Recommender Systems
lecture 2: neighbourhood-based models

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Recap: taxonomy

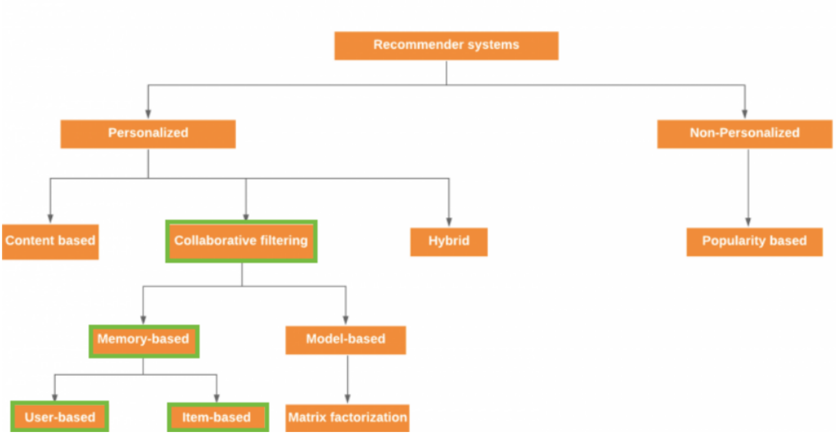
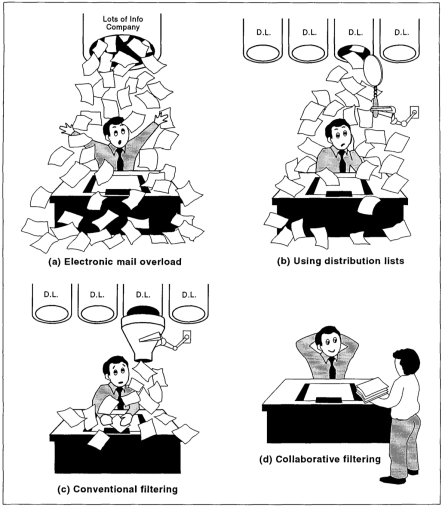


image credit: https://thingsolver.com/introduction-to-recommender-systems



(a) Electronic mail overload

(b) Using distribution lists

(c) Conventional filtering

(d) Collaborative filtering

Idea:

«Collaborative filters help people make choices based on the opinions of other people.»

[Paul Resnick et al. "Grouplens" 1994]

«Collaborative filtering simply means that people collaborate to help one another perform filtering by recording their reactions to documents they read. Such reactions may be that a document was particularly interesting (or particularly uninteresting).

These reactions, more generally called annotations, can be accessed by others' filters.»

[David Goldberg et al. "Tapestry" 1992]

Memory-based vs Model-based

- memory-based (neighbourhood-based, heuristic-based) algorithms make recommendations based on the entire collection of previously rated items by the users.
- model-based algorithms use the collection of ratings to learn a model which is then used to make recommendations.

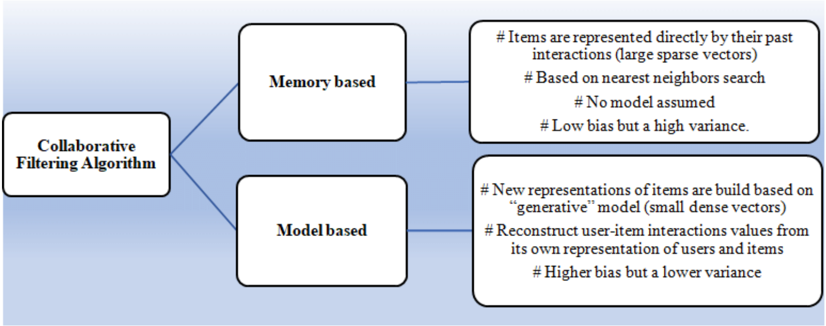
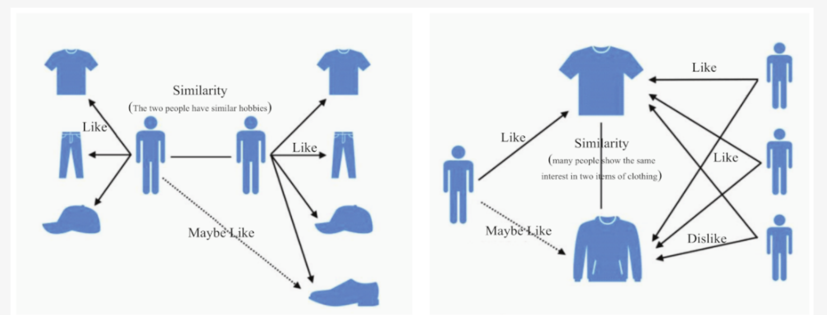


