Chapter - 1

Introduction to Signals and systems

Signal: A signal is a function representing a physical quantity or variable and typically it contains information about the behavior or nature of the phenomenon.

or

A signal is defined as a function that conveys usefull information about the state or behaviour of a physical phenomenon. Signal is typically the variation with respect to an independent quantity like time, spatial variables (length, width, height etc)

Example

(1) Speech signal - plot of amplitude with respect to time [$x(t)$]

(2) Image - plot of intensity with spatial variable [$x(l,b)$] $l \rightarrow$ length, $b \rightarrow$ breadth

(3) video - plot of intensity with respect to spatial variables and time [$x(l,b,t)$]

* speech signal is one dimensional signal (1-D) as only one independent variable (t).
* Image - 2 dimensional signal (2-D)
* video - 3-D signal.

Classification of signals

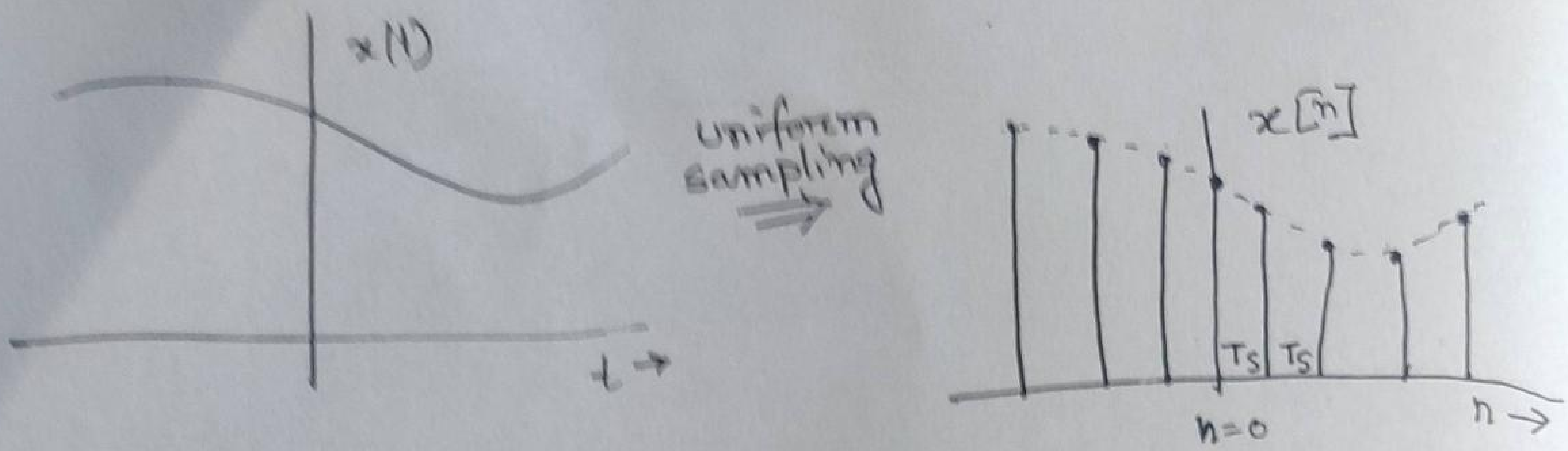
(A) Continuous time (CT) signal and discrete time (DT) signal.

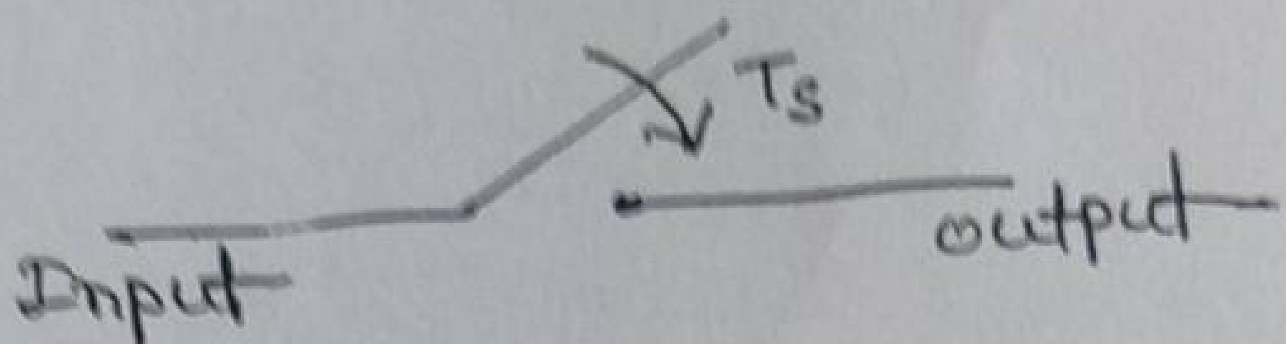
A signal $x(t)$ is a continuous time signal if t is a continuous variable (t can take any value between $-\infty$ to $+\infty$)

A signal $x[n]$ is called a discrete time signal if n takes specified or distinct values. Generally n is an integer.

Example CT - Voltage ($v(t)$), current ($I(t)$)
DT - daily stock market data.

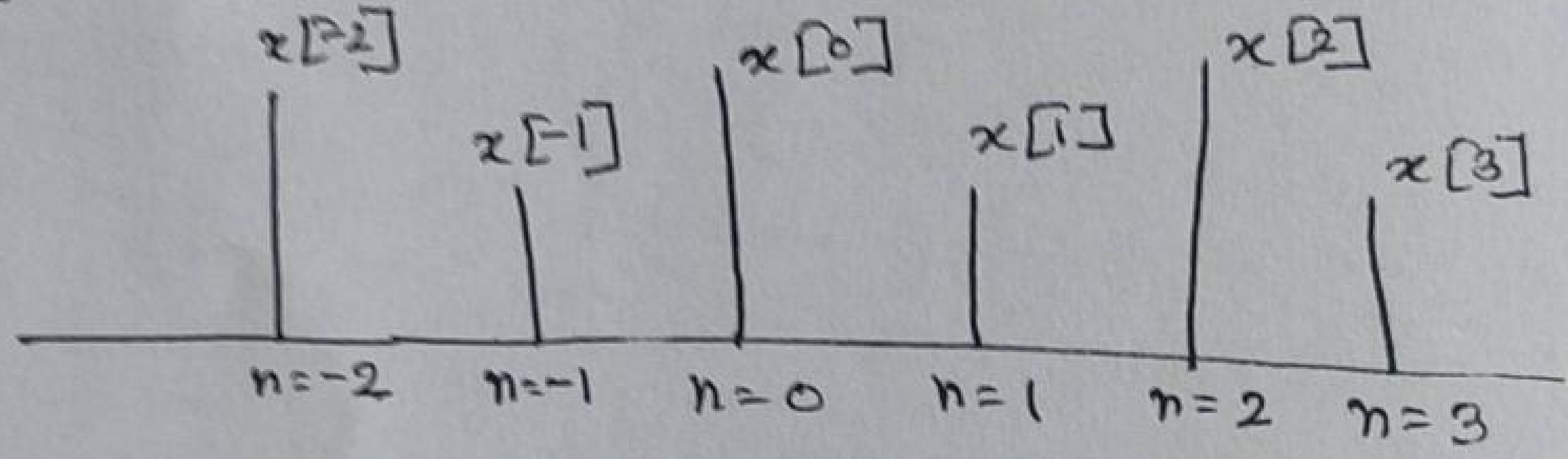
A discrete time signal ($x[n]$) may be obtained by the process called sampling of continuous time signal $x(t)$





$$x(t) \xrightarrow{\text{Sampling}} x[nT_s] = x[n]$$

T_s is fixed for uniform sampling and looses its significance in $x[nT_s]$, so we do not write T_s in DT signal. Rather we write $x[n]$, $n \to$ integers.



Here

$$x[n] = \{x[-2], x[-1], x[0], x[1], x[2], x[3]\}$$

$x[-2], x[-1], x[0]$ . . . . are called the samples of $x[n]$

$T_s \to$ sampling interval.

we can define $x[n]$ in two possible ways.

(1) specify a rule to calculate nth value of the sequence

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(2) we can list the values as a sequence

$$x[n] = \{ \text{-----} \; 0, 0, \underset{\uparrow}{1}, \tfrac{1}{2}, \tfrac{1}{4}, \text{----} \}$$

So, DT signals are some time called discrete time sequence.

(B) Analog signal and digital signal

If a continuous time (CT) signal $x(t)$ takes any value in the continuous interval (a,b), where $a > -\infty$ and $b < \infty$, then the continuous time signal $x(t)$ is called an analog signal.

Example $\quad x(t) = \sin(\omega t + \phi)$

If a discrete time signal $x[n]$ takes only a finite number of distinct values, then the discrete time (DT) signal is called a digital signal.
Digital signal is obtained from discrete time signal by quantization process.

Example $\quad x[n] = \sin(\omega n + \phi)$

(C) Real and complex signal

A signal $x(t)$ is a real signal if its all values are real numbers.

A signal $x(t)$ is a complex signal if at least one value is a complex number.

A complex signal $x(t)$ is of the form

$$x(t) = x_1(t) + j\, x_2(t)$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$

(D) Deterministic and random signal.

Deterministic signals are those signals whose values are completely specified for any given time. It is modeled by a known function of time t.

Random signals are those signals that take random values at any given time and must have probabilistic/statistical analysis.

(E) Even and odd signal

A signal $x(t)$ or $x[n]$ is an even signal if

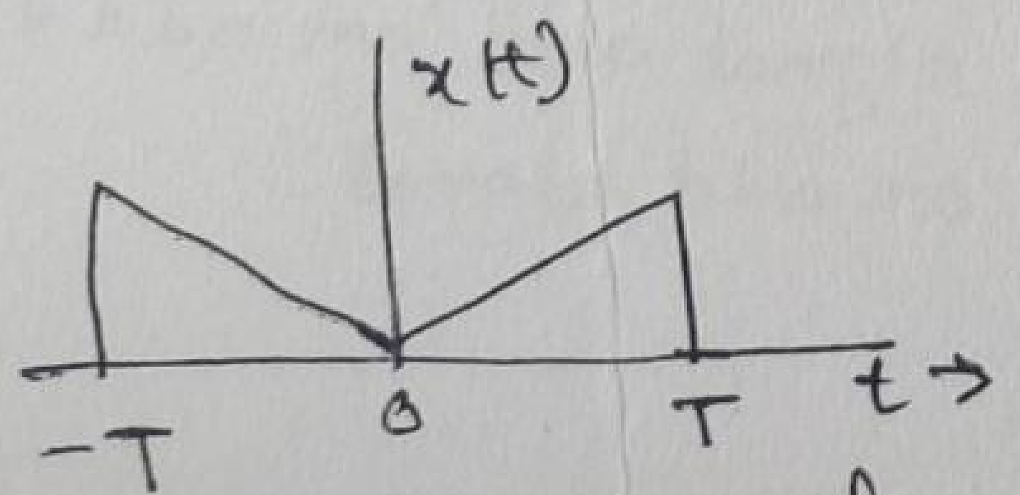
$$x(t) = x(-t) \quad - \text{CT}$$
$$x[n] = x[-n] \quad - \text{DT}$$

Example: $x(t) = \cos t$

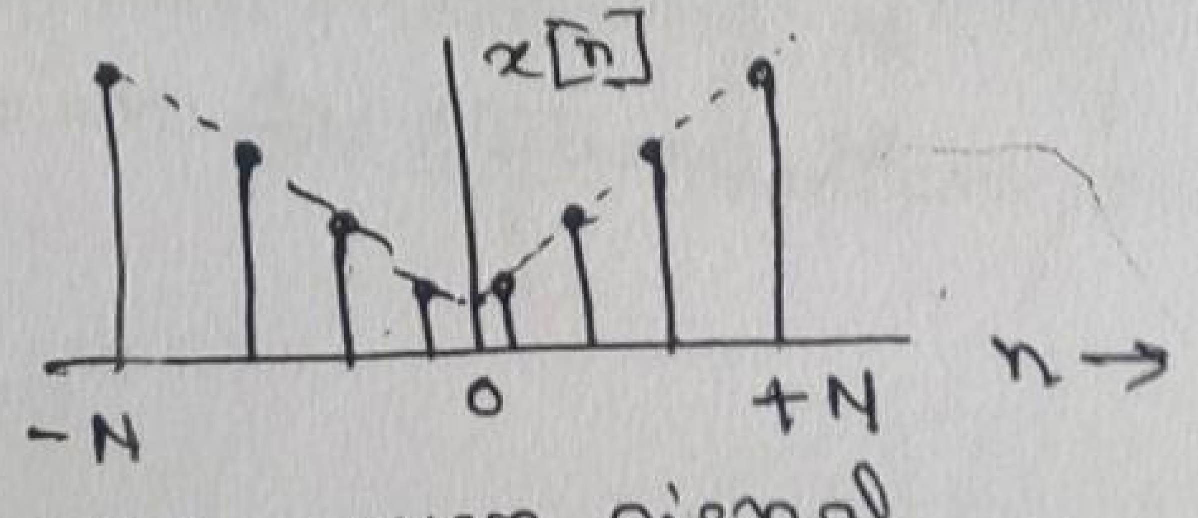
A signal $x(t)$ or $x[n]$ is an odd signal if

$$x(t) = -x(-t) \quad - \text{CT}$$
$$x[n] = -x[-n] \quad - \text{DT}$$

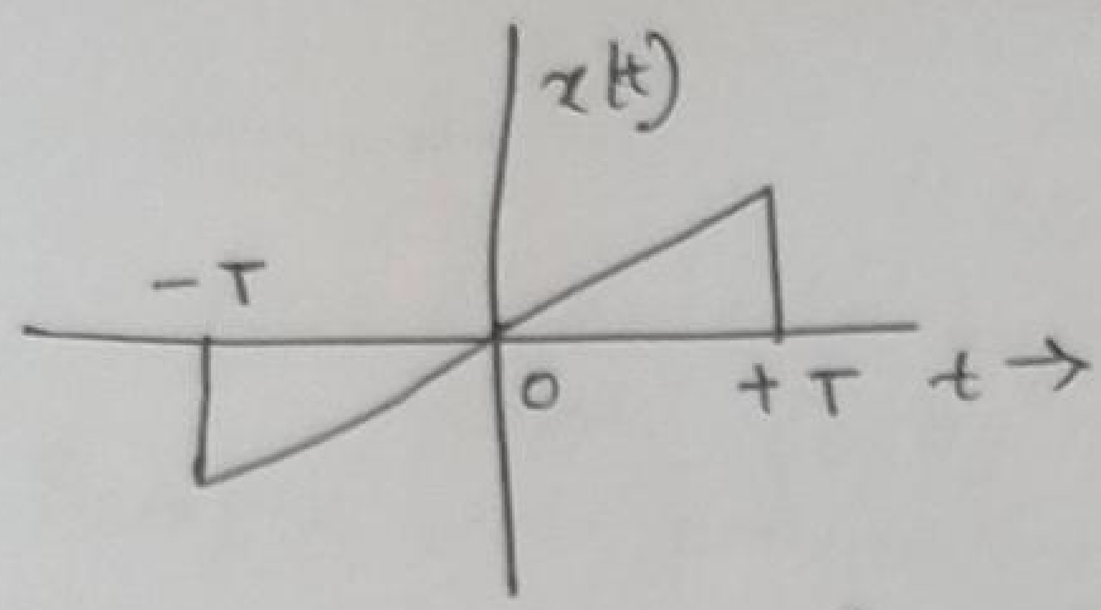
Example: $x(t) = \sin t$.



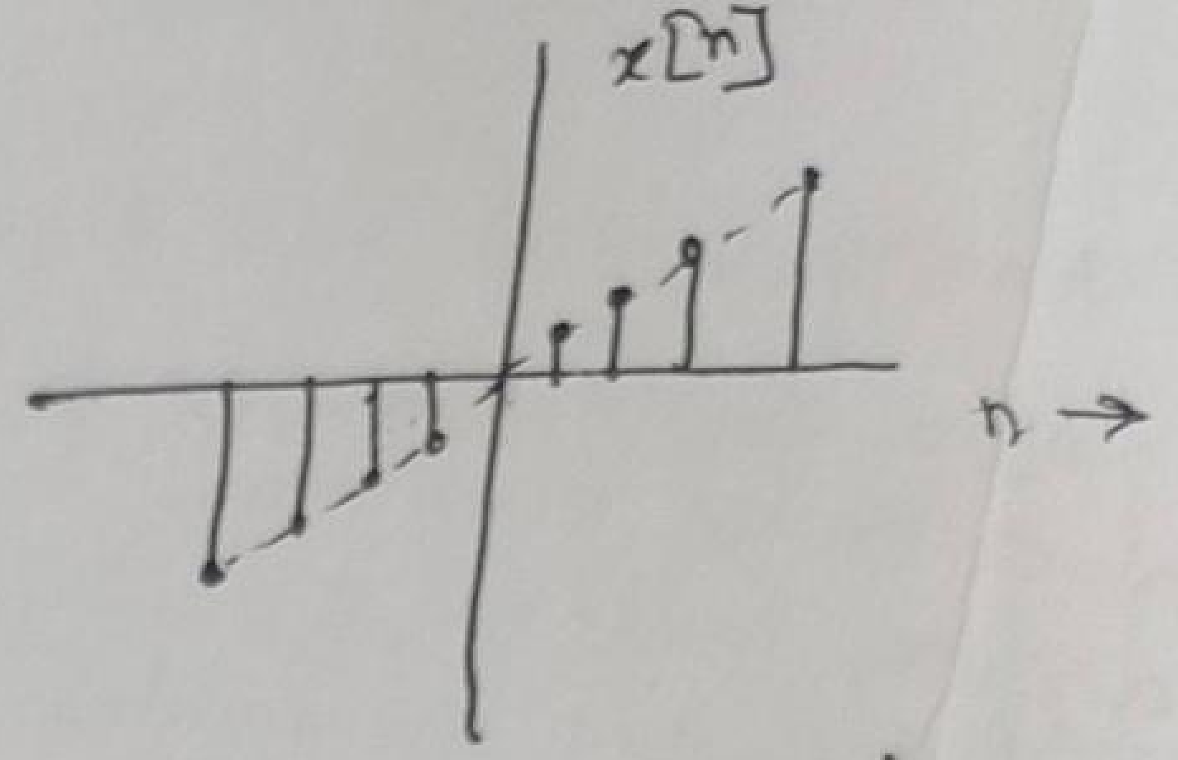
CT- even signal



DT - even signal



CT- odd signal



DT-odd signal.

Any real signal $x(t)$ or $x[n]$ can be expressed as a sum of two signals, one is an even signal and one is an odd signal.

$$x(t) = x_e(t) + x_o(t) \quad - CT$$
$$x[n] = x_e[n] + x_o[n] \quad - DT$$

$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ even part of $x(t)$

$x_e[n] = \frac{1}{2}\{x[n] + x[-n]\}$ even part of $x[n]$

$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$ odd part of $x(t)$

$x_o[n] = \frac{1}{2}\{x[n] - x[-n]\}$ odd part of $x[n]$

(F) Conjugate symmetry and conjugate skew symmetry

A signal $x(t)$ or $x[n]$ is said to be conjugate symmetric if $x(t) = x^*(-t)$ or $x[n] = x^*[-n]$, for all n

Also $x(t)$ is called conjugate skew symmetric if $x(t) = -x^*(-t)$ or $x[n] = -x^*[-n]$

Thus

$$x(t) = x_e(t) + x_o(t)$$

$x_e(t) = \frac{1}{2}[x(t) + x^*(-t)] = x_e^*(-t) \rightarrow$ conjugate symmetric part.

