运动·力学·刚体

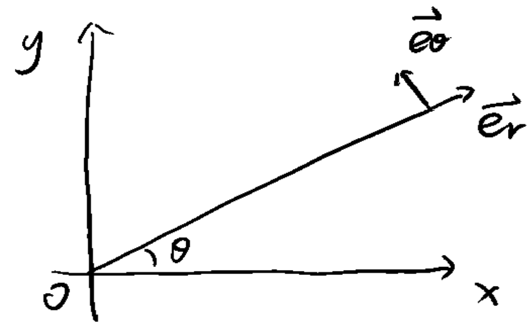
△ 平面极坐标系及其运动分解

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$



曲率半径 $\rho = \dfrac{(1+f'^2)^{\frac{3}{2}}}{|f''|}$

$$\vec{r} = r\vec{e_r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta}$$

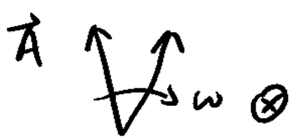
$$\vec{a} = \ddot{r}\vec{e_r} + 2\dot{r}\dot{\theta}\vec{e_\theta} + r\ddot{\theta}\vec{e_\theta} - r\dot{\theta}^2\vec{e_r}$$

$$\vec{e_r} = \cos\theta\,\vec{e_x} + \sin\theta\,\vec{e_y}$$

$$\vec{e_\theta} = -\sin\theta\,\vec{e_x} + \cos\theta\,\vec{e_y}$$

$$\frac{d\vec{e_r}}{dt} = \frac{d\vec{e_r}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\vec{e_\theta}$$

$$\frac{d\vec{e_\theta}}{dt} = -\dot{\theta}\vec{e_r}$$



$|\vec{A}|$不变

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

△ 非惯性系

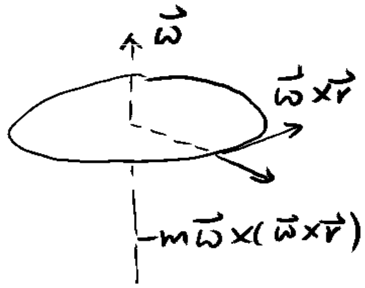
平移惯性力　S系中有 $\vec{F} = m\vec{a}$

S'系中　$\vec{a} = \vec{a}_{0'} + \vec{a}'$

$\vec{F} - m\vec{a}_{0'} = m\vec{a}'$　引入 $\vec{f} = -m\vec{a}_{0'}$ 惯性力

惯性离心力　静止在S'系中的物体在S系中看来

$$\vec{F} = m\vec{a} = m\frac{d\vec{\omega}}{dt} + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

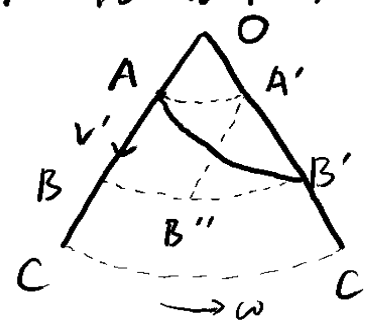


$$= m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$　若$\vec{\omega}$为常矢量

因此在S'系看来必须受 $\vec{f} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

惯性离心力垂直转轴并指向离开转轴的方向

若物体相对于转动参考系作相对运动，则除惯性离心力外还应受到科里奥利力的作用



相对圆盘运动 v'：A→B

圆盘转动 ω · B→B''

B''→B'：$\Delta s = \frac{1}{2}a_{cor}(\Delta t)^2$

$$\vec{f}_{cor} = -m\vec{a}_{cor} = -2m\vec{v}'\times\vec{\omega}$$

刚体.