image credit: https://iopscience.iop.org/article/10.1088/1757-899X/1022/1/012057 a **User-based:** compute similarity between pairs of **users** based on thair past interactions with items

b **Item-based:** compute similarity between pairs of **items** based on thair past interactions with users. The similarity between items is typically more stable than the similarity between the users.



(a) (b)

image credit: https://www.mdpi.com/2076-3417/11/20/9554

Recap: problem statement

Given:

- $U = \{u_j|,\ j \in 1, \dots n_{users}\}$ — set of users
- $I = \{i_j|\ ,\ j \in 1, \dots n_{items}\}$ — set of items.
- $R = \|r_{ui}\|$ — relation matrix of shape $n_{users} \times n_{items}$,

$$r_{ui} \in \left[\begin{array}{c} \text{typically } \{0,1\}\text{— implicit feedback} \\ \text{typically } \{1,2,3,4,5\}\text{— explicit feedback} \end{array} \right.$$

Possible tasks:

Rating prediction

- Predict unknown r_{ui} — regression or (multi-class) classification

Top-k ranking

- Rank top-k recommendations for users — **item2user (course focus)**

Neighbourhood-based recommendations

$$r_{ui} = \left[\begin{array}{l} \underset{u' \in U_i}{\text{aggr}}\ r_{u'i} \text{ — user-based} \\ \underset{i' \in I_u}{\text{aggr}}\ r_{ui'} \text{ — item-based} \end{array} \right.$$

Next we will focus on **item-based** (if not specified explicitly). Examples of the aggregation functions are:

$$r_{ui} = \frac{1}{N} \sum_{i' \in I_u} r_{ui'}$$

$$r_{ui} = k \sum_{i' \in I_u} \text{sim}(i, i') \times r_{ui'}$$

$$r_{ui} = \overline{r_i} + k \sum_{i' \in I_u} \text{sim}(i, i') \times (r_{ui'} - \overline{r_i}),$$

k serves as a normalization factor and usually used as $k = \frac{1}{\sum\limits_{i' \in I_u} |\text{sim}(i, i')|}$

Similarity weight computation

Possible variants for $\mathrm{sim}(i,j)$:

$$Cosine(i,j) = \frac{\sum\limits_{u \in U_{ij}} r_{ui} \cdot r_{uj}}{\sqrt{\sum\limits_{u \in U_{ij}} r_{ui}^2}\sqrt{\sum\limits_{u \in U_{ij}} r_{uj}^2}}$$

$$PC(i,j) = \frac{\sum\limits_{u \in U_{ij}} (r_{ui} - \bar{r}_i)(r_{ui} - \bar{r}_j)}{\sqrt{\sum\limits_{u \in U_{ij}} (r_{ui} - \bar{r}_i)^2 \sum\limits_{u \in U_{ij}} (r_{uj} - \bar{r}_j)^2}}$$

Additional weighting (optional)

IDF, BM25, etc.

To reduce the influence of popular items in the similarity measure and to give more weight to less popular items that a more relevant to a particular user, one may use common weighting from NLP. For example reweight cosine similarity with IDF:

$$\lambda_u = \log \frac{|I|}{|I_u|}; \quad Cosine(i,j) = \frac{\sum\limits_{u \in U_{ij}} \lambda_u r_{ui} \cdot r_{uj}}{\sqrt{\sum\limits_{u \in U_{ij}} \lambda_u r_{ui}^2} \sqrt{\sum\limits_{u \in U_{ij}} \lambda_u r_{uj}^2}}$$

Shrink coefficient

When weight is computed using only a few ratings, one may use shrinkage where a weak or biased estimator can be improved if it is “shrunk” toward a null-value (typical value for β is 100):

$$\mathrm{sim}'(i,j) = \frac{|U_{ij}|}{|U_{ij}| + \beta} \mathrm{sim}(i,j)$$

ItemKNN

$$r_{ui} = \frac{\sum\limits_{i' \in J} \mathrm{sim}(i, i') \times r_{ui'}}{\sum\limits_{i' \in J} \mathrm{sim}(i, i')},$$

where J — set of k most similar items to i.

