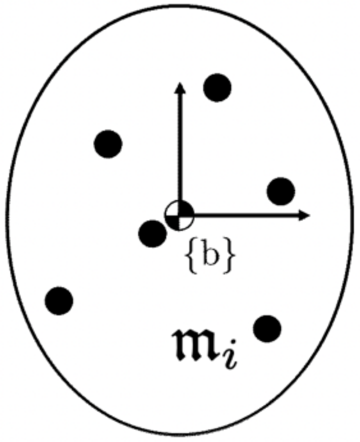
Important concepts, symbols, and equations

dynamics of a rigid body: $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b \dot{\omega}_b + [\omega_b]\mathcal{I}_b \omega_b \\ \mathfrak{m}(\dot{v}_b + [\omega_b]v_b) \end{bmatrix}$



$$\begin{aligned}
\mathcal{I}_b &= \begin{bmatrix} \sum \mathfrak{m}_i(y_i^2 + z_i^2) & -\sum \mathfrak{m}_i x_i y_i & -\sum \mathfrak{m}_i x_i z_i \\ -\sum \mathfrak{m}_i x_i y_i & \sum \mathfrak{m}_i(x_i^2 + z_i^2) & -\sum \mathfrak{m}_i y_i z_i \\ -\sum \mathfrak{m}_i x_i z_i & -\sum \mathfrak{m}_i y_i z_i & \sum \mathfrak{m}_i(x_i^2 + y_i^2) \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}
\end{aligned}$$

inertia matrix

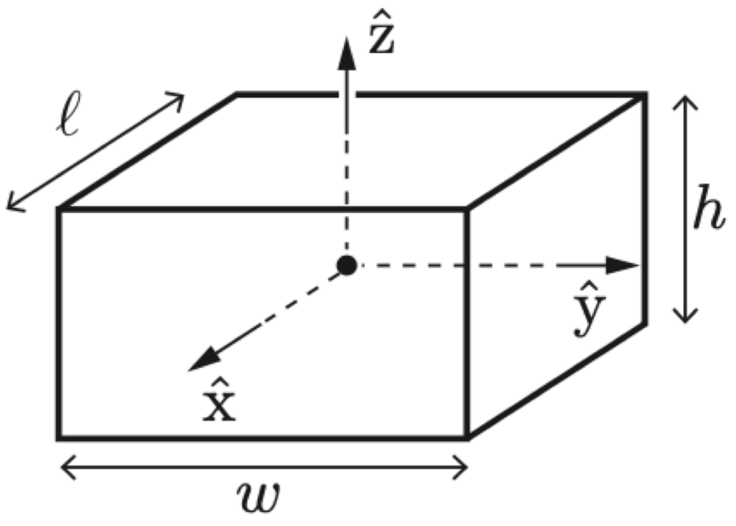
symmetric, positive definite

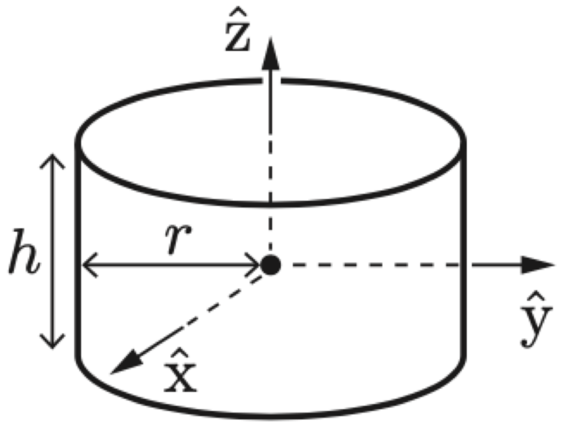
($x^{\mathrm{T}} I_b x > 0$ for all $x \neq 0$)