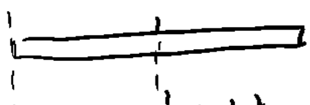
$$I=\sum_i m_i R_i^2$$

v, a	m	$\frac{1}{2}mv^2$	mvl	$F=ma$
ω β	I	$\frac{1}{2}I\omega^2$	$I\omega$	$M=I\beta$

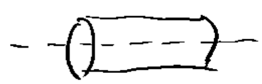
刚体平面平行运动中动能:

$$E=\frac{1}{2}mv_c^2+\frac{1}{2}I\omega^2$$

转动惯量



$\frac{1}{3}mL^2$ $\frac{1}{12}mL^2$



$\frac{1}{2}mR^2$

球壳 $\frac{2}{3}mR^2$

球体 $\frac{2}{5}mR^2$

圆环 mR^2

圆盘(绕中心轴) $\frac{1}{2}mR^2$

合运动 → 质心平动+绕质心转动

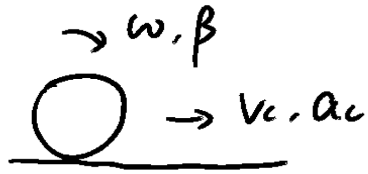
合外力对质心所做的功等于质心平动动能的变化量

相对于质心的合外力矩所做的功等于刚体相对质心转动动能的变化量.

M 瞬心.（两速度方向垂线交点）.

$$E_k = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2 = \frac{1}{2} I_M \omega^2$$

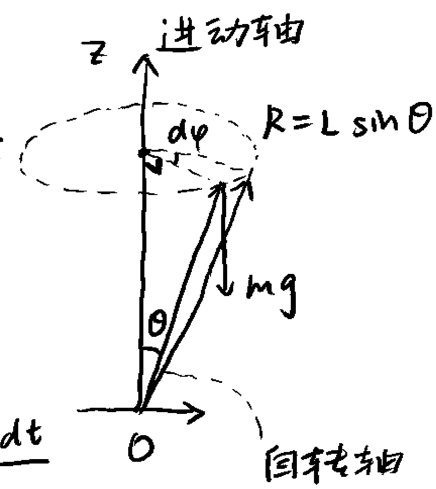
平面纯滚运动



$v_c = \omega R.$

$a_c = \beta R$

刚体进动（自转轴绕另一个轴转动）



对O点 物体只受重力矩作用

$$\vec{M} = \Sigma(\vec{r_i} \times m_i \vec{g}) = (\Sigma m_i \vec{r_i}) \times \vec{g}$$

$$= \vec{r_c} \times mg$$

由 $d\vec{L} = \vec{M} dt$，$dL = L \sin\theta \, d\varphi$

$$d\varphi = \frac{dL}{L\sin\theta} = \frac{M dt}{L \sin\theta} = \frac{r_c mg \sin\theta \, dt}{L \sin\theta}$$

$$= \frac{r_c mg}{L} dt$$

其中L是物体自转角动量 $L = I\omega$

$$\Omega = \frac{d\varphi}{dt} = \frac{r_c mg}{I\omega}$$

ω 自转角速度

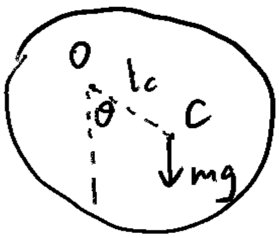
Ω 进动角速度

振动

(力学) $F=-kx=m\ddot{x}$

即 $\ddot{x}+\omega^2 x=0$ $\quad \omega^2=\frac{k}{m}$. $\quad T=2\pi\sqrt{\frac{m}{k}}$

复摆. $\quad M=-mgl_c\sin\theta=I_0\beta$.



