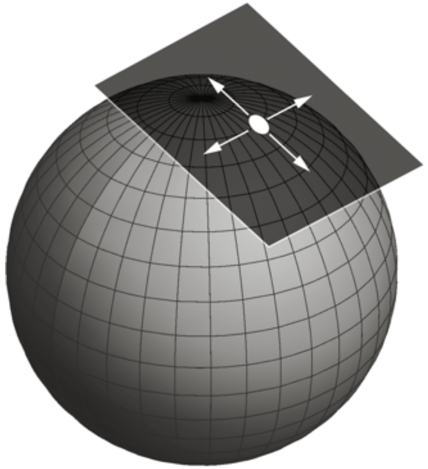
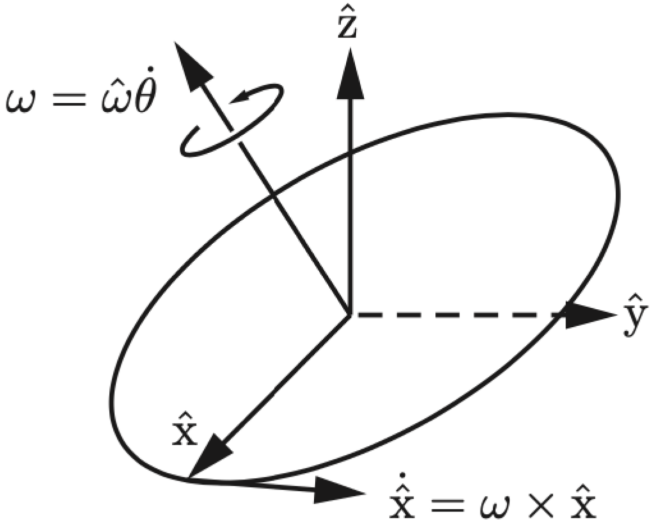
Important concepts, symbols, and equations

- $SO(3)$ is a curved 3-dimensional space, but the feasible velocities at any point of $SO(3)$ form a flat 3-dimensional **vector space** (the "tangent space").



Another example: the tangent space at a point of S^2.



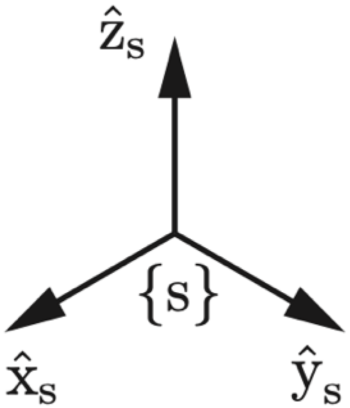
- Any rotational velocity can be expressed as an **angular velocity** $\omega \in \mathbb{R}^3$, which can be considered the product of a unit axis (in S^2) and a speed (a scalar).

Important concepts, symbols, and equations (cont.)

- Given $p \in \mathbb{R}^3$ and ω defined in the same reference frame, $\dot{p} = \omega \times p$.

- Linear algebra notation: $\dot{p} = \omega \times p = [\omega]\, p$, where

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$ $\in so(3)$, the $3{\times}3$ real skew-symmetric matrices (satisfying $[x] = -[x]^{\mathrm{T}}$).



$so(3)$ describes the possible $\dot{R}$ when $R = I$, and it is called the **Lie algebra** of the Lie group $SO(3)$.

Important concepts, symbols, and equations (cont.)

- If $R_{sb} = [p_1 \;\; p_2 \;\; p_3]$, then $\dot{R}_{sb} = [\,[\omega_s]\, p_1 \;\;\; [\omega_s]\, p_2 \;\;\; [\omega_s]\, p_3\,] = [\omega]\, R_{sb}$.

- Expressing the angular velocity in a different frame:

$$\omega_b = R_{b\cancel{s}}\; \omega_{\cancel{s}} = R^{-1}{}_{sb}\; \omega_s = R^{\mathrm{T}}{}_{sb}\; \omega_s \qquad\qquad \omega_s = R_{sb}\; \omega_b$$

- The $so(3)$ representations:

$$[\omega_b] = R^{-1}{}_{sb}\, \dot{R} = R^{\mathrm{T}}{}_{sb}\, \dot{R} \qquad\qquad [\omega_s] = \dot{R}\, R^{-1}{}_{sb} = \dot{R}\, R^{\mathrm{T}}{}_{sb}$$

- **Exponential coordinate** (axis-angle) representation of orientation: $\hat{\omega}\theta$

Important concepts, symbols, and equations (cont.)

- Scalar first-order linear diffeq:

$$\dot{x}(t) = ax(t) \Longrightarrow x(t) = e^{at}x_0$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

- Vector first-order linear diffeq:

$$\dot{x}(t) = Ax(t) \Longrightarrow x(t) = e^{At}x_0$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

matrix exponential

Important concepts, symbols, and equations (cont.)

- Integrating an angular velocity

$$\dot{p} = \hat{\omega} \times p = [\hat{\omega}]p \longrightarrow p(t) = e^{[\hat{\omega}]t}p(0)$$

$$p(\theta) = e^{[\hat{\omega}]\theta}p(0)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Rodrigues' formula

- Matrix exponential and matrix log:

$$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$$

$$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$$