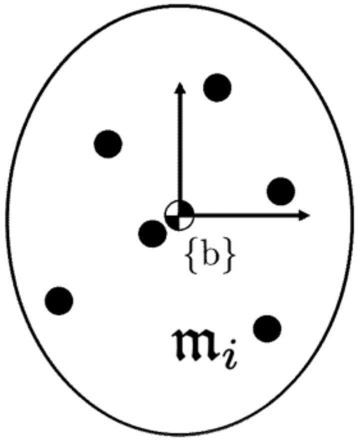
Important concepts, symbols, and equations

dynamics of a rigid body: $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b \dot{\omega}_b + [\omega_b]\mathcal{I}_b \omega_b \\ \mathfrak{m}(\dot{v}_b + [\omega_b] v_b) \end{bmatrix}$



$$\mathcal{I}_b = \begin{bmatrix} \sum \mathfrak{m}_i (y_i^2 + z_i^2) & -\sum \mathfrak{m}_i x_i y_i & -\sum \mathfrak{m}_i x_i z_i \\ -\sum \mathfrak{m}_i x_i y_i & \sum \mathfrak{m}_i (x_i^2 + z_i^2) & -\sum \mathfrak{m}_i y_i z_i \\ -\sum \mathfrak{m}_i x_i z_i & -\sum \mathfrak{m}_i y_i z_i & \sum \mathfrak{m}_i (x_i^2 + y_i^2) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

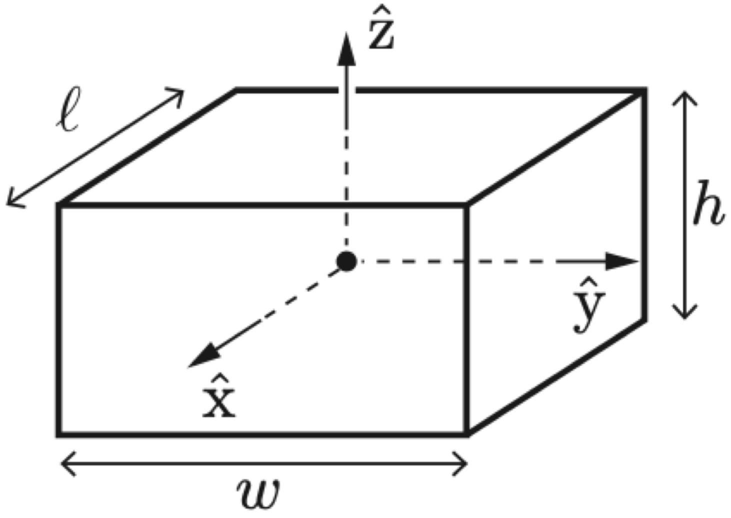
inertia matrix

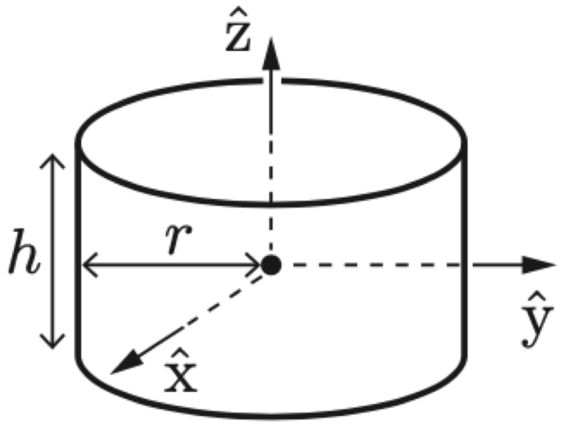
symmetric, positive definite

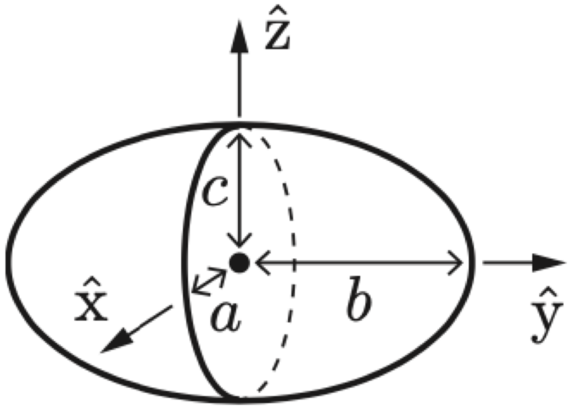
($x^{\mathrm{T}} I_b x > 0$ for all $x \neq 0$)