$x_o(t) = \frac{1}{2}[x(t) - x^*(-t)] = -x_o^*(-t) \rightarrow$ conjugate skew symmetric part.

* product of two even signals is an even signal
* product of two odd signals is an even signal
* product of one even signal and an odd signal is an odd signal.

(G). Periodic and non-periodic signal.

A CT signal $x(t)$ is said to be periodic with fundamental period T if there is a positive non-zero value of T for which

$$x(t+T) = x(t) \quad \text{for all } t.$$

More generally

$$x(t+mT) = x(t) \quad m \rightarrow \text{any integer}$$

The fundamental period T is the smallest positive value after which signal gets repeated.

If there is no such T, then the signal is called non-periodic (or aperiodic) signal.

Example

$x(t) = \sin t$ or $x(t) = \cos t$ — periodic

$x(t) = e^{t}$ or e^{-t} — non-periodic

For DT signal

$$x[n] = x[n+mN] \quad m \rightarrow \text{integer}$$

$N \rightarrow$ fundamental period.

* A sequence obtained by uniform sampling of a periodic continuous time signal may not be periodic.
* Sum of two continuous time periodic signals may not be periodic.
* Sum of two periodic sequences is always periodic.
* Even a non-periodic signal can be called periodic with period ∞.

CT

fundamental Period = T

Fundamental frequency $f = \frac{1}{T}$ cycles/sec or Hz.

Fundamental angular frequency $\Omega = \frac{2\pi}{T}$ radians/sec.

DT

Fundamental period = N

Fundamental frequency $F = \frac{1}{N}$ cycles/sample

Fundamental angular frequency $\omega = \frac{2\pi}{N}$ radians/sample

Note 1

If $x_1(t)$ and $x_2(t)$ are periodic signals with periods T_1 and T_2 respectively, then $x(t) = x_1(t) + x_2(t)$ is periodic if (if and only if) T_1/T_2 is a rational number and period of $x(t)$ is least common multiple (LCM) of T_1 and T_2.

Note-2

If $x_1[n]$ is periodic with fundamental period N and $x_2[n]$ is periodic with fundamental period M then $x[n] = x_1[n] + x_2[n]$ is always periodic with fundamental period equal to the least common multiple LCM of M and N.

Nature of the signal	Property	Condition
complex, CT	conjugate symmetry	$x(t) = x^*(-t)$
complex, CT	conjugate skew symmetry	$x(t) = -x^*(-t)$
Real, CT	Even signal	$x(t) = x(-t)$
Real CT	odd signal	$x(t) = -x(-t)$
complex, DT	conjugate Symmetry	$x[n] = x^*[-n]$
complex, DT	conjugate skew Symmetry	$x[n] = -x^*[-n]$
Real, DT	Even signal	$x[n] = x[-n]$
Real, DT	odd signal	$x[n] = -x[-n]$

(4) Energy and Power Signal

Signal	Energy and Power
CT, non-periodic	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$, $P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \lvert x(t)\rvert^2 dt$
CT, periodic	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$, $P = \frac{1}{T}\int_{-T/2}^{T/2} \lvert x(t)\rvert^2 dt$
DT, non-periodic	$E = \lim_{N\to\infty} \sum_{n=-N}^{N} \lvert x[n]\rvert^2$, $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n)\rvert^2$
DT, Periodic	$E = \sum_{n=-N}^{N} \lvert x(n)\rvert^2$, $P = \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n)\rvert^2$

A signal is called energy signal if $0<E<\infty$ and $P=0$
A signal is called Power signal if $0<P<\infty$ and $E\to\infty$

* Energy signal has zero average Power
* Power signal has infinite energy.
* Usually periodic signals and random signals are power signals.
* Usually deterministic and non periodic signals are energy signals.

Basic operations on Signal

Depending on nature of operation, different basic operations can be applied on dependent and independent variables of a signal.

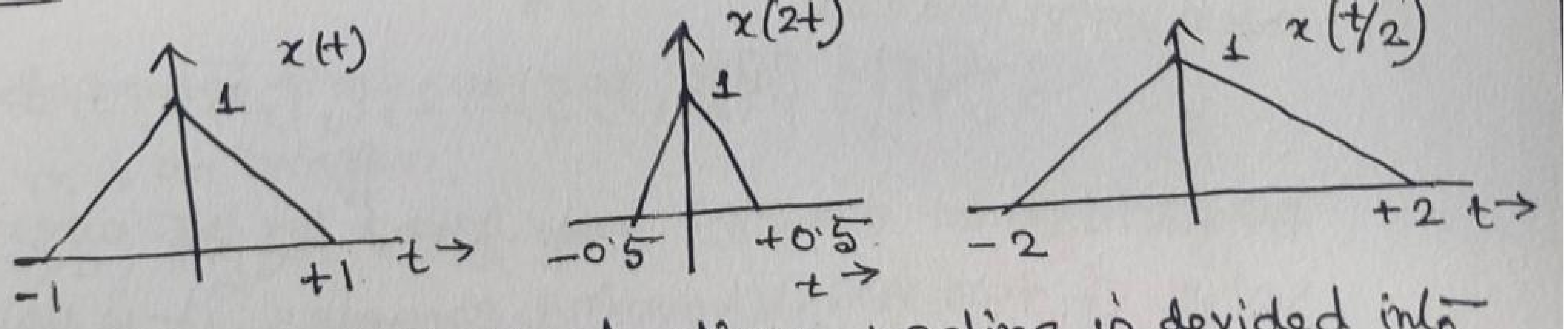
Operation	Continuous-time signal	Discrete time signal
Amplitude scaling	$y(t) = c\,x(t)$	$y[n] = c\,x[n]$
Addition	$y(t) = x_1(t) + x_2(t)$	$y[n] = x_1[n] + x_2[n]$
Multiplication	$y(t) = x_1(t)\,x_2(t)$	$y[n] = x_1[n]\,x_2[n]$
Differentiation	$y(t) = \frac{d}{dt}\{x(t)\}$	$y[n] = x[n] - x[n-1]$
Integration	$y(t) = \int_{T_1}^{T_2} x(t)\,dt$	$y[n] = \sum_{n=N_1}^{N_2} x[n]$

These are operations on dependent variable. Similarly, operations can be performed on independent variable of a signal.

I. Time scaling

For continuous time signals, $y(t)=x(at)$. If $a>1$, $y(t)$ is obtained by compressing the signal $x(t)$ along the time axis by 'a'. If $a<1$, $y(t)$ is obtained by expanding the signal $x(t)$ along time axis by 'a'.

Example



In discrete time (DT) signal, time scaling is devided into two parts. one is decimation and the other is interpolation.

For DT signal $x[n]$, the scaled version of it is given by $y(n)$, where $y(n) = x[kn]$ and k is scaling factor.

(1) **Decimation:** If value of $K > 1$, it would lead to reduction of samples from the original DT sequence $x[n]$. This process is known as Decimation or Down Sampling.

Example:

$$x[n] = \{4, 3, 2, \underset{\uparrow}{1}, 2, 3, 4\}$$

$$x[2n] = \{3, \underset{\uparrow}{1}, 3\}$$

$$x[3n] = \{4, \underset{\uparrow}{1}, 4\}$$

From the above example, it is clear that $x[2n]$ and $x[3n]$ are decimated sequences formed by every second and every third sample respectively of $x[n]$ starting from $n=0$.

Note: In the process of decimation, care has to be taken that sequence is not affected by Aliasing.

(2) **Interpolation:** If the value of $K < 1$, it would lead to increase in samples when compared to original sequence. This process is known as interpolation or Up Sampling.

$$x[n] = \{4, 3, 2, \underset{\uparrow}{1}, 2, 3, 4\}$$

$$x\left[\tfrac{n}{2}\right] = \{4, -, 3, -, 2, -, \underset{\uparrow}{1}, -, 2, -, 3, -, 4\}$$

If the extra samples are added by zero, it is known as zero interpolation.

$$x\left[\tfrac{n}{2}\right] = \{4, 0, 3, 0, 2, 0, \underset{\uparrow}{1}, 0, 2, 0, 3, 0, 4\}$$

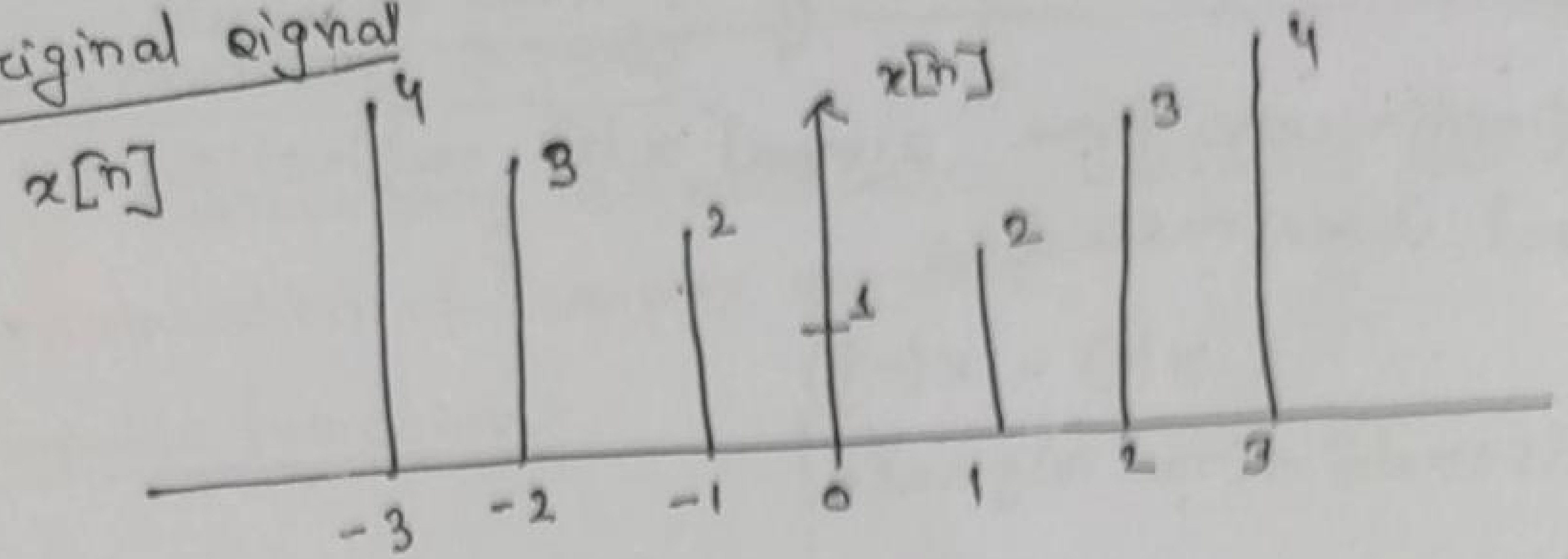
If extra samples retain previous value, it is called step interpolation.

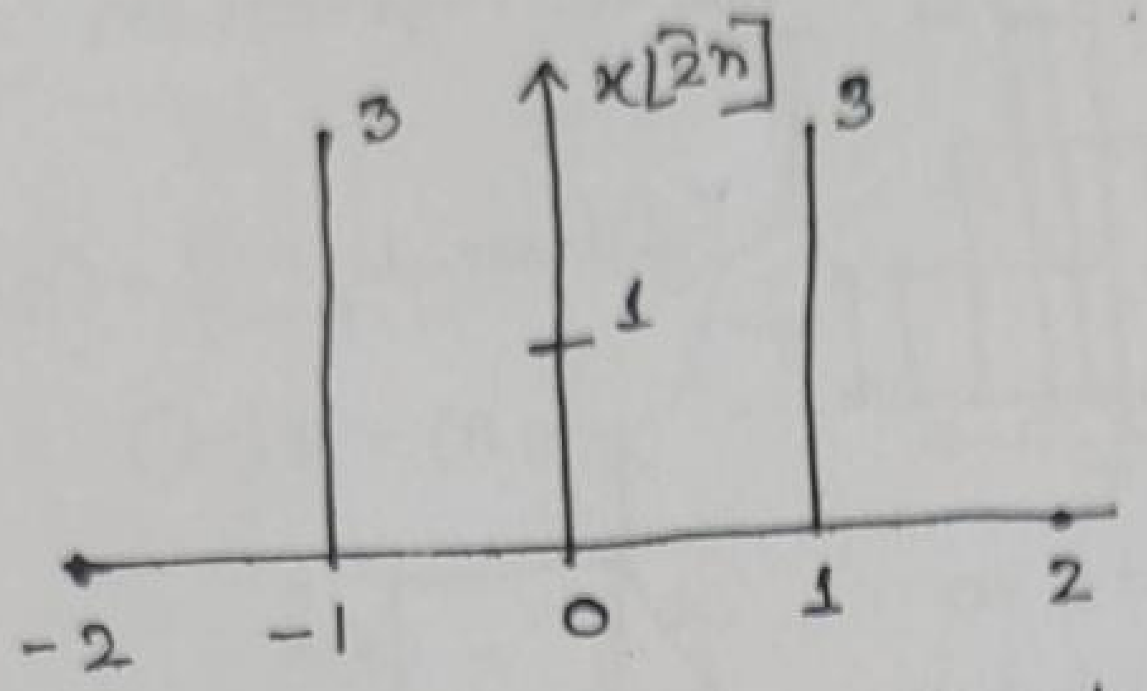
$$x[n/2] = \{4, 4, 3, 3, 2, 2, \underset{\uparrow}{1}, 1, 2, 2, 3, 3, 4\}$$

If the extra samples are filled by the average value of two neighbouring samples, it is called average or linear interpolation.

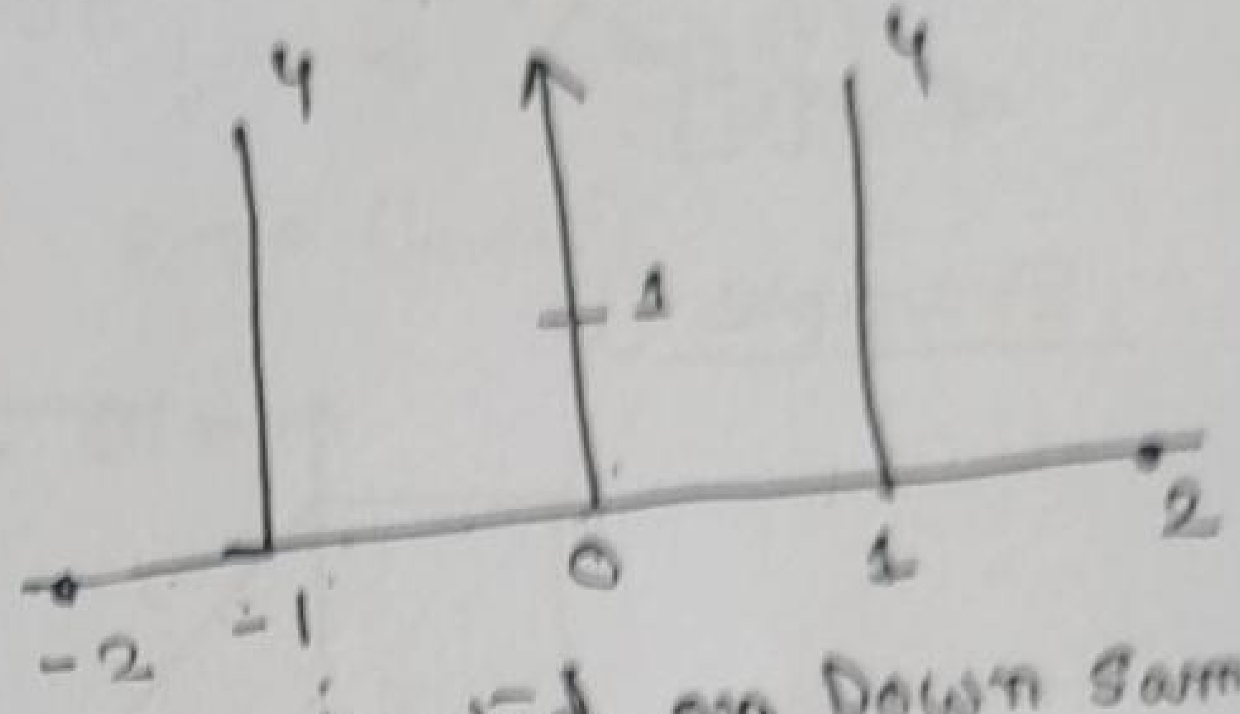
$$x\left[\tfrac{n}{2}\right] = \{4, 3.5, 3, 2.5, 2, 1.5, \underset{\uparrow}{1}, 1.5, 2, 2.5, 3, 3.5, 4\}$$

original signal

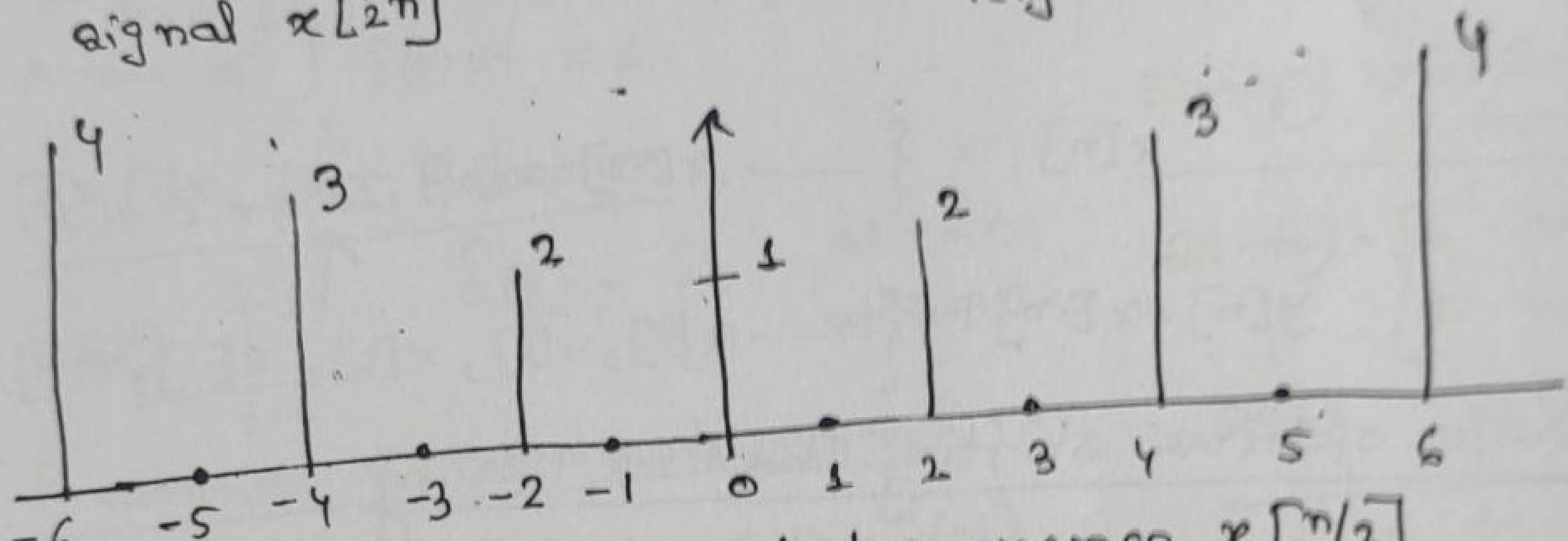




Decimated or Down Sampled signal $x[2n]$



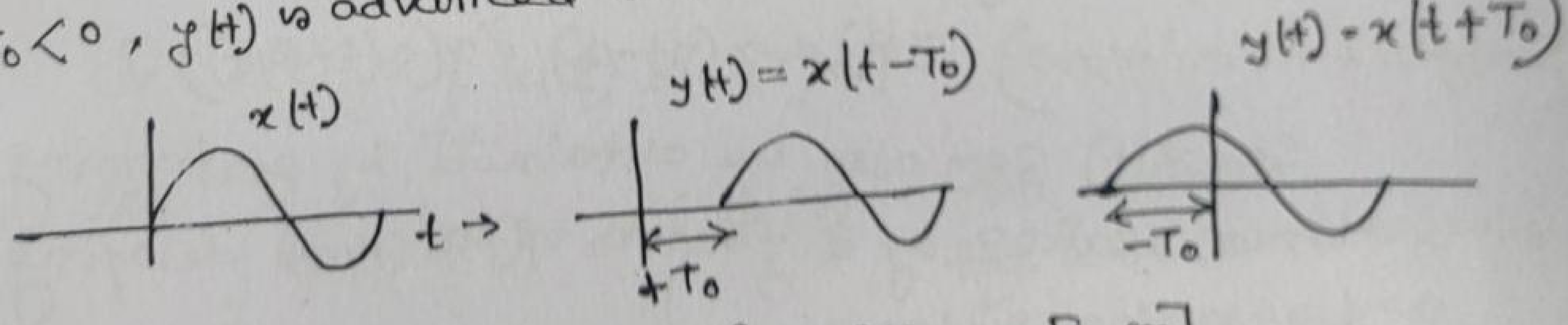
Decimated or Down Sampled signal $x[3n]$



Interpolated / Up sampled sequence $x[n/2]$
(Here is shown zero interpolation)

II. Time Shifting

For continuous time signal, $y(t) = x(t - T_0)$.
If $T_0 > 0$, $y(t)$ is delayed version of $x(t)$ and if ~~$T_0 < 0$~~
$T_0 < 0$, $y(t)$ is advanced version of $x(t)$.



