Important concepts, symbols, and equations

- The **special Euclidean group** $SE(3)$ is a matrix Lie group also known as the group of rigid-body motions or **homogeneous transformation matrices** in $\mathbb{R}^3$.

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \qquad R \in SO(3) \text{ and } p \in \mathbb{R}^3$$

- The inverse of $T \in SE(3)$ is

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

Important concepts, symbols, and equations (cont.)

- Three uses of HT matrices:

1. Represent a configuration. T_{ab} represents frame $\{\mathrm{b}\}$ relative to $\{\mathrm{a}\}$.
2. Change the reference frame of a vector or frame.

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$

$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a$$

v should be written in **homogeneous coordinates**, $v = [v_1 \;\; v_2 \;\; v_3 \;\; 1]^{\mathrm{T}}$.

3. Displace a vector or frame. $T = (R, p) = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta)$

$$\mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}(\hat{\omega},\theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

Important concepts, symbols, and equations (cont.)

Space-frame transformation:

$$T_{sb''} = T\,T_{sb} = \quad \mathrm{Trans}(p) \quad \mathrm{Rot}(\hat{\omega},\theta) \quad T_{sb}$$

2. translate $\{b'\}$ by p in $\{s\}$ to get $\{b''\}$

1. rotate $\{b\}$ by θ about $\hat{\omega}$ in $\{s\}$ to get $\{b'\}$ (moves $\{b\}$ origin)

Body-frame transformation:

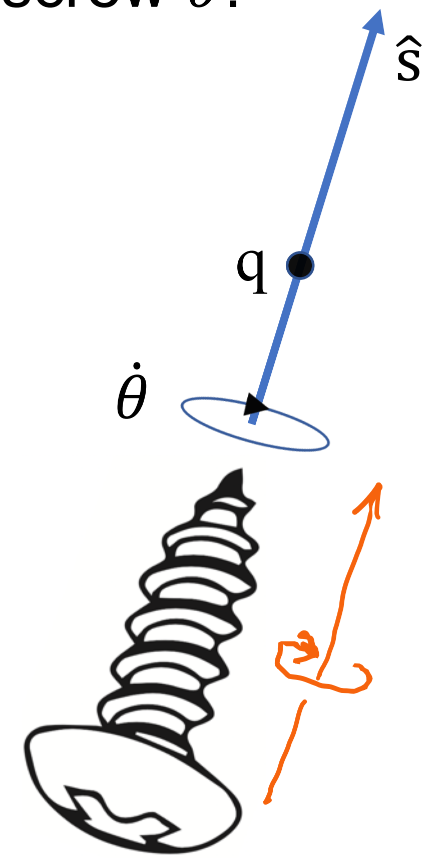
$$T_{sb''} = T_{sb}\,T = \quad T_{sb} \quad \mathrm{Trans}(p) \quad \mathrm{Rot}(\hat{\omega},\theta)$$

1. translate $\{b\}$ by p in $\{b\}$ to get $\{b'\}$

2. rotate $\{b'\}$ by θ about $\hat{\omega}$ in $\{b'\}$ to get $\{b''\}$

Important concepts, symbols, and equations (cont.)

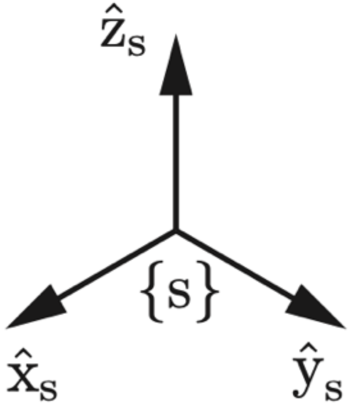
Any rigid-body velocity can represented as a **screw axis** (a direction $\hat{s} \in S^2$, a point $q \in \mathbb{R}^3$ on the screw, and the **pitch** (linear speed/angular speed) of the screw h), plus the speed along the screw $\dot{\theta}$.



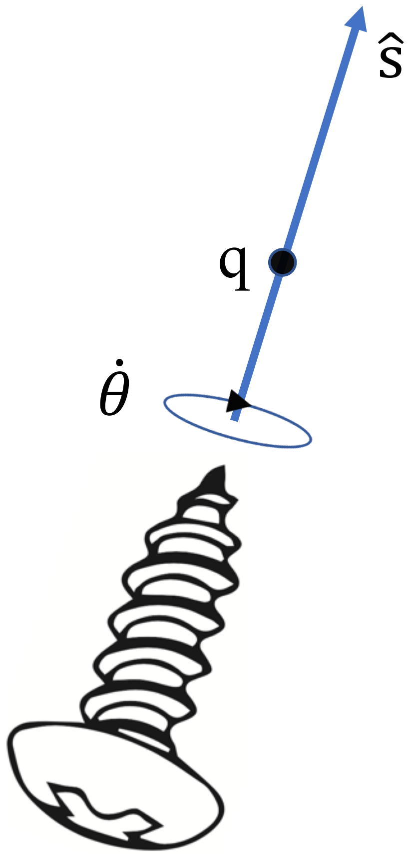
If h is infinite, $\dot{\theta}$ is the linear speed. Otherwise, it is the angular speed.

