Important concepts, symbols, and equations

- **forward dynamics** (for simulation): $\theta, \dot{\theta}, \tau \rightarrow \ddot{\theta}$
- **inverse dynamics** (for control): $\theta, \dot{\theta}, \ddot{\theta} \rightarrow \tau$
- two equivalent approaches to computing the dynamics:
  - Lagrangian (variational, based on energy)
  - Newton-Euler (“f = ma” for the rigid bodies)

Important concepts, symbols, and equations (cont.)

Lagrangian approach:

Lagrangian $\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$ kinetic minus potential energy

eqs of motion $$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} \in \mathbb{R}^n$$

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i}$$ in components

<u>Important concepts, symbols, and equations</u> (cont.)

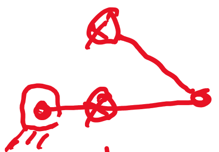
Standard forms of dynamic equations:

$$
\begin{aligned}
\tau &= M(\theta)\ddot{\theta} + h(\theta,\dot{\theta}) \\
&= M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta) \\
&= M(\theta)\ddot{\theta} + \dot{\theta}^{\mathrm{T}}\Gamma(\theta)\dot{\theta} + g(\theta) \\
&= M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta)
\end{aligned}
$$

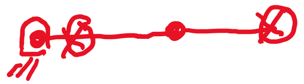
$$+J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$$

depends on config θ!

$M(\theta)$	$n \times n$ symmetric positive definite **mass matrix**
$g(\theta)$	gravity (potential) terms
$c(\theta,\dot{\theta})$	velocity-product terms
$\Gamma(\theta)$	$n \times n \times n$ tensor of **Christoffel symbols** (due to nonzero $\partial M/\partial\theta$)
$C(\theta,\dot{\theta})$	**Coriolis matrix**



small inertia about joint 1



large inertia

Important concepts, symbols, and equations (cont.)

Velocity-product terms ($c(\theta,\dot{\theta})$, $C(\theta,\dot{\theta})\dot{\theta}$, $\dot{\theta}^{\mathrm{T}}\Gamma(\theta)\dot{\theta}$) consist of

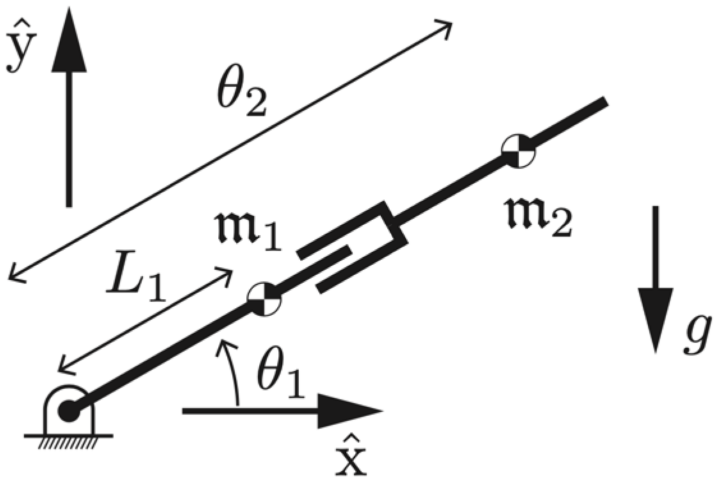
- **centripetal** terms, e.g.,

$$\mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2$$

- **Coriolis** terms, e.g.,

$$-\mathfrak{m}_2 L_1 L_2 \sin\theta_2 (2\dot{\theta}_1\dot{\theta}_2)$$

RP in gravity



$$\mathcal{P}_1 = \mathfrak{m}_1 g y_1 = \mathfrak{m}_1 g L_1 \sin\theta_1$$

$$\mathcal{P}_2 = \mathfrak{m}_2 g y_2 = \mathfrak{m}_2 g \theta_2 \sin\theta_1$$

$$\mathcal{K}_1 = \frac{1}{2}\mathfrak{m}_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}\mathcal{I}_1\dot{\theta}_1^2 = \frac{1}{2}(\mathcal{I}_1 + \mathfrak{m}_1 L_1^2)\dot{\theta}_1^2$$

$$\mathcal{K}_2 = \frac{1}{2}\mathfrak{m}_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}\mathcal{I}_2\dot{\theta}_1^2 = \frac{1}{2}\left((\mathcal{I}_2 + \mathfrak{m}_2\theta_2^2)\dot{\theta}_1^2 + \mathfrak{m}_2\dot{\theta}_2^2\right)$$