For discrete time signal, $y[n] = x[n-m]$
where m is an integer.
If $m > 0$, signal $x[n]$ gets shifted to right by m (delayed by m samples)
If $m < 0$, signal $x[n]$ gets shifted left by m (advanced by m samples)

III Reflection or Transposing of Time variable

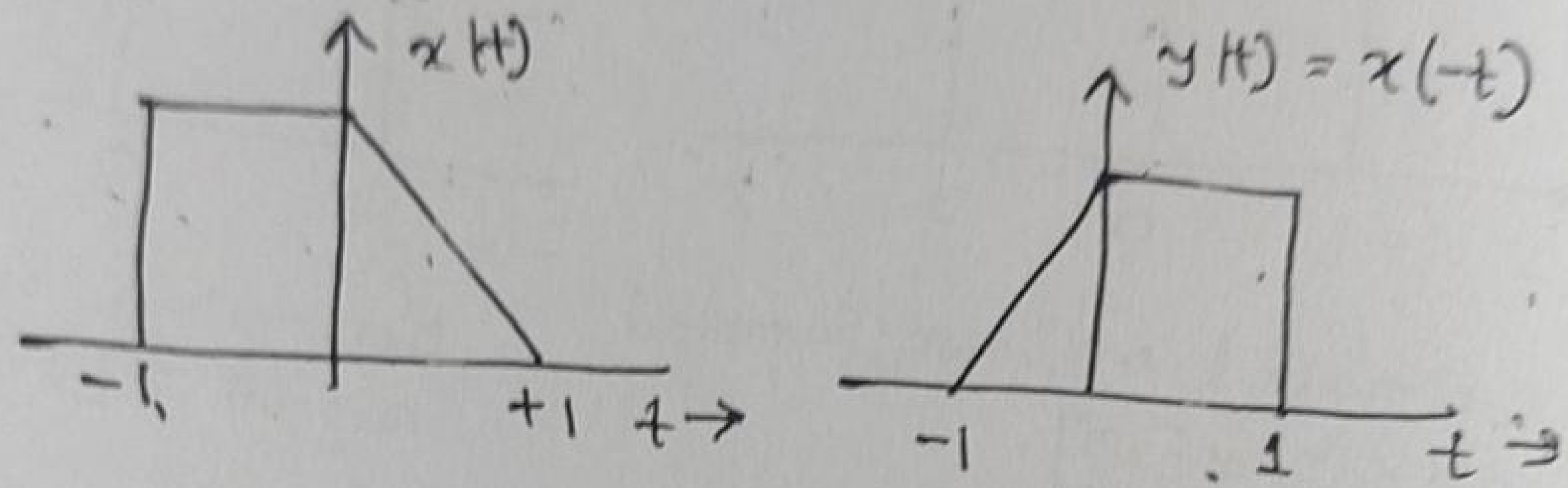
For continuous-time signal $x(t)$, reflection or Reversal is expressed as

$$y(t) = x(-t)$$

For discrete time signals.

$y[n] = x[-n]$ is the reflection of $x[n]$:

Example (1)



(2) $x[n] = \{ \cdots\cdots x[-2], x[-1], x[0], x[1], x[2] \cdots \}$
$\uparrow$

$y[n] = x[-n] = \{ \cdots\cdots x[2], x[1], x[0], x[-1], x[-2], \cdots \}$

Rule of time shifting and time scaling

Consider a signal $y(t)$ which is derived version of $x(t)$,

$$y(t) = x(at-b).$$

Here $y(t)$ is obtained by delaying $x(t)$ by 'b' samples and then performing time scaling by 'a'.
Alternatively, $y(t)$ can be written as

$$y(t) = x(at-b) = x(a(t-b/a)).$$

So, $y(t)$ can also be obtained by performing time scaling by a factor of 'a' and delaying by 'b/a' samples.

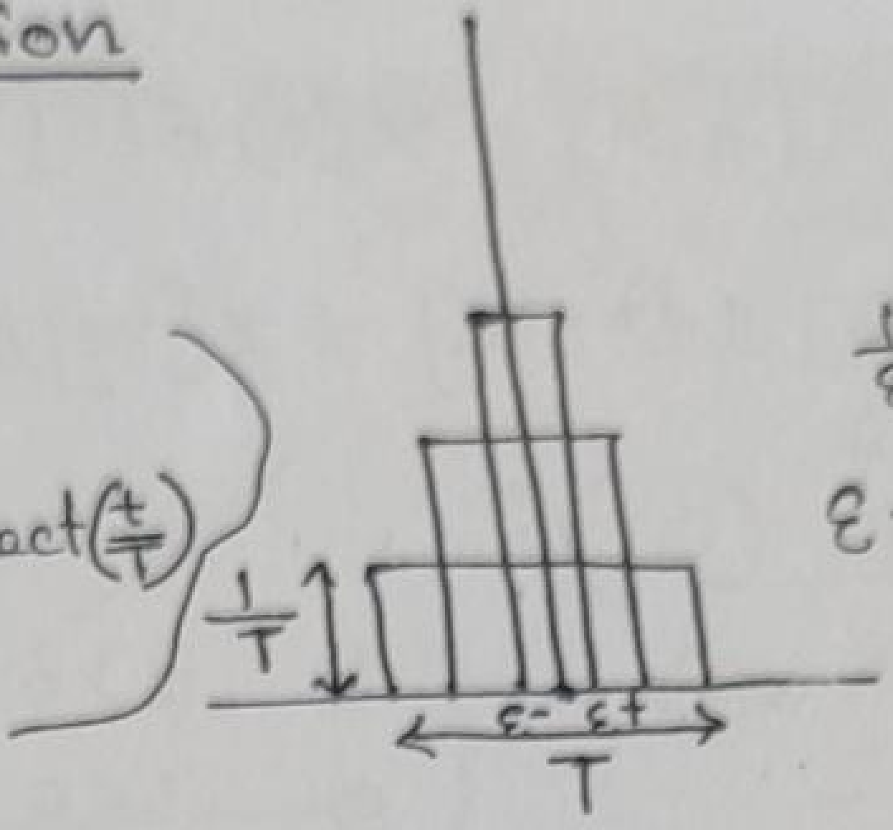
The same applies to discrete time signals also

Elementary Signals

Elementary signals are building blocks for the construction of complex signals.

Impulse function

$\delta(t) = \lim_{T \to 0} \frac{1}{T} \text{rect}\left(\frac{t}{T}\right)$



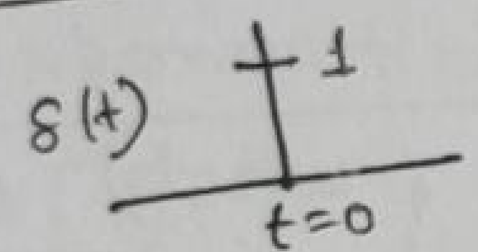
$\frac{1}{\varepsilon} \to \infty$ (height)

$\varepsilon \to 0$ (width)

$$\delta(t) = \begin{cases} \infty & \text{at } t = 0 \\ 0 & \text{at } t \neq 0 \end{cases}$$

$$\text{Area} = \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1.$$

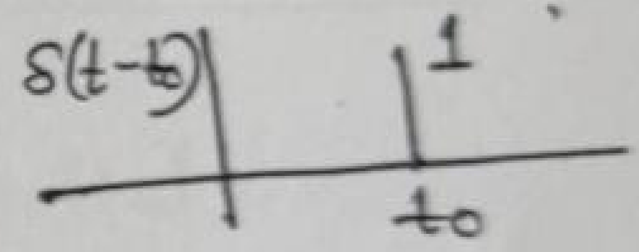
unit Impulse function



$$\delta(t) = \begin{cases} 1 & \text{at } t = 0 \\ 0 & \text{at } t \neq 0 \end{cases}$$

shifted unit Impulse function

$$\delta(t-t_0) = \begin{cases} 1 & \text{at } t = t_0 \\ 0 & \text{at } t \neq t_0 \end{cases}$$



Properties of impulse function

(1) $\delta(t)$ is even : $\delta(t) = \delta(-t)$

(2) $\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)\delta(t) = \phi(0)$

(3) $\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)\,dt = \phi(t_0)\delta(t-t_0) = \phi(t_0)$

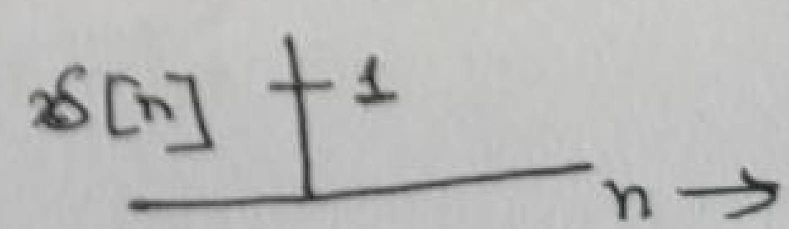
(4) $\delta(at) = \frac{1}{|a|}\delta(t)$

(5) $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$ (convolution Integral)

(6) $\int_a^b \phi(t)\,\delta(t) = \begin{cases} \phi(0) & \text{for } a<0<b \\ 0 & \text{for } a<b<0 \text{ or } 0<a<b \\ \text{undefined} & \text{for } a=0 \text{ or } b=0. \end{cases}$

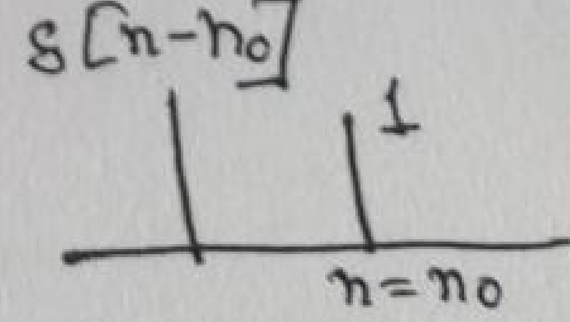
Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Shifted unit Impulse Sequence

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$



Note: Any arbitrary sequence can be generated by a sum of scaled, delayed impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \quad \text{(Convolution Sum)}$$

Properties

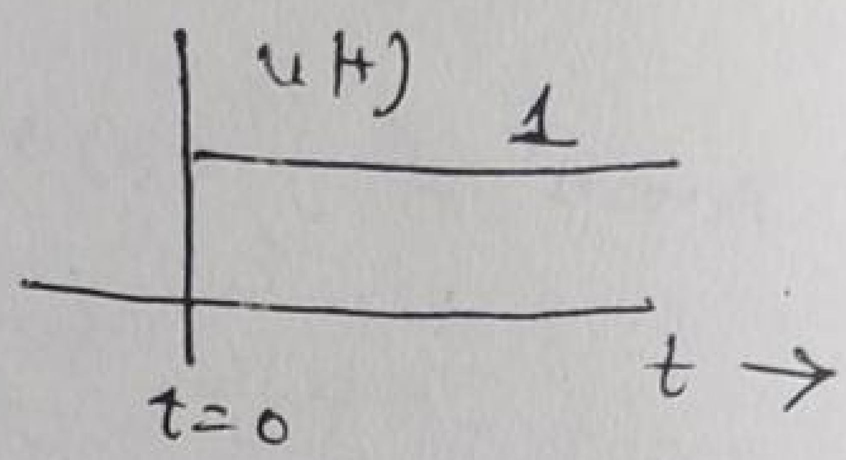
(1) $x[n]\,\delta[n] = x(0)\,\delta[n] = x[0]$

(2) $x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0] = x[n_0]$

Unit step function (CT):

The unit step function $u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

Note: $u(t)$ is discontinuous at $t=0$ and its value of $u(t)$ at $t=0$ is undefined.



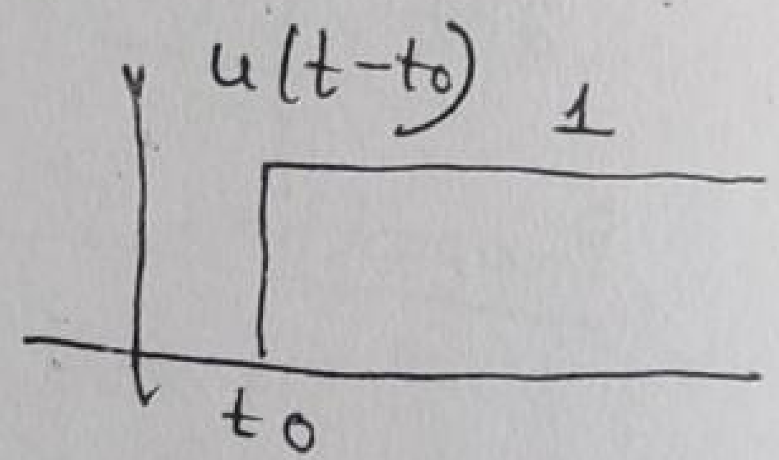
Property

(1) $\delta(t) = \frac{d}{dt}\{u(t)\}$

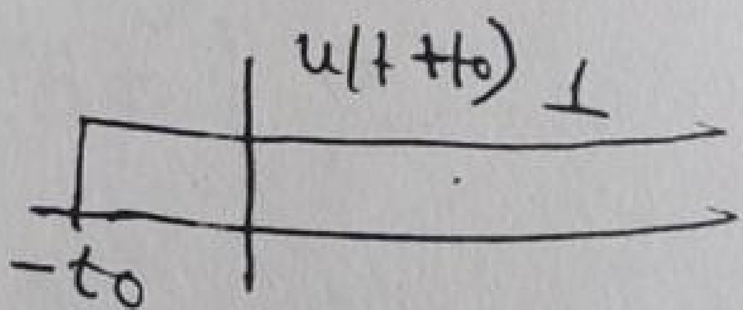
(2) $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau.$

Shifted unit step function

$u(t-t_0) = \begin{cases} 1 & t>t_0 \\ 0 & t<t_0 \end{cases}$

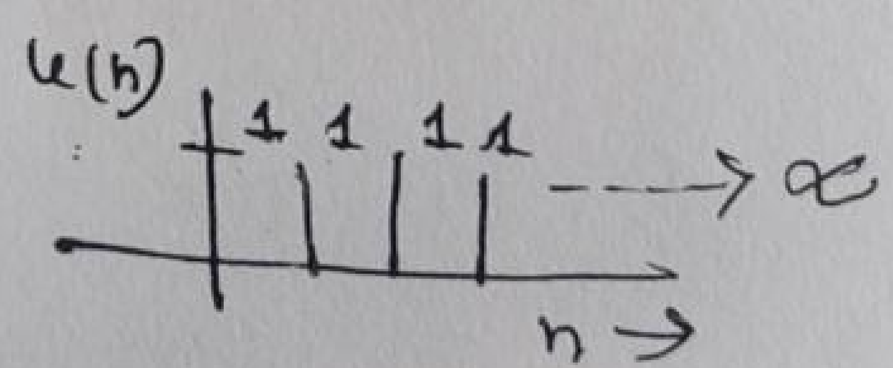


$u(t+t_0) = \begin{cases} 1 & t>-t_0 \\ 0 & t<-t_0 \end{cases}$



Unit step sequence (DT)

$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$



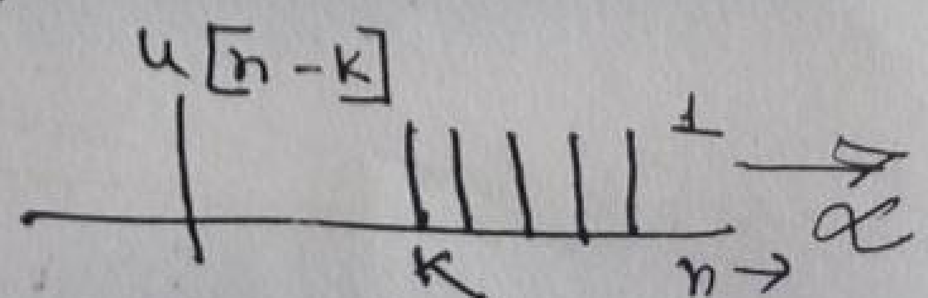
Property

(1) $\delta[n] = u[n] - u[n-1]$

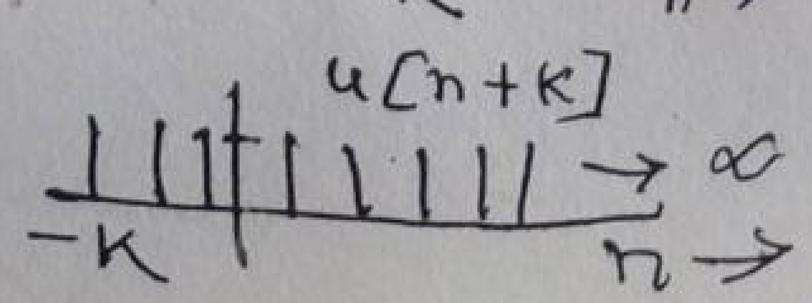
(2) $u[n] = \sum_{k=-\infty}^{n} \delta[k]$

Shifted unit step sequence

$u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n<k \end{cases}$



$u[n+k] = \begin{cases} 1 & n \geq -k \\ 0 & n<k \end{cases}$



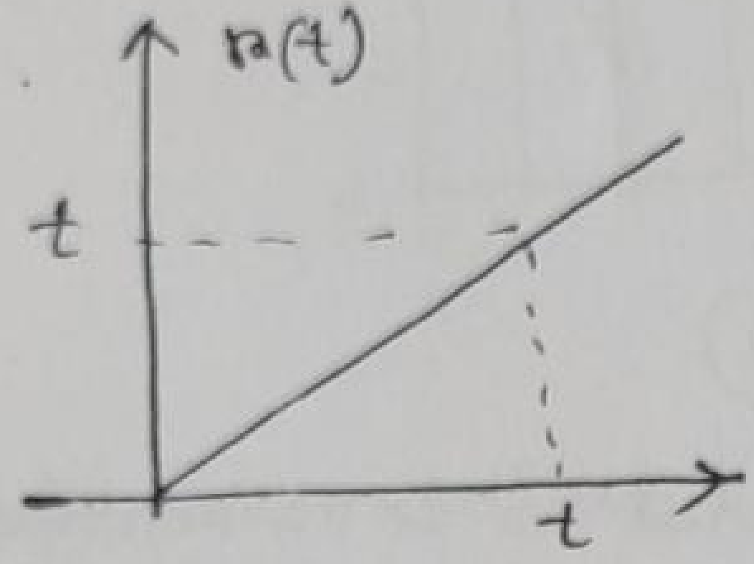
Sum)

