Thirteen

Complex Numbers

CHAPTER OUTLINE

- Introduction
- Complex Numbers
- De Moivre's Theorem
- Roots of an Algebraic Equation
- Expansion of Trigonometric Functions
- Circular and Hyperbolic Functions of Complex Numbers
- Logarithmic Functions of Complex Numbers

13.1 INTRODUCTION

Complex numbers are an extension of real numbers obtained by introducing an imaginary unit i, where $i = \sqrt{-1}$. The operations of addition, subtraction, multiplication and division are applicable on complex numbers. A negative real number can be obtained by squaring a complex number. With a complex number, it is always possible to find solutions to polynomial equations of degree more than one. Complex numbers are used in many applications, such as control theory, signal analysis, quantum mechanics, relativity, etc.

13.2 COMPLEX NUMBERS

A complex number z is an ordered pair (x, y) of real numbers x and y. It is written as $z = (x, y)$ or $z = x + iy$, where $i = \sqrt{-1}$ is known as the imaginary unit. Here, x is called the real part of z and is written as "$Re(z)$" and y is called the imaginary part of z and is written as "$Im(z)$".

If $x = 0$ and $y \neq 0$ then $z = 0 + iy = iy$ which is purely imaginary.
If $x \neq 0$ and $y = 0$ then $z = x + i0 = x$ which is real.

Hence, z is *purely imaginary*, if its real part is zero and is *real*, if its imaginary part is zero.

This shows that every real number can be written in the form of a complex number by taking its imaginary part as zero. Hence, the set of real numbers is contained in the set of complex numbers.

The even power of i is either 1 or -1 and odd power of i is either i or $-i$.

$$i^2 = i.i = -1$$
$$i^3 = i^2.i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$
$$i^5 = i.i^4 = i, \text{ etc.}$$

Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal. If $z = x + iy$ is a complex number then its *conjugate* or *complex conjugate* is defined as $\bar{z} = x - iy$.

13.2.1 Geometrical Representation of Complex Numbers (Argand's Diagram)

Any complex number $z = x + iy$ can be represented as a point $P(x, y)$ in the xy-plane with reference to the rectangular x and y axes. The plot of a given complex number $z = x + iy$, as the point $P(x, y)$ in the xy-plane is known as **Argand's diagram** (Fig. 13.1). The x-axis is called the real axis, y-axis is called the imaginary axis and the xy-plane is called the complex plane.

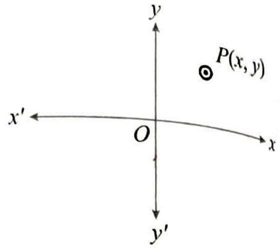

Fig. 13.1 Argand's diagram

HISTORICAL DATA

Jean-Robert Argand (1768–1822) was a gifted amateur mathematician. In 1806, while managing a bookstore in Paris, he published the idea of geometrical interpretation of complex numbers known as the Argand diagram. In 1813, it was republished in the French journal *Annales de Mathématiques*. It discussed a method of graphing complex numbers via analytical geometry. It proposed the interpretation of the value i as a rotation of 90 degrees in the Argand plane. In this essay, Argand was also the first to propose the idea of modulus to indicate the magnitude of vectors and complex numbers, as well as the notation for vectors $\overrightarrow{ab}$. Argand is also renowned for delivering a proof of the fundamental theorem of algebra in his 1814 work *Réflexions sur la nouvelle théorie d'analyse* (Reflections on the new theory of analysis). It was the first complete and rigorous proof of the theorem, and was also the first proof to generalize the fundamental theorem of algebra to include polynomials with complex coefficients.

13.2.2 Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

(a) *Addition:* $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i\,(y_1 + y_2)$

(b) *Subtraction:* $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i\,(y_1 - y_2)$

(c) *Multiplication:* $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i\,(x_2 y_1 + y_2 x_1)$ $\quad [\because i^2 = -1]$

(d) *Division:* $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)}{(x_2 + iy_2)} \cdot \dfrac{(x_2 - iy_2)}{(x_2 - iy_2)} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\left(\dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right)$

13.2.3 Different Forms of Complex Numbers

1. Cartesian or Rectangular Form If x and y are real numbers then $z = x + iy$ is called the Cartesian form of the complex number.

2. Polar Form The complex number $z = x + iy$ can be represented by the point P whose Cartesian coordinates are (x, y) (Fig. 13.2). If polar coordinates of the same point P are (r, θ) then $x = r\cos\theta$ and $y = r\sin\theta$.

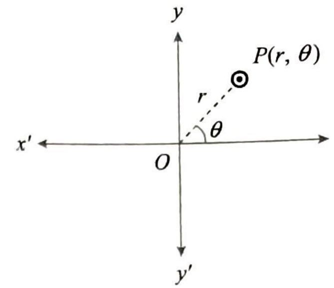

Fig. 13.2 Polar-form representation

Hence, the polar form of z is

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Polar form can also be written as $r\angle\theta$.

3. Exponential Form $e^{i\theta} = \cos\theta + i\sin\theta$

Using polar form, $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

This is called the exponential form or Euler's form of a complex number z.

Note $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Hence, $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$

HISTORICAL DATA

Leonhard Euler (1707–1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.

He is also renowned for his work in mechanics, fluid dynamics, optics, astronomy, and music theory. Euler is considered the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He is also one of the most prolific mathematicians; his collected works fill 60-80 quarto volumes. He spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia.

13.2.4 Modulus and Argument (or Amplitude) of a Complex Number

Let z be a complex number such that $z = x + iy = r(\cos\theta + i\sin\theta)$

where $\qquad x = r\cos\theta,\ y = r\sin\theta$

Then $\qquad r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Here, 'r' is called the modulus or absolute value of z and is denoted by $|z|$ or mod (z), and θ is called argument or amplitude of z and is denoted by arg (z) or amp (z).

Hence, $|z| = r = \sqrt{x^2 + y^2}$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note The value of θ which satisfies both the equations $x = r\cos\theta$ and $y = r\sin\theta$, gives the argument of z.
Argument θ has infinite number of values. The value of θ lying between $-\pi$ and π is called the *principal value* of the argument.

13.2.5 Properties of Complex Numbers

Let $z = x + iy$ and $\bar{z} = x - iy$.

(a) $\text{Re}(z) = x = \frac{1}{2}(z + \bar{z})$

(b) $\text{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$

(c) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(d) $\overline{(z_1 z_2)} = \bar{z}_1\bar{z}_2$

(e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(f) $z\bar{z} = |z|^2 = |\bar{z}|^2 \qquad [\because |z| = |\bar{z}| = \sqrt{x^2 + y^2}]$

(g) $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(h) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

EXAMPLE 13.1 *Find the value of $\sqrt{-5+12i}$.*

Solution: Let

$$x + iy = \sqrt{-5+12i}$$
$$(x + iy)^2 = -5 + 12i$$
$$(x^2 - y^2) + i(2xy) = -5 + 12i$$

Comparing real and imaginary parts,

$$x^2 - y^2 = -5, \qquad \dots (1)$$

and $$2xy = 12, \; xy = 6$$

Putting $y = \frac{6}{x}$ in Eq. (1),

$$x^2 - \frac{36}{x^2} = -5$$
$$x^4 + 5x^2 - 36 = 0$$
$$(x^2 + 9)(x^2 - 4) = 0$$
$$x^2 = -9, \; x^2 = 4$$

Since x is real, $x = \pm 2$

When $x = 2, \; y = \frac{6}{2} = 3$

When $x = -2, \; y = \frac{6}{-2} = -3$

Hence, $\sqrt{-5+12i} = \pm 2 \pm 3i$

EXAMPLE 13.2 *If $u = \frac{z+i}{z+2}$ and $z = x = iy$ then show that*

(i) locus of (x, y) is a straight line, if u is real

(ii) locus of (x, y) is a circle, if u is purely imaginary

Find the centre and radius of the circle.

Solution: $u = \frac{z+i}{z+2}$ and $z = x + iy$

$$u = \frac{x+iy+i}{x+iy+2} = \frac{x+i(y+1)}{(x+2)+iy}\cdot\frac{(x+2)-iy}{(x+2)-iy} = \frac{[x(x+2)+y(y+1)]+i[(y+1)(x+2)-xy]}{(x+2)^2+y^2}$$

$$\text{Re}(u) = \frac{x(x+2)+y(y+1)}{(x+2)^2+y^2}$$

$$\text{Im}(u) = \frac{(y+1)(x+2)-xy}{(x+2)^2+y^2} = \frac{x+2y+2}{(x+2)^2+y^2}$$

(i) If u is real then

$$\text{Im}(u) = 0$$

$$\frac{x+2y+2}{(x+2)^2+y^2} = 0$$

$$x + 2y + 2 = 0$$

Hence, locus of (x, y) is $x + 2y + 2 = 0$, which represents a straight line.

(ii) If u is purely imaginary then

$$\text{Re}(u) = 0$$

$$\frac{x(x+2)+y(y+1)}{(x+2)^2+y^2} = 0$$

$$x^2 + y^2 + 2x + y = 0$$

Hence, locus of (x, y) is $x^2+y^2+2x+y=0$, which represents a circle with centre at $\left(-1, -\frac{1}{2}\right)$ and radius of $\frac{\sqrt{5}}{2}$ units.

EXAMPLE 13.3 *Show that* $\left|\frac{z}{|z|}-1\right| \le |\arg(z)|$.

Solution: Let $z = re^{i\theta}$, where $|z| = r$ and $\arg(z) = \theta$

$$\left|\frac{z}{|z|}-1\right| = \left|\frac{re^{i\theta}}{r}-1\right| = |e^{i\theta}-1| = |\cos\theta + i\sin\theta - 1| = |\cos\theta - 1 + i\sin\theta|$$

$$= \left|-2\sin^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right| = \left|2\sin\frac{\theta}{2}\right|\left|-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right|$$

$$= \left|2\sin\frac{\theta}{2}\right|\sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = 2\left|\sin\frac{\theta}{2}\right| \le 2\left|\frac{\theta}{2}\right| \qquad \left[\because \frac{\sin\theta}{\theta} \le 1\right]$$

$$\le |\theta| \le |\arg(z)|$$

EXAMPLE 13.4 *If* $(a_1+ib_1)(a_2+ib_2)\ldots\ldots(a_n+ib_n) = A+iB$, *prove that*

(i) $(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots(a_n^2+b_n^2) = A^2+B^2$

(ii) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$.

Solution: $(a_1+ib_1)(a_2+ib_2)\ldots\ldots(a_n+ib_n) = A+iB$ ... (1)

(i) Taking modulus of Eq. (1),

$$|(a_1+ib_1)(a_2+ib_2)\ldots\ldots(a_n+ib_n)| = |A+iB|$$

$$|a_1+ib_1|\,|a_2+ib_2|\ldots\ldots|a_n+ib_n| = |A+iB|$$

$$\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}\ldots\ldots\sqrt{a_n^2+b_n^2} = \sqrt{A^2+B^2}$$

Squaring both the sides,

$$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots(a_n^2+b_n^2) = A^2+B^2$$

(ii) Taking argument of Eq. (1),

$$\arg[(a_1+ib_1)(a_2+ib_2)\ldots\ldots(a_n+ib_n)] = \arg(A+iB)$$

$$\arg(a_1+ib_1) + \arg(a_2+ib_2) + \ldots\ldots + \arg(a_n+ib_n) = \arg(A+iB)$$

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

EXAMPLE 13.5 *Prove that* $e^{2ai\cot^{-1}b}\left(\frac{bi-1}{bi+1}\right)^{-a} = 1$.

Solution:

$$\frac{bi-1}{bi+1} = \frac{bi+i^2}{bi-i^2} = \frac{b+i}{b-i}$$

Let $b+i = re^{i\theta}$ then $b-i = re^{-i\theta}$

$$r = |b+i| = \sqrt{b^2+1} \text{ and } \theta = \arg(b+i) = \tan^{-1}\left(\frac{1}{b}\right) = \cot^{-1}b$$

$$\frac{bi-1}{bi+1} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta} = e^{2i\cot^{-1}b}$$

$$e^{2ai\cot^{-1}b}\left(\frac{bi-1}{bi+1}\right)^{-a} = e^{2ai\cot^{-1}b}(e^{2i\cot^{-1}b})^{-a} = e^0 = 1$$

Hence,
$$e^{2ai\cot^{-1}b}\left(\frac{bi-1}{bi+1}\right)^{-a} = 1$$

EXERCISE 13.1

1. Find the modulus and principal value of the arguments of

   (i) $\dfrac{1+2i}{1-3i}$ (ii) $\tan\alpha - i$

   $\left[\textbf{Ans.:}\ \text{(i)}\ \dfrac{1}{\sqrt{2}}, \dfrac{3\pi}{4} \quad \text{(ii)}\ \sec\alpha,\ \alpha-\dfrac{\pi}{2}\right]$

2. Express in polar form:

   (i) $\sqrt{3}-i$ (ii) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

   $\left[\textbf{Ans.:}\ \text{(i)}\ 2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right) \quad \text{(ii)}\ 2\left(\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right)\right]$

3. Find the value of $\sqrt{3-4i}$.

   [Ans. : $2-i$ or $-2+i$**]**

4. Find z if $\arg(z+2i) = \dfrac{\pi}{4}$, $\arg(z-2i) = \dfrac{3\pi}{4}$.

   [Ans. : 2**]**

5. Find the locus of z if

   (i) $\left|\dfrac{z-1}{z+1}\right| = 1$ (ii) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$

   $\left[\textbf{Ans.:}\ \text{(i)}\ x=0 \quad \text{(ii) circle}\ x^2+y^2-2y-1=0\right]$

6. Find two numbers whose sum is 4 and product is 8.

   [Ans. : $2 \pm 2i$**]**

7. If $x+iy = \sqrt{a+ib}$, prove that $(x^2+y^2)^2 = a^2+b^2$.

8. If $a = e^{i2\alpha}$, $b = e^{i2\beta}$, $c = e^{i2\gamma}$ then prove that
$$\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha+\beta-\gamma).$$

9. If $\alpha = a + ib$ $\beta = c + id$, show that if
$$\alpha = \frac{\beta-1}{\beta+1} \text{ then } a^2+b^2 = \frac{(c-1)^2+d^2}{(c+1)^2+d^2}.$$

10. If $x^2+y^2 = 1$ then prove that
$$\frac{1+x+iy}{1+x-iy} = x+iy.$$

11. If $(1+ai)(1+bi)(1+ci) = p+iq$, prove that

    (i) $p\tan[\tan^{-1}a + \tan^{-1}b + \tan^{-1}c] = q$

    (ii) $(1+a^2)(1+b^2)(1+c^2) = p^2+q^2$.