$$\begin{aligned} \mathcal{I}_{xx} &= \int_{\mathcal{B}} (y^2 + z^2)\rho(x,y,z)\, dV & \mathcal{I}_{xy} &= -\int_{\mathcal{B}} xy\rho(x,y,z)\, dV \\ \mathcal{I}_{yy} &= \int_{\mathcal{B}} (x^2 + z^2)\rho(x,y,z)\, dV & \mathcal{I}_{xz} &= -\int_{\mathcal{B}} xz\rho(x,y,z)\, dV \\ \mathcal{I}_{zz} &= \int_{\mathcal{B}} (x^2 + y^2)\rho(x,y,z)\, dV & \mathcal{I}_{yz} &= -\int_{\mathcal{B}} yz\rho(x,y,z)\, dV \end{aligned}$$

Important concepts, symbols, and equations (cont.)







rectangular parallelepiped:

$$\text{volume} = hlw,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$$

circular cylinder:

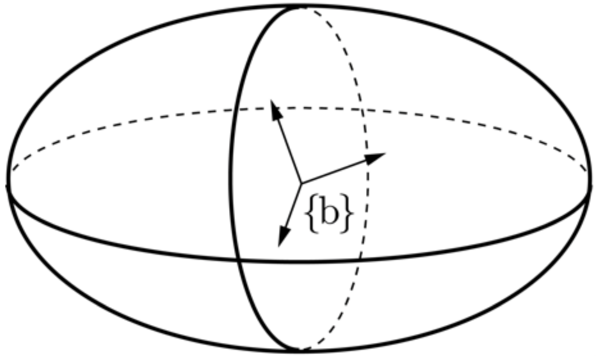
$$\text{volume} = \pi r^2 h,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$$

ellipsoid:

$$\text{volume} = 4\pi abc/3,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$$

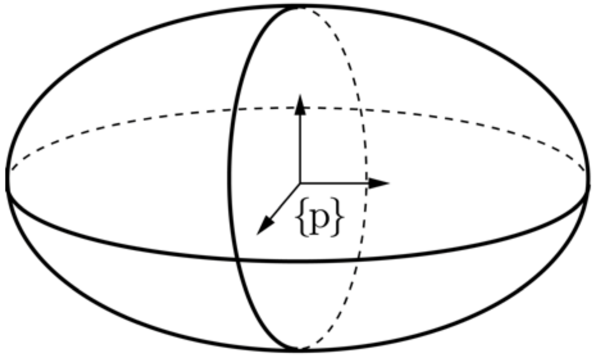
Important concepts, symbols, and equations (cont.)

kinetic energy of a rotating body: $\mathcal{K} = \frac{1}{2}\omega_b^{\mathrm{T}} \mathcal{I}_b \omega_b$



$$\mathcal{I}_b = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

e-vals: $\lambda_1, \lambda_2, \lambda_3$
e-vecs: v_1, v_2, v_3



$$\mathcal{I}_p = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

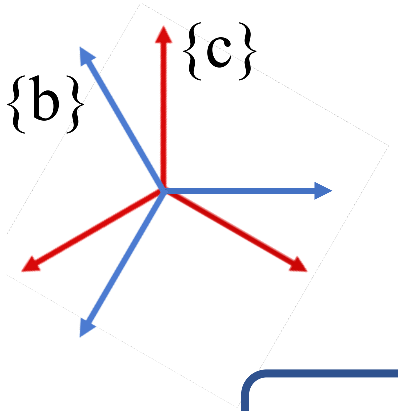
principal moments of inertia

for coordinate axes along $v_1, v_2, v_3,$
the **principal axes of inertia**

Important concepts, symbols, and equations (cont.)

