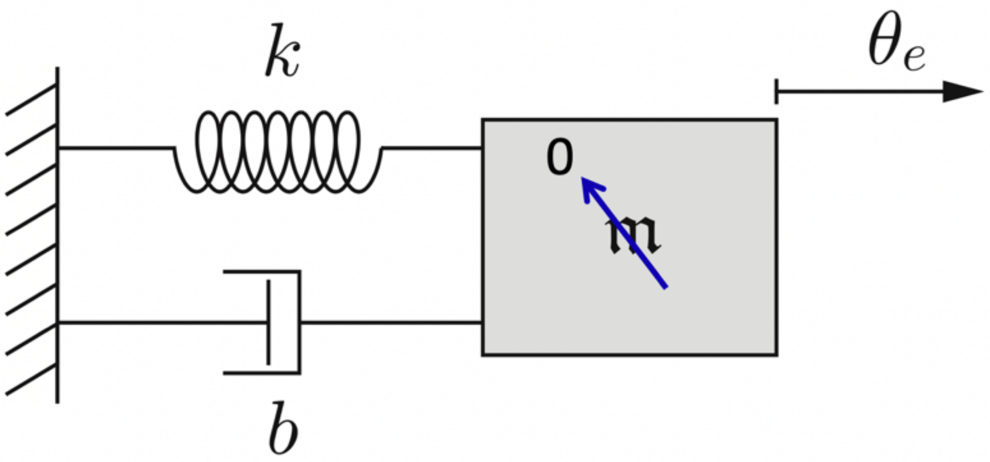
Important concepts, symbols, and equations

First-order error dynamics



$$\cancelto{0}{\mathfrak{m}}\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e = 0$$

$$\dot{\theta}_e(t) + \frac{k}{b}\theta_e(t) = 0$$

standard first-order form

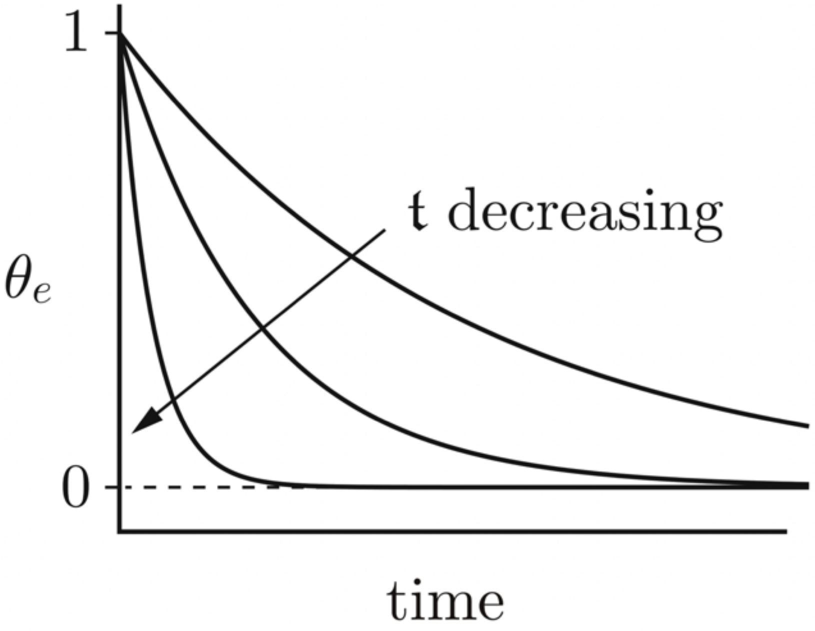
time constant

$$\mathfrak{t} = b/k$$

$$\dot{\theta}_e(t) + \frac{1}{\mathfrak{t}}\theta_e(t) = 0$$

Important concepts, symbols, and equations (cont.)

