Chapter - 2

Linear Time - Invariant System (LTI System)

or

Linear Shift - Invariant System (LSI System)

Two most important attributes of systems are linearity and time invariance. In this chapter, we will develop the fundamental input-output relationship for systems having these attributes. The input output relationship of LTI system can be described by convolution operation. The importance of the convolution operation in LTI system is that the knowledge of the response of the LTI system to the unit impulse input allows us to find its output to any input signal. Some class of LTI system can be described by differential (for CT) and difference (for DT) equations also.

(A) Impulse response

The impulse response $h(t)$ of a continuous time LTI system is defined to be the response of the system when the input is $\delta(t)$

$$h(t) = T\{\delta(t)\}$$

(B) Response to an Arbitrary Input

The input can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau.$$

Since the system is linear, the response $y(t)$ of the system to an arbitrary input can be expressed as

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\, T\{\delta(t-\tau)\}\, d\tau.$$

Since the system is time-invariant, we have

$$h(t-\tau) = T\{\delta(t-\tau)\}$$

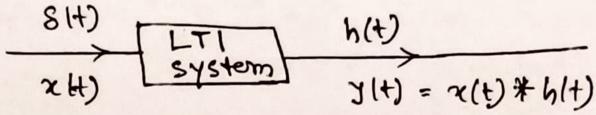
Then, $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

Note: A continuous time LTI system can be completely characterized by its impulse response $h(t)$.

© Convolution Integral

The convolution of two continuous time signals $x(t)$ and $h(t)$ is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$



The output of any continuous time LTI system is the convolution of the input signal ($x(t)$) with the impulse response $h(t)$ of the system.

Ⓓ Properties of Convolution Integral

1. Commutative $\qquad x(t) * h(t) = h(t) * x(t)$

Here,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

and

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau.$$

2. Distributive

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

3. Associative

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

Ⓔ Convolution Integral operation

Applying the commutative property.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau.$$

$$= x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$

Step-1

The impulse response is time reversed to obtain $h(-\tau)$ and then shifted by t to get $h(t-\tau)$

Step-2

The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of τ with t fixed at some value.

Step-3

The product $x(\tau)\,h(t-\tau)$ is integrated over all τ to produce a ~~sig~~ single output value of $y(t)$

Step-4

Step 1 to 3 are repeated as t varies over $-\infty$ to $+\infty$ to produce the entire output $y(t)$.

(F) **Step response**

The step response $s(t)$ of a Continuous time LTI system is defined to be the response of the system when the input is $u(t)$.

$$s(t) = T\{u(t)\}$$

In many applications, the step response $s(t)$ is also a useful characterization of the system.

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)\,u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau.$$

Thus, the step response $s(t)$ can be obtained by integrating the impulse response.

$$h(t) = s'(t) = \frac{ds(t)}{dt}.$$

Thus, the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

* **Properties of Continuous time LTI system**

(A) **System with or without memory**

Since the output $y(t)$ of a memoryless system depends on only the present input $x(t)$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$= h(0)\,x(t) = K\,x(t)$$

$K \rightarrow$ gain or Constant.

Then, $h(t) = K\,\delta(t)$

$$\text{If } \begin{cases} h(t_0) \neq 0 & \text{for } t_0 = 0 \\ h(t_0) = 0 & \text{for } t_0 \neq 0 \end{cases} \quad \text{memoryless}$$

But if $h(t_0) \neq 0$ for $t_0 \neq 0$, the C.T system has memory.

(B) **Causality**

A causal system does not respond to an input event untill the event actually occurs. Therefore, for a causal continuous time LTI system,

$$h(t) = 0 \qquad t < 0$$

Applying the causality condition, the output of a causal continuous time LTI system is

$$y(t) = \int_{0}^{\infty} h(\tau)\,x(t-\tau)\,d\tau.$$

or

$$y(t) = \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau.$$

Any signal $x(t)$ is said to be causal if

$$x(t) = 0 \qquad t < 0$$

and anti causal if

$$x(t) = 0 \qquad t > 0.$$

Thus, the response $y(t)$ of a causal input of a causal continuous time LTI system is

$$y(t) = \int_{0}^{t} h(\tau)\,x(t-\tau)\,d\tau = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau.$$

(C) **Stability**

The BIBO (Bounded input Bounded output) stability of an LTI system is readily ascertained from its impulse response.

$$|x(t)| \leq B_x < \infty \qquad \text{for all } t.$$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)\,x(t-\tau)|\,d\tau.$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau.$$

$$|B_y| \leq \int_{-\infty}^{\infty} |h(\tau)|\,B_x\,d\tau.$$

Thus, for BIBO stability, impulse response has to be absolutely integrable.

i.e. $\int_{-\infty}^{\infty} |h(\tau)|\,d\tau = B_h < \infty$

* Eigenfunctions of CT LTI system

$$T\{e^{st}\} = \lambda e^{st}$$

$\lambda \rightarrow$ eigen value.

let, $x(t) = e^{st}$.

$$y(t) = T\{e^{st}\} = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau\right] e^{st}$$

$$= H(s)\, e^{st} = \lambda\, e^{st}.$$

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau.$$

$$y(0) = \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau\right] e^{s.0} = H(s)$$

Thus, the eigen value of a CT LTI system associated with the eigen function e^{st} is given by $H(s)$ which is a complex constant

* System Described by Differential equation

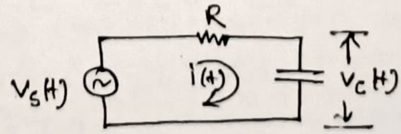
A general N th order linear constant - coefficient differential equation (LCCDE) is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

a_k, b_k are real constant.

Such differential equations play a central role in describing the input-output relationships of a wide variety of electrical, mechanical, chemical, Biological systems.

