Fourteen

Complex Variables

CHAPTER OUTLINE

14.1 INTRODUCTION

The theory of functions of a complex variable plays a very important role in solving a large number of problems in the field of engineering and science. The functions of a complex variable is particularly concerned with the analytic function of complex variable.

14.2 COMPLEX VARIABLES

Any variable of the form $z = x + iy$ is known as a complex variable, where x and y are real and $i = \sqrt{-1}$.

Functions of a Complex Variable If $z = x + iy$ be a complex variable then $w = f(z) = u + iv$ is known as function of complex variable z and is denoted by $w = f(z)$.

Basic Definitions

1. Distance $|z - z_0|$ represents distance between two points z and z_0.

2. Circle $|z - z_0| = r$ represents a circle with centre at the point z_0 and radius r.

3. Interior of a circle $|z - z_0| < r$ represents interior of the circle.

4. Exterior of a circle $|z-z_0|>r$ represents exterior of the circle.

5. Neighbourhood The set of all points for which $|z-z_0|<r$ is known as the neighbourhood of z_0.

6. Limit If $w=f(z)$ be a single-valued function defined at all points in some neighbourhood of a point z_0 then the limit of $f(z)$ as z approaches z_0 is

$$\lim_{z\to z_0} f(z)=w_0$$

7. Continuity $f(z)$ is said to be continuous at $z=z_0$ if

$$\lim_{z\to z_0} f(z)=f(z_0)$$

8. Differentiability A single-valued function $f(z)$ is said to be differentiable at a point z_0 if

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\delta z)-f(z_0)}{\delta z} \text{ exists.}$$

Derivative of $f(z)$ at a point z_0 can also be written as

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

9. Entire Function A function $f(z)$ is said to be an entire function if it is analytic everywhere in the finite plane (complex plane). An entire function is not analytic at $z=\infty$, e.g., cos z, sin z, e^z, and polynomial functions etc.

14.3 ANALYTIC FUNCTIONS

A function $f(z)$ is said to be analytic function if it is defined and differentiable at each point of a region R. This function is also known as *regular* or *holomorphic function.*

14.3.1 Necessary Conditions for f(z) to be Analytic (Cauchy-Riemann Equations)

The necessary conditions for a function $f(z)=u+iv$ to be an analytic function at all the points in a region R are

(i) $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

Proof Let $f(z)=u+iv$ be an analytic function and δu, δv be the increments in u and v corresponding to increments δx, δy in x and y respectively.

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\lim_{\delta z\to 0}\frac{[(u+\delta u)+i(v+\delta v)]-(u+iv)}{\delta z}=\lim_{\delta z\to 0}\frac{\delta u+i\delta v}{\delta z} \quad \ldots(14.1)$$

HISTORICAL DATA

Georg Friedrich Bernhard Riemann (1826–1866) was an influential German mathematician who made lasting contributions to analysis, number theory, and differential geometry, some of them enabling the later development of general relativity.

He made some famous contributions to modern analytic number theory. In a single short paper (the only one he published on the subject of number theory), he investigated the Riemann zeta function and established its importance for understanding the distribution of prime numbers. He made a series of conjectures about properties of the zeta function, one of which is the well-known Riemann hypothesis.

He applied the Dirichlet principle from variational calculus to great effect; this was later seen to be a powerful heuristic rather than a rigorous method. Its justification took at least a generation. His work on monodromy and the hypergeometric function in the complex domain made a great impression, and established a basic way of working with functions by considering only their singularities.

He was also the first to suggest using dimensions higher than merely three or four in order to describe physical reality—an idea that was ultimately vindicated with Einstein's contribution in the early 20th century.

Since $f(z)$ is differentiable at each point of the region R, δz may approach zero along any path. Consider two paths as follows:

(i) **Parallel to x-axis:** $\delta y = 0$

$$\delta z = \delta x + i\delta y = \delta x$$

Substituting in Eq. (14.1),

$$f'(z) = \lim_{\delta x \to 0} \frac{\delta u + i\delta v}{\delta x} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + i \lim_{\delta x \to 0} \frac{\delta v}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{...(14.2)}$$

(ii) **Parallel to y-axis:** $\delta x = 0$

$$\delta z = \delta x + i\delta y = i\delta y$$

Substituting in Eq. (14.1),

$$f'(z) = \lim_{\delta y \to 0} \frac{\delta u + i\delta v}{i\delta y} = \lim_{\delta y \to 0} \frac{\delta u}{i\delta y} + \lim_{\delta y \to 0} \frac{\delta v}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{...(14.3)}$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.

Equating Eqs (14.2) and (14.3),

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

These equations are known as *Cauchy-Riemann(C-R)* equations.

14.3.2 Sufficient Conditions for $f(z)$ to be Analytic

For a function $f(z)=u+iv$, if the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in the region R and $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ then the function of $f(z)$ is analytic.

Proof Let $f(z) = u + iv$ be a single valued function possessing continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in the region R satisfying C–R equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$f(z+\delta z)=u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y)$$

$$=\left[u(x, y)+\left(\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y\right)+\cdots\right]+i\left[v(x, y)+\left(\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y\right)+\cdots\right]$$

[By Taylor's theorem]

$$=u(x, y)+iv(x, y)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta y$$

[Neglecting terms with power more than one]

$$=f(z)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y \quad \text{[Using C-R equations]}$$

$$f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(i^2\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z$$

$$\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial v}{\partial y} \quad \text{[Using C-R equations]}$$

Hence, using Eqs 14.2 and 14.3, $f(z)$ is analytic at all points in the region R.

EXAMPLE 14.1 *Test the analyticity of the function $w = \sin z$.*

Solution:

$$w=\sin z$$

$$u+iv=\sin(x+iy)=\sin x\cos iy+\cos x\sin iy$$

$$=\sin x\cosh y+i\cos x\sinh y$$

Comparing real and imaginary parts,

$$u=\sin x\cosh y, \qquad v=\cos x\sinh y$$

$$\frac{\partial u}{\partial x}=\cos x\cosh y, \qquad \frac{\partial v}{\partial x}=-\sin x\sinh y$$

$$\frac{\partial u}{\partial y}=\sin x\sinh y, \qquad \frac{\partial v}{\partial y}=\cos x\cosh y$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

C-R equations are satisfied. Hence, $\sin z$ is analytic.

