3.323…

Now, as we stated above this example was more about the evaluation process than the graph so let's go through the first one to make sure that you can do these.

$$\begin{aligned} g(-1) &= 5\mathbf{e}^{1-(-1)} - 4 \\ &= 5\mathbf{e}^{2} - 4 \\ &= 5(7.389) - 4 \end{aligned}$$

Notice that when evaluating exponential functions we first need to actually do the exponentiation before we multiply by any coefficients (5 in this case). Also, we used only 3 decimal places here since we are only graphing. In many applications we will want to use far more decimal places in these computations.

Here is a sketch of the graph.



Notice that this graph violates all the properties we listed above. That is okay. Those properties are only valid for functions in the form $f(x) = b^x$ or $f(x) = \mathbf{e}^x$. We've got a lot more going on in this function and so the properties, as written above, won't hold for this function.

Section 6-2 : Logarithm Functions

In this section we now need to move into logarithm functions. This can be a tricky function to graph right away. There is going to be some different notation that you aren't used to and some of the properties may not be all that intuitive. Do not get discouraged however. Once you figure these out you will find that they really aren't that bad and it usually just takes a little working with them to get them figured out.

Here is the definition of the logarithm function.

If *b* is any number such that $b > 0$ and $b \neq 1$ and $x > 0$ then,

$$y = \log_b x \qquad \text{is equivalent to} \qquad b^y = x$$

We usually read this as "log base *b* of *x*".

In this definition $y = \log_b x$ is called the **logarithm form** and $b^y = x$ is called the **exponential form**.

Note that the requirement that $x > 0$ is really a result of the fact that we are also requiring $b > 0$. If you think about it, it will make sense. We are raising a positive number to an exponent and so there is no way that the result can possibly be anything other than another positive number. It is very important to remember that we can't take the logarithm of zero or a negative number.

Now, let's address the notation used here as that is usually the biggest hurdle that students need to overcome before starting to understand logarithms. First, the "log" part of the function is simply three letters that are used to denote the fact that we are dealing with a logarithm. They are not variables and they aren't signifying multiplication. They are just there to tell us we are dealing with a logarithm.

Next, the *b* that is subscripted on the "log" part is there to tell us what the base is as this is an important piece of information. Also, despite what it might look like there is no exponentiation in the logarithm form above. It might look like we've got b^x in that form, but it isn't. It just looks like that might be what's happening.

It is important to keep the notation with logarithms straight, if you don't you will find it very difficult to understand them and to work with them.

Now, let's take a quick look at how we evaluate logarithms.

Example 1 Evaluate each of the following logarithms.

(a) $\log_4 16$

(b) $\log_2 16$

(c) $\log_6 216$

(d) $\log_5 \dfrac{1}{125}$

(e) $\log_{\frac{1}{3}} 81$

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$

Solution

Now, the reality is that evaluating logarithms directly can be a very difficult process, even for those who really understand them. It is usually much easier to first convert the logarithm form into exponential form. In that form we can usually get the answer pretty quickly.

(a) $\log_4 16$

Okay what we are really asking here is the following.

$$\log_4 16 = ?$$

As suggested above, let's convert this to exponential form.

$$\log_4 16 = ? \qquad\qquad \Rightarrow \qquad\qquad 4^{?} = 16$$

Most people cannot evaluate the logarithm $\log_4 16$ right off the top of their head. However, most people can determine the exponent that we need on 4 to get 16 once we do the exponentiation. So, since,

$$4^2 = 16$$

we must have the following value of the logarithm.

$$\log_4 16 = 2$$

(b) $\log_2 16$

This one is similar to the previous part. Let's first convert to exponential form.

$$\log_2 16 = ? \qquad\qquad \Rightarrow \qquad\qquad 2^{?} = 16$$

If you don't know this answer right off the top of your head, start trying numbers. In other words, compute 2^2, 2^3, 2^4, *etc* until you get 16. In this case we need an exponent of 4. Therefore, the value of this logarithm is,

$$\log_2 16 = 4$$

Before moving on to the next part notice that the base on these is a very important piece of notation. Changing the base will change the answer and so we always need to keep track of the base.

(c) $\log_6 216$

We'll do this one without any real explanation to see how well you've got the evaluation of logarithms down.

$$\log_6 216 = 3 \qquad \text{because} \qquad 6^3 = 216$$

(d) $\log_5 \dfrac{1}{125}$

Now, this one looks different from the previous parts, but it really isn't any different. As always let's first convert to exponential form.

$$\log_5 \frac{1}{125} = ? \qquad \Rightarrow \qquad 5^? = \frac{1}{125}$$

First, notice that the only way that we can raise an integer to an integer power and get a fraction as an answer is for the exponent to be negative. So, we know that the exponent has to be negative.

Now, let's ignore the fraction for a second and ask $5^? = 125$. In this case if we cube 5 we will get 125.

So, it looks like we have the following,

$$\log_5 \frac{1}{125} = -3 \qquad \text{because} \qquad 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

(e) $\log_{\frac{1}{3}} 81$

Converting this logarithm to exponential form gives,

$$\log_{\frac{1}{3}} 81 = ? \qquad \Rightarrow \qquad \left(\frac{1}{3}\right)^? = 81$$

Now, just like the previous part, the only way that this is going to work out is if the exponent is negative. Then all we need to do is recognize that $3^4 = 81$ and we can see that,

$$\log_{\frac{1}{3}} 81 = -4 \qquad \text{because} \qquad \left(\frac{1}{3}\right)^{-4} = \left(\frac{3}{1}\right)^4 = 3^4 = 81$$

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$

Here is the answer to this one.

$$\log_{\frac{3}{2}} \frac{27}{8} = 3 \qquad \text{because} \qquad \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$$

Hopefully, you now have an idea on how to evaluate logarithms and are starting to get a grasp on the notation. There are a few more evaluations that we want to do however, we need to introduce some special logarithms that occur on a very regular basis. They are the **common logarithm** and the **natural logarithm**. Here are the definitions and notations that we will be using for these two logarithms.

$$\text{common logarithm}: \qquad \log x = \log_{10} x$$
$$\text{natural logarithm}: \qquad \ln x = \log_{\mathbf{e}} x$$

So, the common logarithm is simply the log base 10, except we drop the “base 10” part of the notation. Similarly, the natural logarithm is simply the log base **e** with a different notation and where **e** is the same number that we saw in the previous section and is defined to be $\mathbf{e} = 2.718281827\ldots$.

Let’s take a look at a couple more evaluations.

Example 2 Evaluate each of the following logarithms.

(a) $\log 1000$

(b) $\log \dfrac{1}{100}$

(c) $\ln \dfrac{1}{\mathbf{e}}$

(d) $\ln \sqrt{\mathbf{e}}$

(e) $\log_{34} 34$

(f) $\log_8 1$

Solution

To do the first four evaluations we just need to remember what the notation for these are and what base is implied by the notation. The final two evaluations are to illustrate some of the properties of all logarithms that we’ll be looking at eventually.

(a) $\log 1000 = 3$ because $10^3 = 1000$.

(b) $\log \dfrac{1}{100} = -2$ because $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$.

(c) $\ln \dfrac{1}{\mathbf{e}} = -1$ because $\mathbf{e}^{-1} = \dfrac{1}{\mathbf{e}}$.

(d) $\ln \sqrt{\mathbf{e}} = \dfrac{1}{2}$ because $\mathbf{e}^{\frac{1}{2}} = \sqrt{\mathbf{e}}$. Notice that with this one we are really just acknowledging a change of notation from fractional exponent into radical form.

(e) $\log_{34} 34 = 1$ because $34^1 = 34$. Notice that this one will work regardless of the base that we’re using.

(f) $\log_8 1 = 0$ because $8^0 = 1$. Again, note that the base that we’re using here won’t change the answer.

So, when evaluating logarithms all that we’re really asking is what exponent did we put onto the base to get the number in the logarithm.

Now, before we get into some of the properties of logarithms let's first do a couple of quick graphs.

Example 3 Sketch the graph of the common logarithm and the natural logarithm on the same axis system.

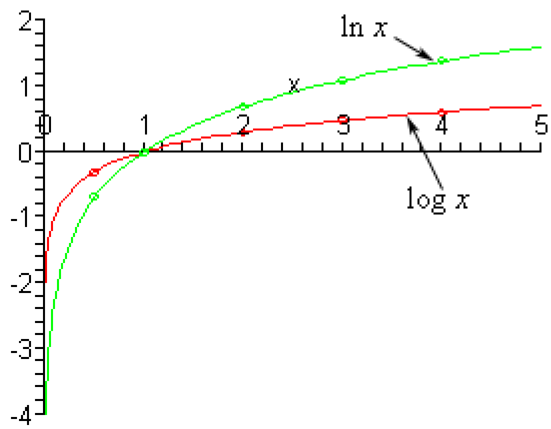
Solution

This example has two points. First, it will familiarize us with the graphs of the two logarithms that we are most likely to see in other classes. Also, it will give us some practice using our calculator to evaluate these logarithms because the reality is that is how we will need to do most of these evaluations.

Here is a table of values for the two logarithms.

x	$\log x$	$\ln x$
$\frac{1}{2}$	-0.3010	-0.6931
1	0	0
2	0.3010	0.6931
3	0.4771	1.0986
4	0.6021	1.3863

Here is a sketch of the graphs of these two functions.



Now let's start looking at some properties of logarithms. We'll start off with some basic evaluation properties.

Properties of Logarithms

1. $\log_b 1 = 0$. This follows from the fact that $b^0 = 1$.
2. $\log_b b = 1$. This follows from the fact that $b^1 = b$.
3. $\log_b b^x = x$. This can be generalized out to $\log_b b^{f(x)} = f(x)$.
4. $b^{\log_b x} = x$. This can be generalized out to $b^{\log_b f(x)} = f(x)$.

Properties 3 and 4 leads to a nice relationship between the logarithm and exponential function. Let's first compute the following function compositions for $f(x) = b^x$ and $g(x) = \log_b x$.

$$(f \circ g)(x) = f\left[g(x)\right] = f(\log_b x) = b^{\log_b x} = x$$
$$(g \circ f)(x) = g\left[f(x)\right] = g\left[b^x\right] = \log_b b^x = x$$

Recall from the section on inverse functions that this means that the exponential and logarithm functions are inverses of each other. This is a nice fact to remember on occasion.

We should also give the generalized version of Properties 3 and 4 in terms of both the natural and common logarithm as we'll be seeing those in the next couple of sections on occasion.

$$\ln \mathbf{e}^{f(x)} = f(x) \qquad \log 10^{f(x)} = f(x)$$
$$\mathbf{e}^{\ln f(x)} = f(x) \qquad 10^{\log f(x)} = f(x)$$

Now, let's take a look at some manipulation properties of the logarithm.

More Properties of Logarithms

For these properties we will assume that $x > 0$ and $y > 0$.

5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b\left(x^r\right) = r\log_b x$
8. If $\log_b x = \log_b y$ then $x = y$.

We won't be doing anything with the final property in this section; it is here only for the sake of completeness. We will be looking at this property in detail in a couple of sections.

The first two properties listed here can be a little confusing at first since on one side we've got a product or a quotient inside the logarithm and on the other side we've got a sum or difference of two logarithms. We will just need to be careful with these properties and make sure to use them correctly.

Also, note that there are no rules on how to break up the logarithm of the sum or difference of two terms. To be clear about this let's note the following,

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

Be careful with these and do not try to use these as they simply aren't true.

Note that all of the properties given to this point are valid for both the common and natural logarithms. We just didn't write them out explicitly using the notation for these two logarithms, the properties do hold for them nonetheless

Now, let's see some examples of how to use these properties.

Example 4 Simplify each of the following logarithms.

(a) $\log_4\left(x^3 y^5\right)$

(b) $\log\left(\dfrac{x^9 y^5}{z^3}\right)$

(c) $\ln\sqrt{xy}$

(d) $\log_3\left(\dfrac{\left(x+y\right)^2}{x^2+y^2}\right)$

Solution

The instructions here may be a little misleading. When we say simplify we really mean to say that we want to use as many of the logarithm properties as we can.

(a) $\log_4\left(x^3 y^5\right)$

Note that we can't use Property 7 to bring the 3 and the 5 down into the front of the logarithm at this point. In order to use Property 7 the whole term in the logarithm needs to be raised to the power. In this case the two exponents are only on individual terms in the logarithm and so Property 7 can't be used here.

We do, however, have a product inside the logarithm so we can use Property 5 on this logarithm.

$$\log_4\left(x^3 y^5\right)=\log_4\left(x^3\right)+\log_4\left(y^5\right)$$

Now that we've done this we can use Property 7 on each of these individual logarithms to get the final simplified answer.

$$\log_4\left(x^3 y^5\right)=3\log_4 x+5\log_4 y$$

(b) $\log\left(\dfrac{x^9 y^5}{z^3}\right)$

In this case we've got a product and a quotient in the logarithm. In these cases it is almost always best to deal with the quotient before dealing with the product. Here is the first step in this part.

$$\log\left(\frac{x^9 y^5}{z^3}\right)=\log\left(x^9 y^5\right)-\log z^3$$

Now, we'll break up the product in the first term and once we've done that we'll take care of the exponents on the terms.

$$\log\left(\frac{x^9 y^5}{z^3}\right) = \log\left(x^9 y^5\right) - \log z^3$$
$$= \log x^9 + \log y^5 - \log z^3$$
$$= 9\log x + 5\log y - 3\log z$$

(c) $\ln\sqrt{xy}$

For this part let's first rewrite the logarithm a little so that we can see the first step.

$$\ln\sqrt{xy} = \ln\left(xy\right)^{\frac{1}{2}}$$

Written in this form we can see that there is a single exponent on the whole term and so we'll take care of that first.

$$\ln\sqrt{xy} = \frac{1}{2}\ln\left(xy\right)$$

Now, we will take care of the product.

$$\ln\sqrt{xy} = \frac{1}{2}\left(\ln x + \ln y\right)$$

Notice the parenthesis in this the answer. The $\frac{1}{2}$ multiplies the original logarithm and so it will also need to multiply the whole "simplified" logarithm. Therefore, we need to have a set of parenthesis there to make sure that this is taken care of correctly.