12. If $(\alpha+i\beta) = \dfrac{1}{a+ib}$, prove that

    $(\alpha^2+\beta^2)(a^2+b^2) = 1$.

13. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, where $0 < \alpha, \beta < \dfrac{\pi}{2}$ then find the polar form of $\dfrac{1+a^2}{1-iab}$.

    $\left[\textbf{Ans.:}\ \cos\alpha\sec\left(\dfrac{\pi}{4}-\dfrac{\alpha+\beta}{2}\right)e^{i\left[\frac{\pi}{4}+\left(\frac{\alpha-\beta}{2}\right)\right]}\right]$

14. If $x^r = \cos\dfrac{\pi}{3^r} + i\sin\dfrac{\pi}{3^r}$, prove that

    (i) $x_1 \cdot x_2 \cdot x_3 \ldots\ldots \infty = i$

    (ii) $x_0 x_1 x_2 \ldots\ldots \infty = -i$

13.3 DE MOIVRE'S THEOREM

Statement

For any real number n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof

Case I n is a positive integer

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ……, $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \end{aligned}$$

Similarly,

$$\begin{aligned} z_1 z_2 \ldots\ldots z_n &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \ldots\ldots r_n(\cos\theta_n + i\sin\theta_n) \\ &= (r_1 r_2 \ldots\ldots r_n)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots\ldots (\cos\theta_n + i\sin\theta_n) \\ &= (r_1 r_2 \ldots\ldots r_n)[\cos(\theta_1+\theta_2+\ldots\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots\ldots+\theta_n)] \end{aligned} \quad \ldots (13.1)$$

If $z_1 = z_2 = \ldots\ldots = z_n = z = r(\cos\theta + i\sin\theta)$ then Eq. (13.1) reduces to

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, where n is a positive integer.

Case II n is a negative integer

Let $n = -m$, where m is a positive integer.

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{(\cos m\theta + i\sin m\theta)} \quad \text{[Using Case I]}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta \quad [\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$$

$$= \cos n\theta + i\sin n\theta, \quad \text{where } n \text{ is a negative integer.}$$

Case III n is a rational number

Let $n = \frac{p}{q}$, where p and q are integers and $q \neq 0$.

$$\text{Consider}\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q\cdot\frac{\theta}{q}\right) + i\sin\left(q\cdot\frac{\theta}{q}\right) \quad \text{[Using Cases I and II]}$$

$$= \cos\theta + i\sin\theta$$

Hence,

$$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = (\cos\theta + i\sin\theta)^{\frac{p}{q}}$$

$$\cos\left(p\cdot\frac{\theta}{q}\right) + i\sin\left(p\cdot\frac{\theta}{q}\right) = (\cos\theta + i\sin\theta)^{\frac{p}{q}} \quad \text{[Using Cases I and II]}$$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, where n is a rational number.

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for any real number n.

HISTORICAL DATA

Abraham de Moivre (1667–1754) was a French mathematician known for de Moivre's formula, one of those that link complex numbers and trigonometry, and for his work on the normal distribution and probability theory. He was a friend of Isaac Newton, Edmund Hailey, and James Stirling. De Moivre wrote a book on probability theory, *The Doctrine of Chances*, said to have been prized by gamblers. De Moivre first discovered Binet's formula, the dosed-form expression for Fibonacci numbers linking the ηth power of the golden ratio φ to the ηth Fibonacci number.

EXAMPLE 13.6 *Prove that* $\left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$.

Solution:

$$\left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right) - i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^n$$

$$= \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$

$$= \left[\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n = \left[\frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}\right]^n = \left[e^{2i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n = e^{i\left(\frac{n\pi}{2}-n\alpha\right)}$$

$$= \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$$

EXAMPLE 13.7 *Prove that* $(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = 2(x^2+y^2)^{\frac{m}{2n}}\cos\left[\frac{m}{n}\tan^{-1}\left(\frac{y}{x}\right)\right]$.

Solution: Let

$$x + iy = r(\cos\theta + i\sin\theta)$$

$$r = |x+iy| = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

$$\theta = \arg(x+iy) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{m}{n}} + [r(\cos\theta - i\sin\theta)]^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n} + \cos\frac{m\theta}{n} - i\sin\frac{m\theta}{n}\right) = r^{\frac{m}{n}}\left(2\cos\frac{m\theta}{n}\right)$$

Substituting the values of r and θ,

$$(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = 2(x^2+y^2)^{\frac{m}{2n}}\cos\left[\frac{m}{n}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

EXAMPLE 13.8 *If* $\sin\alpha + \sin\beta + \sin\gamma = 0$ *and* $\cos\alpha + \cos\beta + \cos\gamma = 0$, *prove that*

(i) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$

(ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(iii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(iv) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

Solution:

$$\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$$

$$(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0+i\cdot 0$$

$$(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)+(\cos\gamma+i\sin\gamma)=0$$

$$e^{i\alpha}+e^{i\beta}+e^{i\gamma}=0$$

Let $$x=e^{i\alpha},\ y=e^{i\beta},\ z=e^{i\gamma}$$

Then $$x+y+z=0 \quad \dots(1)$$

Also, $$(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0-i\cdot 0$$

$$(\cos\alpha-i\sin\alpha)+(\cos\beta-i\sin\beta)+(\cos\gamma-i\sin\gamma)=0$$

$$e^{-i\alpha}+e^{-i\beta}+e^{-i\gamma}=0$$

$$\frac{1}{e^{i\alpha}}+\frac{1}{e^{i\beta}}+\frac{1}{e^{i\gamma}}=0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$$

$$\frac{yz+zx+xy}{xyz}=0$$

$$xy+yz+zx=0 \quad \dots(2)$$

(i) From Eq. (2),

$$xy+yz+zx=0$$

$$e^{i\alpha}e^{i\beta}+e^{i\beta}e^{i\gamma}+e^{i\gamma}e^{i\alpha}=0$$

$$e^{i(\alpha+\beta)}+e^{i(\beta+\gamma)}+e^{i(\gamma+\alpha)}=0$$

$$[\cos(\alpha+\beta)+i\sin(\alpha+\beta)]+[\cos(\beta+\gamma)+i\sin(\beta+\gamma)]+[\cos(\gamma+\alpha)+i\sin(\gamma+\alpha)]=0+i\cdot 0$$

Comparing the real parts,

$$\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)=0$$

(ii) From Eq. (1),

$$x+y+z=0$$

$$(x+y+z)^2=0$$

$$x^2+y^2+z^2+2(xy+yz+zx)=0$$

$$x^2+y^2+z^2=0 \quad \text{[Using Eq. (2)]}$$

$$(e^{i\alpha})^2+(e^{i\beta})^2+(e^{i\gamma})^2=0$$

$$e^{2i\alpha}+e^{2i\beta}+e^{2i\gamma}=0$$

$$(\cos 2\alpha+i\sin 2\alpha)+(\cos 2\beta+i\sin 2\beta)+(\cos 2\gamma+i\sin 2\gamma)=0+i\cdot 0$$

Comparing the real parts,

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0 \quad \dots(3)$$

(iii) From Eq. (3),

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$$

$$(2\cos^2\alpha-1)+(2\cos^2\beta-1)+(2\cos^2\gamma-1)=0$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$$

(iv) From Eq. (1),

$$x+y+z=0$$

$$x+y=-z$$

$$(x+y)^3=(-z)^3$$

$$x^3+y^3+3xy(x+y)=-z^3$$

$$x^3+y^3+z^3=-3xy(x+y)=-3xy(-z)=3xyz$$

$$(e^{i\alpha})^3+(e^{i\beta})^3+(e^{i\gamma})^3=3e^{i\alpha}e^{i\beta}e^{i\gamma}$$

$$e^{3i\alpha}+e^{3i\beta}+e^{3i\gamma}=3e^{i(\alpha+\beta+\gamma)}$$

$$(\cos 3\alpha+i\sin 3\alpha)+(\cos 3\beta+i\sin 3\beta)+(\cos 3\gamma+i\sin 3\gamma)=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$$

Comparing the imaginary parts,

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

EXAMPLE 13.9 *Prove that* $\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n$ *has the value* -1, *if* $n=3k\pm 1$ *(not a multiple of* 3) *and* 2, *if* $n=3k$ *(multiple of* 3), *where k is an integer.*

Solution: Let $$\frac{-1+i\sqrt{3}}{2}=r(\cos\theta+i\sin\theta)$$

$$r=\left|-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

and $$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)=\tan^{-1}\left(-\sqrt{3}\right)=\frac{2\pi}{3} \quad \left[\because \text{Point } \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) \text{ lies in the second quadrant}\right]$$

$$\frac{-1+i\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=e^{\frac{2i\pi}{3}}$$

and $$\frac{-1-i\sqrt{3}}{2}=\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}=e^{-\frac{2i\pi}{3}}$$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n=\left(e^{\frac{2i\pi}{3}}\right)^n+\left(e^{-\frac{2i\pi}{3}}\right)^n=e^{\frac{2in\pi}{3}}+e^{-\frac{2in\pi}{3}}=2\cos\left(\frac{2n\pi}{3}\right)$$

If $n = 3k \pm 1$,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k \pm 1) = 2\cos\left(2k\pi \pm \frac{2\pi}{3}\right)$$

$$= 2\cos\left(\frac{2\pi}{3}\right) \qquad [\because \cos(2k\pi \pm \theta) = \cos\theta]$$

$$= 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

If $n = 3k$,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k) = 2\cos 2k\pi = 2$$

EXERCISE 13.2

1. Simplify
$$\frac{(\cos 5\theta - i\sin 5\theta)^2(\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9(\cos\theta + i\sin\theta)^5}.$$
[**Ans.** : 1]

2. Express in polar form:
$$\frac{(1+i)^6(\sqrt{3}-i)^4}{(1-i)^8(\sqrt{3}+i)^5}$$
$\left[\textbf{Ans.}: \frac{i}{4}\right]$

3. Evaluate
$[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)]^n$
$+ [(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)]^n$.
$\left[\textbf{Ans.}: 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\pi+\theta+\phi}{2}\right)\right]$

4. If $x + \frac{1}{x} = 2\cos\theta,\ y + \frac{1}{y} = 2\cos\phi$, prove that
$$x^r y^r + \frac{1}{x^r y^r} = 2\cos(r\theta + r\phi).$$

5. If $2\cos\theta = x + \frac{1}{x}$, prove that
$$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}.$$

6. If $\sin\alpha + \sin\beta = 0 = \cos\alpha + \cos\beta$, prove that
   (i) $\cos 2\alpha + \cos 2\beta = 2\cos(\pi + \alpha + \beta)$
   (ii) $\sin 2\alpha + \sin 2\beta = 2\sin(\pi + \alpha + \beta)$

7. If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0 = \cos\alpha + 2\cos\beta + 3\cos\gamma$, prove that
   (i) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$
   (ii) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$

8. If $x_n + iy_n = \left(1 + i\sqrt{3}\right)^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

9. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2\cdot 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$. Hence, show that $\alpha^8 + \beta^8 = 32$.

10. If α, β are the roots of the equation $x^2 - \sqrt{3}x + 1 = 0$, prove that $\alpha^n + \beta^n = 2\cos\frac{n\pi}{6}$. Hence, show that $\alpha^{12} + \beta^{12} = 2$.

13.4 ROOTS OF AN ALGEBRAIC EQUATION

De Moivre's theorem can be used to find the roots of an algebraic equation.

Let the equation be $\qquad z^n = \cos\theta + i\sin\theta$

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = [\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{\frac{1}{n}} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$

Putting $k = 0, 1, 2, \ldots\ldots n-1$, all n roots of the equation are obtained.

Notes

(i) Complex roots always occur in conjugate pairs if all the coefficients of the equation including constant terms are real.

(ii) The product of all the roots of the equation is known as the *continued product*.

EXAMPLE 13.10 *Find the continued product of all the values of* $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$.

Solution: Let $\qquad z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

where $\qquad r = \left|\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

and $\qquad \theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = \left(\cos 3\cdot\frac{\pi}{3} + i\sin 3\cdot\frac{\pi}{3}\right)^{\frac{1}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}} = \left[\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}\right]$$

Putting $k = 0, 1, 2, 3$, all the 4 roots are obtained.

$$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}$$

$$z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{i\frac{3\pi}{4}}$$

$$z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{i\frac{5\pi}{4}}$$

$$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{i\frac{7\pi}{4}}$$

The continued product is

$$z_0 z_1 z_2 z_3 = e^{\frac{i\pi}{4}} \cdot e^{i\frac{3\pi}{4}} \cdot e^{i\frac{5\pi}{4}} \cdot e^{i\frac{7\pi}{4}} = e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)} = e^{i\frac{16\pi}{4}} = e^{i4\pi} = \cos 4\pi + i\sin 4\pi = 1 + i\cdot 0$$

Hence, the continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ is 1.

EXAMPLE 13.11 *Show that the n^{th} roots of unity form a geometric progression with common ratio $\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)$ and show that the continued product of all n^{th} roots is $(-1)^{n+1}$.*

Solution: To find nth roots of unity, consider the equation

$$x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

Putting $k = 0, 1, 2, \ldots\ldots n-1$, all n roots are obtained.

$$x_0 = \cos 0 + i\sin 0 = 1$$

$$x_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2i\pi}{n}} = \omega, \text{say}$$

$$x_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = e^{\frac{4i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^2 = \omega^2$$

$$x_3 = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n} = e^{\frac{6i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^3 = \omega^3$$

....

....

$$x_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = e^{\frac{2(n-1)i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^{n-1} = \omega^{n-1}$$

Hence, $1, \omega, \omega^2, \omega^3 \ldots\ldots, \omega^{n-1}$ represent n roots of unity which are in geometric progression with common ratio $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

The continued product of the roots is

$$x_0 \cdot x_1 \cdot x_2 \cdots\cdots x_{n-1} = 1\cdot e^{\frac{2i\pi}{n}} \cdot e^{\frac{4i\pi}{n}} \cdot e^{\frac{6i\pi}{n}} \cdots\cdots e^{\frac{2(n-1)i\pi}{n}} = e^{\frac{2i\pi}{n}\{1+2+3+\ldots\ldots+(n-1)\}}$$

$$= e^{\frac{2i\pi}{n}\frac{n(n-1)}{2}} \qquad \text{[Using sum of AP]}$$

$$= e^{i\pi(n-1)} = e^{in\pi}\cdot e^{-i\pi} = (\cos n\pi + i\sin n\pi)(\cos\pi - i\sin\pi) = (-1)^n(-1) = (-1)^{n+1}$$

Hence, the product of all n^{th} roots of unity is $(-1)^{n+1}$.

