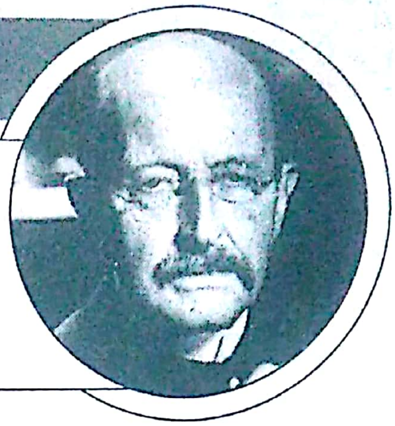
Chapter-10

Origin of Quantum Theory & Particle Properties of Waves



10.1. Introduction

Classical mechanics (*i.e.* classical theory of physics) is based on some basic laws. These are

① Newton's laws of motion
② Gravitational law of attraction between two bodies
③ Coulomb's law of attraction or repulsion between two electrically charged bodies.
④ Lorentz force equation related to the force on a moving charge particle passing through a magnetic field.

It has been observed that classical mechanics can explain successfully the motion of the celestial bodies or other earthly observable objects. **But when the classical concepts were applied to the particles of atomic dimensions like electrons, protons, neutrons, nucleus etc., they failed to explain their motion.**

The failure of classical mechanics to explain the energy distribution of black body radiation, put forward Max Plank (in 1900) to propose **the quantum hypothesis.** This was the origin of quantum theory.

10.2. Inadequacy of Classical Mechanics

- **Stability of the atom** According to Rutherford, an atom consists of a positively charged heavy nucleus surrounded by negatively charged electrons. These negative charged electrons are revolving around the nucleus in circular orbits and they feel strong attractive force by the positively charged heavy nucleus. As a result, they should come closer to the nucleus. Secondly, the energy of these negatively charged moving electrons should decrease continuously because an accelerated charge particle always radiates energy in the form of electromagnetic waves. Due to this continuous loss of energy, an orbital electron should come closer and closer until it collapses with nucleus. This shows the instability of an atom. But, it contradicts the observed fact of the stability of atom. So, **the classical mechanics failed to explain the stability of the atom** [Fig. 1].

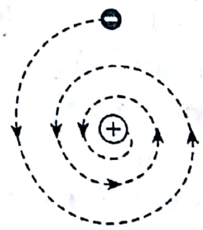


Fig. 1 ▷ Orbital electron comes closer & closer until it collapses with nucleus. This contradicts the stability of the atom

Spectrum of the hydrogen atom According to classical theory, an excited hydrogen atom radiates energy of all wavelengths continuously. But experimentally, it has been observed that the excited atom of hydrogen emits electromagnetic radiation of certain definite wavelengths due to transition of an electron from higher energy state to lower energy state corresponding to Lyman $(n_1 = 1)$, Balmer $(n_1 = 2)$, Paschen $(n_1 = 3)$, Bracket $(n_1 = 4)$ and Pfund $(n_1 = 5)$ series. These discrete set of lines are represented by

$$\bar{\nu}\left(=\frac{1}{\lambda}\right) = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \qquad \cdots(10.1)$$

where, $\bar{\nu}$ = wave number, R = Rydberg constant, n_1 and n_2 are two integers such that $n_2 > n_1$.

So, it can be concluded that **classical theory does not hold good in the region of atomic dimensions.**

10.2.1. Particle Nature of Radiation

Radiation of energy is usually characterized by its wave like nature. But many experimental observation such as black body radiation, photoelectric effect, Compton effect, pair production etc can not be explained properly only by considering the wave nature of radiation. These phenomena along with any natural phenomena (from elementary particle level to cosmological level) can be properly explained on the basis of particle aspect of radiation (i.e., quantum theory of radiation). In that case the waves of electromagnetic radiation behave like particles of definite momentum and energy.

10.3. Black Body Radiation

10.3.1. Radiation

Any heated body emits electromagnetic radiation. If the temperature of the body is increased, the wavelength of the emitted radiation decreases.

Similarly, a heated body can also absorb radiation incident on it.

Emissive power (e_λ) ❶ The emissive power of a body at a certain temperature (T) for a certain wavelength (λ) is defined as the energy radiated normally in vacuum from its surface per unit area per unit second and per unit range of wavelength.

Absorptive power (a_λ) The absorptive power of a body is defined as **the ratio of the amount of heat (Q') absorbed by the body to the total amount of heat** (Q) **falling on it.** Therefore,

$$a_\lambda = \frac{Q'}{Q} \qquad \cdots(10.2)$$

Kirchhoff's law of heat radiation It states that the **ratio of emissive power** (e_λ) **of a body to its absorptive power (a_λ) for radiation of a given wavelength** at a **particular temperature is always constant.**

❶ The emissive power may be defined as the rate of emission of radiant energy from unit area through unit solid angle of a blackbody.

Thus, $\dfrac{e_\lambda}{a_\lambda} = \text{constant} = (k)$ ...(10.3)

where k is an arbitrary constant depending on temperature

In case of black body For black body radiations, absorptive power $a_\lambda = 1$.

So, emissive power, e_λ = constant.

10.3.2. Black Body

The substance or a body which can absorb all the (heat) radiations of all wavelengths incident on it and does not transmit or reflect at all is known as a perfectly black body. Since, *it neither reflects nor transmits any of the incident radiations, it appears black whatever may be the colour of incident radiations.*

We know, a good absorber of radiation is also a good emitter of radiation. So, a perfectly black body (in the technical sense) emits radiations of all possible wavelengths when it is heated.

Thus, a **black body is a substance (or body) which can absorb radiations of all wavelengths incident on it and emits radiations of all wavelengths when it is heated.**

A perfectly black body is an ideal concept. Lamp black, platinum black etc. are considered as almost perfect blackbodies for practical purposes because these can absorb more than 95% of the incident radiations.

10.3.3. Construction of Black Body and Black Body Radiation

For experimental purposes, a black body is made up by a hollow spherical (copper) enclosure with a small hole (called orifice) in its surface. The inner surface of it is coated with lamp black. When any radiation enters the spherical enclosure through this hole, it suffers multiple reflections at the inner surface [Fig. 2(a)] until it is totally absorbed. In this way the body acts as a black body absorber.[2]

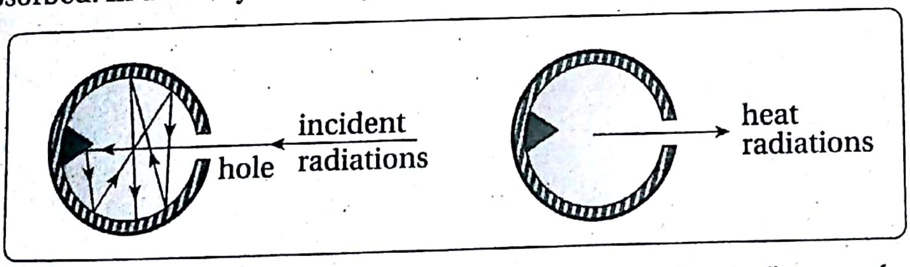


Fig. 2(a) ▷ Schematic diagram of black body absorber

Fig. 2(b) ▷ Schematic diagram of black body emitter

When a black body (*i.e.* hollow enclosure) is heated to a uniform temperature, the radiation of all possible wavelengths coming out through its fine hole. It is called black body emitter. [Fig. 2(b)]

10.3.4. Characteristics of Black Body Radiation

The important characteristics of a black body are—

(1) The energy density (*i.e.* energy radiated per unit area per second) does not depend upon the nature or shape of the walls of enclosure. It depends only upon the temperature of the enclosure.

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[2] The conical projection prevents direct reflection towards the hole from the cavity walls.

② The radiations are isotropic in nature.

③ The radiations are homogeneous in nature.

④ All bodies placed inside the enclosure also emit black body radiations.

10.3.5. Comparison between Thermal Radiation and Light Rays

① Both are electromagnetic waves.

② Both are transverse waves.

③ Radiant heat like light rays travel in straight line.

④ Both of them move with same speed $(3 \times 10^8 \mathrm{m \cdot s^{-1}})$.

⑤ Both of them obey the laws of reflection and refraction.

⑥ Both can travel in vacuum.

⑦ The only difference is that the wavelength of visible light [is smaller than that of thermal radiation].

10.4. Energy Distribution of the Spectrum of a Black Body

From the experimental results as obtained by Lummer and Pringsheim (1899), a graph is drawn between **emissive power** and **wavelength** (λ) **of a black body radiation** for the different tempratures [Fig. 4].

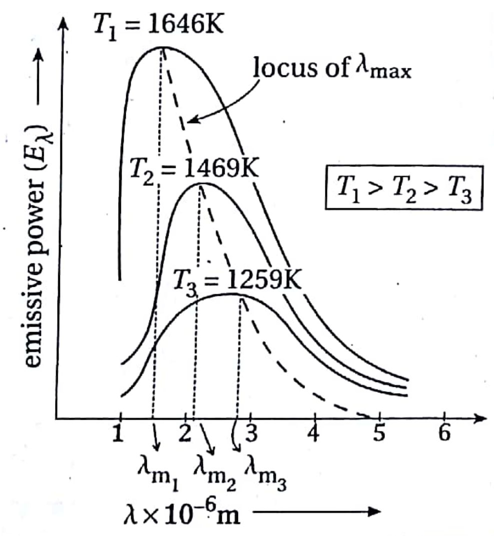


Fig. 3 ▷ The variation of emissive power (*i.e.* intensity of radiation) with the wavelength at different temperatures.

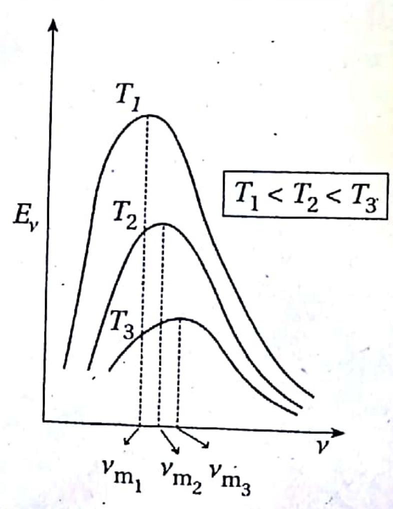


Fig. 4 ▷ The variation of emissive power (*i.e.* intensity of radiation) with the frequency at different temperatures.