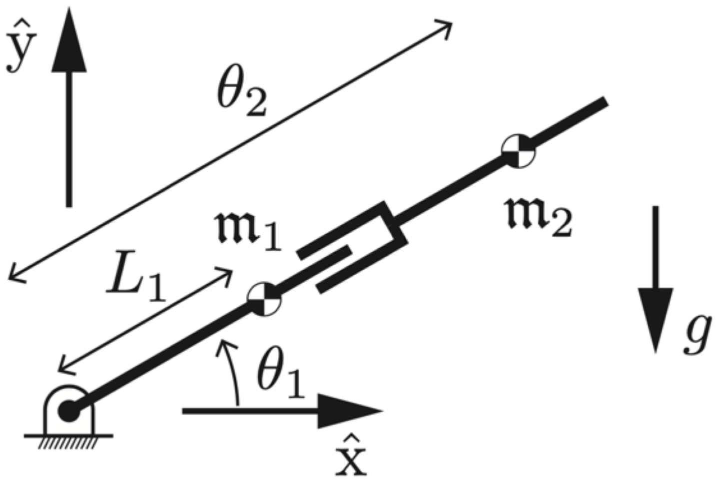
$$\mathcal{L} = \mathcal{K}_1 + \mathcal{K}_2 - \mathcal{P}_1 - \mathcal{P}_2 = \frac{1}{2}\mathfrak{m}_2\theta_2^2\dot{\theta}_1^2 \; + \ldots$$

$$= \mathcal{L}_1 + \ldots$$

product rule: $(fg)' = f'g + fg'$

chain rule: $\frac{df\big(g(t)\big)}{dt} = \left(\frac{\partial f}{\partial g}\right)\frac{dg}{dt}$

RP in gravity



$$\mathcal{L}_1 = \frac{1}{2}\mathfrak{m}_2\theta_2^2\dot{\theta}_1^2$$

Contribution to τ_2:

$$\tau_2 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} - \frac{\partial \mathcal{L}}{\partial \theta_2}$$

$$= 0 - m_2\theta_2\dot{\theta}_1^2$$ centripetal term

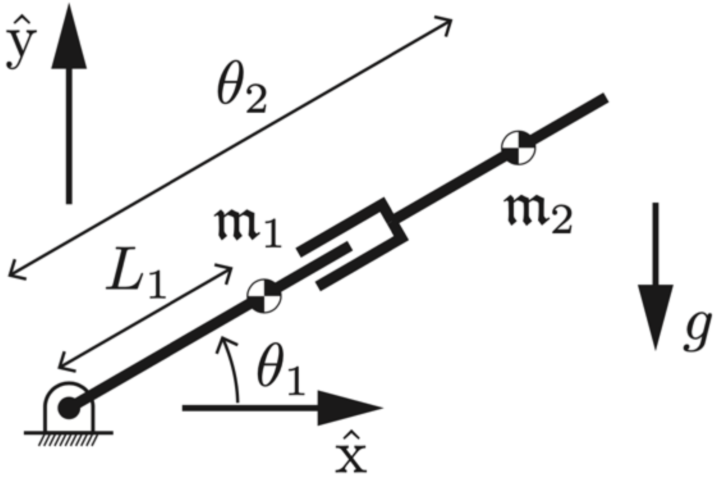
Contribution to τ_1:

$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \cancel{\frac{\partial \mathcal{L}}{\partial \theta_1}} \rightarrow 0$$

$$= \frac{d}{dt}\left(m_2\theta_2^2\dot{\theta}_1\right)$$

$$= \underbrace{2m_2\theta_2\dot{\theta}_1\dot{\theta}_2}_{\text{Coriolis}} + \underbrace{m_2\theta_2^2\ddot{\theta}_1}_{\text{mass matrix term}}$$

RP in gravity



$$\tau = M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta)$$

$$M(\theta) = \begin{bmatrix} \mathcal{I}_1 + \mathcal{I}_2 + \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} 2\mathfrak{m}_2\theta_2\dot{\theta}_1\dot{\theta}_2 \\ -\mathfrak{m}_2\theta_2\dot{\theta}_1^2 \end{bmatrix}$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2) g\cos\theta_1 \\ \underbrace{\mathfrak{m}_2 g \sin\theta_1}_{} \end{bmatrix}$$

torque to keep joint 1 stationary in gravity

force to keep joint 2 stationary in gravity

Explain the velocity-product terms.

Assume $\ddot{\theta} = 0$, $g = 0$

$\dot{\theta}_2 = 0$: $\tau_2 = -m_2\theta_2\dot{\theta}_1^2$ is a force needed to prevent m_2 flying out tangent to the circle of motion

$\tau_1 = 2m_2\dot{\theta}_1\dot{\theta}_2$: Assume $\dot{\theta}_1 > 0$.

If $\dot{\theta}_2 < 0$: inertia of arm about joint 1 is decreasing, so $\tau_1 < 0$ to prevent $\dot{\theta}_1$ from increasing like a skater does (due to conservation of angular momentum) when she pulls her arms in