EXAMPLE 14.2 *Show that $f(z)=z|z|$ is not analytic anywhere.*

Solution:

$$f(z)=z|z|$$

$$u+iv=(x+iy)\sqrt{x^2+y^2}=x\sqrt{x^2+y^2}+iy\sqrt{x^2+y^2}$$

Comparing real and imaginary parts,

$$u=x\sqrt{x^2+y^2}, \qquad v=y\sqrt{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\sqrt{x^2+y^2}+x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x, \qquad \frac{\partial v}{\partial x}=y\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x$$

$$= \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$= \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y,$$

$$\frac{\partial v}{\partial y} = \sqrt{x^2 + y^2} + y \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y$$

$$= \frac{xy}{\sqrt{x^2 + y^2}}$$

$$= \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied. Hence, $f(z)$ is not analytic anywhere.

EXAMPLE 14.3 *Find the analytic region of $f(z) = (x - y)^2 + 2i(x + y)$.*

Solution:

$$f(z) = (x - y)^2 + 2i(x + y)$$
$$u + iv = (x - y)^2 + 2i(x + y)$$

Comparing real and imaginary parts,

$$u = (x - y)^2, \qquad v = 2(x + y)$$

$$\frac{\partial u}{\partial x} = 2(x - y), \qquad \frac{\partial v}{\partial x} = 2$$

$$\frac{\partial u}{\partial y} = -2(x - y), \qquad \frac{\partial v}{\partial y} = 2$$

All the partial derivatives are continuous.
For $f(z)$ to be analytic, u and v should satisfy C–R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2(x - y) = 2, \qquad -2(x - y) = -2$$

$$x - y = 1, \qquad x - y = 1$$

$f(z)$ is analytic at all points satisfying $x - y = 1$. Hence, the analytic region is the line $x - y = 1$.

EXAMPLE 14.4 *Find the values of a and b such that the function $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is analytic. Also, find $f'(z)$.*

Solution:

$$f(z) = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$$
$$u + iv = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$$

Comparing real and imaginary parts,

$$u = x^2 + ay^2 - 2xy, \qquad v = bx^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y, \qquad \frac{\partial v}{\partial x} = 2bx + 2y$$

$$\frac{\partial u}{\partial y} = 2ay - 2x, \qquad \frac{\partial v}{\partial y} = -2y + 2x$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2x - 2y = -2y + 2x \quad \ldots(1), \quad 2ay - 2x = -2bx - 2y \quad \ldots(2)$$

Comparing coefficients of x and y in Eqs (1) and (2),

$$2a = -2, \qquad -2 = -2b$$

$$a = -1, \qquad b = 1$$

$$\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i(2bx + 2y) \\
&= (2x - 2y) + i(2x + 2y) \qquad [\because b = 1] \\
&= 2(x + i^2 y + ix + iy) \qquad [\because i^2 = -1] \\
&= 2[x(1+i) + iy(1+i)] = 2(1+i)(x+iy) = 2(1+i)z
\end{aligned}$$

EXAMPLE 14.5 *If $f(z)$ and $\overline{f(z)}$ are both analytic then show that $f(z)$ is constant.*

Solution: Let $f(z) = u + iv$

$$\overline{f(z)} = u - iv$$

$f(z)$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots(2)$$

$\overline{f(z)}$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y} = -\frac{\partial v}{\partial y} \qquad \ldots(3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x} = \frac{\partial v}{\partial x} \qquad \ldots(4)$$

Adding Eqs (1) and (3),

$$2\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

Substracting Eq.(2) from Eq.(4),

$$0 = 2\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i \cdot 0 = 0$$

$$\therefore f(z) = \text{constant}$$

EXAMPLE 14.6 *Show that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\bar{z}$.*

Solution:

$$z = x + iy, \qquad \bar{z} = x - iy$$

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

$\therefore$ u and v can be considered as functions of z and $\bar{z}$. Hence, w also depends on z and $\bar{z}$ (Fig 14.1).

$$w = u + iv$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left[\frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right] + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

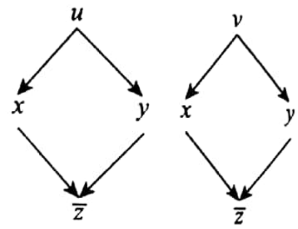

Fig. 14.1

$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$= \left[\frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}\right] \quad \text{[Using C–R equations]}$$

$$= 0$$

$\therefore$ w does not depend on $\bar{z}$.

Hence, w can be expressed as function of z alone, not as a function of $\bar{z}$.

EXAMPLE 14.7 *For what values of z, the function w defined by $z=e^{-v}(\cos u+i\sin u)$ ceases to be analytic.*

Solution:

$$z=e^{-v}(\cos u+i\sin u)=e^{-v}e^{iu}=e^{i^2v}e^{iu}=e^{i(iv+u)}=e^{iw}$$

$$\log z=\log e^{iw}=iw$$

$$w=\frac{1}{i}\log z=-i\log z$$

$$\frac{dw}{dz}=-\frac{i}{z}$$

w is not analytic, where $\frac{dw}{dz}$ does not exist, i.e., $\frac{dw}{dz}\to\infty$

At $z=0$ $\frac{dw}{dz}\to\infty$

Hence, w is not analytic at $z=0$.

EXAMPLE 14.8 *Prove that*

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\quad z\neq 0$$
$$=0,\qquad z=0$$

satisfy C-R equations at the origin but $f'(0)$ does not exist.

Solution:

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\quad z\neq 0$$

and $f(0)=0$

$$u+iv=\left(\frac{x^3-y^3}{x^2+y^2}\right)+i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

Comparing real and imaginary parts,

$$u=\frac{x^3-y^3}{x^2+y^2},\quad v=\frac{x^3+y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\lim_{\delta x\to 0}\frac{u(x+\delta x,y)-u(x,y)}{\delta x}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)}=\lim_{\delta x\to 0}\frac{u(\delta x,0)-u(0,0)}{\delta x}=\lim_{\delta x\to 0}\frac{\delta x-0}{\delta x}=1$$

$$\frac{\partial u}{\partial y}=\lim_{\delta y\to 0}\frac{u(x,y+\delta y)-u(x,y)}{\delta y}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{\delta y \to 0} \frac{u(0,\delta y) - u(0,0)}{\delta y} = \lim_{\delta y \to 0} \frac{-(\delta y) - 0}{\delta y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{\delta x \to 0} \frac{v(\delta x, 0) - v(0,0)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x - 0}{\delta x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{\delta y \to 0} \frac{v(0,\delta y) - v(0,0)}{\delta y} = \lim_{\delta y \to 0} \frac{\delta y}{\delta y} = 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C–R equations are satisfied.