(d) $\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right)$

We'll first take care of the quotient in this logarithm.

$$\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right) = \log_3\left(x+y\right)^2 - \log_3\left(x^2+y^2\right)$$

We now reach the real point to this problem. The second logarithm is as simplified as we can make it. Remember that we can't break up a log of a sum or difference and so this can't be broken up any farther. Also, we can only deal with exponents if the term as a whole is raised to the exponent. The fact that both pieces of this term are squared doesn't matter. It needs to be the whole term squared, as in the first logarithm.

So, we can further simplify the first logarithm, but the second logarithm can't be simplified any more. Here is the final answer for this problem.

$$\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right) = 2\log_3\left(x+y\right) - \log_3\left(x^2+y^2\right)$$

Now, we need to work some examples that go the other way. This next set of examples is probably more important than the previous set. We will be doing this kind of logarithm work in a couple of sections.

Example 5 Write each of the following as a single logarithm with a coefficient of 1.

(a) $7\log_{12} x + 2\log_{12} y$

(b) $3\log x - 6\log y$

(c) $5\ln\left(x+y\right) - 2\ln y - 8\ln x$

Solution

The instruction requiring a coefficient of 1 means that the when we get down to a final logarithm there shouldn't be any number in front of the logarithm.

Note as well that these examples are going to be using Properties 5 – 7 only we'll be using them in reverse. We will have expressions that look like the right side of the property and use the property to write it so it looks like the left side of the property.

(a) The first step here is to get rid of the coefficients on the logarithms. This will use Property 7 in reverse. In this direction, Property 7 says that we can move the coefficient of a logarithm up to become a power on the term inside the logarithm.

Here is that step for this part.

$$7\log_{12} x + 2\log_{12} y = \log_{12} x^7 + \log_{12} y^2$$

We've now got a sum of two logarithms both with coefficients of 1 and both with the same base. This means that we can use Property 5 in reverse. Here is the answer for this part.

$$7\log_{12} x + 2\log_{12} y = \log_{12}\left(x^7 y^2\right)$$

(b) Again, we will first take care of the coefficients on the logarithms.

$$3\log x - 6\log y = \log x^3 - \log y^6$$

We now have a difference of two logarithms and so we can use Property 6 in reverse. When using Property 6 in reverse remember that the term from the logarithm that is subtracted off goes in the denominator of the quotient. Here is the answer to this part.

$$3\log x - 6\log y = \log\left(\frac{x^3}{y^6}\right)$$

(c) In this case we've got three terms to deal with and none of the properties have three terms in them. That isn't a problem. Let's first take care of the coefficients and at the same time we'll factor a minus sign out of the last two terms. The reason for this will be apparent in the next step.

$$5\ln\left(x+y\right) - 2\ln y - 8\ln x = \ln\left(x+y\right)^5 - \left(\ln y^2 + \ln x^8\right)$$

Now, notice that the quantity in the parenthesis is a sum of two logarithms and so can be combined into a single logarithm with a product as follows,

$$5\ln\left(x+y\right) - 2\ln y - 8\ln x = \ln\left(x+y\right)^5 - \ln\left(y^2 x^8\right)$$

Now we are down to two logarithms and they are a difference of logarithms and so we can write it as a single logarithm with a quotient.

$$5\ln\left(x+y\right)-2\ln y-8\ln x=\ln\left(\frac{\left(x+y\right)^5}{y^2x^8}\right)$$

The final topic that we need to discuss in this section is the **change of base** formula.

Most calculators these days are capable of evaluating common logarithms and natural logarithms. However, that is about it, so what do we do if we need to evaluate another logarithm that can't be done easily as we did in the first set of examples that we looked at?

To do this we have the change of base formula. Here is the change of base formula.

$$\log_a x=\frac{\log_b x}{\log_b a}$$

where we can choose *b* to be anything we want it to be. In order to use this to help us evaluate logarithms this is usually the common or natural logarithm. Here is the change of base formula using both the common logarithm and the natural logarithm.

$$\log_a x=\frac{\log x}{\log a} \qquad\qquad \log_a x=\frac{\ln x}{\ln a}$$

Let's see how this works with an example.

Example 6 Evaluate $\log_5 7$.

Solution

First, notice that we can't use the same method to do this evaluation that we did in the first set of examples. This would require us to look at the following exponential form,

$$5^? = 7$$

and that's just not something that anyone can answer off the top of their head. If the 7 had been a 5, or a 25, or a 125, *etc*. we could do this, but it's not. Therefore, we have to use the change of base formula.

Now, we can use either one and we'll get the same answer. So, let's use both and verify that. We'll start with the common logarithm form of the change of base.

$$\log_5 7=\frac{\log 7}{\log 5}=\frac{0.845098040014}{0.698970004336}=1.20906195512$$

Now, let's try the natural logarithm form of the change of base formula.

$$\log_5 7=\frac{\ln 7}{\ln 5}=\frac{1.94591014906}{1.60943791243}=1.20906195512$$

So, we got the same answer despite the fact that the fractions involved different answers.

Section 6-3 : Solving Exponential Equations

Now that we've seen the definitions of exponential and logarithm functions we need to start thinking about how to solve equations involving them. In this section we will look at solving exponential equations and we will look at solving logarithm equations in the next section.

There are two methods for solving exponential equations. One method is fairly simple but requires a very special form of the exponential equation. The other will work on more complicated exponential equations but can be a little messy at times.

Let's start off by looking at the simpler method. This method will use the following fact about exponential functions.

$$\text{If} \quad b^x = b^y \quad \text{then} \quad x = y$$

Note that this fact does require that the base in both exponentials to be the same. If it isn't then this fact will do us no good.

Let's take a look at a couple of examples.

Example 1 Solve each of the following.

(a) $5^{3x} = 5^{7x-2}$

(b) $4^{t^2} = 4^{6-t}$

(c) $3^z = 9^{z+5}$

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

Solution

(a) $5^{3x} = 5^{7x-2}$

In this first part we have the same base on both exponentials so there really isn't much to do other than to set the two exponents equal to each other and solve for *x*.

$$\begin{aligned} 3x &= 7x - 2 \\ 2 &= 4x \\ \frac{1}{2} &= x \end{aligned}$$

So, if we were to plug $x = \dfrac{1}{2}$ into the equation then we would get the same number on both sides of the equal sign.

(b) $4^{t^2} = 4^{6-t}$

Again, there really isn't much to do here other than set the exponents equal since the base is the same in both exponentials.

$$\begin{aligned} t^2 &= 6 - t \\ t^2 + t - 6 &= 0 \\ (t+3)(t-2) &= 0 \qquad\qquad \Rightarrow \qquad t = -3,\ t = 2 \end{aligned}$$

In this case we get two solutions to the equation. That is perfectly acceptable so don't worry about it when it happens.

(c) $3^z = 9^{z+5}$

Now, in this case we don't have the same base so we can't just set exponents equal. However, with a little manipulation of the right side we can get the same base on both exponents. To do this all we need to notice is that $9 = 3^2$. Here's what we get when we use this fact.

$$3^z = \left(3^2\right)^{z+5}$$

Now, we still can't just set exponents equal since the right side now has two exponents. If we recall our exponent properties we can fix this however.

$$3^z = 3^{2(z+5)}$$

We now have the same base and a single exponent on each base so we can use the property and set the exponents equal. Doing this gives,

$$\begin{aligned} z &= 2(z+5) \\ z &= 2z + 10 \\ z &= -10 \end{aligned}$$

So, after all that work we get a solution of $z = -10$.

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

In this part we've got some issues with both sides. First the right side is a fraction and the left side isn't. That is not the problem that it might appear to be however, so for a second let's ignore that. The real issue here is that we can't write 8 as a power of 4 and we can't write 4 as a power of 8 as we did in the previous part.

The first thing to do in this problem is to get the same base on both sides and to so that we'll have to note that we can write both 4 and 8 as a power of 2. So let's do that.

$$\begin{aligned} \left(2^2\right)^{5-9x} &= \frac{1}{\left(2^3\right)^{x-2}} \\ 2^{2(5-9x)} &= \frac{1}{2^{3(x-2)}} \end{aligned}$$

It's now time to take care of the fraction on the right side. To do this we simply need to remember the following exponent property.

$$\frac{1}{a^n} = a^{-n}$$

Using this gives,

$$2^{2(5-9x)} = 2^{-3(x-2)}$$

So, we now have the same base and each base has a single exponent on it so we can set the exponents equal.

$$\begin{aligned} 2(5-9x) &= -3(x-2) \\ 10-18x &= -3x+6 \\ 4 &= 15x \\ x &= \frac{4}{15} \end{aligned}$$

And there is the answer to this part.

Now, the equations in the previous set of examples all relied upon the fact that we were able to get the same base on both exponentials, but that just isn't always possible. Consider the following equation.

$$7^x = 9$$

This is a fairly simple equation however the method we used in the previous examples just won't work because we don't know how to write 9 as a power of 7. In fact, if you think about it that is exactly what this equation is asking us to find.

So, the method we used in the first set of examples won't work. The problem here is that the *x* is in the exponent. Because of that all our knowledge about solving equations won't do us any good. We need a way to get the *x* out of the exponent and luckily for us we have a way to do that. Recall the following logarithm property from the last section.

$$\log_b a^r = r \log_b a$$

Note that to avoid confusion with *x*'s we replaced the *x* in this property with an *a*. The important part of this property is that we can take an exponent and move it into the front of the term.

So, if we had,

$$\log_b 7^x$$

we could use this property as follows.

$$x \log_b 7$$

The *x* in now out of the exponent! Of course, we are now stuck with a logarithm in the problem and not only that but we haven't specified the base of the logarithm.

The reality is that we can use any logarithm to do this so we should pick one that we can deal with. This usually means that we'll work with the common logarithm or the natural logarithm.

So, let's work a set of examples to see how we actually use this idea to solve these equations.

Example 2 Solve each of the following equations.

(a) $7^x = 9$

(b) $2^{4y+1} - 3^y = 0$

(c) $\mathbf{e}^{t+6} = 2$

(d) $10^{5-x} = 8$

(e) $5\mathbf{e}^{2z+4} - 8 = 0$

Solution

(a) $7^x = 9$

Okay, so we say above that if we had a logarithm in front the left side we could get the *x* out of the exponent. That's easy enough to do. We'll just put a logarithm in front of the left side. However, if we put a logarithm there we also must put a logarithm in front of the right side. This is commonly referred to as **taking the logarithm of both sides**.

We can use any logarithm that we'd like to so let's try the natural logarithm.

$$\begin{aligned} \ln 7^x &= \ln 9 \\ x \ln 7 &= \ln 9 \end{aligned}$$

Now, we need to solve for *x*. This is easier than it looks. If we had $7x = 9$ then we could all solve for *x* simply by dividing both sides by 7. It works in exactly the same manner here. Both ln7 and ln9 are just numbers. Admittedly, it would take a calculator to determine just what those numbers are, but they are numbers and so we can do the same thing here.

$$\begin{aligned} \frac{x \ln 7}{\ln 7} &= \frac{\ln 9}{\ln 7} \\ x &= \frac{\ln 9}{\ln 7} \end{aligned}$$

Now, that is technically the exact answer. However, in this case it's usually best to get a decimal answer so let's go one step further.

$$x = \frac{\ln 9}{\ln 7} = \frac{2.19722458}{1.94591015} = 1.12915007$$

Note that the answers to these are decimal answers more often than not.

Also, be careful here to not make the following mistake.

$$1.12915007 = \frac{\ln 9}{\ln 7} \neq \ln\left(\frac{9}{7}\right) = 0.2513144283$$

The two are clearly different numbers.

Finally, let's also use the common logarithm to make sure that we get the same answer.

$$\begin{aligned} \log 7^x &= \log 9 \\ x \log 7 &= \log 9 \\ x &= \frac{\log 9}{\log 7} = \frac{0.954242509}{0.845098040} = 1.12915007 \end{aligned}$$

So, sure enough the same answer. We can use either logarithm, although there are times when it is more convenient to use one over the other.

(b) $2^{4y+1} - 3^y = 0$

In this case we can't just put a logarithm in front of both sides. There are two reasons for this. First on the right side we've got a zero and we know from the previous section that we can't take the logarithm of zero. Next, in order to move the exponent down it has to be on the whole term inside the logarithm and that just won't be the case with this equation in its present form.

So, the first step is to move on of the terms to the other side of the equal sign, then we will take the logarithm of both sides using the natural logarithm.

$$\begin{aligned} 2^{4y+1} &= 3^y \\ \ln 2^{4y+1} &= \ln 3^y \\ (4y+1)\ln 2 &= y \ln 3 \end{aligned}$$

Okay, this looks messy, but again, it's really not that bad. Let's look at the following equation first.

$$\begin{aligned} 2(4y+1) &= 3y \\ 8y+2 &= 3y \\ 5y &= -2 \\ y &= -\frac{2}{5} \end{aligned}$$

We can all solve this equation and so that means that we can solve the one that we've got. Again, the ln2 and ln3 are just numbers and so the process is exactly the same. The answer will be messier than this equation, but the process is identical. Here is the work for this one.

$$\begin{aligned} (4y+1)\ln 2 &= y \ln 3 \\ 4y \ln 2 + \ln 2 &= y \ln 3 \\ 4y \ln 2 - y \ln 3 &= -\ln 2 \\ y(4\ln 2 - \ln 3) &= -\ln 2 \\ y &= -\frac{\ln 2}{4\ln 2 - \ln 3} \end{aligned}$$

So, we get all the terms with *y* in them on one side and all the other terms on the other side. Once this is done we then factor out a *y* and divide by the coefficient. Again, we would prefer a decimal answer so let's get that.

$$y = -\frac{\ln 2}{4\ln 2 - \ln 3} = -\frac{0.693147181}{4\left(0.693147181\right) - 1.098612289} = -0.414072245$$

(c) $\mathbf{e}^{t+6} = 2$

Now, this one is a little easier than the previous one. Again, we'll take the natural logarithm of both sides.

$$\ln \mathbf{e}^{t+6} = \ln 2$$

Notice that we didn't take the exponent out of this one. That is because we want to use the following property with this one.

$$\ln \mathbf{e}^{f(x)} = f\left(x\right)$$

We saw this in the previous section (in more general form) and by using this here we will make our life significantly easier. Using this property gives,

$$t + 6 = \ln 2$$
$$t = \ln\left(2\right) - 6 = 0.69314718 - 6 = -5.30685202$$

Notice the parenthesis around the 2 in the logarithm this time. They are there to make sure that we don't make the following mistake.

$$-5.30685202 = \ln\left(2\right) - 6 \neq \ln\left(2-6\right) = \ln\left(-4\right) \text{ can't be done}$$

Be very careful with this mistake. It is easy to make when you aren't paying attention to what you're doing or are in a hurry.

