Motivation of baye's theorem

Consider this: - We can easily calculate the probability of having the word "dear" given I know it is spam - baye's theorem allows

Baye's theorem

For any events E and F where $P(E) > 0$ and $P(F) > 0$

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

Why does this work?

$P(E|F)P(F) = P(F \cap F)$

and by definition we know that $P(E|F) = \frac{P(E \cap F)}{P(F)}$

Further expansion can happen

$$P(E) = (P(E|F)P(F) + P(E|F^c)P(F^c))$$

Monty Hall problem statement

- there are three doors
- one door leads to a car, the other two leads to goats.
- you can pick a door *without* opening it
- then host opens a door
- if the host always opens a goat door, is it wise to change the door.

The decision of changing choice feels like a 0.5 chance and that it hardly matters.

Exhaustive counting solution

2 Solution to Game Show: Choice Tree, Conditional Probability

Let us look into all possible (exhaustive) cases:

Door You Choose	Prize in Door	Host Opens	Stay	Switch
1	1	2/3	win	loose
1	2	3	loose	win
1	3	2	loose	win
2	1	3	loose	win
2	2	1/3	win	loose
2	3	1	loose	win
3	1	2	loose	win
3	2	1	loose	win
3	3	1/2	win	loose

Table: Exhaustive list of possibilities

Conclusion

If you switch, the probability that you win a car is 2/3,

Bayes theorem solution

- Let H be the hypothesis "door 1 has a car behind it" and E is the evidence that Monty has revealed a door with a goat behind it.
- Then problem can be restated as finding the value of $P(H|E)$ because that will determine the decision of switching.

$$P(H) = \frac{1}{3}$$

$$P(H^c) = 1\frac{1}{3} = \frac{2}{3}$$

$$P(E) = 1$$

Also,

$$P(E|H) = 1$$

(Note: $P(E)$ and $P(E|H) = 1$ are 1 because we know that the host will always open a door with the goat, no matter what.)

By Bayes theorem we can say that

$$\begin{aligned} P(H|E) &= \frac{P(E|H)P(H)}{P(E)} \\ &= \frac{1 \cdot P(H)}{1} \\ &= P(H) \\ &= \frac{1}{3} \end{aligned}$$

This basically tells us that no matter the evidence, the probability I had chosen the right door the first time, doesn't change. Meaning that the chance that I picked the right door is $\frac{1}{3}$. Thus it is more likely that my door is wrong. Since one door has been removed from commission, we can say that the left door has a $\frac{2}{3}$ probability of having a car.

Note: you don't have to pick door one the first time round, you can pick any without the loss of generality for the proof.