Now,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\dfrac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)} - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$.

$$f'(0) = \lim_{z \to 0} \frac{(x^3 - m^3 x^3) + i(x^3 + m^3 x^3)}{(x^2 + m^2 x^2)(x + imx)} = \lim_{x \to 0} \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

Since limit depends on m (i.e., on the path), it is not unique.

Therefore, $f'(0)$ does not exist.

Hence, $f(z)$ is not analytic at the origin.

14.3.3 Cauchy–Riemann Equations in Polar Form

If $f(z) = u + iv$ is an analytic function, where u and v are functions of r, θ and $z = re^{i\theta}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof $f(z) = u + iv$, $z = re^{i\theta}$

$$u + iv = f(re^{i\theta}) \qquad \text{...(14.4)}$$

Differentiating Eq. (14.4) partially w.r.t. r,

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta})\cdot e^{i\theta}=f'(z)\cdot e^{i\theta}$$

$$f'(z)=\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \qquad \text{...(14.5)}$$

Differentiating Eq. (14.4) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\cdot ire^{i\theta}=f'(z)\,ire^{i\theta}$$

$$f'(z)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right) \qquad \text{...(14.6)}$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.
Equating Eqs (14.5) and (14.6),

$$\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=-\frac{i}{r}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

EXAMPLE 14.9 *Find p such that the function $f(z)=r^2\cos 2\theta+ir^2\sin p\theta$ is analytic.*

Solution:

$$f(z)=r^2\cos 2\theta+ir^2\sin p\theta$$

$$u+iv=r^2\cos 2\theta+ir^2\sin p\theta$$

Comparing real and imaginary parts,

$$u=r^2\cos 2\theta, \qquad v=r^2\sin p\theta$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

From the second equation,

$$r^2(-2\sin 2\theta)=-r(2r\sin p\theta)$$

$$\sin 2\theta=\sin p\theta$$

On comparing,

$$p=2$$

EXAMPLE 14.10 *Show that $f(z)=\log z$ is analytic everywhere except at the origin and find its derivative.*

Solution: Let

$$z = re^{i\theta}$$
$$f(z) = \log z$$
$$u+iv = \log\left(re^{i\theta}\right) = \log r + \log e^{i\theta} = \log r + i\theta$$

Comparing real and imaginary parts,

$$u = \log r, \qquad v = \theta$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{\partial v}{\partial r} = 0$$
$$\frac{\partial u}{\partial \theta} = 0, \qquad \frac{\partial v}{\partial \theta} = 1$$
$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

C–R equations are satisfied.

But $\frac{\partial u}{\partial r} = \frac{1}{r}$ is not continuous at $r=0$ (i.e., $z=0$).

Hence, $\log z$ is analytic everywhere except at the origin ($z=0$).

Now, $f(z) = u + iv$

Differentiating w.r.t. r,

$$f'(z)\cdot\frac{\partial z}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$
$$f'(z)\cdot e^{i\theta} = \frac{1}{r} + i(0)$$
$$f'(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

EXERCISE 14.1

1. Show that an analytic function with a constant real part is constant.
2. Determine which of the following functions are analytic.
   (i) $xy+iy$ (ii) $e^x(\cos y - i\sin y)$
   (iii) z^2+z (iv) z^3
   (v) $z+2\bar{z}$

   [**Ans.:** (i) not analytic (ii) not analytic (iii) analytic (iv) analytic (v) not analytic]
3. Find the constants a, b, c, d and e if the function $f(z) = \left(ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2\right) + i\left(4x^3y - exy^3 + 4xy\right)$ is analytic.

   [**Ans.:** $a=1, b=-6, c=1, d=2, e=4$]

4. Determine p such that the function

$$f(z)=\frac{1}{2}\log\left(x^2+y^2\right)+\tan^{-1}\left(\frac{px}{y}\right)$$

is analytic. [**Ans.:** $p=-1$]

5. For what values of z does the function case to be analytic?

(i) $\frac{z^2-4}{z^2+1}$ (ii) $\frac{z}{z^2-1}$ (iii) $\tan^2 z$

$$\left[\textbf{Ans.:} \text{(i) } z=\pm i \text{ (ii) } z=\pm 1 \quad \text{(iii) } z=\frac{(2n-1)\pi}{2}, n=1,2,\ldots\ldots\right]$$

6. Show that $f(z)=\frac{xy^2(x+iy)}{x^2+y^4}, \quad z\neq 0$

$= 0, \qquad z=0$

is not analytic at the origin although C-R equations are satisfied at the origin.

14.4 HARMONIC FUNCTIONS

A real function ϕ of two variables x and y is said to be a harmonic function in a region R if it has continuous second-order partial derivatives and satisfies Laplace equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$$\nabla^2\phi=0, \text{where } \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

14.5 PROPERTIES OF ANALYTIC FUNCTIONS

(1) If $f(z)=u+iv$ is an analytic function then u and v are harmonic functions.

Proof $f(z)=u+iv$ is an analytic function.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \ldots(14.7)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad \ldots(14.8)$$

Differentiating Eq. (14.7) w.r.t. x and Eq. (14.8) w.r.t. y,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\,\partial y} \qquad \ldots(14.9)$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\,\partial x} \qquad \ldots(14.10)$$

Adding Eqs (14.9) and (14.10),

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial y\,\partial x}$$

$$=0 \qquad \left[\because \frac{\partial^2 v}{\partial x\,\partial y}=\frac{\partial^2 v}{\partial y\,\partial x}\right]$$

Hence, u is a harmonic function.

Differentiating Eq. (14.7) w.r.t. y and Eq. (14.8) w.r.t. x,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \ldots(14.11)$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \ldots(14.12)$$

Subtracting Eq. (14.12) from Eq. (14.11),

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

Hence, v is a harmonic function.

Note u and v of an analytic function $u + iv$ are called *conjugate harmonic functions* of each other.

(2) If $f(z) = u + iv$ is an analytic function then the family of curves $u(x, y) = c_1$ and the family of curves $v(x, y) = c_2$ cut orthogonally.

Proof $f(z) = u + iv$ is an analytic function (Fig 14.2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots(14.13)$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots(14.14)$$

$$u(x, y) = c_1$$

$$\frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1, \quad \text{say}$$

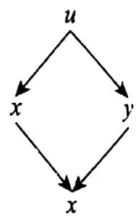

Fig. 14.2

which represents the slope of the family of curves $u(x, y) = c_1$

$$v(x, y) = c_2$$

$$\frac{\partial v}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}=m_2, \quad \text{say}$$

which represents the slope of the family of curves $v(x,y)=c_2$

$$m_1m_2=\left[-\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}\right]$$

$$=\left[\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[\frac{-\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}\right] \qquad \text{[Using equations (14.13) and (14.14)]}$$

$$=-1$$

Hence, both the families of curves cut orthogonally.

