Chapter-11

Matter Waves & Uncertainty Principle

11.1. Introduction

Wave-particle duality In the previous chapter, we have discussed the phenomena like photoelectric effect, Compton effect or the emission of energy from a hot cavity (body). All these phenomena are explained on the basis of quantum theory of radiation. According to quantum theory of radiation, an electromagnetic radiation (e.g. light) behaves like quanta or photon particle of relativistic mass $\frac{h\nu}{c^2}$, velocity c and momentum $\frac{h\nu}{c}$.

But, the photon theory failed to explain the phenomenon of interference or diffraction etc. These types of phenomena can be explained on the basis of wave theory of light.

Thus, we can conclude that light and other electromagnetic radiations sometimes act like waves and sometimes like particles (photons). So, light has a dual nature. It behaves like a wave in time of transmission and as a particle when it interacts with matter.

Matter, evidently, has particle properties. Therefore by analogy with radiation, particles should be expected to show wave like properties. In 1924, French theoretical physicist de Broglie proposed that material particles behave like waves on some occasions and it was experimentally verified by Davisson and Germer (1927) by electron diffraction from nickel crystal.

11.2. de Broglie Hypothesis

de Broglie proposed that a wave is always associated with every moving particle (like electron) and its corresponding wavelength (called de Broglie wavelength) is given by

$$\boxed{\lambda = \frac{h}{p}} \qquad \cdots(11.1)$$

where, p is the momentum of the material particle and h is Planck's constant.

This wave which is associated with a moving particle is known as matter wave. The wavelength of matter wave is called de Broglie wavelength. **The equation (11.1) is called de Broglie wave equation (or relation).**

Proof of de Broglie equation Let us consider the case of a photon with its mass m and rest mass m_0. According to Planck's quantum theory, the relation between energy (E) of a photon and the frequency ν of the associated wave is given by

$$E = h\nu \qquad \cdots(11.2)$$

Now, for a relativistic particle, the square of relativistic energy

$$E^2 = p^2c^2 + m_0^2c^4, \text{ where } c \text{ is the velocity of light} \qquad \cdots(11.3)$$

or, $E^2 = p^2c^2$ [$\because$ for a photon with rest mass $m_0 = 0$]

or, $E = pc \qquad \cdots(11.4)$

Thus, we get from equation (11.2) and equation (11.3),

$$E = pc = h\nu \quad \text{or,} \quad p = \frac{h\nu}{c}$$

or, $p = \frac{h}{\lambda}$ [$\because \nu = \frac{c}{\lambda}$ and the speed of the photon in free space $= c$]

or, $\boxed{\lambda = \frac{h}{p}}$

> **Special Note :**
> de Broglie assumed that the above relation is hold good for elementary particles like electrons. Thus for an electron of mass m and velocity v, **the de Broglie wavelength is**
> $\boxed{\lambda = \frac{h}{mv}}$, [$\because p = mv$] $\cdots(1)$

11.2.1. de Broglie Wavelength of an Electron Subjected to a Potential Difference V

Let us consider an electron of mass m and charge e. If the particle is subjected to a potential difference V so that it acquires a velocity v, its kinetic energy,

$$E = \frac{1}{2}mv^2$$

In this case, $eV = \frac{1}{2}mv^2 \; (= E) \qquad \cdots(11.5)$

or, $v = \sqrt{\frac{2eV}{m}} \quad \cdots(11.6a) \quad$ or, $v = \sqrt{\frac{2E}{m}} \quad \cdots(11.6b)$

If the de Broglie wavelength of this moving electron of momentum p is λ then,

$$\lambda = \frac{h}{p} \quad \text{or,} \quad \lambda = \frac{h}{mv} \qquad \cdots(11.7)$$

After substituting the value of velocity v from equation (11.6a) to equation (11.7), we have

$$\lambda = \frac{h}{m\sqrt{\frac{2eV}{m}}}$$

or, $= \boxed{\frac{h}{\sqrt{2meV}}} \qquad \cdots(11.8)$

We can write the another from of de Broglie wavelength using equation (11.6b),

$$\boxed{\lambda = \frac{h}{\sqrt{2mE}}} \quad [\because E = eV] \qquad \cdots(11.9)$$

Substituting the value of $h = 6.62 \times 10^{-34} \text{J} \cdot \text{s}$

$e = 1.6 \times 10^{-19}$ coulomb

m = mass of electron = 9.11×10^{-31} kg

in equation (11.8), we get de Broglie wavelength

$$\lambda = \frac{6.6 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 1.6 \times 10^{-19}\, V}} \text{ m} = \frac{12.26 \times 10^{-10}}{\sqrt{V}} \text{ m}$$

or, $$\boxed{\lambda = \frac{12.26}{\sqrt{V}} \text{ Å}} \qquad \cdots(11.10)$$

Special Note :

The equivalence between the wave and particle properties of a matter are given below :

	Particle Property	Wave Property
1.	momentum	wavelength
2.	energy	frequency

Problem 1

Calculate the de Broglie wavelength of a particle of mass 10g moving with a speed of $300\,\text{m} \cdot \text{s}^{-1}$.

Solution Here, mass of the particle $m = 10\text{g}$,

speed of the particle $v = 300\text{m} \cdot \text{s}^{-1}$

de Broglie wavelength $\lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34}}{10 \times 10^{-3} \times 300} \text{m} = 2.206 \times 10^{-34} \text{Å}$

Problem 2

A proton is moving freely with kinetic energy 43.9eV. Calculate its de Broglie wavelength.

Solution Here, the energy of the proton $E = 43.9\text{eV} = 43.9 \times 1.6 \times 10^{-19}\text{J}$

Mass of the proton = $m_p = 1.67 \times 10^{-27}\text{kg}$

∴ de Broglie wavelength

$$\lambda = \frac{h}{\sqrt{2m_p E}} = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 1.67 \times 10^{-27} \times 43.9 \times 1.6 \times 10^{-19}}} \text{m}$$

$$= 4.32 \times 10^{-12}\text{m}$$

Problem 3

If an electron beam in a television receiver tube is accelerated through a potential difference of 10000 volts, calculate the de Broglie wavelength of the electron.

Solution Here, potential difference $V = 10000$ volts

So, de Broglie wavelength $\lambda = \dfrac{12.26}{\sqrt{10000}}\ \text{Å} = 0.1226\ \text{Å}$

Problem 4

In between a photon of 100 eV and an electron of 100 eV, which one has shorter wavelength ?