(d) $10^{5-x} = 8$

The equation in this part is similar to the previous part except this time we've got a base of 10 and so recalling the fact that,

$$\log 10^{f(x)} = f\left(x\right)$$

it makes more sense to use common logarithms this time around.

Here is the work for this equation.

$$\begin{aligned} \log 10^{5-x} &= \log 8 \\ 5 - x &= \log 8 \\ 5 - \log 8 &= x \qquad \Rightarrow \qquad x = 5 - 0.903089987 = 4.096910013 \end{aligned}$$

This could have been done with natural logarithms but the work would have been messier.

(e) $5\mathbf{e}^{2z+4} - 8 = 0$

With this final equation we've got a couple of issues. First, we'll need to move the number over to the other side. In order to take the logarithm of both sides we need to have the exponential on one side by itself. Doing this gives,

$$5\mathbf{e}^{2z+4} = 8$$

Next, we've got to get a coefficient of 1 on the exponential. We can only use the facts to simplify this if there isn't a coefficient on the exponential. So, divide both sides by 5 to get,

$$\mathbf{e}^{2z+4} = \frac{8}{5}$$

At this point we will take the logarithm of both sides using the natural logarithm since there is an **e** in the equation.

$$\begin{aligned} \ln \mathbf{e}^{2z+4} &= \ln\left(\frac{8}{5}\right) \\ 2z+4 &= \ln\left(\frac{8}{5}\right) \\ 2z &= \ln\left(\frac{8}{5}\right) - 4 \\ z &= \frac{1}{2}\left(\ln\left(\frac{8}{5}\right) - 4\right) = \frac{1}{2}\left(0.470003629 - 4\right) = -1.76499819 \end{aligned}$$

Note that we could have used this second method on the first set of examples as well if we'd wanted to although the work would have been more complicated and prone to mistakes if we'd done that.

Section 6-4 : Solving Logarithm Equations

In this section we will now take a look at solving logarithmic equations, or equations with logarithms in them. We will be looking at two specific types of equations here. In particular we will look at equations in which every term is a logarithm and we also look at equations in which all but one term in the equation is a logarithm and the term without the logarithm will be a constant. Also, we will be assuming that the logarithms in each equation will have the same base. If there is more than one base in the logarithms in the equation the solution process becomes much more difficult.

Before we get into the solution process we will need to remember that we can only plug positive numbers into a logarithm. This will be important down the road and so we can't forget that.

Now, let's start off by looking at equations in which each term is a logarithm and all the bases on the logarithms are the same. In this case we will use the fact that,

$$\text{If } \log_b x = \log_b y \text{ then } x = y$$

In other words, if we've got two logs in the problem, one on either side of an equal sign and both with a coefficient of one, then we can just drop the logarithms.

Let's take a look at a couple of examples.

Example 1 Solve each of the following equations.

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

(b) $\log x + \log\left(x-1\right) = \log\left(3x+12\right)$

(c) $\ln 10 - \ln\left(7-x\right) = \ln x$

Solution

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

With this equation there are only two logarithms in the equation so it's easy to get on one either side of the equal sign. We will also need to deal with the coefficient in front of the first term.

$$\log_9\left(\sqrt{x}\right)^2 = \log_9\left(6x-1\right)$$
$$\log_9 x = \log_9\left(6x-1\right)$$

Now that we've got two logarithms with the same base and coefficients of 1 on either side of the equal sign we can drop the logs and solve.

$$x = 6x - 1$$
$$1 = 5x \qquad \Rightarrow \qquad x = \frac{1}{5}$$

Now, we do need to worry if this solution will produce any negative numbers or zeroes in the logarithms so the next step is to plug this into the **original** equation and see if it does.

$$2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(6\left(\frac{1}{5}\right)-1\right)=2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(\frac{1}{5}\right)=0$$

Note that we don't need to go all the way out with the check here. We just need to make sure that once we plug in the *x* we don't have any negative numbers or zeroes in the logarithms. Since we don't in this case we have the solution, it is $x=\frac{1}{5}$.

(b) $\log x+\log\left(x-1\right)=\log\left(3x+12\right)$

Okay, in this equation we've got three logarithms and we can only have two. So, we saw how to do this kind of work in a set of examples in the previous section so we just need to do the same thing here. It doesn't really matter how we do this, but since one side already has one logarithm on it we might as well combine the logs on the other side.

$$\log\left(x\left(x-1\right)\right)=\log\left(3x+12\right)$$

Now we've got one logarithm on either side of the equal sign, they are the same base and have coefficients of one so we can drop the logarithms and solve.

$$\begin{aligned} x\left(x-1\right)&=3x+12\\ x^2-x-3x-12&=0\\ x^2-4x-12&=0\\ \left(x-6\right)\left(x+2\right)&=0 \qquad\Rightarrow\qquad x=-2,\ x=6\end{aligned}$$

Now, before we declare these to be solutions we MUST check them in the original equation.

$x=6$:

$$\begin{aligned}\log 6+\log\left(6-1\right)&=\log\left(3\left(6\right)+12\right)\\ \log 6+\log 5&=\log 30\end{aligned}$$

No logarithms of negative numbers and no logarithms of zero so this is a solution.

$x=-2$:

$$\log\left(-2\right)+\log\left(-2-1\right)=\log\left(3\left(-2\right)+12\right)$$

We don't need to go any farther, there is a logarithm of a negative number in the first term (the others are also negative) and that's all we need in order to exclude this as a solution.

Be careful here. We are not excluding $x=-2$ because it is negative, that's not the problem. We are excluding it because once we plug it into the original equation we end up with logarithms of negative numbers. It is possible to have negative values of *x* be solutions to these problems, so don't mistake the reason for excluding this value.

Also, along those lines we didn't take $x = 6$ as a solution because it was positive, but because it didn't produce any negative numbers or zero in the logarithms upon substitution. It is possible for positive numbers to not be solutions.

So, with all that out of the way, we've got a single solution to this equation, $x = 6$.

(c) $\ln 10 - \ln\left(7 - x\right) = \ln x$

We will work this equation in the same manner that we worked the previous one. We've got two logarithms on one side so we'll combine those, drop the logarithms and then solve.

$$\begin{aligned} \ln\left(\frac{10}{7-x}\right) &= \ln x \\ \frac{10}{7-x} &= x \\ 10 &= x\left(7-x\right) \\ 10 &= 7x - x^2 \\ x^2 - 7x + 10 &= 0 \\ \left(x-5\right)\left(x-2\right) &= 0 \qquad \Rightarrow \qquad x = 2,\ x = 5 \end{aligned}$$

We've got two possible solutions to check here.

$x = 2$:

$$\begin{aligned} \ln 10 - \ln\left(7-2\right) &= \ln 2 \\ \ln 10 - \ln 5 &= \ln 2 \end{aligned}$$

This one is okay.

$x = 5$:

$$\begin{aligned} \ln 10 - \ln\left(7-5\right) &= \ln 5 \\ \ln 10 - \ln 2 &= \ln 5 \end{aligned}$$

This one is also okay.

In this case both possible solutions, $x = 2$ and $x = 5$, end up actually being solutions. There is no reason to expect to always have to throw one of the two out as a solution.

Now we need to take a look at the second kind of logarithmic equation that we'll be solving here. This equation will have all the terms but one be a logarithm and the one term that doesn't have a logarithm will be a constant.

In order to solve these kinds of equations we will need to remember the exponential form of the logarithm. Here it is if you don't remember.

$$y = \log_b x \qquad \Rightarrow \qquad b^y = x$$

We will be using this conversion to exponential form in all of these equations so it's important that you can do it. Let's work some examples so we can see how these kinds of equations can be solved.

Example 2 Solve each of the following equations.

(a) $\log_5(2x+4)=2$

(b) $\log x = 1-\log(x-3)$

(c) $\log_2(x^2-6x)=3+\log_2(1-x)$

Solution

(a) $\log_5(2x+4)=2$

To solve these we need to get the equation into exactly the form that this one is in. We need a single log in the equation with a coefficient of one and a constant on the other side of the equal sign. Once we have the equation in this form we simply convert to exponential form.

So, let's do that with this equation. The exponential form of this equation is,

$$2x+4=5^2=25$$

Notice that this is an equation that we can easily solve.

$$2x=21 \qquad \Rightarrow \qquad x=\frac{21}{2}$$

Now, just as with the first set of examples we need to plug this back into the **original** equation and see if it will produce negative numbers or zeroes in the logarithms. If it does it can't be a solution and if it doesn't then it is a solution.

$$\begin{aligned}\log_5\left(2\left(\frac{21}{2}\right)+4\right)&=2\\ \log_5(25)&=2\end{aligned}$$

Only positive numbers in the logarithm and so $x=\frac{21}{2}$ is in fact a solution.

(b) $\log x = 1-\log(x-3)$

In this case we've got two logarithms in the problem so we are going to have to combine them into a single logarithm as we did in the first set of examples. Doing this for this equation gives,

$$\begin{aligned}\log x+\log(x-3)&=1\\ \log(x(x-3))&=1\end{aligned}$$

Now, that we've got the equation into the proper form we convert to exponential form. Recall as well that we're dealing with the common logarithm here and so the base is 10.

Here is the exponential form of this equation.

$$x(x-3) = 10^1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0 \qquad \Rightarrow \qquad x = -2,\ x = 5$$

So, we've got two potential solutions. Let's check them both.

$x = -2$:

$$\log(-2) = 1 - \log(-2-3)$$

We've got negative numbers in the logarithms and so this can't be a solution.

$x = 5$:

$$\log 5 = 1 - \log(5-3)$$
$$\log 5 = 1 - \log 2$$

No negative numbers or zeroes in the logarithms and so this is a solution.

Therefore, we have a single solution to this equation, $x = 5$.

Again, remember that we don't exclude a potential solution because it's negative or include a potential solution because it's positive. We exclude a potential solution if it produces negative numbers or zeroes in the logarithms upon substituting it into the equation and we include a potential solution if it doesn't.

(c) $\log_2(x^2 - 6x) = 3 + \log_2(1-x)$

Again, let's get the logarithms onto one side and combined into a single logarithm.

$$\log_2(x^2 - 6x) - \log_2(1-x) = 3$$
$$\log_2\left(\frac{x^2 - 6x}{1-x}\right) = 3$$

Now, convert it to exponential form.

$$\frac{x^2 - 6x}{1-x} = 2^3 = 8$$

Now, let's solve this equation.

$$x^2 - 6x = 8(1-x)$$
$$x^2 - 6x = 8 - 8x$$
$$x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0 \qquad \Rightarrow \qquad x = -4,\ x = 2$$

Now, let's check both of these solutions in the original equation.

$x = -4$:

$$\log_2\left(\left(-4\right)^2 - 6\left(-4\right)\right) = 3 + \log_2\left(1 - \left(-4\right)\right)$$
$$\log_2\left(16 + 24\right) = 3 + \log_2\left(5\right)$$

So, upon substituting this solution in we see that all the numbers in the logarithms are positive and so this IS a solution. Note again that it doesn't matter that the solution is negative, it just can't produce negative numbers or zeroes in the logarithms.

$x = 2$:

$$\log_2\left(2^2 - 6\left(2\right)\right) = 3 + \log_2\left(1 - 2\right)$$
$$\log_2\left(4 - 12\right) = 3 + \log_2\left(-1\right)$$

In this case, despite the fact that the potential solution is positive we get negative numbers in the logarithms and so it can't possibly be a solution.

Therefore, we get a single solution for this equation, $x = -4$.

Section 6-5 : Applications

In this final section of this chapter we need to look at some applications of exponential and logarithm functions.

Compound Interest

This first application is compounding interest and there are actually two separate formulas that we'll be looking at here. Let's first get those out of the way.

If we were to put *P* dollars into an account that earns interest at a rate of *r* (written as a decimal) for *t* years (yes, it must be years) then,

1. if interest is compounded *m* times per year we will have
$$A = P\left(1 + \frac{r}{m}\right)^{t\,m}$$
dollars after *t* years.

2. if interest is compounded continuously then we will have
$$A = P\mathbf{e}^{r\,t}$$
dollars after *t* years.

Let's take a look at a couple of examples.

Example 1 We are going to invest \$100,000 in an account that earns interest at a rate of 7.5% for 54 months. Determine how much money will be in the account if,

(a) interest is compounded quarterly.
(b) interest is compounded monthly.
(c) interest is compounded continuously.

Solution

Before getting into each part let's identify the quantities that we will need in all the parts and won't change.

$$P = 100,000 \qquad\qquad r = \frac{7.5}{100} = 0.075 \qquad\qquad t = \frac{54}{12} = 4.5$$

Remember that interest rates must be decimals for these computations and *t* must be in years! Now, let's work the problems.

(a) Interest is compounded quarterly.

In this part the interest is compounded quarterly and that means it is compounded 4 times a year. After 54 months we then have,

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\ &= 100000(1.01875)^{18} \\ &= 100000(1.39706686207) \\ &= 139706.686207 = \$139,706.69 \end{aligned}$$

Notice the amount of decimal places used here. We didn't do any rounding until the very last step. It is important to not do too much rounding in intermediate steps with these problems.