Note In polar form, the condition of orthogonality for the families of curves $u(r,\theta)=c_1$ and $v(r,\theta)=c_2$ is

$$\left(r\frac{d\theta}{dr}\right)_{u=c_1}\cdot\left(r\frac{d\theta}{dr}\right)_{v=c_2}=-1$$

EXAMPLE 14.11 *Prove that* $u=x^2-y^2$ and $v=-\dfrac{y}{x^2+y^2}$ *are harmonic but* $u+iv$ *is not regular.*

Solution:

$$u=x^2-y^2, \qquad v=-\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=2x, \qquad \frac{\partial v}{\partial x}=\frac{y}{\left(x^2+y^2\right)^2}(2x)=\frac{2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2}=2, \qquad \frac{\partial^2 v}{\partial x^2}=2y\left[\frac{\left(x^2+y^2\right)^2-x\cdot 2\left(x^2+y^2\right)\cdot 2x}{\left(x^2+y^2\right)^4}\right]=\frac{2y\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial u}{\partial y}=-2y \qquad \frac{\partial v}{\partial y}=-\left[\frac{\left(x^2+y^2\right)-y(2y)}{\left(x^2+y^2\right)^2}\right]=-\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial^2 v}{\partial y^2} = \frac{2y\left(x^2+y^2\right)^2 - \left(y^2-x^2\right)\cdot 2\left(x^2+y^2\right)\cdot 2y}{\left(x^2+y^2\right)^4}$$

$$= \frac{2y\left(x^2+y^2-2y^2+2x^2\right)}{\left(x^2+y^2\right)^3} = \frac{2y\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2=0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} = 0$$

Hence, u and v are harmonic.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.
Hence, $u+iv$ is not regular.

EXAMPLE 14.12 *If $u(x,y)$ and $v(x,y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ is an analytic function of $z=x+iy$.*

Solution: Let $U+iV = \left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

Comparing real and imaginary parts,

$$U = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \qquad V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}, \qquad \frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x}, \qquad \frac{\partial V}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}, \qquad = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}$$

Since u and v are harmonic functions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

$$\therefore \frac{\partial U}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y}$$

and

$$\frac{\partial V}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Hence, $U + iV$, i.e., the given function, is analytic.

EXAMPLE 14.13 *Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = e^{z^2}$.*

Solution:

$$w = e^{z^2}$$

$$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2} e^{i2xy} = e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$$

Comparing real and imaginary parts,

$$u = e^{x^2-y^2}\cos 2xy, \quad v = e^{x^2-y^2}\sin 2xy$$

Slope of families of curves $u = c_1$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = -\frac{e^{x^2-y^2}\cdot 2x\cos 2xy + e^{x^2-y^2}(-2y\sin 2xy)}{e^{x^2-y^2}(-2y)\cos 2xy + e^{x^2-y^2}(-2x\sin 2xy)} = \frac{x\cos 2xy - y\sin 2xy}{y\cos 2xy + x\sin 2xy} = m_1, \text{ say}$$

Slope of families of curves $v = c_2$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = -\frac{e^{x^2-y^2}\cdot 2x\sin 2xy + e^{x^2-y^2}(2y\cos 2xy)}{e^{x^2-y^2}(-2y)\sin 2xy + e^{x^2-y^2}(2x\cos 2xy)} = -\frac{x\sin 2xy + y\cos 2xy}{-y\sin 2xy + x\cos 2xy} = m_2, \text{say}$$

$$m_1 m_2 = \left(\frac{x\cos 2xy - y\sin 2xy}{x\sin 2xy + y\cos 2xy}\right)\left(-\frac{x\sin 2xy + y\cos 2xy}{x\cos 2xy - y\sin 2xy}\right) = -1$$

Hence, the families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

EXAMPLE 14.14 *Show that the families of the curves $r^n = a\sec n\theta$ and $r^n = b\,\text{cosec}\, n\theta$ cut orthogonally.*

Solution: Let

$$u = r^n - a\sec n\theta = 0 \qquad ...(1)$$

and

$$v = r^n - b\,\text{cosec}\, n\theta = 0 \qquad ...(2)$$

Differentiating Eq. (1) w.r.t. r,

$$nr^{n-1} - a(\sec n\theta \tan n\theta)\cdot n\frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{r^{n-1}}{a\sec n\theta \tan n\theta}$$

$$r\frac{d\theta}{dr} = \frac{r^{n}}{a\sec n\theta \tan n\theta} = \frac{a\sec n\theta}{a\sec n\theta \tan n\theta} = \cot n\theta = m_1, \text{ say}$$

Differentiating Eq. (2) w.r.t. r,

$$nr^{n-1} - b(-\operatorname{cosec} n\theta \cot n\theta)n\frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{-r^{n-1}}{b\operatorname{cosec} n\theta \cot n\theta}$$

$$r\frac{d\theta}{dr} = -\frac{r^{n}}{b\operatorname{cosec} n\theta \cot n\theta} = -\frac{r^{n}}{r^{n}\cot n\theta} = -\frac{1}{\cot n\theta} = m_2, \text{say}$$

$$m_1 m_2 = \cot n\theta\left(-\frac{1}{\cot n\theta}\right) = -1$$

Hence, the families of the given curves cut orthogonally.

EXERCISE 14.2

1. Prove that $\tan^{-1}\frac{y}{x}$ is harmonic.
2. Prove that $\log\sqrt{x^2+y^2}$ is harmonic.
3. If $f(z)$ is analytic, prove that

(i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2$

(ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Im } f(z)|^2 = 2|f'(z)|^2$

(iii) $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

4. Prove that $\log\left\{(x-1)^2+(y-2)^2\right\}$ is harmonic in every region which does not include the point (1, 2).

14.6 CONSTRUCTION OF ANALYTIC FUNCTIONS: MILNE-THOMSON METHOD

Case I If the real part u of $f(z)$ is given

Step 1: Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Step 2 : Find $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ [Using C–R equations]

Step 3 : Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4 : Integrate $f'(z)$ to obtain $f(z)$.

Case II If imaginary part v of $f(z)$ is given

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Step 2: Find $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ [Using C–R equations]

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z)$ to obtain $f(z)$.