EXAMPLE 13.12 *Solve the following equations:*

(i) $x^6 - i = 0$ (ii) $x^{10} + 11x^5 + 10 = 0$

Solution:

(i)

$$x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos\left(2k\pi+\frac{\pi}{2}\right) + i\sin\left(2k\pi+\frac{\pi}{2}\right)$$

$$x = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{\frac{1}{6}}$$

$$= \cos(4k+1)\frac{\pi}{12} + i\sin(4k+1)\frac{\pi}{12} \qquad \text{[Using De Moivre's theorem]}$$

Putting $k = 0, 1, 2, 3, 4, 5$, all the 6 roots of the given equation are obtained.

(ii)

$$x^{10} + 11x^5 + 10 = 0$$
$$x^{10} + 10x^5 + x^5 + 10 = 0$$
$$x^5(x^5+10) + 1(x^5+10) = 0$$
$$(x^5+1)(x^5+10) = 0$$

All the roots of $x^{10} + 11x^5 + 10 = 0$ are the roots of $x^5 + 1 = 0$ and $x^5 + 10 = 0$.

$$x^5 + 1 = 0$$

$$x^5 = -1 = \cos\pi + i\sin\pi = \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$$

$$x = [\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi]^{\frac{1}{5}} = \cos(2k_1+1)\frac{\pi}{5} + i\sin(2k_1+1)\frac{\pi}{5} = e^{i(2k_1+1)\frac{\pi}{5}}$$

Putting $k_1 = 0, 1, 2, 3, 4$, all the 5 roots of $x^5 + 1 = 0$ are obtained.

Also, $x^5 + 10 = 0$

$$x^5 = -10 = 10(\cos\pi + i\sin\pi) = 10[\cos(2k_2\pi + \pi) + i\sin(2k_2\pi + \pi)]$$

$$x = \left[10\{\cos(2k_2+1)\pi + i\sin(2k_2+1)\pi\}\right]^{\frac{1}{5}} = (10)^{\frac{1}{5}}\left[\cos(2k_2+1)\frac{\pi}{5} + i\sin(2k_2+1)\frac{\pi}{5}\right] = (10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$$

Putting $k_2 = 0, 1, 2, 3, 4$, all the 5 roots of $x^5 + 10 = 0$ are obtained.

The roots of the equation $x^{10} + 11x^5 + 10 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{5}}$ and $(10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$

where $k_1 = k_2 = 0, 1, 2, 3, 4$.

EXAMPLE 13.13 *Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm\, i\cot\frac{k\pi}{7}$, $k = 1, 2, 3$.*

Solution: $(x+1)^7 = (x-1)^7$

$$\left(\frac{x+1}{x-1}\right)^7 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\frac{x+1}{x-1} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{7}} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right) = e^{i\frac{2k\pi}{7}}$$

where $k = 0, 1, 2, 3, 4, 5, 6$

But for $k = 0$,

$$\frac{x+1}{x-1} = \cos 0 + i\sin 0 = 1$$

$$x+1 = x-1$$

$$1 = -1, \text{ absurd}$$

Thus, $k \neq 0$

Hence, $\frac{x+1}{x-1} = e^{i\frac{2k\pi}{7}}$, where $k = 1, 2, 3, 4, 5, 6$

$$\frac{x+1}{x-1} = \frac{e^{i\theta}}{1}, \quad \text{where } \theta = \frac{2k\pi}{7}$$

Applying componendo–dividendo,

$$\frac{2x}{2} = \frac{e^{i\theta}+1}{e^{i\theta}-1}$$

$$x = \frac{(e^{i\theta}+1)e^{-\frac{i\theta}{2}}}{(e^{i\theta}-1)e^{-\frac{i\theta}{2}}} = \frac{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}} = \frac{2\cos\frac{\theta}{2}}{2i\sin\frac{\theta}{2}} = -i\cot\frac{\theta}{2} \qquad \left[\because \frac{1}{i} = \frac{i}{i^2} = -i\right]$$

$$= -i\cot\frac{k\pi}{7} \qquad \text{[Resubstituting } \theta]$$

Putting $k = 1, 2, 3, 4, 5, 6$, all the roots of $(x+1)^7 = (x-1)^7$ are obtained.

$$x_1 = -i\cot\frac{\pi}{7}$$

$$x_2 = -i\cot\frac{2\pi}{7}$$

$$x_3 = -i\cot\frac{3\pi}{7}$$

$$x_4 = -i\cot\frac{4\pi}{7} = -i\cot\left(\pi - \frac{3\pi}{7}\right) = i\cot\frac{3\pi}{7} = \bar{x}_3 \qquad [\because \cot(\pi-\theta) = -\cot\theta]$$

$$x_5 = -i\cot\frac{5\pi}{7} = -i\cot\left(\pi - \frac{2\pi}{7}\right) = i\cot\frac{2\pi}{7} = \bar{x}_2,$$

$$x_6 = -i\cot\frac{6\pi}{7} = -i\cot\left(\pi - \frac{\pi}{7}\right) = i\cot\frac{\pi}{7} = \bar{x}_1$$

Hence, all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot\frac{k\pi}{7}$, where $k = 1, 2, 3$.

EXAMPLE 13.14 *Find all the roots of $x^{12} - 1 = 0$ and identify the roots which are also the roots of $x^4 - x^2 + 1 = 0$.*

Solution:

$$x^{12} - 1 = 0$$

$$(x^6)^2 - 1 = 0$$

$$(x^6+1)(x^6-1) = 0$$

All the roots of $x^6 + 1 = 0$ and $x^6 - 1 = 0$ are the roots of $x^{12} - 1 = 0$.

$$x^6 = -1 = \cos\pi + i\sin\pi = \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$$

$$x = [\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi]^{\frac{1}{6}} = \cos(2k_1+1)\frac{\pi}{6} + i\sin(2k_1+1)\frac{\pi}{6}$$

$$= e^{i(2k_1+1)\frac{\pi}{6}}$$

Putting $k_1 = 0, 1, 2, 3, 4, 5$, all the roots of $x^6 + 1 = 0$ are obtained.

Also, $x^6 - 1 = 0$

$$x^6 = 1 = \cos 0 + i\sin 0 = \cos 2k_2\pi + i\sin 2k_2\pi$$

$$x = (\cos 2k_2\pi + i\sin 2k_2\pi)^{\frac{1}{6}}$$

$$= \cos\frac{2k_2\pi}{6} + i\sin\frac{2k_2\pi}{6} = \cos\frac{k_2\pi}{3} + i\sin\frac{k_2\pi}{3} = e^{i\frac{k_2\pi}{3}}$$

Putting $k_2 = 0, 1, 2, 3, 4, 5$, all the roots of $x^6 - 1 = 0$ are obtained.

Hence, all the roots of $x^{12} - 1 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{6}}$ and $e^{i\frac{k_2\pi}{3}}$, where $k_1 = k_2 = 0, 1, 2, 3, 4, 5$.

Now,

$$x^6 + 1 = 0$$

$$(x^2)^3 + 1 = 0$$

$$(x^2+1)[(x^2)^2 - x^2 + 1] = 0 \qquad [\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$$

$$(x^2+1)(x^4 - x^2 + 1) = 0$$

Thus, all the roots of $x^6 + 1 = 0$, except $x = \pm i$, are the roots of $x^4 - x^2 + 1 = 0$.

The roots of $x^6 + 1 = 0$ are

$$x_0 = e^{\frac{i\pi}{6}}$$

$$x_1 = e^{\frac{i3\pi}{6}} = e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$$

$$x_2 = e^{\frac{i5\pi}{6}}$$

$$x_3 = e^{\frac{i7\pi}{6}}$$

$$x_4 = e^{\frac{i9\pi}{6}} = e^{\frac{i3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i = -i$$

$$x_5 = e^{\frac{i11\pi}{6}}$$

Hence, x_0, x_2, x_3 and x_5 are the roots of the equation $x^4 - x^2 + 1 = 0$.

EXAMPLE 13.15 *Show that* $x^5 - 1 = (x-1)\left[x^2 + 2x\cos\frac{\pi}{5} + 1\right]\left[x^2 + 2x\cos\frac{3\pi}{5} + 1\right]$.

Solution:

$$x^5 - 1 = 0$$

$$x^5 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} = e^{i\frac{2k\pi}{5}}$$

Putting $k = 0, 1, 2, 3, 4$, all the roots of $x^5 - 1 = 0$ are obtained.

$$x_0 = 1$$

$$x_1 = e^{i\frac{2\pi}{5}} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$x_2 = e^{i\frac{4\pi}{5}} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$x_3 = e^{i\frac{6\pi}{5}} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right) = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} = e^{-i\frac{4\pi}{5}} \quad \left[\begin{array}{l}\because \cos(2\pi - \theta) = \cos\theta \\ \sin(2\pi - \theta) = -\sin\theta\end{array}\right]$$

$$= \overline{x_2}$$

$$x_4 = e^{i\frac{8\pi}{5}} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = e^{-i\frac{2\pi}{5}} = \overline{x_1}$$

Now,

$$x^5 - 1 = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$= (x-1)\left(x - e^{\frac{2i\pi}{5}}\right)\left(x - e^{\frac{4i\pi}{5}}\right)\left(x - e^{-\frac{4i\pi}{5}}\right)\left(x - e^{-\frac{2i\pi}{5}}\right)$$

$$= (x-1)\left(x - e^{\frac{2i\pi}{5}}\right)\left(x - e^{-\frac{2i\pi}{5}}\right)\left(x - e^{\frac{4i\pi}{5}}\right)\left(x - e^{-\frac{4i\pi}{5}}\right)$$

$$= (x-1)\left[x^2 - x\left(e^{\frac{2i\pi}{5}} + e^{-\frac{2i\pi}{5}}\right) + 1\right]\left[x^2 - x\left(e^{\frac{4i\pi}{5}} + e^{-\frac{4i\pi}{5}}\right) + 1\right]$$

$$= (x-1)\left[x^2 - x\left(2\cos\frac{2\pi}{5}\right) + 1\right]\left[x^2 - x\left(2\cos\frac{4\pi}{5}\right) + 1\right]$$

$$= (x-1)\left[x^2 - 2x\cos\left(\pi - \frac{3\pi}{5}\right) + 1\right]\left[x^2 - 2x\cos\left(\pi - \frac{\pi}{5}\right) + 1\right]$$

$$= (x-1)\left[x^2 + 2x\cos\frac{3\pi}{5} + 1\right]\left[x^2 + 2x\cos\frac{\pi}{5} + 1\right] \quad [\because \cos(\pi - \theta) = -\cos\theta]$$

EXAMPLE 13.16 *If* $\alpha, \alpha^2, \alpha^3, \alpha^4$ *are roots of* $x^5 - 1 = 0$ *then show that* $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Solution: One root of $x^5 - 1 = 0$ is obviously 1, the remaining roots are given as $\alpha, \alpha^2, \alpha^3, \alpha^4$.

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$[\because \ x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots\ldots + 1)]$$

$$x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

Putting $x = 1$ on both the sides,

$$1 + 1 + 1 + 1 + 1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

EXERCISE 13.3

1. Find all the values of the following:

   (i) $\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}}$ (ii) $\left(\frac{2+3i}{1+i}\right)^{\frac{1}{4}}$

   $\left[\begin{array}{l}\textbf{Ans.:} \text{ (i) } 2^{\frac{4}{3}}\cos(6k+1)^{\frac{\pi}{6}}, k = 0, 1, 2 \\ \text{(ii) } \left(\frac{13}{2}\right)^{\frac{1}{8}} e^{i\left(2k\pi + \tan^{-1}\frac{1}{15}\right)}, k = 0, 1, 2, 3\end{array}\right]$

2. Find the continued product of all the values of the following:

   (i) $(i)^{\frac{2}{3}}$ (ii) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$

   [Ans. : (i) –1 (ii) 1]

3. Solve the equations:

   (i) $x^{12} - 1 = 0$

   (ii) $x^5 + \sqrt{3} = i$

   (iii) $x^7 + x^4 + x^3 + 1 = 0$

   (iv) $(1 + x)^6 + x^6 = 0$ (v) $(2z - 1)^5 = 32z^5$

   (vi) $x^{14} + 127x^7 - 128 = 0$

Ans.:

(i) $\pm 1, \pm i, \pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right), \pm\left(\frac{\sqrt{3}}{2} \pm \frac{i}{2}\right)$

(ii) $2^{\frac{1}{5}} e^{i\left(\frac{12k+5}{30}\right)\pi}, k = 0, 1, 2, 3, 4$

(iii) $-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \pm\left(\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right)$

(iv) $-\frac{1}{2} - i\cot(2k+1)\frac{\pi}{12}, k = 0, 1, 2, 3, 4, 5$

(v) $\frac{1}{4}\left(1 + i\cos\frac{k\pi}{5}\right), k = 1, 2, 3, 4$

(vi) $e^{i\frac{2k_1\pi}{7}}, e^{i(2k_2+1)\frac{\pi}{7}}, k_1 = k_2 = 0, 1, 2, 3, 4, 5, 6$

4. Solve the equations:

   (i) $x^4 - x^2 + 1 = 0$

   (ii) $x^4 + x^2 + 1 = 0$

Ans.: (i) $e^{i(2k+1)\frac{\pi}{6}}, k = 0, 2, 3, 5$

(ii) $e^{i\frac{k\pi}{3}}, k = 1, 2, 4, 5$

5. Solve the equation: $x^2 + x^{-2} = i$.

Ans.:

$$x = \frac{(1 \pm \sqrt{5})}{2}\left\{\cos\left(2k\pi + \frac{\pi}{2}\right)\frac{1}{2} + i\sin\left(2k\pi + \frac{\pi}{2}\right)\frac{1}{2}\right\}$$

6. Prove that $(x^2 - x^3)(x^4 - x) = \sqrt{5}$, where $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

7. Prove that $(-1 + i)^7 = -8(1 + i)$.

8. Prove that

   (i) $1 + \omega + \omega^2 = 0$

   (ii) $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$, where ω is a complex cube root of unity.

9. Prove that

   (i) $\alpha^{3n} + \beta^{3n} = 2$, n is an integer

   (ii) $\alpha e^{\alpha x} + \beta e^{\beta x}$

   $$= -e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right],$$ where

   α, β are complex cube roots of unity.

10. Prove that

   (i) $\sqrt[n]{a+ib} + \sqrt[n]{a-ib}$ has n real roots.

   (ii) $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}}$

   $$= 2\left(\sqrt{a^2+b^2}\right)^{\frac{m}{n}}\cos\left(\frac{m\,\theta}{n}\right)$$

   where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

11. Prove that all the roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by $-i\cot\frac{(2k+1)\pi}{12}, k = 0, 1, 2, 3, 4, 5.$

12. Prove that the points representing the roots of the equation $z^3 = i(z - 1)^3$ on the Argand's diagram are collinear.