From these curves [Fig. 4], the following important conclusions can be drawn :

① For a particular temperature, the emissive power e_λ of a perfectly black body increases with an increase in wavelength and becomes maximum at a particular wavelength λ_m. With further increase in wavelength, the emissive power (*i.e.* intensity of radiation) decreases.

(2) The wavelength (λ_m) of radiation for which the intensity of radiation is maximum, **shifts towards shorter wavelength region** (*i.e.* λ_m decreases) with a increase of temperature of the body. The relation between wavelength λ_m and the absolute temperature T is given by, $\lambda_m T = \text{constant}$.

10.5. Theoretical Explanations of the Spectra Obtained from Black Body Radiation

On the basis of classical physics, number of attempts were made to explain the observed spectral (energy) distribution [Fig. 3] of a black body as a function of wavelength. Though, these attempts were not very successful, we shall discuss only three such well known classical laws in this context.

10.5.1. Wien's Radiation Formula

In order to explain the observed spectral distribution, Wien first showed that the energy density of radiation of **wavelength λ and $\lambda + d\lambda$ from a cavity** (*i.e.* black body) of temperature T is

$$\boxed{E_\lambda d\lambda = \frac{A}{\lambda^5} e^{-B/\lambda T} d\lambda} \quad \text{where } A \text{ and } B \text{ are two constants} \quad \cdots(10.4)$$

Limitation of the formula Wien's radiation formula explains only the experimental results fairly well (*i.e.* makes good fitting of experimental curve) for low wavelength region [Fig. 5].

> ***Special Note :***
> But for high wavelength region, the values of E_λ is lower than that of experimental results.

10.5.2. Wien's Displacement Law

It states that **the maximum energy density (or intensity) of radiation emitted from a black body is displaced towards the shorter wavelength for the rise of temperature (T) of the black body.**

Therefore, the wavelength $\lambda_{\max}$ at the maximum energy distribution of a black body radiation changes in the inverse ratio of its temperature (TK).

$$\text{Thus,} \quad \lambda_{\max} \propto \frac{1}{T} \quad \text{or,} \quad \lambda_{\max} T = \text{constant} \quad \cdots(10.5)$$

10.5.3. Rayleigh-Jeans Law

Rayleigh and Jeans applied the classical law of equipartition of energy to this electromagnetic black body radiation. They found, the energy density of radiation of wavelength range λ and $\lambda + d\lambda$ from a black body of temperature T is

$$\boxed{E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda} \quad \cdots(10.6)$$

So, it states that **the energy density** (E_λ) **of a black body radiation** (of **wavelength** λ) **is inversely proportional to the fourth power of wavelength** (λ)

$$\text{i.e. } \boxed{E_\lambda \propto \frac{1}{\lambda^4}} \qquad \cdots(10.7)$$

Thus, the energy density (E_λ) is increasing with the decrease of wavelength.

Assumption and derivation Lord Rayleigh and James Jeans considered few assumptions to overcome the failure of Wien's distribution law to explain the experimental result for longer wavelength region. The assumptions are—

1. The thermal radiation in the cavity is full of standing waves of electromagnetic radiations of all wavelengths. The walls of the cavity are perfectly reflecting.
2. The number of oscillators (modes of standing waves) per unit volume within the frequency range ν and $\nu + d\nu$ is given by $n_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu$, where c is the velocity of light in free space.
3. The average energy per degree of freedom of an entity (harmonic oscillator, gas molecule) can be written from Maxwell's law of equipartition of energy as $\bar{\epsilon} = \frac{1}{2}kT$, where k is the Boltzmann constant $= 1.38 \times 10^{-23}$ $(\mathrm{J \cdot K^{-1}})$ and T is the absolute temperature of the system which contains the entity.

   We know that the electromagnetic standing wave in the cavity of black body radiation is a linear harmonic oscillator.

   So, it has two degrees of freedom corresponding to its kinetic energy and potential energy. Thus, the average energy of each oscillator in thermal equilibrium at an absolute temperature T is given by $\bar{\epsilon} = 2 \times \frac{1}{2}kT = kT$.

   Thus, the radiant energy density (i.e. the amount of radiant energy per unit volume) within the frequency range ν and $\nu + d\nu$ is given by

$$E_\nu d\nu = \bar{\epsilon}(n_\nu d\nu) = kT\left(\frac{8\pi\nu^2}{c^3} d\nu\right) = \frac{8\pi\nu^2}{c^3} kT d\nu$$

Now, $\nu = \frac{c}{\lambda}$ i.e. $d\nu = \left|-\frac{c}{\lambda^2} d\lambda\right|$: so the above equation can be written in terms of wavelength as $E_\lambda d_\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$.

This is Rayleigh-Jeans law.

Limitation of the law It was found that the above equation explains only the experimental result for high wavelength region [Fig. 5]. But for **low wavelength** region, Rayleigh-Jeans formula fails completely to explain the experimental result and thereby leads to what is known as **ultraviolet catastrophe**. Thus, this law also fails to explain black body radiation.

Ultraviolet catastrophe According to Rayleigh-Jeans law, the energy density of the black body radiation within the wavelength range λ and $\lambda + d\lambda$ is

$$E_\lambda\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda$$

From this equation, we can see, as the wavelength of the radiation spectrum decreases, the energy density increases. Thus, $\lambda \longrightarrow 0$ gives $E_\lambda \longrightarrow \infty$ [Fig. 6]. But the experimental result shows that when the wavelength of the spectrum decreases, the energy density decreases. In this case $\lambda \longrightarrow 0$ implies $E_\lambda \longrightarrow 0$.

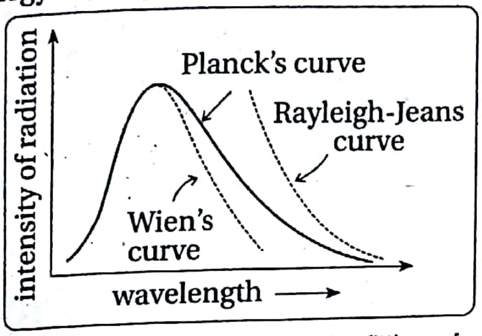


Fig. 5 ▷ Graph indicating the fitting of the experimental curve with Wien's law, Rayleigh-Jeans law and Planck's law.

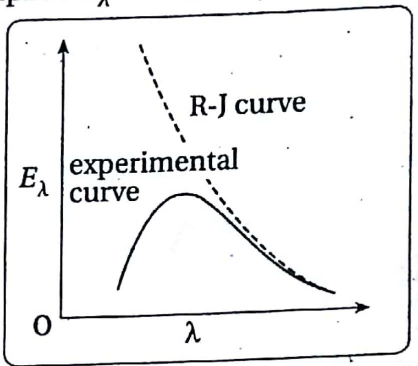


Fig. 6 ▷ The variation of experimental curve with Rayleigh-Jeans law

The discrepancy between theoretical conclusion of Rayleigh-Jeans law (energy density, $E_\lambda \longrightarrow \infty$) and experimental results (energy density $E_\lambda \longrightarrow 0$) of black body radiation for low wavelength (*i.e.* $\lambda \longrightarrow 0$) region is known as ultraviolet catastrophe.

This indicates the limitations of classical mechanics on the basis of which the law of equipartition of energy is derived. This discrepancy has come due to the assumption that energy can be absorbed or emitted by the oscillators[3] continuously in any amount.

10.5.4. Stefan-Boltzmann's Law

In 1879 Josef Stefan experimentally and in 1884 Ludwig Boltzmann theoretically (from thermodynamics) deduced a law relating the total power radiated per unit area of a black body to its absolute temperature. This is known as Stefan-Boltzmann's law.

This law states that **the total energy (E') emitted per unit area per second from a perfectly black body is proportional to the fourth power of its absolute temperature (T).**

Mathematically, it is written as

$$E' \propto T^4 \quad \text{or,} \quad \boxed{E' = \sigma T^4} \qquad \cdots(10.8)$$

where σ = Stefan's constant = $5.672 \times 10^{-5} \text{erg} \cdot \text{cm}^2 \cdot \text{s}^{-1} \cdot \text{K}^{-4}$.

If a perfectly black body at temperature TK is completely surrounded by an enclosure at T_0K ($T > T_0$), the amount of radiant energy per unit area per second is given by

$$\boxed{E' = \sigma(T^4 - T_0^4)} \qquad \cdots(10.9)$$

[3] For oscillator model of radiation read Article 10.6.

10.6. Planck's Hypothesis and Radiation Law

In order to explain the distribution of energy in the spectrum of a black body, Max-Planck in 1900, established the quantum theory of radiation. He derived the radiation law on the basis of following assumptions—

① All bodies placed inside an enclosure can also emit black body radiation.

② The atoms in the walls of a black body radiator behave like simple harmonic oscillators and each has a characteristic frequency of oscillation (*i.e.* vibrate with all possible frequencies).

③ The oscillator of the black body can not have any arbitrary amount of energy but has a discrete energy E_n given by $E_n = nh\nu$, where n is an integer $(n = 0, 1, 2, \cdots)$ and ν is the frequency of oscillation.
This relation shows that the total energy of an oscillator is quantised.

④ The oscillator can radiate or absorb energy in quantas (packets) of $h\nu$ (*i.e.* in discrete set of values $0, h\nu, 2h\nu, \cdots nh\nu$) only when the oscillator jumps from one energy state to another.