Solution For a photon, the energy $E = h\nu$ or, $E = \dfrac{hc}{\lambda}$

$$\text{or,}\quad \lambda = \frac{hc}{E} = \frac{6.62\times10^{-34}\times3\times10^{8}}{100\times1.6\times10^{-19}} = 124\times10^{-10}\ \text{m} = 124\ \text{Å}$$

But for an electron like particle, de Broglie wavelength

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.62\times10^{-34}}{[2\times9.1\times10^{-31}\times100\times1.6\times10^{-19}]^{1/2}}$$

$$= 1.2\times10^{-10}\ \text{m} = 1.2\ \text{Å}$$

Hence, **the de Broglie wavelength of the electron is much smaller than that of a photon for their same energy.**

Problem 5

If a proton and an electron have the same wavelength, which one has more energy ?

Solution Let the mass of an electron and proton be m_e and m_p respectively.

Then, the de Broglie wavelength of an electron moving with velocity v_e is

$$\lambda_e = \frac{h}{m_e v_e} \qquad \ldots(1)$$

Similarly, the de Broglie wavelength of a proton moving with velocity v_p is

$$\lambda_p = \frac{h}{m_p v_p}$$

$$\because\quad \lambda_e = \lambda_p,\ \text{we can write}\ \frac{h}{m_e v_e} = \frac{h}{m_p v_p} \quad \text{or,}\quad m_e v_e = m_p v_p \qquad \ldots(2)$$

Now, the kinetic energy of an electron,

$$= \frac{1}{2} m_e v_e^2 = \frac{1}{2}(m_e v_e)v_e = \frac{1}{2}(m_p v_p)\cdot\left(\frac{m_p v_p}{m_e}\right)$$

[$\because$ from equation (2) we have $m_e v_e = m_p v_p$ and $v_e = \dfrac{m_p v_p}{m_e}$]

$$= \frac{m_p}{m_e} \times \left(\frac{1}{2} m_p v_p^2\right)$$

$$= \frac{m_p}{m_e} \times \text{energy of a proton} \quad [\because \text{ energy of a proton} = \frac{1}{2} m_p v_p^2]$$

Since $m_p > m_e$, the energy of an electron is greater than that of a proton corresponding to same wavelength.

Problem 6

What is the de Broglie wavelength of a thermal neutron at 400 K ?

Solution We know, the kinetic energy of a particle at equilibrium absolute temperature T,

$$E = \frac{3}{2} kT$$

Now, de Broglie wavelength of a neutron with energy E

$$\lambda_n = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2m \cdot \frac{3}{2} kT}} = \boxed{\frac{h}{\sqrt{3mkT}}}$$

[here mass of neutron = 1.675×10^{-27} kg
Boltzmann's constant $k = 1.38 \times 10^{-23}\ \text{J} \cdot \text{K}^{-1}$
$T = 400$ K; $h = 6.62 \times 10^{-34}\text{J} \cdot \text{s}$]

$$\therefore \quad \lambda_n = \frac{6.62 \times 10^{-34}}{\sqrt{3 \times (1.675 \times 10^{-27}) \times (1.38 \times 10^{-23}) \times (400)}} \text{ m}$$

$$= 1.25 \times 10^{-10} \text{ m} = 1.25 \text{ Å}$$

Problem 7

If an electron has a wavelength of 1 Å, find the energy and momentum.

Solution For a particle of mass m and energy E, the de Broglie wavelength

$$\lambda = \frac{h}{\sqrt{2mE}}$$

or, $$E = \frac{h^2}{2m\lambda^2}$$

or, $$E = \frac{(6.62 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31} \times (10^{-10})^2} \text{ J}$$

$$= \frac{(6.62)^2 \times 10^{-68} \times 10^{31} \times 10^{20}}{2 \times 9.1} \text{ J}$$

$$= \frac{(6.62)^2 \times 10^{-17}}{18.2 \times 1.6 \times 10^{-19}} \text{ eV} = \frac{(6.62)^2}{18.2 \times 1.6} \times 10^2 \text{ eV} = 150 \text{ eV (approx)}$$

Now, an electron of momentum p has de Broglie wavelength

$$\lambda = \frac{h}{p}$$

or, $$p = \frac{h}{\lambda} = \frac{6.62 \times 10^{-34}}{10^{-10}} = 6.62 \times 10^{-24}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}$$

11.2.2. Davisson and Germer's Experiment—(Experimental Evidences in Favour of Matter Waves)

In 1924, Louis de Broglie proposed that a wave is associated with every moving particle. These waves are called de Broglie waves or matter waves. But there was no experimental proof of this theory.

In 1927, two American physicists Davisson and Germer performed an experiment which gave first experimental evidences in favour of de Broglie's hypothesis of matter waves. They were **studying the scattering of electrons from a nickel crystal target**. The experimental arrangement in shown is Fig.1.

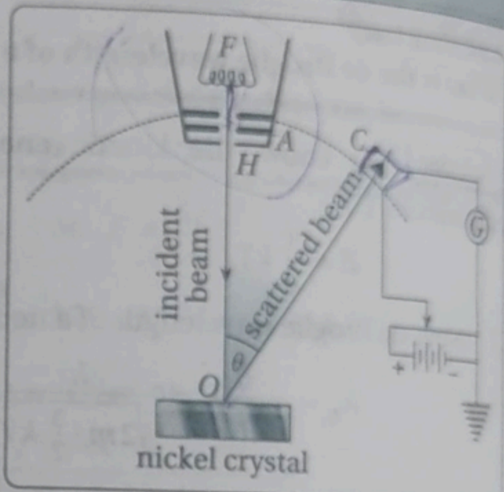

Fig. 1 ▷ An experimental arrangement of Davisson and Germer experiment

It consists of tungsten filament F coated with zinc oxide. A beam of electrons from a heated filament F are accelerated by a potential difference V between anode (A) and the filament (F). These electrons are colliminated by passing through a system of narrow slits and finally a fine beam of electrons emerge through a fine narrow hole (H) of the anode. The whole arrangement is known as an **electron gun**. The apparatus is enclosed in an evacuated chamber.