(b) Interest is compounded monthly.
Here we are compounding monthly and so that means we are compounding 12 times a year. Here is how much we'll have after 54 months.

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{12}\right)^{(12)(4.5)} \\ &= 100000(1.00625)^{54} \\ &= 100000(1.39996843023) \\ &= 139996.843023 = \$139,996.84 \end{aligned}$$

So, compounding more times per year will yield more money.

(c) Interest is compounded continuously.
Finally, if we compound continuously then after 54 months we will have,

$$\begin{aligned} A &= 100000\mathbf{e}^{(0.075)(4.5)} \\ &= 100000(1.40143960839) \\ &= 140143.960839 = \$140,143.96 \end{aligned}$$

Now, as pointed out in the first part of this example it is important to not round too much before the final answer. Let's go back and work the first part again and this time let's round to three decimal places at each step.

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\ &= 100000(1.019)^{18} \\ &= 100000(1.403) \\ &= \$140,300.00 \end{aligned}$$

This answer is off from the correct answer by $593.31 and that's a fairly large difference. So, how many decimal places should we keep in these? Well, unfortunately the answer is that it depends. The larger the initial amount the more decimal places we will need to keep around. As a general rule of thumb, set your calculator to the maximum number of decimal places it can handle and take all of them until the final answer and then round at that point.

Let's now look at a different kind of example with compounding interest.

Example 2 We are going to put \$2500 into an account that earns interest at a rate of 12%. If we want to have \$4000 in the account when we close it how long should we keep the money in the account if,

(a) we compound interest continuously.

(b) we compound interest 6 times a year.

Solution

Again, let's identify the quantities that won't change with each part.

$$A = 4000 \qquad P = 2500 \qquad r = \frac{12}{100} = 0.12$$

Notice that this time we've been given *A* and are asking to find *t*. This means that we are going to have to solve an exponential equation to get at the answer.

(a) Compound interest continuously.

Let's first set up the equation that we'll need to solve.

$$4000 = 2500\mathbf{e}^{0.12t}$$

Now, we saw how to solve these kinds of equations a couple of sections ago. In that section we saw that we need to get the exponential on one side by itself with a coefficient of 1 and then take the natural logarithm of both sides. Let's do that.

$$\begin{aligned} \frac{4000}{2500} &= \mathbf{e}^{0.12t} \\ 1.6 &= \mathbf{e}^{0.12t} \\ \ln 1.6 &= \ln \mathbf{e}^{0.12t} \\ \ln 1.6 &= 0.12t \qquad \Rightarrow \qquad t = \frac{\ln 1.6}{0.12} = 3.917 \end{aligned}$$

We need to keep the amount in the account for 3.917 years to get \$4000.

(b) Compound interest 6 times a year.

Again, let's first set up the equation that we need to solve.

$$\begin{aligned} 4000 &= 2500\left(1 + \frac{0.12}{6}\right)^{6t} \\ 4000 &= 2500\left(1.02\right)^{6t} \end{aligned}$$

We will solve this the same way that we solved the previous part. The work will be a little messier, but for the most part it will be the same.

$$\frac{4000}{2500} = (1.02)^{6t}$$
$$1.6 = (1.02)^{6t}$$
$$\ln 1.6 = \ln(1.02)^{6t}$$
$$\ln 1.6 = 6t\ln(1.02)$$
$$t = \frac{\ln 1.6}{6\ln(1.02)} = \frac{0.470003629246}{6(0.019802627296)} = 3.956$$

In this case we need to keep the amount slightly longer to reach $4000.

Exponential Growth and Decay

There are many quantities out there in the world that are governed (at least for a short time period) by the equation,

$$Q = Q_0 \mathbf{e}^{kt}$$

where Q_0 is positive and is the amount initially present at $t = 0$ and *k* is a non-zero constant. If *k* is positive then the equation will grow without bound and is called the **exponential growth** equation. Likewise, if *k* is negative the equation will die down to zero and is called the **exponential decay** equation.

Short term population growth is often modeled by the exponential growth equation and the decay of a radioactive element is governed the exponential decay equation.

Example 3 The growth of a colony of bacteria is given by the equation,

$$Q = Q_0 \mathbf{e}^{0.195t}$$

If there are initially 500 bacteria present and *t* is given in hours determine each of the following.

(a) How many bacteria are there after a half of a day?

(b) How long will it take before there are 10000 bacteria in the colony?

Solution

Here is the equation for this starting amount of bacteria.

$$Q = 500\,\mathbf{e}^{0.195t}$$

(a) How many bacteria are there after a half of a day?

In this case if we want the number of bacteria after half of a day we will need to use $t = 12$ since *t* is in hours. So, to get the answer to this part we just need to plug *t* into the equation.

$$Q = 500\,\mathbf{e}^{0.195(12)} = 500(10.3812365627) = 5190.618$$

So, since a fractional population doesn't make much sense we'll say that after half of a day there are 5190 of the bacteria present.

(b) How long will it take before there are 10000 bacteria in the colony?

Do NOT make the mistake of assuming that it will be approximately 1 day for this answer based on the answer to the previous part. With exponential growth things just don't work that way as we'll see. In order to answer this part we will need to solve the following exponential equation.

$$10000 = 500\,\mathbf{e}^{0.195\,t}$$

Let's do that.

$$\frac{10000}{500} = \mathbf{e}^{0.195\,t}$$
$$20 = \mathbf{e}^{0.195\,t}$$
$$\ln 20 = \ln \mathbf{e}^{0.195\,t}$$
$$\ln 20 = 0.195t \qquad \Rightarrow \qquad t = \frac{\ln 20}{0.195} = 15.3627$$

So, it only takes approximately 15.4 hours to reach 10000 bacteria and NOT 24 hours if we just double the time from the first part. In other words, be careful!

Example 4 Carbon 14 dating works by measuring the amount of Carbon 14 (a radioactive element) that is in a fossil. All living things have a constant level of Carbon 14 in them and once they die it starts to decay according to the formula,

$$Q = Q_0\mathbf{e}^{-0.000124t}$$

where *t* is in years and Q_0 is the amount of Carbon 14 present at death and for this example let's assume that there will be 100 milligrams present at death.

(a) How much Carbon 14 will there be after 1000 years?
(b) How long will it take for half of the Carbon 14 to decay?

Solution

(a) How much Carbon 14 will there be after 1000 years?

In this case all we need to do is plug in *t=1000* into the equation.

$$Q = 100\mathbf{e}^{-0.000124(1000)} = 100\left(0.883379840883\right) = 88.338 \text{ milligrams}$$

So, it looks like we will have around 88.338 milligrams left after 1000 years.

(b) How long will it take for half of the Carbon 14 to decay?

So, we want to know how long it will take until there is 50 milligrams of the Carbon 14 left. That means we will have to solve the following equation,

$$50 = 100\mathbf{e}^{-0.000124t}$$

Here is that work.

$$\frac{50}{100} = \mathbf{e}^{-0.000124t}$$
$$\frac{1}{2} = \mathbf{e}^{-0.000124t}$$
$$\ln\frac{1}{2} = \ln \mathbf{e}^{-0.000124t}$$
$$\ln\frac{1}{2} = -0.000124t$$
$$t = \frac{\ln\frac{1}{2}}{-0.000124} = \frac{-0.69314718056}{-0.000124} = 5589.89661742$$

So, it looks like it will take about 5589.897 years for half of the Carbon 14 to decay. This number is called the **half-life** of Carbon 14.

We've now looked at a couple of applications of exponential equations and we should now look at a quick application of a logarithm.

Earthquake Intensity

The **Richter scale** is commonly used to measure the intensity of an earthquake. There are many different ways of computing this based on a variety of different quantities. We are going to take a quick look at the formula that uses the energy released during an earthquake.

If *E* is the energy released, measured in joules, during an earthquake then the magnitude of the earthquake is given by,

$$M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$$

where $E_0 = 10^{4.4}$ joules.

Example 5 If 8×10^{14} joules of energy is released during an earthquake what was the magnitude of the earthquake?

Solution

There really isn't much to do here other than to plug into the formula.

$$M = \frac{2}{3}\log\left(\frac{8\times10^{14}}{10^{4.4}}\right) = \frac{2}{3}\log\left(8\times10^{9.6}\right) = \frac{2}{3}\left(10.50308999\right) = 7.002$$

So, it looks like we'll have a magnitude of about 7.

Example 6 How much energy will be released in an earthquake with a magnitude of 5.9?

Solution

In this case we will need to solve the following equation.

$$5.9 = \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right)$$

We saw how solve these kinds of equations in the previous section. First we need the logarithm on one side by itself with a coefficient of one. Once we have it in that form we convert to exponential form and solve.

$$\begin{aligned} 5.9 &= \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right) \\ 8.85 &= \log\left(\frac{E}{10^{4.4}}\right) \\ 10^{8.85} &= \frac{E}{10^{4.4}} \\ E &= 10^{8.85}\left(10^{4.4}\right) = 10^{13.25} \end{aligned}$$

So, it looks like there would be a release of $10^{13.25}$ joules of energy in an earthquake with a magnitude of 5.9.

Chapter 7 : Systems of Equations

This is a fairly short chapter devoted to solving systems of equations. A system of equations is a set of equations each containing one or more variable.

We will focus exclusively on systems of two equations with two unknowns and three equations with three unknowns although the methods looked at here can be easily extended to more equations. Also, with the exception of the last section we will be dealing only with systems of linear equations.

Here is a list of the topics in this section.

Section 7-1 : Linear Systems with Two Variables

A linear system of two equations with two variables is any system that can be written in the form.

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

where any of the constants can be zero with the exception that each equation must have at least one variable in it.

Also, the system is called linear if the variables are only to the first power, are only in the numerator and there are no products of variables in any of the equations.

Here is an example of a system with numbers.

$$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

Before we discuss how to solve systems we should first talk about just what a solution to a system of equations is. A solution to a system of equations is a value of *x* and a value of *y* that, when substituted into the equations, satisfies both equations at the same time.

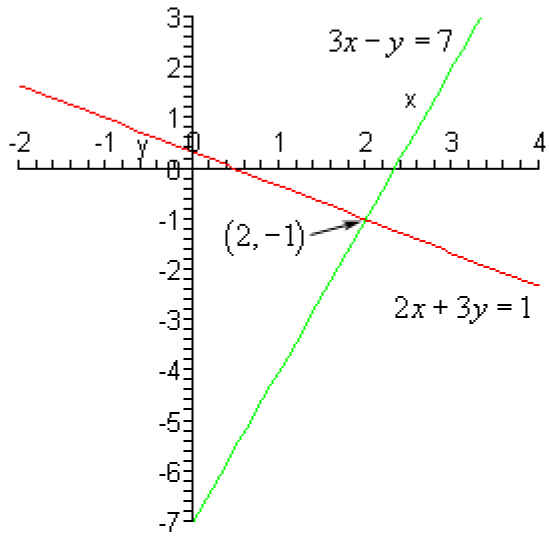
For the example above $x = 2$ and $y = -1$ is a solution to the system. This is easy enough to check.

$$\begin{aligned} 3(2) - (-1) &= 7 \\ 2(2) + 3(-1) &= 1 \end{aligned}$$

So, sure enough that pair of numbers is a solution to the system. Do not worry about how we got these values. This will be the very first system that we solve when we get into examples.

Note that it is important that the pair of numbers satisfy both equations. For instance, $x = 1$ and $y = -4$ will satisfy the first equation, but not the second and so isn't a solution to the system. Likewise, $x = -1$ and $y = 1$ will satisfy the second equation but not the first and so can't be a solution to the system.

Now, just what does a solution to a system of two equations represent? Well if you think about it both of the equations in the system are lines. So, let's graph them and see what we get.



As you can see the solution to the system is the coordinates of the point where the two lines intersect. So, when solving linear systems with two variables we are really asking where the two lines will intersect.

We will be looking at two methods for solving systems in this section.

The first method is called the **method of substitution**. In this method we will solve one of the equations for one of the variables and substitute this into the other equation. This will yield one equation with one variable that we can solve. Once this is solved we substitute this value back into one of the equations to find the value of the remaining variable.

In words this method is not always very clear. Let's work a couple of examples to see how this method works.

Example 1 Solve each of the following systems.

(a) $$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

(b) $$\begin{aligned} 5x + 4y &= 1 \\ 3x - 6y &= 2 \end{aligned}$$

Solution

(a) $$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

So, this was the first system that we looked at above. We already know the solution, but this will give us a chance to verify the values that we wrote down for the solution.

Now, the method says that we need to solve one of the equations for one of the variables. Which equation we choose and which variable that we choose is up to you, but it's usually best to pick an equation and variable that will be easy to deal with. This means we should try to avoid fractions if at all possible.

In this case it looks like it will be really easy to solve the first equation for *y* so let's do that.

$$3x - 7 = y$$

Now, substitute this into the second equation.

$$2x + 3\left(3x - 7\right) = 1$$

This is an equation in *x* that we can solve so let's do that.

$$\begin{aligned} 2x + 9x - 21 &= 1 \\ 11x &= 22 \\ x &= 2 \end{aligned}$$

So, there is the *x* portion of the solution.

Finally, do NOT forget to go back and find the *y* portion of the solution. This is one of the more common mistakes students make in solving systems. To so this we can either plug the *x* value into one of the original equations and solve for *y* or we can just plug it into our substitution that we found in the first step. That will be easier so let's do that.

$$y = 3x - 7 = 3\left(2\right) - 7 = -1$$

So, the solution is $x = 2$ and $y = -1$ as we noted above.

(b) $\begin{aligned} 5x + 4y &= 1 \\ 3x - 6y &= 2 \end{aligned}$

With this system we aren't going to be able to completely avoid fractions. However, it looks like if we solve the second equation for *x* we can minimize them. Here is that work.

$$\begin{aligned} 3x &= 6y + 2 \\ x &= 2y + \frac{2}{3} \end{aligned}$$

Now, substitute this into the first equation and solve the resulting equation for *y*.

$$5\left(2y+\frac{2}{3}\right)+4y=1$$

$$10y+\frac{10}{3}+4y=1$$

$$14y=1-\frac{10}{3}=-\frac{7}{3}$$

$$y=-\left(\frac{7}{3}\right)\left(\frac{1}{14}\right)$$

$$y=-\frac{1}{6}$$

Finally, substitute this into the original substitution to find *x*.

$$x=2\left(-\frac{1}{6}\right)+\frac{2}{3}=-\frac{1}{3}+\frac{2}{3}=\frac{1}{3}$$

So, the solution to this system is $x=\frac{1}{3}$ and $y=-\frac{1}{6}$.