First-order error dynamics



$$\theta_e(t) = e^{-t/\mathfrak{t}}\theta_e(0)$$

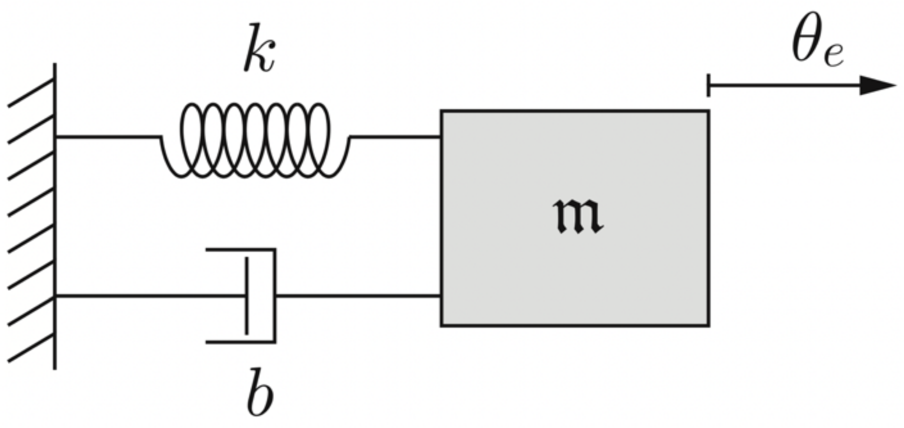
$$\theta_e(0) = 1$$

$$\frac{\theta_e(t)}{\theta_e(0)} = 0.02 = e^{-t/\mathfrak{t}}$$

$$\ln 0.02 = -t/\mathfrak{t} \quad \rightarrow \quad t = 3.91\mathfrak{t}$$

Important concepts, symbols, and equations (cont.)

Second-order error dynamics



$$\ddot{\theta}_e(t) + \frac{b}{\mathfrak{m}}\dot{\theta}_e(t) + \frac{k}{\mathfrak{m}}\theta_e(t) = 0$$

natural frequency **damping ratio**

$$\omega_n = \sqrt{k/\mathfrak{m}} \qquad \zeta = b/(2\sqrt{k\mathfrak{m}})$$

$$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2\theta_e(t) = 0$$

standard second-order form

Important concepts, symbols, and equations (cont.)

Second-order error dynamics

$$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2\theta_e(t) = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