Derivation of the law Let N be the total number of Planck's oscillators and E be their total energy. So, the energy per oscillator $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{E}{N} \qquad \cdots(10.10)$$

If $N_0, N_1, N_2, N_3 \cdots N_r \cdots$ etc. are the number of oscillators with energies 0, $\varepsilon, 2\varepsilon, 3\varepsilon \cdots r\varepsilon \cdots$ etc. respectively, we can write,

$$N = N_0 + N_1 + N_2 + N_3 + \cdots + N_r + \cdots \qquad \cdots(10.11)$$

and

$$E = 0 \cdot N_0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + \cdots + r\varepsilon N_r + \cdots$$

$$= 0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + \cdots + r\varepsilon N_r + \cdots \quad \text{where } \varepsilon = h\nu \qquad \cdots(10.12)$$

From Maxwell's distribution formula, the number of oscillators with energy $r\varepsilon$ is

$$N_r = N_0 e^{-r\varepsilon/kT} \text{ where, } k \text{ is Boltzmann's constant} \qquad \cdots(10.13)$$

$$\therefore \quad N_1 = N_0 e^{-\varepsilon/kT},\ N_2 = N_0 e^{-2\varepsilon/kT},\ N_3 = e^{-3\varepsilon/kT} \text{ and so on.}$$

Substituting the values of $N_1, N_2, N_3 \cdots$ in equation number (10.11), we get the total number of Planck's oscillators,

$$N = N_0 + N_0 e^{-(\varepsilon/kT)} + N_0 e^{-(2\varepsilon/kT)} + N_0 e^{-(3\varepsilon/kT)} + N_0 e^{(-4\varepsilon/kT)} + \cdots + N_0 e^{-r\varepsilon/kT} + \cdots$$

$$= N_0 [1 + x + x^2 + \cdots] \quad \text{where } x = e^{-\varepsilon/kT}$$

$$= N_0 \frac{1}{1-x} \quad \left[\because \ 1 + x + x^2 + \cdots = \frac{1}{1-x}\right]$$

$$= \frac{N_0}{1 - e^{-\varepsilon/kT}} \qquad \cdots(10.14)$$

Similarly, the total energy E of the oscillators can be written from equation (10.12).

$$E = 0 + \varepsilon N_0 e^{-\varepsilon/kT} + 2\varepsilon N_0 e^{-2\varepsilon/kT} + 3\varepsilon N_0 e^{-3\varepsilon/kT} + \cdots$$

$$= \varepsilon N_0 e^{-\varepsilon/kT}(1 + 2e^{-\varepsilon/kT} + 3e^{-2\varepsilon/kT} + \cdots)$$

$$= \varepsilon N_0 e^{-\varepsilon/kT}(1+2x+3x^2+\cdots) \quad \text{where } x = e^{-\varepsilon/kT}$$

$$= \varepsilon N_0 e^{-\varepsilon/kT}\left(\frac{1}{1-x}\right)^2 \quad \left[\because\ 1+2x+3x^2+\cdots = \frac{1}{(1-x)^2}\right]$$

$$= \frac{\varepsilon N_0 e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2} \qquad \cdots(10.15)$$

On the basis of the quantum theory, the average energy of Planck's oscillator can be written using equation (10.14) and (10.15) as

$$\bar{\varepsilon} = \frac{E}{N}$$

$$= \frac{\varepsilon N_0\, e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2} \times \frac{1-e^{-\varepsilon/kT}}{N_0} = \frac{\varepsilon\, e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}} \qquad \cdots(10.16)$$

or,
$$\bar{\varepsilon} = \frac{\varepsilon}{e^{\varepsilon/kT}-1} \qquad \cdots(10.17)$$

or,
$$\boxed{\bar{\varepsilon} = \frac{h\nu}{e^{h\nu/kT}-1}} \quad [\because\ \varepsilon = h\nu] \qquad \cdots(10.18)$$

We know, the number of oscillators per unit volume within the frequency range ν and $\nu + d\nu$ is given by

$$N = \frac{8\pi\nu^2}{c^3}\, d\nu \qquad \cdots(10.19)$$

Hence, the radiant energy density between the freequency range ν and $\nu + d\nu$ is given as

$$E_\nu d\nu = N\bar{\varepsilon} = \frac{8\pi\nu^2}{c^3}\,\frac{h\nu}{e^{h\nu/kT}-1}\, d\nu$$

or,
$$\boxed{E_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\,\frac{1}{e^{h\nu/kT}-1}\, d\nu} \qquad \cdots(10.20)$$

This relation is known as **Planck's radiation law** or **Planck's law for energy density of black body** or **cavity radiation.**

To obtain **Planck's radiation law in terms of wavelength** between the range λ and $\lambda + d\lambda$, we put $\nu = \frac{c}{\lambda}$ and $d\nu = \left|-\frac{c}{\lambda^2}d\lambda\right|$ in equation (10.20).

Thus, we have

$$\boxed{E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1}\, d\lambda} \qquad \cdots(10.21)$$

The **equations** (10.20) **and** (10.21) **are two forms of Planck's radiation law.** This equation explains successfully the experimental curves throughout the whole range of wavelength.

With the help of Planck's radiation law, we can derive Wien's law, Wien's displacement law, Rayleigh-Jeans law and Stefan-Boltzmann's law.

Problem 1

How many photons are presented per unit volume of radiation in thermal equilibrium at 1500K? Calculate the average energy of a photon at this temperature.

Solution Let n be the number of photons per unit volume of radiation in thermal equilibrium at 1500K

$\therefore$ Energy of photons per unit volume of radiation $= nh\nu$.

Now, we can write from Planck's radiation law—

$$(nh\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu$$

So, total no of photons,

$$n = \int_0^\infty \frac{8\pi\nu^2}{c^3(e^{h\nu/kT}-1)}d\nu = \frac{8\pi}{c^3}\times\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}-1}d\nu$$

$$= \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_0^\infty \frac{x^2}{e^x-1}dx \quad \left[\text{putting } \frac{h\nu}{kT} = x\right]$$

$$= \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3[2.404] \quad \left[\because \int_0^\infty \frac{x^2}{e^x-1}dx = 2.404\right] = 6.83\times 10^{16}$$

Now, average energy of a photon is

$$\frac{\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu}{n} = \frac{\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu}{\frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}-1}d\nu}$$

$$= \frac{\frac{8\pi(kT)^4}{c^3h^3}\times 6.4938}{\frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\times 2.404} \quad \left[\because \int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15} = 6.4938\right] = 2.7012kT = 0.349 \text{ eV}.$$

10.6.1. Wien's Radiation Law from Planck's Radiation Law

For *small values of temperature* or *in the region of low wavelength* $\frac{hc}{\lambda kT}$ of euqation (10.21) becomes very large ($\frac{hc}{\lambda kT} \gg 1$). Hence, 1 can be neglected in the denominator of right hand side of equation (10.21) in comparison with exponential term. Thus we get, $E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}e^{-hc/\lambda kT}d\lambda$

$$\therefore \quad \boxed{E_\lambda = \frac{A}{\lambda^5}e^{-B/\lambda T}} \qquad \cdots(10.22)$$

where, $A = 8\pi hc$ and $B = \frac{hc}{k}$ are the constants

This equation (10.22) is **Wien's radiation law.**

10.6.2. Wien's Displacement Law from Planck's Radiation Law

Planck's law shows that for a particular temperature the emissive power e_λ for a perfectly black body increases with an increase in wavelength and becomes maximum at a particular wavelength λ_m.

So, the denominator of equation (10.21) should be minimum.

Let $Z = \lambda^5(e^{hc/\lambda kT} - 1)$

For Z to be minimum, $\frac{dZ}{d\lambda} = 0$ at $\lambda = \lambda_m$

$$\therefore \quad 5\lambda_m^4 (e^{hc/\lambda_m kT} - 1) + \lambda_m^5 (e^{hc/\lambda_m kT}) \cdot \left(\frac{-hc}{kT\lambda_m^2}\right) = 0 \quad [\because \lambda = \lambda_m] \qquad \cdots(10.23)$$

$$\text{or,} \quad 5\lambda_m^4 e^{hc/\lambda_m kT} - 5\lambda_m^4 - \frac{hc}{kT}\lambda_m^3 e^{hc/\lambda_m kT} = 0$$

$$\text{or,} \quad 1 - e^{-hc/\lambda_m kT} - \frac{hc}{5\lambda_m kT} = 0 \quad [\text{dividing both sides by } 5\lambda_m^4 e^{hc/\lambda_m kT}]$$

$$\text{or,} \quad 1 - \exp(-x) - \frac{x}{5} = 0, \text{ where } x = \frac{hc}{\lambda_m kT}$$

$$\text{or,} \quad \boxed{1 - \exp(-x) = \frac{x}{5}} \qquad \cdots(10.24)$$

This is a transcendental equation which cannot be solved analytically. The only way to solve this equation is by graphical method. The point of intersection of the two graphs $y = \frac{x}{5}$ and $y = 1 - e^{-x}$ gives the solution which turns out to be $x = 4.9651$

$$\text{Now,} \quad x = \frac{hc}{\lambda_m kT} \quad \text{or,} \quad \lambda_m T = \frac{hc}{kx}$$

$$\text{or,} \quad \lambda_m T = \frac{hc}{4.9651\,k} = 0.0029 \text{ m} \cdot \text{K} \qquad \cdots(10.25)$$

$$\therefore \quad \boxed{\lambda_m T = \text{constant}} \qquad \cdots(10.26)$$

This is Wien's displacement law.

10.6.3. Rayleigh-Jeans Law from Planck's Radiation Law

For large values of temperature or in the region of high wavelengths, $\frac{hc}{\lambda kT}$ is very small. Thus, the exponential term of equation (10.21) can be expanded $\left(e^x = 1 + x + \frac{x^2}{2} + \cdots\right)$ and we get,

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda kT} \quad \text{(neglecting the higher terms)}$$

Hence, we get from equation (10.21),

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\left[\frac{1}{\left(1 + \frac{hc}{\lambda kT}\right) - 1}\right] d\lambda = \frac{8\pi hc}{\lambda^5} \times \frac{\lambda kT}{hc}\, d\lambda = \boxed{\frac{8\pi kT}{\lambda^4}\, d\lambda} \qquad \cdots(10.27)$$

This is the Rayleigh-Jeans law.