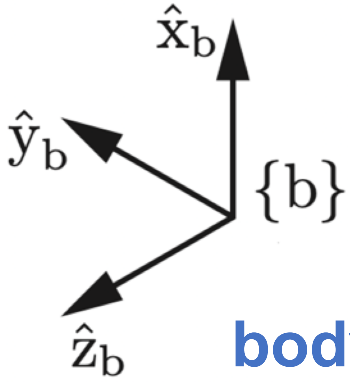
Important concepts, symbols, and equations (cont.)

The **twist** $\mathcal{V}_a = (\omega_a, v_a) \in \mathbb{R}^6$ is the angular velocity expressed in $\{a\}$ and the linear velocity of the origin of $\{a\}$ expressed in $\{a\}$.



spatial twist $\mathcal{V}_s$





body twist $\mathcal{V}_b$

Important concepts, symbols, and equations (cont.)

- To transform a twist from one frame to another,

$$\cancel{\underset{}{\mathcal{V}_a = \underset{4\times4}{T_{ab}}\ \underset{6\times1}{\mathcal{V}_b}}}$$

$\mathcal{V}_a = [Ad_{T_{ab}}] \mathcal{V}_b$

$\mathcal{S}_a = [Ad_{T_{ab}}] \mathcal{S}_b$

$\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\,\mathcal{V}_b$, where the **adjoint representation** of $T = (R\,,\,p)$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Important concepts, symbols, and equations (cont.)

Matrix representation of twists:

4x4 matrix

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3) \quad (T_{b\cancel{s}}\dot{T}_{\cancel{s}b})$$

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3) \quad (\dot{T}_{s\cancel{b}}T_{\cancel{b}s})$$

where $se(3)$ is the Lie algebra of $SE(3)$ (the set of all possible $\dot{T}$ when $T = I$).

Important concepts, symbols, and equations (cont.)

Screws and twists:

- for a screw axis $\{q, \hat{s}, h\}$ with finite h,
$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$

- "unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$,
  where either (i) $\|\omega\| = 1$ or
  (ii) $\omega = 0$ and $\|v\| = 1$

- $\mathcal{V} = \mathcal{S}\dot{\theta}$

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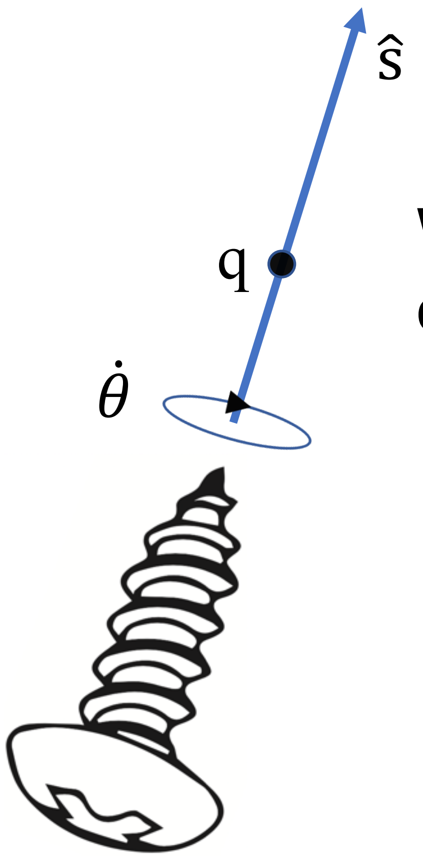
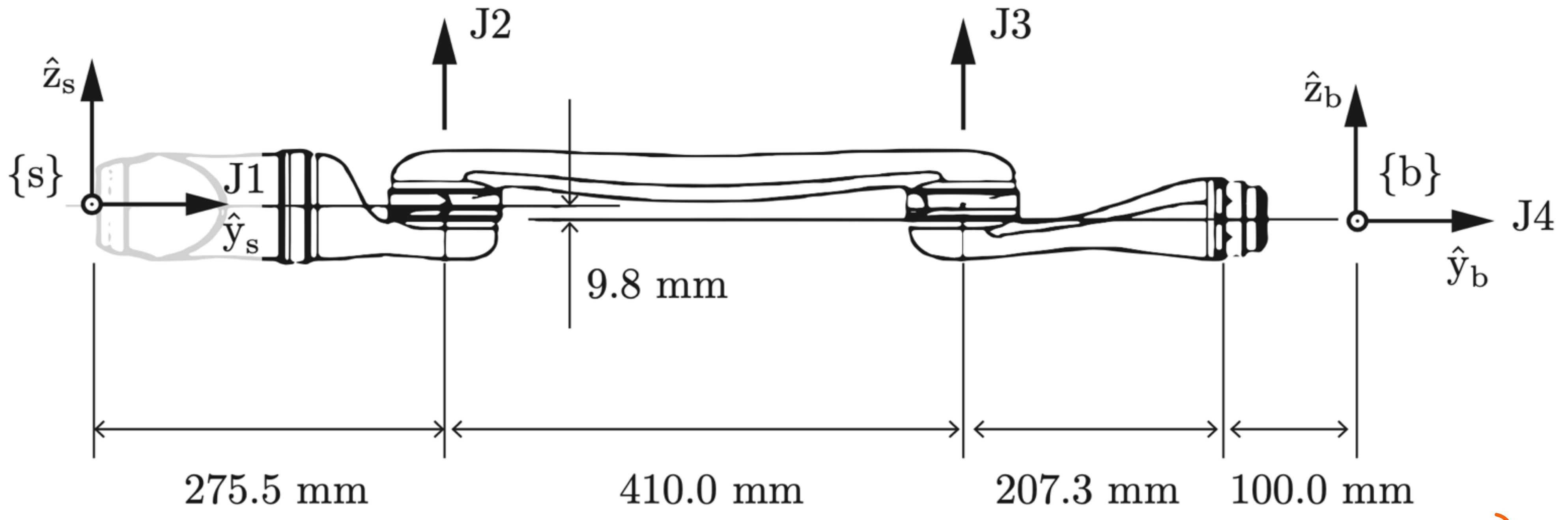