The fine beam of electrons coming out of the fine hole are allowed to strike normally on a nickel crystal. The electrons are scattered in all directions by the atoms of the parallel atomic planes of the crystal. The scattered rays take part in constructive interference of the electron waves. *The intensity of the electron beam scattered in a given direction is received with the help of Faraday cup (detector) C and is measured with the help of galvanometer G*. The Faraday cup (C) is designed in such a way that it can detect only the electrons scattered elastically by nickel scatterer S.

By rotating the detector about an axis through the point O, the intensity of the scattered beam can be measured for different values of scattering angle θ (the angle between incident and scattered direction of electron beam. For different increasing values of accelerating potential V (*i.e.* increase in electron energy) the variation of intensity (I) of scattered electron beam was plotted as a function of scattering angle θ in a polar graph as shown in Fig. (2).● The intensity of electron beam in given direction (in each graph) is proportional to the distance of the curve from the point 'O' (origin).

● The current measured by the galvanometer is directly proportional to the number of scattered electrons entering the chamber in 1 s. So, the intensity of the scattered beam is measured as a function of the scattering angle θ.

From the experiment, it is observed that a kink begins to appear in the curve at 44 volt of accelerating potential. This spur (or kink) becomes maximum at $V = 54\text{V}$ and for scattering angle $\theta = 50°$. Beyond 54 volts, the spur again decreases. **This observation indicates that electrons with kinetic energy 54 eV for its scattering angle 50°, suffer maximum scattering. The spur at 54 volts offers the evidence for the existence of matter waves (*i.e.*, electron waves)**

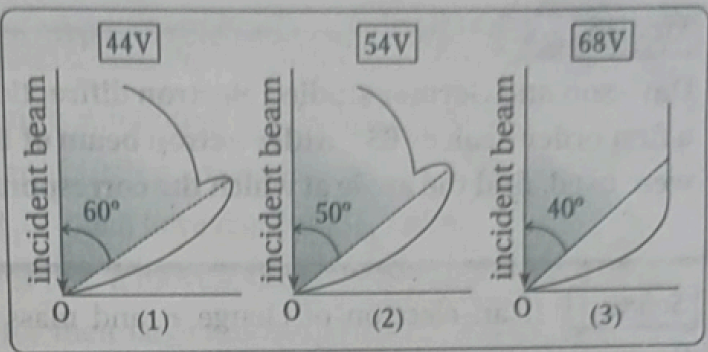

Fig. 2 ▷ The variation of intensity of scattered electron as a function of scattering angle for different accelerating potentials (V)

Discussion and inferences

The de Broglie wavelength associated with an electron accelerated through a potential difference of V volts is given by

$$\lambda = \frac{12.26}{\sqrt{V}}\ \text{Å}$$

Thus for $V = 54$ volt,

$$\lambda = \frac{12.26}{\sqrt{54}} = 1.66\ \text{Å}$$

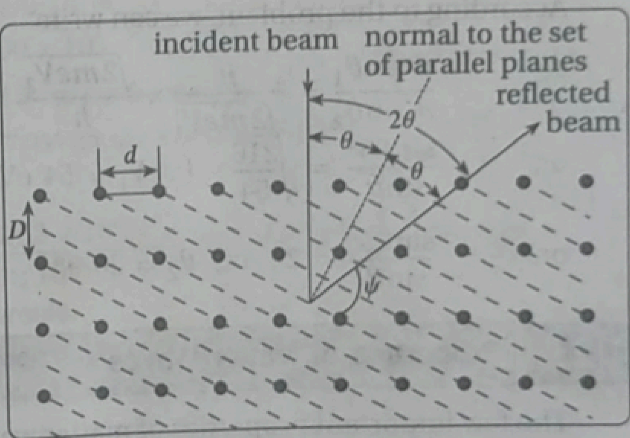

Fig. 3 ▷ Bragg reflection of electron waves from nickel crystal at normal incidence

Applying Bragg's law for 1st order maximum (from the parallel atomic planes of nickel crystal [Fig.3]; we get

$$2D\sin\psi = 1\lambda \quad \text{or,}\quad 2D\cos\theta = \lambda \quad [\because \psi = 90-\theta]$$

or, $$2d\sin\theta\cos\theta = \lambda \quad [\because D = d\sin\theta]$$

or, $$d\sin 2\theta = \lambda \quad \text{or,}\quad \boxed{d\sin\theta' = \lambda} \qquad \cdots(11.11)$$

where D = spacing between two atomic layers (*i.e.*, interplaner spacing) of nickel.

d = distance between the atoms in the surface layer

$\theta' = 2\theta$ = angle between the incident and reflected beam.

In Davisson and Germer's experiment, $d = 2.15\text{Å}$, $\theta' = 50°$ at accelerating potential 54 volt. Thus, we get from equation (11.11), $2.15\sin 50° = \lambda$ or, $\lambda = 1.65$ Å.

So, this result is in excellent agreement with the observed wavelength. Hence, it **verifies experimentally de Broglie's hypothesis of the wave nature of moving particles.**

Problem 1

Davisson and Germer studied electron diffraction with nickel crystal and found a first order peak at 65° with electron beam of 54 eV. If instead a 216 eV beam were used, find the angle at which the corresponding peak will be seen.

[W.B.U.T. Short Question, 2007]

Solution If an electron of charge e and mass m is accelerated by a potential difference of V volt, the corresponding de Broglie wavelength is

$$\lambda = \frac{h}{\sqrt{2meV}}$$

If d is the distance between the atoms in the surface layer of nickel crystal, Bragg's equation for 1st order reflection with diffraction angle θ is

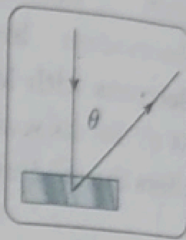

Fig. 4

$$d \sin\theta = \lambda$$

According to the problem, we can write

$$\frac{d \sin\theta_1}{d \sin\theta_2} = \frac{h}{\sqrt{2meV_1}} \times \frac{\sqrt{2meV_2}}{h} \quad \text{or,} \quad \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{V_2}{V_1}}$$

or, $$\frac{\sin 65^\circ}{\sin\theta_2} = \sqrt{\frac{216}{54}} \quad [\because V_1 = 54 \text{ eV}, \theta_1 = 65^\circ \text{ and } V_2 = 216 \text{ eV}]$$

or, $$\frac{\sin 65^\circ}{\sin\theta_2} = 2 \quad \text{or,} \quad \theta_2 = 26.9^\circ$$