As with single equations we could always go back and check this solution by plugging it into both equations and making sure that it does satisfy both equations. Note as well that we really would need to plug into both equations. It is quite possible that a mistake could result in a pair of numbers that would satisfy one of the equations but not the other one.

Let's now move into the next method for solving systems of equations. As we saw in the last part of the previous example the method of substitution will often force us to deal with fractions, which adds to the likelihood of mistakes. This second method will not have this problem. Well, that's not completely true. If fractions are going to show up they will only show up in the final step and they will only show up if the solution contains fractions.

This second method is called the **method of elimination**. In this method we multiply one or both of the equations by appropriate numbers (*i.e.* multiply every term in the equation by the number) so that one of the variables will have the same coefficient with opposite signs. Then next step is to add the two equations together. Because one of the variables had the same coefficient with opposite signs it will be eliminated when we add the two equations. The result will be a single equation that we can solve for one of the variables. Once this is done substitute this answer back into one of the original equations.

As with the first method it's much easier to see what's going on here with a couple of examples.

Example 2 Solve each of the following systems of equations.

(a) $\begin{aligned} 5x+4y&=1 \\ 3x-6y&=2 \end{aligned}$

(b) $\begin{aligned} 2x+4y&=-10 \\ 6x+3y&=6 \end{aligned}$

Solution

(a) $\begin{aligned} 5x+4y&=1 \\ 3x-6y&=2 \end{aligned}$

This is the system in the previous set of examples that made us work with fractions. Working it here will show the differences between the two methods and it will also show that either method can be used to get the solution to a system.

So, we need to multiply one or both equations by constants so that one of the variables has the same coefficient with opposite signs. So, since the *y* terms already have opposite signs let's work with these terms. It looks like if we multiply the first equation by 3 and the second equation by 2 the *y* terms will have coefficients of 12 and -12 which is what we need for this method.

Here is the work for this step.

$$\begin{array}{rcr} 5x+4y=1 & \underrightarrow{\times 3} & 15x+12y=3 \\ 3x-6y=2 & \underrightarrow{\times 2} & \underline{6x-12y=4} \\ & & 21x \qquad =7 \end{array}$$

So, as the description of the method promised we have an equation that can be solved for *x*. Doing this gives, $x=\frac{1}{3}$ which is exactly what we found in the previous example. Notice however, that the only fraction that we had to deal with to this point is the answer itself which is different from the method of substitution.

Now, again don't forget to find *y*. In this case it will be a little more work than the method of substitution. To find *y* we need to substitute the value of *x* into either of the original equations and solve for *y*. Since *x* is a fraction let's notice that, in this case, if we plug this value into the second equation we will lose the fractions at least temporarily. Note that often this won't happen and we'll be forced to deal with fractions whether we want to or not.

$$\begin{aligned} 3\left(\frac{1}{3}\right)-6y&=2 \\ 1-6y&=2 \\ -6y&=1 \\ y&=-\frac{1}{6} \end{aligned}$$

Again, this is the same value we found in the previous example.

(b) $$\begin{aligned} 2x + 4y &= -10 \\ 6x + 3y &= 6 \end{aligned}$$

In this part all the variables are positive so we're going to have to force an opposite sign by multiplying by a negative number somewhere. Let's also notice that in this case if we just multiply the first equation by -3 then the coefficients of the *x* will be -6 and 6.

Sometimes we only need to multiply one of the equations and can leave the other one alone. Here is this work for this part.

$$\begin{aligned} 2x + 4y &= -10 \quad & \xrightarrow{\times -3} \quad & -6x - 12y = 30 \\ 6x + 3y &= 6 \quad & \xrightarrow{\text{same}} \quad & \underline{\;\; 6x + 3y = 6} \\ & & & \quad -9y = 36 \\ & & & \qquad y = -4 \end{aligned}$$

Finally, plug this into either of the equations and solve for *x*. We will use the first equation this time.

$$\begin{aligned} 2x + 4(-4) &= -10 \\ 2x - 16 &= -10 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

So, the solution to this system is $x = 3$ and $y = -4$.

There is a third method that we'll be looking at to solve systems of two equations, but it's a little more complicated and is probably more useful for systems with at least three equations so we'll look at it in a later section.

Before leaving this section we should address a couple of special case in solving systems.

Example 3 Solve the following systems of equations.

$$\begin{aligned} x - y &= 6 \\ -2x + 2y &= 1 \end{aligned}$$

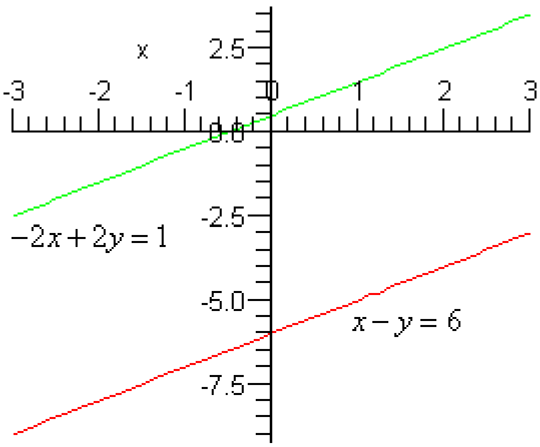
Solution

We can use either method here, but it looks like substitution would probably be slightly easier. We'll solve the first equation for *x* and substitute that into the second equation.

$$x = 6 + y$$

$$-2(6 + y) + 2y = 1$$
$$-12 - 2y + 2y = 1$$
$$-12 = 1 \;\; ??$$

So, this is clearly not true and there doesn't appear to be a mistake anywhere in our work. So, what's the problem? To see let's graph these two lines and see what we get.



It appears that these two lines are parallel (can you verify that with the slopes?) and we know that two parallel lines with different *y*-intercepts (that's important) will never cross.

As we saw in the opening discussion of this section solutions represent the point where two lines intersect. If two lines don't intersect we can't have a solution.

So, when we get this kind of nonsensical answer from our work we have two parallel lines and there is **no solution** to this system of equations.

The system in the previous example is called **inconsistent**. Note as well that if we'd used elimination on this system we would have ended up with a similar nonsensical answer.

Example 4 Solve the following system of equations.

$$2x + 5y = -1$$
$$-10x - 25y = 5$$

Solution

In this example it looks like elimination would be the easiest method.

$$\begin{array}{rcr} 2x + 5y = -1 & \xrightarrow{\times 5} & 10x + 25y = -5 \\ -10x - 25y = 5 & \xrightarrow{\text{same}} & \underline{-10x - 25y = 5} \\ & & 0 = 0 \end{array}$$

On first glance this might appear to be the same problem as the previous example. However, in that case we ended up with an equality that simply wasn't true. In this case we have 0=0 and that is a true equality and so in that sense there is nothing wrong with this.

However, this is clearly not what we were expecting for an answer here and so we need to determine just what is going on.

We'll leave it to you to verify this, but if you find the slope and *y*-intercepts for these two lines you will find that both lines have exactly the same slope and both lines have exactly the same *y*-intercept. So, what does this mean for us? Well if two lines have the same slope and the same *y*-intercept then the graphs of the two lines are the same graph. In other words, the graphs of these two lines are the same graph. In these cases any set of points that satisfies one of the equations will also satisfy the other equation.

Also, recall that the graph of an equation is nothing more than the set of all points that satisfies the equation. In other words, there is an infinite set of points that will satisfy this set of equations.

In these cases we do want to write down something for a solution. So, what we'll do is solve one of the equations for one of the variables (it doesn't matter which you choose). We'll solve the first for *y*.

$$\begin{aligned} 2x + 5y &= -1 \\ 5y &= -2x - 1 \\ y &= -\frac{2}{5}x - \frac{1}{5} \end{aligned}$$

Then, given any *x* we can find a *y* and these two numbers will form a solution to the system of equations. We usually denote this by writing the solution as follows,

$$\begin{aligned} x &= t \\ y &= -\frac{2}{5}t - \frac{1}{5} \end{aligned} \qquad \text{where } t \text{ is any real number}$$

To show that these give solutions let's work through a couple of values of *t*.

$t = 0$

$$x = 0 \qquad\qquad y = -\frac{1}{5}$$

To show that this is a solution we need to plug it into both equations in the system.

$$\begin{aligned} 2(0) + 5\left(-\frac{1}{5}\right) &\stackrel{?}{=} -1 \\ -1 &= -1 \end{aligned} \qquad\qquad \begin{aligned} -10(0) - 25\left(-\frac{1}{5}\right) &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$$

So, $x = 0$ and $y = -\frac{1}{5}$ is a solution to the system. Let's do another one real quick.

$t = -3$

$$x = -3 \qquad\qquad y = -\frac{2}{5}\left(-3\right) - \frac{1}{5} = \frac{6}{5} - \frac{1}{5} = 1$$

Again we need to plug it into both equations in the system to show that it's a solution.

$$\begin{aligned} 2\left(-3\right) + 5\left(1\right) &\stackrel{?}{=} -1 \\ -1 &= -1 \end{aligned} \qquad\qquad \begin{aligned} -10\left(-3\right) - 25\left(1\right) &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$$

Sure enough $x = -3$ and $y = 1$ is a solution.

So, since there are an infinite number of possible *t*'s there must be an **infinite number of solutions** to this system and they are given by,

$$\begin{aligned} x &= t \\ y &= -\frac{2}{5}t - \frac{1}{5} \end{aligned} \qquad \text{where } t \text{ is any real number}$$

Systems such as those in the previous examples are called **dependent**.

We've now seen all three possibilities for the solution to a system of equations. A system of equation will have either no solution, exactly one solution or infinitely many solutions.

Section 7-2 : Linear Systems with Three Variables

This is going to be a fairly short section in the sense that it's really only going to consist of a couple of examples to illustrate how to take the methods from the previous section and use them to solve a linear system with three equations and three variables.

So, let's get started with an example.

Example 1 Solve the following system of equations.

$$\begin{aligned} x-2y+3z&=7\\ 2x+y+z&=4\\ -3x+2y-2z&=-10 \end{aligned}$$

Solution

We are going to try and find values of *x*, *y*, and a *z* that will satisfy all three equations at the same time. We are going to use elimination to eliminate one of the variables from one of the equations and two of the variables from another of the equations. The reason for doing this will be apparent once we've actually done it.

The elimination method in this case will work a little differently than with two equations. As with two equations we will multiply as many equations as we need to so that if we start adding pairs of equations we can eliminate one of the variables.

In this case it looks like if we multiply the second equation by 2 it will be fairly simple to eliminate the *y* term from the second and third equation by adding the first equation to both of them. So, let's first multiply the second equation by two.

$$\begin{array}{rcr} x-2y+3z=7 & \xrightarrow{\text{same}} & x-2y+3z=7\\ 2x+y+z=4 & \xrightarrow{\times 2} & 4x+2y+2z=8\\ -3x+2y-2z=-10 & \xrightarrow{\text{same}} & -3x+2y-2z=-10 \end{array}$$

Now, with this new system we will replace the second equation with the sum of the first and second equations and we will replace the third equation with the sum of the first and third equations.

Here is the resulting system of equations.

$$\begin{aligned} x-2y+3z&=7\\ 5x\qquad+5z&=15\\ -2x\qquad+z&=-3 \end{aligned}$$

So, we've eliminated one of the variables from two of the equations. We now need to eliminate either *x* or *z* from either the second or third equations. Again, we will use elimination to do this. In this case we will multiply the third equation by -5 since this will allow us to eliminate *z* from this equation by adding the second onto is.

$$\begin{aligned} x-2y+3z&=7 \\ 5x \qquad +5z&=15 \\ -2x \qquad +z&=-3 \end{aligned} \quad \begin{array}{c} \xrightarrow{\text{same}} \\ \xrightarrow{\text{same}} \\ \xrightarrow{\times -5} \end{array} \quad \begin{aligned} x-2y+3z&=7 \\ 5x \qquad +5z&=15 \\ 10x \qquad -5z&=15 \end{aligned}$$

Now, replace the third equation with the sum of the second and third equation.

$$\begin{aligned} x-2y+3z&=7 \\ 5x \qquad +5z&=15 \\ 15x \qquad\qquad &=30 \end{aligned}$$

Now, at this point notice that the third equation can be quickly solved to find that $x=2$. Once we know this we can plug this into the second equation and that will give us an equation that we can solve for *z* as follows.

$$\begin{aligned} 5(2)+5z&=15 \\ 10+5z&=15 \\ 5z&=5 \\ z&=1 \end{aligned}$$

Finally, we can substitute both *x* and *z* into the first equation which we can use to solve for *y*. Here is that work.

$$\begin{aligned} 2-2y+3(1)&=7 \\ -2y+5&=7 \\ -2y&=2 \\ y&=-1 \end{aligned}$$

So, the solution to this system is $x=2$, $y=-1$ and $z=1$.

That was a fair amount of work and in this case there was even less work than normal because in each case we only had to multiply a single equation to allow us to eliminate variables.

In the next section we'll be looking at a third method for solving systems that is basically a shorthand method for what we did in the previous example. The work using that method will be messy as well, but it will be slightly easier to do once you get the hang of it.

In the previous example all we did was use the method of elimination until we could start solving for the variables and then just back substitute known values of variables into previous equations to find the remaining unknown variables.

Not every linear system with three equations and three variables uses the elimination method exclusively so let's take a look at another example where the substitution method is used, at least partially.

Example 2 Solve the following system of equations.

$$\begin{aligned} 2x-4y+5z&=-33\\ 4x-y&=-5\\ -2x+2y-3z&=19 \end{aligned}$$

Solution

Before we get started on the solution process do not get excited about the fact that the second equation only has two variables in it. That is a fairly common occurrence when we have more than two equations in the system.