HISTORICAL DATA

Louis Melville Milne-Thomson, CBE (1891–1974) was an English applied mathematician who wrote several classic textbooks on applied mathematics, including *The Calculus of Finite Differences.* In the mid 1930s, Milne-Thomson developed an interest in hydrodynamics and later in aerodynamics. This led to publication of two popular textbooks: *Theoretical Hydrodynamics and Theoretical Aerodynamics.* He is also known for developing several mathematical tables such as Jacobian Elliptic Function Tables. The Milne-Thomson circle theorem is named after him. Milne-Thomson was made a Commander of the Order of the British Empire (CBE) in 1952.

EXAMPLE 14.15 *Find the analytic function $w = u + iv$ whose imaginary part is given by $v = e^x(x\sin y + y\cos y)$.*

Solution:

$$v = e^x(x\sin y + y\cos y)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y)$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$

$$= e^x(x\cos y + \cos y - y\sin y) + ie^x(x\sin y + y\cos y + \sin y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = e^z(z\cos 0 + \cos 0 - 0) + ie^z(z\sin 0 + 0 + \sin 0) = e^z(z+1)$$

Integrating w.r.t. z,

$$f(z) = \int (z+1)e^z\,dz = (z+1)e^z - (1)e^z + c = ze^z + c$$

EXAMPLE 14.16 *Find the analytic function $f(z) = u + iv$, whose real part u is $\dfrac{x}{x^2+y^2}$.*

Solution:

$$u = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = -\frac{x}{(x^2+y^2)^2}\cdot 2y$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \qquad\qquad = -\frac{2xy}{(x^2+y^2)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} - i\left[-\frac{2xy}{(x^2+y^2)^2}\right]$$

Putting $x = z, y = 0$,

$$f'(z) = \frac{-z^2}{z^4} + i(0) = -\frac{1}{z^2}$$

Integrating w.r.t. z,

$$f(z) = \int -\frac{1}{z^2}\,dz = -\left(\frac{z^{-1}}{-1}\right) + c = \frac{1}{z} + c$$

EXAMPLE 14.17 *Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Find also its conjugate.*

Solution:

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x, \qquad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{(x^2+y^2)}\cdot 2y$$

$$= \frac{x}{x^2+y^2} \qquad\qquad = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2},\qquad \frac{\partial^2 u}{\partial y^2}=\frac{1(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}\qquad\qquad =\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0$$

Hence, u is a harmonic function.

Now,
$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad [\text{Using C–R equations}]$$

$$=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$

Putting $x=z$, $y=0$,

$$f'(z)=\frac{z}{z^2}-i(0)=\frac{1}{z}$$

Integrating w.r.t. z,

$$f(z)=\int\frac{1}{z}dz=\log z+\alpha+i\beta$$

$$u+iv=\log(x+iy)+\alpha+i\beta=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}+\alpha+i\beta$$

Comparing imaginary parts,

$$v=\tan^{-1}\frac{y}{x}+\beta$$

EXAMPLE 14.18 *Find the analytic function* $f(z)=u+iv$ *if* $u+v=\dfrac{x}{x^2+y^2}$ *and* $f(1)=1$.

Solution:

$$f(z)=u+iv \qquad ...(1)$$

$$i\,f(z)=iu+i^2v=iu-v \qquad ...(2)$$

Adding Eqs (1) and (2),

$$f(z)+i\,f(z)=(u+iv)+(iu-v)$$

$$(1+i)f(z)=(u-v)+i(u+v)$$

$$F(z)=U+iV$$

where $F(z)=(1+i)f(z)$, $U=u-v$, $V=u+v$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$V=u+v=\frac{x}{x^2+y^2}$$

$$\frac{\partial V}{\partial x}=\frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y}=-\frac{x}{(x^2+y^2)^2}\cdot 2y=\frac{-2xy}{(x^2+y^2)^2}$$

$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x}$$

$$=\frac{-2xy}{(x^2+y^2)^2}+i\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{-2xy+i(y^2-x^2)}{(x^2+y^2)^2}$$

Putting $x=z,\ y=0$,

$$F'(z)=-\frac{iz^2}{z^4}=-\frac{i}{z^2}$$

Integrating w.r.t. z,

$$F(z)=\int-\frac{i}{z^2}\mathrm{d}z=\frac{i}{z}+c$$

$$(1+i)f(z)=\frac{i}{z}+c$$

Putting $z=1$,

$$(1+i)(1)=i+c \qquad [\because f(1)=1]$$

$$c=1$$

$$\therefore\ (1+i)f(z)=\frac{i}{z}+1$$

$$f(z)=\frac{i}{z(1+i)}+\frac{1}{1+i}=\frac{1+i}{2z}+\frac{1-i}{2}$$

EXAMPLE 14.19 *Show that $2x(1-y)$ can be the imaginary part of an analytic function.*

Solution: Since real and imaginary parts of an analytic function are harmonic functions, they satisfy the Laplace equation.

Let $v=2x(1-y)$.

$$\frac{\partial v}{\partial x}=2(1-y),\qquad \frac{\partial v}{\partial y}=-2x$$

$$\frac{\partial^2 v}{\partial x^2}=0,\qquad \frac{\partial^2 v}{\partial y^2}=0$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Hence, v can be the imaginary part of an analytic function.

EXAMPLE 14.20 *Find the orthogonal trajectory of the family of the curves $3x^2y - y^3 = a$.*

Solution: Let $u = 3x^2y - y^3$ and $f(z) = u + iv$ is an analytic function.

Then $v = b$ will be the orthogonal trajectory to $u = a$.

$$u = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= 6xy - i(3x^2 - 3y^2)$$

Putting $x = z, y = 0$,

$$f'(z) = -i\left(3z^2\right)$$

Integrating w.r.t. z,

$$f(z) = -i\int 3z^2\, dz = -iz^3 + \alpha + i\beta$$

$$u + iv = -i(x+iy)^3 + \alpha + i\beta = -i\left(x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2\right) + \alpha + i\beta$$

$$= -ix^3 - i^4y^3 - 3x^2i^2y - 3i^3xy^2 + \alpha + i\beta$$

$$= \left(-y^3 + 3x^2y\right) + i\left(-x^3 + 3xy^2\right) + \alpha + i\beta$$

Comparing imaginary part,

$$v = 3xy^2 - x^3 + \beta$$

Hence, orthogonal trajectory is

$$3xy^2 - x^3 + \beta = b$$

$$3xy^2 - x^3 = c', \text{ where } c' = b - \beta$$

EXERCISE 14.3

1. Prove that the function $v = e^{-x}(x\cos y + y\sin y)$ is harmonic and determine the corresponding analytic function $f(z) = u + iv$.
2. Construct the analytic function whose real part is

