Qs.3 Analytic solution:-

Given PDE → (1D wave equation) -

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad u(x,t) = X(x)T(t) \quad \text{(separation of variables)}$$

$$\frac{X(x)\frac{dT}{dt} + cT(t)\frac{dX}{dx}}{XT} = \frac{0}{XT}$$

$$\Rightarrow \frac{1}{T(t)}\frac{dT}{dt} + \frac{c}{X(x)}\frac{dX}{dx} = 0$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = -\frac{c}{X(x)}\frac{dX}{dx} = k \text{ (constant)}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = k, \quad \frac{c}{X}\frac{dX}{dx} = k \quad —\textcircled{1}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = k \Rightarrow \frac{dT}{dt} = kT(t) \Rightarrow T(t) = T(0)\exp(kt)$$

$$\Rightarrow \frac{c}{X}\frac{dX}{dx} = -k \Rightarrow \frac{dX}{dx} = -\frac{k}{c}X(x)$$

$$\Rightarrow X(x) = X(0)\exp\left(-\frac{k}{c}x\right)$$

$$\Rightarrow \boxed{u(x,t) = X(x)T(t) = \underbrace{X(0)T(0)}_{= u(0)}\exp\left(kt - \frac{k}{c}x\right)}$$

$$\Rightarrow u(x,t) = X(x)T(t) = u(0)\exp\left(kt - \frac{k}{c}x\right)$$

$$\Rightarrow X(0)T(0) = u(0)\exp\left[-\frac{k}{c}(x-ct)\right]$$

assume $\left(-\frac{k}{c} = 1\right)$

analytic solution:-

$$\boxed{u(x,t) = u_0 \exp(x-ct)} \Big\} \to \text{decaying wave}$$