10.6.4. Stefan's Law from Planck's Radiation Law

From Planck's radiation law [of equation (10.20)] $E_\nu d\nu = \dfrac{8\pi h\nu^3}{c^3(e^{h\nu/kT}-1)} d\nu$

So, the total radiation energy comprising of all frequencies of a black body enclosure at temperature T is given by

$$E = \int_0^\infty E_\nu d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT}-1} = \frac{8\pi (kT)^4}{c^3 h^3}\int_0^\infty \frac{x^3}{e^x-1}dx$$

$$\left[\text{putting } x = \frac{h\nu}{kT}\right] \quad \cdots(10.28)$$

But $\displaystyle\int_0^\infty \frac{x^3 dx}{e^x-1} = \frac{\pi^4}{15}$ $\cdots(10.29)$

Thus, we get from equation (10.28)

$$E = \frac{8\pi k^4 T^4}{c^3 h^3}\times\frac{\pi^4}{15} \quad \cdots(10.30a)$$

It can be shown that for any black body radiation, the amount of radiant energy passing through unit area in space filled with radiation in unit time at the same temperature (*i.e.* the total emissive power)

$$E' = E\left(\frac{c}{4}\right) = \frac{2\pi^5 k^4}{15c^2h^3}T^4 = \sigma T^4 \quad \cdots(10.30b)$$

where σ = Stefan's constant $= \dfrac{2\pi^5 k^4}{15c^2h^3} = 5.67\times10^{-8}\ \mathrm{W\cdot m^{-2}\cdot K^{-4}}$

This equation (10.30b) is the **Stefan Boltzmann's law of radiation.**

> **Special Note :**
>
> It has been observed that the experimental distribution curve for black body radiation agrees excellently with that of theoretical distribution curve for all possible wavelengths. This success of Planck's hypothesis to explain the distribution of energy spectrum of a black body radiation was the beginning of quantum mechanics.

10.7. Planck's Quantum Theory (of Radiations) and Photon

There exist a good number of experimental findings such as photoelectric effect, Compton effect, Raman effect etc. that can not be explained on the basis of wave theory of light. Historically, this failure of classical wave theory of light was first encountered in the attempt to explain black body radiation by Max Planck in 1900.

In order to explain the mode of radiation emitted by hot black bodies, Max Planck proposed quantum theory of radiation according to which—

1. a body can not emit or absorb energy in a continuous manner.
2. absorption or emission of radiation takes place in discrete bundles of energy equal to $nh\nu$ $(n = 0, 1, 2, \cdots)$ (*i.e.* integral multiple of energy $h\nu$). These bundles or packets of radiant energy are called quanta.

③ the energy associated with each quantum of a particular radiation is given by $E = h\nu$, where h = Planck's constant = 6.626×10^{-34} J · s and ν = frequency of radiation.

In 1905, Einstein while explaining the photoelectric effect, extended the idea of Planck and suggested that **light composed of stream of discrete quanta (*i.e.* discrete energy packet) called photons. These photons move through space with the velocity of light.** Photons are electrically neutral.

The **energy** of a **photon** $\boxed{E = h\nu = \frac{hc}{\lambda}}$ ...(10.31)

and its **relativistic mass** $\boxed{m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{h}{c\lambda}}$ ...(10.32)

where the **rest mass** of the photon $(m_0) = 0$.

The **momentum** of the **photon** $\boxed{p = mc = \frac{h\nu}{c^2} \times c = \frac{h\nu}{c} = \frac{E}{c}}$ ...(10.33)

Problem 1

A photon particle has energy of 85 eV. Find its i frequency, ii wavelength, iii momentum, iv mass and v number required to produced 1 joule of energy.

Solution

i Energy of photon $E = 85$ eV $= 85 \times 1.6 \times 10^{-19}$ J $= 136 \times 10^{-19}$ J

[= K.E. of the photon as its rest mass energy = 0]

If ν is the frequency of the photon, energy of a photon, $E = h\nu$

$$\text{or,}\quad \nu = \frac{136 \times 10^{-19}}{6.62 \times 10^{-34}} = 20.5 \times 10^{15} \text{ Hz}$$

ii Wavelength $\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{20.5 \times 10^{15}} = 1.46 \times 10^{-8}\text{m} = 146 \times 10^{-10}\text{ m} = 146$ Å

iii If $p(= mc)$ is the momentum of a photon,

$$p = \frac{E}{c} = \frac{136 \times 10^{-19}}{3 \times 10^8} = 4.5 \times 10^{-26} \text{ kg} \cdot \text{m} \cdot \text{s}^{-1}$$

iv If m is the relativistic mass of a photon,

$$p = \frac{E}{c} \quad [\because \text{ photon moves with velocity of light}]$$

$$\text{or,}\quad mc = \frac{E}{c} \quad \text{or,}\quad m = \frac{E}{c^2} = \frac{136 \times 10^{-19}}{(3 \times 10^8)^2} = 1.5 \times 10^{-34} \text{ kg}$$

v The number of photons to make 1 joule of energy

$$= \frac{1}{136 \times 10^{-19}} = 7.35 \times 10^{16}$$

The relativistic mass (m) is related to rest mass (m_0) as,

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \cdots(10.37)$$

10.9.2. Mass Energy Equivalence

According to Einstein the mass of a body can be expressed as a kind of energy. If a particle of mass m whether at rest or motion, then it must has an energy

$$\boxed{E = mc^2} \quad \text{where } c \text{ is the velocity of light} \quad \cdots(10.38)$$

This relation is a fundamental equation of Einstein's mass energy equivalence.

10.9.3. Energy Momentum Relation

If a particle of rest mass m_0 has a momentum p, then the relativistic relation between energy and momentum can be written as

$$\boxed{E^2 = p^2c^2 + m_0^2c^4} \quad \cdots(10.39)$$

Physical significance This equation indicates that the inertial mass (m_0) may be a property of energy (rest mass energy $E = m_0c^2$) rather than the quantity of matter.

10.10. Compton Effect

When a photon of high energy $h\nu$ collides with the free (or loosely bound) electron of the scatterer (target-electron) at rest, it transfers some energy to the electron. The scattered photon will therefore have a smaller energy $(h\nu')$ and consequently a greater wavelength than that of the incident photon [Fig. 14]. The observed change in wavelength of scattered photon by considering the elastic collisions between the incident photon and the free electron of the scattered material is known as Compton effect.

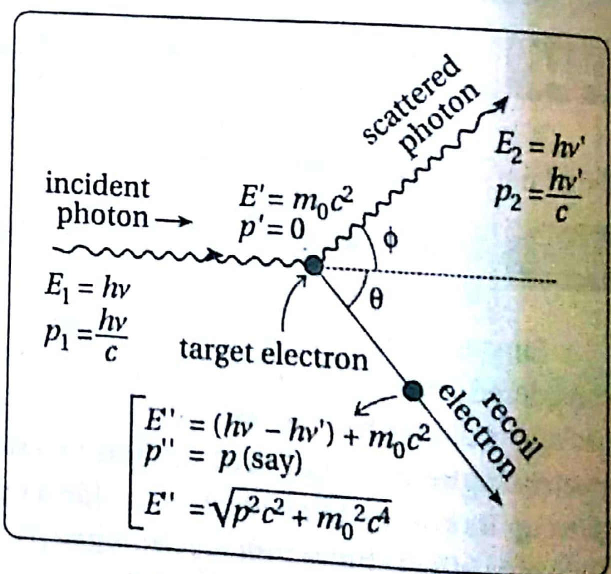


Fig. 14 ▷ Compton scattering

Let a photon of energy $h\nu$ collides with a free or loosely bound electron of the scatterer. The photon is scattered at an angle ϕ while the electron recoils at an angle θ. The process of recoiling of electron and scattering of photon is shown in Fig 13. Applying the laws of conservation of energy and momentum, we can find out the change in wavelength of the scattered photon.

Now **applying the law of momentum conservation** along and perpendicular to the direction of incidence, we get,

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\phi + p\cos\theta \quad \text{(along the direction)} \qquad \cdots(10.40a)$$

$$0 = \frac{h\nu'}{c}\sin\phi - p\sin\theta \quad \text{(perpendicular to the direction)} \qquad \cdots(10.40b)$$

Here, p is the momentum of the recoil electron.

From equation $(10.40a)$ and equation $(10.40b)$ we have,

$$pc\cos\theta = h\nu - h\nu'\cos\phi \qquad \cdots(10.41a)$$

$$pc\sin\theta = h\nu'\sin\phi \qquad \cdots(10.41b)$$

Squaring and adding we get,

$$(pc)^2 = (h\nu)^2 - 2(h\nu)(h\nu')\cos\phi + (h\nu')^2 \qquad \cdots(10.42).$$

Again, **applying the law of energy conservation** we get the energy of the recoil electron,

$$\begin{aligned} E'' &= \text{K. E. of recoil electron + its rest mass energy} \\ &= \text{energy transferred to electron by photon + its rest mass energy} \\ &= (h\nu - h\nu') + m_0c^2 \end{aligned} \qquad \cdots(10.43)$$

Again, for a relativistic recoil electron

$$E'' = \sqrt{p^2c^2 + m_0^2c^4} \qquad \cdots(10.44)$$

Hence, we get from equations (10.43) and (10.44)

$$p^2c^2 + m_0^2c^4 = [(h\nu - h\nu') + m_0c^2]^2$$

or, $$p^2c^2 + m_0^2c^4 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2(m_0c^2)(h\nu - h\nu') + m_0^2c^4$$

or, $$p^2c^2 = (h\nu)^2 - 2(h\nu)(h\nu') + (h\nu')^2 + 2m_0c^2(h\nu - h\nu') \qquad \cdots(10.45)$$

So, we get from equation (10.42) and (10.45)

$$2m_0c^2(h\nu - h\nu') = 2(h\nu)(h\nu') - 2(h\nu)(h\nu')\cos\phi$$

or, $$2h(\nu - \nu')m_0c^2 = 2h^2\nu\nu'(1 - \cos\phi)$$

or, $$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1 - \cos\phi)$$

or, $$\boxed{\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1 - \cos\phi)} \qquad \cdots(10.46a)$$

or, $$\frac{1}{c/\lambda'} - \frac{1}{c/\lambda} = \frac{h}{m_0c^2}(1 - \cos\phi)$$

or, $$\boxed{\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\phi)} \qquad \cdots(10.46b)$$

Equation (10.46b) gives the change (*i.e.* increase) in wavelength or Compton shift of the scattered photon by a particle of rest mass m_0.

Compton wavelength Since, the maximum value of $\cos\phi = 1$, the wavelength of the scattered photon is always greater than the incident photon. **This change is independent of the wavelength λ of the incident photon and the quantity $\frac{h}{m_0c} (= \lambda_c)$ is known as Compton wavelength** of the scattering particle.

The equation (10.46b) gives the **following conclusions**:

Case 1 When $\phi = 0$, then $\lambda' - \lambda = 0$ *i.e.* there is no scattering along the direction of the incident photon.