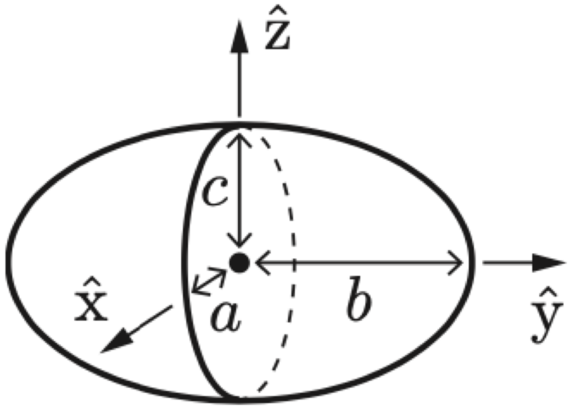
$$\begin{aligned}
\mathcal{I}_{xx} &= \int_{\mathcal{B}} (y^2 + z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xy} &= -\int_{\mathcal{B}} xy\rho(x,y,z)\,dV \\
\mathcal{I}_{yy} &= \int_{\mathcal{B}} (x^2 + z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xz} &= -\int_{\mathcal{B}} xz\rho(x,y,z)\,dV \\
\mathcal{I}_{zz} &= \int_{\mathcal{B}} (x^2 + y^2)\rho(x,y,z)\,dV & \mathcal{I}_{yz} &= -\int_{\mathcal{B}} yz\rho(x,y,z)\,dV
\end{aligned}$$

Important concepts, symbols, and equations (cont.)

uniform density







rectangular parallelepiped:

$$\text{volume} = hlw,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$$

circular cylinder:

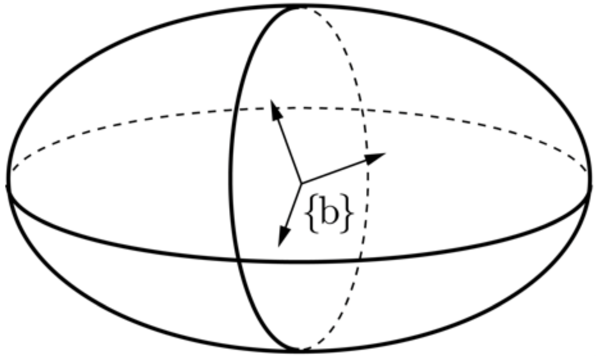
$$\text{volume} = \pi r^2 h,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$$

ellipsoid:

$$\text{volume} = 4\pi abc/3,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$$

Important concepts, symbols, and equations (cont.)

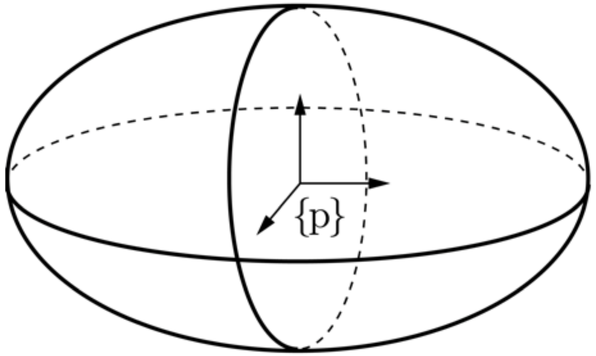
kinetic energy of a rotating body: $\mathcal{K} = \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b$



$$\mathcal{I}_b = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

e-vals: $\lambda_1, \lambda_2, \lambda_3$

e-vecs: v_1, v_2, v_3



$$\mathcal{I}_p = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

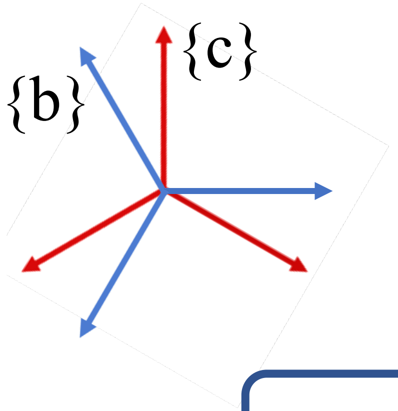
principal moments of inertia

for coordinate axes along v_1, v_2, v_3, the **principal axes of inertia**

Important concepts, symbols, and equations (cont.)