In fact, we're going to take advantage of the fact that it only has two variables and one of them, the *y*, has a coefficient of -1. This equation is easily solved for *y* to get,

$$y=4x+5$$

We can then substitute this into the first and third equation as follows,

$$\begin{aligned} 2x-4\left(4x+5\right)+5z&=-33\\ -2x+2\left(4x+5\right)-3z&=19 \end{aligned}$$

Now, if you think about it, this is just a system of two linear equations with two variables (*x* and *z*) and we know how to solve these kinds of systems from our work in the previous section.

First, we'll need to do a little simplification of the system.

$$\begin{aligned} 2x-16x-20+5z&=-33\\ -2x+8x+10-3z&=19 \end{aligned} \qquad \to \qquad \begin{aligned} -14x+5z&=-13\\ 6x-3z&=9 \end{aligned}$$

The simplified version looks just like the systems we were solving in the previous section. Well, it's almost the same. The variables this time are *x* and *z* instead of *x* and *y*, but that really isn't a difference. The work of solving this will be the same.

We can use either the method of substitution or the method of elimination to solve this new system of two linear equations.

If we wanted to use the method of substitution we could easily solve the second equation for *z* (you do see why it would be easiest to solve the second equation for *z* right?) and substitute that into the first equation. This would allow us to find *x* and we could then find both *z* and *y*.

However, to make the point that often we use both methods in solving systems of three linear equations let's use the method of elimination to solve the system of two equations. We'll just need to multiply the first equation by 3 and the second by 5. Doing this gives,

$$\begin{aligned} -14x+5z&=-13 & \xrightarrow{\times 3} && -42x+15z&=-39 \\ 6x-3z&=9 & \xrightarrow{\times 5} && 30x-15z&=45 \\ &&&& \hline -12x\qquad &=6 \end{aligned}$$

We can now easily solve for *x* to get $x=-\frac{1}{2}$. The coefficients on the second equation are smaller so let's plug this into that equation and solve for *z*. Here is that work.

$$\begin{aligned} 6\left(-\frac{1}{2}\right)-3z&=9 \\ -3-3z&=9 \\ -3z&=12 \\ z&=-4 \end{aligned}$$

Finally, we need to determine the value of *y*. This is very easy to do. Recall in the first step we used substitution and in that step we used the following equation.

$$y=4x+5$$

Since we know the value of *x* all we need to do is plug that into this equation and get the value of *y*.

$$y=4\left(-\frac{1}{2}\right)+5=3$$

Note that in many cases where we used substitution on the very first step the equation you'll have at this step will contain both *x*'s and *z*'s and so you will need both values to get the third variable.

Okay, to finish this example up here is the solution : $x=-\frac{1}{2}$, $y=3$ and $z=-4$.

As we've seen with the two examples above there are a variety of paths that we could choose to take when solving a system of three linear equations with three variables. That will always be the case. There is no one true path for solving these. However, having said that there is often a path that will allow you to avoid some of the mess that can arise in solving these types of systems. Once you work enough of these types of problems you'll start to get a feel for a "good" path through the solution process that will (hopefully) avoid some of the mess.

Interpretation of solutions in these cases is a little harder in some senses. All three of these equations in the examples above are equations of planes in three dimensional space and solution to this systems in the examples above is the one point that all three of the planes have in common.

Note as well that it is completely possible to have no solutions to these systems or infinitely many systems as we saw in the previous section with systems of two equations. We will look at these cases once we have the next section out of the way.

Section 7-3 : Augmented Matrices

In this section we need to take a look at the third method for solving systems of equations. For systems of two equations it is probably a little more complicated than the methods we looked at in the first section. However, for systems with more equations it is probably easier than using the method we saw in the previous section.

Before we get into the method we first need to get some definitions out of the way.

An **augmented matrix** for a system of equations is a matrix of numbers in which each row represents the constants from one equation (both the coefficients and the constant on the other side of the equal sign) and each column represents all the coefficients for a single variable.

Let's take a look at an example. Here is the system of equations that we looked at in the previous section.

$$\begin{aligned} x-2y+3z&=7\\ 2x+y+z&=4\\ -3x+2y-2z&=-10 \end{aligned}$$

Here is the augmented matrix for this system.

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 7\\ 2 & 1 & 1 & 4\\ -3 & 2 & -2 & -10 \end{array}\right]$$

The first row consists of all the constants from the first equation with the coefficient of the *x* in the first column, the coefficient of the *y* in the second column, the coefficient of the *z* in the third column and the constant in the final column. The second row is the constants from the second equation with the same placement and likewise for the third row. The dashed line represents where the equal sign was in the original system of equations and is not always included. This is mostly dependent on the instructor and/or textbook being used.

Next, we need to discuss **elementary row operations**. There are three of them and we will give both the notation used for each one as well as an example using the augmented matrix given above.

1. **Interchange Two Rows**. With this operation we will interchange all the entries in row *i* and row *j*. The notation we'll use here is $R_i \leftrightarrow R_j$. Here is an example.

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 7\\ 2 & 1 & 1 & 4\\ -3 & 2 & -2 & -10 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \rightarrow \end{array} \left[\begin{array}{ccc:c} -3 & 2 & -2 & -10\\ 2 & 1 & 1 & 4\\ 1 & -2 & 3 & 7 \end{array}\right]$$

So, we do exactly what the operation says. Every entry in the third row moves up to the first row and every entry in the first row moves down to the third row. Make sure that you move all the entries. One of the more common mistakes is to forget to move one or more entries.

2. **Multiply a Row by a Constant.** In this operation we will multiply row *i* by a constant *c* and the notation will use here is cR_i. Note that we can also divide a row by a constant using the notation $\frac{1}{c}R_i$. Here is an example.

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -3 & 2 & -2 & -10 \end{array}\right] \xrightarrow{-4R_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 12 & -8 & 8 & 40 \end{array}\right]$$

So, when we say we will multiply a row by a constant this really means that we will multiply every entry in that row by the constant. Watch out for signs in this operation and make sure that you multiply every entry.

3. **Add a Multiple of a Row to Another Row**. In this operation we will replace row *i* with the sum of row *i* and a constant, *c*, times row *j*. The notation we'll use for this operation is $R_i + cR_j \to R_i$. To perform this operation we will take an entry from row *i* and add to it *c* times the corresponding entry from row *j* and put the result back into row *i*. Here is an example of this operation.

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -3 & 2 & -2 & -10 \end{array}\right] \xrightarrow{R_3 - 4R_1 \to R_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -7 & 10 & -14 & -38 \end{array}\right]$$

Let's go through the individual computation to make sure you followed this.

$$\begin{aligned} -3-4(1) &= -7 \\ 2-4(-2) &= 10 \\ -2-4(3) &= -14 \\ -10-4(7) &= -38 \end{aligned}$$

Be very careful with signs here. We will be doing these computations in our head for the most part and it is very easy to get signs mixed up and add one in that doesn't belong or lose one that should be there.

It is very important that you can do this operation as this operation is the one that we will be using more than the other two combined.

Okay, so how do we use augmented matrices and row operations to solve systems? Let's start with a system of two equations and two unknowns.

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

We first write down the augmented matrix for this system,

$$\left[\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right]$$

and use elementary row operations to convert it into the following augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & h \\ 0 & 1 & k \end{array}\right]$$

Once we have the augmented matrix in this form we are done. The solution to the system will be $x = h$ and $y = k$.

This method is called **Gauss-Jordan Elimination**.

Example 1 Solve each of the following systems of equations.

(a) $$\begin{aligned} 3x - 2y &= 14 \\ x + 3y &= 1 \end{aligned}$$

(b) $$\begin{aligned} -2x + y &= -3 \\ x - 4y &= -2 \end{aligned}$$

(c) $$\begin{aligned} 3x - 6y &= -9 \\ -2x - 2y &= 12 \end{aligned}$$

Solution

(a) $$\begin{aligned} 3x - 2y &= 14 \\ x + 3y &= 1 \end{aligned}$$

The first step here is to write down the augmented matrix for this system.

$$\left[\begin{array}{cc|c} 3 & -2 & 14 \\ 1 & 3 & 1 \end{array}\right]$$

To convert it into the final form we will start in the upper left corner and work in a counter-clockwise direction until the first two columns appear as they should be.

So, the first step is to make the red three in the augmented matrix above into a 1. We can use any of the row operations that we'd like to. We should always try to minimize the work as much as possible however.

So, since there is a one in the first column already it just isn't in the correct row let's use the first row operation and interchange the two rows.

$$\left[\begin{array}{cc|c} 3 & -2 & 14 \\ 1 & 3 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & -2 & 14 \end{array}\right]$$

The next step is to get a zero below the 1 that we just got in the upper left hand corner. This means that we need to change the red three into a zero. This will almost always require us to use third row operation. If we add -3 times row 1 onto row 2 we can convert that 3 into a 0. Here is that operation.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & -2 & 14 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \rightarrow R_2 \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -11 & 11 \end{array}\right]$$

Next, we need to get a 1 into the lower right corner of the first two columns. This means changing the red -11 into a 1. This is usually accomplished with the second row operation. If we divide the second row by -11 we will get the 1 in that spot that we need.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -11 & 11 \end{array}\right] \underset{\rightarrow}{-\frac{1}{11}R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Okay, we're almost done. The final step is to turn the red three into a zero. Again, this almost always requires the third row operation. Here is the operation for this final step.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \end{array}\right] \underset{\rightarrow}{R_1 - 3R_2 \rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

We have the augmented matrix in the required form and so we're done. The solution to this system is $x = 4$ and $y = -1$.

(b) $\begin{aligned} -2x + y &= -3 \\ x - 4y &= -2 \end{aligned}$

In this part we won't put in as much explanation for each step. We will mark the next number that we need to change in red as we did in the previous part.

We'll first write down the augmented matrix and then get started with the row operations.

$$\left[\begin{array}{cc|c} -2 & 1 & -3 \\ 1 & -4 & -2 \end{array}\right] \underset{\rightarrow}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -4 & -2 \\ -2 & 1 & -3 \end{array}\right] \underset{\rightarrow}{R_2 + 2R_1 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & -4 & -2 \\ 0 & -7 & -7 \end{array}\right]$$

Before proceeding with the next step let's notice that in the second matrix we had one's in both spots that we needed them. However, the only way to change the -2 into a zero that we had to have as well was to also change the 1 in the lower right corner as well. This is okay. Sometimes it will happen and trying to keep both ones will only cause problems.

Let's finish the problem.

$$\left[\begin{array}{cc|c} 1 & -4 & -2 \\ 0 & -7 & -7 \end{array}\right] \underset{\rightarrow}{-\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & -4 & -2 \\ 0 & 1 & 1 \end{array}\right] \underset{\rightarrow}{R_1 + 4R_2 \rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

The solution to this system is then $x = 2$ and $y = 1$.

(c) $\begin{aligned} 3x - 6y &= -9 \\ -2x - 2y &= 12 \end{aligned}$

Let's first write down the augmented matrix for this system.

$$\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & -2 & 12 \end{array}\right]$$

Now, in this case there isn't a 1 in the first column and so we can't just interchange two rows as the first step. However, notice that since all the entries in the first row have 3 as a factor we can divide the first row by 3 which will get a 1 in that spot and we won't put any fractions into the problem.

Here is the work for this system.

$$\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & -2 & 12 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & -2 & -3 \\ -2 & -2 & 12 \end{array}\right] \xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & -6 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & -6 & 6 \end{array}\right] \xrightarrow{-\frac{1}{6}R_2} \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -1 \end{array}\right]$$

The solution to this system is $x = -5$ and $y = -1$.

It is important to note that the path we took to get the augmented matrices in this example into the final form is not the only path that we could have used. There are many different paths that we could have gone down. All the paths would have arrived at the same final augmented matrix however so we should always choose the path that we feel is the easiest path. Note as well that different people may well feel that different paths are easier and so may well solve the systems differently. They will get the same solution however.

For two equations and two unknowns this process is probably a little more complicated than just the straight forward solution process we used in the first section of this chapter. This process does start becoming useful when we start looking at larger systems. So, let’s take a look at a couple of systems with three equations in them.

In this case the process is basically identical except that there’s going to be more to do. As with two equations we will first set up the augmented matrix and then use row operations to put it into the form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right]$$

Once the augmented matrix is in this form the solution is $x = p$, $y = q$ and $z = r$. As with the two equations case there really isn’t any set path to take in getting the augmented matrix into this form. The usual path is to get the 1’s in the correct places and 0’s below them. Once this is done we then try to get zeroes above the 1’s.

Let’s work a couple of examples to see how this works.

Example 2 Solve each of the following systems of equations.

(a)
$$\begin{aligned}3x+y-2z&=2\\x-2y+z&=3\\2x-y-3z&=3\end{aligned}$$

(b)
$$\begin{aligned}3x+y-2z&=-7\\2x+2y+z&=9\\-x-y+3z&=6\end{aligned}$$

Solution

(a)
$$\begin{aligned}3x+y-2z&=2\\x-2y+z&=3\\2x-y-3z&=3\end{aligned}$$

Let's first write down the augmented matrix for this system.

$$\left[\begin{array}{ccc:c}3&1&-2&2\\1&-2&1&3\\2&-1&-3&3\end{array}\right]$$

As with the previous examples we will mark the number(s) that we want to change in a given step in red. The first step here is to get a 1 in the upper left hand corner and again, we have many ways to do this. In this case we'll notice that if we interchange the first and second row we can get a 1 in that spot with relatively little work.

$$\left[\begin{array}{ccc:c}3&1&-2&2\\1&-2&1&3\\2&-1&-3&3\end{array}\right]\begin{array}{c}R_1\leftrightarrow R_2\\\rightarrow\end{array}\left[\begin{array}{ccc:c}1&-2&1&3\\3&1&-2&2\\2&-1&-3&3\end{array}\right]$$

The next step is to get the two numbers below this 1 to be 0's. Note as well that this will almost always require the third row operation to do. Also, we can do both of these in one step as follows.

$$\left[\begin{array}{ccc:c}1&-2&1&3\\3&1&-2&2\\2&-1&-3&3\end{array}\right]\begin{array}{c}R_2-3R_1\rightarrow R_2\\R_3-2R_1\rightarrow R_3\\\rightarrow\end{array}\left[\begin{array}{ccc:c}1&-2&1&3\\0&7&-5&-7\\0&3&-5&-3\end{array}\right]$$

Next, we want to turn the 7 into a 1. We can do this by dividing the second row by 7.

$$\left[\begin{array}{ccc:c}1&-2&1&3\\0&7&-5&-7\\0&3&-5&-3\end{array}\right]\begin{array}{c}\frac{1}{7}R_2\\\rightarrow\end{array}\left[\begin{array}{ccc:c}1&-2&1&3\\0&1&-\frac{5}{7}&-1\\0&3&-5&-3\end{array}\right]$$

So, we got a fraction showing up here. That will happen on occasion so don't get all that excited about it. The next step is to change the 3 below this new 1 into a 0. Note that we aren't going to bother with the -2 above it quite yet. Sometimes it is just as easy to turn this into a 0 in the same step. In this case however, it's probably just as easy to do it later as we'll see.