characteristic equation

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta > 1$: **Overdamped**

$\zeta = 1$: **Critically damped**

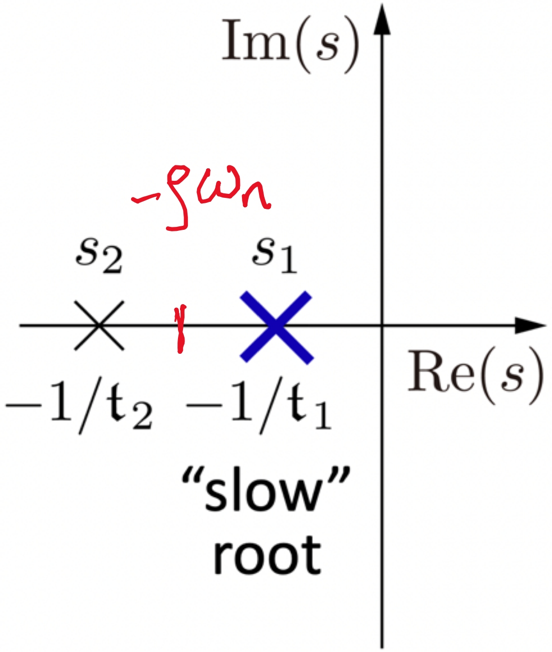
$\zeta < 1$: **Underdamped**

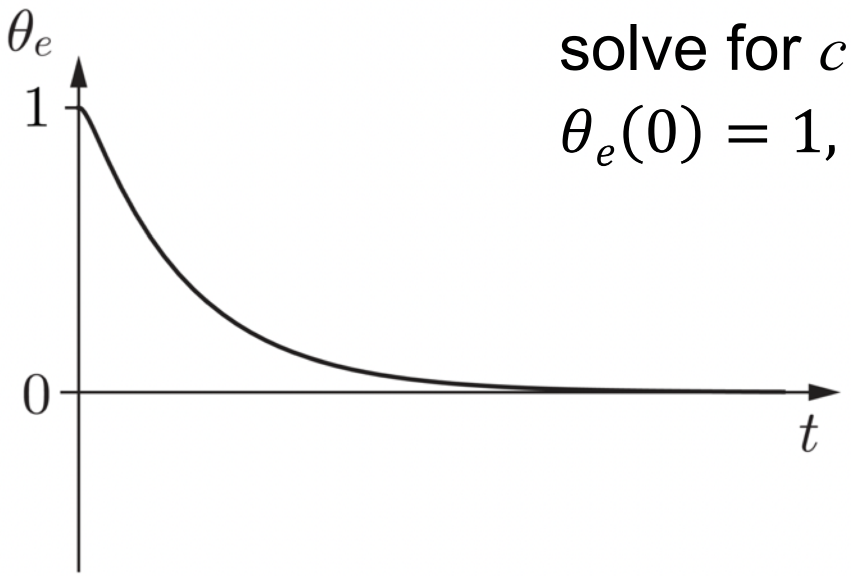
Important concepts, symbols, and equations (cont.)

$\zeta > 1:$ Overdamped

$$\theta_e(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$



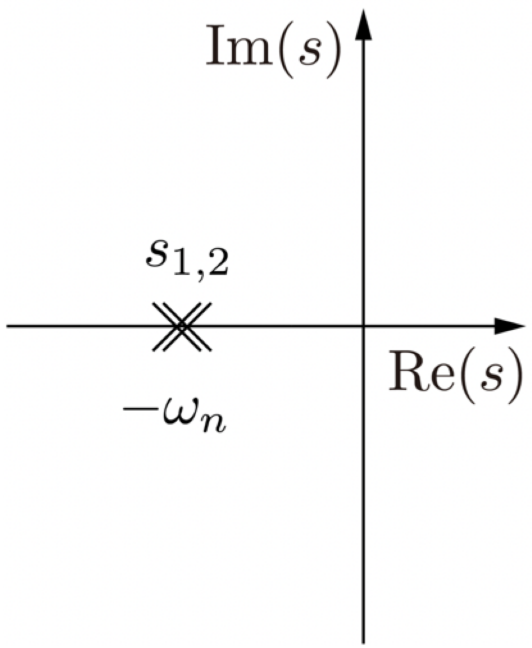


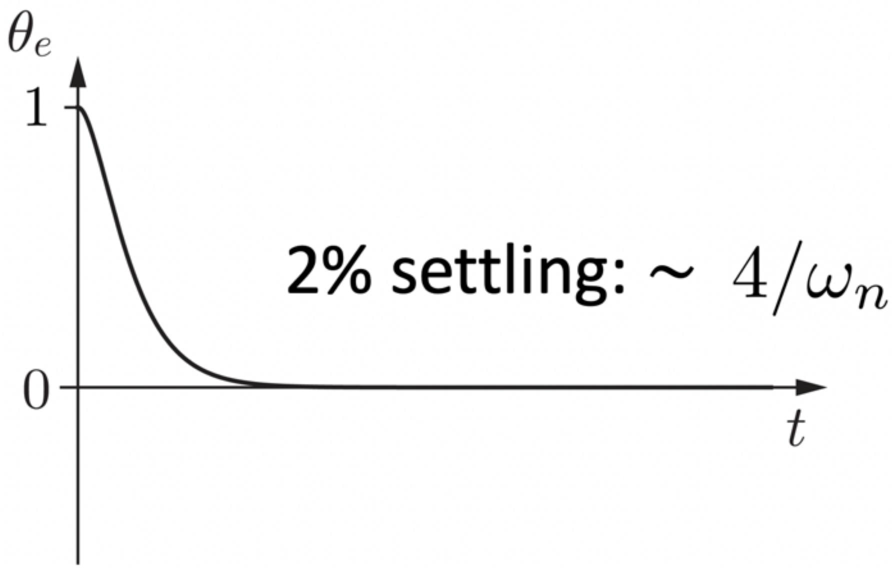
solve for c_1 and c_2 using
$\theta_e(0) = 1, \dot{\theta}_e(0) = 0$

Important concepts, symbols, and equations (cont.)