Example: Consider the RC circuit,



$x(t) = V_s(t)$

$y(t) = V_c(t)$

KVL

$$V_s(t) = V_R(t) + V_c(t)$$

$$V_s(t) = R\, i(t) + V_c(t) \rightarrow (\text{i})$$

Now, $$i(t) = C \frac{dV_c(t)}{dt} \rightarrow (\text{ii})$$

$$V_s(t) = RC \frac{dV_c(t)}{dt} + V_c(t)$$

$$x(t) = RC \frac{dy(t)}{dt} + y(t)$$

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Thus, the input-output relationship of RC circuit is described by a first order linear differential equation with constant coefficients.

The general solution

$$y(t) = y_p(t) + y_h(t)$$

$y_p(t) \rightarrow$ particular solution.

$y_h(t) \rightarrow$ homogeneous or complementary solution.

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

It gives homogeneous solution.

Impulse response of LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k h(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}$$

with the initial rest condition.

(*) Response of a discrete time LTI system and Convolution Sum

A. Impulse Response

The impulse response (or unit sample response) $h[n]$ of a discrete time LTI system is defined to be the response of the system when the input is $\delta[n]$

$$h[n] = T\{\delta[n]\}$$

B. Response to an Arbitrary Input

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

Since the system is linear, the response $y[n]$ of the system to an arbitrary input $x[n]$ is

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, T\{\delta(n-k)\}$$

Since the system is time-invariant,

$$h[n-k] = T\{\delta[n-k]\}$$

Then,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$$

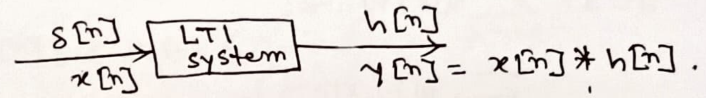
Note: A discrete time LTI system is Completely Characterized by its impulse Response $h[n]$.

(c) Convolution Sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

It is called Convolution Sum.

The output of any discrete time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$ of the system.



(D) Properties of convolution sum

(1) Commutative

$$x[n] * h[n] = h[n] * x[n]$$

where

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

and

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

(2) Distributive

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

(3) Associative

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * [h_1[n] * h_2[n]]$$

(E) Convolution Sum operation

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Step-1

The impulse response $h[k]$ is time reversed (i.e reflected about the origin) to obtain $h[-k]$, then shifted by n to form $h[n-k]$

Step-2

Two sequences $x[k]$ and $h[n-k]$ are multiplied together for all values of k, with n fixed at some value.

Step-3

The product $x[k]\, h[n-k]$ is summed over all k to produce a single output sample $y[n]$.

Step-4

Step 1 to 3 are repeated as n varies over $-\infty$ to $+\infty$ to produce entire output $y[n]$.

(F) Step response

The step response $s[n]$ of a discrete time LTI system with impulse response $h[n]$ is

$$s[n] = h[n] * u[n] = \sum_{k=-\alpha}^{\alpha} h[k]u[n-k]$$

$$= \sum_{k=-\alpha}^{n} h[k]$$

and $h[n] = s[n] - s[n-1]$

Properties of Discrete time LTI system

A. System with or without memory.

Since the output $y[n]$ of a memoryless system depends on only the present input $x[n]$.

$$y[n] = \sum_{k=-\alpha}^{\alpha} x(k)\, h(n-k)$$

$$= \sum_{k=-\alpha}^{\alpha} h(k)\, x(n-k)$$

$$= h[0]x[n] = K x[n]$$

where K is gain or constant.

Thus, impulse response

$$h[n] = K\delta[n]$$

For memory less

$$\begin{cases} h[n_0] \neq 0 & \text{for } n_0 = 0 \\ h[n_0] = 0 & \text{for } n_0 \neq 0 \end{cases}$$

Therefore,

if $h[n_0] \neq 0$ for $n_0 \neq 0$, the discrete time LTI system has memory.

B. Causality

$y[n]$ is causal if $h[n] = 0 \quad n<0$

$y[n]$ is anti-causal if $h[n] = 0 \quad n \geq 0$.

$y[n]$ is non-causal if $h[n] = 0 \quad |n| > N$

Applying the causality condition on system.

$$y[n] = \sum_{k=0}^{\alpha} h[k]\, x[n-k]$$

Applying causality condition on signal

$$y[n] = \sum_{k=0}^{n} h[k]\, x[n-k] = \sum_{k=0}^{n} x[k]\, h[n-k]$$

The signal $x[n]$ is causal if $x[n] = 0 \quad n<0$

The signal $x[n]$ is anti-causal if $x[n] = 0$ for $n \geq 0$

c. Stability

The input $x[n]$ is bounded if
$$|x[n]| \le B_x < \infty$$

Then,
$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x(n-k)\right|$$
$$\le \sum_{k=-\infty}^{\infty} |h[k]\, x[n-k]|$$
$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]|\, B_x$$

Therefore, if impulse response is absolutely summable,

i.e. $\sum_{k=-\infty}^{\infty} |h[k]| = K < \infty$

we have
$$|y[n]| \le K B_x < \infty$$

and the system is BIBO stable.

Eigenfunctions of DT LTI system

let $x[n] = z^n$ where z is a complex variable.

Then, $y[n] = T\{x[n]\} = T\{z^n\} = \lambda z^n$

where λ is called eigen value of the eigen function z^n.

$$y[n] = T\{z^n\} = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$
$$= \left[\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\right] z^n$$
$$y[n] = H(z)\, z^n$$

where, $\lambda = H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$ → a complex constant.

System Described by Difference equation

The role of differential equation in CT System is played by difference equation for DT system.

A. Linear Constant Coefficient Difference equation. (LCCDE)

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \longrightarrow (1)$$

where a_k, b_k are real constants.

The system is causal and LTI if the system is initially at rest.