13. If $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the roots of $x^7 - 1 = 0$, prove that $(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4)(1 - \alpha^5)(1 - \alpha^6) = 7$.

14. If $1 + 2i$ is one root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.

   Ans.: $1 - 2i, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

15. Prove that

$$x^7 + 1 = (x + 1)\left(x^2 - 2x\cos\frac{\pi}{7} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{7} + 1\right)\left(x^2 - 2x\cos\frac{5\pi}{7} + 1\right).$$

13.5 EXPANSION OF TRIGONOMETRIC FUNCTIONS

13.5.1 Type I Expansion of $\sin^n\theta$, $\cos^n\theta$ in terms of $\sin n\theta$, $\cos n\theta$, where n is a Positive Integer

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

$$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

Hence, $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

Again, $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$

$$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

To expand $\cos^n\theta$ and $\sin^n\theta$, write $\cos^n\theta = \frac{1}{2^n}\left(x + \frac{1}{x}\right)^n$ and $\sin^n\theta = \frac{1}{(2i)^n}\left(x - \frac{1}{x}\right)^n$ and expand the RHS using binomial expansion

$$(x + a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + a^n$$

EXAMPLE 13.17 *Prove that* $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

Solution: Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$\cos^6\theta = \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^6 = \frac{1}{2^6}\left(x + \frac{1}{x}\right)^6$$

$$= \frac{1}{2^6}\left(x^6 + 6x^5 \cdot \frac{1}{x} + 15x^4\frac{1}{x^2} + 20x^3 \cdot \frac{1}{x^3} + 15x^2 \cdot \frac{1}{x^4} + 6x \cdot \frac{1}{x^5} + \frac{1}{x^6}\right)$$

$$= \frac{1}{2^6}\left[\left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20\right]$$

$$= \frac{1}{2^6}(2\cos 6\theta + 6 \cdot 2\cos 4\theta + 15 \cdot 2\cos 2\theta + 20) \qquad \left[\because x^n + \frac{1}{x^n} = 2\cos n\theta\right]$$

$$= \frac{1}{2^5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10) \qquad \ldots(1)$$

$$\sin^6\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^6$$

$$= \frac{1}{(2i)^6}\left[x^6 + 6x^5\left(-\frac{1}{x}\right) + 15x^4\left(-\frac{1}{x}\right)^2 + 20x^3\left(-\frac{1}{x}\right)^3 + 15x^2\left(-\frac{1}{x}\right)^4 + 6x\left(-\frac{1}{x}\right)^5 + \left(-\frac{1}{x}\right)^6\right]$$

$$= -\frac{1}{2^6}\left[\left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20\right] \qquad [\because i^6 = (i^2)^3 = -1]$$

$$= -\frac{1}{2^6}(2\cos 6\theta - 6\cdot 2\cos 4\theta + 15\cdot 2\cos 2\theta - 20)$$

$$= -\frac{1}{2^5}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10) \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$\cos^6\theta + \sin^6\theta = \frac{1}{2^5}(12\cos 4\theta + 20) = \frac{1}{8}(3\cos 4\theta + 5)$$

EXAMPLE 13.18 *If $\sin^4\theta\cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$, prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$.*

Solution: Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$\sin^4\theta\cos^3\theta = \left[\frac{1}{2i}\left(x-\frac{1}{x}\right)\right]^4\left[\frac{1}{2}\left(x+\frac{1}{x}\right)\right]^3$$

$$= \frac{1}{(2i)^4}\left(x-\frac{1}{x}\right)\left(x-\frac{1}{x}\right)^3\cdot\frac{1}{2^3}\left(x+\frac{1}{x}\right)^3 = \frac{1}{2^4 i^4}\cdot\frac{1}{2^3}\left(x-\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^3$$

$$= \frac{1}{2^7}\left(x-\frac{1}{x}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right) = \frac{1}{2^7}\left(x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}\right)$$

$$= \frac{1}{2^7}\left(x^7+\frac{1}{x^7}\right) - \left(x^5+\frac{1}{x^5}\right) - 3\left(x^3+\frac{1}{x^3}\right) + 3\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{2^7}(2\cos 7\theta - 2\cos 5\theta - 3\cdot 2\cos 3\theta + 3\cdot 2\cos\theta)$$

$$= \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$$

$$= \frac{1}{64}(3\cos\theta - 3\cos 3\theta - \cos 5\theta + \cos 7\theta) \qquad \ldots(1)$$

Given $\sin^4\theta\cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$ ...(2)

Comparing Eqs (1) and (2),

$$a_1 = \frac{3}{64},\quad a_3 = -\frac{3}{64},\quad a_5 = -\frac{1}{64},\quad a_7 = \frac{1}{64}$$

$$\therefore a_1 + 9a_3 + 25a_5 + 49a_7 = \frac{3}{64} + 9\left(-\frac{3}{64}\right) + 25\left(-\frac{1}{64}\right) + 49\left(\frac{1}{64}\right) = 0$$

13.5.2 Type II Expansion of sin $n\theta$, cos $n\theta$ in Powers of sin θ, cos θ

By De Moivre's theorem,

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\cdot i\sin\theta + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots$$

$$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$$

Comparing real and imaginary parts,

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots$$

EXAMPLE 13.19 *Expand $\dfrac{\sin 5\theta}{\sin\theta}$ in powers of $\cos\theta$ only.*

Solution: $(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$ (Using De Moivre's theorem)

$$= \cos^5\theta + 5\cos^4\theta\cdot i\sin\theta + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i\,(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Comparing imaginary parts,

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta = \sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)$$

$$\frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 16\cos^4\theta - 12\cos^2\theta + 1$$

EXAMPLE 13.20 *Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ and hence, deduce that $1 - \tan^2\dfrac{\pi}{10} + 5\tan^4\dfrac{\pi}{10} = 0$.*

Solution: Comparing real and imaginary parts in Example 13.19,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing the numerator and denominator by $\cos^5\theta$,

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1-\tan^2\theta + 5\tan^4\theta}$$

Putting $\theta = \frac{\pi}{10}$,

$$\tan 5\cdot\frac{\pi}{10} = \frac{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}{1-\tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}} \quad \cdots (1)$$

$$\frac{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}{1-\tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}} = \tan\frac{\pi}{2} = \infty = \frac{1}{0}$$

Hence, $1-\tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10} = 0$

EXAMPLE 13.21 *If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, find the values of a, b, c.*

Solution: $(\cos 6\theta + i\sin 6\theta) = (\cos\theta + i\sin\theta)^6$

$$\begin{aligned} &= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 \\ &\quad + 20\cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4 \\ &\quad + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6 \\ &= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta) \\ &\quad + i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta) \end{aligned}$$

Comparing imaginary parts,

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \quad ...(1)$$

Given $\quad \sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta \quad ...(2)$

Comparing Eqs (1) and (2),

$$a = 6, b = -20, c = 6.$$

EXAMPLE 13.22 *Prove that* $\frac{1+\cos 6\theta}{1+\cos 2\theta} = 16\cos^4\theta - 24\cos^2\theta + 9.$

Solution: Comparing real parts in Example 13.21,

$$\begin{aligned} \cos 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \\ &= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3 \\ &= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) \\ &\quad - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) \\ &= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 \end{aligned}$$

$$\frac{1+\cos 6\theta}{1+\cos 2\theta} = \frac{32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta}{2\cos^2\theta} = 16\cos^4\theta - 24\cos^2\theta + 9$$

EXAMPLE 13.23 *Using De Moivre's theorem, prove that*

$$\frac{1+\cos 9\theta}{1+\cos\theta} = (x^4 - x^3 - 3x^2 = 2x = 1)^2, \text{ where } x = 2\cos\theta.$$

Solution:
$$\frac{1+\cos 9\theta}{1+\cos\theta} = \frac{2\cos^2\frac{9\theta}{2}}{2\cos^2\frac{\theta}{2}}\cdot\frac{2\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{\left(2\cos\frac{9\theta}{2}\sin\frac{\theta}{2}\right)^2}{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2} = \frac{(\sin 5\theta - \sin 4\theta)^2}{\sin^2\theta} \quad [\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

From Example 13.20,

$$\begin{aligned} \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= \sin\theta[5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2] \\ &= \sin\theta[5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta] \\ &= \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1] \\ \sin 4\theta &= \sin 2(2\theta) \\ &= 2\sin 2\theta\cdot\cos 2\theta \\ &= 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1) \\ &= \sin\theta(8\cos^3\theta - 4\cos\theta) \\ \sin 5\theta - \sin 4\theta &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1 - 8\cos^3\theta + 4\cos\theta) \\ &= \sin\theta(16\cos^4\theta - 8\cos^3\theta - 12\cos^2\theta + 4\cos\theta + 1) \end{aligned}$$

$$\begin{aligned} \frac{1+\cos 9\theta}{1+\cos\theta} &= \frac{(\sin 5\theta - \sin 4\theta)^2}{\sin^2\theta} \\ &= \frac{\sin^2\theta(16\cos^4\theta - 8\cos^3\theta - 12\cos^2\theta + 4\cos\theta + 1)^2}{\sin^2\theta} \\ &= (x^4 - x^3 - 3x^2 + 2x + 1)^2, \text{ where } x = 2\cos\theta \end{aligned}$$

EXERCISE 13.4

1. Prove that
$\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$.

2. Prove that
$\cos^4\theta\sin^3\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - \sin\theta)$.

3. Prove that $-2^{12}\cos^6\theta\sin^7\theta = (\sin 13\theta - \sin 11\theta - 6\sin 9\theta + 6\sin 7\theta + 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta)$.

4. Prove that $-256\sin^7\theta\cos^2\theta = \cos 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta$.

5. Expand $\dfrac{\sin 7\theta}{\sin\theta}$ in powers of $\sin\theta$ only.

$\left[\textbf{Ans.:} 7-56\sin^2\theta+112\sin^4\theta-64\sin^6\theta\right]$

6. Prove that $\tan 7\theta=$
$$\frac{7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$$
and hence, deduce that
$$1-21\tan^2\frac{\pi}{14}+35\tan^4\frac{\pi}{14}-7\tan^6\frac{\pi}{14}=0.$$

7. Prove that $1-\cos 10\theta=2\,(16\sin^5\theta-20\sin^3\theta+5\sin\theta)^2$.

8. Prove that $\dfrac{1+\cos 7\theta}{1+\cos\theta}=(x^3-x^2-2x+1)^2$, where $x=2\cos\theta$.

13.5.3 Type III Summation of Sine and Cosine Series

De Moivre's theorem can also be used to find the sum of sine and cosine series of the form

$$a_0\sin\alpha+a_1\sin(\alpha+\beta)+a_2\sin(\alpha+2\beta)+\ldots\ldots \quad \ldots(13.2)$$
$$a_0\cos\alpha+a_1\cos(\alpha+\beta)+a_2\cos(\alpha+2\beta)+\ldots\ldots \quad \ldots(13.3)$$

where $a_0, a_1, a_2, \ldots$ are either constants or some standard functions.

Working Rule to Find the Sum of Sine and Cosine Series

(i) Denote the given series by S if it is a sine series and by C if it is a cosine series.

(ii) Write the cosine series C if the sine series S is known, by replacing sine terms with cosine terms. Similarly, write the sine series S if the cosine series C is known, by replacing cosine terms with sine terms in the given series.

(iii) Multiply the sine series by i and add to the cosine series to obtain $C+iS$. Find the sum of the series $C+iS$ by using any one of the following series and then separate its real and imaginary parts to obtain C and S.

1. Geometric series: (i) $a+ar+ar^2+\ldots\ldots+ar^{n-1}=\dfrac{a(1-r^n)}{1-r},\ |r|<1$

   (ii) $a+ar+ar^2+\ldots\ldots\infty=\dfrac{a}{1-r},\ |r|<1.$

2. Exponential series: $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\ldots\infty=e^x.$

3. Logarithmic series: (i) $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots\ldots\infty=\log(1+x)$

   (ii) $-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\ldots\ldots\infty=\log(1-x).$

4. Trigonometric series:

   (i) $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots\ldots\infty=\sin x$ (ii) $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots\ldots\infty=\cos x$

   (iii) $x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots\ldots\infty=\sinh x$ (iv) $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots\ldots\infty=\cosh x$

5. Binomial series: $1+nx+\dfrac{n(n-1)}{2!}x^2+\ldots\ldots\infty=(1+x)^n.$

6. Gregory series: (i) $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\ldots\infty=\tan^{-1}x$

   (ii) $x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots\ldots\infty=\tanh^{-1}x=\dfrac{1}{2}\log\dfrac{1+x}{1-x}$

EXAMPLE 13.24 *Find the sum of the series* $\cos\alpha+\dfrac{1}{2}\cos 3\alpha+\dfrac{1}{2^2}\cos 5\alpha+\ldots\ldots$

Solution: Let $C=\cos\alpha+\dfrac{1}{2}\cos 3\alpha+\dfrac{1}{2^2}\cos 5\alpha+\ldots\ldots$

$$S=\sin\alpha+\frac{1}{2}\sin 3\alpha+\frac{1}{2^2}\sin 5\alpha+\ldots\ldots$$

$$C+iS=(\cos\alpha+i\sin\alpha)+\frac{1}{2}(\cos 3\alpha+i\sin 3\alpha)+\frac{1}{2^2}(\cos 5\alpha+i\sin 5\alpha)+\ldots\ldots$$

$$=e^{i\alpha}+\frac{1}{2}e^{i3\alpha}+\frac{1}{2^2}e^{i5\alpha}+\ldots\ldots\infty=\frac{e^{i\alpha}}{1-\frac{1}{2}e^{i2\alpha}} \qquad \text{[Sum of GP]}$$

$$=\frac{e^{i\alpha}\left(1-\frac{1}{2}e^{-i2\alpha}\right)}{\left(1-\frac{1}{2}e^{i2\alpha}\right)\left(1-\frac{1}{2}e^{-i2\alpha}\right)}=\frac{e^{i\alpha}-\frac{1}{2}e^{-i\alpha}}{1-\frac{1}{2}e^{-i2\alpha}-\frac{1}{2}e^{i2\alpha}+\frac{1}{4}}$$

$$=\frac{(\cos\alpha+i\sin\alpha)-\frac{1}{2}(\cos\alpha-i\sin\alpha)}{\frac{5}{4}-\cos 2\alpha}=\frac{\frac{1}{2}\cos\alpha+i\frac{3}{2}\sin\alpha}{\frac{5}{4}-\cos 2\alpha}$$