11.2.3. Properties of Matter Waves

The few important properties of matter waves are :

① The de Broglie's wavelength $\left(\lambda = \frac{h}{\sqrt{2meV}}\right)$ of a particle is inversely proportional to the mass of the moving particle. **Lighter the particle, greater is its de Broglie wavelength.**

② Matter waves are **not visible.**

③ *The faster the particle moves, smaller is its de Broglie wavelength* ($\because \lambda = \frac{h}{mv}$)

④ The de Broglie's wave guides the particle. So, these waves are also called **pilot waves.**

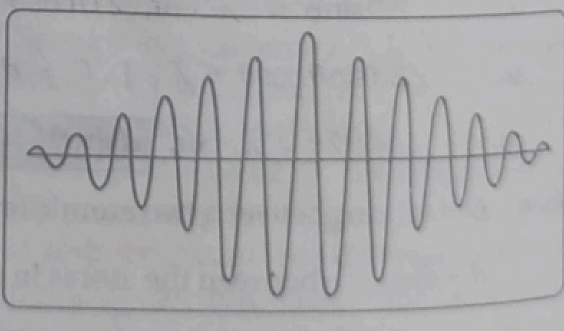
Fig. 5 ▷ Group of waves

⑤ The amplitude of the de Broglie waves associated with a moving particle, measures the probability of finding the particle in space at a particular instant. So, *the wave associated with large amplitude means a large probability of finding the particle at that particular position* [Fig. 5.].

or, $\frac{E_0 + K}{E_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ [$\because$ from equation (1), $mc^2 = E_0 + K$]

or, $\frac{(E_0 + K)^2}{E_0^2} = \frac{1}{1 - \frac{v^2}{c^2}}$ or, $E_0^2 = (E_0 + K)^2 - \frac{v^2}{c^2}(E_0 + K)^2$

or, $E_0^2 = (E_0 + K)^2 - \frac{v^2}{c^2}(mc^2)^2$ [$\because$ from equation (1), $E_0 + K = mc^2$]

or, $E_0^2 = (E_0 + K)^2 - m^2 v^2 c^2$ or, $m^2 c^2 v^2 = E_0^2 + 2E_0 K + K^2 - E_0^2$

or, $m^2 c^2 v^2 = K(2E_0 + K)$ or, $mv = \frac{\sqrt{K(2E_0 + K)}}{c}$ ...(2)

Therefore, the de Broglie wavelength

$$\lambda = \frac{h}{mv}$$

or, $\boxed{\lambda = \frac{hc}{\sqrt{K(2E_0 + K)}}}$ $\left[\because \text{from equation (2) } mv = \frac{\sqrt{K(2E_0 + K)}}{c}\right]$ ...(3)

Now substituting the value of

$$h = 6.62 \times 10^{-34} \text{J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s} \quad \text{and} \quad c = 3 \times 10^8 \text{ m} \cdot \text{s}^{-1}$$

We get, $hc = (4.136 \times 10^{-15}) \times (3 \times 10^8) \text{ eV} \cdot \text{m} = 1.241 \times 10^{-6} \text{ eV} \cdot \text{m}$

Hence, we get from equation (3),

$$\lambda = \frac{1.241 \times 10^{-6}}{\sqrt{K(2E_0 + K)}} \text{ metre}$$

11.3. Phase and Group Velocity

11.3.1. Concept of Phase or Wave Velocity

A progressive wave of amplitude 'a' propagating with velocity 'v' along the positive direction of X-axis can be represented by

$$y = a \sin\omega\left(t - \frac{x}{v}\right) \quad \text{...(11.12)}$$

where ω is the angular frequency. Usually, this wave velocity 'v' is also known as phase velocity (v_p).

Explanation of phase velocity with its magnitude We have from equation (11.12), the phase ϕ of the wave at any time t at position x is

$$\phi(x, t) = \omega\left(t - \frac{x}{v}\right) \quad \text{...(11.13)}$$

Differentiating both sides of equation (11.13) w.r.t. 't', we have

$$\frac{\partial \phi}{\partial t} = \omega\left(1 - \frac{1}{v}\frac{dx}{dt}\right) \quad \cdots(11.14)$$

Now, for a point of constant phase, we have

$$\left(\frac{\partial \phi}{\partial t}\right) = 0 \quad \text{or,} \quad \left[1 - \frac{1}{v}\left(\frac{dx}{dt}\right)_{\phi = \text{constant}}\right] = 0$$

or, $v = \left(\frac{dx}{dt}\right)_\phi$ = the velocity with which a wave having a constant phase moves forward. This is called phase velocity v_p

$$\therefore \quad v = v_p \quad \cdots(11.15)$$

Phase velocity **The velocity with which a wave advances in the medium is called the phase or wave velocity.**

Now, the equation (11.12) can be written, in the form of phase velocity v_p as

$$y = a \sin\omega\left(t - \frac{x}{v_p}\right) = a \sin\left(\omega t - \frac{\omega}{v_p}x\right) = a \sin(\omega t - kx) \quad \cdots(11.16)$$

where $k = \frac{\omega}{v_p}$ = propagation constant (or phase constant) of the wave.

$$\therefore \quad \text{phase velocity } \boxed{v_p = \frac{\omega}{k}} \quad \cdots(11.17)$$

11.3.2. Concept of Group Velocity (v_g)

The superposition of several waves (harmonic) [Fig. 6a] will form a complex wave called a wave group or wave packet [Fig. 6b]. **In a wave packet, the different components move with different phase velocities but the whole group advances through the medium with a constant velocity. This constant velocity with which a group of waves travels through the medium is called group velocity.** *The energy of the group of waves is transported with group velocity through a medium.*