B. Recursive formulation

Rearranging the equation ①, we get

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

It needs present input as well as past input and output to compute present output. If previous output is required to calculate present output is called recursive solution. (The equation is called recursive equation)

If $N=0$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

→ non-recursive equation.

(Since previous outputs are not required to calculate present output)

(C) Impulse response

① $$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k] - \sum_{k=1}^{N} a_k h[n-k]\right\}$$

(Recursive equation)

② $$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k]\right\}$$

(Non-recursive equation)

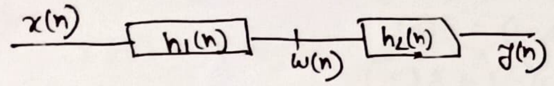
$$= \begin{cases} b_n/a_0 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Impulse response of equation ② has finite terms. i.e. it is non-zero for only a finite time duration. Because of that, the system is called finite impulse response (FIR) system.

On the other hand, a system whose impulse response is non-zero for an infinite duration time is called Infinite impulse Response (IIR) System.

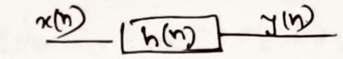
Some properties of LTI System

(1)

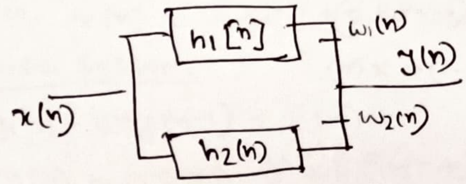


$$y(n) = w(n) * h_2(n) = [x(n) * h_1(n)] * h_2(n)$$
$$= x[n] * [h_1(n) * h_2(n)]$$
$$= x[n] * h(n)$$

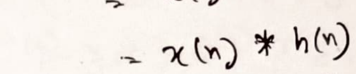
Then, $h(n) = h_1(n) * h_2(n)$



(2)



$$y(n) = w_1(n) + w_2(n)$$
$$= h_1(n) * x(n) + h_2(n) * x(n)$$
$$= x(n) * [h_1(n) + h_2(n)]$$
$$= x(n) * h(n)$$



$$h(n) = h_1(n) + h_2(n)$$

(3) $\delta(n) * x(n) = x(n)$

(4) convolution yields the zero state response of an LTI system.

(5) The response of LTI System to periodic signal is also periodic with identical period.

$$y(n) = h(n) * x(n) = \sum_{k=-\alpha}^{\alpha} h(k)\, x(n-k)$$

$$y(n+N) = \sum_{k=-\alpha}^{\alpha} h(k)\, x(n-k+N)$$

Put $n-k=m$

$$= \sum_{m=-\alpha}^{\alpha} h(n-m)\, x(m+N)$$

$$= \sum_{m=-\alpha}^{\alpha} h(n-m)\, x(m)$$

Put $m=k$

$$= \sum_{k=-\alpha}^{\alpha} h(n-k)\, x(k) = y(n) \quad \text{(Proved)}$$

Q. $y(n) - 0.4y(n-1) = x(n)$. Find the causal impulse response ($h(n) = 0,\ n<0$)

Ans: $h(n) - 0.4h(n-1) = \delta(n)$

$h(n) = 0.4h(n-1) + \delta(n)$

$n=0$ $\quad h(0) = 0.4h(-1) + \delta(0) = 1.$

$n=1$ $\quad h(1) = 0.4h(0) + \delta(1) = 0.4$

$n=2$ $\quad h(2) = 0.4h(1) + \delta(2) = 0.4^2$

Hence, $\quad h(n) = 0.4^n \quad$ for $n \geq 0$

Q. $y(n) - 0.4y(n-1) = x(n)$. Find the anti causal Impulse response ($h(n) = 0$ for $n \geq 0$)

Ans: $y(n) - 0.4y(n-1) = x(n)$

when $x(n) = \delta(n)$

$h(n) - 0.4h(n-1) = \delta(n)$

$h(n-1) = 2.5[h(n) - \delta(n)]$

$n=0$ $\quad h(-1) = 2.5[h(0) - \delta(0)] = -2.5$

$n=-1$ $\quad h(-2) = 2.5[h(-1) - \delta(-1)] = -2.5^2$

Hence, $h(n) = -2.5^n \quad$ for $n \leq -1$

Q. $y(n) = \{\underset{\uparrow}{1}, 2, 3\}$, $x(n) = \{\underset{\uparrow}{3}, 4\}$ Obtain the difference equation from input output information.

Ans: $y(n) + 2y(n-1) + 3y(n-2) = 3x(n) + 4x(n-1)$

Q. $x(n) = \{\underset{\uparrow}{4}, 4\}$, $y(n) = x(n) - 0.5x(n-1)$

Find the difference equation of the inverse system. Sketch the realization of each system and find the output of each system.

Ans: The original system $\quad y(n) = x(n) - 0.5x(n-1)$

The inverse system $\quad x(n) = y(n) - 0.5y(n-1)$

$y(n) = x(n) - 0.5x(n-1)$

z-transform

$Y(z) = X(z) - 0.5z^{-1}X(z)$.

$\dfrac{Y(z)}{X(z)} = 1 - 0.5z^{-1}$

The Inverse system.

$$x(n) = y(n) - 0.5\,y(n-1)$$

$\xrightarrow{z\text{-transform}}$

$$X(z) = Y(z) - 0.5\,z^{-1}Y(z)$$

$$\frac{Y(z)}{X(z)} = \left(1-0.5z^{-1}\right)^{-1}$$

The output of the original system

$$y(n) = 4\delta(n) - 2\delta(n-1) + 4\delta(n-1) - 2\delta(n-2)$$

$$= 4\delta(n) + 2\delta(n-1) - 2\delta(n-2) \longrightarrow ①$$

equation ① is the input of inverse system.