$\theta \ll 1 \Rightarrow \ddot{\theta}+\frac{mgl_c}{I_0}\theta=0$

当质量集于C时即为单摆. $I_0=ml_c^2$

$\ddot{\theta}+\frac{g}{l}\theta=0$. $\quad T=2\pi\sqrt{\frac{l}{g}}$

$x=A\cos(\omega t+\varphi_0)$ $\quad E=\frac{1}{2}kA^2$

(能量) $\frac{1}{2}mv^2+\frac{1}{2}kx^2=Const$

$\Rightarrow mv\dot{v}+kx\cdot\dot{x}=0$

$\Rightarrow m\ddot{x}+kx=0$

对复摆来说, $mgl_c(1-\cos\theta)+\frac{1}{2}I_0\omega^2=Const$

$\theta \ll 1 \quad \cos\theta \to 1-\frac{\theta^2}{2}$

$\Rightarrow mgl_c\theta\cdot\dot{\theta}+I_0\omega\cdot\dot{\omega}=0$

$\Rightarrow mgl_c\theta+I_0\ddot{\theta}=0$

牛顿定律

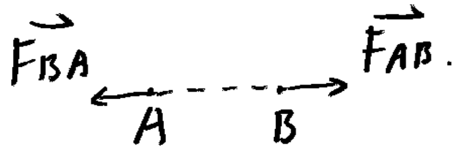
1、惯性力. 惯性系 S $\vec{F} = m\vec{a} = m(\vec{a'}+\vec{a_0})$

非惯性系 S' ($\vec{a_0}$)

$\vec{a'}$为相对 S'系加速度. $\vec{a'}+\vec{a_0}=\vec{a}$

定义 $\vec{F}_{惯} = -m\vec{a_0}$ 则 $\vec{F} + \vec{F}_{惯} = m\vec{a'}$

2. 两体约化质量.



A系中: $\vec{F_B}' = m_B \vec{a_{BA}} = m_B(\vec{a_B}-\vec{a_A})$

$$= \vec{F_{AB}} + m_B \frac{\vec{F_{AB}}}{m_A}$$

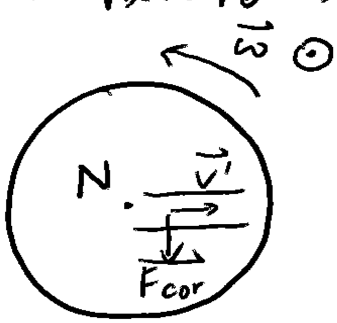
$$= \frac{m_A+m_B}{m_A}\vec{F_{AB}}$$

$\vec{F_{AB}} = \mu_B \vec{a_{BA}}$ $\mu_B = \frac{m_A m_B}{m_A+m_B} = \mu_A = \mu$.

$W_{内} = \frac{1}{2}\mu V_{相对}^2$ ⊕ $\overset{}{\underset{r}{-}}$ ⊕ $\frac{kq^2}{r} = \mu\cdot\frac{v^2}{r}$

v为相对速度

3. 惯心离心力.



$F_{惯}$ $\begin{cases} 离心力 & m\omega^2\vec{r} \\ 科氏力 & 2m\vec{v'}\times\vec{\omega} \quad 不做功 \end{cases}$

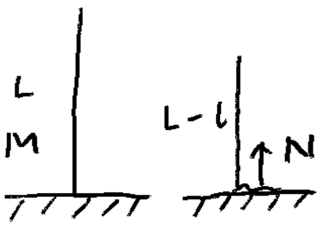
离心力产生势能

$E_p(r) = -\frac{1}{2}m\omega^2 r^2$ (中心为势能零点)

动量.

$$\vec{F}\,dt = d(m\vec{v})$$

过程中 $\vec{F}_{合外}$ 恒为 0，则质点系动量 $\vec{P}$ 守恒.

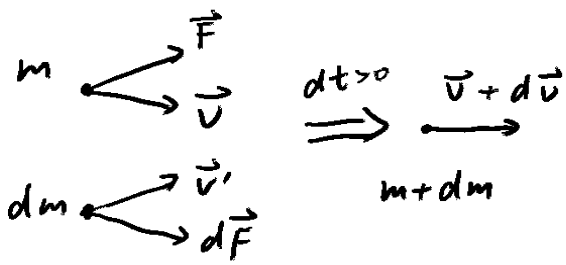


$$\left(N - \frac{l}{L}Mg\right)\Delta t = \frac{v\Delta t}{L} M v \qquad v = \sqrt{2gl}$$