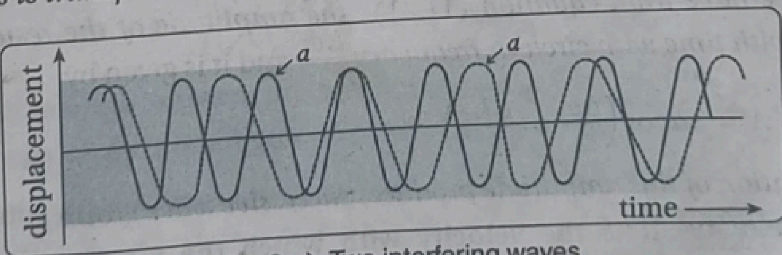

Fig. 6a ▷ Two interfering waves

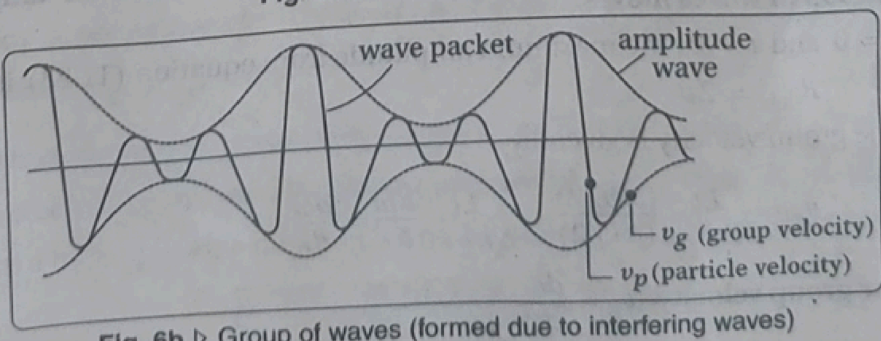

Fig. 6b ▷ Group of waves (formed due to interfering waves)

> **Special Note :**
> *In a pure harmonic wave, the group velocity and the phase velocity are the same. When a group of waves form a complex wave, the phase velocity and the group velocity are two separate entities.*

Explanation of group velocity Let us consider that a wave group arises from the combination of two harmonic waves with slightly differing angular frequencies by $\Delta\omega$ and wave numbers by Δk.

Thus, the two interfering waves are

$y_1 = a\sin(\omega t - kx)$ ···(11.18) and $y_2 = a[\sin(\omega+\Delta\omega)t - (k+\Delta k)x]$ ···(11.19)

So, the resultant displacement (y) at any time at any position is the sum of y_1 and y_2.

So, the resultant displacement (or the *equation of the wave packet* or the *wave group formed*) is given by

$$y = (y_1 + y_2) = a\sin(\omega t - kx) + a\sin[(\omega+\Delta\omega)t - (k+\Delta k)x]$$

$$= 2a\cos\left(-\frac{\Delta\omega}{2}t + \frac{\Delta k}{2}x\right)\sin\left[\left(\frac{2\omega+\Delta\omega}{2}\right)t - \left(\frac{2k+\Delta k}{2}\right)x\right] \quad \cdots(11.20)$$

$$\left[\because \sin c + \sin d = 2\cos\frac{c-d}{2}\sin\frac{c+d}{2}\right]$$

Now, since $\Delta\omega$ and Δk are small compared to ω and k we can neglect $\Delta\omega$ and Δk in the sine terms.

Thus, we have $y = 2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\sin(\omega t - kx)$ ···(11.21)

This equation (11.21) *represents that the resultant wave is an amplitude modulated wave which is travelling with phase velocity*

$\boxed{v_p = \frac{\omega}{k}}$ [$\because$ the sine factor represents a carrier wave] ...(11.22)

Again, *we have from equation* (11.21), *the amplitude of the resultant group of waves varies with time with circular frequency* $\frac{\Delta\omega}{2}$ and it is given by

$$A = 2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \quad \cdots(11.23)$$

The variation of this amplitude produces successive wave group.

Group velocity **It is the velocity with which the maximum amplitude of resultant group of waves moves.**

At $x = 0$ and $t = 0$, the maximum amplitude from equation (11.23) is

$$R_{max} = 2a \quad \cdots(11.24)$$

$\therefore$ The group velocity is given by

$$v_g = \underset{\Delta k \to 0}{Lt}\frac{(\Delta\omega/2)}{(\Delta k/2)} = \underset{\Delta k \to 0}{Lt}\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

So, the group velocity $v_g = \frac{d\omega}{dk}$ ...(11.25)

> **Special Note :**
> Since, energy is directly proportional to the square of amplitude, the velocity of propagation of amplitude and energy through the medium is equal to the group velocity (v_g) of the wave packet.

11.3.3. Relation between Phase Velocity and Group Velocity

We know, that phase velocity is

$$v_p = \frac{\omega}{k};$$ where ω and k are the angular frequency and propagation constant of a wave respectively

or, $$\omega = kv_p$$

Now, group velocity

$$v_g = \frac{d}{dk}(\omega) = \frac{d}{dk}(kv_p) \quad [\because \omega = kv_p]$$

$$= v_p + k\frac{dv_p}{dk} = v_p + k\frac{dv_p}{d\lambda}\frac{d\lambda}{dk} \qquad \cdots(11.26)$$

Again propagation constant $k = \frac{2\pi}{\lambda}$

$\therefore$ $$dk = -\frac{2\pi}{\lambda^2}\, d\lambda$$

$$\frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi} = -\frac{4\pi^2}{k^2} \times \frac{1}{2\pi} \quad \left[\because \lambda = \frac{2\pi}{k}\right]$$

or, $$\frac{d\lambda}{dk} = -\frac{2\pi}{k^2} \qquad \cdots(11.27)$$

So, we can write from equation (11.26) by using equation (11.27)

$$v_g = v_p + k\left(\frac{dv_p}{d\lambda}\right)\left(-\frac{2\pi}{k^2}\right) \quad \text{or,} \quad \boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}} \qquad \cdots(11.28)$$

This is the relation between phase velocity and group velocity.

Now, we can study this equation for dispersive and non-dispersive medium.

Case 1 **For dispersive medium** (*when v_p is a function of λ*), $v_g > v_p$. (*i.e.* group velocity is greater than phase velocity)

Case 2 For a non-dispersive medium, waves of all wavelengths move with same velocity. So, $\frac{dv_p}{d\lambda} = 0$.

Thus, for a non-dispersive medium, $v_p = v_g$ (*i.e.* group velocity = phase velocity). This is valid for electromagnetic waves in vacuum and elastic waves in a homogeneous medium. Thus, for light wave in vacuum where $\frac{dv_p}{d\lambda} = 0$

$$\boxed{v_p = v_g = c \text{ (velocity of light)}}$$

11.4. Phase Velocity, Group Velocity and Particle Velocity of a de Broglie Wave

11.4.1. Phase Velocity (v_p)

According to de Broglie, the wave associated with a moving particle of mass 'm' and velocity v (i.e. particle velocity v) is in the form of a group of waves (or a wave packet) [Fig. 3.]. The velocity with which an individual wave of a wave packet travels is called phase velocity or wave velocity.

