Chapter 2 Configuration Space
Chapter 3 Rigid-Body Motions
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## Important concepts, symbols, and equations

- The **special Euclidean group** $SE(3)$ is a matrix Lie group also known as the group of rigid-body motions or **homogeneous transformation matrices** in $\mathbb{R}^3$.

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \qquad R \in SO(3) \text{ and } p \in \mathbb{R}^3$$

- The inverse of $T \in SE(3)$ is

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

## Important concepts, symbols, and equations (cont.)

- Three uses of HT matrices:

1. Represent a configuration. $T_{ab}$ represents frame $\{\mathrm{b}\}$ relative to $\{\mathrm{a}\}$.
2. Change the reference frame of a vector or frame.

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$

$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a$$

$v$ should be written in **homogeneous coordinates**, $v = [v_1 \;\; v_2 \;\; v_3 \;\; 1]^{\mathrm{T}}$.

3. Displace a vector or frame. $T = (R, p) = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta)$

$$\mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{Rot}(\hat{\omega},\theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

## Important concepts, symbols, and equations (cont.)

Space-frame transformation:

$$T_{sb''} = T\,T_{sb} = \quad \mathrm{Trans}(p) \quad \mathrm{Rot}(\hat{\omega},\theta) \quad T_{sb}$$

2. translate $\{\mathrm{b}'\}$ by $p$ in $\{\mathrm{s}\}$ to get $\{\mathrm{b}''\}$

1. rotate $\{\mathrm{b}\}$ by $\theta$ about $\hat{\omega}$ in $\{\mathrm{s}\}$ to get $\{\mathrm{b}'\}$ (moves $\{\mathrm{b}\}$ origin)

Body-frame transformation:

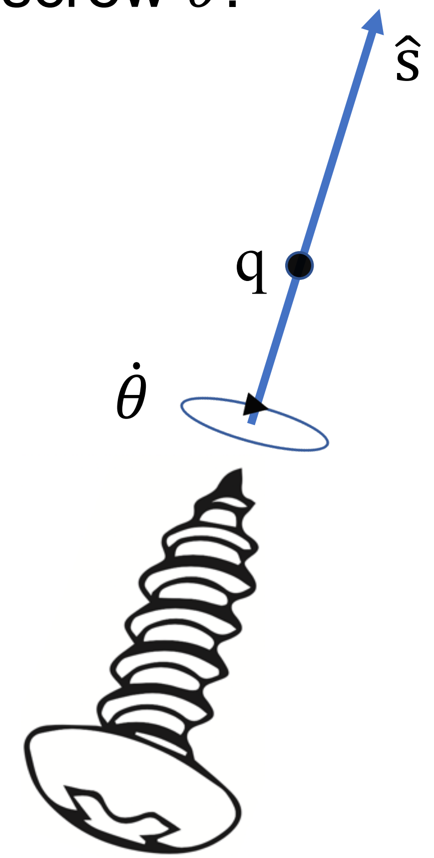
$$T_{sb''} = T_{sb}\,T = \quad T_{sb} \quad \mathrm{Trans}(p) \quad \mathrm{Rot}(\hat{\omega},\theta)$$

1. translate $\{\mathrm{b}\}$ by $p$ in $\{\mathrm{b}\}$ to get $\{\mathrm{b}'\}$

2. rotate $\{\mathrm{b}'\}$ by $\theta$ about $\hat{\omega}$ in $\{\mathrm{b}'\}$ to get $\{\mathrm{b}''\}$

# Important concepts, symbols, and equations (cont.)

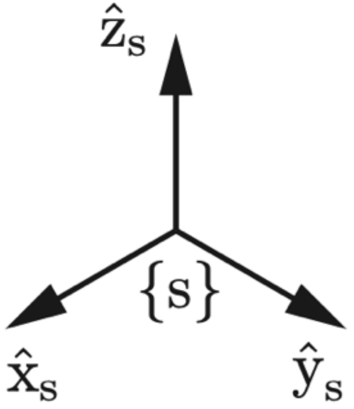
Any rigid-body velocity can represented as a **screw axis** (a direction $\hat{s} \in S^2$, a point $q \in \mathbb{R}^3$ on the screw, and the **pitch** (linear speed/angular speed) of the screw $h$), plus the speed along the screw $\dot{\theta}$.