The inertia matrix of a compound body is the sum of their inertias when expressed in a common frame.

Frame at different orientation:

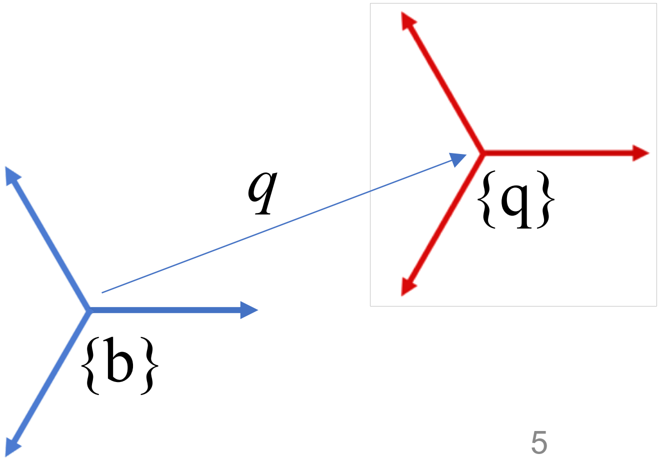


$$
\begin{aligned}
\frac{1}{2}\omega_c^{\mathrm{T}}\mathcal{I}_c\omega_c &= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b \\
&= \frac{1}{2}(R_{bc}\omega_c)^{\mathrm{T}}\mathcal{I}_b(R_{bc}\omega_c) \\
&= \frac{1}{2}\omega_c^{\mathrm{T}}(R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc})\omega_c
\end{aligned}
$$

$$\boxed{\mathcal{I}_c = R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc}}$$

Aligned frame at q in $\{\mathrm{b}\}$ (**Steiner's theorem**):

$$\boxed{\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}}qI - qq^{\mathrm{T}})}$$



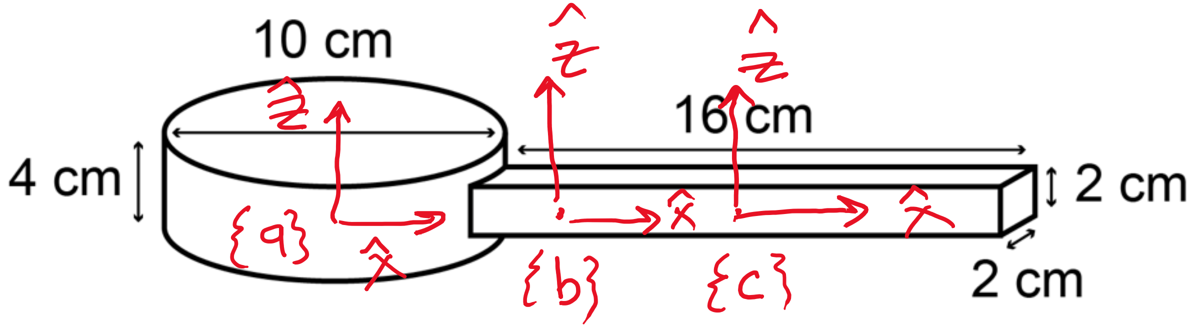
Important concepts, symbols, and equations (cont.)

- **spatial inertia matrix**: $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m} I \end{bmatrix} \in \mathbb{R}^{6\times 6}$
- kinetic energy: $\frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$
- rigid-body dynamics: $\mathcal{F}_b = \mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$

  where the "little adjoint" of a twist is $[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$
- spatial inertia matrix in a frame $\{\mathrm{a}\}$:

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]$$

- the form of the dynamics is independent of the frame!



