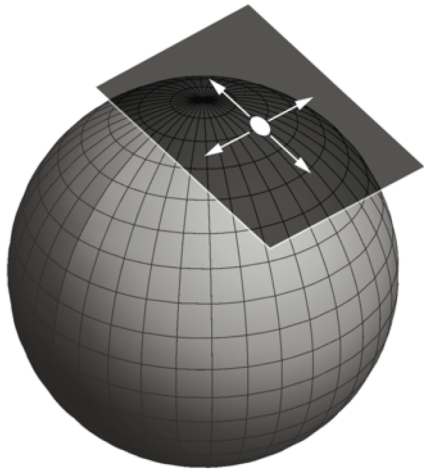


Chapter 2	Configuration Space
Chapter 3	Rigid-Body Motions
	3.2 Rotations and Angular Velocities
Chapter 4	Forward Kinematics
Chapter 5	Velocity Kinematics and Statics
Chapter 6	Inverse Kinematics
Chapter 7	Kinematics of Closed Chains
Chapter 8	Dynamics of Open Chains
Chapter 9	Trajectory Generation
Chapter 10	Motion Planning
Chapter 11	Robot Control
Chapter 12	Grasping and Manipulation
Chapter 13	Wheeled Mobile Robots

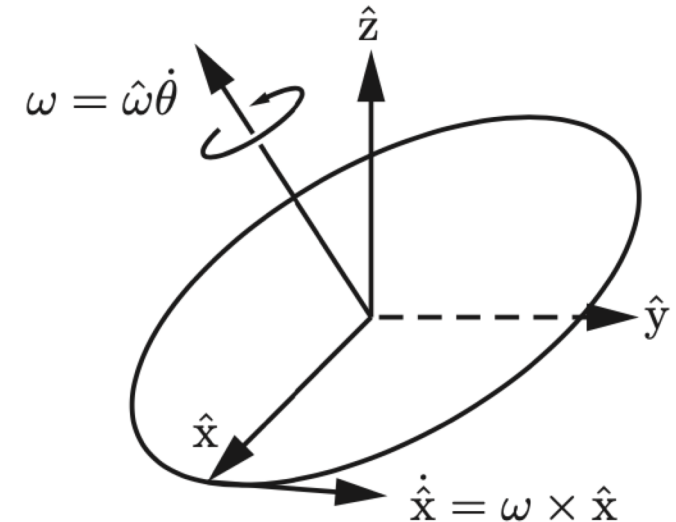
Important concepts, symbols, and equations

- $SO(3)$ is a curved 3-dimensional space, but the feasible velocities at any point of $SO(3)$ form a **flat** 3-dimensional **vector space** (the “tangent space”).



Another example: the tangent space at a point of S^2 .

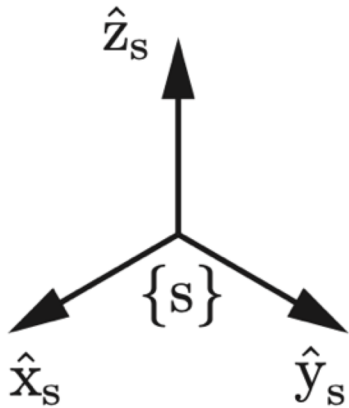
- Any rotational velocity can be expressed as an **angular velocity** $\omega \in \mathbb{R}^3$, which can be considered the product of a unit axis (in S^2) and a speed (a scalar).



Important concepts, symbols, and equations (cont.)

- Given $p \in \mathbb{R}^3$ and ω defined in the same reference frame, $\dot{p} = \omega \times p$.
- Linear algebra notation: $\dot{p} = \omega \times p = [\omega] p$, where

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in so(3), \text{ the } 3 \times 3 \text{ real skew-symmetric} \\ \text{matrices (satisfying } [x] = -[x]^T \text{)}.$$



$so(3)$ describes the possible \dot{R} when $R = I$, and it is called the **Lie algebra** of the Lie group $SO(3)$.

Important concepts, symbols, and equations (cont.)

- If $R_{sb} = [p_1 \ p_2 \ p_3]$, then $\dot{R}_{sb} = [[\omega_s] p_1 \ [\omega_s] p_2 \ [\omega_s] p_3] = [\omega] R_{sb}$.

- Expressing the angular velocity in a different frame:

$$\omega_b = R_{b\cancel{s}} \omega_{\cancel{s}} = R_{sb}^{-1} \omega_s = R_{sb}^T \omega_s \qquad \omega_s = R_{sb} \omega_b$$

- The $so(3)$ representations:

$$[\omega_b] = R_{sb}^{-1} \dot{R} = R_{sb}^T \dot{R} \qquad [\omega_s] = \dot{R} R_{sb}^{-1} = \dot{R} R_{sb}^T$$

- **Exponential coordinate** (axis-angle) representation of orientation: $\hat{\omega}\theta$

Important concepts, symbols, and equations (cont.)

- Scalar first-order linear diffeq:

$$\dot{x}(t) = ax(t) \longrightarrow x(t) = e^{at}x_0$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$$

- Vector first-order linear diffeq:

$$\dot{x}(t) = Ax(t) \longrightarrow x(t) = e^{At}x_0$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

matrix exponential

Important concepts, symbols, and equations (cont.)

- Integrating an angular velocity

$$\dot{p} = \hat{\omega} \times p = [\hat{\omega}]p \quad \longrightarrow \quad \begin{aligned} p(t) &= e^{[\hat{\omega}]t} p(0) \\ p(\theta) &= e^{[\hat{\omega}]\theta} p(0) \end{aligned}$$

$$\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta) [\hat{\omega}]^2 \in SO(3)$$

Rodrigues' formula

- Matrix exponential and matrix log:

$$\exp : [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$$

$$\log : R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$$