Hence, $x(n) = 4\delta(n) + 2\delta(n-1) - 2\delta(n-2)$

The inverse system

$$y(n) - 0.5\,y(n-1) = x(n)$$

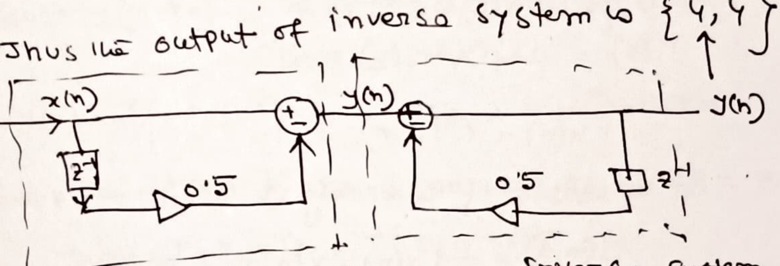
$$y(n) = 0.5\,y(n-1) + 4\delta(n) + 2\delta(n-1) - 2\delta(n-2)$$

$n=0$ $\quad y(0) = 0.5\,y(-1) + 4\delta(0) = 4$

$n=1$ $\quad y(1) = 0.5\,y(0) + 4\delta(1) + 2\delta(0) = 4$

$n=2$ $\quad y(2) = 0.5\,y(1) - 2\delta(0) = 0$

Thus the output of inverse system is $\{\underset{\uparrow}{4}, 4\}$



Invertible System. $\qquad$ Inverse System.

*

Non-Recursive filters	Recursive filters
$y(n) = \sum_{k=-\infty}^{\infty} a_k x(n-k)$	$y(n) = \sum_{k=0}^{M} a_k x(n-k) - \sum_{k=1}^{N} b_k y(n-k)$
for causal system $y(n) = \sum_{k=0}^{\infty} a_k x(n-k)$	present response is a function of the present and past M values of exitations as well as past N values of responses. It gives IIR output but not always.
for causal Input Sequence $y(n) = \sum_{k=0}^{M} a_k x(n-k)$	Ex $y(n) - y(n-1) = x(n) - x(n-3)$
Present Response depends only on present input and previous inputs but not future inputs. It gives FIR output.	

Q $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

find the given system is stable.

Ans

Let $x(n) = \delta(n)$

Then, $h(n) = \frac{1}{3}[\delta(n+1) + \delta(n) + \delta(n-1)]$

$h(0) = \frac{1}{3}[\delta(1) + \delta(0) + \delta(-1)] = \frac{1}{3}$

$h(-1) = \frac{1}{3}[\delta(0) + \delta(-1) + \delta(-2)] = \frac{1}{3}$

$h(-2) = \frac{1}{3}[\delta(-1) + \delta(-2) + \delta(-3)] = 0$

$h(1) = \frac{1}{3}[\delta(2) + \delta(1) + \delta(0)] = \frac{1}{3}$

$h(2) = \frac{1}{3}[\delta(3) + \delta(2) + \delta(1)] = 0$

$S = \sum h(n) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 < \infty$, therefore the system is stable.

Q. $y(n) = ay(n-1) + x(n)$, given $y(-1) = 0$

find the condition for stability.

Let, $x(n) = \delta(n)$

$h(n) = ah(n-1) + \delta(n)$

$h(0) = ah(-1) + \delta(0) = 1$

$h(1) = ah(0) + \delta(1) = a$

$h(2) = ah(1) + \delta(2) = a^2$

Hence,

$h(n) = a^n u(n)$

It is stable if and only if $a < 1$.

$y(n-1) = \frac{1}{a}[y(n) - x(n)]$

$y(-1) = \frac{1}{a}[y(0) - x(0)]$

$h(-1) = \frac{1}{a}[h(0) - \delta(0)] = 0$

$h(-2) = \frac{1}{a}[h(-1) - \delta(-1)] = 0$

Q. $y(n) = \frac{1}{n+1} y(n-1) + x(n)$ for $n \geq 0$

$= 0$ else.

find the given system is time invariant or not.

let, $x(n) = \delta(n)$

$$h(0) = 1y(-1) + \delta(0) = 1 = y(0)$$
$$h(1) = \frac{1}{2}h(0) + \delta(1) = \frac{1}{2} = y(1)$$
$$h(2) = \frac{1}{3}h(1) + \delta(2) = \frac{1}{6} = y(2)$$
$$h(3) = \frac{1}{4}h(2) + \delta(3) = \frac{1}{24} = y(3)$$

If $x(n) = \delta(n-1)$, then $y(n) = h(n-1)$

$$h(n-1) = \frac{1}{n+1}h(n-2) + \delta(n-1)$$

$n=0 \quad h(-1) = 1h(-2) + \delta(-1) = 0 = y(0)$

$n=1 \quad h(0) = \frac{1}{2}h(-1) + \delta(0) = 1 = y(1)$

$n=2 \quad h(1) = \frac{1}{3}h(0) + \delta(1) = \frac{1}{3} = y(2)$

$n=3 \quad h(2) = \frac{1}{4}h(1) + \delta(2) = \frac{1}{12} = y(3)$

$\therefore h(n,0) \neq h(n,1)$, hence TV.

Q. $y(n) = 2n\,x(n)$ find TV or TIV

Ans $y_1(n)$ is the response of $x_1(n) = x(n-n_0)$

$$y_1(n) = T\{x_1(n)\} = T\{x(n-n_0)\}$$
$$= 2n\,x(n-n_0)$$

But $y(n-n_0) = 2(n-n_0)\,x(n-n_0) \neq y_1(n)$

Hence TV.

Q. $y(n) = T\{x(k_0 n)\}$ find TV or TIV

Suppose $y_1(n)$ is the response of $x_1(n) = x(n-n_0)$

$$y_1(n) = T\{x_1(n)\} = T\{x(n-n_0)\}$$
$$= x(k_0 n - n_0)$$

But, $y(n-n_0) = x[k_0(n-n_0)] \neq y_1(n)$

Hence TV.