$$\therefore N = 3\frac{l}{L}Mg$$

变质量物体运动 $\frac{dm}{dt} = \alpha(t)$

1. 增质型

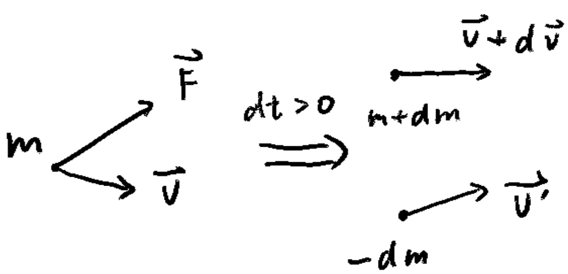


$$(\vec{F}+d\vec{F})\,dt = (m+dm)(\vec{v}+d\vec{v}) - m\vec{v} - dm\,\vec{v}'$$

$$\therefore \vec{F}\,dt = m\,d\vec{v} + dm(\vec{v}-\vec{v}')$$

$$\vec{F} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}(\vec{v}-\vec{v}')$$

2. 减质型



(dm < 0)

$$\vec{F}\,dt = (m+dm)(\vec{v}+d\vec{v}) + (-dm)\vec{v}' - m\vec{v}'$$

$$\therefore \vec{F}\,dt = m\,d\vec{v} + dm(\vec{v}-\vec{v}')$$

$$\vec{F} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}(\vec{v}-\vec{v}')$$

3. 火箭发射

初始质量 m_0，燃料 m_1，喷射速度常量 u_0，$\frac{dm}{dt} = -\alpha < 0$

末速度 V_e （以向上为正）

$$-mg = m\frac{dv}{dt} + u_0\frac{dm}{dt}$$

$$\therefore \frac{dv}{dt} = \frac{u_0\alpha}{m} - g$$

$$\boxed{\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt}} = -\alpha\frac{dv}{dm}$$

$$\Big\} \; dv = \left(\frac{g}{\alpha} - \frac{u_0}{m}\right)dm$$

$$V_e = \int_0^{V_e} dv = \int_{m_0}^{m_0-m_1}\left(\frac{g}{\alpha} - \frac{u_0}{m}\right)dm = u_0\ln\frac{m_0}{m_0-m_1} - \frac{m_1 g}{\alpha}$$

能量

势能. $E_p(\vec{r}) = \int_{\vec{r}}^{\vec{r_0}} \vec{F} \cdot d\vec{l}$ / $-F(x)dx = dE_p$

1. 弹性势能 (x=0 零点) $E_p(x) = \int_x^0 -kx\,dx = \frac{1}{2}kx^2$.

$$F(x) = \frac{-dE_p}{dx} = -kx$$

2. 万有引力势能 ($r=\infty$ 零点) $E_p(r) = \int_r^\infty -\frac{GMm}{r^2}dr = -\frac{GMm}{r}$

$$F(x) = \frac{-dE_p}{dx} = -\frac{GMm}{r^2}$$

3 离心势能 (中心为零点) $E_p(r) = -\frac{1}{2}m\omega^2 r^2$

4. 若受到平方反比吸引力 $F(r) = -\frac{\alpha}{r^2}$

则 $E_p(r) = -\frac{\alpha}{r}$

△ 质心 $\boxed{\text{不受外力，则质心不动}}$

质点系动量定理 $\vec{F} = \sum \vec{F_i} = \frac{d}{dt}(\sum m_i \vec{v_i})$

其中 $\vec{v_i} = \frac{d\vec{r_i}}{dt}$ $\quad \vec{r_c} = \frac{\sum m_i \vec{r_i}}{\sum m_i}$

$$\vec{F} = \frac{d^2}{dt^2}(\sum m_i \vec{r_i}) = \sum m_i \frac{d^2}{dt^2}\left(\frac{\sum m_i \vec{r_i}}{\sum m_i}\right) = m \frac{d^2 \vec{r_c}}{dt^2}$$

即质点系中存在点C满足 $\vec{F} = m\vec{a_c}$ 称为质心

$$\vec{P} = \sum m_i \vec{v_i} = \sum m_i \dot{\vec{r_i}} = m \frac{d}{dt} \frac{\sum m_i \vec{r_i}}{m} = m\dot{\vec{r_c}} = m\vec{v_c}$$