Case 2 When $\phi = \frac{\pi}{2}$, then $\lambda' - \lambda = \frac{h}{m_0 c} = \lambda_c =$ Compton wavelength

$$\therefore \quad \lambda_c = \frac{h}{m_0 c} = \frac{6.627 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^8} = 0.02427 \text{ Å}$$

$$\therefore \quad \text{Compton shift } \Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\phi) = 0.02427\,(1 - \cos\phi)$$

Case 3 When $\phi = 180°$ then $\lambda' - \lambda = \frac{2h}{m_0 c}$ *i.e.* the wavelength change will be twice of the Compton wavelength λ_c and this is the maximum possible shift.

10.10.1. Direction of the Recoil Electron

In Compton scattering when a high energy photon collides with free or loosely bound electron, it recoils at an angle θ,

Now, dividing equations (10.41b) by (10.41a) we get,

$$\tan\theta = \frac{h\nu' \sin\phi}{h\nu - h\nu' \cos\phi} = \frac{\nu' \sin\phi}{\nu - \nu' \cos\phi} \qquad \cdots(10.47)$$

Now, from equation (10.46a), we can write,

$$\frac{1}{\nu'} = \frac{1}{\nu} + \frac{h}{m_0 c^2}(1 - \cos\phi) \quad \text{or,} \quad \frac{\nu}{\nu'} = 1 + \frac{h\nu}{m_0 c^2} \cdot 2\sin^2\frac{\phi}{2}$$

So, the frequency of scattered photon

$$\boxed{\nu' = \frac{\nu}{1 + \frac{h\nu}{m_0 c^2} 2\sin^2\frac{\phi}{2}}} \qquad \cdots(10.48)$$

So, from equation (10.47) we get,

$$\tan\theta = \frac{\left(\frac{\nu \sin\phi}{1 + 2\alpha \sin^2\frac{\phi}{2}}\right)}{\left[\nu - \left(\frac{\nu}{1 + 2\alpha \sin^2\frac{\phi}{2}}\right)\cos\phi\right]} \quad \text{where } \alpha = \frac{h\nu}{m_0 c^2}$$

$$\tan\theta = \frac{\sin\phi}{1 + 2\alpha \sin^2\frac{\phi}{2} - \cos\phi} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\alpha \sin^2\frac{\phi}{2} + (1 - \cos\phi)}$$

$$= \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2} + 2\alpha \sin^2\frac{\phi}{2}} \qquad \left[\because\ 1 - \cos\phi = 2\sin^2\frac{\phi}{2}\right]$$

$$= \frac{\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\sin^2\frac{\phi}{2}(1 + \alpha)} = \frac{\cot\frac{\phi}{2}}{(1 + \alpha)} = \frac{\cot\frac{\phi}{2}}{1 + \frac{h\nu}{m_0 c^2}} \qquad \cdots(10.49)$$

This gives the direction of the recoil electron in terms of the direction and frequency of the scattered photon.

10.10.2. Energy of the Recoil Electron

The kinetic energy of the **recoil electron** (E) **must be equal to the decrease in energy of the incident photon.** So, $E = h\nu - h\nu'$ ...(10.50)

If m_0c^2 and mc^2 are the **rest mass energy and the recoil energy of the electron respectively then kinetic energy of the recoil electron** $= mc^2 - m_0c^2$.

This **energy must be equal to the decrease in energy of the incident photon.**

So, kinetic energy of the recoil electron

$$E = h(\nu - \nu') = h\nu\left(1 - \frac{\nu'}{\nu}\right) = h\nu\left(1 - \frac{\lambda}{\lambda'}\right) \quad \left[\because \nu = \frac{c}{\lambda} \text{ and } \nu' = \frac{c}{\lambda'}\right]$$

$$= h\nu\left(\frac{\lambda' - \lambda}{\lambda'}\right) \qquad \text{...(10.51)}$$

We know, the change in wavelength of the scattered photon (*i.e.* Compton shift) from Compton theory as, $\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\phi)$ [from equation (10.46b)]

Thus we get from equation (10.51) by substituting the values of $\lambda' - \lambda$ and λ'

$$E = \frac{h\nu \cdot \frac{h}{m_0c}(1-\cos\phi)}{\lambda + \frac{h}{m_0c}(1-\cos\phi)} = \frac{h\nu \cdot \frac{h}{m_0c}(1-\cos\phi)}{\lambda\left[1 + \frac{h}{m_0c\lambda}(1-\cos\phi)\right]}$$

$$= \frac{h\nu \cdot \frac{h}{m_0c\lambda}(1-\cos\phi)}{1 + \frac{h}{m_0c\lambda}(1-\cos\phi)} = \frac{\frac{h^2\nu^2}{m_0c^2}(1-\cos\phi)}{1 + \frac{h\nu}{m_0c^2}(1-\cos\phi)} \quad \left[\therefore \nu = \frac{c}{\lambda}\right] \qquad \text{...(10.52)}$$

This equation gives the kinetic energy of the recoil electron.

Let us consider the **following cases**:

Case 1 If $\phi = 0$, $E = 0$, so no electron will recoil for incident photon.

Case 2 If $\phi = \frac{\pi}{2}$ then $E = \frac{h^2\nu^2/m_0c^2}{1 + \frac{h\nu}{m_0c^2}} = \frac{h\nu\alpha}{1+\alpha}$, taking $\alpha = \frac{h\nu}{m_0c^2}$

Case 3 If $\phi = \pi$, $E = \frac{2h^2\nu^2/m_0c^2}{1 + \frac{2h\nu}{m_0c^2}} = \frac{2h\nu}{1+2\alpha}\alpha$

Hence, the minimum energy transferred by incident photon to electron is zero [for $\phi = 0$].

And maximum energy transfer to recoil electron is $E = h\nu \cdot \frac{2\alpha}{1+2\alpha}$ [for $\phi = 180°$]

Now, $E < 1$ as $\frac{2\alpha}{1+2\alpha} < 1$

So, the incident photon cannot transfer its whole energy to the recoil electron in Compton scattering.

So, the total K.E. (E'') of the recoil electron

$$E'' = \text{K.E} + \text{rest mass energy} = \boxed{\frac{\frac{h^2\nu^2}{m_0c^2}(1-\cos\phi)}{1+\frac{h\nu}{m_0c^2}(1-\cos\phi)} + m_0c^2} \quad \cdots(10.53)$$

> **Special Note :**
> Since, the work function is negligibly small as compared to the energy of the incident photon, so bound electron (electron bound to the nucleus) will become free by taking a small fraction of the energy $h\nu$ of the incident photon. Hence, **the electron is treated practically free.**

Problem 1

In Compton scattering experiment a beam of γ-radiation having wavelength (of photon) 2.426×10^{-12} m is incident on a foil of aluminium.

i **Find the Compton shift, for the scattering angle $45°$.**

ii **Find the wavelength of the scattered radiation if the scattered radiation are viewed at an angle $45°$ to the direction of the incident beam.**

iii **Find Compton wavelength.**

iv **What is the energy of incident photon ?**

v **What is the energy of scattered photon ?**

vi **Find the energy lost by the photon ?**

vii **Find how much kinetic energy is imparted to the recoil electron.**

viii **Find the direction of the emission of the corresponding recoil electron (assuming scattering angle $\phi = 90°$).**

ix **Find the total energy of the recoil electron.**

x **Find the fraction of energy loss.**

Solution

i We know Compton shift $\lambda' - \lambda (= \Delta\lambda) = \frac{h}{m_0c}(1-\cos\phi)$

As the radiation is viewed at an angle $45°$ to the direction of incident beam we assume the scattering angle $\phi = 45°$. So,

$$\therefore \quad \lambda' - \lambda = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.1 \times 10^{-31}\ \text{kg}) \times (3 \times 10^{8}\ \text{m}\cdot\text{s}^{-1})}(1-\cos 45°)$$

$$= \mathbf{7.1 \times 10^{-13}\ m}$$

ii The wavelength of the scattered radiation for the above scattering angle

$$\lambda' = \lambda + \Delta\lambda = 2.426 \times 10^{-12}\ \text{m} + 7.1 \times 10^{-13}\ \text{m} = \mathbf{3.1 \times 10^{-12}\ m}$$

iii For scattering angle $\phi = \frac{\pi}{2}$, we called Compton shift as Compton wavelength (λ_c).

$$\therefore \quad \lambda_c = (\Delta\lambda)_{\phi = 90°} = \frac{h}{m_0c} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}\ \text{m} = \mathbf{0.024\ Å}$$

iv Energy of incident photon

$$E_i = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34})\times(3\times10^8)}{2.426\times10^{-12}}\ \text{J} = 8.198\times10^{-14}\ \text{J} \simeq 0.51\times10^6\ \text{eV}$$

$[\because\ 1\ \text{eV} = 1.6\times10^{-19}\ \text{J}]$

$= \mathbf{0.51\ MeV}$

v Energy of scattered photon

$$E_s = \frac{hc}{\lambda'} = \frac{(6.63\times10^{-34})\times(3\times10^8)}{3.1\times10^{-12}}\ \text{J} = \mathbf{0.40\ MeV}$$

vi Energy loss by incident photon

$= E_i - E_s = 0.51\ \text{MeV} - 0.40\ \text{MeV} = \mathbf{0.11\ MeV}$

vii Kinetic energy of the recoil electron = energy loss by incident photon.

$E = E_i - E_s = \mathbf{0.11\ MeV}$

viii The direction (θ) of emission of the corresponding recoil electron

$$\tan\theta = \frac{\cot\frac{\phi}{2}}{1+\frac{h\nu}{m_0c^2}} \qquad \left[\phi = \text{scattering angle} = 90^\circ,\ \nu = \frac{c}{\lambda} = \frac{3\times10^8}{2.426\times10^{-12}}\ \text{Hz}\right]$$

$$= \frac{\cot 45^\circ}{1+\frac{6.63\times10^{-34}\times3\times10^8}{2.426\times10^{-12}\times9.1\times10^{-31}\times(3\times10^8)^2}} \simeq \frac{1}{2}$$

$\therefore\ \theta = \mathbf{26.56^\circ}$

ix The total energy of recoil electron

= kinetic energy of recoil electron + rest mass energy of recoil electron

$$= E + m_0c^2 = (0.11\ \text{MeV}) + \frac{9.1\times10^{-31}\times(3\times10^8)^2}{1.6\times10^{-19}\times10^6}\ \text{MeV}$$

$= 0.11\ \text{MeV} + 0.51\ \text{MeV} = \mathbf{0.62\ MeV}$

x The fraction of energy loss $= \frac{E_i - E_s}{E_i} = \frac{0.11\ \text{MeV}}{0.51\ \text{MeV}} = \mathbf{0.21}$

10.10.3. Compton Effect Can not be Observed with Visible Light but Can be Observed due to X-Rays or γ-Rays

The wavelength of visible light is $\lambda \simeq 6000\times10^{-10}$ m

$$\therefore\ \text{Energy of visible light } E = \frac{hc}{\lambda} = \frac{6.627\times10^{-34}\times3\times10^8}{6000\times10^{-10}\times1.6\times10^{-19}}\ \text{eV} \simeq 2\ \text{eV}$$

The binding energy of an electron in the atoms $\simeq 10$ eV. For example, the binding energy of an electron for simple hydrogen $\simeq 13.6$ eV.