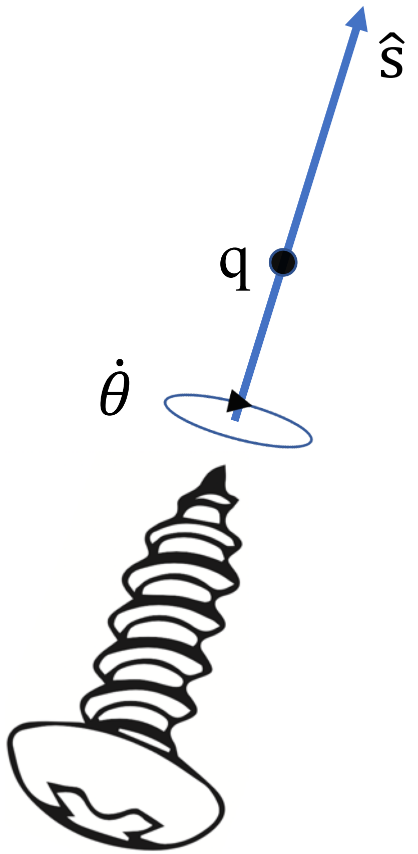

If $h$ is infinite, $\dot{\theta}$ is the linear speed. Otherwise, it is the angular speed.

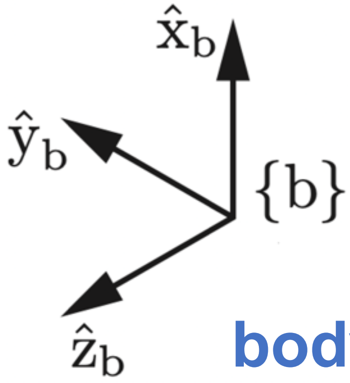
# Important concepts, symbols, and equations (cont.)

The **twist** $\mathcal{V}_a = (\omega_a, v_a) \in \mathbb{R}^6$ is the angular velocity expressed in $\{a\}$ and the linear velocity of the origin of $\{a\}$ expressed in $\{a\}$.




**spatial twist** $\mathcal{V}_s$




**body twist** $\mathcal{V}_b$

## Important concepts, symbols, and equations (cont.)

- To transform a twist from one frame to another,

$$\cancel{\underset{}{\mathcal{V}_a = \underset{4\times 4}{T_{ab}}\ \underset{6\times 1}{\mathcal{V}_b}}}$$

$\mathcal{V}_a = [\mathrm{Ad}_{\mathrm{Tab}}]\,\mathcal{V}_b$ , where the **adjoint representation** of $T = (R\,,\,p)$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

## Important concepts, symbols, and equations (cont.)

Matrix representation of twists:

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

where $se(3)$ is the Lie algebra of $SE(3)$ (the set of all possible $\dot{T}$ when $T = I$).

## Important concepts, symbols, and equations (cont.)

Screws and twists:

- for a screw axis $\{q, \hat{s}, h\}$ with finite $h$,
$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$

- "unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$,
  where either (i) $\|\omega\| = 1$ or
  (ii) $\omega = 0$ and $\|v\| = 1$