Q. $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$ linear.

Q. $y(n) = 12x(n-1) + 11x(n-2)$ TIV

Q. $y(n) = 7x^2(n-1)$ non-linear.

Q $y(n) = x^2(n)$ non-linear.

Q. $y(n) = n^2 x(n+2)$ linear.

Q. $y(n) = x(n^2)$ linear.

Q. $y(n) = e^{x(n)}$ non-linear.

Q $y(n) = 2^{x(n)}$ non-linear, TIV

Q $y(n) = \frac{1}{-} [\ldots$

Note: The roots of characteristic equation are of magnitude less than unity. It is the necessary and sufficient condition for stability.

Non-recursive system or FIR filters are always stable.

Q. $y(n) + 2y^2(n) = 2x(n) - x(n-1)$, non-linear, TIV

Q $y(n) - 2y(n-1) = 2^{x(n)} x(n)$ non-linear, TIV

Q $y(n) + 4y(n)y(2n) = x(n)$ non-linear, TV

Q $y(n+1) - y(n) = x(n+1)$ causal.

Q $y(n) - 2y(n-2) = x(n)$ causal.

Q. $y(n) - 2y(n-2) = x(n+1)$ non-causal.

Q $y(n+1) - y(n) = x(n+2)$ non-causal

Q $y(n-2) = 3x(n-2)$ static or Instantaneous

Q $y(n) = 3x(n-2)$ dynamic

Q. $y(n+4) - y(n+3) = x(n+2)$ causal & dynamic

Q. $y(n) = 2x(\alpha n)$

If $\alpha = 1$ causal, static

$\alpha < 1$ causal, dynamic

$\alpha > 1$ non-causal, dynamic

$\alpha \neq 1$ TV

Q $y(n) = 2(n+1)x(n)$ causal, static but TV

Q $y(n) = x(-n)$ TV.

Q $y(n) - 3y(n-1) - 4y(n-2) = 0$, Determine the zero input response of the system, given $y(-2) = 0$ & $y(-1) = 5$

Answer:- Let solution to the homogeneous equation be

$$y_h(n) = \lambda^n$$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

$$\lambda = -1, 4$$

$$y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n = c_1(-1)^n + c_2 4^n$$

$n = 0,$

$$y(0) = 3y(-1) + 4y(-2)$$
$$= 15 + 4.0 = 15.$$
$$c_1(-1)^0 + c_2 4^0 = 15$$
$$c_1 + c_2 = 15 \longrightarrow (1)$$

$$y(1) = 3y(0) + 4y(-1) = 3\times15 + 4\times5 = 65.$$
$$c_1(-1)^1 + c_2 4^1 = 65$$
$$-c_1 + 4c_2 = 65 \longrightarrow (11)$$

From (1) & (11)

$$5c_2 = 80$$
$$c_2 = 16$$
$$c_1 = 15 - 16 = -1.$$

Then, $y(n) = (-1)(-1)^n + 16\,4^n$
$$= (-1)^{n+1} + (4)^{n+2}$$

If it contains multiple roots

$$y_h(n) = c_1\lambda_1^n + c_2 n \lambda_1^n + c_2 n^2 \lambda_1^n$$

or

$$\lambda_1^n[c_1 + n c_2 + n^2 c_3]$$

Q. Determine the particular solution of
$$y(n) + a_1 y(n-1) = x(n)$$

Let $x(n) = u(n)$

Then, $y_p(n) = K u(n)$

$$K u(n) + a_1 K u(n-1) = u(n)$$

To determine the value of K, we must evaluate this equation for any $n \geq 1$

$$K + a_1 K = 1$$
$$K = \frac{1}{1+a_1}$$
$$y_p(n) = \frac{1}{1+a_1} u(n)$$

$x(n)$	$y_p(n)$
A	K
$A m^n$	$K m^n$
$A n^m$	$K_0 n^m + K_1 n^{m-1} + \dots + K_m$
$A \cos\omega_0 n$ or $A \sin\omega_0 n$	$K_1 \cos\omega_0 n + K_2 \sin\omega_0 n$
$\delta[n]$	0

Q. $y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$

$x(n) = 2^n. \quad n \geq 0$

let,

$y_p(n) = K 2^n u(n)$

$x(n) = 2^n \quad n \geq 0$

$= 2^n u(n)$

Then,

$K 2^n u(n) = \frac{5}{6} K 2^{n-1} u(n-1)$

$- \frac{1}{6} K 2^{n-2} u(n-2) + 2^n u(n)$

for $n \geq 2$

$4K = \frac{5}{6}(2K) - \frac{1}{6} K + 4$

$K = 8/5$

$$\boxed{y_p(n) = \frac{8}{5} 2^n u(n)}$$

Q. $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$. Find the $h(n)$ for recursive system. $[y(-1) = y(-2) = 0]$

we know that

$y_h(n) = c_1(-1)^n + c_2 4^n$ (According to Solution of page 16)

let $x(n) = \delta(n)$

then $y_p(n) = 0$ and $y_h[n] = h[n]$

When Input is Impulse signal, output is Impulse response Hence, $y_h[n] = h[n]$

for $n=0$

$y(0) - 3y(-1) - 4y(-2) = \delta(0) + 2\delta(-1)$

$y(0) = 1.$

Also, $y_h(0) = c_1(-1)^0 + c_2 4^0$

$c_1 + c_2 = 1 \longrightarrow$ (i)

for $n=1$ $\quad y(1) - 3y(0) - 4y(-1) = \delta(1) + 2\delta(0)$

$y(1) = 3 + 2 = 5.$

Also, $y_h(1) = c_1(-1)^1 + c_2 4^1$

$-c_1 + 4c_2 = 5 \longrightarrow$ (ii)

Hence, $c_1 = -\frac{1}{5}$, $c_2 = 6/5$.