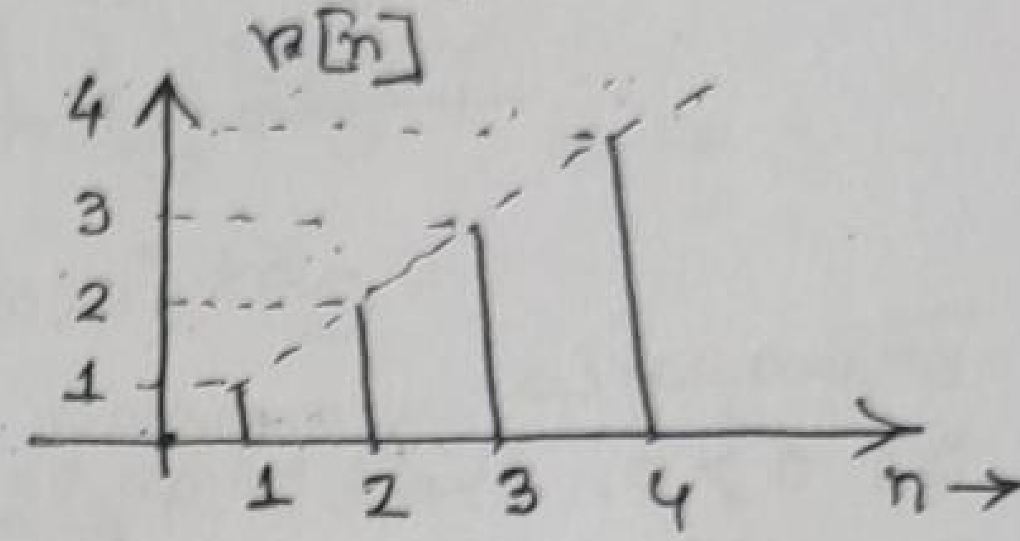
Ramp Function

$$r(t) = t\,u(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}$$

Discrete time ramp function

$$r[n] = n\,u[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$





Note

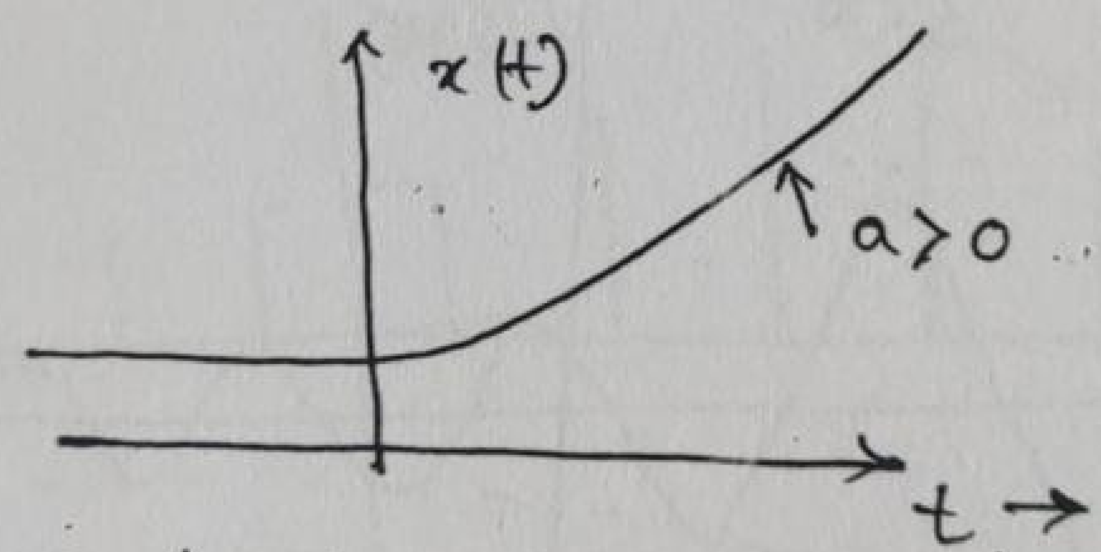
(1) $\frac{d}{dt}\{r(t)\} = u(t)$ and $\int_0^t u(t)\,dt = r(t)$

(2) $\frac{d^2}{dt^2}\{r(t)\} = \frac{d}{dt}\{u(t)\} = \delta(t)$

Exponential Signal

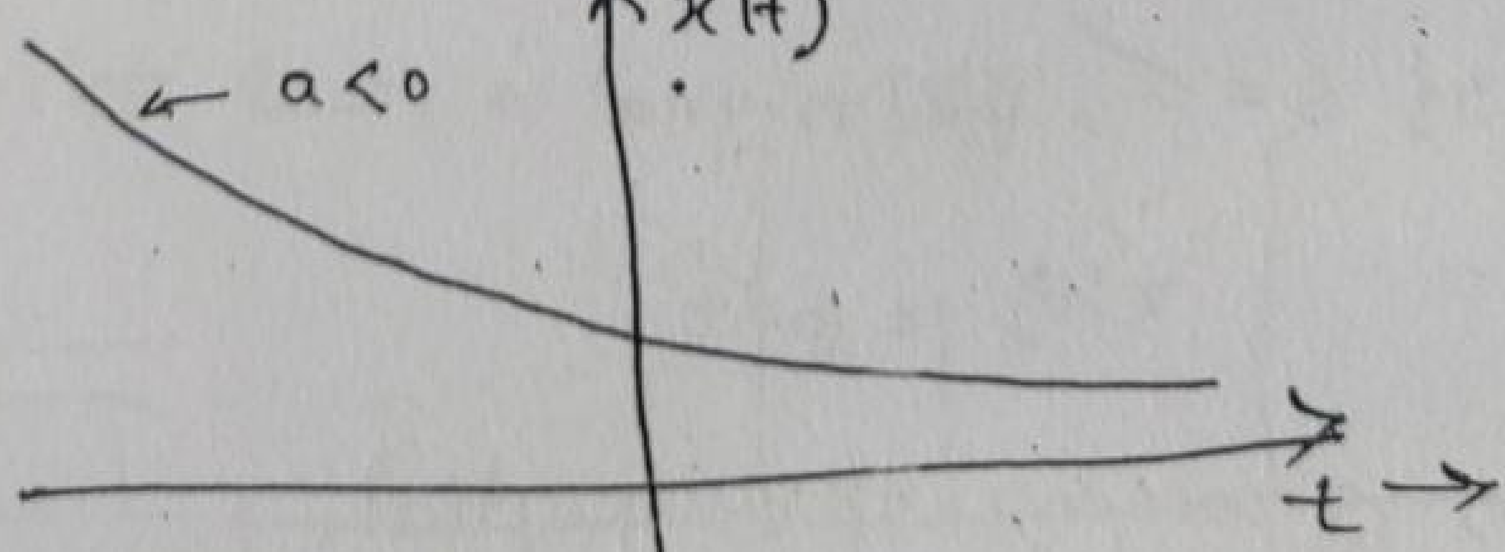
[CT] $x(t) = B\,e^{at}$, where B and a are real numbers.

If $a > 0$, $x(t)$ is growing exponential.



If $a < 0$, $x(t)$ is decaying exponential.

Here $\frac{1}{|a|}$ is called time constant.



Complex exponential Signal

$$x(t) = e^{j\omega t}$$

using Euler's formula

$$x(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Note : Complex exponential signals are periodic.
The fundamental period

$$T = \frac{2\pi}{\omega}$$

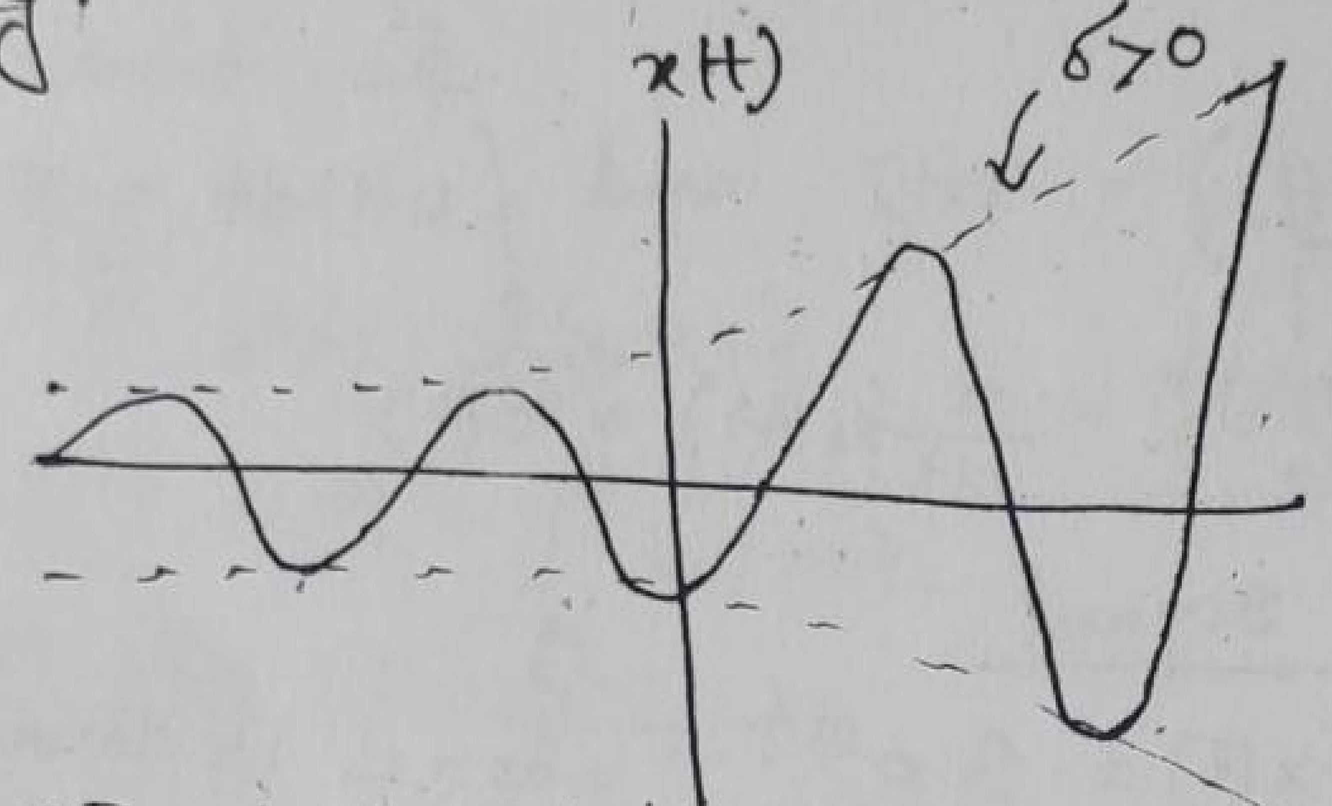
General complex exponential signal

$x(t) = e^{st}$ $s \rightarrow$ laplace variable.

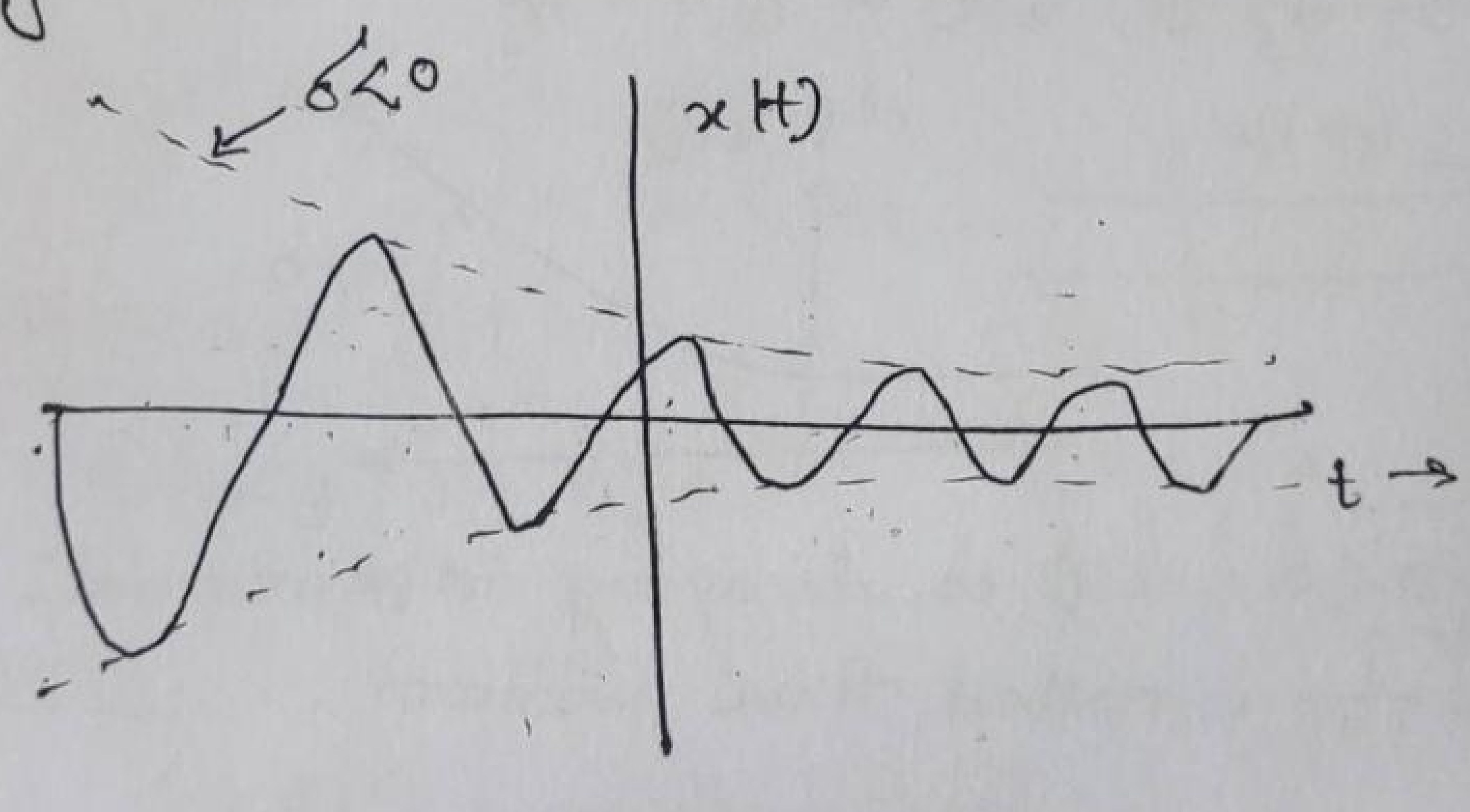
we know $s = (\sigma + j\omega)$

$$x(t) = e^{(\sigma + j\omega)t}$$
$$= e^{\sigma t} e^{j\omega t}$$
$$\therefore = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

If $\sigma > 0$, the sinusoidal signal is exponentially increasing.

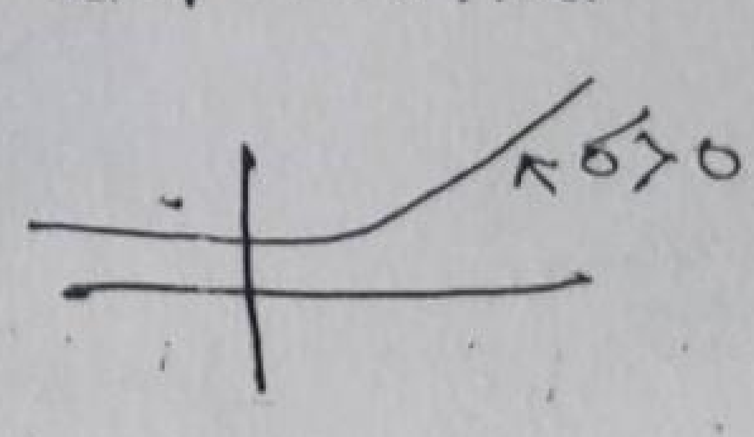


If $\sigma < 0$, the sinusoidal signal exponentially decreasing

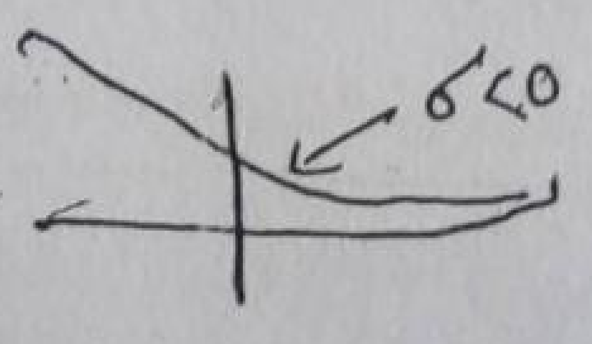


Note If $s = \sigma$, the signal is real exponential signal.

$$x(t) = e^{\sigma t}$$



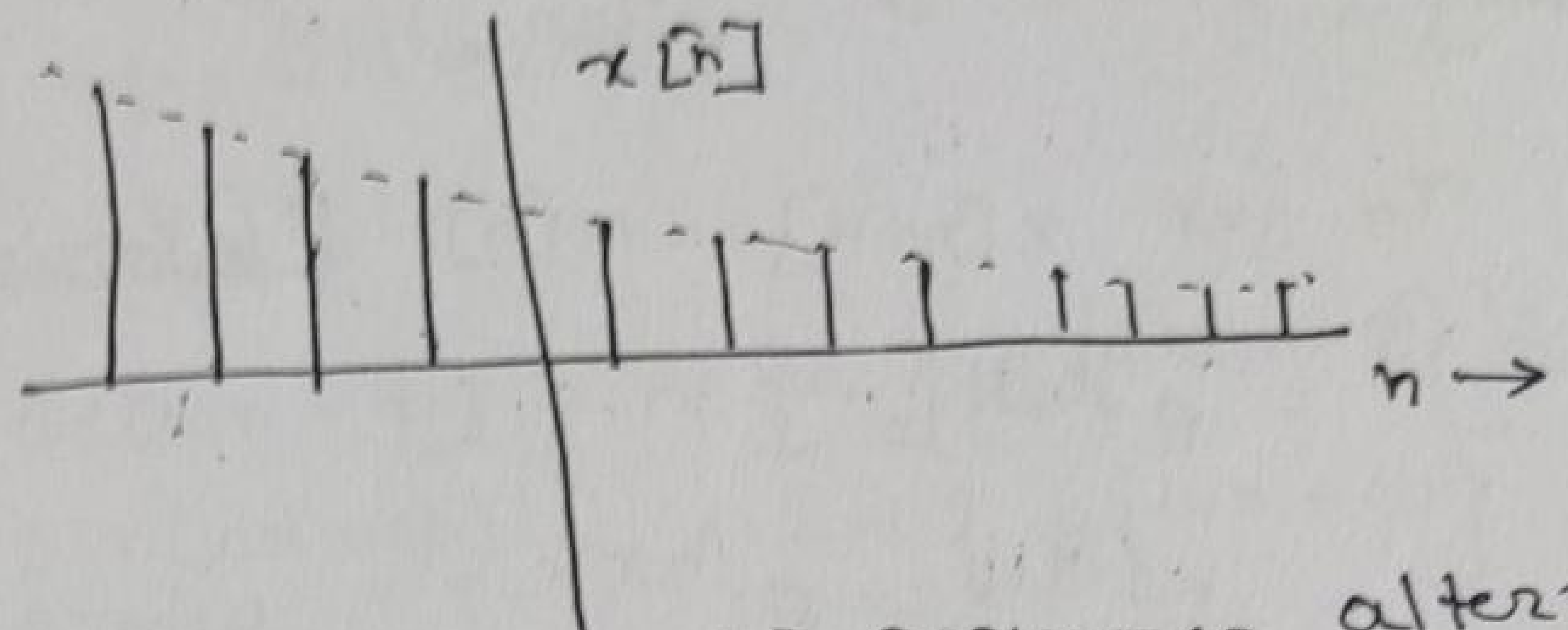
$\sigma > 0$, growing exponential.
$\sigma < 0$, decaying exponential



