Chapter 2 Configuration Space
Chapter 3 Rigid-Body Motions
3.2 Rotations and Angular Velocities

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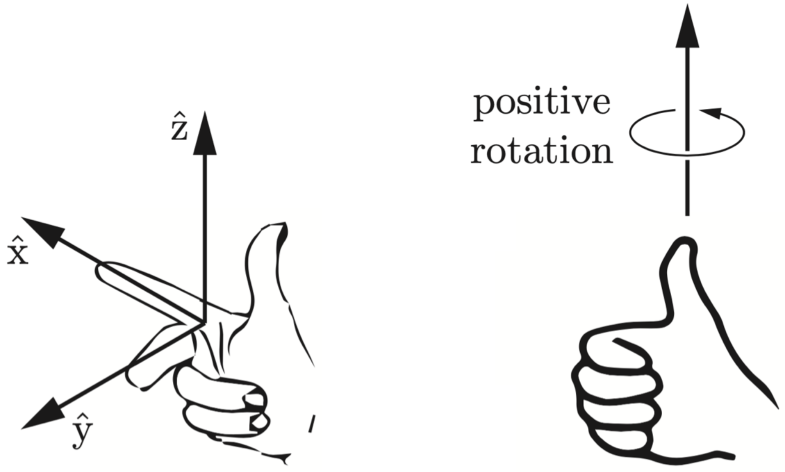
## Important concepts, symbols, and equations

- We often define a fixed **space frame** $\{s\}$ and a **body frame** $\{b\}$ attached to some body of interest. All frames are *instantaneously stationary*.

- Right-handed frames, and right-hand rule for positive rotation.




orthogonal special

- **Special orthogonal group** $SO(3)$: matrices $R$ in $\mathbb{R}^{3\times 3}$ where $R^{\mathrm{T}}R = I$, $\det R = 1$. $R$ is a **rotation matrix**. Implicit representation with 9 numbers for 3 dof.

# Important concepts, symbols, and equations (cont.)

- A **group** is a set of elements $G = \{a, b, c \ldots\}$ and a binary operation • satisfying

| | |
|---|---|
| closure | $a \bullet b \in G$ for all $a, b \in G$ |
| associativity | $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ |
| identity element exists | there is an $I \in G$ such that $a \bullet I = I \bullet a = a$ for each $a \in G$ |
| inverse exists | for each $a \in G$, there exists $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = I$ |

Integers under addition? Nonnegative integers under addition? Square real matrices under multiplication? What is a **Lie group**?

## Important concepts, symbols, and equations (cont.)

- $SO(3)$ is a **matrix (Lie) group** (the group operation is matrix multiplication).

| | |
|---|---|
| closure: | $R_1R_2 \in SO(3)$ |
| associative: | $(R_1R_2)R_3 = R_1(R_2R_3)$ (*not commutative!* $R_1R_2 \neq R_2R_1$ generally) |
| identity: | identity matrix $I$ |
| inverse: | matrix inverse |

$R^{\mathrm{T}}R = I$, so $R^{-1} = R^{\mathrm{T}}$.

For $x \in \mathbb{R}^3$, $\|x\| = \|Rx\|$.

## Important concepts, symbols, and equations (cont.)

- Uses of a rotation matrix:

1. Represent an orientation. $R_{ab}$ represents orientation of $\{\mathrm{b}\}$ in $\{\mathrm{a}\}$.
2. Change the reference frame of a vector or frame.

subscript cancellation:

$$R_{ab}R_{bc} = R_{a\cancel{b}}R_{\cancel{b}c} = R_{ac}$$

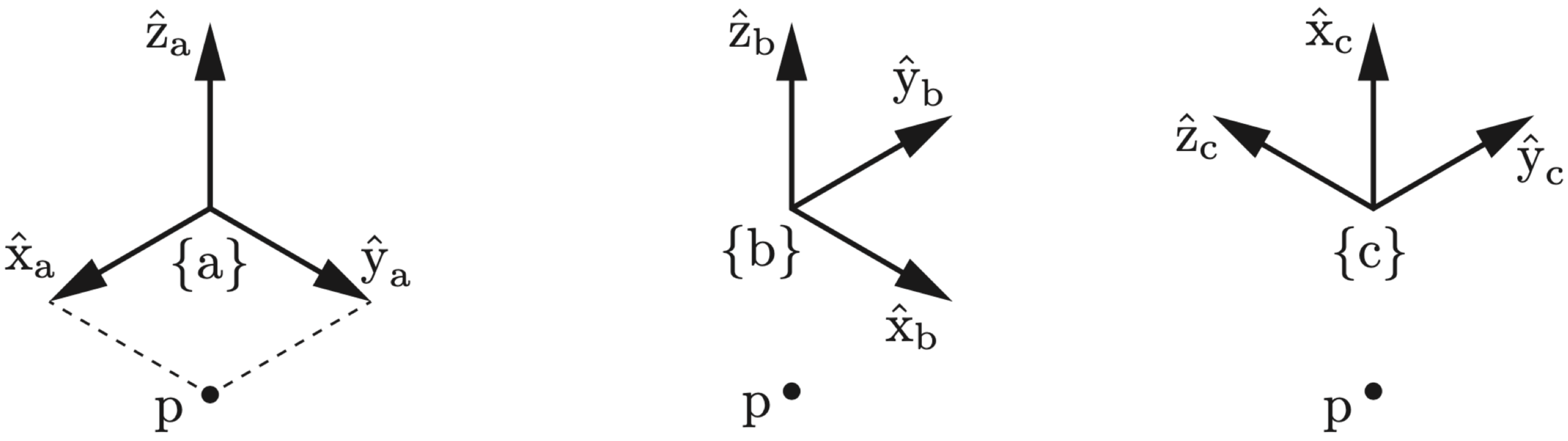
$$R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a$$