$\zeta = 1$: Critically damped

$$\theta_e(t) = (c_1 + c_2 t)e^{-\omega_n t} \qquad s_{1,2} = -\omega_n$$





Important concepts, symbols, and equations (cont.)

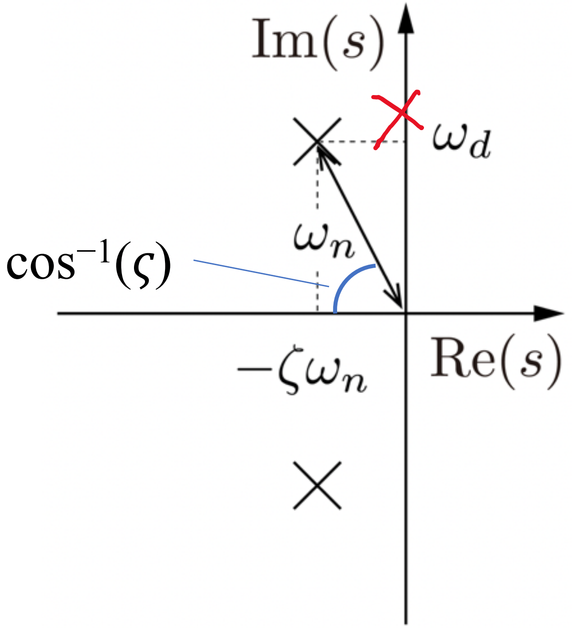
$\zeta < 1$: Underdamped

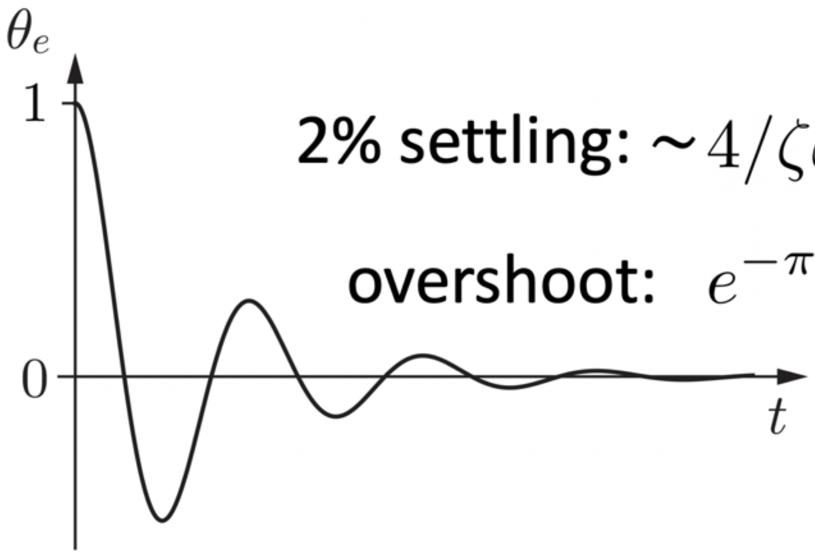
$$\theta_e(t) = (c_1 \cos \omega_d t + c_2 \sin \omega_d t)\, e^{-\zeta \omega_n t}$$

damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$s_{1,2} = -\zeta \omega_n \pm j \omega_d$$