   (i) $\dfrac{y}{x^2 + y^2}$

   (ii) $e^{2x}(x\cos 2y - y\sin 2y)$

   (iii) $\cos x \cosh y$

   (iv) $e^x\left[\left(x^2 - y^2\right)\cos y - 2xy\sin y\right]$

   [**Ans.:** (i) $\log z + c$ (ii) $ze^{2z} + c$ (iii) $\cos z + c$ (iv) $x^2e^z + c$]

3. Construct the analytic function whose imaginary part is

(i) $\dfrac{(x-y)}{(x^2+y^2)}$

(ii) $-\sin x \sinh y$

(iii) $e^{-x}(x\sin y - y\cos y)$

(iv) $\dfrac{2\sin x \sin y}{\cos 2x + \cosh 2y}$

$$\left[\textbf{Ans.:} \text{(i)} \frac{1+i}{z} + c \ \text{(ii)} \cos z + c \right.$$

$$\left. \text{(iii)} \bar{z} e^{\bar{z}} + c \ \text{(iv)} \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} + c \right]$$

4. Show that the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic and find the analytic function $f(z) = u + iv$.

$$\left[\textbf{Ans.:} (1-2i)(\sin z + z^2) + c\right]$$

5. Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$.

$$\left[\textbf{Ans.:} \frac{(-1+5i)e^z}{13} + c\right]$$

6. Find the analytic function $f(z) = u + iv$, given that $2u + v = e^{2x}\{(2x+y)\cos 2y + (x-2y)\sin 2y\}$.

$$\left[\textbf{Ans.:} f(z) = ze^{2z} + c\right]$$

7. Find the analytic function $z = u + iv$ if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$.

$$\left[\textbf{Ans.:} \frac{1}{2} - \cot\frac{z}{2}\right]$$

8. Show that the function $u = e^{-2xy}(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z.

$$\left[\textbf{Ans.:} v = -e^{-2xy}\cos(x^2 - y^2) + c, f(z) = -ie^{iz^2} + c\right]$$

9. Find the orthogonal trajectories of the given family of curves:

(i) $x^3y - xy^3 = c$ (ii) $e^x - \cos y - xy = c$

$$\left[\textbf{Ans.:} \text{(i)} x^4 - 6x^2y^2 + y^4 = c \quad \text{(ii)} x^2 - y^2 + 2e^x - \sin y = c\right]$$

14.7 CONFORMAL MAPPING

Conformal mapping transforms curves from one complex plane to the other with respect to size and orientation. If the point $z\ (x, y)$ describes some curve C in the z-plane, the point $w\ (u, v)$ describes a corresponding curve C' in the w-plane. Hence, for each point (x, y) in z-plane, there corresponds a point (u, v) in the w-plane. Thus, a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the mapping $w = f(z)$. A mapping or transformation $w = f(z)$ is said to be conformal at a point z_0 if it preserves the angle between any two curves passing through z_0 in magnitude and sense.

Isogonal Transformation A transformation which preserves the angle between any two curves passing through a point only in magnitude and not in sense is known as isogonal transformation at that point.

Conformal Property of Analytic Function If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of the z-plane then the mapping $w = f(z)$ is conformal at all points of the region R.

Critical Points A point of the transformation $w = f(z)$ at which $f'(z) = 0$ is known as critical point. At this point, mapping is not conformal.

14.7.1 Some Standard Transformations

1. Translation The transformation $w = z + c$, where c is a complex constant represents translation.
Let $z = x + iy, w = u + iv, c = a + ib$

$$w = z + c$$
$$u + iv = (x + iy) + (a + ib) = (x + a) + i(y + b)$$

Comparing real and imaginary parts,

$$u = x + a, \qquad v = y + b$$

Therefore, the image of the point (x, y) in the z-plane is the point $(x + a, y + b)$ in the w-plane. Hence, $w = z + c$ is simply a translation of the axes and preserves shape and size of the region z-plane.

EXAMPLE 14.21 *Find the image of $|z| = 2$ under the mapping $w = z + 3 + 2i$.*

Solution: $w = z + 3 + 2i$

$$u + iv = x + iy + 3 + 2i = (x + 3) + i(y + 2)$$

Comparing real and imaginary parts,

$$u = x + 3, \qquad v = y + 2$$
$$x = u - 3, \qquad y = v - 2$$

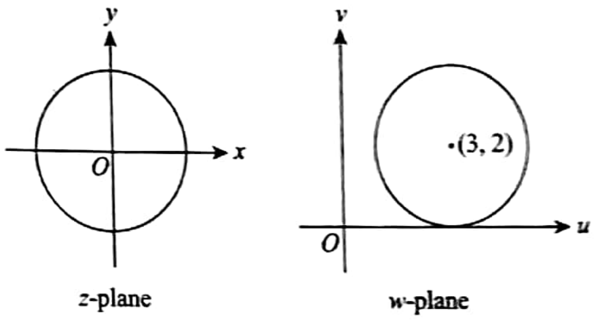

Fig. 14.3

Given $\qquad |z| = 2$

$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$
$$(u - 3)^2 + (v - 2)^2 = 4$$

Hence, the circle $|z| = 2$ in the z-plane is mapped onto a circle with centre $(3, 2)$ and radius 2 in the w-plane (Fig 14.3).

EXAMPLE 14.22 *Find the image of the triangular region whose vertices are $i, 1 + i, 1 - i$ under the transformation $w = z + 4 - 2i$. Show the region graphically.*

Solution: $w = z + 4 - 2i$

$$u + iv = x + iy + 4 - 2i = (x + 4) + i(y - 2)$$

Comparing real and imaginary parts,

$$u = x + 4, \qquad v = y - 2$$

(i) For vertex i,

$$z = i$$
$$x + iy = i$$

Comparing real and imaginary parts,

$$x = 0, y = 1$$
$$u = 4, v = -1$$

Hence, the point A (0,1) is mapped onto the point A' (4,−1).

(ii) For vertex $1 + i$,

$$z = 1 + i$$
$$x + iy = 1 + i$$

Comparing real and imaginary parts,

$$x = 1, y = 1$$
$$u = 5, v = -1$$

Hence, the point B (1,1) is mapped onto the point B'(5,−1).

(iii) For vertex $1 - i$,

$$z = 1 - i$$
$$x + iy = 1 - i$$

Comparing real and imaginary parts,

$$x = 1, \quad y = -1$$
$$u = 5, \quad v = -3$$

Hence, the point C $(1, -1)$ is mapped onto the point C' $(5, -3)$.