Comparing real parts,

$$C=\frac{\frac{1}{2}\cos\alpha}{\frac{5-4\cos 2\alpha}{4}}=\frac{2\cos\alpha}{5-4\cos 2\alpha}$$

EXAMPLE 13.25 *Find the sum of the series* $1+x\cos\alpha+x^2\cos 2\alpha+x^3\cos 3\alpha+\ldots\ldots n$ *terms, where* $x<1$. *Also, find the sum to infinity.*

Solution: Let $C=1+x\cos\alpha+x^2\cos 2\alpha+x^3\cos 3\alpha+\ldots\ldots n$ terms

$S=0+x\sin\alpha+x^2\sin 2\alpha+x^3\sin 3\alpha+\ldots\ldots n$ terms

$[\because \text{First term}=x^2\sin(0\cdot\alpha)=\sin 0=0]$

$$C + iS = (1 + i0) + x(\cos\alpha + i\sin\alpha) + x^2(\cos 2\alpha + i\sin 2\alpha) + x^3(\cos 3\alpha + i\sin 3\alpha) + \ldots\ldots n \text{ terms}$$

$$= 1 + xe^{i\alpha} + x^2 e^{i2\alpha} + x^3 e^{i3\alpha} + \ldots\ldots n \text{ terms}$$

[Sum of n terms of GP]

$$= \frac{1-(xe^{i\alpha})^n}{1-xe^{i\alpha}}$$

$$= \frac{1-x^n e^{in\alpha}}{1-xe^{i\alpha}} = \frac{1-x^n e^{in\alpha}}{1-xe^{i\alpha}} \cdot \frac{1-xe^{-i\alpha}}{1-xe^{-i\alpha}} = \frac{1-xe^{-i\alpha}-x^n e^{in\alpha}+x^{n+1}e^{i(n-1)\alpha}}{1-xe^{-i\alpha}-xe^{i\alpha}+x^2}$$

$$= \frac{1-x(\cos\alpha - i\sin\alpha) - x^n(\cos n\alpha + i\sin n\alpha) + x^{n+1}[\cos(n-1)\alpha + i\sin(n-1)\alpha]}{1-2x\cos\alpha + x^2}$$

Equating real parts,

$$C = \frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha+x^2}$$

To find sum to infinity, taking limit $n \to \infty$ in the above expression,

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha+x^2}$$

Since, $x < 1$, $\lim_{n\to\infty} x^n = 0$ and $\lim_{n\to\infty} x^{n+1} = 0$

$$\lim_{n\to\infty} C = \frac{1-x\cos\alpha}{1-2x\cos\alpha+x^2}$$

Hence, $1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \ldots\ldots \infty = \dfrac{1-x\cos\alpha}{1-2x\cos\alpha+x^2}$

EXAMPLE 13.26 *Find the sum of the series* $x\sin\theta + \dfrac{x^2}{2!}\sin 2\theta + \dfrac{x^3}{3!}\sin 3\theta + \ldots\ldots \infty$.

Solution: Let $S = x\sin\theta + \dfrac{x^2}{2!}\sin 2\theta + \dfrac{x^3}{3!}\sin 3\theta + \ldots\ldots$

$$C = 1 + x\cos\theta + \frac{x^2}{2!}\cos 2\theta + \frac{x^3}{3!}\cos 3\theta + \ldots$$

[$\because$ first term $= x^0\cos(0.\theta) = \cos 0 = 1$]

$$C + iS = 1 + x(\cos\theta + i\sin\theta) + \frac{x^2}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{x^3}{3!}(\cos 3\theta + i\sin 3\theta) + \ldots\ldots$$

$$= 1 + xe^{i\theta} + \frac{x^2}{2!}e^{i2\theta} + \frac{x^3}{3!}e^{i3\theta} + \ldots\ldots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots, \text{ where } z = xe^{i\theta}$$

$$= e^z = e^{xe^{i\theta}} = e^{x(\cos\theta + i\sin\theta)}$$

$$= e^{x\cos\theta} \cdot e^{ix\sin\theta} = e^{x\cos\theta}[\cos(x\sin\theta) + i\sin(x\sin\theta)]$$

Comparing imaginary parts,

$$S = e^{x\cos\theta} \cdot \sin(x\sin\theta)$$

EXAMPLE 13.27 *Find the sum of the series* $a\cos^2\alpha - \dfrac{1}{3}a^3\cos^2 3\alpha + \dfrac{1}{5}a^5\cos^2 5\alpha - \ldots\ldots \infty$.

Solution: Let $C = a\cos^2\alpha - \dfrac{1}{3}a^3\cos^2 3\alpha + \dfrac{1}{5}a^5\cos^2 5\alpha - \ldots\ldots \infty$

$$= a\frac{(1+\cos 2\alpha)}{2} - \frac{a^3}{3}\frac{(1+\cos 6\alpha)}{2} + \frac{a^5}{5}\frac{(1+\cos 10\alpha)}{2} + \ldots\ldots \infty$$

$$= \frac{1}{2}\left(a - \frac{a^3}{3} + \frac{a^5}{5} - \ldots\ldots \infty\right) + \frac{1}{2}\left(a\cos 2\alpha - \frac{a^3}{3}\cos 6\alpha + \frac{a^5}{5}\cos 10\alpha + \ldots\ldots \infty\right)$$

$$= \frac{1}{2}\tan^{-1} a + \frac{1}{2}\left(a\cos 2\alpha - \frac{a^3}{3}\cos 6\alpha + \frac{a^5}{5}\cos 10\alpha + \ldots\right)$$

$$\left[\because\ a - \frac{a^3}{3} + \frac{a^5}{5} - \ldots = \tan^{-1} a\right]$$

$$= \frac{1}{2}\tan^{-1} a + \frac{1}{2}C_1 \qquad \ldots (1)$$

where $C_1 = a\cos 2\alpha - \dfrac{a^3}{3}\cos 6\alpha + \dfrac{a^5}{5}\cos 10\alpha + \ldots$

$$S_1 = a\sin 2\alpha - \frac{a^3}{3}\sin 6\alpha + \frac{a^5}{5}\sin 10\alpha + \ldots$$

$$C_1 + iS_1 = a(\cos 2\alpha + i\sin 2\alpha) - \frac{a^3}{3}(\cos 6\alpha + i\sin 6\alpha) + \frac{a^5}{5}(\cos 10\alpha + i\sin 10\alpha) + \ldots\ldots$$

$$= ae^{i2\alpha} - \frac{a^3}{3}e^{i6\alpha} + \frac{a^5}{5}e^{i10\alpha} + \ldots\ldots$$

$$= ae^{i2\alpha} - \frac{(ae^{i2\alpha})^3}{3} + \frac{(ae^{i2\alpha})^5}{5} - \ldots\ldots$$

$$= \tan^{-1}(ae^{i2\alpha}) = \tan^{-1}[a(\cos 2\alpha + i\sin 2\alpha)]$$

Let $\tan^{-1}(ae^{i2\alpha}) = x + iy$

$$\tan^{-1}(ae^{-i2\alpha}) = x - iy$$

Adding both the equations,

$$2x = \tan^{-1}(ae^{i2\alpha}) + \tan^{-1}(ae^{-i2\alpha}) = \tan^{-1}\left[\frac{ae^{i2\alpha} + ae^{-i2\alpha}}{1 - ae^{i2\alpha}.ae^{-i2\alpha}}\right]$$

$$x = \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

$$\therefore C_1 + iS_1 = \tan^{-1}(a\cos 2\alpha + ia\sin 2\alpha) = x + iy = \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right) + iy$$

Comparing real parts,

$$C_1 = \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

From Eq. (1),

$$C = \frac{1}{2}\tan^{-1} a + \frac{1}{4}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

EXAMPLE 13.28 *Find the sum of the series* $a \sin \alpha + \frac{a^3}{3}\sin 3\alpha + \frac{a^5}{5}\sin 5\alpha + \ldots\ldots$.

Solution: Let $$S = a\sin\alpha + \frac{a^3}{3}\sin 3\alpha + \frac{a^5}{5}\sin 5\alpha + \ldots\ldots$$

and $$C = a\cos\alpha + \frac{a^3}{3}\cos 3\alpha + \frac{a^5}{5}\cos 5\alpha + \ldots\ldots$$

$$C + iS = a(\cos\alpha + i\sin\alpha) + \frac{a^3}{3}(\cos 3\alpha + i\sin 3\alpha) + \frac{a^5}{5}(\cos 5\alpha + i\sin 5\alpha) + \ldots\ldots \infty$$

$$= ae^{i\alpha} + \frac{a^3}{3}e^{i3\alpha} + \frac{a^5}{5}e^{i5\alpha} + \ldots\ldots$$

$$= \frac{1}{2}\log\frac{1+ae^{i\alpha}}{1-ae^{i\alpha}} \qquad \left[\because x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \frac{1}{2}\log\frac{1+x}{1-x}\right]$$

$$= \frac{1}{2}[\log\{(1+a\cos\alpha) + ia\sin\alpha\} - \log\{(1-a\cos\alpha) - ia\sin\alpha\}]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{(1+a\cos\alpha)^2 + a^2\sin^2\alpha\right\} + i\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right) - \frac{1}{2}\log\left\{(1-a\cos\alpha)^2 + a^2\sin^2\alpha\right\} - i\tan^{-1}\left(\frac{-a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$\left[\because \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Comparing imaginary parts,

$$S = \frac{1}{2}\left[\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right) - \tan^{-1}\left(\frac{-a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right) + \tan^{-1}\left(\frac{a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left\{\frac{\frac{a\sin\alpha}{1+a\cos\alpha} + \frac{a\sin\alpha}{1-a\cos\alpha}}{1-\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right)\cdot\left(\frac{a\sin\alpha}{1-a\cos\alpha}\right)}\right\}\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2a\sin\alpha}{1-a^2}\right)$$

EXERCISE 13.5

Find the sum of the series:

1. $\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^3}\sin 3\theta + \frac{1}{2^4}\sin 4\theta + \ldots$

   $\left[\textbf{Ans.: } \frac{2\sin\theta}{5-\cos\theta}\right]$

2. $1 + \cos\alpha\cos\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha \cos 3\alpha + \ldots\ldots$

   [**Ans.:** $\sin^2\alpha$]

3. $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{3!}\cos 3\beta + \ldots\ldots$

   [**Ans.:** $e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\cos\beta)$]

4. $\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \ldots + \sin\left[\frac{2(n-1)\pi}{n}\right]$

   [**Ans.:** 0]

5. $x\sin\alpha - \frac{1}{2}x^2\sin 2\alpha + \frac{1}{3}x^3\sin 3\alpha - \ldots\infty$

   $\left[\textbf{Ans.: } \tan^{-1}\left(\frac{x\sin\alpha}{1+x\cos\alpha}\right)\right]$

6. $e^{\alpha}\cos\beta - \frac{1}{3}e^{3\alpha}\cos 3\beta + \frac{1}{5}e^{5\alpha}\cos^5\beta + \ldots\infty$

   $\left[\textbf{Ans.: } \frac{1}{2}\tan^{-1}\frac{\cos\beta}{\sinh\alpha}\right]$

7. $1 - \frac{1}{2}\cos\alpha + \frac{1.3}{2.4}\cos 2\alpha - \frac{1.3.5}{2.4.6}\cos 3\alpha + \ldots\ldots\infty$

   $\left[\textbf{Ans.: } (2\cos\alpha)^{-\frac{1}{2}}\cos\frac{\alpha}{4}\right]$

8. $\frac{1}{2}\sin\alpha + \frac{1.3}{2.4}\sin 2\alpha + \frac{1.3.5}{2.4.6}\sin 3\alpha + \ldots\ldots (\alpha \neq n\pi)$

   $\left[\textbf{Ans.: } \left(2\sin\frac{\alpha}{2}\right)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4} - \frac{\alpha}{4}\right)\right]$

13.6 CIRCULAR AND HYPERBOLIC FUNCTIONS OF COMPLEX NUMBERS

13.6.1 Circular Functions

From Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

If $z = x + iy$ is a complex number then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

These are called circular functions of complex numbers.

13.6.2 Hyperbolic Functions

If z is a complex number then the sine hyperbolic of z is denoted by $\sinh z$ and is given as

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

and the cosine hyperbolic of z is denoted by $\cosh z$ and is given as

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

From these expressions, other hyperbolic functions can also be obtained using

$$\tanh z = \frac{\sinh z}{\cosh z}, \operatorname{cosech} z = \frac{1}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z} \quad \text{and } \coth z = \frac{1}{\tanh z}$$

From the above definitions of sinh z, cosh z, tanh z, the following range of hyperbolic functions is obtained:

z	$-\infty$	0	∞
$\sinh z$	$-\infty$	0	∞
$\cosh z$	∞	1	∞
$\tanh z$	-1	0	1

Note $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$

13.6.3 Relation between Circular and Hyperbolic Functions

(i) sin iz = i sinh z and sinh z = –i sin iz

Proof By Euler's formula,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Replacing z by iz,

$$\sin iz = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = -i\frac{\left(e^{-z} - e^{z}\right)}{2} \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= i\frac{\left(e^{z} - e^{-z}\right)}{2} = i \sinh z$$

and

$$\sinh z = \frac{1}{i} \sin iz = -i \sin iz$$

(ii) cos iz = cosh z

Proof By Euler's formula,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Replacing z by iz,

$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z$$

(iii) tan iz = i tanh z and tanh z = –i tan iz

Proof

$$\tan iz = \frac{\sin iz}{\cos iz} = \frac{i \sinh z}{\cosh z} = i \tanh z$$

$$\tanh z = \frac{1}{i} \tan iz = -i \tan iz$$

13.6.4 Formulae on Hyperbolic Functions

A. (i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\coth^2 z - \operatorname{cosech}^2 z = 1$

(iii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

B. (iv) $\sinh 2z = 2 \sinh z \cosh z$

(v) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

(vi) $\tanh 2z = \dfrac{2 \tanh z}{1 + \tanh^2 z}$

C. (vii) $\sinh 3z = 3 \sinh z + 4 \sinh^3 z$

(viii) $\cosh 3z = 4\cosh^3 z - 3 \cosh z$

(ix) $\tanh 3z = \dfrac{3 \tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

D. (x) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

E. (xiii) $\sinh z_1 + \sinh z_2 = 2 \sinh \dfrac{z_1 + z_2}{2} \cosh \dfrac{z_1 - z_2}{2}$

(xiv) $\sinh z_1 - \sinh z_2 = 2 \cosh \dfrac{z_1 + z_2}{2} \sinh \dfrac{z_1 - z_2}{2}$

(xv) $\cosh z_1 + \cosh z_2 = 2\cosh\frac{z_1+z_2}{2}\cosh\frac{z_1-z_2}{2}$

(xvi) $\cosh z_1 - \cosh z_2 = 2\sinh\frac{z_1+z_2}{2}\sinh\frac{z_1-z_2}{2}$

F. (xvii) $2\sinh z_1 \cosh z_2 = \sinh(z_1+z_2) + \sinh(z_1-z_2)$

(xviii) $2\cosh z_1 \sinh z_2 = \sinh(z_1+z_2) - \sinh(z_1-z_2)$

(xix) $2\cosh z_1 \cosh z_2 = \cosh(z_1+z_2) + \cosh(z_1-z_2)$

(xx) $2\sinh z_1 \sinh z_2 = \cosh(z_1+z_2) - \cosh(z_1-z_2)$

13.6.5 Inverse Hyperbolic Functions

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called the sine hyperbolic inverse of x, where x is real. Similarly, $\cosh^{-1} x$, $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$ and $\operatorname{cosech}^{-1} x$ are defined.