So, using the third row operation we get,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 3 & -5 & -3 \end{array}\right] \xrightarrow{R_3 - 3R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & -\frac{20}{7} & 0 \end{array}\right]$$

Next, we need to get the number in the bottom right corner into a 1. We can do that with the second row operation.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & -\frac{20}{7} & 0 \end{array}\right] \xrightarrow{-\frac{7}{20}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now, we need zeroes above this new 1. So, using the third row operation twice as follows will do what we need done.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - R_3 \to R_1]{R_2 + \frac{5}{7}R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Notice that in this case the final column didn't change in this step. That was only because the final entry in that column was zero. In general, this won't happen.

The final step is then to make the -2 above the 1 in the second column into a zero. This can easily be done with the third row operation.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, we have the augmented matrix in the final form and the solution will be,

$$x = 1, \quad y = -1, \quad z = 0$$

This can be verified by plugging these into all three equations and making sure that they are all satisfied.

$$3x + y - 2z = -7$$
(b) $$2x + 2y + z = 9$$
$$-x - y + 3z = 6$$

Again, the first step is to write down the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & -7 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right]$$

We can't get a 1 in the upper left corner simply by interchanging rows this time. We could interchange the first and last row, but that would also require another operation to turn the -1 into a 1. While this isn't difficult it's two operations. Note that we could use the third row operation to get a 1 in that spot as follows.

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & -7 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right]$$

Now, we can use the third row operation to turn the two red numbers into zeroes.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right] \xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 4 & 7 & 41 \\ 0 & -2 & 0 & -10 \end{array}\right]$$

The next step is to get a 1 in the spot occupied by the red 4. We could do that by dividing the whole row by 4, but that would put in a couple of somewhat unpleasant fractions. So, instead of doing that we are going to interchange the second and third row. The reason for this will be apparent soon enough.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 4 & 7 & 41 \\ 0 & -2 & 0 & -10 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & -2 & 0 & -10 \\ 0 & 4 & 7 & 41 \end{array}\right]$$

Now, if we divide the second row by -2 we get the 1 in that spot that we want.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & -2 & 0 & -10 \\ 0 & 4 & 7 & 41 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 4 & 7 & 41 \end{array}\right]$$

Before moving onto the next step let's think notice a couple of things here. First, we managed to avoid fractions, which is always a good thing, and second this row is now done. We would have eventually needed a zero in that third spot and we've got it there for free. Not only that, but it won't change in any of the later operations. This doesn't always happen, but if it does that will make our life easier.

Now, let's use the third row operation to change the red 4 into a zero.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 4 & 7 & 41 \end{array}\right] \begin{array}{c} R_3 - 4R_2 \to R_3 \\ \to \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 7 & 21 \end{array}\right]$$

We now can divide the third row by 7 to get that the number in the lower right corner into a one.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 7 & 21 \end{array}\right] \begin{array}{c} \frac{1}{7}R_3 \\ \to \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Next, we can use the third row operation to get the -3 changed into a zero.

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 + 3R_3 \to R \\ \to \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The final step is to then make the -1 into a 0 using the third row operation again.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 + R_2 \to R \\ \to \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The solution to this system is then,

$$x = -2, \;\; y = 5, \;\; z = 3$$

Using Gauss-Jordan elimination to solve a system of three equations can be a lot of work, but it is often no more work than solving directly and is many cases less work. If we were to do a system of four equations (which we aren't going to do) at that point Gauss-Jordan elimination would be less work in all likelihood that if we solved directly.

Also, as we saw in the final example worked in this section, there really is no one set path to take through these problems. Each system is different and may require a different path and set of operations to make. Also, the path that one person finds to be the easiest may not by the path that another person finds to be the easiest. Regardless of the path however, the final answer will be the same.

Section 7-4 : More on the Augmented Matrix

In the first section in this chapter we saw that there were some special cases in the solution to systems of two equations. We saw that there didn't have to be a solution at all and that we could in fact have infinitely many solutions. In this section we are going to generalize this out to general systems of equations and we're going to look at how to deal with these cases when using augmented matrices to solve a system.

Let's first give the following fact.

Fact

Given any system of equations there are exactly three possibilities for the solution.

1. There will not be a solution.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

This is exactly what we found the possibilities to be when we were looking at two equations. It just turns out that it doesn't matter how many equations we've got. There are still only these three possibilities.

Now, let's see how we can identify the first and last possibility when we are using the augmented matrix method for solving. In the previous section we stated that we wanted to use the row operations to convert the augmented matrix into the following form,

$$\left[\begin{array}{cc|c} 1 & 0 & h \\ 0 & 1 & k \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right]$$

depending upon the number of equations present in the system. It turns out that we should have added the qualifier, "if possible" to this instruction, because it isn't always possible to do this. In fact, if it isn't possible to put it into one of these forms then we will know that we are in either the first or last possibility for the solution to the system.

Before getting into some examples let's first address how we knew what the solution was based on these forms of the augmented matrix. Let's work with the two equation case.

Since,

$$\left[\begin{array}{cc|c} 1 & 0 & h \\ 0 & 1 & k \end{array}\right]$$

is an augmented matrix we can always convert back to equations. Each row represents an equation and the first column is the coefficient of *x* in the equation while the second column is the coefficient of the *y* in the equation. The final column is the constant that will be on the right side of the equation.

So, if we do that for this case we get,

$$\begin{array}{lcl} (1)x + (0)y = h & \Rightarrow & x = h \\ (0)x + (1)y = k & \Rightarrow & y = k \end{array}$$

and this is exactly what we said the solution was in the previous section.

This idea of turning an augmented matrix back into equations will be important in the following examples.

Speaking of which, let's go ahead and work a couple of examples. We will start out with the two systems of equations that we looked at in the first section that gave the special cases of the solutions.

Example 1 Use augmented matrices to solve each of the following systems.

(a)
$$\begin{aligned} x - y &= 6 \\ -2x + 2y &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 5y &= -1 \\ -10x - 25y &= 5 \end{aligned}$$

Solution

(a)
$$\begin{aligned} x - y &= 6 \\ -2x + 2y &= 1 \end{aligned}$$

Now, we've already worked this one out so we know that there is no solution to this system. Knowing that let's see what the augmented matrix method gives us when we try to use it.

We'll start with the augmented matrix.

$$\left[\begin{array}{rr|r} 1 & -1 & 6 \\ -2 & 2 & 1 \end{array}\right]$$

Notice that we've already got a 1 in the upper left corner so we don't need to do anything with that. So, we next need to make the -2 into a 0.

$$\left[\begin{array}{rr|r} 1 & -1 & 6 \\ -2 & 2 & 1 \end{array}\right] \begin{array}{c} R_2 + 2R_1 \to R_2 \\ \to \end{array} \left[\begin{array}{rr|r} 1 & -1 & 6 \\ 0 & 0 & 13 \end{array}\right]$$

Now, the next step should be to get a 1 in the lower right corner, but there is no way to do that without changing the zero in the lower left corner. That's a problem, because we must have a zero in that spot as well as a one in the lower right corner. What this tells us is that it isn't possible to put this augmented matrix form.

Now, go back to equations and see what we've got in this case.

$$\begin{aligned} x - y &= 6 \\ 0 &= 13 \quad ??? \end{aligned}$$

The first row just converts back into the first equation. The second row however converts back to nonsense. We know this isn't true so that means that there is no solution. Remember, if we reach a point where we have an equation that just doesn't make sense we have no solution.

Note that if we'd gotten

$$\left[\begin{array}{rr|r} 1 & -1 & 6 \\ 0 & 1 & 0 \end{array}\right]$$

we would have been okay since the last row would return the equation $y=0$ so don't get confused between this case and what we actually got for this system.

(b)
$$\begin{aligned} 2x+5y&=-1 \\ -10x-25y&=5 \end{aligned}$$

In this case we know from the first section that there are infinitely many solutions to this system. Let's see what we get when we use the augmented matrix method for the solution.

Here is the augmented matrix for this system.

$$\left[\begin{array}{cc|c} 2 & 5 & -1 \\ -10 & -25 & 5 \end{array}\right]$$

In this case we'll need to first get a 1 in the upper left corner and there isn't going to be any easy way to do this that will avoid fractions so we'll just divide the first row by 2.

$$\left[\begin{array}{cc|c} 2 & 5 & -1 \\ -10 & -25 & 5 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & \frac{5}{2} & -\frac{1}{2} \\ -10 & -25 & 5 \end{array}\right]$$

Now, we can get a zero in the lower left corner.

$$\left[\begin{array}{cc|c} 1 & \frac{5}{2} & -\frac{1}{2} \\ -10 & -25 & 5 \end{array}\right] \begin{array}{c} R_2+10R_1 \rightarrow R_2 \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

Now, as with the first part we are never going to be able to get a 1 in place of the red zero without changing the first zero in that row. However, this isn't the nonsense that the first part got. Let's convert back to equations.

$$\begin{aligned} x+\frac{5}{2}y&=-\frac{1}{2} \\ 0&=0 \end{aligned}$$

That last equation is a true equation and so there isn't anything wrong with this. In this case we have infinitely many solutions.

Recall that we still need to do a little work to get the solution. We solve one of the equations for one of the variables. Note however, that if we use the equation from the augmented matrix this is very easy to do.

$$x=-\frac{5}{2}y-\frac{1}{2}$$

We then write the solution as,

$$\begin{aligned} x&=-\frac{5}{2}t-\frac{1}{2} \qquad \text{where } t \text{ is any real number} \\ y&=t \end{aligned}$$

We get solutions by picking t and plugging this into the equation for x. Note that this is NOT the same set of equations we got in the first section. That is okay. When there are infinitely many solutions there are more than one way to write the equations that will describe all the solutions.

Let's summarize what we learned in the previous set of examples. First, if we have a row in which all the entries except for the very last one are zeroes and the last entry is NOT zero then we can stop and the system will have no solution.

Next, if we get a row of all zeroes then we will have infinitely many solutions. We will then need to do a little more work to get the solution and the number of equations will determine how much work we need to do.

Now, let's see how some systems with three equations work. The no solution case will be identical, but the infinite solution case will have a little work to do.

Example 2 Solve the following system of equations using augmented matrices.

$$\begin{aligned} 3x-3y-6z&=-3\\ 2x-2y-4z&=10\\ -2x+3y+z&=7 \end{aligned}$$

Solution

Here's the augmented matrix for this system.

$$\left[\begin{array}{rrr|r} 3 & -3 & -6 & -3\\ 2 & -2 & -4 & 10\\ -2 & 3 & 1 & 7 \end{array}\right]$$

We can get a 1 in the upper left corner by dividing by the first row by a 3.

$$\left[\begin{array}{rrr|r} 3 & -3 & -6 & -3\\ 2 & -2 & -4 & 10\\ -2 & 3 & 1 & 7 \end{array}\right] \begin{array}{c} \frac{1}{3}R_1\\ \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 2 & -2 & -4 & 10\\ -2 & 3 & 1 & 7 \end{array}\right]$$

Next, we'll get the two numbers under this one to be zeroes.

$$\left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 2 & -2 & -4 & 10\\ -2 & 3 & 1 & 7 \end{array}\right] \begin{array}{c} R_2-2R_1\rightarrow R_2\\ R_3+2R_1\rightarrow R_3\\ \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 0 & 0 & 0 & 12\\ 0 & 1 & -3 & 5 \end{array}\right]$$

And we can stop. The middle row is all zeroes except for the final entry which isn't zero. Note that it doesn't matter what the number is as long as it isn't zero.

Once we reach this type of row we know that the system won't have any solutions and so there isn't any reason to go any farther.

Okay, let's see how we solve a system of three equations with an infinity number of solutions with the augmented matrix method. This example will also illustrate an interesting idea about systems.

Example 3 Solve the following system of equations using augmented matrices.

$$\begin{aligned} 3x-3y-6z&=-3\\ 2x-2y-4z&=-2\\ -2x+3y+z&=7 \end{aligned}$$

Solution

Notice that this system is almost identical to the system in the previous example. The only difference is the number to the right of the equal sign in the second equation. In this system it is -2 and in the previous example it was 10. Changing that one number completely changes the type of solution that we're going to get. Often this kind of simple change won't affect the type of solution that we get, but in some rare cases it can.