- $\mathcal{V} = \mathcal{S}\dot{\theta}$

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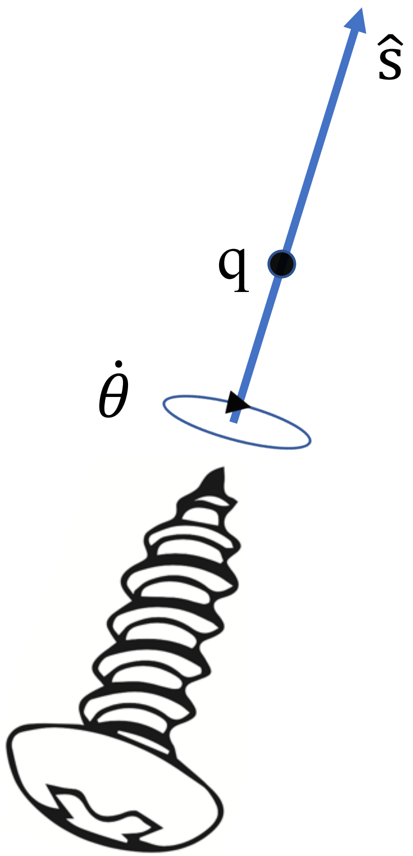
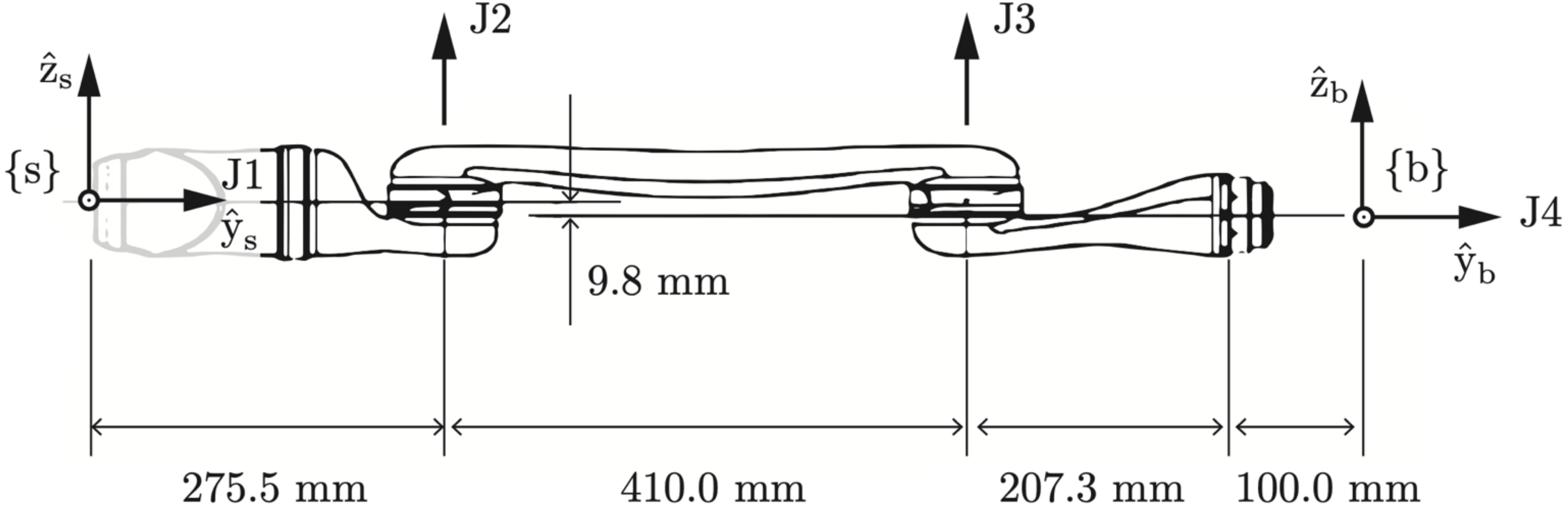


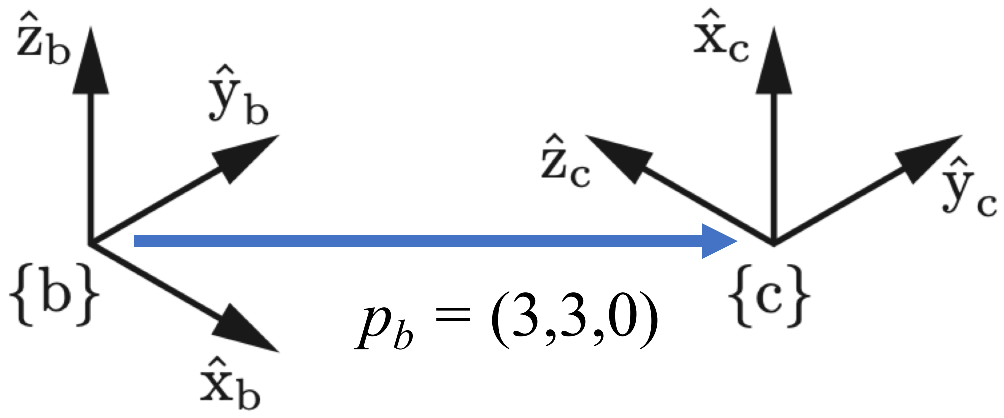

What is the dimension of the space of screws?

# Kinova lightweight 4-dof arm




For J4, what is the screw axis $S_b$? $S_s$?

For J2?




Write $T_{bc}$.

A camera with frame $\{\mathrm{c}\}$ tracks the optical marker on a tool to get its frame $\{\mathrm{t}\}$ relative to $\{\mathrm{c}\}$. This transformation is $T_1$. A space frame $\{\mathrm{s}\}$ is attached to the floor of the room, and the camera observes its configuration relative to $\{\mathrm{c}\}$ as $T_2$. A robot arm has a mounting frame $\{\mathrm{m}\}$ which has been measured relative to $\{\mathrm{s}\}$ as $T_3$. The arm's encoders and the robot's kinematics tell us the gripper's frame $\{\mathrm{e}\}$ relative to $\{\mathrm{m}\}$. This is represented as $T_4$. The gripper should be at the frame $\{\mathrm{g}\}$ relative to $\{\mathrm{t}\}$ to be able to close on the tool and pick it up. This configuration is represented as $T_5$.

Write $T_{eg}$, the configuration of the grasping frame $\{\mathrm{g}\}$ relative to the current end-effector frame $\{\mathrm{e}\}$, in terms of $T_1$, $T_2$, $T_3$, $T_4$, and $T_5$.