The inertia matrix of a compound body is the sum of their inertias when expressed in a common frame.

Frame at different orientation:

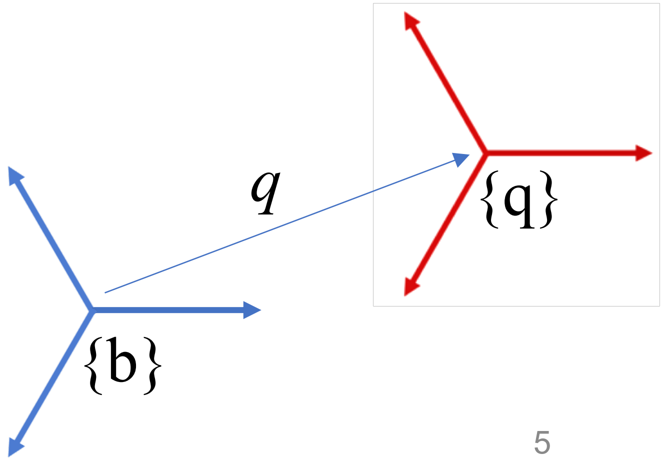


$$
\begin{aligned}
\frac{1}{2}\omega_c^{\mathrm{T}}\mathcal{I}_c\omega_c &= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b \\
&= \frac{1}{2}(R_{bc}\omega_c)^{\mathrm{T}}\mathcal{I}_b(R_{bc}\omega_c) \\
&= \frac{1}{2}\omega_c^{\mathrm{T}}(R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc})\omega_c
\end{aligned}
$$

$$\boxed{\mathcal{I}_c = R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc}}$$

Aligned frame at q in $\{\mathrm{b}\}$ (**Steiner's theorem**):

$$\boxed{\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}}qI - qq^{\mathrm{T}})}$$



Important concepts, symbols, and equations (cont.)

- **spatial inertia matrix**: $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m} I \end{bmatrix} \in \mathbb{R}^{6\times 6}$

- kinetic energy: $\frac{1}{2}\mathcal{V}_b^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b$

- rigid-body dynamics: $\mathcal{F}_b = \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b$

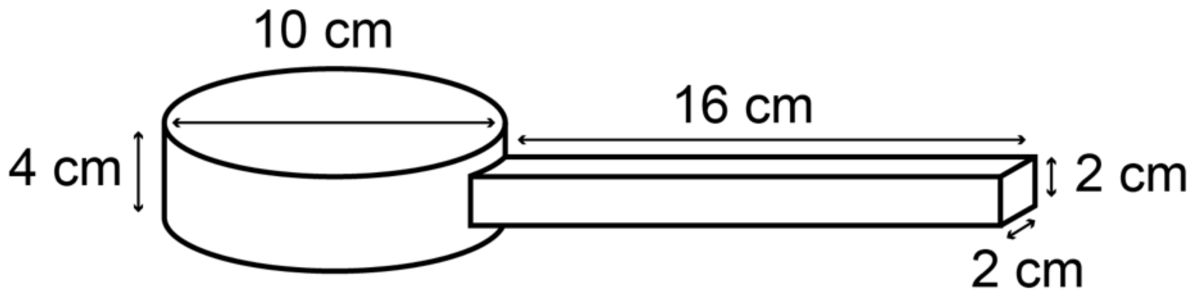
  where the "little adjoint" of a twist is $[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$

- spatial inertia matrix in a frame $\{\mathrm{a}\}$:

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]$$

- the form of the dynamics is independent of the frame!

A compound object consists of a uniform-density cylinder and a uniform-density rectangular prism. The mass of the cylinder is 2 kg and the mass of the prism is 1 kg. A frame $\{a\}$ is defined at the center of the cylinder, with the x-axis along the prism and the z-axis vertical.



Where is the CM of the compound object in $\{a\}$?

In a frame $\{b\}$ at the CM, aligned with $\{a\}$, what is the inertia of the compound object?

Derive

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]$$

using equivalence of kinetic energy in different frames.