So when visible light will fall on a target, it can not liberate electrons of the scatterer. So we can not see compton effect with visible light.

But the energy of an X-rays of wavelength $1\text{Å} \simeq 2 \times 10^3$ eV and that for γ-rays $= 10$ eV.

So when X-rays or γ-rays will fall on the target, they can liberate electrons of the scatterer at rest. Hence, we can see Compton effect with γ-rays or X-rays.

10.10.4. Compton Shift is Independent of the Nature of the Scatterers

The shift in the wavelength of the modified line (Compton line)

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi)$$

Here, $\Delta\lambda$ depends only on the rest mass of the electron (m_0) and the scattering angle ϕ.

Now, the rest mass of an electron is constant for all substances. Therefore, the Compton shift is totally independent of the nature of the scatterer.

10.10.5. The Intensity (Energy) of Compton Lines Depends upon the Nature of the Scatterer

From the interpretation of primary lines we know that the intensity of parent line is larger for heavier elements than that of the modified line, while for lighter elements the intensity is lower than that of the modified line. Hence, only the intensity or energy of the Compton lines is dependent on the nature of the scatterer.

10.10.6. Modified and Unmodified Lines (Compton Shift)

The presence of the modified line (Compton line) is due to the collision of X-ray photon with free electrons. (This assumption of free electrons is justified in the sense that the energy given to the electron by the high energy photon is much greater than the work function of the metal). But it is always probable that the electron is bound to the atom. In this case, when the collision occurs between the photon and the atom as a whole the rest mass m_0 of eletron must be replaced by M (the mass of an atom) in the equation

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi)$$

So, $\Delta\lambda$ will become too small to be detected for the large value of M and under this condition this will given rise to **unmodified line**. Thus, modified beam of line is showing increase in λ but unmodified beam has the same wavelength as that of the primary.

For one and same scatterer there may be a certain numbers of loosely bound electrons and some tightly bound electrons. Therefore, under the collision of photon with the scatterer there will be some modified and unmodified lines for the same scatterer.

For heavier elements, there are a small number of electrons which are free and for light elements a large number of electrons are free. Hence, the intensity of unmodified line is greater than Compton lines (modified lines) for heavier elements and the intensity of unmodified line is smaller than Compton lines for lighter elements.

The theory of Compton effect is also verified experimentally by Compton himself. It was seen that there is a good agreement between theory and experiment. It proves the quantum or particle nature of radiation.

10.10.7. Comparison between Photoelectric Effect and Compton Effect

The photoelectric effect and Compton effect both are occured due to the action of photon on a electron but the two effects are not the same.

In **photoelectric effect**, *the incident photon is completely absorbed by the bound electron.* After absorbing energy, this bound electron will be emitted from the metal surface with its maximum kinetic energy which is equal to the difference of energy between the incident photon and the work function of the metal.

But in **Compton effect**, the incident photon will strike a loosely bound electron of the atom. *A part of this incident energy of photon will be taken by loosely bound electron and it recoils. The incident photon with its reduced energy will be scattered by the scatterer.*

10.11. Pair Production

The process in which a gamma (γ) ray photon (can be considered as high energy photon) having energy more than 1.02 MeV while passing nearby a nucleus disappears and is converted to an electron-positron[5] pair is called pair production [Fig. 15].

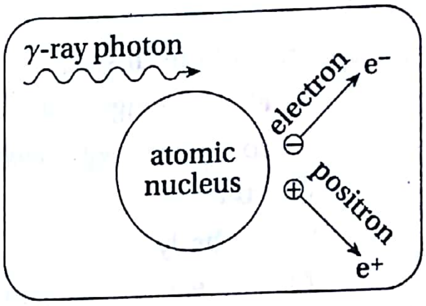


Fig. 15 ▷ γ-ray photon is converted to electron positron pair

The rest mass energy of an electron = the rest mass energy of a positron = m_0c^2, where c = velocity of light.

When a γ-ray photon of energy $h\nu$ disappears and then produces electron-positron pair in the intense coulomb field of an atomic nucleus, this incident energy ($h\nu$) is spent to form an electron-positron pair and imparting kinetic energy to them as per the following reaction.

$$h\nu = 2m_0c^2 + E^- + E^+ \quad \cdots(10.54)$$

where E^- = kinetic energy of electron and E^+ = kinetic energy of positron.

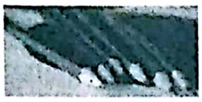
This indicates *the energy of incident γ-ray photon must be equal or greater than $2m_0c^2$ for pair production. The minimum energy of γ-ray photon for which electron-positron pair has zero kinetic energy* = $2m_0c^2 = 2\times0.51\text{MeV} = 1.02\text{MeV}$. The corresponding incident wavelength (λ) = 1.2×10^{-12}m is called threshold wavelength of pair production.

The probability of **pair production is high for high energy incident ray photon.** But if the incident radiation is in the **ultraviolet region, the probability of photoelectric effect is large.** On the otherhand if the incident radiation is in the X-rays region (i.e., intermediate frequency range), **the probability of Compton effect is higher in comparison to the other two process.**

[5] Basically, positron (e^+) is an antiparticle of electron (e^-). So its electric charge and other intrinsic quantum numbers are opposite to those of the electron.

Annihilation of electron and positron Electron positron annihilation occurs when a positron and an electron interact. The result of interaction is the annihilation of the electron positron pair and production of γ ray photon.

Pair production can not seen in vacuum Basically, in pair production the conservation of momentum and energy are valid in the vicinity of a nucleus. But in vacuum, there is nothing to help the conservation of momentum and energy during the pair production process.




Exercise

Multiple Choice Questions

1. In the region of atomic dimensions which theory is applicable?
   Ⓐ classical theory Ⓑ quantum theory Ⓒ none **Ans.** Ⓑ
2. The absorptive power of a black body is—
   Ⓐ 1 Ⓑ 0 Ⓒ 2 **Ans.** Ⓐ
3. A black body absorbs—
   Ⓐ all wavelengths of incident light
   Ⓑ some selective wavelengths depending on the nature of black body.
   Ⓒ none **Ans.** Ⓐ
4. A black body—
   Ⓐ can reflect or transmit
   Ⓑ cannot reflect Ⓒ cannot transmit **Ans.** Ⓐ
5. All bodies placed inside an enclosure acts as a—
   Ⓐ black body Ⓑ photo cell Ⓒ none **Ans.** Ⓐ
6. Black body radiation is—
   Ⓐ an E. M. wave Ⓑ a mechanical wave Ⓒ an elastic wave **Ans.** Ⓐ
7. The wavelength of visible light in comparison to that of thermal radiation is—
   Ⓐ smaller Ⓑ greater Ⓒ equal **Ans.** Ⓐ
8. 'The maximum energy density of radiations of a black body at temperature $T = 0$ K is displaced towards the shorter wavelength'. This law is known as—
   Ⓐ Wien's radiation law
   Ⓑ Rayleigh-Jeans law Ⓒ Wien's displacement law **Ans.** Ⓒ
9. The wavelength of black body changes with its absolute temperature as—
   Ⓐ $\lambda \propto \frac{1}{T}$ Ⓑ $\lambda \propto T$ Ⓒ $\lambda \propto \frac{1}{T^2}$ **Ans.** Ⓐ
10. In Rayleigh-Jeans law the energy density of a black body is—
    Ⓐ inversely proportional to the fourth power of wavelength
    Ⓑ directly proportional to the fourth power of wavelength
    Ⓒ none **Ans.** Ⓐ

Spectrum of the hydrogen atom According to classical theory, an excited hydrogen atom radiates energy of all wavelengths continuously. But experimentally, it has been observed that the excited atom of hydrogen emits electromagnetic radiation of certain definite wavelengths due to transition of an electron from higher energy state to lower energy state corresponding to Lyman $(n_1 = 1)$, Balmer $(n_1 = 2)$, Paschen $(n_1 = 3)$, Bracket $(n_1 = 4)$ and Pfund $(n_1 = 5)$ series. These discrete set of lines are represented by

$$\bar{\nu}\left(=\frac{1}{\lambda}\right) = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \quad \cdots(10.1)$$

where, $\bar{\nu}$ = wave number, R = Rydberg constant, n_1 and n_2 are two integers such that $n_2 > n_1$.

So, it can be concluded that **classical theory does not hold good in the region of atomic dimensions.**

10.2.1. Particle Nature of Radiation

Radiation of energy is usually characterized by its wave like nature. But many experimental observation such as black body radiation, photoelectric effect, Compton effect, pair production etc can not be explained properly only by considering the wave nature of radiation. These phenomena along with any natural phenomena (from elementary particle level to cosmological level) can be properly explained on the basis of particle aspect of radiation (i.e., quantum theory of radiation). In that case the waves of electromagnetic radiation behave like particles of definite momentum and energy.

10.3. Black Body Radiation

10.3.1. Radiation

Any heated body emits electromagnetic radiation. If the temperature of the body is increased, the wavelength of the emitted radiation decreases.

Similarly, a heated body can also absorb radiation incident on it.

Emissive power (e_λ) ❶ The emissive power of a body at a certain temperature (T) for a certain wavelength (λ) is defined as the energy radiated normally in vacuum from its surface per unit area per unit second and per unit range of wavelength.