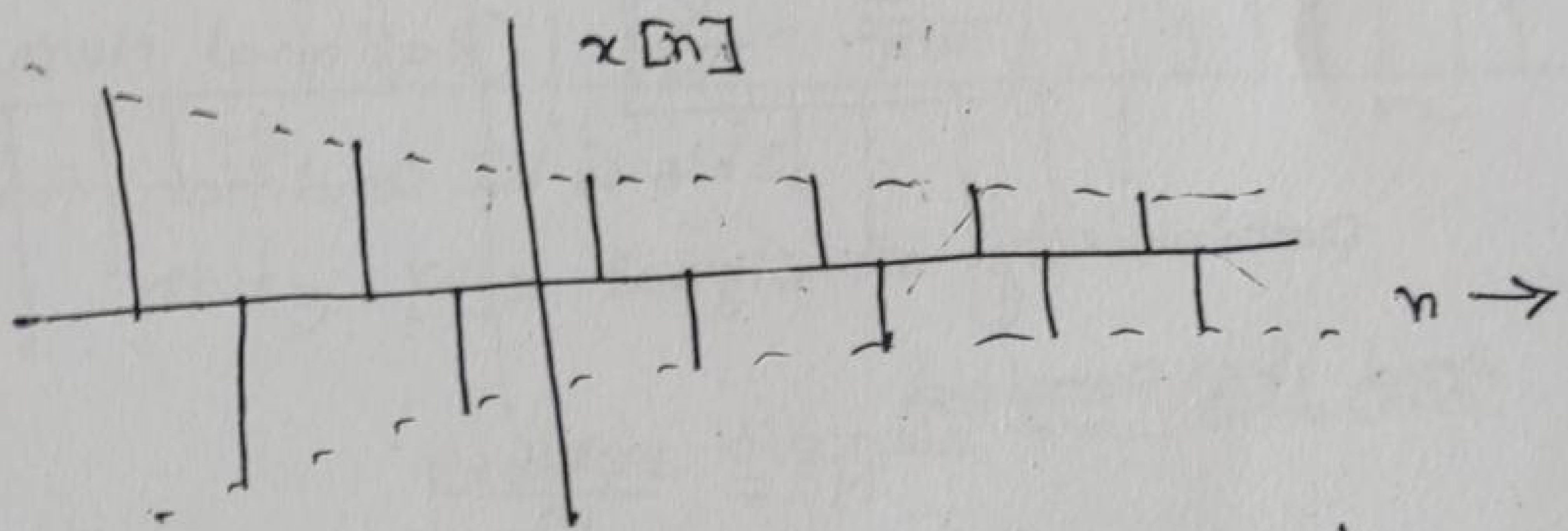
Exponential sequence (ii)

$$x[n] = B r^n$$

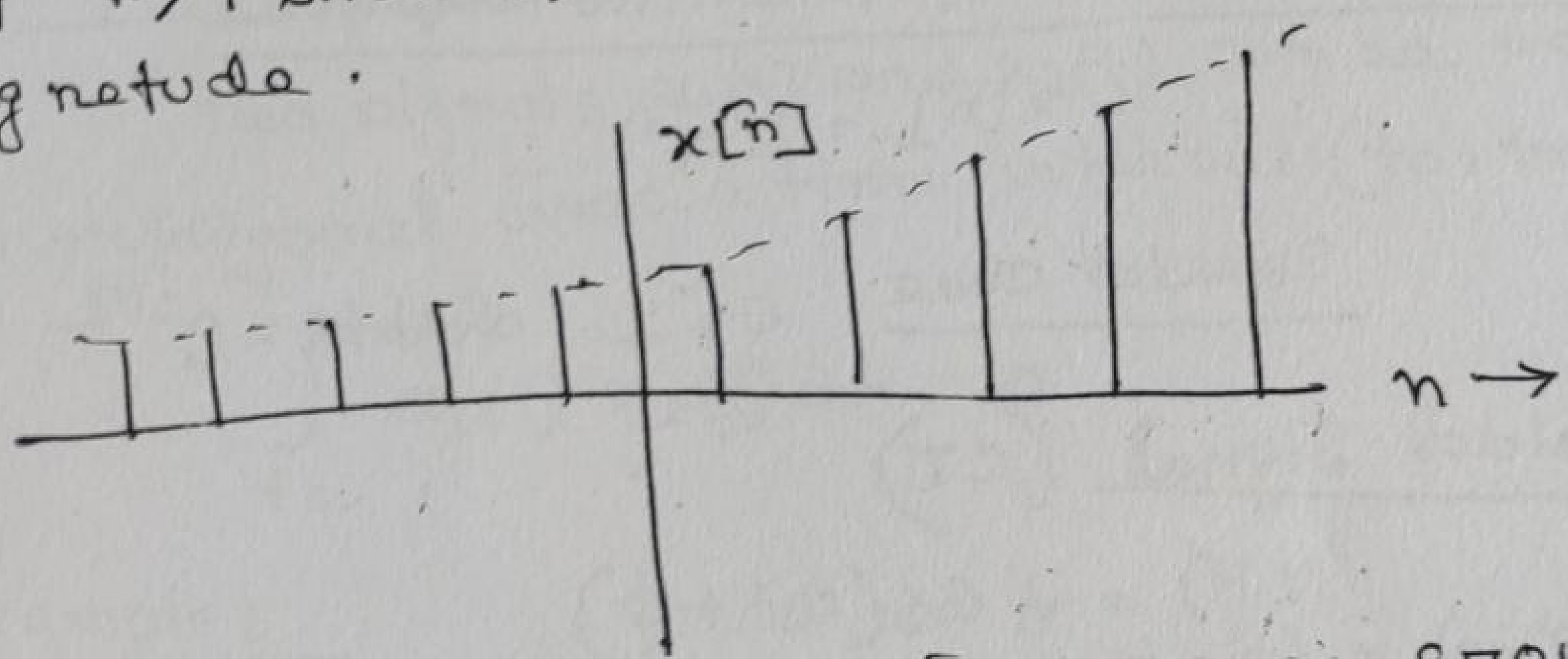
If $0 < r < 1$ and $B > 0$, then $x[n]$ is positive and decreases with increasing n.



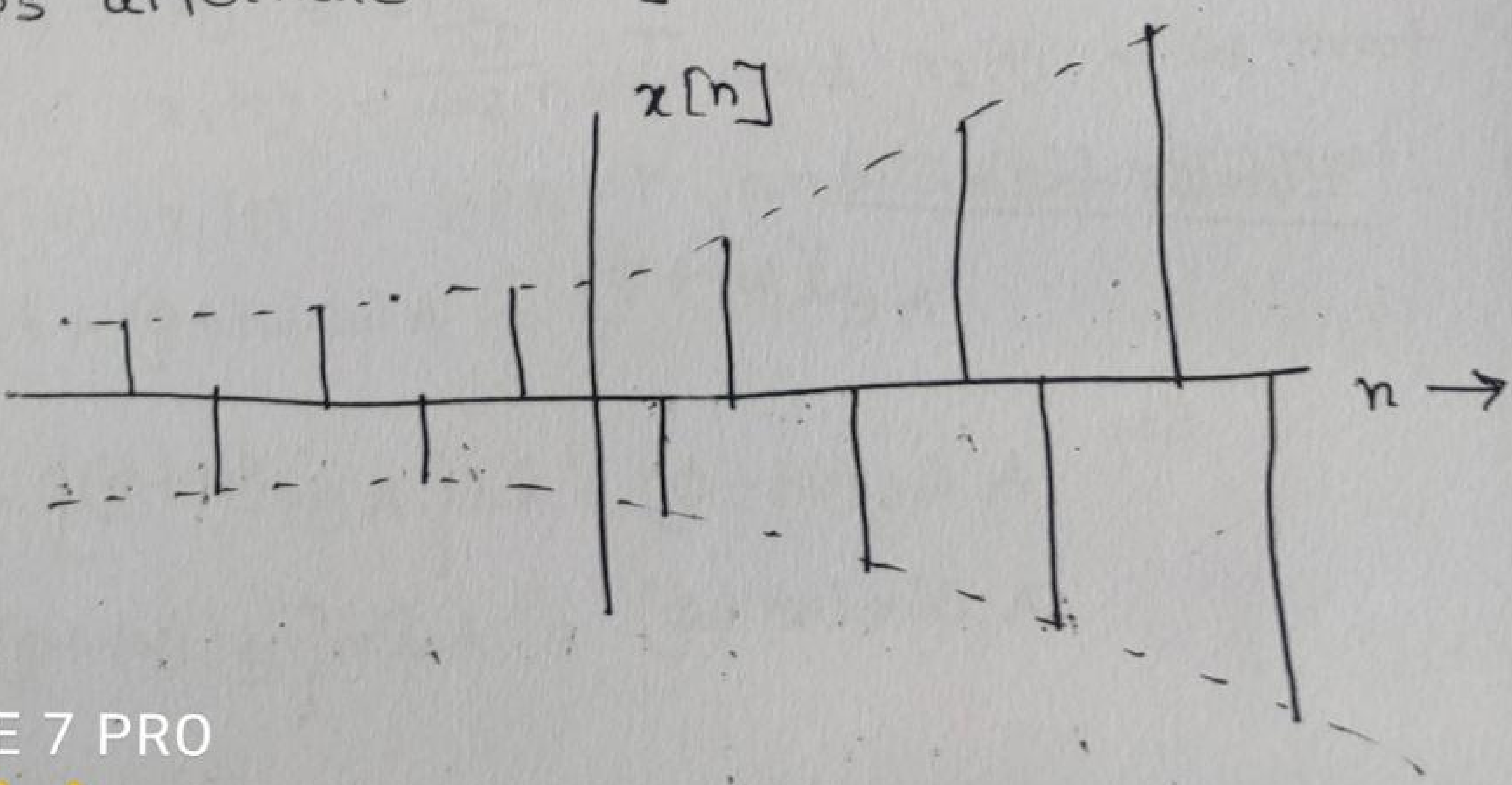
If $-1 < r < 0$ and $B > 0$, the sequence alternates in sign but decreases in magnitude.



If $r > 1$ and $B > 0$, the sequence grows in magnetude.



If $r < -1$ and $B > 0$, the sequence grows with samples alternate in sign.



Complex exponential sequence

$$x[n] = e^{j\Omega n}$$

Euler's formula,

$$x[n] = e^{j\Omega n} = \cos\Omega n + j\sin\Omega n$$

$$x[n+N] = e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}$$

To get $x[n+N] = x[n]$ (Periodic Condition)

$$e^{j\Omega N} = 1$$

$$e^{j\Omega N} = e^{j2\pi K}$$ where $K \rightarrow$ integer.

So, $\Omega N = 2\pi K$

$$\boxed{\frac{\Omega}{2\pi} = \frac{K}{N}}$$ (Rational Number)

This is the condition for periodicity of signal $x[n] = e^{j\Omega n}$.

And its period

$$N = \frac{2\pi K}{\Omega}$$

General complex exponential sequence

$$x[n] = C\alpha^n$$

Special case $C = 1$ and $\alpha = e^{j\Omega}$

Sinusoidal Signal (CT)

$$x(t) = A\cos(\Omega t + \phi)$$

$x(t)$ is periodic with fundamental period

$$T = \frac{2\pi}{\Omega}.$$

Euler's formula

$$Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + A\sin(\omega t+\phi)$$

So,

$$A\cos(\omega t+\phi) = Re\{Ae^{j(\omega t+\phi)}\} = A\,Re\{e^{j(\omega t+\phi)}\}$$

$$A\sin(\omega t+\phi) = A\,Im\{e^{j(\omega t+\phi)}\}$$

Discrete time Sinusoid

$$x[n] = A\cos(\omega n + \phi)$$

For $x[n]$ to be periodic it must satisfy

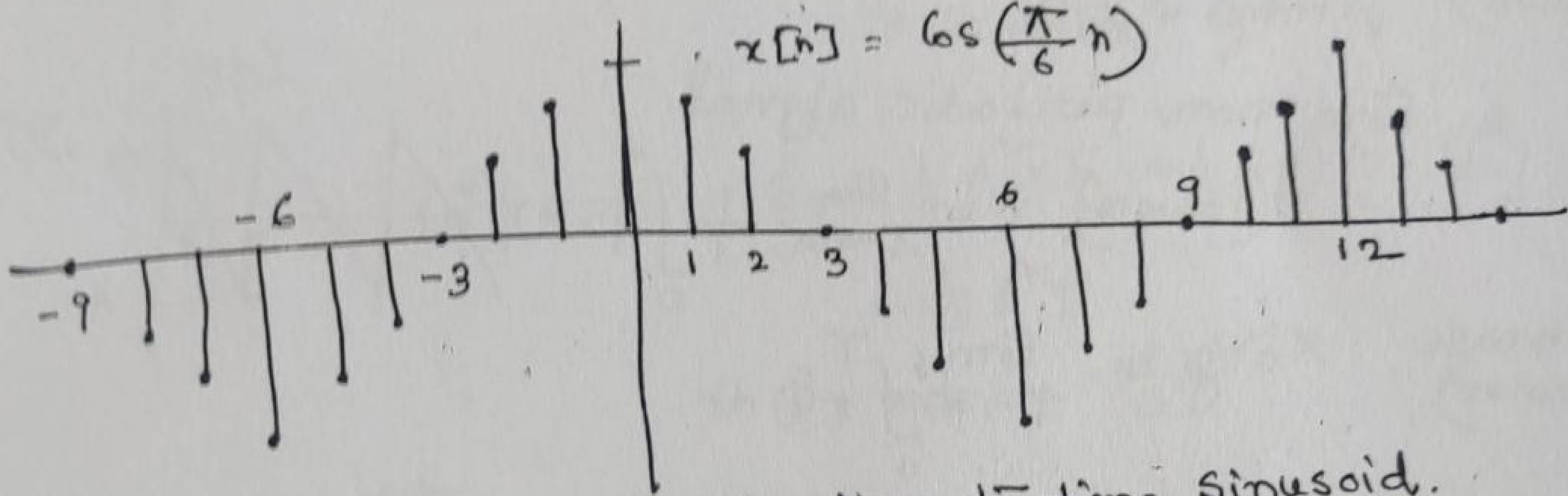
$x[n] = x[n+N]$

$\Rightarrow \omega N = 2\pi K$

~~A cos (wn +~~

$$A\cos(\omega n+\phi) = A\,\mathrm{Re}\{e^{j(\omega n+\phi)}\}$$

$$A\sin(\omega n+\phi) = A\,\mathrm{Im}\{e^{j\omega n+\phi}\}$$



Example discrete time sinusoid.

Orthogonal signals

Two signals $x_1(t)$ and $x_2(t)$ can be said to be orthogonal over a time interval (t_0, t_0+T) if.

$$\int_{t_0}^{t_0+T} x_1(t)\, x_2(t)\, dt = 0$$

Examples

(i) $x_1(t) = \sin n\omega t$ and $x_2(t) = \cos m\omega t$.

(ii) $x_1(t) = \sin n\omega t$ and $x_2(t) = \sin m\omega t$

(iii) $x_1(t) = \cos n\omega t$ and $x_2(t) = \cos m\omega t$

(iv) $x_1(t) = e^{jn\omega t}$ and $x_2(t) = e^{jn\omega t}$.

Note : Neither energy nor power signal

(1) $E = \infty$, $P = 0$ Example: $\frac{1}{\sqrt{t}}$ $t \geq 1$

(2) $E = \infty$, $P = \infty$ Example: t^n

Piecewise Continuous: posses different expressions over different intervals.

* For periodic signal

(Power) $P = \frac{1}{T}\int_0^T |x(t)|^2 dt.$

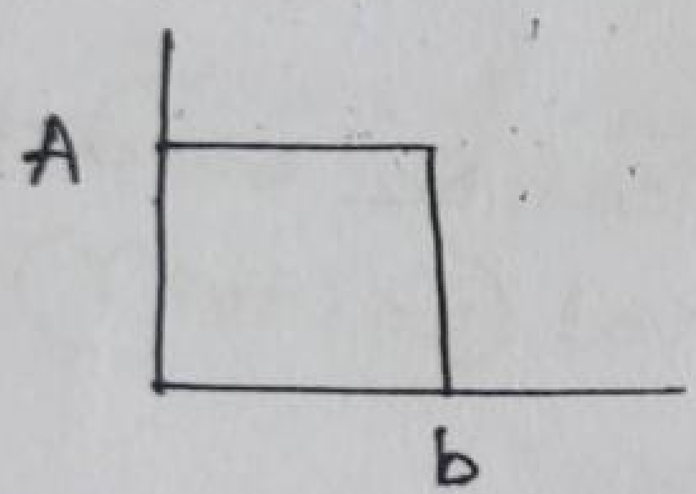
(r.m.s value) $X_{rms} = \sqrt{P}$

* For non-periodic signal

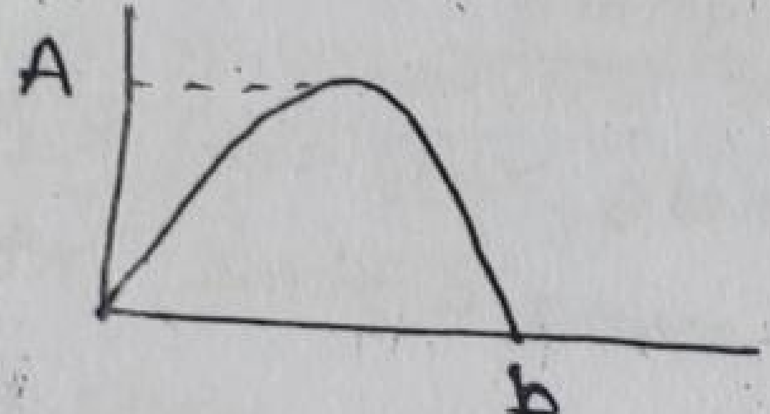
(Power). $P = \lim_{T\to\infty} \frac{1}{T}\int_0^T |x(t)|^2 dt.$

(average value) $X_{avg} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\, dt.$