The inverse hyperbolic functions are many-valued functions but their principal values are only considered.

(i) $\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$, where x is real

Proof Let

$$\sinh^{-1} x = y$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$$

$$e^{2y} - 2x\,e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \log\left(x \pm \sqrt{x^2+1}\right)$$

But $x - \sqrt{x^2+1} < 0$ and log of negative number is not defined.

$$y = \log\left(x+\sqrt{x^2+1}\right)$$

$$\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$$

(ii) $\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$, where x is real

Proof Let

$$\cosh^{-1} x = y$$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$y = \log\left(x \pm \sqrt{x^2-1}\right) \qquad \text{... (13.4)}$$

Consider

$$y = \log\left(x - \sqrt{x^2-1}\right)$$

$$e^y = x - \sqrt{x^2-1} \qquad \text{... (13.5)}$$

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1} = x+\sqrt{x^2-1}$$

$$-y = \log\left(x+\sqrt{x^2-1}\right)$$

$$y = -\log\left(x+\sqrt{x^2-1}\right) \qquad \text{... (13.6)}$$

Equating Eqs (13.5) and (13.6),

$$\log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right) \qquad \text{... (13.7)}$$

From Eqs (13.4) and (13.7),

$$y = \pm\log\left(x+\sqrt{x^2-1}\right) \qquad \text{... (13.8)}$$

$$\cosh^{-1} x = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$x = \cosh\left[\pm\log\left(x+\sqrt{x^2-1}\right)\right]$$

$$= \cosh\left[\log\left(x+\sqrt{x^2-1}\right)\right] \quad [\because \cosh(-z) = \cosh z]$$

$$\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, where x is real

Proof Let

$$\tanh^{-1} x = y$$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo–dividendo,

$$\frac{1+x}{1-x}=\frac{e^y+e^{-y}+e^y-e^{-y}}{e^y+e^{-y}-e^y+e^{-y}}=\frac{2e^y}{2e^{-y}}=e^{2y}$$

$$e^{2y}=\frac{1+x}{1-x}$$

$$2y=\log\left(\frac{1+x}{1-x}\right)$$

$$y=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

EXAMPLE 13.29 *Solve the equation* $17\cosh x+18\sinh x=1$ *for real values of* x.

Solution:

$$17\cosh x+18\sinh x=1$$

$$17\left(\frac{e^x+e^{-x}}{2}\right)+18\left(\frac{e^x-e^{-x}}{2}\right)=1$$

$$35e^x-e^{-x}=2$$

$$35e^{2x}-1=2e^x$$

$$35e^{2x}-2e^x-1=0$$

This equation is quadratic in e^x.

$$e^x=\frac{2\pm\sqrt{4+140}}{70}=\frac{2\pm12}{70}=\frac{14}{70},-\frac{10}{70}$$

For real values of x, e^x should be positive.

$$\therefore\quad e^x=\frac{14}{70}=\frac{1}{5}$$

$$x=\log\frac{1}{5}=-\log 5$$

EXAMPLE 13.30 *If* $\cos\alpha\cosh\beta=\frac{x}{2}$, $\sin\alpha\sinh\beta=\frac{y}{2}$, *prove that*

(i) $\sec(\alpha-i\beta)+\sec(\alpha+i\beta)=\dfrac{4x}{x^2+y^2}$

(ii) $\sec(\alpha-i\beta)-\sec(\alpha+i\beta)=-\dfrac{4iy}{x^2+y^2}$.

Solution: $\sec(\alpha-i\beta)=\dfrac{1}{\cos(\alpha-i\beta)}=\dfrac{1}{\cos\alpha\cos i\beta+\sin\alpha\sin i\beta}=\dfrac{1}{\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta}$

$$=\frac{1}{\frac{x}{2}+i\frac{y}{2}}=\frac{2}{(x+iy)}\cdot\frac{(x-iy)}{(x-iy)}=\frac{2(x-iy)}{x^2+y^2}\qquad\ldots(1)$$

Similarly, $\sec(\alpha+i\beta)=\dfrac{1}{\cos(\alpha+i\beta)}=\dfrac{1}{\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta}=\dfrac{1}{\frac{x}{2}-i\frac{y}{2}}$

$$=\frac{2}{x-iy}\cdot\frac{(x+iy)}{(x+iy)}=\frac{2(x+iy)}{x^2+y^2}\qquad\ldots(2)$$

Adding Eqs (1) and (2),

$$\sec(\alpha-i\beta)+\sec(\alpha+i\beta)=\frac{4x}{x^2+y^2}$$

Subtracting Eq. (2) from Eq. (1),

$$\sec(\alpha-i\beta)-\sec(\alpha+i\beta)=-\frac{4iy}{x^2+y^2}$$

EXAMPLE 13.31 *Prove that* $\tanh^{-1}(\sin\theta)=\cosh^{-1}(\sec\theta)$.

Solution: $\tanh^{-1}(\sin\theta)=\dfrac{1}{2}\log\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)=\dfrac{1}{2}\log\left[\dfrac{(1+\sin\theta)\,(1+\sin\theta)}{(1-\sin\theta)\,(1+\sin\theta)}\right]$

$$=\frac{1}{2}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)^2=\frac{1}{2}\cdot2\log(\sec\theta+\tan\theta)$$

$$=\log\left(\sec\theta+\sqrt{\sec^2\theta-1}\right)=\cosh^{-1}(\sec\theta)\qquad\left[\because\cosh^{-1}x=\log\left(x+\sqrt{x^2-1}\right)\right]$$

EXAMPLE 13.32 *If* $\cosh x=\sec\theta$, *prove that*

(i) $x=\log(\sec\theta+\tan\theta)$ (ii) $\theta=\frac{\pi}{2}-2\tan^{-1}(e^{-x})$

(iii) $\sinh x=\tan\theta$ (iv) $\tanh x=\sin\theta$ (v) $\tanh\frac{x}{2}=\pm\tan\frac{\theta}{2}$.

Solution:

(i) $x=\cosh^{-1}(\sec\theta)=\log\left(\sec\theta+\sqrt{\sec^2\theta-1}\right)=\log(\sec\theta+\tan\theta)$

(ii) From (i), $e^x = \sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

$$= \left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}\right] = \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right] = \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$e^{-x} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\tan^{-1}(e^{-x}) = \frac{\pi}{4}-\frac{\theta}{2}$$

$$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(iii) $\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\sec^2\theta - 1}$ $\qquad [\because \cosh x = \sec\theta]$

$= \tan\theta$

(iv) $\tanh x = \sqrt{1-\text{sech}^2 x} = \sqrt{1-\cos^2\theta}$ $\qquad [\because \cosh x = \sec\theta]$

$= \sin\theta$

(v) $\cosh x = \sec\theta$

$$\frac{1+\tanh^2\frac{x}{2}}{1-\tanh^2\frac{x}{2}} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

Applying componendo–dividendo,

$$\tanh^2\frac{x}{2} = \tan^2\frac{\theta}{2}$$

$$\tanh\frac{x}{2} = \pm\tan\frac{\theta}{2}$$

EXAMPLE 13.33 *Prove that* $\tan^{-1}\left[i\left(\dfrac{x-a}{x+a}\right)\right] = -\dfrac{i}{2}\log\left(\dfrac{a}{x}\right)$.

Solution: Let

$$\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$$

$$i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$$

$$\frac{x-a}{x+a} = \frac{-e^{i\theta}+e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$$

Applying componendo–dividendo,

$$\frac{x+a+x-a}{x+a-x+a} = \frac{e^{i\theta}+e^{-i\theta}-e^{i\theta}+e^{-i\theta}}{e^{i\theta}+e^{-i\theta}+e^{i\theta}-e^{-i\theta}}$$

$$\frac{x}{a} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

$$\frac{a}{x} = e^{2i\theta}$$

$$2i\theta = \log\frac{a}{x}$$

$$\theta = \frac{1}{2i}\log\frac{a}{x} = -\frac{i}{2}\log\frac{a}{x}$$

$$\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$$

EXERCISE 13.6

1. Prove that $\left(\dfrac{1+\tanh x}{1-\tanh x}\right)^n = \cosh 2nx + \sinh 2nx$.

2. Prove that $\text{cosec}\, x + \coth x = \coth\dfrac{x}{2}$.

3. Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = -\sinh^2 x$.

4. If $\cosh^6 x = a\cosh 6x + b\cosh 4x + c\cosh 2x + d$, prove that $5a - 5b + 3c - 4d = 0$.

5. If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} x$ then prove that $a\sqrt{b^2-1} + b\sqrt{a^2-1} = \sqrt{x^2-1}$.

6. If $6\sinh x + 2\cosh x + 7 = 0$, find $\tanh x$.

$\left[\textbf{Ans.: } \dfrac{3}{5}, -\dfrac{15}{17}\right]$

7. Find the value of $\tanh \log\sqrt{5}$.

$\left[\textbf{Ans.: } \dfrac{2}{3}\right]$

8. If $\sin\alpha\cosh\beta = \dfrac{x}{2}$, $\cos\alpha\sinh\beta = \dfrac{y}{2}$, show that

   (i) $\text{cosec}\,(\alpha - i\beta) + \text{cosec}\,(\alpha + i\beta) = \dfrac{4x}{x^2+y^2}$

   (ii) $\text{cosec}\,(\alpha - i\beta) - \text{cosec}\,(\alpha + i\beta) = \dfrac{4iy}{x^2+y^2}$

9. Prove that $\cos^{-1}x = -i\log\left(x \pm \sqrt{1-x^2}\right)$.

10. Prove that $\sin^{-1}ix = 2n\pi + i\log\left(x+\sqrt{1+x^2}\right)$.

11. Prove that $\sinh^{-1}(\tan x) = \log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.

12. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tan 2\phi}{\tan 2\theta - \tan 2\phi}\right) + \tan^{-1}\left(\frac{\tan\theta - \tan\phi}{\tan\theta + \tan\phi}\right) = \tan^{-1}(\cot\theta\coth\phi).$$

13. Prove that

(i) $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\operatorname{cosec}\theta)$
(ii) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$

14. Prove that $\cosh^{-1}\left(\frac{3i}{4}\right) = \log 2 + \frac{i\pi}{2}$.

15. If $\cosh^{-1}(x+iy) + \cosh^{-1}(x-iy) = \cosh^{-1}a$, prove that $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$.

16. Prove that

(i) $\sinh^{-1}x = \operatorname{cosech}^{-1}\left(\frac{x}{2x\sqrt{1-x^2}}\right)$

(ii) $\tanh^{-1}x = \cosh^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(iii) $\coth^{-1}x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$

13.6.6 Separation Into Real and Imaginary Parts

To separate real and imaginary parts of a complex number, the following results are used:

(i) $\sin(x \pm iy) = \sin x \cos iy \pm \cos x \sin iy = \sin x \cosh y \pm i\cos x \sinh y$

(ii) $\cos(x \pm iy) = \cos x \cos iy \mp \sin x \sin iy = \cos x \cosh y \mp i \sin x \sinh y$

(iii) $\tan(x \pm iy) = \frac{2\sin(x \pm iy)}{2\cos(x \pm iy)} \cdot \frac{\cos(x \mp iy)}{\cos(x \mp iy)} = \frac{\sin 2x \pm \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x \pm i \sinh 2y}{\cos 2x + \cosh 2y}$

(iv) $\sinh(x \pm iy) = \sinh x \cosh iy \pm \cosh x \sinh iy = \sinh x \cos iiy \pm \cosh x(-i \sin iiy)$
$= \sinh x \cos(-y) \pm \cosh x[-i\sin(-y)] = \sinh x \cos y \pm i \cosh x \sin y$

(v) $\cosh(x \pm iy) = \cosh x \cosh iy \pm \sinh x \sinh iy = \cosh x \cos y \pm i \sinh x \sin y$

(vi) $\tanh(x \pm iy) = \frac{2\sinh(x \pm iy)}{2\cosh(x \pm iy)} \cdot \frac{\cosh(x \mp iy)}{\cosh(x \mp iy)} = \frac{\sinh 2x \pm \sinh 2iy}{\cosh 2x + \cosh 2iy} = \frac{\sinh 2x \pm i \sin 2y}{\cosh 2x + \cos 2y}$

EXAMPLE 13.34 *Separate real and imaginary parts of*

(i) $\cos^{-1}\left(\frac{3i}{4}\right)$ (ii) $\sin^{-1}(\operatorname{cosec}\theta)$.