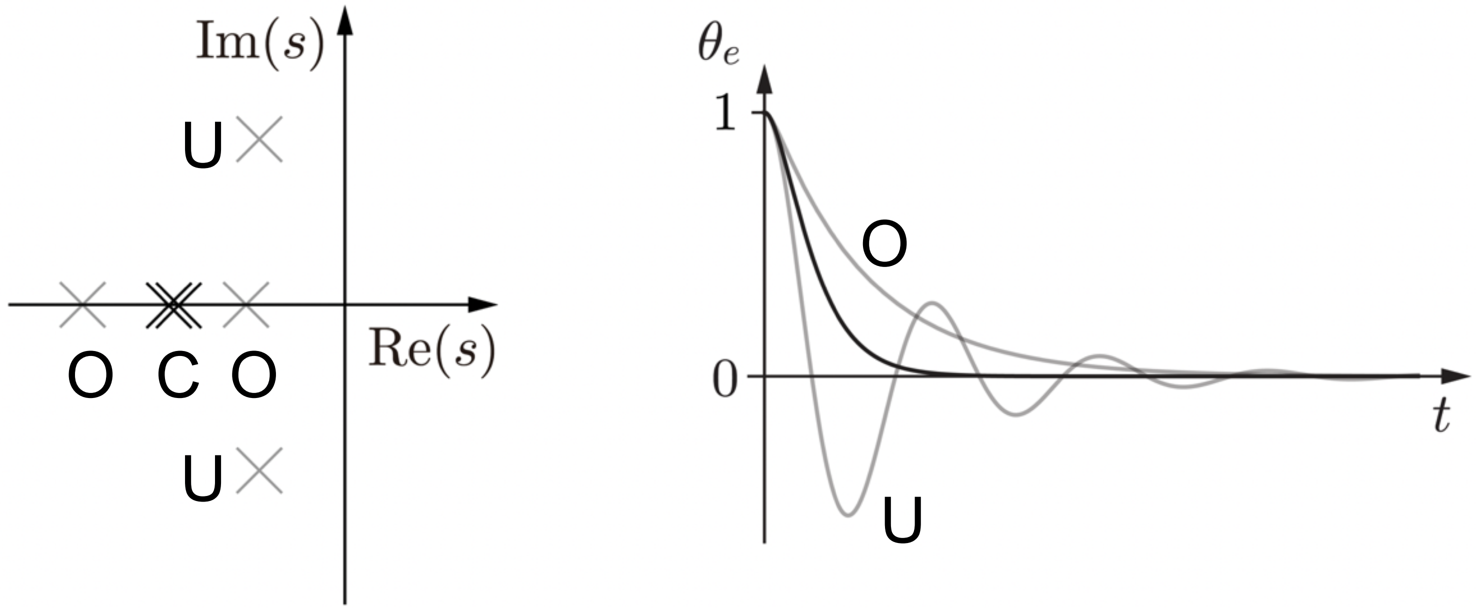


2% settling: $\sim 4/\zeta\omega_n$

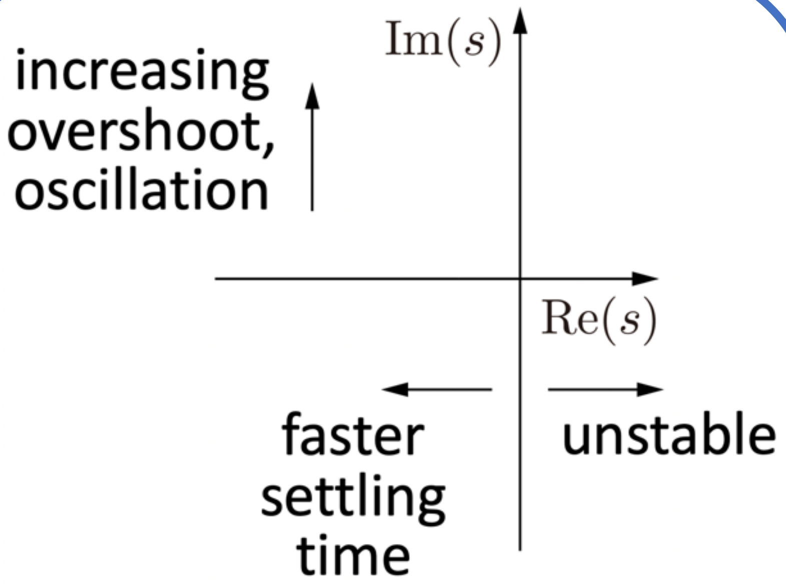
overshoot: $e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$