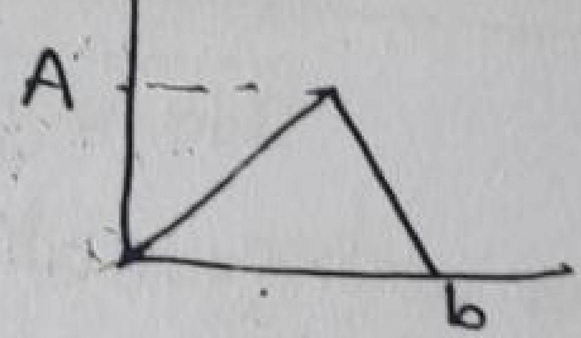
Signal Energy



$E = A^2 b$

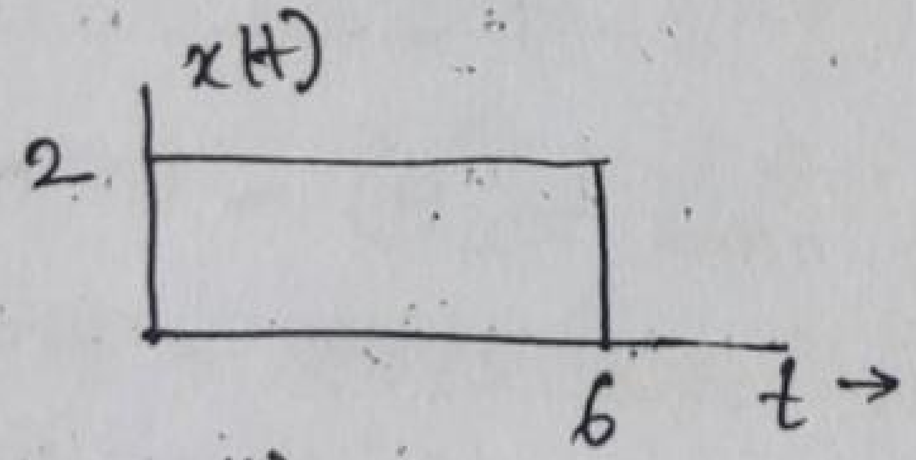


$E = \frac{1}{2}A^2 b$



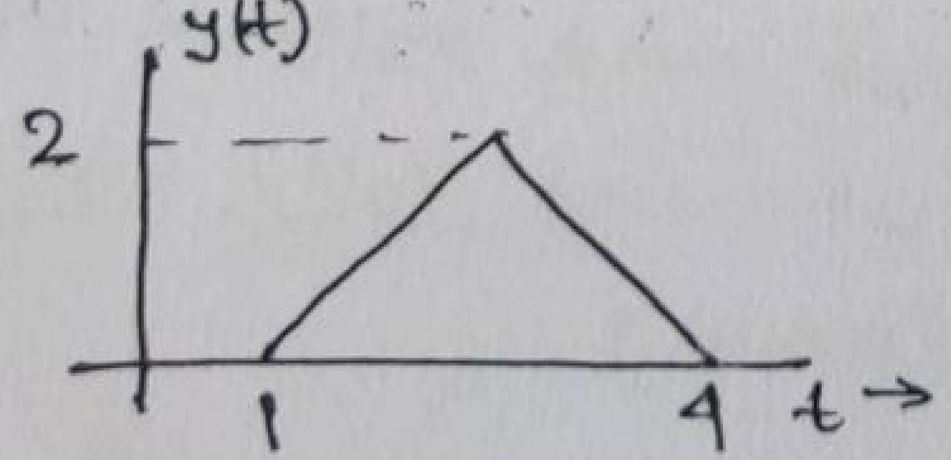
$E = \frac{1}{3}A^2 b$

Q.



$E = 2^2 \times 6\, J$
$= 24 J.$

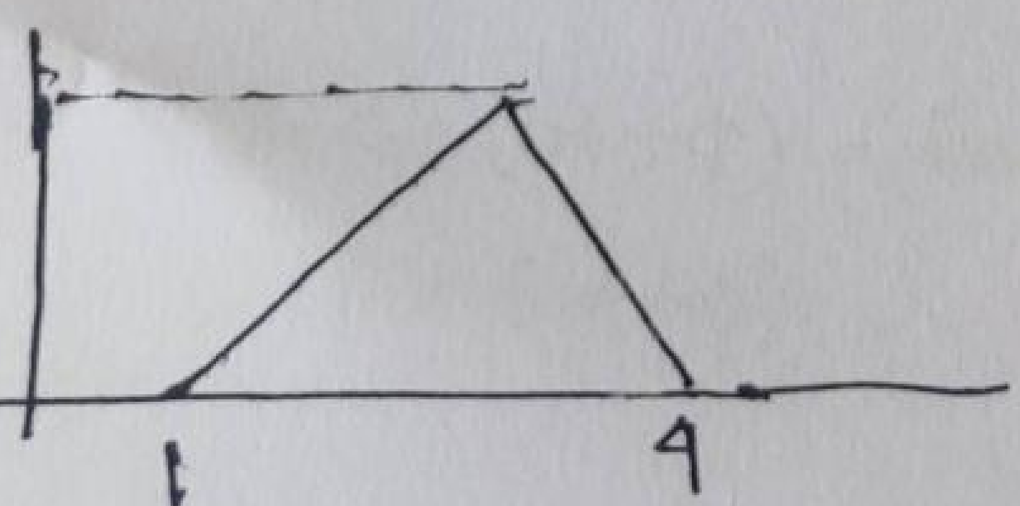
Q.



$E = \frac{1}{3} 2^2 \times (4-1)\, J$
$= 4 J.$

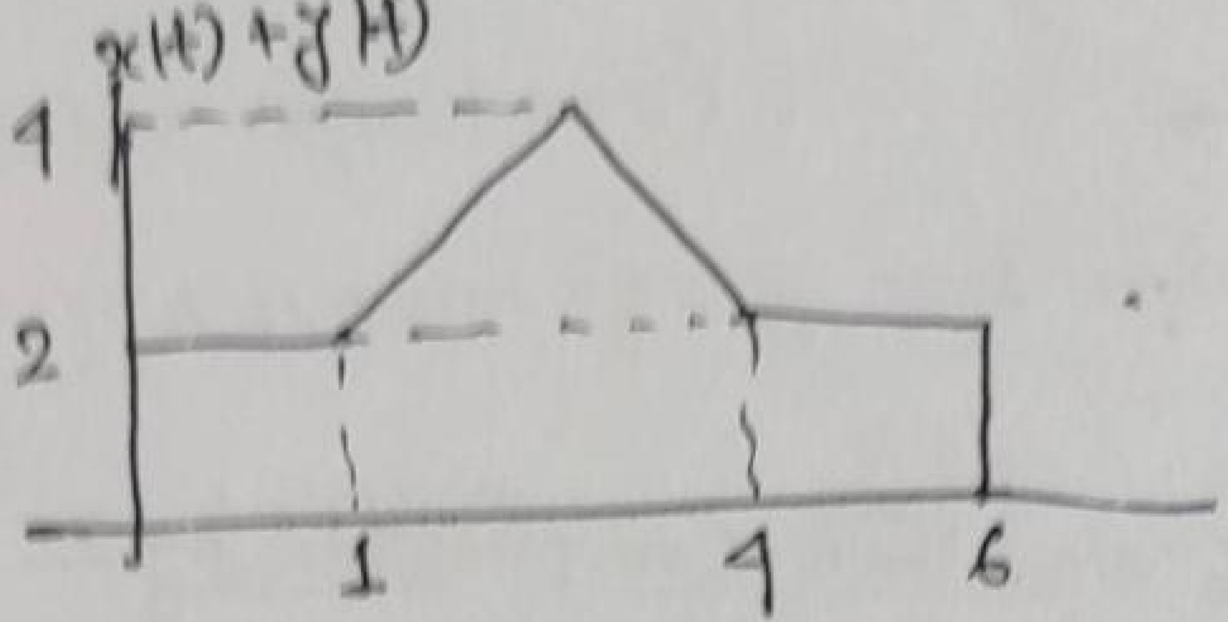
Q 4

$x(t)y(t)$



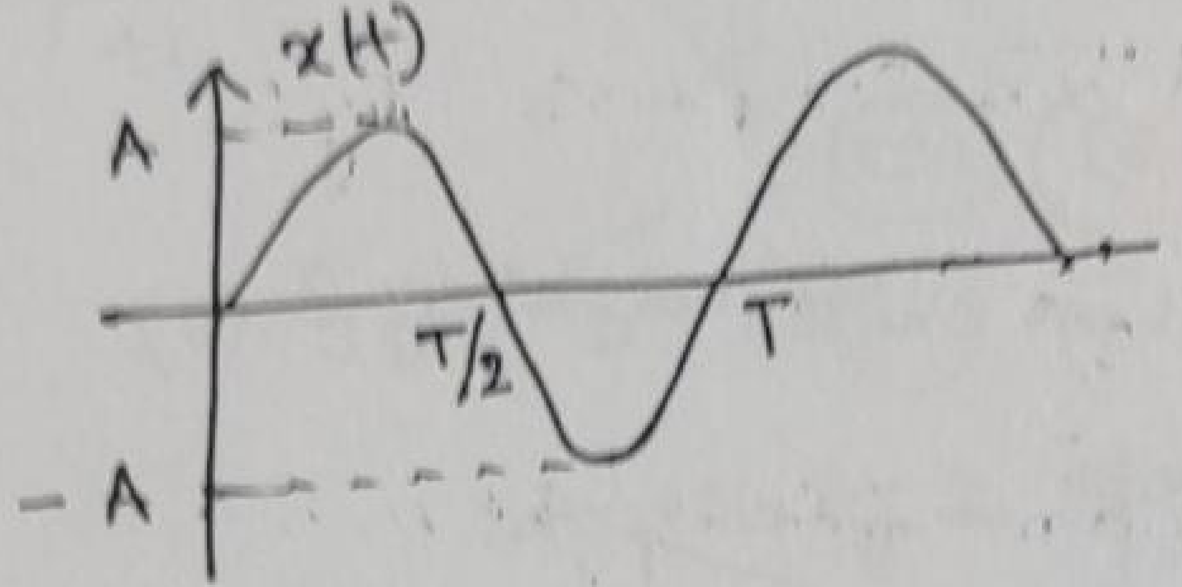
$\cancel{E = \frac{1}{3} \times 6^2 \times (4-1)\, J}$
$= \cancel{36}$
$E = \frac{1}{3} \times 4^2 \times (4-1)\, J$
$= 16 J.$

Q.



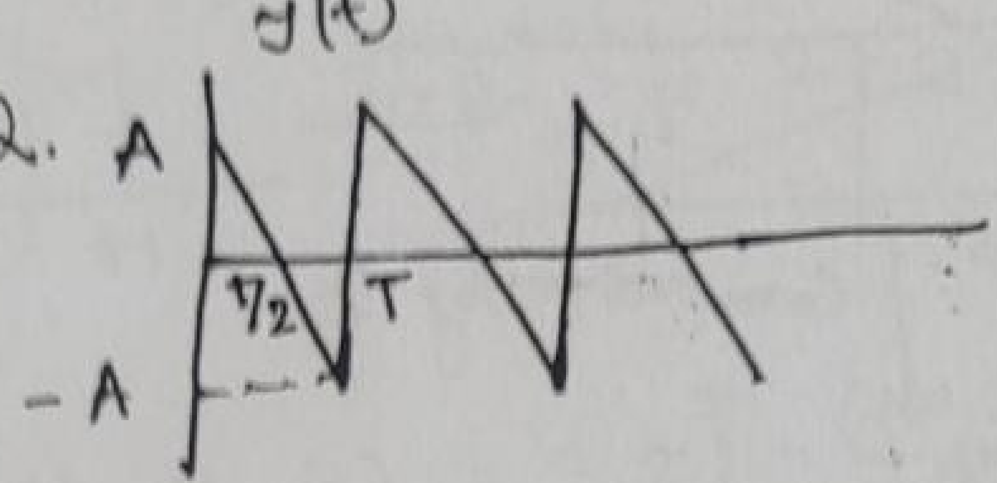
$$E = 2^2 \times 6 + \frac{1}{3}(4-2)^2(4-1)$$
$$= 24 + 4 \text{ J}$$
$$= 28 \text{ J}.$$

Q.



$$E_x = \frac{1}{2}A^2 \times \frac{T}{2} + \frac{1}{2}(-A)^2\left(T - \frac{T}{2}\right)$$
$$= \frac{1}{2}A^2 T$$
$$P_x = \frac{1}{T}E_x = \frac{1}{2}A^2$$

Q.



$$E_y = \frac{1}{3}A^2 \times \frac{T}{2} + \frac{1}{3}(-A)^2(T - T/2)$$
$$= \frac{1}{3}A^2 T$$
$$P_y = \frac{1}{T}E_y = \frac{1}{3}A^2$$

Q. $x(t) = Ae^{j\omega t}$, find power.

Ans: $x(t)$ is alway periodic

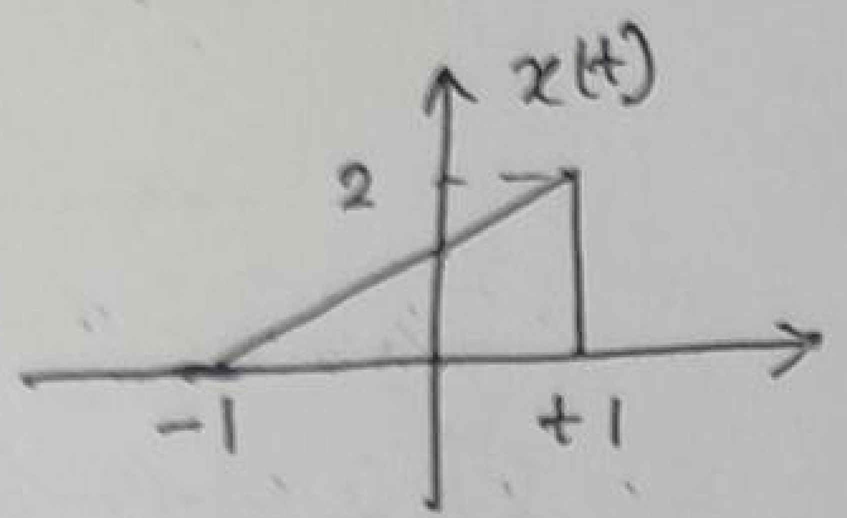
Hence $P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_0^T |Ae^{j\omega t}|^2 dt$.

$$= \frac{1}{T}\int_0^T A^2 dt$$
$$= \frac{1}{T}A^2(T-0) = A^2/\cancel{T} \times \cancel{T}$$
$$= A^2$$

(*) $|e^{j\omega t}|^2 = |\cos\omega t + j\sin\omega t|^2 \longrightarrow$ Uler's formula.

$$= \left|\sqrt{\cos^2\omega t + \sin^2\omega t}\right|^2$$
$$= 1.$$

(*) Uler's formula equates complex exponential with complex sinusoidal.



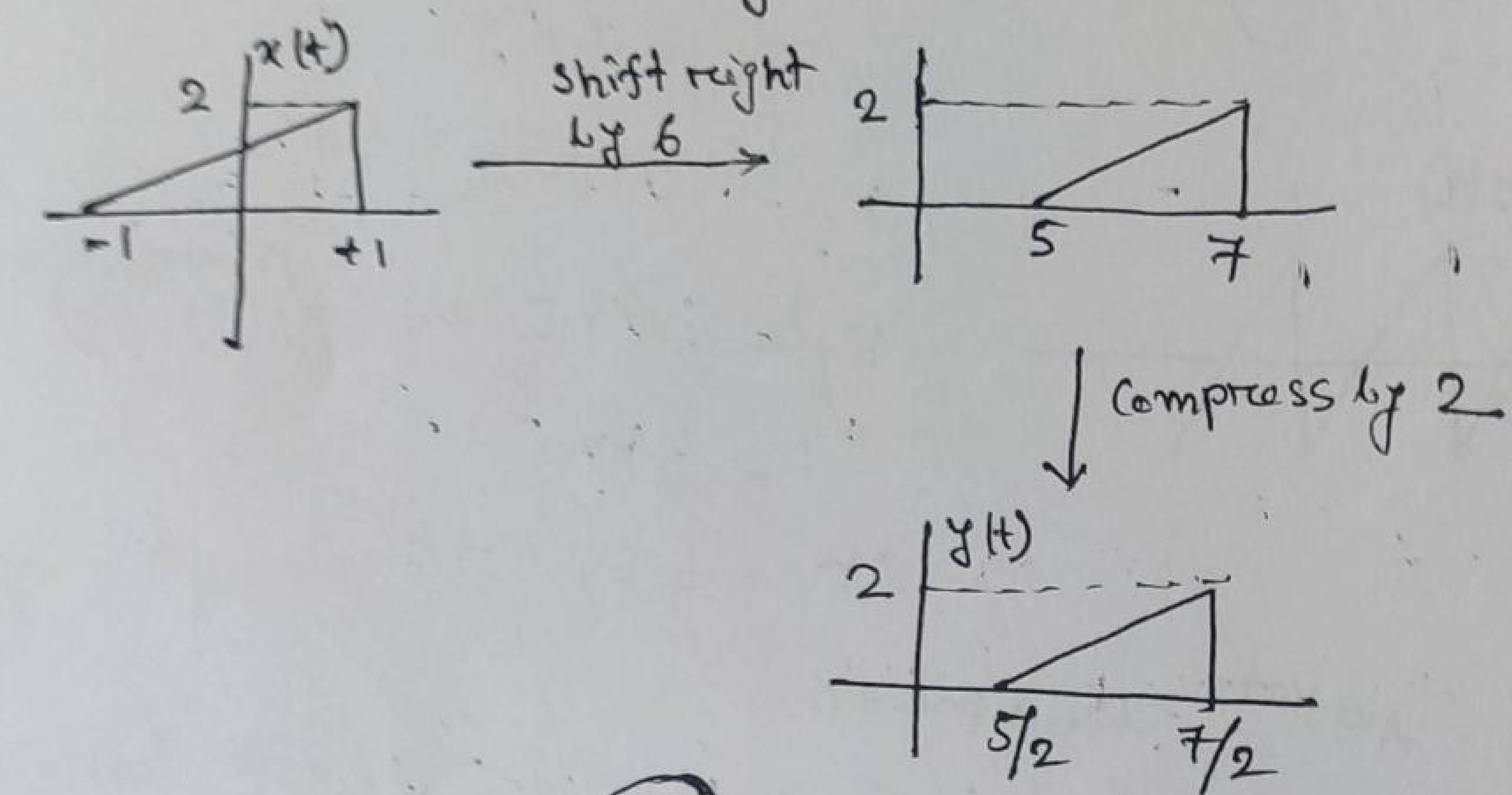
(1) $y(t) = x(2t-6)$

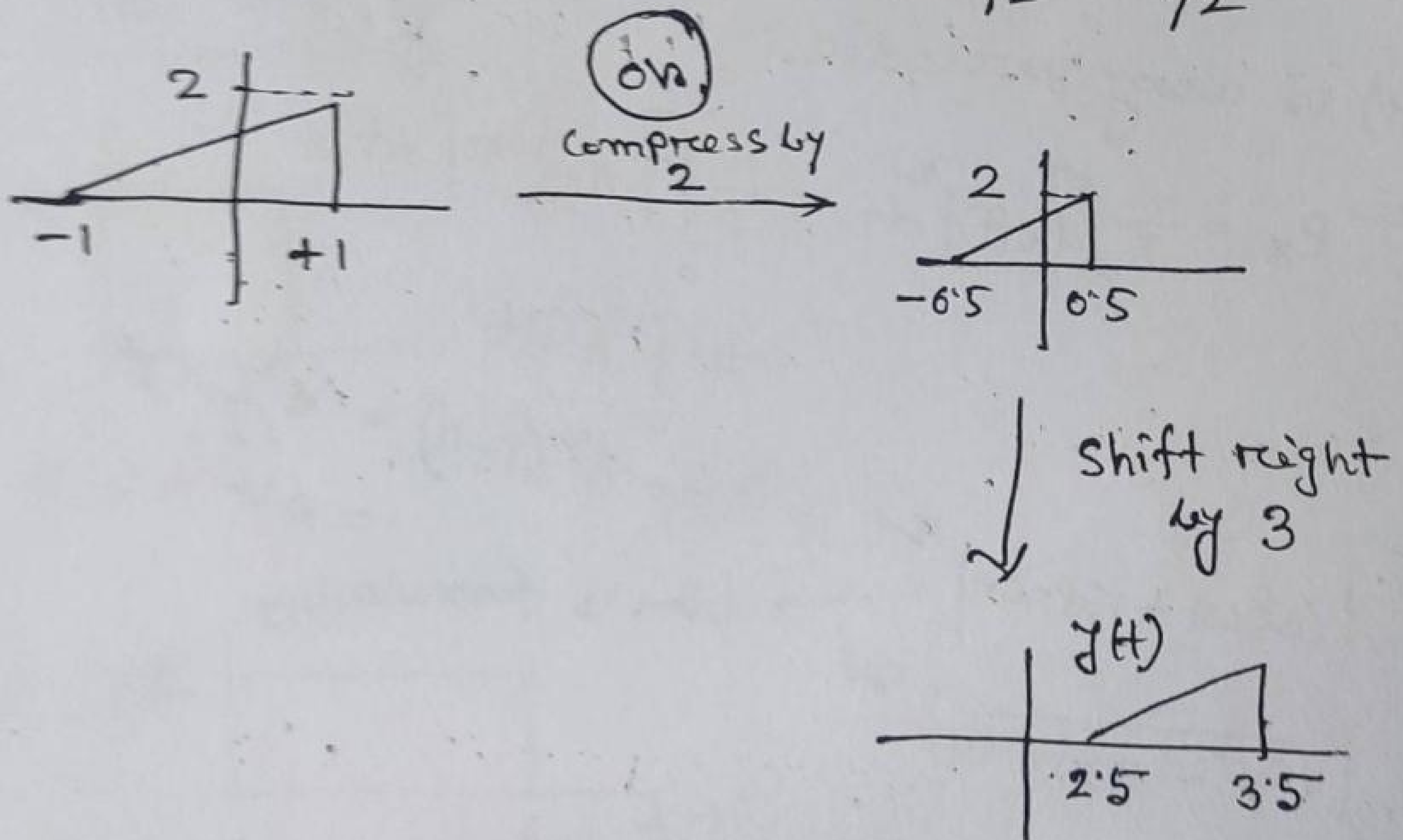
(i) shift right $x(t)$ by 6 and then compressed by 2