Now, the energy of a wave as per quantum mechanics

$$E = h\nu = \frac{h}{2\pi} 2\pi\nu$$

or, $$E = \hbar\omega \text{ where } \hbar = \frac{h}{2\pi} \text{ and } \omega = 2\pi\nu$$

or, $$\boxed{\omega = \frac{E}{\hbar}} \quad \cdots(11.29)$$

The de Broglie wavelength (λ) corresponding to a moving particle of mass 'm' and momentum p,

$$\lambda = \frac{h}{p} \text{ or, } p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} \text{ or, } p = \hbar k \text{ where } k = \text{propagation constant}$$

or, $$\boxed{k = \frac{p}{\hbar}} \quad \cdots(11.30)$$

So, we know from equation (11.22), the phase velocity of the associated wave

$$v_p = \frac{\omega}{k}$$

Now, substituting the values of k and v_p from equations (11.29) and (11.30) we get

$$v_p = \frac{E/\hbar}{p/\hbar} \text{ or, } \boxed{v_p = \frac{E}{p}} \quad \cdots(11.31)$$

Special Note :

For a relativistic particle, whose velocity v (*i.e.* particle velocity) is comparable to the velocity (c) of light, the energy $E = mc^2$

The phase velocity $v_p = \frac{E}{p}$ [from equation (3.31)]

$$= \frac{mc^2}{mv} = \frac{c^2}{v} = \frac{c}{v/c}$$

$\therefore \quad v_p > c \quad [\because \frac{v}{c}$ is less than 1]

But, we know from the basic postulates of relativity, that a particle cannot move with a velocity greater than the velocity of light. So, there is no physical meaning of phase velocity here. Thus it indicates that **the motion of the relativistic particle cannot be represented by a single wave.**

11.4.2. Group Velocity (v_g)

We know, the energy (E) of a wave as per quantum mechanics is

$$E = h\nu$$

or, $$E = \frac{h}{2\pi}2\pi\nu = \hbar\omega\text{, where angular frequency } \omega = 2\pi\nu$$

or, $$\omega = \frac{E}{\hbar} \quad \cdots(11.32a)$$

Differentiating both sides, we have

$$\boxed{d\omega = \frac{dE}{\hbar}} \quad \cdots(11.32b)$$

Similarly, the de Broglie wavelength of a moving particle of momentum p

$$\lambda = \frac{h}{p} \quad \text{or,} \quad p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k\text{, where } k = \text{wave number.}$$

or, $$k = \frac{p}{\hbar} \quad \cdots(11.33a)$$

Differentiating, we have $\boxed{dk = \frac{dp}{\hbar}}$ $\cdots(11.33b)$

Again, the group velocity $v_g = \frac{d\omega}{dk}$

Substituting the value of $d\omega$ and dk from equation (11.32b) and (11.33b), we have

$$v_g = \frac{dE/\hbar}{(dp/\hbar)} \quad \text{or,} \quad \boxed{v_g = \frac{dE}{dp}} \quad \cdots(11.34)$$

We can study this expression for relativistic and non-relativistic case.

Case 1 For, relativistic particle,

$$E^2 = p^2c^2 + m_0^2c^4 \quad \text{and} \quad p = mv,$$

where v = velocity with which the particle travels *i.e.* particle velocity

Differentiating both sides *w. r. t.* p, we have

$$2E\frac{dE}{dp} = 2pc^2 \quad \text{or,} \quad \frac{dE}{dp} = \frac{pc^2}{E} = \frac{(mv)c^2}{mc^2} = v$$

or, $$\boxed{v_g = v} \quad \left[\because v_g = \text{group velocity} = \frac{dE}{dp}\right] \quad \cdots(11.35)$$

Case 2 For, non-relativistic particle

$$E = \frac{p^2}{2m} \quad \text{and} \quad p = mv$$

$$\therefore \quad \frac{dE}{dp} = \frac{p}{m}$$

Now, $v_g = \frac{dE}{dp} = \frac{p}{m} = \frac{mv}{m} = v$

or, $\boxed{v_g = v = \text{particle velocity}}$ ...(11.36)

Hence, **in both cases the group velocity of de Broglie (group) waves is same as that of the particle velocity v.**

11.4.3. Relation between Group Velocity (v_g), Phase Velocity (v_p) and the Velocity of Light

We know, phase velocity $v_p = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar}$ $\left[\because \omega = \frac{E}{\hbar} \text{ and } k = \frac{p}{\hbar}\right]$

or, $v_p = \frac{E}{p}$

or, $v_p = \frac{mc^2}{mv_{\text{particle}}}$ $[\because E = mc^2$ and momentum $p = mv]$

or, $v_p = \frac{c^2}{v_{\text{particle}}}$ or, $v_p = \frac{c^2}{v_g}$

$[\because$ group velocity (v_g) of de Broglie wave = particle velocity $v_{\text{particle}}]$

or, $\boxed{v_p \cdot v_g = c^2}$ ...(11.37)

This is the required relation between v_p, v_g and velocity of light, c.

11.4.4. Rest Mass (m_0) of a Photon

For a relativistic particle of mass m, momentum p, the total relativistic energy E is given by

$$E^2 = p^2c^2 + m_0^2c^4$$

Now, phase velocity, $v_p = \frac{\omega}{k} = \frac{E}{p} = \left[\frac{p^2c^2 + m_0^2c^4}{p^2}\right]^{\frac{1}{2}}$

or, $v_p = c\left[1 + \frac{m_0^2c^2}{p^2}\right]^{\frac{1}{2}}$ ...(11.38)

Now, the velocity of a photon is c and it corresponds to electromagnetic waves. So, phase velocity $v_p = c$.

Thus, we get from equation (11.38),

$c\left[1 + \frac{m_0^2c^2}{h^2/\lambda^2}\right]^{\frac{1}{2}} = c$ [from de Broglie hypothesis, momentum of a photon $p = \frac{h}{\lambda}$]

or, $\frac{m_0^2c^2\lambda^2}{h^2} = 0$ or, $m_0 = 0$

Hence, the rest mass (m_0) of a photon is zero.