Note

Consider $W \in \mathbb{R}^{|I| \times |I|}$ — item-item weight-matrix,
$R_{u,\cdot}$ — row u; $W_{\cdot,i}$ — column i.

r_{ui} might also be computed using $R_{u,\cdot}$ and $W_{\cdot,i}$

Exercise

The UserKNN algorithm also exists. Derive its formulas.

SLIM: Sparse LInear Methods

r_{ui} is calculated as a sparse aggregation of items, user u interacted with:

$$r_{ui} = R_{u,\cdot} \cdot W_{\cdot,i}$$

Optimization problem

Learning W for SLIM:

$$\begin{aligned} \underset{W}{\text{minimize}} \quad & ||R - RW||_F^2 + \frac{\beta}{2}||W||_F^2 + \lambda||W||_1 \\ \text{subject to} \quad & W \geq 0 \\ & \text{diag}(W) = 0, \end{aligned}$$

where $||W||_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}|w_{ij}|$ is the entry-wise l_1-norm of W,

and $||\cdot||_F$ is the matrix Frobenius norm.

EASE

r_{ui} is calculated similar to SLIM:

$$r_{ui} = R_{u,\cdot} \cdot W_{\cdot,i}$$

Optimization problem

Learning W for EASE:

$$\begin{aligned} \underset{W}{\text{minimize}} \quad & ||R - RW||_F^2 + \lambda ||W||_F^2 \\ \text{subject to} \quad & \text{diag}(W) = 0 \end{aligned}$$

- Square loss allows for a closed-form solution (next slide). Training with other loss functions, however, might result in improved ranking accuracy.
- The constraint of a zero diagonal, $\text{diag}(W) = 0$, is crucial for ranking accuracy (in contrast to $W \geq 0$).

EASE: closed-form solution derivation

Lagrangian:

$$L = ||R - RW||_F^2 + \lambda ||W||_F^2 + 2 \cdot \gamma^T \cdot \text{diag}(W),$$

where $\gamma = (\gamma_1, \dots, \gamma_{|I|})^T$ — Lagrange multipliers.
Setting derivative to zero yields the estimate of the weight matrix:

$$W = \left(R^T R + \lambda E\right)^{-1} \left(R^T R - \text{diagMat}(\gamma)\right)$$

Defining (for sufficiently large λ):

$$P = \left(R^T R + \lambda E\right)^{-1}$$

Substituting into the previous equation:

$$\begin{aligned} W &= \left(R^T R + \lambda E\right)^{-1} \left(R^T R - \text{diagMat}(\gamma)\right) \\ &= P\left(P^{-1} - \lambda E - \text{diagMat}(\gamma)\right) \\ &= E - P\left(\lambda E + \text{diagMat}(\gamma)\right) \\ &= E - P \cdot \text{diagMat}(\tilde{\gamma}), \ \ (\text{where } \tilde{\gamma} = \lambda \vec{1} + \gamma) \end{aligned}$$

EASE: closed-form solution

$$0 = \mathrm{diag}(W) = \vec{1} - \mathrm{diag}(P) \odot \tilde{\gamma}$$

$$\Rightarrow \tilde{\gamma} = \vec{1}/\mathrm{diag}(P),$$

where / — elementwise division.

Substituting into previous slide gives

$$W = E - P \cdot \mathrm{diagMat}(\vec{1}/\mathrm{diag}(P))$$

EASE closed-form solution

$$P = \left(R^T R + \lambda E\right)^{-1}$$

$$W_{ij} = \begin{cases} 0, \text{if } i = j \\ -\frac{P_{ij}}{P_{jj}}, \text{otherwise} \end{cases} .$$

Literature

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3. [Paul Resnick et al. "Grouplens" 1994]
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