or

$y(t) = x(2t-6) = x\left(2(t-3)\right)$

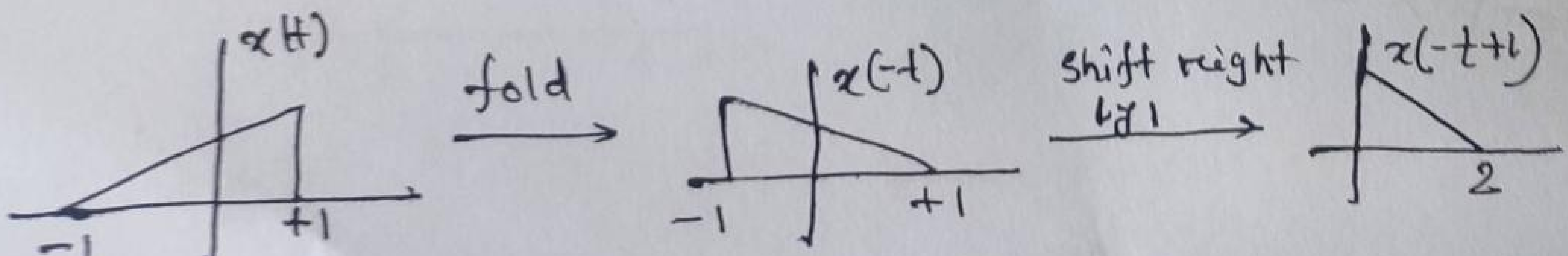
compress $x(t)$ by 2, then shift right by 3





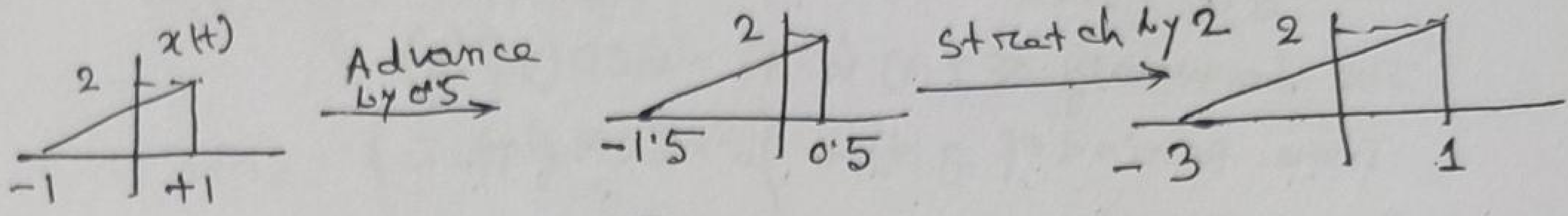
(2) $y_1(t) = x(1-t) = x(-t+1)$

Fold $x(t)$ to get $x(-t)$, then shift right by 1



(3) $y_2(t) = x(0.5t + 0.5)$

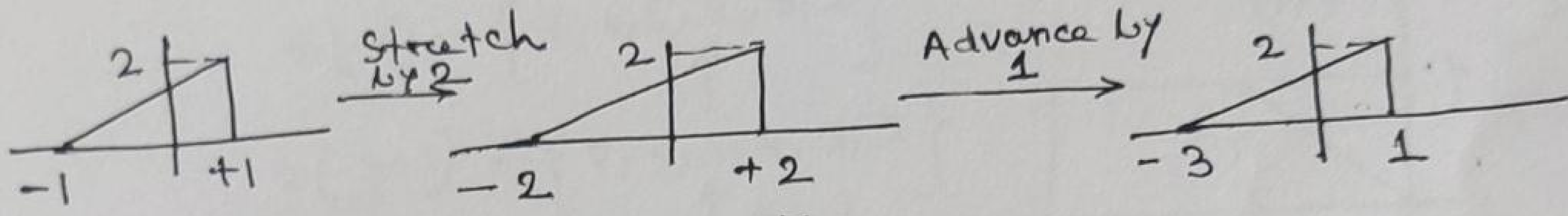
(i) Advance by 0.5 and then stretch by 2



(OR)

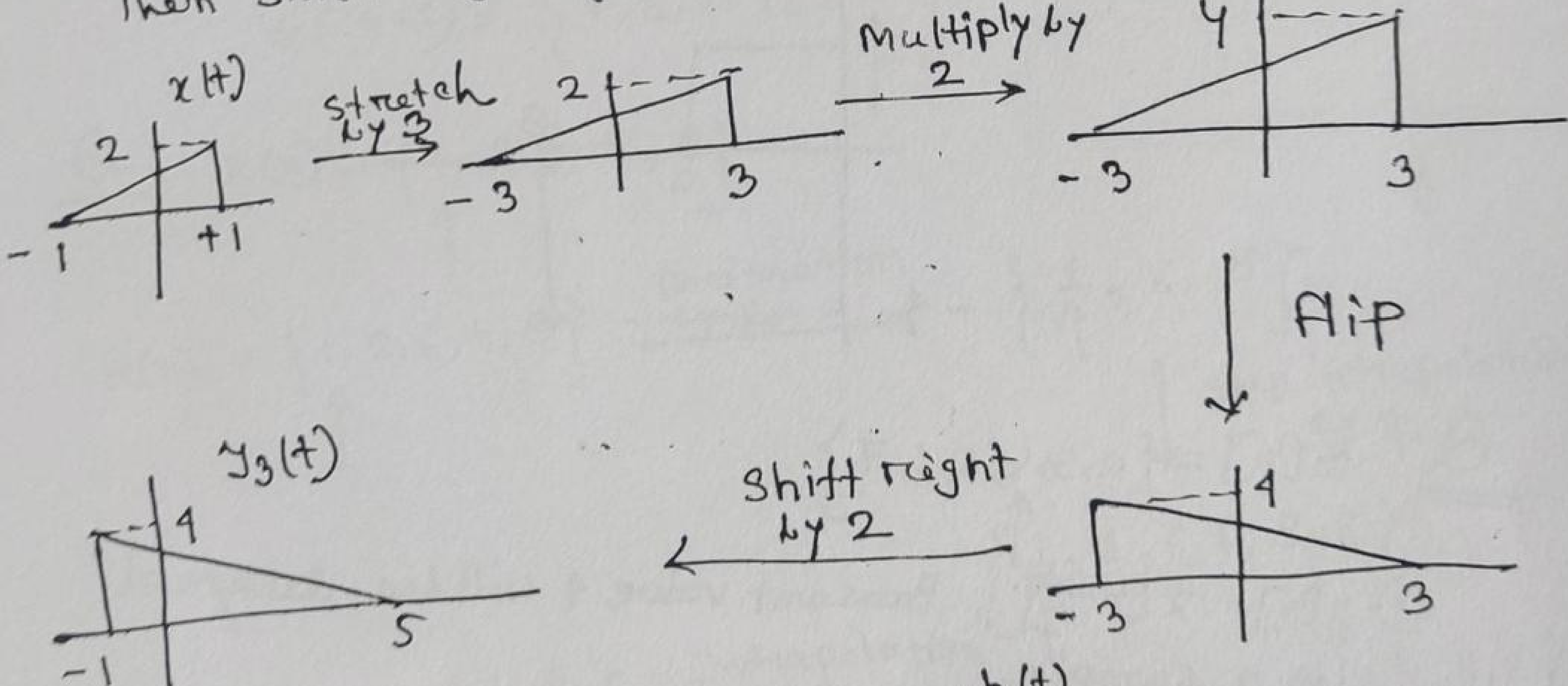
(ii) $y_2(t) = x\left(\frac{t+1}{2}\right)$

stretch by 2 and advance by 1.

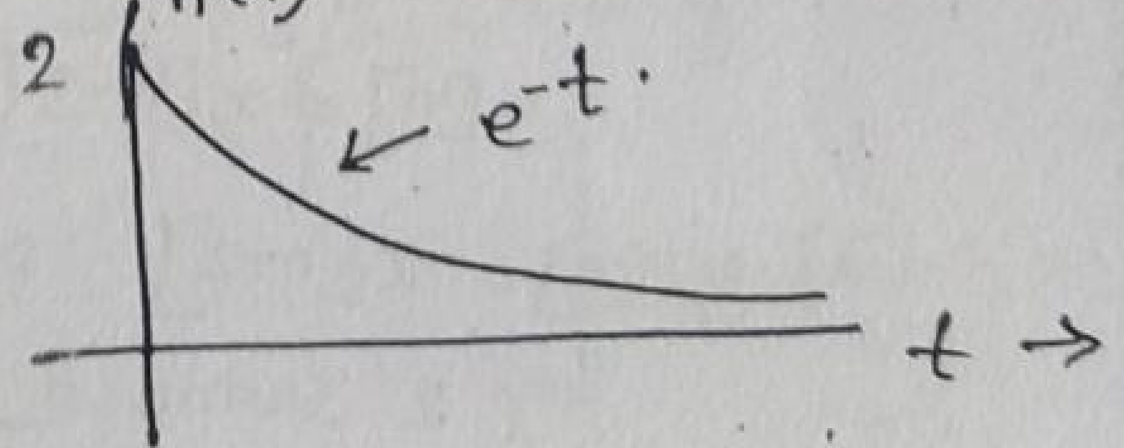


(4) $y_3(t) = 2x\left[-\frac{(t-2)}{3}\right]$

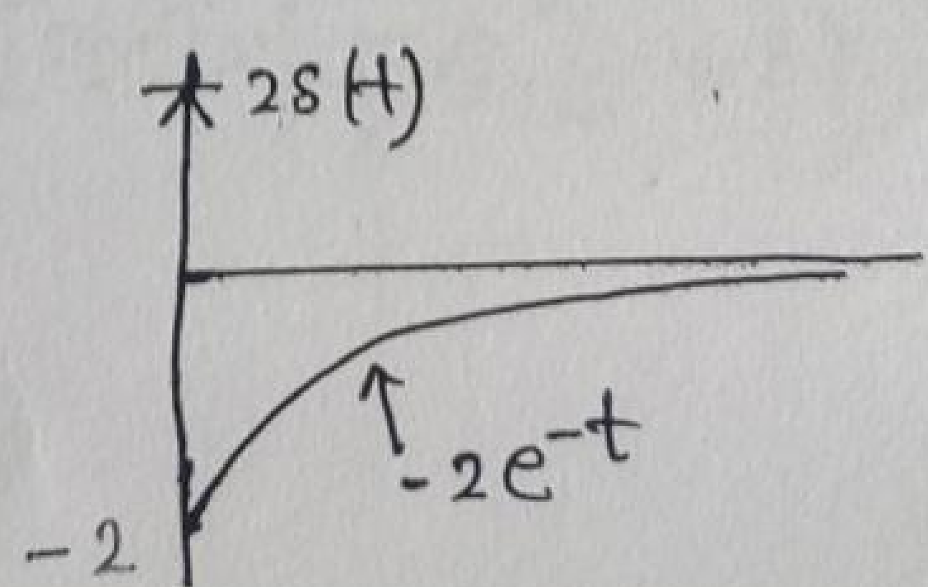
Stretch $x(t)$ by 3 to get $x(t/3)$; Multiply $x(t)$ by 2 to get $2x(t/3)$, Flip ~~2(~~ $2x(t/3)$ to get $2x(-t/3)$, then shift right by 2 to get $2x\left(-\frac{t-2}{3}\right)$



Q. $h(t) = 2e^{-t}u(t)$



$$h'(t) = \frac{d}{dt}(h(t))$$
$$= \frac{d}{dt}(2e^{-t}u(t))$$
$$= 2\delta(t) - 2e^{-t}u(t)$$



(*) If two signals are added of different frequencies

say $x_1(t) \longrightarrow$ has frequency f_1

$x_2(t) \longrightarrow$ has frequency f_2

Then $y(t) = x_1(t) + x_2(t)$

The frequency of $y(t)$ is $f_0 = GCD(f_1, f_2)$

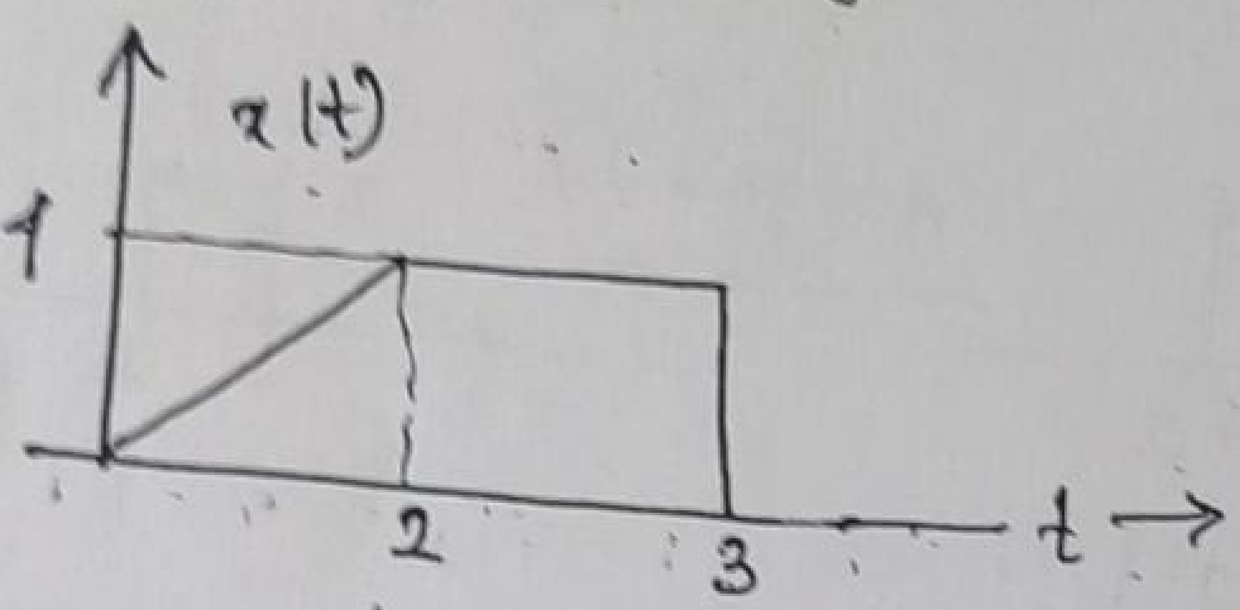
Time period of $y(t)$ is $T_0 = LCM(T_1, T_2)$

Power of $y(t)$

$$P_y = P_{x_1} + P_{x_2}$$

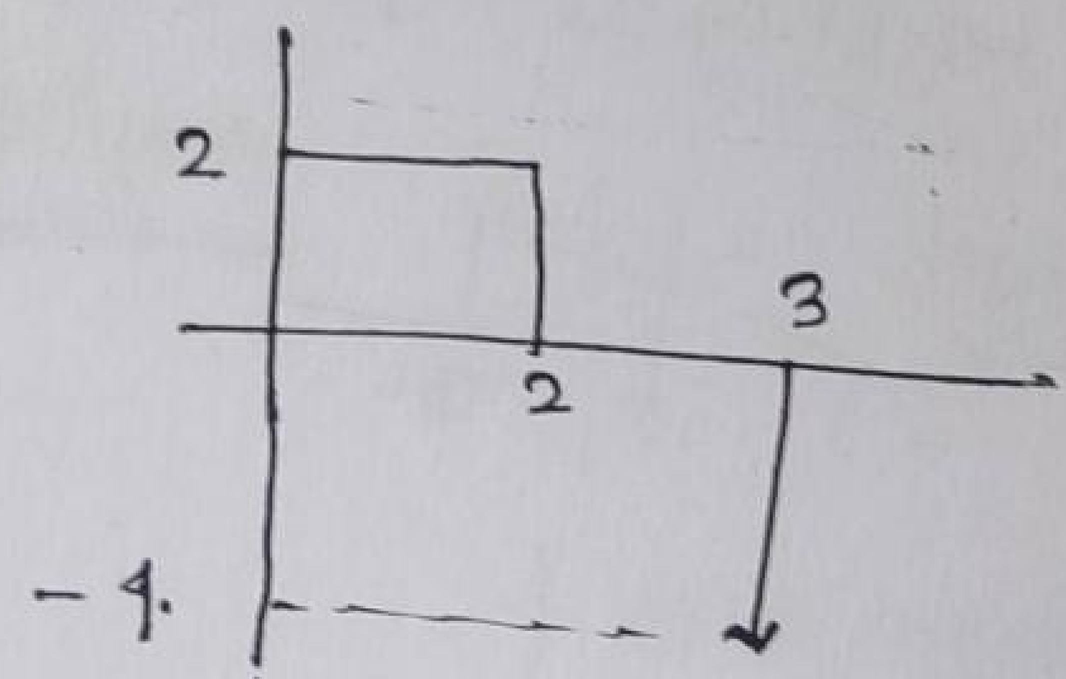
$$y(t)_{rms} = \sqrt{P_y}.$$

Q



$$x(t) = 2r(t) - 2r(t-2) - 4u(t-3)$$

$$x'(t) = 2u(t) - 2u(t-2) - 4\delta(t-3)$$



Q $x[n] = \{2, 3, \underset{\uparrow}{4}, 5, 6, 7\}$

(i) $y[n] = x[n-3]$, Present value 4 will be delayed by 3 sample.

$$y[n] = x[n-3] = \{\underset{\uparrow}{0}, 2, 3, 4, 5, 6, 7\}$$

(ii) $y_1[n] = x[n+2]$, Present value 4 will be advanced by 2 sample

$$y_1[n] = x[n+2] = \{2, 3, 4, 5, \underset{\uparrow}{6}, 7\}$$

(iii) $x(-n)$

$x(n)$ will be flipped around the present value 4

$$x(n) = \{2, 3, \underset{\uparrow}{4}, 5, 6, 7\}$$

$$x(-n) = \{7, 6, 5, \underset{\uparrow}{4}, 3, 2\}$$

(iv) $x(-n+1)$

when n is $-ve$, $+K$ will be delay (Just opposite when n is $+ve$)

$$x(-n) = \{7, 6, 5, \underset{\uparrow}{4}, 3, 2\}$$

$$x(-n+1) = \{7, 6, 5, \underset{\uparrow}{4}, 3, 2\}$$

(v) $x(-n-2)$

$x(-n)$ will be advanced by 2

$$x(-n) = \{7, 6, 5, \underset{\uparrow}{4}, 3, 2\}$$

$$x(-n-2) = \{7, 6, 5, 4, 3, \underset{\uparrow}{2}\}$$

Q. $x(n) = \{\underset{\uparrow}{1}, 2, 6, 4, 8\}$

$$x(n) = \{\underset{\uparrow}{1}, 2, 6, 4, 8\} \xrightarrow[\text{by } 2]{\text{Decimation}} \{\underset{\uparrow}{1}, 6, 8\}$$

Step interpolation by 2

$$\{\underset{\uparrow}{1}, 1, 6, 6, 8, 8\}$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 6, 4, 8\} \xrightarrow[\text{by } 2]{\text{Interpolation}} \{\underset{\uparrow}{1}, 1, 2, 2, 6, 6, 4, 4, 8, 8\}$$

Decimation by 2

$$\{\underset{\uparrow}{1}, 2, 6, 4, 8\}$$

Q. $x(n) = \{\underset{\uparrow}{1}, 1, 2, 5, -1\}$

$x(n/3) = \{1, 0, 0, 2, 0, 0, 5, 0, 0, -1, 0, 0\}$ Zero Interpolation.