11.5. Uncertainty Principle

In classical mechanics, we can determine the position and momentum of macroscopic bodies simultaneously with perfect (*i.e.* same) accuracy from its initial position, momentum and the forces acting upon it. However, in quantum mechanics, each moving microparticle (e.g. subatomic particle electron, proton etc.) is associated with a wave packet that is extending throughout a region of space. Thus, when a microparticle is in motion (with a velocity comparable to the velocity of light), its position can be anywhere within the wave packet. Hence, there will be an uncertainty in specifying the position of the particle. At the same time, a wave packet consists of a range of wavelengths. Thus, from the de Broglie relation ($p = \frac{h}{\lambda}$), there will also be an uncertainty in the measurements of momentum of the microparticle. Therefore, the **momentum and position of a moving microparticle cannot be measured simultaneously with perfect (*i.e.* same) accuracy.**

On the basis of these considerations, Werner Heisenberg, in 1927, enunciated the *principle of uncertainty.*

Heisenberg's uncertainty principle states that **the product of uncertainties in the simultaneous measurement of the position and momentum of a particle is equal to or greater than the $\hbar$ $(=\frac{h}{2\pi})$, where h is the Planck's constant.**

i.e. $$\boxed{\Delta x \Delta p_x \geqslant \hbar} \qquad \cdots(11.39)$$

This is the **position-momentum Heisenberg's uncertainty relation.** Here, for motion along X-axis, Δx is the uncertainty in the determination of position of the particle and Δp_x is the uncertainty in the determination of the corresponding momentum of the particle. *Position and momentum are conjugated variables.*

Similarly, **the product of uncertainties in the simultaneous measurement of the energy and time (at which the measurement was done) of a particle is equal to or greater than $\hbar$**

i.e. $$\boxed{\Delta E \Delta t \geqslant \hbar} \qquad \cdots(11.40)$$

This is the **energy-time Heisenberg's uncertainty relation.**

Here, *energy and time are two conjugated variables.*

Special Note :

① According to Heisenberg's principle, the position and momentum of a particle at any time cannot be measured simultaneously with accuracy. So, if we desire to reduce the error in the measurement of position (*i.e.* to reduce error Δx), then this can be possible at the expense of accuracy for determining the momentum (*i.e.* the error involved in measurement of Δp will increase).

(2) ***Exact statement of uncertainty principle :*** From a more accurate calculations where we consider a group consisting of very large number of waves of continuously varying wavelengths, we get the product of the uncertainties in the measurement of position and momentum of the particle is equal to or greater than $\frac{\hbar}{2}$.

i.e. $$\boxed{\Delta x \Delta p_x \geqslant \frac{\hbar}{2}}$$

Similarly $$\boxed{\Delta E \Delta t \geqslant \frac{\hbar}{2}}$$

Problem 1

A 20g particle is moving with a speed $30\,\text{m}\cdot\text{s}^{-1}$. At certain instant of time its position was determined with uncertainty of 2mm. Calculate the fractional uncertainty in its linear momentum.

Solution Here, mass of the particle = m = 20g, velocity of the particle = v = $30\,\text{m}\cdot\text{s}^{-1}$

uncertainty in position = Δx = 2mm = 2×10^{-3} m

Now, we can write from Heisenberg uncertainty principle

$$\Delta x \times \Delta p = \frac{h}{4\pi}$$

or, $$\Delta p = \frac{6.63 \times 10^{-34}}{4 \times 3.14 \times 2 \times 10^{-3}} = 2.63 \times 10^{-32}$$

$\therefore$ The fractional uncertainty in its linear momentum

$$\frac{\Delta p}{p} = \frac{2.63 \times 10^{-32}}{20 \times 10^{-3} \times 30} = 4.398 \times 10^{-32}$$

Problem 2

An electron remains in excited state for 10^{-11} s. (i) What is the minimum uncertainty in the energy of an excited state ? [W.B.U.T. 2004 (S)] (ii) What is the physical interpretation of this uncertainty in measurement of energy ? (iii) Find the uncertainty in the frequency of light emitted at 10^{-11} s.

Solution

(i) From, Heisenberg energy-time uncertainty principle we have

$$\Delta E \geqslant \frac{\hbar}{2\Delta t}$$ [considering exact statement of uncertainty principle]

Hence, the minimum uncertainty in the energy of the excited state

$$\Delta E = \frac{h}{4\pi\Delta t} = \frac{6.62 \times 10^{-34}}{4 \times 3.14 \times 10^{-11}}\ \text{J}$$

$$= \frac{6.62 \times 10^{-34}}{4 \times 3.14 \times 10^{-11} \times (1.6 \times 10^{-19})}\ \text{eV} = 3.29 \times 10^{-5}\ \text{eV}$$

Problem 5

Compute the smallest possible uncertainty in the position of an electron moving with velocity $3 \times 10^7 \text{m} \cdot \text{s}^{-1}$. The rest mass of electron is 9.1×10^{-31} kg.

[W.B.U.T. 2009]

Solution From Heisenberg's position-momentum uncertainty principle,

$$(\Delta x) = \frac{\hbar}{2\,\Delta p_x}, \text{ where } \hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}\ \text{J} \cdot \text{s}^{-1}$$

Now, Δp_x = maximum uncertainty in momentum

$$= p = mv = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\, v$$

$$= \frac{9.1 \times 10^{-31} \times 3 \times 10^7}{\sqrt{1 - \left(\frac{3 \times 10^7}{3 \times 10^8}\right)^2}} = 27.4 \times 10^{-24}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}$$

$$\therefore \quad (\Delta x)_{\min} = \frac{\hbar}{2(\Delta p_x)_{\max}} = \frac{1.054 \times 10^{-34}}{2 \times 27.4 \times 10^{-24}}$$

$$= 0.019 \times 10^{-10}\ \text{m}$$

$$= 0.019\ \text{Å}$$

11.5.1. Physical Interpretation of Heisenberg's Uncertainty Relation

Heisenberg's position-momentum uncertainty relation states $\Delta x \Delta p_x \geq \frac{\hbar}{2}$. It draws the following inference.

(1) If we can measure the position of a particle accurately (*i.e.* $\Delta x = 0$), the uncertainty in measurement of momentum (*i.e.* Δp_x) at the same instant becomes infinite.