A compound object consists of a uniform-density cylinder and a uniform-density rectangular prism. The mass of the cylinder is 2 kg and the mass of the prism is 1 kg. A frame $\{a\}$ is defined at the center of the cylinder, with the x-axis along the prism and the z-axis vertical.

Where is the CM of the compound object in $\{a\}$?

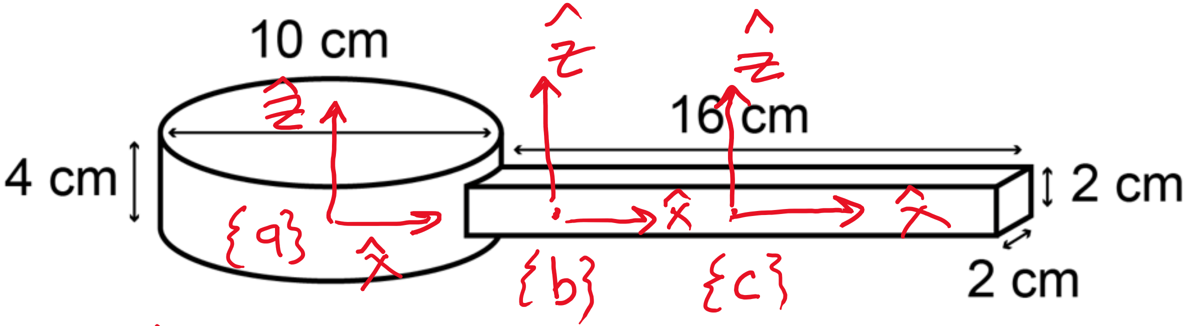
1 kg at (13, 0, 0) / 3 kg total

so CM at $\left(\frac{13}{3}, 0, 0\right)$ cm

In a frame $\{b\}$ at the CM, aligned with $\{a\}$, what is the inertia of the compound object?

Cylinder:

$$I_a = \begin{bmatrix} 15\tfrac{1}{6} & 0 & 0 \\ 0 & 15\tfrac{1}{6} & 0 \\ 0 & 0 & 25 \end{bmatrix} \text{kg cm}^2$$



$$I_b^{cyl} = \begin{bmatrix} 15.2 & 0 & 0 \\ 0 & 15.2 & 0 \\ 0 & 0 & 25 \end{bmatrix} + 2\left(q^T q I - q q^T\right) \quad \text{where } q = (4.3, 0, 0)$$

$$+ 2\left(\begin{bmatrix} 18.7 & 0 & 0 \\ 0 & 18.7 & 0 \\ 0 & 0 & 18.7 \end{bmatrix} - \begin{bmatrix} 18.7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 15.2 & 0 & 0 \\ 0 & 52.6 & 0 \\ 0 & 0 & 62.4 \end{bmatrix} \text{kg cm}^2$$

Steiner's thm

$$\boxed{I_b^{tot} = I_b^{cyl} + I_b^{prism} = \begin{bmatrix} 15.9 & 0 & 0 \\ 0 & 133.6 & 0 \\ 0 & 0 & 143.4 \end{bmatrix} \text{kg cm}^2}$$

prism:

$$I_c = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 21\tfrac{2}{3} & 0 \\ 0 & 0 & 21\tfrac{2}{3} \end{bmatrix} \text{kg cm}^2$$

$$q = (-7.7, 0, 0)$$

$$I_b^{prism} = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

Derive

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]$$

using equivalence of kinetic energy in different frames.

$$\frac{1}{2} \mathcal{V}_a^{\mathrm{T}} \mathcal{G}_a \mathcal{V}_a = \frac{1}{2} \mathcal{V}_b^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b \qquad \mathcal{V}_b = [\mathrm{Ad}_{T_{ba}}] \mathcal{V}_a$$

$$= \frac{1}{2} \mathcal{V}_a^{\mathrm{T}} \underbrace{[\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]}_{\mathcal{G}_a} \mathcal{V}_a$$