Hence, the image of the triangular region ABC in the z-plane is the triangular region $A'B'C'$ with vertices $(4, -1), (5, -1)$, and $(5, -3)$ in the w-plane (Fig. 14.4).

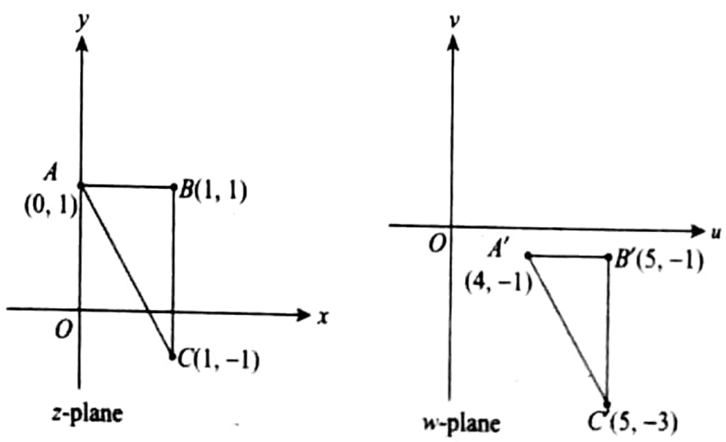

Fig. 14.4

2. Magnification and Rotation The transformation $w = cz$, where c is a complex constant represents magnification and rotation.

Let

$$w = Re^{i\phi},\ z = re^{i\theta},\ c = \rho e^{i\alpha}$$

$$w = cz$$

$$Re^{i\phi} = \left(\rho e^{i\alpha}\right)\left(re^{i\theta}\right) = \rho r e^{i(\alpha+\theta)}$$

Comparing both the sides,

$$R = \rho r,\ \phi = \theta + \alpha$$

Therefore, this transformation maps the point (r, θ) in the z-plane onto the point $(\rho r, \theta+\alpha)$ in the w-plane. The radius vector r is magnified by ρ and is rotated through an angle α. Hence, geometrically it maps any figure in the z-plane into a similar figure in the w-plane. In particular, this transformation maps circles into circles.

EXAMPLE 14.23 *Determine the region in w-plane into which the region bounded by $x = 0, y = 0, x = 1, y = 2$ in z-plane is mapped under the transformation $w = (1+i)z + (2-i)$.*

Solution:

$$w = (1+i)z + (2-i)$$

$$u + iv = (1+i)(x+iy) + (2-i) = x + iy + ix - y + 2 - i = (x - y + 2) + i(x + y - 1)$$

Comparing real and imaginary parts,

$$u = x - y + 2, \qquad v = x + y - 1$$

$$x - y = u - 2 \qquad \text{...(1)}$$

$$x + y = v + 1 \qquad \text{...(2)}$$

Adding Eqs (1) and (2),

$$2x = u + v - 1$$

$$x = \frac{1}{2}(u + v - 1)$$

Subtracting Eq. (1) from Eq. (2),

$$2y = (v+1) - (u-2)$$

$$y = \frac{1}{2}(v - u + 3)$$

Let $ABCD$ be the given region in the z-plane.

(i) When $x = 0,\ u + v = 1$

The line $x = 0$ is mapped onto the line $u + v = 1$.

(ii) When $y = 0,\ u - v = 3$

The line $y = 0$ is mapped onto the line $u - v = 3$.

(iii) When $x = 1$, $u + v = 3$

The line $x = 1$ is mapped onto the line $u + v = 3$.

(iv) When $y = 2$, $u - v = -1$

The line $y = 2$ is mapped onto the line $u - v = -1$.

Hence, the region $ABCD$ in the z-plane mapped onto the region $A'B'C'D'$ in the w-plane (Fig 14.5).

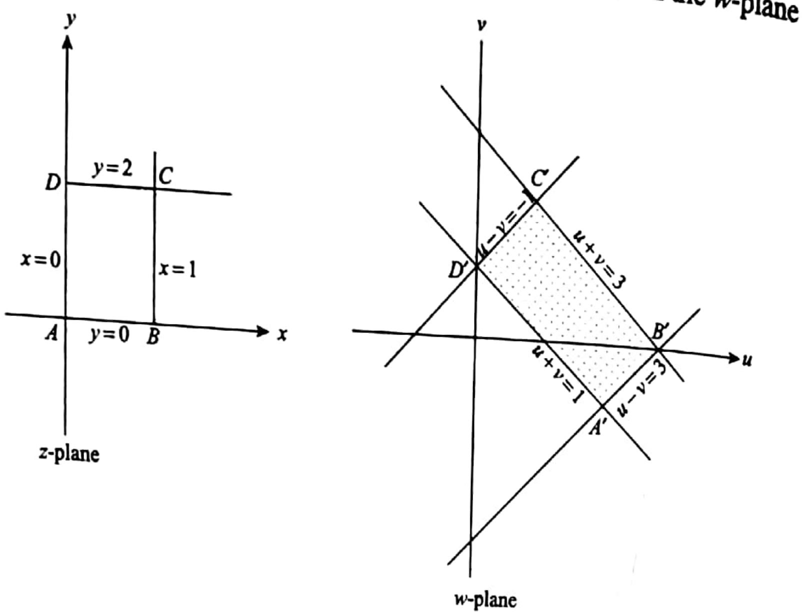

Fig. 14.5

3. Inversion The transformation $w = \frac{1}{z}$ represents inversion and reflection.

This transformation is conformal for all $z \neq 0$.

Let $w = Re^{i\phi}, z = re^{i\theta}$

where

$$R = |w|, \phi = \arg(w), r = |z|, \theta = \arg(z)$$

$$w = \frac{1}{z}$$

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

Comparing both the sides,

$$R = \frac{1}{r}, \qquad \phi = -\theta$$

$$|w| = \frac{1}{|z|}, \qquad \arg(w) = -\arg(z)$$

Hence, this transformation maps the point (r, θ) in the z-plane onto the point $\left(\frac{1}{r}, -\theta\right)$ in the w-plane.

Thus, this transformation represents inversion with respect to the unit circle $|z| = 1$ followed by the reflection in the real axis.

If $|z|<1, \quad |w|>1$

and $|z|>1, \quad |w|<1$

Hence, this transformation maps interior of the circle $|z|=1$ onto the exterior of the circle $|w|=1$ and exterior of the circle $|z|=1$ onto the interior of the circle $|w|=1$. Thus, the unit circle $|z|=1$ in the z-plane is mapped onto the unit circle $|w|=1$ in the w-plane.