What is the dimension of the space of screws?

5

Kinova lightweight 4-dof arm



For J4, what is the screw axis S_b? S_s?

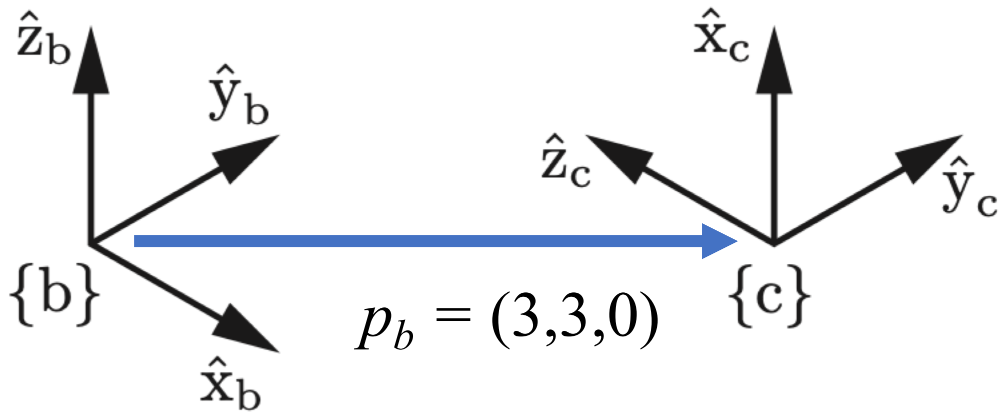
$$\mathcal{S}_b = (\underbrace{0,1,0}_{\omega_b}, \underbrace{0,0,0}_{v_b})$$

$$\mathcal{S}_s = (0,1,0, 9.8\text{ mm/s}, 0, 0)$$

For J2?

$$\mathcal{S}_b = (0,0,1,-717.3,0,0)$$

$$\mathcal{S}_s = (0,0,1,275.5,0,0)$$



Write T_{bc}.

$$T_{bc} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A camera with frame $\{c\}$ tracks the optical marker on a tool to get its frame $\{t\}$ relative to $\{c\}$. This transformation is T_1. A space frame $\{s\}$ is attached to the floor of the room, and the camera observes its configuration relative to $\{c\}$ as T_2. A robot arm has a mounting frame $\{m\}$ which has been measured relative to $\{s\}$ as T_3. The arm's encoders and the robot's kinematics tell us the gripper's frame $\{e\}$ relative to $\{m\}$. This is represented as T_4. The gripper should be at the frame $\{g\}$ relative to $\{t\}$ to be able to close on the tool and pick it up. This configuration is represented as T_5.

Write T_{eg}, the configuration of the grasping frame $\{g\}$ relative to the current end-effector frame $\{e\}$, in terms of T_1, T_2, T_3, T_4, and T_5.

$T_1 = T_{ct}$ $\quad T_3 = T_{sm}$ $\quad T_5 = T_{tg}$

$T_2 = T_{cs}$ $\quad T_4 = T_{me}$

$$T_{eg} = T_{em} T_{ms} T_{sc} T_{ct} T_{tg}$$
$$= T_4^{-1} T_3^{-1} T_2^{-1} T_1 T_5$$