(2) If we can measure the momentum of a particle accurately (*i.e.* $\Delta p_x = 0$), the uncertainty in measurement of position (*i.e.* Δx) at the same instant becomes infinite.

(3) For a particle of mass m moving with velocity v, the position-momentum uncertainty relation becomes

$$\Delta x (m \Delta v) \geq \frac{\hbar}{2} \quad \text{or,} \quad \Delta x \Delta v \geq \frac{\hbar}{2m} \qquad \text{...(11.41)}$$

So, for a heavy body, $\frac{\hbar}{m} \simeq 0$.

Therefore, we get from equation (11.41), $\Delta x \Delta v = 0$

This indicates that the uncertainty in measurement vanishes. Hence, **we can measure the position and momentum of a heavy body with perfect accuracy.** This is also the result coming out from classical mechanics.

Thus, classical mechanics is applicable for macroparticles (heavy body) whereas quantum mechanics associated with uncertainty principle is applicable for microparticles (e.g. electron, proton, neutron etc).

(4) From energy-time uncertainty relation we have $\Delta E\,\Delta t \geqslant \frac{\hbar}{2}$.

We can measure energy accurately ($\Delta E = 0$) only when the measurement is made over an infinite period of time (*i.e.* $\Delta t = \infty$).

11.6. Applications of Uncertainty Principle

11.6.1. Non-existence of Electrons and Existence of Protons and Neutrons inside the Nucleus of an Atom

The radius (r) of the nucleus of an atom is in the order of 10^{-14} m. So, if the electron is considered inside the nucleus, the uncertainty in position of the electron

$$\Delta x = \text{diameter of the nucleus} \simeq 2r \simeq 2 \times 10^{-14} \text{ m}$$

From, uncertainty principle, the uncertainty in momentum of an electron is,

$$\Delta p_x \geqslant \frac{h}{2\pi \Delta x}$$

or,

$$\Delta p_x \geqslant \frac{6.62 \times 10^{-34}}{2 \times 3.14 \times (2 \times 10^{-14})} \text{ kg} \cdot \text{m} \cdot \text{s}^{-1}$$

$$\geqslant 5.27 \times 10^{-21} \text{ kg} \cdot \text{m} \cdot \text{s}^{-1}$$

It means that if the elementary particle is inside the nucleus, its minimum momentum must be

$$p_{\min} = 5.27 \times 10^{-21} \text{ kg} \cdot \text{m} \cdot \text{s}^{-1}$$

(1) **For electrons** The minimum energy of an electron of mass m is obtained from relativistic formula of energy

$$E^2 = p^2c^2 + m_0^2c^4 \simeq p^2c^2$$

[$\because$ rest mass energy (m_0c^2) of the electron $m_0c^2 \simeq 0.511$ MeV and is negligible in comparison to p^2c^2.]

$$\therefore \quad E = pc$$

$$= 5.27 \times 10^{-21} \times 3 \times 10^{8} \text{ J}$$

$$= \frac{5.27 \times 10^{-21} \times 3 \times 10^{8}}{1.6 \times 10^{-19}} \text{ eV}$$

$$\simeq 10 \text{ MeV}$$

So, if an electron is inside a nucleus, its energy must be of the order of 10 MeV. But from experimental data, we know that the electrons emitted by radioactive nucleus during β decay have energy only 3 to 4 MeV. Therefore, electrons cannot be present within the nucleus.

(2) **For protons and neutrons** For protons and neutrons, their rest mass

$$m_0 \simeq 1.67 \times 10^{-27} \text{ kg}$$

So, the corresponding value of energy (kinetic energy) is given by

$$E_k = \frac{p^2}{2m_0} = \frac{(5.27 \times 10^{-21})^2}{2 \times (1.67 \times 10^{-27})} \text{ J}$$

$$= \frac{(5.27 \times 10^{-21})^2}{2 \times (1.67 \times 10^{-27}) \times 1.6 \times 10^{-19}} \text{ eV} = 52 \text{ keV}$$

Since this energy E_k is smaller than the energies carried by these particle emitted from a nucleus, both these particles can exist inside the nucleus.

11.6.2. Binding Energy of an Electron in Atom

When an electron revolves round the nucleus of an atom in an orbit of radius r the uncertainty in the position of the electron Δx is in the order of r

$$\therefore \quad \Delta x = 2r$$

Now, we can write from position-momentum uncertainty principle

$$\Delta p_x \Delta x \geqslant \frac{h}{2\pi} \quad \text{or,} \quad \Delta p_x \geqslant \frac{h}{2\pi \Delta x} \quad \text{or,} \quad \Delta p_x \geqslant \frac{h}{2\pi (2r)}$$

Hence, minimum value of the momentum of the particle

$$p = \Delta p_x = \frac{h}{4\pi r}$$

Taking $r \simeq 10^{-10}$ m,

$$p \simeq \frac{h}{4\pi \times 10^{-10}} \simeq 0.527 \times 10^{-24} \text{ kg} \cdot \text{m} \cdot \text{s}^{-1}$$

The kinetic energy corresponding to this non-relativistic momentum for an electron

$$E_k = \frac{p^2}{2m_0} = \frac{(.527 \times 10^{-24})^2}{2 \times 9.1 \times 10^{-31}} = 1.5 \times 10^{-19} \simeq 1 \text{ eV (approx)}$$

Now, the potential energy of an electron in the field of nucleus with atomic number Z is given by

$$V = -\frac{Ze^2}{4\pi \epsilon_0 r} = -\frac{Z(1.6 \times 10^{-19})^2}{4 \times 3.14 \times 8.85 \times 10^{-12} \times (10^{-10})} \text{ J} = -14.4Z \text{ eV}$$

So, the total energy of the electron in its orbit will be

$$E = E_k + V = (1 - 14.4\ Z) \text{ eV}$$

or, $\boxed{E = (1 - 14.4\ Z)}$ eV

So, for **hydrogen**, $Z = 1$, $E = 1 - 14.4 = -13.4$ eV

For **helium**, $Z = 2$, $E = (1 - 28.8) = -27.8$ eV

These above two values of binding energy of hydrogen and helium atoms (derived from uncertainty principle) agree closely with the binding energy of the outermost electron/electrons in hydrogen atom (−13.6 eV) and helium atom (−24.6 eV) respectively.

Hence, the value of binding energy derived on the basis of uncertainty principle is acceptable.