Absorptive power (a_λ) The absorptive power of a body is defined as the ratio **of the amount of heat (Q') absorbed by the body to the total amount of heat (Q) falling on it.** Therefore,

$$a_\lambda = \frac{Q'}{Q} \quad \cdots(10.2)$$

Kirchhoff's law of heat radiation It states that the **ratio of emissive power (e_λ) of a body to its absorptive power (a_λ) for radiation of a given wavelength** at a **particular temperature is always constant.**

❶ The emissive power may be defined as the rate of emission of radiant energy from unit area through unit solid angle of a blackbody.

Thus, $\frac{e_\lambda}{a_\lambda}$ = constant = (k) ...(10.3)

where k is an arbitrary constant depending on temperature

In case of black body For black body radiations, absorptive power $a_\lambda = 1$.

So, emissive power, e_λ = constant.

10.3.2. Black Body

The substance or a body which can absorb all the (heat) radiations of all wavelengths incident on it and does not transmit or reflect at all is known as a perfectly black body. Since, *it neither reflects nor transmits any of the incident radiations, it appears black whatever may be the colour of incident radiations.*

We know, a good absorber of radiation is also a good emitter of radiation. So, a perfectly black body (in the technical sense) emits radiations of all possible wavelengths when it is heated.

Thus, a **black body is a substance (or body) which can absorb radiations of all wavelengths incident on it and emits radiations of all wavelengths when it is heated.**

A perfectly black body is an ideal concept. Lamp black, platinum black etc. are considered as almost perfect blackbodies for practical purposes because these can absorb more than 95% of the incident radiations.

10.3.3. Construction of Black Body and Black Body Radiation

For experimental purposes, a black body is made up by a hollow spherical (copper) enclosure with a small hole (called orifice) in its surface. The inner surface of it is coated with lamp black. When any radiation enters the spherical enclosure through this hole, it suffers multiple reflections at the inner surface [Fig. 2(a)] until it is totally absorbed. In this way the body acts as a black body absorber.[2]

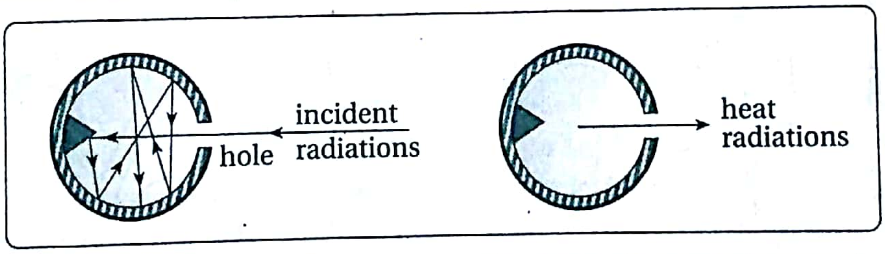


Fig. 2(a) ▷ Schematic diagram of black body absorber

Fig. 2(b) ▷ Schematic diagram of black body emitter

When a black body (*i.e.* hollow enclosure) is heated to a uniform temperature, the radiation of all possible wavelengths coming out through its fine hole. It is called black body emitter. [Fig. 2(b)]

10.3.4. Characteristics of Black Body Radiation

The important characteristics of a black body are—

① The energy density (*i.e.* energy radiated per unit area per second) does not depend upon the nature or shape of the walls of enclosure. It depends only upon the temperature of the enclosure.

[2] The conical projection prevents direct reflection towards the hole from the cavity walls.

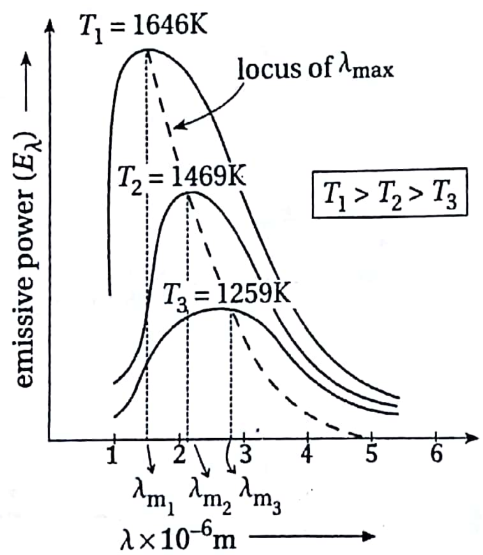
② The radiations are isotropic in nature.
③ The radiations are homogeneous in nature.
④ All bodies placed inside the enclosure also emit black body radiations.

10.3.5. Comparison between Thermal Radiation and Light Rays

① Both are electromagnetic waves.
② Both are transverse waves.
③ Radiant heat like light rays travel in straight line.
④ Both of them move with same speed ($3 \times 10^8 \mathrm{m \cdot s^{-1}}$).
⑤ Both of them obey the laws of reflection and refraction.
⑥ Both can travel in vacuum.
⑦ The only difference is that the wavelength of visible light [is smaller than that of thermal radiation].

10.4. Energy Distribution of the Spectrum of a Black Body

From the experimental results as obtained by Lummer and Pringsheim (1899), a graph is drawn between **emissive power** and **wavelength** (λ) **of a black body radiation** for the different tempratures [Fig. 4].



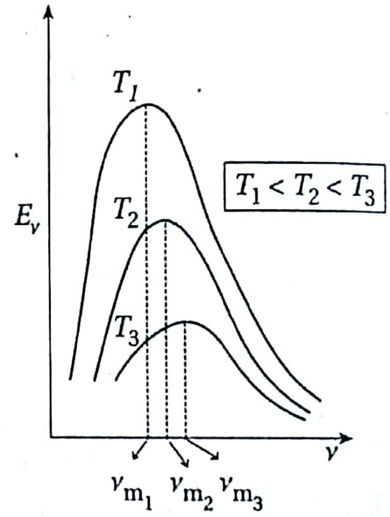


Fig. 3 ▷ The variation of emissive power (*i.e.* intensity of radiation) with the wavelength at different temperatures.

Fig. 4 ▷ The variation of emissive power (*i.e.* intensity of radiation) with the frequency at different temperatures.

From these curves [Fig. 4], the following important conclusions can be drawn:

① For a particular temperature, the emissive power e_λ of a perfectly black body increases with an increase in wavelength and becomes maximum at a particular wavelength λ_m. With further increase in wavelength, the emissive power (*i.e.* intensity of radiation) decreases.

② The wavelength (λ_m) of radiation for which the intensity of radiation is maximum, **shifts towards shorter wavelength region** (*i.e.* λ_m decreases) with a increase of temperature of the body. The relation between wavelength λ_m and the absolute temperature T is given by, $\lambda_m T$ = constant.

10.5. Theoretical Explanations of the Spectra Obtained from Black Body Radiation

On the basis of classical physics, number of attempts were made to explain the observed spectral (energy) distribution [Fig. 3] of a black body as a function of wavelength. Though, these attempts were not very successful, we shall discuss only three such well known classical laws in this context.

10.5.1. Wien's Radiation Formula

In order to explain the observed spectral distribution, Wien first showed that the energy density of radiation of **wavelength λ and $\lambda + d\lambda$ from a cavity** (*i.e.* black body) of temperature T is

$$\boxed{E_\lambda d\lambda = \frac{A}{\lambda^5} e^{-B/\lambda T} d\lambda} \quad \text{where } A \text{ and } B \text{ are two constants} \qquad \cdots(10.4)$$

Limitation of the formula Wien's radiation formula explains only the experimental results fairly well (*i.e.* makes good fitting of experimental curve) for low wavelength region [Fig. 5].

> ***Special Note :***
> But for high wavelength region, the values of E_λ is lower than that of experimental results.

10.5.2. Wien's Displacement Law

It states that **the maximum energy density (or intensity) of radiation emitted from a black body is displaced towards the shorter wavelength for the rise of temperature (T) of the black body.**

Therefore, the wavelength λ_{max} at the maximum energy distribution of a black body radiation changes in the inverse ratio of its temperature (TK).

$$\text{Thus,} \quad \lambda_{max} \propto \frac{1}{T} \quad \text{or,} \quad \lambda_{max} T = \text{constant} \qquad \cdots(10.5)$$

10.5.3. Rayleigh-Jeans Law

Rayleigh and Jeans applied the classical law of equipartition of energy to this electromagnetic black body radiation. They found, the energy density of radiation of wavelength range λ and $\lambda + d\lambda$ from a black body of temperature T is

$$\boxed{E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda} \qquad \cdots(10.6)$$

So, it states that **the energy density (E_λ) of a black body radiation** (of wavelength λ) **is inversely proportional to the fourth power of wavelength** (λ)

$$\text{i.e. } \boxed{E_\lambda \propto \frac{1}{\lambda^4}} \quad \cdots(10.7)$$

Thus, the energy density (E_λ) is increasing with the decrease of wavelength.

Assumption and derivation Lord Rayleigh and James Jeans considered few assumptions to overcome the failure of Wien's distribution law to explain the experimental result for longer wavelength region. The assumptions are—

① The thermal radiation in the cavity is full of standing waves of electromagnetic radiations of all wavelengths. The walls of the cavity are perfectly reflecting.

② The number of oscillators (modes of standing waves) per unit volume within the frequency range ν and $\nu + d\nu$ is given by $n_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu$, where c is the velocity of light in free space.

③ The average energy per degree of freedom of an entity (harmonic oscillator, gas molecule) can be written from Maxwell's law of equipartition of energy as $\bar{\epsilon} = \frac{1}{2}kT$, where k is the Boltzmann constant $= 1.38 \times 10^{-23}$ $(\mathrm{J \cdot K^{-1}})$ and T is the absolute temperature of the system which contains the entity.

We know that the electromagnetic standing wave in the cavity of black body radiation is a linear harmonic oscillator.

So, it has two degrees of freedom corresponding to its kinetic energy and potential energy. Thus, the average energy of each oscillator in thermal equilibrium at an absolute temperature T is given by $\bar{\epsilon} = 2 \times \frac{1}{2}kT = kT$.

Thus, the radiant energy density (i.e. the amount of radiant energy per unit volume) within the frequency range ν and $\nu + d\nu$ is given by

$$E_\nu d\nu = \bar{\epsilon}(n_\nu d\nu) = kT\left(\frac{8\pi\nu^2}{c^3} d\nu\right) = \frac{8\pi\nu^2}{c^3} kT d\nu$$

Now, $\nu = \frac{c}{\lambda}$ i.e. $d\nu = \left|-\frac{c}{\lambda^2} d\lambda\right|$: so the above equation can be written in terms of wavelength as $E_\lambda d_\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$.