$x(n/3) = \{\underset{\uparrow}{1}, 1, 1, 2, 2, 2, 5, 5, 5, -1, -1, -1\}$ Step Interpolation.

$= \{\underset{\uparrow}{1}, \frac{4}{3}, 5/3, 2, 3, 4, 5, 3, 1, -1, -2/3, -\frac{1}{3}\}$

linear Interpolation.

Fractional Delays

It requires interpolation (N), shift (M) and Decimation (N): $x(n - \frac{M}{N}) = x\left(\frac{Nn - M}{N}\right)$.

$x(n) = \{2, 4, \underset{\uparrow}{6}, 8\}$

$g(n) = x\left(\frac{n}{2}\right) = \{2, 2, 4, 4, \underset{\uparrow}{6}, 6, 8, 8\}$ step Interpolation.

$h(n) = g(n-1) = \{2, 2, 4, \underset{\uparrow}{4}, 6, 6, 8, 8\}$

$y(n) = h(2n) = \{2, \underset{\uparrow}{4}, 6, 8\}$

or

$x(n) = \{2, 4, \underset{\uparrow}{6}, 8\}$

$g(n) = x(n/2) = \{2, 3, 4, 5, \underset{\uparrow}{6}, 7, 8, 4\}$ linear Interpolation.

$h(n) = g(n-1) = \{2, 3, 4, \underset{\uparrow}{5}, 6, 7, 8, 4\}$

$y(n) = h(2n) = \{3, \underset{\uparrow}{5}, 7, 4\}$

Classification of Systems

A system is a mathematical model of a physical process that relates the input (or excitation) to the output (or response).

$$x(t) \longrightarrow \boxed{T\{\ \}} \longrightarrow y(t)$$

$y(t) = T\{x(t)\}$. $T \rightarrow$ transformation and gives a mapping to be done on $x(t)$ to get $y(t)$

Example (1) $y(t) = x^2(t)$
(2) $y(t) = e^{x(t)}$
(3) $y(t) = x(t) + x(t-1)$

Above can be applied to discrete time signals too.

Classification

(1) Static or memory less system.

A system is said to be memory less if output at any instant depends on input of the same instant.

Dynamic system or system with memory.

A system which is not memory less is said to have memory or dynamic system.

Example:
$y(t) = x^2(t)$: Memory less.
$y[n] = e^{x[n]}$: Memory less
$y(t) = x(t-1)$: Memory
$y[n] = x[n]\, x[n-2]$: Memory
$y[n] = x[2n]$: Memory.
$y[n] = x[n^2]$: Memory.
$y[n] = a x[n]$: Memory less
$y[n] = n x[n] + b x^3[n]$: Memory less
$y[n] = a(n-1)\, x[n]$: Memory less.

(2) Ideal Delay system

$$y(n) = x(n - n_d)$$

n_d is fixed, positive integer.

(3) Moving average system

$$y(n) = \frac{1}{m_1 + m_2 + 1} \sum_{k=-m_1}^{m_2} x(n-k)$$

If $m_1 = 0$ and $m_2 = 5$.

$$y(7) = \frac{1}{6}\left[\sum_{k=0}^{5} x(7-k)\right]$$

$$= \frac{1}{6}\left[x(7) + x(6) + x(5) + x(4) + x(3) + x(2)\right]$$

$$y(8) = \frac{1}{6}\left[x(8) + x(7) + x(6) + x(5) + x(4) + x(3)\right]$$

(4) Accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$= \sum_{k=-\infty}^{n-1} x(k) + x(n)$$

$$y(n) = y(n-1) + x(n)$$

Ex

$x(n) = \{0, 3, 2, 1, 0, 1, 2, 3, 0\}$

$y(n) = \{0, 3, 5, 6, 6, 7, 9, 12, 12\}$

Ex

$x(n) = n u(n)$, given $y(-1) = 0$ (initially relaxed)

$$y(n) = \sum_{k=-\infty}^{-1} x(k) + \sum_{k=0}^{n} x(k)$$

$$= y(-1) + \sum_{k=0}^{n} x(k)$$

$$= 0 + \sum_{k=0}^{n} n = \frac{n(n+1)}{2}$$

(5) Linear System

A system is linear if it satisfy superposition principle. i.e. weighted sum of inputs when given to a system should give a weighted sum of outputs.

superposition principle has two parts —

1. Additivity:

Given, $T\{x_1(t)\} = y_1(t)$

$T\{x_2(t)\} = y_2(t)$

then $T\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t) \rightarrow (1)$

2. Homogeneity

$T\{a x_1(t)\} = a y_1(t) \rightarrow (2)$

Thus superposition principle is

$T\{a x_1(t) + b x_2(t)\} = a y_1(t) + b y_2(t)$

Any system that does not obey equation (1) and/or equation (2), is a non-linear system.

Example
$y(t) = x^2(t)$ non-linear system.
$y(t) = \cos(t)$ non-linear system.
$y(t) = x(t) + x(t-1)$ linear
$y(t) = e^{x(t)}$ non-linear.
$y[n] = 3x[n]$ linear
$y[n] = 2x^3[n]$ non-linear.
$y[n] = x[n]\,x[n-1]$ non-linear.

(6) Time Invariant (TI) system (or shift-invariant)

A system is said to be time invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal.

$$T\{x(n-k)\} = y(n-k) \rightarrow (1)$$

$$T\{x(t-\tau)\} = y(t-\tau) \rightarrow (2)$$

A system which does not satisfy (1) ~~and~~/or (2), is called a time varying system.

Example
$y(t) = x^2(t)$ TI
$y(t) = x(2t)$ TV
$y(t) = x(k_0 t)$ TV
where $k_0 \neq 1$.
$y[n] = x[n] - x[n-1]$ TI
$y[n] = x[2n]$ TV
$y[n] = x[k_0 n]$ TV
k_0 is an integer and $k_0 \neq 1$.

$x(t) \longrightarrow \boxed{\text{Delay by } T_0} \longrightarrow \boxed{T\{\ \}} \longrightarrow y_1(t)$

$x(t) \longrightarrow \boxed{T\{\ \}} \longrightarrow \boxed{\text{Delay by } T_0} \longrightarrow y_2(t)$

for time invariant system $y_1(t) = y_2(t)$

(7) Continuous time (CT) and discrete time (DT) system.

If input and output signals are continuous time signals, the the system is called continuous time system.

$$x(t) \rightarrow \boxed{\text{system } T} \rightarrow y(t)$$

If input and output signals are discrete time signals, then the system is called discrete time system:

$$x[n] \rightarrow \boxed{\text{system } T} \rightarrow y[n]$$

(8) Causal and non-causal systems

A system is said to be causal if out at any instant depends upon past and present inputs only.

A system is called anti-causal if output at any instant depends on future inputs only,

A system is called non-causal if output at any instant depends upon past, present and future inputs also.

From above, anti-causality implies non-causality but the converse is need not be true.

Note: All memoryless systems are causal but not vice versa.

Example

$y[n] = x[n] - x[n-1]$ causal.

$y[n] = x[n] + x[n+2]$ non-causal

$y[n] = x[n+1] - x[n-1]$ non causal.

$y[n] = x[n+2]$ anti-causal.

$y(t) = x(t+1)$ anti-causal

$y[n] = x[-n]$ non-causal.

(9) Stable and unstable system

A system is bounded input bounded output (BIBO) stable if for any bounded input $x(t)$, the system gives bounded output. otherwise the system is unstable.

For continuous time signal.

$$|x(t)| \leq B_x < \infty \Rightarrow |y(t)| \leq B_y < \infty$$

Example

(1) $y(t) = \int_{-\infty}^{t} x(t)\,dt$ unstable.

(2) $y(t) = \int_{t-T}^{t} x(t)\,dt$ stable.

For discrete time signal,

$$|x[n]| \le B_x < \infty \Rightarrow |y[n]| \le B_y < \infty$$

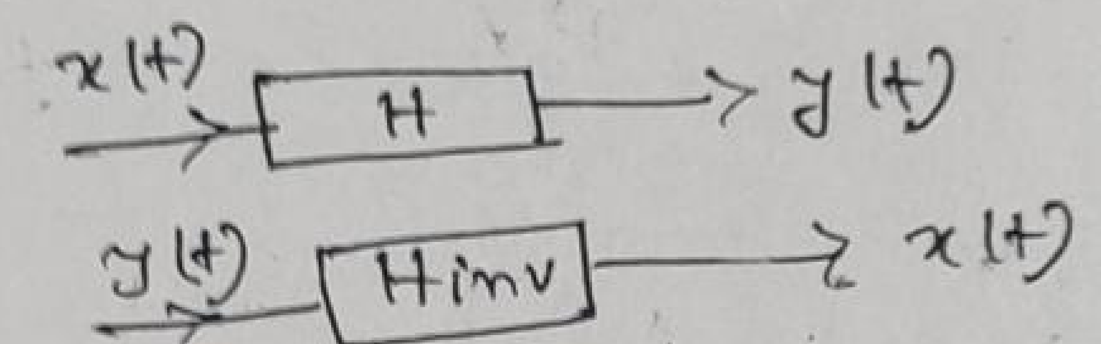
Example

(1) $y[n] = x[n-1]$ stable.

(2) $y[n] = \sum_{m=-\infty}^{n} x[n-m]$ unstable.

(10) Invertible system and Inverse system

A system is said to have inverse, if there exists another system so as to recover the original input from the output of the first system.



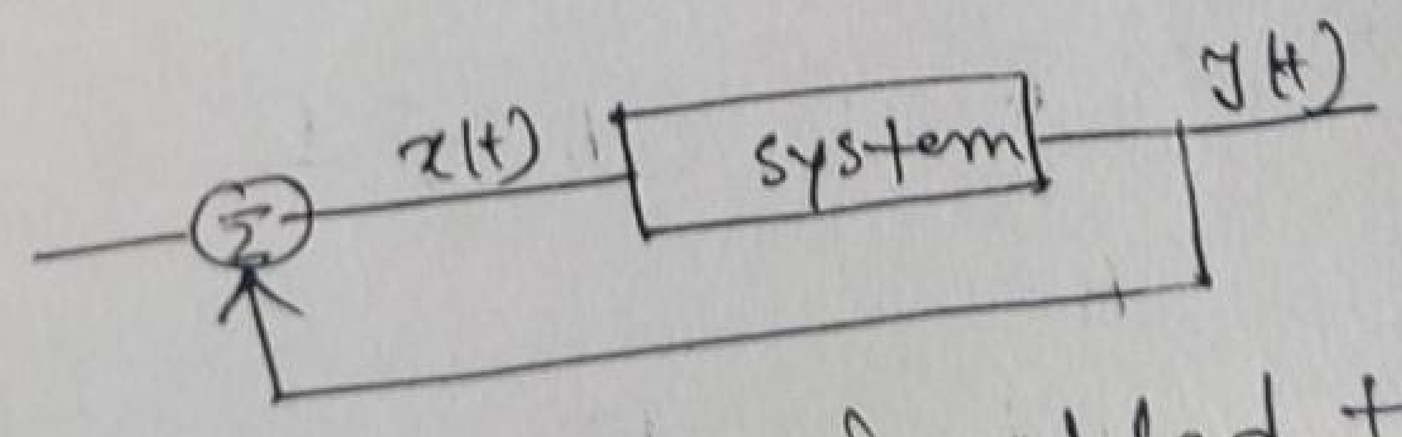
Ex

1. $y(t) = x(t)/2$ is the inverse system of $y(t) = 2x(t)$
2. $y(t) = |x(t)|$ is not invertible due to loss of sign information.

Note only one to one mapped system are invertible.

(11) Feed back System.

A system having feedback from output to input is called a feedback system.

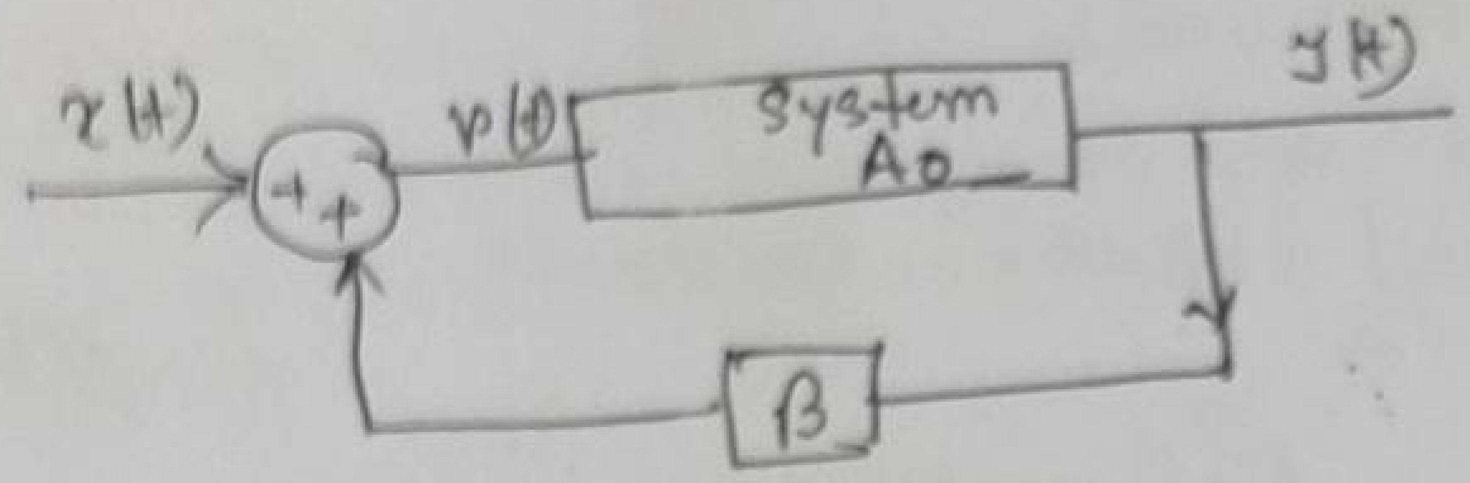


If the feedback is added to the input is called positive feedback (example- oscillator)

If the feedbak is subtracted from the input is called Negative feedback.

-ve feedback — highly stable

+ve feedbak — unstable.



$A_0 \rightarrow$ open loop Gain

$$v(t) = x(t) + \beta y(t)$$

$$y(t) = A_0 v(t)$$

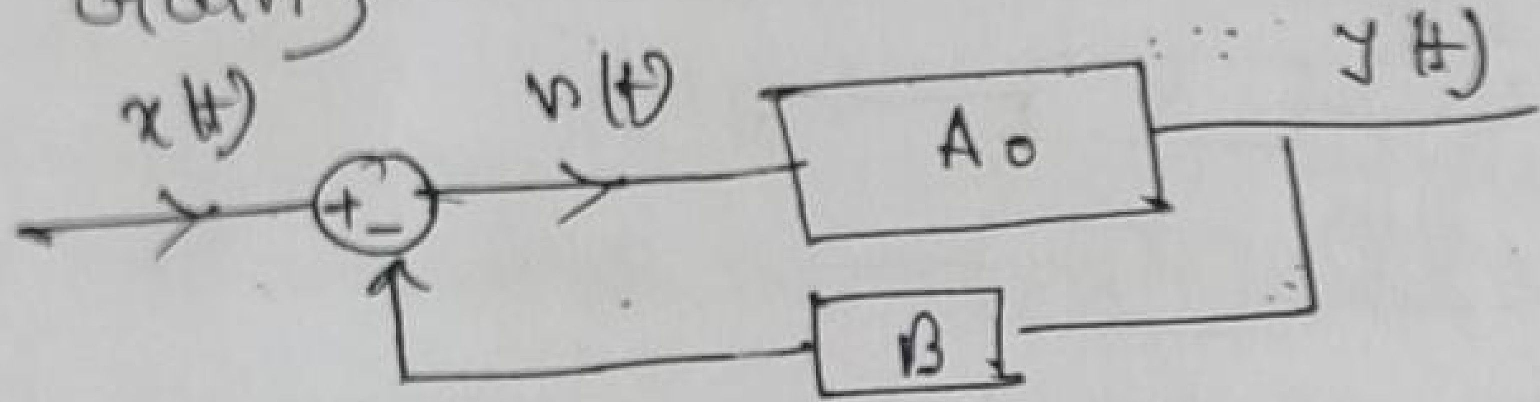
$$y(t) = A_0\{x(t) + \beta y(t)\}$$

$$y(t) = A_0 x(t) + A_0 \beta y(t)$$

$$y(t)[1 - A_0\beta] = A_0 x(t)$$

$$\underset{\text{(Closed loop Gain)}}{A_c} = \frac{y(t)}{x(t)} = \frac{A_0}{1 - \beta A_0}$$

Gain of positive feedback



$A_0 \rightarrow$ open loop Gain

$$v(t) = x(t) - \beta y(t)$$

$$y(t) = A_0 v(t)$$

$$y(t) = A_0 [x(t) - \beta y(t)]$$

$$(1 + \beta A_0) y(t) = A_0 x(t)$$

$$\underset{\text{(Closed loop Gain)}}{A_c} = \frac{y(t)}{x(t)} = \frac{A_0}{1 + \beta A_0}$$

Gain of −ve feedback

when $\beta A_0 \gg 1$

$$A_c \approx \frac{A_0}{\beta A_0} \approx \frac{1}{\beta_0}$$

Thus closed loop gain is independent of open loop gain variation. β_0 is generally a resistance and does not change much. Thus, −ve feed back attains high stability.