柯尼希定理

质心系有 $\vec{v_c}$ $\quad \vec{v_i} = \vec{v_i'} + \vec{v_c}$ （$\vec{v_i'}$ 为相对质心系速度）

地面系：$E_k = \frac{1}{2}m_1 v_1^2 + \cdots + \frac{1}{2}m_n v_n^2$

质心系：$E_{kc} = \frac{1}{2}m_1 v_1'^2 + \cdots + \frac{1}{2}m_n v_n'^2$

$$E_k = \sum \frac{1}{2}m_i (v_i'^2 + v_c^2 + 2\vec{v_i'} \cdot \vec{v_c})$$

$$= \sum \frac{1}{2}m_i v_c^2 + \sum \frac{1}{2}m_i v_i'^2 + \sum m_i \vec{v_i'} \cdot \vec{v_c}$$

$$= E_{kc} + \frac{1}{2}m v_c^2$$

体系相对质心动能 + 质心动能

$$\left(\vec{v_c} = \frac{\sum m_i \vec{v_i}}{\sum m_i} = \frac{\sum m_i (\vec{v_i'} + \vec{v_c})}{\sum m_i} \Rightarrow \sum m_i \vec{v_i'} = 0\right)$$

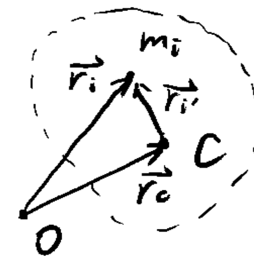
对于两体系统 $\vec{v_c} = \dfrac{m_1\vec{v_1}+m_2\vec{v_2}}{m_1+m_2}$ $\quad \vec{v} = \vec{v_1}-\vec{v_2}$

$$\vec{v_1}' = \vec{v_1}-\vec{v_c} \qquad \vec{v_2}' = \vec{v_2}-\vec{v_c}$$

$$E_{kc} = \frac{1}{2}m_1\vec{v_1}'^2+\frac{1}{2}m_2\vec{v_2}'^2 = \frac{1}{2}\mu\vec{v}^2$$

质心系是零动量系

$$\sum m_i\vec{r_i}' = \sum m_i\vec{r_i}-\sum m_i\vec{r_c} = 0$$



碰撞

$$E_k = \frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$$

$$= \frac{1}{2}(m_1+m_2)v_c^2+\underline{\frac{1}{2}\mu v_{相对}^2}$$

$\mu = \dfrac{m_1m_2}{m_1+m_2}$. 可被损耗，资用能

非弹性碰撞 $v_2-v_1 = e(v_{10}-v_{20})$ $\quad 0\le e<1$

$$E_{k损} = \frac{1}{2}\mu(e^2-1)v_{相对0}^2$$

* 碰撞前后质心动能不变，则只需考虑体系相对质心动能

碰前：$\frac{1}{2}\mu(u_1-u_2)^2$

碰后：$\frac{1}{2}\mu e^2(u_1-u_2)^2$

损失 $\Delta E_k = E_{kc}'-E_{kc} = \frac{1}{2}\mu(e^2-1)(u_1-u_2)^2$

两体问题

$$\vec{f} = \mu \vec{a} \quad （\vec{a}为相对加速度）$$

动能 $E_k = \frac{1}{2}\mu v^2$ （$\vec{v}$为相对速度）

$=$ 两体系统在质心系中的系统总动能

步骤：选定动质点，静质点，将动质点质量视为μ

eg. M, m 开三修正 令m动，M静

$$-G\frac{Mm}{r^2}\frac{\vec{r}}{r} = \mu \ddot{\vec{r}}$$

$$-G\frac{M+m}{r^2}\frac{\vec{r}}{r} = \ddot{\vec{r}}$$

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2}$$

角动量

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{M}$$ 外力矩为0：角动量守恒

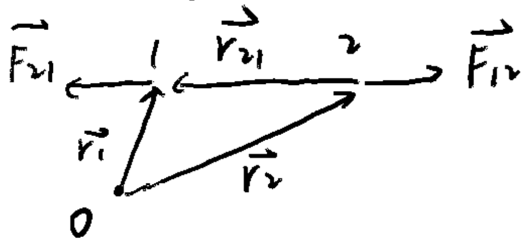
$$\boxed{\int_{t_1}^{t_2} \vec{M}\, dt = \vec{L}_2 - \vec{L}_1}$$