$$\boxed{h[n] = \left[-\frac{1}{5}(-1)^n + \frac{6}{5} 4^n\right] u(n)}$$

Q $y(n) - 0.5y(n-1) = 5\cos 0.5n\pi$ $\quad n \geq 0$ with $y(-1) = 4$

Let

$$y_h(n) = \lambda^n$$

$$\lambda^n - 0.5\lambda^{n-1} = 0$$

$$\lambda^{n-1}(\lambda - 0.5) = 0$$

$$\lambda = 0.5$$

$$\therefore y_h(n) = c(0.5)^n$$

Let

$$y_p(n) = k_1 \cos 0.5n\pi + k_2 \sin 0.5n\pi$$

Then,

$$y_p(n-1) = k_1 \cos 0.5(n-1)\pi + k_2 \sin 0.5(n-1)\pi$$

$$= + k_1 \sin 0.5n\pi - k_2 \cos 0.5n\pi$$

So, $y_p(n) - 0.5y_p(n-1) = 5\cos 0.5n\pi$

$$(k_1 + 0.5k_2)\cos 0.5n\pi - (0.5k_1 - k_2)\sin 0.5n\pi = 5\cos 0.5n\pi$$

Comparing both sides

$$k_1 + 0.5k_2 = 5$$

$$0.5k_1 - k_2 = 0$$

Hence, $k_1 = 4$, $k_2 = 2$

$$y_p(n) = 4\cos 0.5n\pi + 2\sin 0.5n\pi$$

The total response

$$y(n) = c(0.5)^n + 4\cos 0.5n\pi + 2\sin 0.5n\pi$$

let $n = -1$

$$y(-1) = c(0.5)^{-1} + 4\cos 0.5(-\pi) + 2\sin 0.5(-\pi)$$

$$4 = 2c - 2$$

$$c = 3$$

Hence, $y(n) = 3(0.5)^n + 4\cos 0.5n\pi + 2\sin 0.5n\pi$ $\quad$ for $n \geq 0$

Q. $x(n) = a^n u(n)$ and $h(n) = a^n u(n)$ $\quad a < 1$ $\quad$ find $y(n)$

$$y(n) = \sum_{k=0}^{n} a^k a^{n-k} = \sum_{k=0}^{n} a^n = (n+1)a^n u(n).$$

Q. $x(n) = u(n)$ and $h(n) = \alpha^n u(n)$ $\quad \alpha < 1$ $\quad$ find $y(n)$

$$y(n) = \sum_{k=-\alpha}^{\alpha} \alpha^k u(k) u(n-k) = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Note - The Convolution of the left sided signals is also left sided and convolution of two right sided signals also right sided.

Q $x(n) = rect\left(\frac{n}{2N}\right) = 1 \quad |n| \leq N$

$= 0 \quad$ else

$h(n) = rect\left(\frac{n}{2N}\right) = 1 \quad |n| \leq N$

$= 0 \quad$ else

$y(n) = x(n) * h(n)$

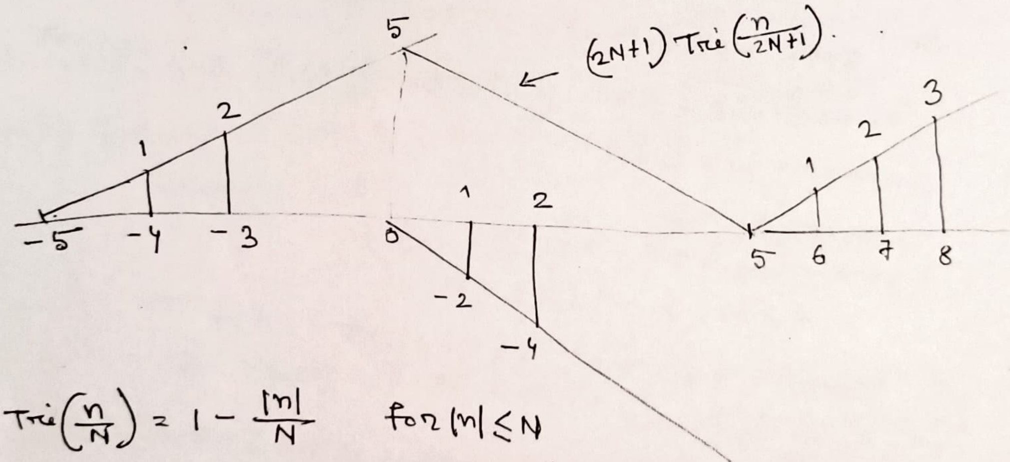
$= \{u(n+N) - u[n-N-1]\} * \{u(n+N) - u(n-N-1)\}$

$= u(n+N) * u(n+N) - u(n+N) * u(n-N-1)$

$\quad - u(n-N-1) * u(n+N) + u(n-N-1) * u(n-N-1)$

$= r(n+2N+1) - r(n) - r(n) + r(n-2N-1)$

for $N = 2$



$\circledast$ $Tri\left(\frac{n}{N}\right) = 1 - \frac{|n|}{N} \quad$ for $|n| \leq N$

$= 0 \quad$ else

Q. $x(n) = \{2, -1, 3\}$

$h(n) = \{1, 2, 2, 3\}$

	1	2	2	3
2	2	4	4	6
-1	-1	-2	-2	-3
3	3	6	6	9

$y(n) = \{2, 3, 5, 10, 3, 9\}$

Q $x(n) = \{4, \underset{\uparrow}{1}, 3\}$, $h(n) = \{2, 5, \underset{\uparrow}{0}, 4\}$

	2	5	0	4
4	8	20	0	16
1	2	5	0	4
3	6	15	0	12

$y(n) = \{8, 22, 11, \underset{\uparrow}{31}, 4, 12\}$

Q. $h(n) = 2\ 5\ 0\ 4$
$x(n) = 4\ 1\ 3$

$$\begin{array}{cccccc} 8 & 20 & 0 & 16 & & \\ & 2 & 5 & 0 & 4 & \\ & & 6 & 15 & 0 & 12 \end{array}$$