Solution: Let

$$\cos^{-1}\left(\frac{3i}{4}\right) = x + iy$$

$$\frac{3i}{4} = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

Comparing real and imaginary parts,

$$\cos x \cosh y = 0 \quad \text{...(1)}$$

$$\sin x \sinh y = -\frac{3}{4} \quad \text{...(2)}$$

From Eq. (1),

$$\cos x = 0 \qquad [\because \cosh y \neq 0]$$

$$x = \frac{\pi}{2}$$

Putting $x = \frac{\pi}{2}$ in Eq. (2),

$$\sin\frac{\pi}{2}\sinh y = \frac{3}{4}$$

$$\sinh y = \frac{3}{4}$$

$$y = \sinh^{-1}\left(\frac{3}{4}\right) = \log\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \log\left(\frac{3}{4}+\frac{5}{4}\right) = \log 2$$

Hence,

$$\cos^{-1}\left(\frac{3i}{4}\right) = \frac{\pi}{2} + i\log 2.$$

(ii) Let $\sin^{-1}(\operatorname{cosec}\theta) = x + iy$

$$\operatorname{cosec}\theta = \sin(x+iy)$$

$$\operatorname{cosec}\theta = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$\operatorname{cosec}\theta = \sin x \cosh y \quad \text{...(1)}$$

$$0 = \cos x \sinh y \quad \text{...(2)}$$

From Eq. (2),

$$\cos x = 0, \; x = \frac{\pi}{2}$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\operatorname{cosec}\theta = \sin\frac{\pi}{2}\cosh y = \cosh y$$

$$y = \cosh^{-1}(\operatorname{cosec}\theta)$$

$$y = \cosh^{-1}(\operatorname{cosec}\theta) = \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right) = \log(\operatorname{cosec}\theta + \cot\theta)$$

$$= \log\left(\frac{1+\cos\theta}{\sin\theta}\right) = \log\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \log\left(\cot\frac{\theta}{2}\right)$$

Hence, $\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$

EXAMPLE 13.35 *If $\sinh(\theta + i\phi) = e^{i\alpha}$, prove that*

(i) $\sinh^4\theta = \cos^2\alpha$ (ii) $\cos^2\phi = \cos^2\alpha$

Solution:

$$\sinh(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\cos\alpha + i\sin\alpha = \sinh\theta\cosh i\phi + \cosh\theta\sinh i\phi$$

$$= \sinh\theta\cos i(i\phi) + \cosh\theta[-i\sin i(i\phi)]$$

$$= \sinh\theta\cos(-\phi) + \cosh\theta[-i\sin(-\phi)]$$

$$= \sinh\theta\cos\phi + i\cosh\theta\sin\phi$$

Comparing real and imaginary parts

$$\cos\alpha = \sinh\theta\cos\phi \quad \text{...(1)}$$

$$\sin\alpha = \cosh\theta\sin\phi \quad \text{...(2)}$$

(i) Eliminating ϕ from Eqs (1) and (2),

$$\cos^2\phi+\sin^2\phi=\frac{\cos^2\alpha}{\sinh^2\theta}+\frac{\sin^2\alpha}{\cosh^2\theta}$$
$$1=\frac{\cos^2\alpha}{\sinh^2\theta}+\frac{\sin^2\alpha}{\cosh^2\theta}$$
$$\sinh^2\theta\cosh^2\theta=\cos^2\alpha\cosh^2\theta+\sin^2\alpha\sinh^2\theta$$
$$\sinh^2\theta(1+\sinh^2\theta)=\cos^2\alpha(1+\sinh^2\theta)+(1-\cos^2\alpha)\sinh^2\theta$$
$$\sinh^2\theta+\sinh^4\theta=\cos^2\alpha+\cos^2\alpha\sinh^2\theta+\sinh^2\theta-\cos^2\alpha\sinh^2\theta$$
$$\sinh^4\theta=\cos^2\alpha \quad \ldots(3)$$

(ii) From Eq. (1),

$$\cos^2\alpha=\sinh^2\theta\cos^2\phi$$
$$\cos^2\phi=\frac{\cos^2\alpha}{\sinh^2\theta}=\frac{\cos^2\alpha}{\cos\alpha} \qquad \text{[Using Eq.(3)]}$$
$$\cos^2\phi=\cos\alpha$$

EXAMPLE 13.36 *If* cosec $(x+iy)=u=iv$, *prove that* $\left(u^2+v^2\right)^2=\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}$.

Solution:

$$\operatorname{cosec}(x+iy)=u+iv$$
$$\sin(x+iy)=\frac{1}{u+iv}$$
$$\sin x\cos iy+i\cos x\sin iy=\frac{1}{u+iv}\cdot\frac{u-iv}{u-iv}$$
$$\sin x\cosh y+i\cos x\sinh y=\frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$\sin x\cosh y=\frac{u}{u^2+v^2} \quad \ldots(1)$$
$$\cos x\sinh y=-\frac{v}{u^2+v^2} \quad \ldots(2)$$

Eliminating y from Eqs (1) and (2),

$$\cosh^2 y-\sinh^2 y=\frac{u^2}{\left(u^2+v^2\right)^2\sin^2 x}-\frac{v^2}{\left(u^2+v^2\right)^2\cos^2 x}$$
$$1=\frac{1}{\left(u^2+v^2\right)^2}\left(\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}\right)$$
$$\left(u^2+v^2\right)^2=\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}$$

EXAMPLE 13.37 *Separate into real and imaginary parts:*

(i) $\tan(x+iy)$ (ii) $\tan^{-1}(e^{i\theta})$

Solution:

(i)
$$\tan(x+iy)=\frac{\sin(x+iy)}{\cos(x+iy)}=\frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$
$$\text{Real part}=\frac{\sin 2x}{\cos 2x+\cosh 2y}$$
$$\text{Imaginary part}=\frac{\sinh 2y}{\cos 2x+\cosh 2y}$$

(ii) Let
$$\tan^{-1}(e^{i\theta})=x+iy \quad \ldots(1)$$
$$\tan^{-1}(e^{-i\theta})=x-iy \quad \ldots(2)$$

Adding Eqs (1) and (2),

$$2x=\tan^{-1}(e^{i\theta})+\tan^{-1}(e^{-i\theta})=\tan^{-1}\left(\frac{e^{i\theta}+e^{-i\theta}}{1-e^{i\theta}\cdot e^{-i\theta}}\right)=\tan^{-1}\infty=n\pi+\frac{\pi}{2}$$
$$x=\frac{n\pi}{2}+\frac{\pi}{4}$$

Subtracting Eq. (2) from Eq. (1),

$$2iy=\tan^{-1}(e^{i\theta})-\tan^{-1}(e^{-i\theta})=\tan^{-1}\left(\frac{e^{i\theta}-e^{-i\theta}}{1+e^{i\theta}\cdot e^{-i\theta}}\right)$$
$$\tan 2iy=\frac{2i\sin\theta}{2}$$
$$i\tanh 2y=i\sin\theta$$
$$2y=\tanh^{-1}(\sin\theta)$$
$$=\frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}=\frac{1}{2}\log\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{2}\log\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$
$$=\frac{1}{2}\log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^2=\log\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=-\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$
$$y=-\frac{1}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

Hence,
$$\tan^{-1}(e^{i\theta})=\left(n+\frac{1}{2}\right)\frac{\pi}{2}-\frac{i}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

EXAMPLE 13.38 *If* $\tan(x+iy)=\alpha+i\beta$, *show that* $\dfrac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2}=\dfrac{\cos 2x}{\cosh 2y}$.

Solution: $\alpha+i\beta=\tan(x+iy)$

$$=\frac{2\sin(x+iy)}{2\cos(x+iy)}\cdot\frac{\cos(x-iy)}{\cos(x-iy)}=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

Taking modulus on both the sides and squaring,

$$\alpha^2+\beta^2=\frac{\sin^2 2x+\sinh^2 2y}{(\cos 2x+\cosh 2y)^2}$$

Applying componendo–dividendo,

$$\frac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2}=\frac{(\cos 2x+\cosh 2y)^2-\sin^2 2x-\sinh^2 2y}{(\cos 2x+\cosh 2y)^2+\sin^2 2x+\sinh^2 2y}$$

$$=\frac{\cos^2 2x+1-\sin^2 2x+2\cos 2x\cosh 2y}{1+\cosh^2 2y+\sinh^2 2y+2\cos 2x\cosh 2y}$$

$$=\frac{2\cos 2x(\cos 2x+\cosh 2y)}{2\cosh 2y(\cosh 2y+\cos 2x)}=\frac{\cos 2x}{\cosh 2y}$$

EXAMPLE 13.39 *If* $\dfrac{x+iy-c}{x+iy+c}=e^{u+iv}$, *show that* $x=\dfrac{-c\sinh u}{\cosh u-\cos v}$, $y=\dfrac{c\sin v}{\cosh u-\cos v}$.

Solution:

$$\frac{x+iy-c}{x+iy+c}=e^{u+iv}$$

Applying componendo–dividendo,

$$\frac{2(x+iy)}{2c}=\frac{(1+e^{u+iv})}{(1-e^{u+iv})}\frac{(1-e^{u-iv})}{(1-e^{u-iv})}$$

[Multiplying and dividing by the conjugate of the denominator]

$$\frac{x+iy}{c}=\frac{(1-e^{u-iv}+e^{u+iv}-e^{2u})}{(1-e^{u-iv}-e^{u+iv}+e^{2u})}\cdot\frac{e^{-u}}{e^{-u}}=\frac{e^{-u}-e^{-iv}+e^{iv}-e^{u}}{e^{-u}-e^{-iv}-e^{iv}+e^{u}}$$

$$=\frac{(e^{-u}-e^{u})+(e^{iv}-e^{-iv})}{(e^{-u}+e^{u})-(e^{iv}+e^{-iv})}=\frac{-2\sinh u+2i\sin v}{2\cosh u-2\cos v}$$

$$\frac{x}{c}+i\frac{y}{c}=\frac{-\sinh u}{\cosh u-\cos v}+i\frac{\sin v}{\cosh u-\cos v}$$

Comparing real and imaginary parts,

$$x=-\frac{c\sinh u}{\cosh u-\cos v},\quad y=\frac{c\sin v}{\cosh u-\cos v}$$

EXERCISE 13.7

1. Separate into real and imaginary parts:
   (i) $\cot(x+iy)$ (ii) $\sec(x+iy)$
   (iii) $\mathrm{cosec}\,(x+iy)$ (iv) $(\sin\theta+i\cos\theta)^i$.

   $\left[\textbf{Ans.:}\ \text{(i)}\ \dfrac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}\quad \text{(ii)}\ \dfrac{2(\cos x\cosh y-i\sin x\sinh y)}{\cos 2x+\cosh 2y}\quad \text{(iii)}\ \dfrac{2(\sin x\cosh y-i\cos x\sinh y)}{\cosh 2y-\cos 2x}\quad \text{(iv)}\ e^{\theta-\frac{\pi}{2}}\right]$

2. If $\sin(\alpha+i\beta)=x+iy$, prove that
   (i) $\dfrac{x^2}{\cosh^2\beta}+\dfrac{y^2}{\sinh^2\beta}=1$ (ii) $\dfrac{x^2}{\sin^2\alpha}-\dfrac{y^2}{\cos^2\alpha}=1$.

3. If $\sin(\theta+i\phi)=p(\cos\alpha+i\sin\alpha)$, prove that $p^2=\dfrac{1}{2}(\cosh 2\phi-\cos 2\theta)$, $\tan\alpha=\tanh\phi\cot\theta$.

4. If $\sin(\theta+i\phi)=\cos\alpha+i\sin\alpha$, prove that
   (i) $\cos^4\theta=\sin^2\alpha=\sinh^4\phi$
   (ii) $\phi=\dfrac{1}{2}\log\dfrac{\cos(x-\theta)}{\cos(x+\theta)}$

5. If $\cosh(\theta+i\phi)=e^{i\alpha}$, prove that $\sin^2\alpha=\sin^4\phi$ $=\sinh^4\theta$.

6. If $\sinh(\theta+i\phi)=x+iy$, prove that $x^2\,\mathrm{cosech}^2\,\theta+y^2\,\mathrm{sech}^2\,\theta=1$ and $y^2\,\mathrm{cosec}^2\,\phi-x^2\sec^2\phi=1$.

7. If $\cos(x+iy)\cos(u+iv)=1$, where x, y, u, v are real then prove that $\tanh^2 v\cosh^2 y=\sin^2 x$.

8. If $x+iy=c\sin(u+iv)$, prove that $u=$ constant represents a family of confocal hyperbolas and $v=$ constant represents a family of confocal ellipses.

9. If $\tan y=\tan\alpha\tanh\beta$ and $\tan z=\cot\alpha\tanh\beta$, prove that $\tan(y+z)=\sinh 2\beta\,\mathrm{cosec}\,2\alpha$.

10. Prove that $\tan\left(\dfrac{u+iv}{2}\right)=\dfrac{\sin u+i\sinh v}{\cos u+\cosh v}$.

11. If $A+iB=C\tan(x+iy)$, prove that $\tan 2x=\dfrac{2CA}{C^2-A^2-B^2}$.

12. If $\tan(\theta+i\phi)=\tan\alpha+i\sec\alpha$ then prove that
    (i) $e^{2\phi}=\cot\dfrac{\alpha}{2}$ (ii) $2\theta=n\pi+\dfrac{\pi}{2}+\alpha$

13. Prove that all solutions of the equation $\sin z=2i\cos z$ are given by $z=\dfrac{n\pi}{2}+\dfrac{i}{2}\log 3$.

14. Prove that one value of $\tan^{-1}\left(\dfrac{x+iy}{x-iy}\right)$ is $\dfrac{\pi}{4}+i\log\left(\dfrac{x+y}{x-y}\right)$, where $x>y>0$.

15. If $\cot\left(\dfrac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2-2x=1$.

16. If $\tanh\left(\alpha+\dfrac{i\pi}{6}\right)=x+iy$, prove that $x^2+y^2+\dfrac{2y}{\sqrt{3}}=1$.

17. If $\tanh(\alpha+i\beta)=x+iy$, prove that
    (i) $x^2+y^2-2x\coth 2\alpha=-1$
    (ii) $x^2+y^2+2y\cot 2\beta=1$

18. If $\cot(\alpha+i\beta)=x+iy$, prove that
    (i) $x^2+y^2-2x\cot 2\alpha=1$
    (ii) $x^2+y^2+2y\coth 2\beta+1=0$

19. Separate real and imaginary parts of $\cos^{-1}(e^{i\theta})$.

    $\left[\textbf{Ans.:}\ \sin^{-1}\sqrt{\sin\theta}+i\log\left(\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\right)\right]$

20. Prove that $\sin^{-1}(ix)=i\log\left(x+\sqrt{x^2+1}\right)+2n\pi$.

21. Prove that $\sin^{-1}(\mathrm{cosec}\,\theta)=\dfrac{\pi}{2}+i\log\cot\dfrac{\theta}{2}$.

13.7 LOGARITHMIC FUNCTIONS OF COMPLEX NUMBERS

If z and w are two complex numbers and $z = e^w$ then $w = \log z$ is called logarithm of the complex number z.

Let $z = x + iy = re^{i\theta}$

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$$\log z = \log(r e^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta \log e = \log r + i\theta = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

Hence,
$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

This is called principal value of $\log(x + iy)$.

The general value of $\log(x + iy)$ is

$$\text{Log}(x+iy) = \log r + i(2n\pi + \theta) = \frac{1}{2}\log(x^2+y^2) + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right] = 2n\pi i + \log(x+iy)$$

The general value of $\log(x + iy)$ is denoted by $\text{Log}(x + iy)$, beginning with capital L to distinguish it from its principal value which is denoted by $\log(x + iy)$.

EXAMPLE 13.40 *Find the values of* (i) $\log_{-3}(-2)$ (ii) $\log(1 + i)$.