This is Rayleigh-Jeans law.

Limitation of the law It was found that the above equation explains only the experimental result for high wavelength region [Fig. 5]. But for **low wavelength** region, Rayleigh-Jeans formula fails completely to explain the experimental result and thereby leads to what is known as **ultraviolet catastrophe**. Thus, this law also fails to explain black body radiation.

Ultraviolet catastrophe According to Rayleigh-Jeans law, the energy density of the black body radiation within the wavelength range λ and $\lambda + d\lambda$ is

$$E_\lambda \, d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

From this equation, we can see, as the wavelength of the radiation spectrum decreases, the energy density increases. Thus, $\lambda \longrightarrow 0$ gives $E_\lambda \longrightarrow \infty$ [Fig. 6]. But the experimental result shows that when the wavelength of the spectrum decreases, the energy density decreases. In this case $\lambda \longrightarrow 0$ implies $E_\lambda \longrightarrow 0$.

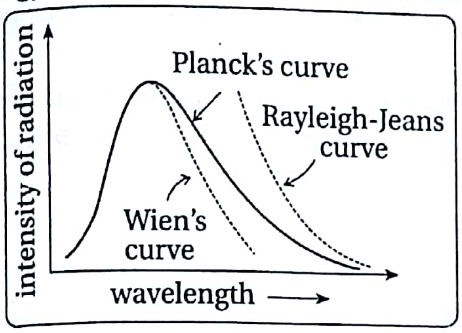


Fig. 5 ▷ Graph indicating the fitting of the experimental curve with Wien's law, Rayleigh-Jeans law and Planck's law.

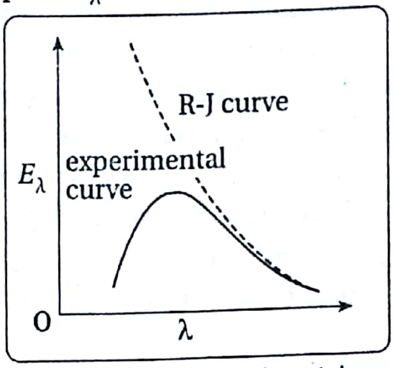


Fig. 6 ▷ The variation of experimental curve with Rayleigh-Jeans law

The discrepancy between theoretical conclusion of Rayleigh-Jeans law (energy density, $E_\lambda \longrightarrow \infty$) and experimental results (energy density $E_\lambda \longrightarrow 0$) of black body radiation for low wavelength (*i.e.* $\lambda \longrightarrow 0$) region is known as ultraviolet catastrophe.

This indicates the limitations of classical mechanics on the basis of which the law of equipartition of energy is derived. This discrepancy has come due to the assumption that energy can be absorbed or emitted by the oscillators[3] continuously in any amount.

10.5.4. Stefan-Boltzmann's Law

In 1879 Josef Stefan experimentally and in 1884 Ludwig Boltzmann theoretically (from thermodynamics) deduced a law relating the total power radiated per unit area of a black body to its absolute temperature. This is known as Stefan-Boltzmann's law.

This law states that **the total energy (E') emitted per unit area per second from a perfectly black body is proportional to the fourth power of its absolute temperature (T).**

Mathematically, it is written as

$$E' \propto T^4 \quad \text{or,} \quad \boxed{E' = \sigma T^4} \qquad \cdots(10.8)$$

where σ = Stefan's constant = $5.672 \times 10^{-5} \text{erg} \cdot \text{cm}^2 \cdot \text{s}^{-1} \cdot \text{K}^{-4}$.

If a perfectly black body at temperature TK is completely surrounded by an enclosure at T_0K ($T > T_0$), the amount of radiant energy per unit area per second is given by

$$\boxed{E' = \sigma(T^4 - T_0^4)} \qquad \cdots(10.9)$$

[3] For oscillator model of radiation read Article 10.6.

10.6. Planck's Hypothesis and Radiation Law

In order to explain the distribution of energy in the spectrum of a black body, Max-Planck in 1900, established the quantum theory of radiation. He derived the radiation law on the basis of following assumptions—

1. All bodies placed inside an enclosure can also emit black body radiation.
2. The atoms in the walls of a black body radiator behave like simple harmonic oscillators and each has a characteristic frequency of oscillation (*i.e.* vibrate with all possible frequencies).
3. The oscillator of the black body can not have any arbitrary amount of energy but has a discrete energy E_n given by $E_n = nh\nu$, where n is an integer ($n = 0, 1, 2, \cdots$) and ν is the frequency of oscillation.
   This relation shows that the total energy of an oscillator is quantised.
4. The oscillator can radiate or absorb energy in quantas (packets) of $h\nu$ (*i.e.* in discrete set of values $0, h\nu, 2h\nu, \cdots nh\nu$) only when the oscillator jumps from one energy state to another.

Derivation of the law Let N be the total number of Planck's oscillators and E be their total energy. So, the energy per oscillator $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{E}{N} \quad \cdots(10.10)$$

If $N_0, N_1, N_2, N_3 \cdots N_r \cdots$ etc. are the number of oscillators with energies $0, \varepsilon, 2\varepsilon, 3\varepsilon \cdots r\varepsilon \cdots$ etc. respectively, we can write,

$$N = N_0 + N_1 + N_2 + N_3 + \cdots + N_r + \cdots \quad \cdots(10.11)$$

and
$$E = 0 \cdot N_0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + \cdots + r\varepsilon N_r + \cdots$$
$$= 0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + \cdots + r\varepsilon N_r + \cdots \text{ where } \varepsilon = h\nu \quad \cdots(10.12)$$

From Maxwell's distribution formula, the number of oscillators with energy $r\varepsilon$ is

$$N_r = N_0 e^{-r\varepsilon/kT} \text{ where, } k \text{ is Boltzmann's constant} \quad \cdots(10.13)$$

$$\therefore \quad N_1 = N_0 e^{-\varepsilon/kT}, \; N_2 = N_0 e^{-2\varepsilon/kT}, \; N_3 = e^{-3\varepsilon/kT} \text{ and so on.}$$

Substituting the values of $N_1, N_2, N_3 \cdots$ in equation number (10.11), we get the total number of Planck's oscillators,

$$N = N_0 + N_0 e^{-(\varepsilon/kT)} + N_0 e^{-(2\varepsilon/kT)} + N_0 e^{-(3\varepsilon/kT)} + N_0 e^{(-4\varepsilon/kT)} + \cdots + N_0 e^{-r\varepsilon/kT} + \cdots$$

$$= N_0 [1 + x + x^2 + \cdots] \text{ where } x = e^{-\varepsilon/kT}$$

$$= N_0 \frac{1}{1-x} \quad \left[\because \; 1 + x + x^2 + \cdots = \frac{1}{1-x}\right]$$

$$= \frac{N_0}{1 - e^{-\varepsilon/kT}} \quad \cdots(10.14)$$

Similarly, the total energy E of the oscillators can be written from equation (10.12).

$$E = 0 + \varepsilon N_0 e^{-\varepsilon/kT} + 2\varepsilon N_0 e^{-2\varepsilon/kT} + 3\varepsilon N_0 e^{-3\varepsilon/kT} + \cdots$$

$$= \varepsilon N_0 e^{-\varepsilon/kT} (1 + 2e^{-\varepsilon/kT} + 3e^{-2\varepsilon/kT} + \cdots)$$

$$= \varepsilon N_0 e^{-\varepsilon/kT}(1 + 2x + 3x^2 + \cdots) \quad \text{where } x = e^{-\varepsilon/kT}$$

$$= \varepsilon N_0 e^{-\varepsilon/kT}\left(\frac{1}{1-x}\right)^2 \quad \left[\because\ 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}\right]$$

$$= \frac{\varepsilon N_0 e^{-\varepsilon/kT}}{(1 - e^{-\varepsilon/kT})^2} \qquad \cdots(10.15)$$

On the basis of the quantum theory, the average energy of Planck's oscillator can be written using equation (10.14) and (10.15) as

$$\bar{\varepsilon} = \frac{E}{N}$$

$$= \frac{\varepsilon N_0\, e^{-\varepsilon/kT}}{(1 - e^{-\varepsilon/kT})^2} \times \frac{1 - e^{-\varepsilon/kT}}{N_0} = \frac{\varepsilon\, e^{-\varepsilon/kT}}{1 - e^{-\varepsilon/kT}} \qquad \cdots(10.16)$$

or,
$$\bar{\varepsilon} = \frac{\varepsilon}{e^{\varepsilon/kT} - 1} \qquad \cdots(10.17)$$

or,
$$\boxed{\bar{\varepsilon} = \frac{h\nu}{e^{h\nu/kT} - 1}} \quad [\because\ \varepsilon = h\nu] \qquad \cdots(10.18)$$

We know, the number of oscillators per unit volume within the frequency range ν and $\nu + d\nu$ is given by

$$N = \frac{8\pi\nu^2}{c^3}\, d\nu \qquad \cdots(10.19)$$

Hence, the radiant energy density between the freequency range ν and $\nu + d\nu$ is given as

$$E_\nu d\nu = N\bar{\varepsilon} = \frac{8\pi\nu^2}{c^3}\, \frac{h\nu}{e^{h\nu/kT} - 1}\, d\nu$$

or,
$$\boxed{E_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\, \frac{1}{e^{h\nu/kT} - 1}\, d\nu} \qquad \cdots(10.20)$$

This relation is known as **Planck's radiation law** or **Planck's law for energy density of black body** or **cavity radiation.**

To obtain **Planck's radiation law in terms of wavelength** between the range λ and $\lambda + d\lambda$, we put $\nu = \frac{c}{\lambda}$ and $d\nu = \left|-\frac{c}{\lambda^2} d\lambda\right|$ in equation (10.20).

Thus, we have

$$\boxed{E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\, \frac{1}{e^{hc/\lambda kT} - 1}\, d\lambda} \qquad \cdots(10.21)$$

The **equations** (10.20) **and** (10.21) **are two forms of Planck's radiation law.** This equation explains successfully the experimental curves throughout the whole range of wavelength.

With the help of Planck's radiation law, we can derive Wien's law, Wien's displacement law, Rayleigh-Jeans law and Stefan-Boltzmann's law.