3. Rotate a vector or frame. $R = R_{cd} = \mathrm{Rot}(\hat{w}, \theta)$, axis $\hat{w}$ expressed in $\{\mathrm{c}\}$.

$p'_c = R_{cd}\, p_c$ (no subscript cancellation)

$R_{ab'} = RR_{ab}$ (after rotating about axis in $\{\mathrm{a}\}$)

$R_{ab''} = R_{ab}R$ (after rotating about axis in $\{\mathrm{b}\}$)




$$R_{ab} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_b =$$

$$p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$R_{ba} p_a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Given $R_1 = R_{ab}$, $R_2 = R_{bc}$, and $R_3 = R_{ad}$,
write $R_{dc}$ in terms of $R_1$, $R_2$, and $R_3$ (no inverses!).

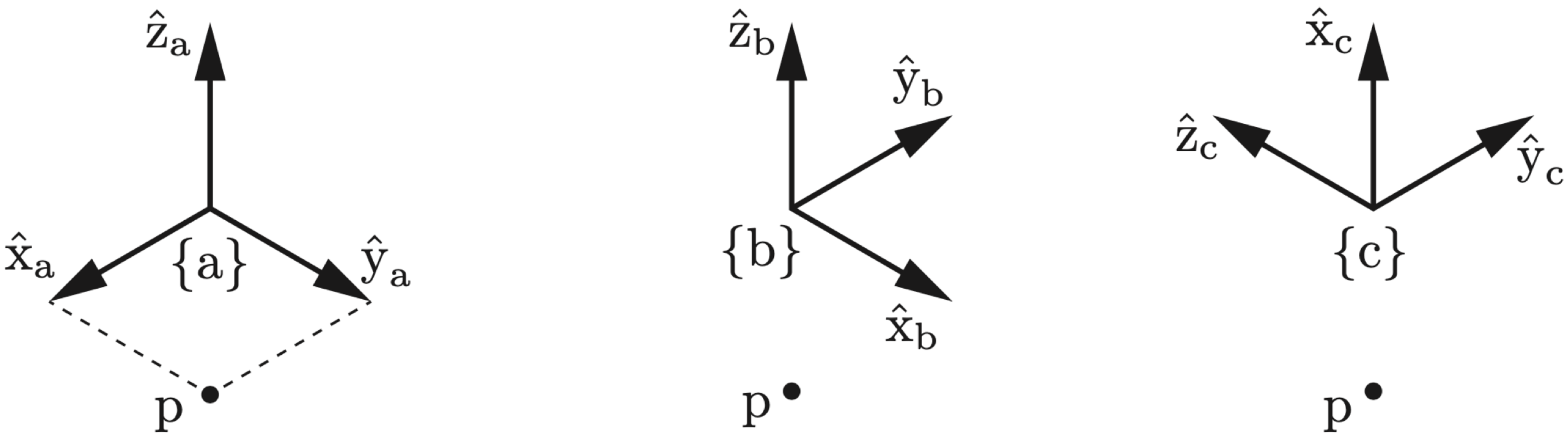
$$R_{dc} = R_{d\cancel{a}} R_{\cancel{a}\cancel{b}} R_{\cancel{b}c}$$

$$= R_3^T R_1 R_2$$

Given $p_b$, what is $p_d$ in terms of $R_1$, $R_2$, and $R_3$ (no inverses)?

$$p_d = R_{d\cancel{a}} R_{\cancel{a}\cancel{b}} p_{\cancel{b}}$$

$$= R_3^T R_1 p_b$$




$R = R_{ba} = \mathrm{Rot}(\hat{w}, \theta)$: $\theta = \pi/2$, axis $\hat{w} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$R_{bc'} = RR_{bc} =$ rotate $\{c\}$ by $\pi/2$ about $-\hat{z}_b$

$R_{bc''} = R_{bc}R =$ rotate $\{c\}$ by $\pi/2$ about $-\hat{z}_c$

| orientation representation | # nums | imp/exp? | pros | cons |
|---|---|---|---|---|
| Euler angles, roll-pitch-yaw | 3 | explicit | minimum #s | singularities (gimbal lock) |
| Unit quaternions<br>$S^3$<br>covers rotations twice | 4 | implicit | no singularities<br>fewer numbers than $SO(3)$<br>easy to interpolate<br>easy to normalize back to $S^3$ | unintuitive |
| Rotation matrices | 9 | implicit | linear algebra<br>linear diffeqs<br>integrate angular velocities<br>unique R for each orientation | lots of numbers<br>more computations to snap back to $SO(3)$ |