开普勒第二定律 $K = \frac{dS}{dt} = const$

面积速率 $\boxed{K = \frac{1}{2} \vec{r} \times \vec{v}} = \frac{L}{2m}$ 以焦点为参考点，角动量守恒.

$$T = \frac{S}{K} = \frac{\pi ab}{K}$$

一对作用力、反作用力 力矩和为0



$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12}$$
$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} = \vec{r}_{21} \times \vec{F}_{21} = 0$$

质点系角动量. $\frac{d\vec{L}}{dt} = \vec{M}_{内} + \vec{M}_{外}$

角动量的柯尼希定理

地面系. $\vec{L} = \sum \vec{r_i} \times m_i \vec{v_i}$ $\qquad \vec{r_i} = \vec{r_c} + \vec{r_i}'$

$\qquad \vec{v_i} = \vec{v_c} + \vec{v_i}'$

$$= \sum (\vec{r_c} + \vec{r_i}') \times m_i (\vec{v_c} + \vec{v_i}')$$

$$= \vec{r_c} \times (\sum m_i) \vec{v_c} + \sum \vec{r_i}' \times m_i \vec{v_i}'$$

$$+ \vec{r_c} \times \sum m_i \vec{v_i}' + (\sum m_i \vec{r_i}') \times \vec{v_c}$$

$$= \vec{r_c} \times m \vec{v_c} + \sum \vec{r_i}' \times m_i \vec{v_i}'$$

质心角动量 + 体系相对质心角动量

部分天体.

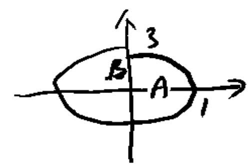
$\boxed{\text{TIPS}}$ 列能量守恒时: 动能 + 势能 = 总能量

轨道能量 $E = -\frac{GMm}{2a}$ (机械能守恒)

$E < 0$ 轨道闭合 (eg. 可列式 $-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{R}$)

$E = 0$ 抛物线

可飞离中心天体 < $E > 0$ 双曲线



$\rho_1 = \frac{B^2}{A}$

$\rho_3 = \frac{A^2}{B}$.

m, m_e, M_s, R_e, r

第一宇宙速度. $\frac{GM_em}{R_e^2} = \frac{mv^2}{R_e}$ $\quad v = \sqrt{\frac{GM_e}{R_e}}$

第二宇宙速度 $\frac{1}{2}mv^2 - \frac{GM_em}{R_e} = 0$ $\quad v = \sqrt{\frac{2GM_e}{R_e}}$

$v = c$ 时所得 R 称为天体引力半径

第三宇宙速度. (v)

① 逃离太阳引力 $V_{2s} = \sqrt{\frac{2GM_s}{r}}$

② 地球公转 $V_{1s} = \sqrt{\frac{GM_s}{r}}$

③ $\frac{1}{2}mv^2 - \frac{GM_em}{R_e} = \frac{1}{2}m(V_{2s} - V_{1s})^2$

开普勒第三定律 $\frac{a^3}{T^2} = \frac{GM}{4\pi^2} \xrightarrow[\text{修正}]{\text{中心天体动}} \frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2}$

准粒子 动能K与动量p满足 $K=\frac{p^2}{2m}+\alpha p$

$$dK = F\,dr = \frac{dp}{dt}\cdot dr = v\cdot dp$$

$$\frac{p}{m}dp + \alpha dp = dK$$

$$\Rightarrow v=\frac{p}{m}+\alpha$$