Solution:

(i) $$\log_{-3}(-2) = \frac{\log_e(-2)}{\log_e(-3)} = \frac{\frac{1}{2}\log 4 + i\tan^{-1}\left(\frac{0}{-2}\right)}{\frac{1}{2}\log 9 + i\tan^{-1}\left(\frac{0}{-3}\right)} = \frac{\log 2 + i\pi}{\log 3 + i\pi}$$

(ii) $$\log(1+i) = \frac{1}{2}\log 2 + i\tan^{-1}\left(\frac{1}{1}\right) = \frac{1}{2}\log 2 + \frac{i\pi}{4}$$

EXAMPLE 13.41 *Prove that* $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x \tanh y)$.

Solution: $$\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x+iy)] - \log[\sin(x-iy)]$$

$$= \log(\sin x\cosh y + i\cos x\sinh y) - \log(\sin x\cosh y - i\cos x\sinh y)$$

$$= \frac{1}{2}\log\left(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\right) + i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

$$-\frac{1}{2}\log\left(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\right) - i\tan^{-1}\left(\frac{-\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= i\tan^{-1}(\cot x\tanh y) + i\tan^{-1}(\cot x\tanh y)$$

$$= 2i\tan^{-1}(\cot x\tanh y)$$

EXAMPLE 13.42 *Prove that* $\text{Log}\left\{\frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right\} = i\left\{2n\pi + \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)\right\}$.

Solution: Let $a - b = x$ and $a + b = y$

$$\text{Log}\left[\frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right] = \text{Log}\left(\frac{x+iy}{y+ix}\right) = 2n\pi i + \log\left(\frac{x+iy}{y+ix}\right)$$

$$= 2n\pi i + \log(x+iy) - \log(y+ix)$$

$$= 2n\pi i + \left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right] - \left[\frac{1}{2}\log(y^2+x^2) + i\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= 2n\pi i + i\left[\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{x}{y}\right)\right] = 2n\pi i + i\tan^{-1}\left(\frac{\frac{y}{x} - \frac{x}{y}}{1 + \frac{y}{x}\cdot\frac{x}{y}}\right)$$

$$= 2n\pi i + i\tan^{-1}\left(\frac{y^2-x^2}{2xy}\right)$$

$$= 2n\pi i + i\tan^{-1}\left[\frac{(a+b)^2-(a-b)^2}{2(a-b)(a+b)}\right] \quad \text{[Resubstituting } x \text{ and } y\text{]}$$

$$= 2n\pi i + i\tan^{-1}\left[\frac{4ab}{2(a^2-b^2)}\right] = i\left[2n\pi + \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)\right]$$

EXAMPLE 13.43 *Prove that* $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$.

Solution: $$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i\log(a+ib) - i\log(a-ib)]$$

$$= \cos i\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a} - \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(-\frac{b}{a}\right)\right]$$

$$= \cos i\left[i\tan^{-1}\left(\frac{b}{a}\right) + i\tan^{-1}\left(\frac{b}{a}\right)\right] = \cos\left[-2\tan^{-1}\left(\frac{b}{a}\right)\right] = \cos\left[2\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$= \cos 2\theta, \quad \text{where} \quad \tan\theta = \frac{b}{a} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}$$

EXAMPLE 13.44 *Separate real and imaginary parts of $(1+i)^i$.*

Solution: Let $x+iy=(1+i)^i$

Taking logarithm on both the sides,

$$\log(x+iy)=i\log(1+i)=i\left[\frac{1}{2}\log 2+i\tan^{-1}(1)\right]=\frac{i}{2}\log 2-\frac{\pi}{4}$$

$$x+iy=e^{\frac{i}{2}\log 2}e^{-\frac{\pi}{4}}=e^{i\log\sqrt{2}}e^{-\frac{\pi}{4}}=e^{-\frac{\pi}{4}}\left[\cos(\log\sqrt{2})+i\sin(\log\sqrt{2})\right]$$

Comparing real and imaginary parts,

$$x=e^{-\frac{\pi}{4}}\cos(\log\sqrt{2})$$

$$y=e^{-\frac{\pi}{4}}\sin(\log\sqrt{2})$$

EXAMPLE 13.45 *Prove that $(1+i\tan\alpha)^{-i}=e^{2m\pi+\alpha}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$.*

Solution: Let $x+iy=(1+i\tan\alpha)^{-i}$

Taking logarithm on both the sides,

$$\text{Log}(x+iy)=-i\,\text{Log}(1+i\tan\alpha)$$

$$=-i[2m\pi i+\log(1+i\tan\alpha)]$$

$$=-i\left[2m\pi i+\frac{1}{2}\log(1+\tan^2\alpha)+i\tan^{-1}(\tan\alpha)\right]$$

$$=-i\left[2m\pi i+\frac{1}{2}\log\sec^2\alpha+i\alpha\right]=2m\pi-\frac{i}{2}\log\sec^2\alpha+\alpha$$

$$=(2m\pi+\alpha)+i\log(\sec^2\alpha)^{-\frac{1}{2}}=(2m\pi+\alpha)+i\log(\cos\alpha)$$

$$x+iy=e^{(2m\pi+\alpha)}e^{i\log\cos\alpha}=e^{(2m\pi+\alpha)}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$$

Hence, $(1+i\tan\alpha)^{-i}=e^{(2m\pi+\alpha)}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

EXAMPLE 13.46 *Prove that if $(1+i\tan\alpha)^{1+i\tan\beta}$ can have only real values, one of them is $(\sec\alpha)^{\sec^2\beta}$ considering only principle value.*

Solution: Let $x=(1+i\tan\alpha)^{1+i\tan\beta}$, where x is real.

Taking logarithm on both the sides,

$$\log x=(1+i\tan\beta)\log(1+i\tan\alpha)$$

$$=(1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha)+i\tan^{-1}(\tan\alpha)\right]$$

$$=(1+i\tan\beta)(\log\sec\alpha+i\alpha)=(\log\sec\alpha-\alpha\tan\beta)+i(\alpha+\tan\beta\log\sec\alpha)$$

Comparing real and imaginary parts,

$$\log x=\log\sec\alpha-\alpha\tan\beta \quad\text{and}\quad \alpha+\tan\beta\log\sec\alpha=0$$

$$x=e^{(\log\sec\alpha-\alpha\tan\beta)} \quad\text{and}\quad \alpha=-\tan\beta\log\sec\alpha$$

Substituting α in x,

$$x=e^{\log\sec\alpha+\tan^2\beta\log\sec\alpha}=e^{\log\sec\alpha(1+\tan^2\beta)}=e^{(\log\sec\alpha)\sec^2\beta}=e^{\sec^2\beta\log\sec\alpha}$$

$$=e^{\log(\sec\alpha)^{\sec^2\beta}}=(\sec\alpha)^{\sec^2\beta}$$

EXAMPLE 13.47 *Separate real and imaginary parts of $(\sqrt{i})^{\sqrt{i}}$.*

Solution: Let $a+ib=\sqrt{i}$

Taking logarithm on both the sides,

$$\log(a+ib)=\log\sqrt{i}=\frac{1}{2}\log i=\frac{1}{2}\cdot\frac{i\pi}{2}\quad\left[\because \log i=\frac{1}{2}\log 1+i\tan^{-1}(\infty)\right]$$

$$a+ib=e^{\frac{i\pi}{4}}$$

Hence, $\sqrt{i}=e^{\frac{i\pi}{4}}$

Let $(\sqrt{i})^{\sqrt{i}}=x+iy$

Taking logarithm on both the sides,

$$\sqrt{i}\log\sqrt{i}=\log(x+iy)$$

$$e^{\frac{i\pi}{4}}\log e^{\frac{i\pi}{4}}=\log(x+iy)$$

$$\log(x+iy)=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\cdot\frac{i\pi}{4}=\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{i\pi}{4}=\frac{i\pi}{4\sqrt{2}}-\frac{\pi}{4\sqrt{2}}$$

$$x+iy=e^{\frac{-\pi}{4\sqrt{2}}}e^{\frac{i\pi}{4\sqrt{2}}}=e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}}+i\sin\frac{\pi}{4\sqrt{2}}\right)$$

Comparing real and imaginary parts,

$$x=e^{-\frac{\pi}{4\sqrt{2}}}\cos\frac{\pi}{4\sqrt{2}}$$

$$y=e^{-\frac{\pi}{4\sqrt{2}}}\sin\frac{\pi}{4\sqrt{2}}$$

EXAMPLE 13.48 *If $i^{i^{\cdots\infty}}=z$, where $z=x+iy$, prove that*

(i) $|z|^2=e^{-(4n+1)\pi y},\ n\in I$ (ii) $\tan\frac{\pi x}{2}=\frac{y}{x}$ and $x^2+y^2=e^{-\pi y}$.

Solution:

$$i^{i^{\cdots\infty}} = x+iy$$

$$i^{x+iy} = x+iy$$

Taking logarithm on both the sides,

$$(x+iy)\,\text{Log}\, i = \text{Log}\,(x+iy)$$

$$\text{Log}(x+iy) = (x+iy)(2n\pi i + \log i)$$

$$\text{Log}(x+iy) = (x+iy)\left(2n\pi i + i\frac{\pi}{2}\right),\quad n \in I = -\left(\frac{4n+1}{2}\right)\pi y + i\left(2n\pi + \frac{\pi}{2}\right)x$$

$$x+iy = e^{-\left(\frac{4n+1}{2}\right)\pi y}\, e^{i(4n+1)\frac{\pi x}{2}} = re^{i\theta},\ \text{say}$$

where

$$r = |x+iy| = |x-iy| = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]$$

$$|x-iy|^2 = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]^2$$

$$|\bar{z}|^2 = e^{-(4n+1)\pi y}$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{4n+1}{2}\right)\pi x$$

For $n = 0$,

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi x}{2}$$

$$\frac{y}{x} = \tan\frac{\pi x}{2}$$

and

$$|\bar{z}|^2 = e^{-\pi y}$$

$$x^2 + y^2 = e^{-\pi y}$$

EXAMPLE 13.49 *Prove that i^i is real and find its principal value. Also, show that the values of i^i form a geometric progression.*

Solution: Let

$$x+iy = i^i$$

$$\log(x+iy) = \log i^i$$

$$\log(x+iy) = i\log i = i\left[\frac{1}{2}\log 1 + i\tan^{-1}\left(\frac{1}{0}\right)\right] = i(0 + i\tan^{-1}\infty) = i\left(\frac{i\pi}{2}\right) = -\frac{\pi}{2}$$

$$x+iy = e^{-\frac{\pi}{2}}$$

Hence, i^i is real and its principal value is $e^{-\frac{\pi}{2}}$.

Let

$$z = x+iy = i^i$$

$$\text{Log}(x+iy) = \text{Log}\, i^i = i\,\text{Log}\, i = i(2n\pi i + \log i) = i\left(2n\pi i + \frac{i\pi}{2}\right) = -2n\pi - \frac{\pi}{2}$$

$$x+iy = e^{-\frac{\pi}{2}} e^{-2n\pi} = e^{-\frac{\pi}{2}}(e^{-2\pi})^n$$

$$\therefore\quad z = k\alpha^n,\ \text{where } k = e^{-\frac{\pi}{2}},\ \alpha = e^{-2\pi}$$

Putting $n = 0, 1, 2, 3, 4. \ldots$

$$z_0 = k,\ z_1 = k\alpha,\ z_2 = k\alpha^2,\ z_3 = k\alpha^3, \ldots$$

Hence, the values of i^i form a geometric progression with a common ratio $\alpha = e^{-2\pi}$.

EXERCISE 13.8

1. Find the general value of

   (i) $\log(-i)$ (ii) $\log_2 5$

   (iii) $\sin(\log i^i)$ (iv) $\cos(\log i^i)$

   Ans.: (i) $i\left(2\pi n - \frac{\pi}{2}\right)$
   (ii) $[(\log 5 \log 2 + 4\pi^2 mn) + i(n\log 2 - m\log 5)2\pi]/[(\log 2)^2 + 4\pi^2 m^2]$
   (iii) -1
   (iv) 0

2. Find the general value of

   $\log\left(1+i\sqrt{3}\right) + \log\left(1-i\sqrt{3}\right)$.

   [Ans. : 2log 2]

3. Prove that $\sin \log_e (i^{-i}) = 1$.

4. Prove that

   $$\log(e^{i\alpha} - e^{i\beta}) = \log\left(2\sin\frac{\alpha-\beta}{2}\right) + i\left(\frac{\pi+\alpha+\beta}{2}\right).$$

5. Show that $\log(-\log i) = \log\frac{\pi}{2} - \frac{i\pi}{2}$.

6. Prove that $\log(1 + i\tan a) = \log\sec a + ia$.

7. Prove that $\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - \frac{i\theta}{2}$.

8. If $\sin^{-1}(x+iy) = \log(A+iB)$, prove that

   $$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1,\ \text{where } A^2 + B^2 = e^{2u}.$$

9. If $(a+ib)^p = m^{x+iy}$, prove that one of the values of $\frac{y}{x}$ is $\dfrac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)}$.

10. Separate $i^{(1-i)}$ into real and imaginary parts.

    $\left[\textbf{Ans.:}\ ie^{\left(2n\pi + \frac{\pi}{2}\right)}\right]$

11. Considering only the principal values, separate real and imaginary parts of $\dfrac{(x+iy)^{\alpha+i\beta}}{(x-iy)^{\alpha-i\beta}}$.

    Ans.: $\cos 2\theta + i\sin 2\theta$, where $\theta = \alpha\tan^{-1}\frac{y}{x} + \beta\log\sqrt{x^2+y^2}$

12. Find the principal value of $(x + iy)^i$ and show that it is real if $\frac{1}{2}\log(x^2 + y^2)$ is a multiple of π.

$$\left[\textbf{Ans.}: e^{\tan^{-1}\left(\frac{y}{x}\right)}[\cos\log(x^2+y^2) + i\sin\log(x^2+y^2)]\right]$$

13. If $i^{\alpha+i\beta} = a + ib$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

14. If $\sqrt{i}^{\sqrt{i}^{\cdots\infty}} = a + ib$, prove that $\alpha^2 + \beta^2 = e^{-\frac{\pi\beta}{2}}$.

15. If $x^{x^{\cdots\infty}} = a(\cos a + i\sin a)$, prove that the general value of x is given by $r(\cos q + i\sin q)$

where $\log r = \dfrac{(2n\pi+\alpha)\sin\alpha + (\cos\alpha)\log a}{a}$

and $\theta = \dfrac{(2n\pi+\alpha)\cos\alpha - (\sin\alpha)\log a}{a}$.

16. Prove that $\log\left[\dfrac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right] = i\left[2n\pi + \tan^{-1}\left(\dfrac{2ab}{a^2-b^2}\right)\right]$.

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