small ζ → large overshoot

<u>Important concepts, symbols, and equations</u> (cont.)



second-order systems



time response relative to root locations for general systems

$\zeta\omega_n = \frac{b}{2\sqrt{km}}\sqrt{\frac{k}{m}} = \frac{b}{2m}$

$$\omega_n = \sqrt{k/\mathfrak{m}} \qquad \zeta = b/(2\sqrt{k\mathfrak{m}})$$

2% settling: $\sim 4/\zeta\omega_n \rightarrow \frac{8m}{b}$

$$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2\theta_e(t) = 0$$

overshoot: $e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$

When controlling a robot joint, what do b, k, and $\mathfrak{m}$ usually correspond to?

m: mass/inertia of link

k: P in PD control

b: D in PD control (+ maybe friction)

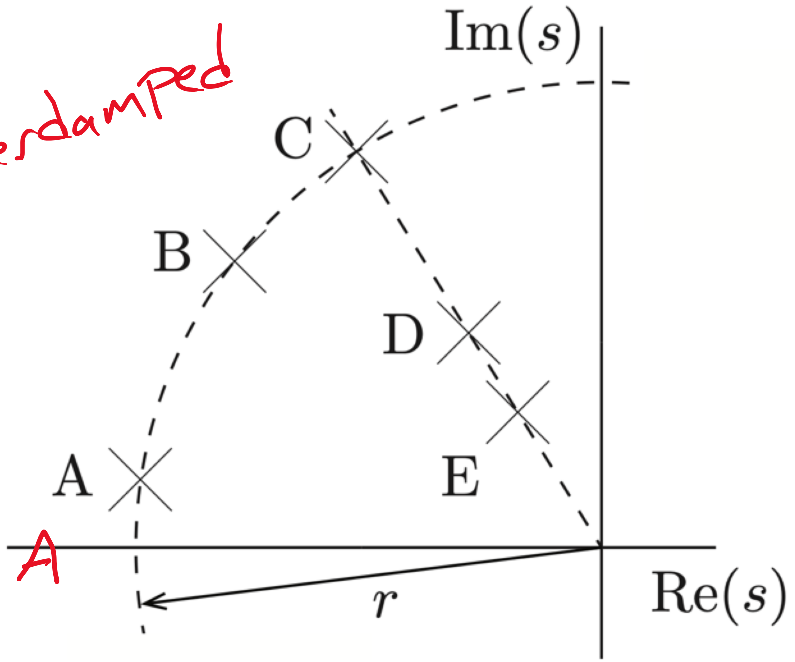
How do you change $\mathfrak{m}$ to decrease settling time? k, b?

m ↓ k no effect b ↑

How do you change $\mathfrak{m}$ to decrease overshoot? k, b?

increase ζ m ↓ k ↓ b ↑

all underdamped



Shown are one of the roots of five different second-order systems, A, B, C, D, and E. List them in the following orders:

1. Natural frequency, highest to lowest. (A, B, C), D, E
2. Damped natural frequency, highest to lowest. C, B, D, E, A
3. Damping ratio, highest to lowest. A, B, (C, D, E)
4. Overshoot in unit step error response, highest to lowest. (C, D, E), B, A
5. Settling time, longest to shortest. E, D, C, B, A

Which has the "best" transient response?

A. Damping ratio is largest, so least overshoot. Furthest left, so fastest settling time.