Since the first two steps of the process are identical to the previous part we won't discuss them. Here they are.

$$\left[\begin{array}{rrr|r} 3 & -3 & -6 & -3\\ 2 & -2 & -4 & -2\\ -2 & 3 & 1 & 7 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 2 & -2 & -4 & -2\\ -2 & 3 & 1 & 7 \end{array}\right] \begin{array}{c} R_2-2R_1\to R_2\\ R_3+2R_1\to R_3\\ \to \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 0 & 0 & 0 & 0\\ 0 & 1 & -3 & 5 \end{array}\right]$$

We've got a row of all zeroes so we instantly know that we've got infinitely many solutions. Unlike the two equation case we aren't going to stop however. It looks like with a couple of row operations we can make the second column look like it is supposed to in the final form so let's do that.

$$\left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 0 & 0 & 0 & 0\\ 0 & 1 & -3 & 5 \end{array}\right] \xrightarrow{R_2\to R_3} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -1\\ 0 & 1 & -3 & 5\\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2\to R_1} \left[\begin{array}{rrr|r} 1 & 0 & -5 & 4\\ 0 & 1 & -3 & 5\\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case we were able to make the second column look like it's supposed to and the third column will never look correct. However, it is possible that the situation could be reversed and it would be the third column that we can make look correct and the second wouldn't look correct. Every system is different.

Once we reach this point we go back to equations.

$$\begin{aligned} x-5z&=4\\ y-3z&=5 \end{aligned}$$

Now, both of these equations contain a *z* and so we'll move that to the other side in each equation.

$$\begin{aligned} x&=5z+4\\ y&=3z+5 \end{aligned}$$

This means that we get to pick the value of *z* for free and we'll write the solution as,

$$x = 5t + 4$$
$$y = 3t + 5 \qquad \text{where } t \text{ is any real number}$$
$$z = t$$

Since there are an infinite number of ways to choose *t* there are an infinite number of solutions to this system.

Section 7-5 : Nonlinear Systems

In this section we are going to be looking at non-linear systems of equations. A non-linear system of equations is a system in which at least one of the variables has an exponent other than 1 and/or there is a product of variables in one of the equations.

To solve these systems we will use either the substitution method or elimination method that we first looked at when we solved systems of linear equations. The main difference is that we may end up getting complex solutions in addition to real solutions. Just as we saw in solving systems of two equations the real solutions will represent the coordinates of the points where the graphs of the two functions intersect.

Let's work some examples.

Example 1 Solve the following system of equations.

$$x^2 + y^2 = 10$$
$$2x + y = 1$$

Solution

In linear systems we had the choice of using either method on any given system. With non-linear systems that will not always be the case. In the first equation both of the variables are squared and in the second equation both of the variables are to the first power. In other words, there is no way that we can use elimination here and so we are must use substitution. Luckily that isn't too bad to do for this system since we can easily solve the second equation for *y* and substitute this into the first equation.

$$y = 1 - 2x$$

$$x^2 + \left(1 - 2x\right)^2 = 10$$

This is a quadratic equation that we can solve.

$$\begin{aligned} x^2 + 1 - 4x + 4x^2 &= 10 \\ 5x^2 - 4x - 9 &= 0 \\ \left(x+1\right)\left(5x-9\right) &= 0 \qquad \Rightarrow \qquad x = -1,\ x = \frac{9}{5} \end{aligned}$$

So, we have two values of *x*. Now, we need to determine the values of *y* and we are going to have to be careful to not make a common mistake here. We determine the values of *y* by plugging *x* into our substitution.

$$x = -1 \qquad \Rightarrow \qquad y = 1 - 2\left(-1\right) = 3$$
$$x = \frac{9}{5} \qquad \Rightarrow \qquad y = 1 - 2\left(\frac{9}{5}\right) = -\frac{13}{5}$$

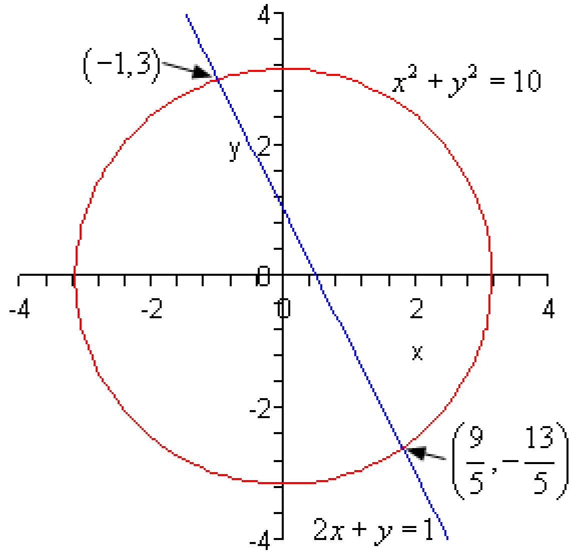
Now, we only have two solutions here. Do not just start mixing and matching all possible values of *x* and *y* into solutions. We get $y = 3$ as a solution ONLY if $x = -1$ and so the first solution is,

$$x = -1, \; y = 3$$

Likewise, we only get $y = -\frac{13}{5}$ ONLY if $x = \frac{9}{5}$ and so the second solution is,

$$x = \frac{9}{5}, \; y = -\frac{13}{5}$$

So, we have two solutions. Now, as noted at the start of this section these two solutions will represent the points of intersection of these two curves. Since the first equation is a circle and the second equation is a line have two intersection points is definitely possible. Here is a sketch of the two equations as a verification of this.



Note that when the two equations are a line and a circle as in the previous example we know that we will have at most two real solutions since it is only possible for a line to intersect a circle zero, one, or two times.

Example 2 Solve the following system of equations.

$$x^2 - 2y^2 = 2$$
$$xy = 2$$

Solution

Okay, in this case we have a hyperbola (the first equation, although it isn't in standard form) and a rational function (the second equation if we solved for *y*). As with the first example we can't use elimination on this system so we will have to use substitution.

The best way is to solve the second equation for either *x* or *y*. Either one will give us pretty much the same work so we'll solve for *y* since that is probably the one that will make the equation look more like those that we've looked at in the past. In other words, the new equation will be in terms of *x* and that is the variable that we are used to seeing in equations.

$$y = \frac{2}{x}$$

$$\begin{aligned} x^2 - 2\left(\frac{2}{x}\right)^2 &= 2 \\ x^2 - 2\frac{4}{x^2} &= 2 \\ x^2 - \frac{8}{x^2} &= 2 \end{aligned}$$

The first step towards solving this equation will be to multiply the whole thing by x^2 to clear out the denominators.

$$\begin{aligned} x^4 - 8 &= 2x^2 \\ x^4 - 2x^2 - 8 &= 0 \end{aligned}$$

Now, this is quadratic in form and we know how to solve those kinds of equations. If we define,

$$u = x^2 \qquad \Rightarrow \qquad u^2 = \left(x^2\right)^2 = x^4$$

and the equation can be written as,

$$\begin{aligned} u^2 - 2u - 8 &= 0 \\ (u-4)(u+2) &= 0 \qquad \Rightarrow \qquad u = -2,\ u = 4 \end{aligned}$$

In terms of x this means that we have the following,

$$\begin{aligned} x^2 &= 4 \qquad \Rightarrow \qquad x = \pm 2 \\ x^2 &= -2 \qquad \Rightarrow \qquad x = \pm\sqrt{2}\ i \end{aligned}$$

So, we have four possible values of x and two of them are complex. To determine the values of y we can plug these into our substitution.

$$\begin{aligned} x = 2 \qquad &\Rightarrow \qquad y = \frac{2}{2} = 1 \\ x = -2 \qquad &\Rightarrow \qquad y = \frac{2}{-2} = -1 \\ x = \sqrt{2}\,i \qquad &\Rightarrow \qquad y = \frac{2}{\sqrt{2}\,i} = \frac{2}{\sqrt{2}\,i}\frac{i}{i} = \frac{2i}{\sqrt{2}\,i^2} = -\frac{2i}{\sqrt{2}} \\ x = -\sqrt{2}\,i \qquad &\Rightarrow \qquad y = -\frac{2}{\sqrt{2}\,i} = -\frac{2}{\sqrt{2}\,i}\frac{i}{i} = -\frac{2i}{\sqrt{2}\,i^2} = \frac{2i}{\sqrt{2}} \end{aligned}$$

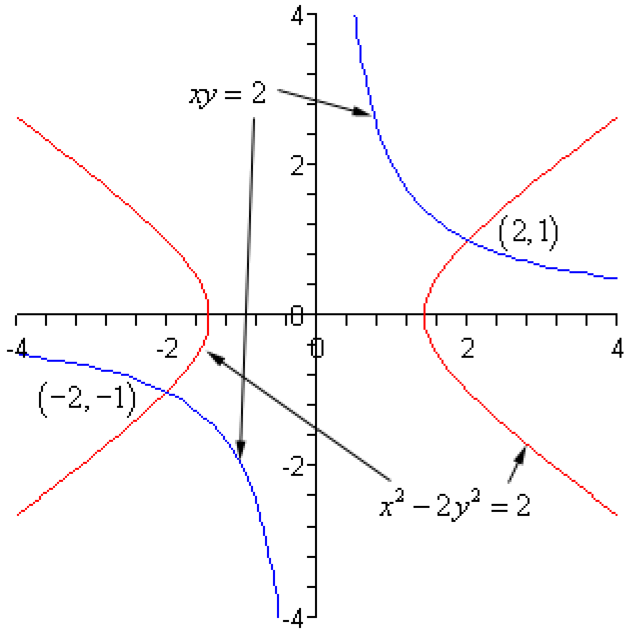
For the complex solutions, notice that we made sure the i was in the numerator. The for solutions are then,

$$x = 2, \quad y = 1 \qquad \text{and} \qquad x = -2, y = -1 \qquad \text{and}$$

$$x = \sqrt{2}\, i, \quad y = -\frac{2i}{\sqrt{2}} \qquad \text{and} \qquad x = -\sqrt{2}\, i, \quad y = \frac{2i}{\sqrt{2}}$$

Two of the solutions are real and so represent intersection points of the graphs of these two equations. The other two are complex solutions and while solutions will not represent intersection points of the curves.

For reference purposes, here is a sketch of the two curves.



Note that there are only two intersection points of these two graphs as suggested by the two real solutions. Complex solutions never represent intersections of two curves.

Example 3 Solve the following system of equations.

$$2x^2 + y^2 = 24$$
$$x^2 - y^2 = -12$$

Solution

This time we have an ellipse and a hyperbola. Neither one are in standard form however.

In the first two examples we've used the substitution method to solve the system and we can use that here as well. Let's notice however, that if we just add the two equations we will eliminate the *y*'s from the system so we'll do it that way.

$$\begin{array}{rl} 2x^2 + y^2 &= 24 \\ x^2 - y^2 &= -12 \\ \hline 3x^2 \quad\quad &= 12 \end{array}$$

This is easy enough to solve for *x*.

$$\begin{aligned} 3x^2 &= 12 \\ x^2 &= 4 \qquad \Rightarrow \qquad x = \pm 2 \end{aligned}$$

To determine the value(s) of the *y*'s we can substitute these into either of the equations. We will use the first since there won't be any minus signs to worry about.

$x = 2$:

$$\begin{aligned} 2(2)^2 + y^2 &= 24 \\ 8 + y^2 &= 24 \\ y^2 &= 16 \qquad \Rightarrow \qquad y = \pm 4 \end{aligned}$$

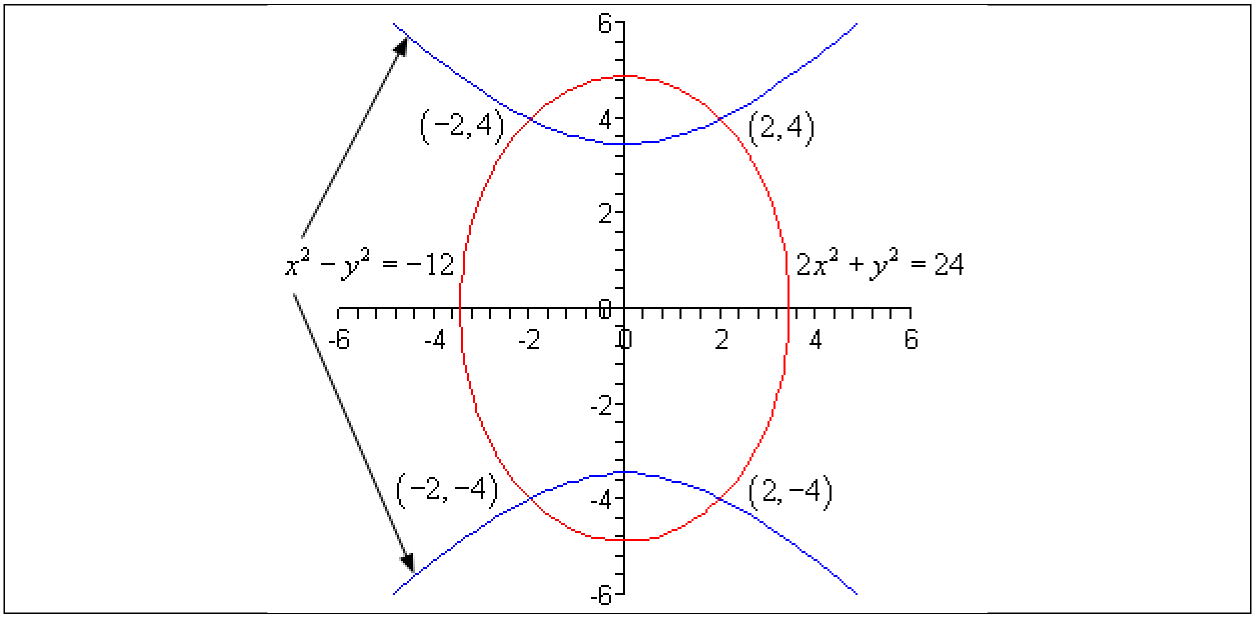
$x = -2$:

$$\begin{aligned} 2(-2)^2 + y^2 &= 24 \\ 8 + y^2 &= 24 \\ y^2 &= 16 \qquad \Rightarrow \qquad y = \pm 4 \end{aligned}$$

Note that for this system, unlike the previous examples, each value of *x* actually gave two possible values of *y*. That means that there are in fact four solutions. They are,

$$(2,4) \quad (2,-4) \quad (-2,4) \quad (-2,-4)$$

This also means that there should be four intersection points to the two curves. Here is a sketch for verification.


$(-2,4)$
$(2,4)$
$x^2 - y^2 = -12$
$2x^2 + y^2 = 24$
$(-2,-4)$
$(2,-4)$
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6