$y(n) = \{8, 22, 11, \underset{\uparrow}{31}, 4, 12\}$

Q. Convolution by sliding rule method

$$\begin{array}{cccccc} & & 2 & 5 & 0 & 4 \\ 3 & 1 & 4 & & & \\ \hline & & 8 & & & \end{array}$$

$$\begin{array}{ccccc} & 2 & 5 & 0 & 4 \\ 3 & 1 & 4 & & \\ \hline & \multicolumn{4}{c}{2 + 20 = 22} \end{array}$$

$$\begin{array}{cccc} 2 & 5 & 0 & 4 \\ 3 & 1 & 4 & \\ \hline \multicolumn{4}{c}{6 + 5 = 11} \end{array}$$

$$\begin{array}{cccc} 2 & 5 & 0 & 4 \\ & 3 & 1 & 4 \\ \hline \multicolumn{4}{c}{15 + 0 + 16 = 31} \end{array}$$

$$\begin{array}{ccccc} 2 & 5 & 0 & 4 & \\ & & 3 & 1 & 4 \\ \hline \multicolumn{5}{c}{0 + 4 = 4} \end{array}$$

$$\begin{array}{cccccc} 2 & 5 & 0 & 4 & & \\ & & & 3 & 1 & 4 \\ \hline & & & 12 & & \end{array}$$

$y(n) = \{8, 22, 11, \underset{\uparrow}{31}, 4, 12\}$

Note: If we insert zeroes between adjacent samples of each signal to be convolved, their convolution corresponding to the original convolution sequence with zeroes inserted between adjacent samples.

Q. $h(n) = \{2, 5, \underset{\uparrow}{0}, 4\}$ $x(n) = \{4, \underset{\uparrow}{1}, 3\}$

$X(z) = (4z^1 + 1.z^0 + 3z^{-1}) = (4z + 3z^{-1})$

$H(z) = (2z^2 + 5z + 4z^{-1})$

$Y(z) = X(z)H(z) = (4z + 3z^{-1})(2z^2 + 5z + 4z^{-1})$

$= 8z^3 + 20z^2 + 16 + 6z + 15 + 12z^{-2} + 2z^2 + 5z + 4z^{-1}$

$= 8z^3 + 22z^2 + 11z + 31 + 4z^{-1} + 12z^{-2}$

$y(n)$

~~{8, 22, 11, 31, 4, 12}~~

$= \{8, 22, 11, \underset{\uparrow}{31}, 4, 12\}$

Correlation of two Sequences

So far we have discussed about the convolution of two signals which is used to find the output $y[n]$ of a System for the given impulse response $h[n]$ and input signal $x[n]$. In this section, we will study a mathematical operation known as correlation that closely resembles convolution. Correlation is basically used to compare two signals. It occupies a significant place in signal processing.

Definition: Correlation is a measure of the degree to which two signals are similar.

The correlation of two signals is divided into

(1) Cross - correlation

(2) Auto - correlation.

(1) Cross - correlation:

The cross correlation between a pair of two signals $x[n]$ and $h[n]$ is given by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \rightarrow ①$$

$l = 0, \pm 1, \pm 2, \ldots$

The index l is the shift (lag) parameter. The order of subscripts xy indicates that $x[n]$ is the reference sequence that remains unshifted in time whereas the sequence $y[n]$ is shifted l units in time with respect to $x[n]$.

If we wish to fix $y[n]$ and shift $x[n]$ then the correlation of two sequence can be

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l) \rightarrow ②$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \rightarrow ③$$

If the time shift $l=0$

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n) y(n)$$

From ① and ②, we get

$\gamma_{xy}(l) = \gamma_{yx}(-l)$, where $\gamma_{xy}(l)$ is the folded version of $\gamma_{xy}(-l)$ about $l=0$.

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(-(l-n)) = x(l) * y(-l)$$

Thus, correlation process is essentially the convolution of two sequences in which one of the sequence is reversed. Therefore, the same algorithm (procedure) can be used to compute convolution and correlation.

(ii) Auto-correlation : The auto correlation of a sequence is correlation of a sequence with itself. It is defined as

$$\gamma_{xx}(l) = \sum_{n=-\alpha}^{\alpha} x(n)\,x(n-l)$$

$$= \sum_{n=-\alpha}^{\alpha} x(n+l)\,x(n)$$

If $l=0$

$$\gamma_{xx}(0) = \sum_{n=-\alpha}^{\alpha} x^2(n)$$

(iii) Properties of cross-correlation and Auto-correlation

The linear combination of two sequences (finite energy) $x(n)$ and $y(n)$ is

$a_1, a_2 \rightarrow$ constant
$l \rightarrow$ time shift

$$a_1 x(n) + a_2\, y(n-l)$$

The energy of the signal is

$$E = \sum_{n=-\alpha}^{\alpha} |a_1 x(n) + a_2 y(n-l)|^2$$

$$= a_1^2 \sum_{n=-\alpha}^{\alpha} x^2(n) + 2a_1a_2 \sum_{n=-\alpha}^{\alpha} x(n)y(n-l) + a_2^2 \sum_{n=-\alpha}^{\alpha} y^2(n-l)$$

$$= a_1^2\,\gamma_{xx}(0) + 2a_1a_2\,\gamma_{xy}(l) + a_2^2\,\gamma_{yy}(0)$$

Let, $E_x = \gamma_{xx}(0)$ and $E_y = \gamma_{yy}(0)$

If E is finite

$$a_1^2\gamma_{xx}(0) + 2a_1a_2\,\gamma_{xy}(l) + a_2^2\,\gamma_{yy}(0) \geq 0$$

$$\left(\frac{a_1}{a_2}\right)^2\gamma_{xx}(0) + 2\frac{a_1}{a_2}\,\gamma_{xy}(l) + \gamma_{yy}(0) \geq 0$$

$$K^2\gamma_{xx}(0) + 2K\,\gamma_{xy}(l) + \gamma_{yy}(0) \geq 0$$

$$\begin{bmatrix} K & 1 \end{bmatrix} \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xy}(l) \\ \gamma_{xy}(l) & \gamma_{yy}(0) \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0$$

for finite value of K, the determinant is

$$\begin{vmatrix} \gamma_{xx}(0) & \gamma_{xy}(l) \\ \gamma_{xy}(l) & \gamma_{yy}(0) \end{vmatrix} \geq 0$$

$$\gamma_{xx}(0)\,\gamma_{yy}(0) - \gamma_{xy}^2(l) \geq 0$$

$$\gamma_{xy}(l) \leq \sqrt{\gamma_{xx}(0)\,\gamma_{yy}(0)} = \sqrt{E_x E_y}$$

If $y(n) = x(n)$

$$\gamma_{xy}(l) \leq \sqrt{E_x^2} = E_x = \gamma_{xx}(0)$$

At zero lag, the auto-correlation of a sequence attains its maximum value.

The normalized expression for $\gamma_{xx}(l)$ is

$$\rho_{xx}(l) = \frac{\gamma_{xx}(l)}{\gamma_{xx}(0)}$$

and
$$\rho_{xy}(l) = \frac{\gamma_{xy}(l)}{\sqrt{\gamma_{xx}(0)\,\gamma_{yy}(0)}}$$

$\rho_{xy}(l)$ is known as cross-correlation coefficient. Its value always lies between +1 to -1. A value 0 for cross correlation means no correlation.

(iv) Computation of Correlation:

1. Obtain sequence $y(n-l)$ by shifting the sequence right by a time lag l.
2. Multiply $y(n-l)$ with $x(n)$ and sum all values to obtain $\gamma_{xy}(l)$
3. Repeat 1 and 2 for all values of the lag l.

EX

Find cross-correlation of $x(n) = \{1, 2, 1, 1\}$ and $y(n) = \{1, 1, 2, 1\}$

Answer $x(n) = \{1, 2, 1, 1\}$, $y(l) = \{1, 1, 2, 1\}$, $y(-l) = \{1, 2, 1, 1\}$

	1	2	1	1
1	1	2	1	1
2	2	4	2	2
1	1	2	1	1
1	1	2	1	1

$\gamma_{xy}(l) = \{1, 4, 6, 6, 5, 2, 1\}$

Inverse system and deconvolution

A system is said to be invertible if the input to the system can be recovered from its output when inverse system is cascaded with the original system, the output is equal to the input of the original system.

$$\xrightarrow{x(n)} \boxed{\text{System } h[n]} \xrightarrow{y(n)} \boxed{\text{Inverse system } h^{-1}[n]} \xrightarrow{w(n)=x(n)}$$

EX
$$y(n) = a\,x(n)$$
$$w(n) = \frac{1}{a}\,y(n)$$

$$y(n) = x(n) * h(n)$$
$$w(n) = y(n) * h^{-1}(n)$$
$$= x(n) * h(n) * h^{-1}(n)$$
$$= x(n) * \delta(n)$$

where $h(n) * h^{-1}(n) = \delta(n)$

De-convolution

In certain applications, the knowledge of impulse response $h[n]$ and output $y[n]$ may be known and we may have to find the input applied to the system. The process of recovering $x[n]$ from $y[n] = x[n] * h[n]$ is known as deconvolution.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Assuming $y[n]$ and $h[n]$ are one sided finite sequences,

$$y[n] = \sum_{k=0}^{N} x[k]\, h[n-k]$$

$$y[0] = h[0]\, x[0]$$
$$y[1] = h[1]\, x[0] + h[0]\, x[1]$$
$$y[2] = h[2]\, x[0] + h[1]\, x[1] + h[0]\, x[2]$$
$$\vdots$$

$$\underbrace{\begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} h[0] & 0 & 0 & 0 & \cdots & 0 \\ h[1] & h[0] & 0 & 0 & \cdots & 0 \\ h[2] & h[1] & h[0] & 0 & \cdots & 0 \\ \cdots & & & & & \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \end{bmatrix}}_{X}$$

$$\therefore Y = HX$$
$$\Rightarrow X = H^{-1} Y$$

$$y[0] = h[0]\, x[0]$$
$$x[0] = \frac{Y[0]}{h[0]}$$

$$Y[1] = h[1]\, x[0] + h[0]\, x[1]$$
$$x[1] = \frac{Y[1] - h[1]\, x[0]}{h[0]}$$

$$x(2) = \frac{Y[2] - x[0]\, h[2] - x[1]\, h[1]}{h[0]}$$

Ex What is the input signal $x[n]$ that will generate the output sequence $y[n] = \{\underset{\uparrow}{1}, 5, 10, 11, 8, 4, 1\}$ for a system with impulse response $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$

Answer let length $h[n] = N_1 = 3$

and length $x[n] = N_2$

then length $y[n] = N_1 + N_2 - 1 = 7$

then, $N_2 = 5$.

$$x[n] = \frac{y[n] - \sum_{k=0}^{N-1} x[k]\, h[n-k]}{h[0]}$$

$$x[0] = \frac{y[0]}{h[0]} = \frac{1}{1} = 1$$

$$x[1] = \frac{y[1] - x[0]\, h[1]}{h[0]} = \frac{5 - 1\times 2}{1} = 3$$

$$x[2] = \frac{y[2] - \sum_{k=0}^{1} x[k]\, h[2-k]}{h[0]}$$
$$= \frac{y[2] - x[0]\, h[2] - x[1]\, h[1]}{h[0]} = \frac{10 - 1\times 1 - 3\times 2}{1} = 3$$