EXAMPLE 14.24 *Find the image of the line $x-y=1$ under the transformation $w=\frac{1}{z}$.*

Solution:

$$w=\frac{1}{z}$$

$$z=\frac{1}{w}=\frac{1}{u+iv}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x=\frac{u}{u^2+v^2}, \quad y=-\frac{v}{u^2+v^2}$$

Given $x-y=1$

Substituting x and y,

$$\frac{u}{u^2+v^2}-\left(-\frac{v}{u^2+v^2}\right)=1$$

$$u+v=u^2+v^2$$

$$u^2+v^2-u-v=0$$

which represents a circle with centre at $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius of $\frac{1}{\sqrt{2}}$.

Hence, the image of the line $x-y=1$ in the z-plane is a circle $u^2+v^2-u-v=0$ in the w-plane (Fig. 14.6).

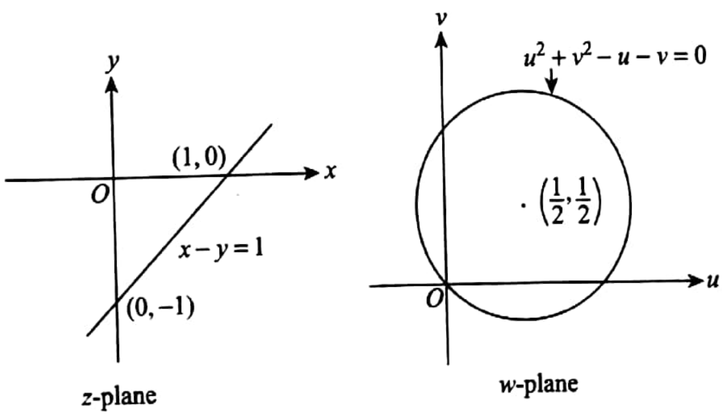

Fig. 14.6

EXAMPLE 14.25

Find the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.
Also, show the regions graphically.

Solution:

$$w = \frac{1}{z}$$

$$z = \frac{1}{w} = \frac{1}{u+iv}$$

$$x + iy = \frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \qquad y = -\frac{v}{u^2+v^2}$$

Given

$$\frac{1}{4} \le y \le \frac{1}{2}$$

$$\frac{1}{4} \le -\frac{v}{u^2+v^2} \le \frac{1}{2}$$

$$u^2+v^2 \le -4v \le 2\left(u^2+v^2\right) \qquad \left[\text{Multiplying by } 4\left(u^2+v^2\right)\right]$$

(i) $u^2+v^2 < -4v$

$u^2+v^2+4v < 0$

which represents interior of a circle with centre at $(0,-2)$ and radius of 2.

(ii) $-4v < 2\left(u^2+v^2\right)$

$0 < u^2+v^2+2v$

$u^2+v^2+2v > 0$

which represents exterior of a circle with centre at $(0,-1)$ and radius of 1.

Hence, the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ in the z-plane is the region between the circles $u^2+v^2+2v=0$ and $u^2+v^2+4v=0$ in the w-plane (Fig. 14.7).

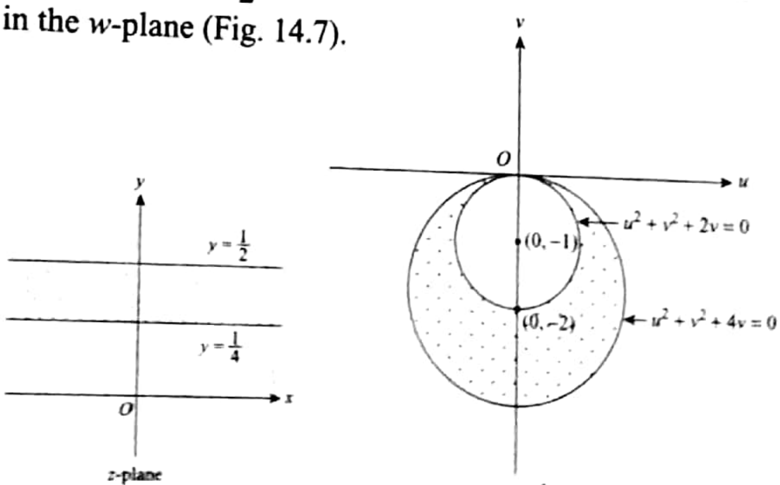

Fig. 14.7

14.7.2 Some Special Transformations

1. $w = z^2$

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

Comparing real and imaginary parts,

$$u = x^2 - y^2, \qquad v = 2xy$$

(i) Let $x = \text{constant} = a$, say

$$u = a^2 - y^2, \qquad v = 2ay$$

$$y = \frac{v}{2a}$$

Substituting y in u,

$$u = a^2 - \frac{v^2}{4a^2}$$

$$\frac{v^2}{4a^2} = a^2 - u$$

$$v^2 = -4a^2\left(u - a^2\right)$$

which represents a parabola with vertex $\left(a^2, 0\right)$.

Hence, the lines parallel to the y-axis in the z-plane are mapped into the parabolas in the w-plane.

(ii) Let $y = \text{constant} = b$, say

$$u = x^2 - b^2, \qquad v = 2xb$$

$$x = \frac{v}{2b}$$

Substituting x in u,

$$u = \frac{v^2}{4b^2} - b^2$$

$$v^2 = 4b^2\left(u + b^2\right)$$

which represents a parabola with vertex $\left(-b^2, 0\right)$.

Hence, the lines parallel to the x-axis in the z-plane are mapped into the parabolas in the w-plane.

EXAMPLE 14.26 *Find the image of the triangular region bounded by the lines $x = 1, y = 1, x + y = 1$ under the transformation $w = z^2$.*

Solution:

$$w = z^2$$

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

Comparing real and imaginary parts,

$$u = x^2 - y^2, \qquad v = 2xy$$

(i) When $x = 1$,

$$u = 1 - y^2, \quad v = 2y$$

$$y = \frac{v}{2}$$

Substituting y in u,

$$u = 1 - \frac{v^2}{4}$$

$$4u = 4 - v^2$$

$$v^2 = -4(u-1)$$

which represents a parabola with vertex (1, 0) and opening to the left of the vertex. The image of the line $x = 1$ is the parabola $v^2 = -4(u-1)$.

(ii) When $y = 1$,

$$u = x^2 - 1, \qquad v = 2x$$

$$x = \frac{v}{2}$$

Substituting x in u,

$$u = \frac{v^2}{4} - 1$$

$$4u = v^2 - 4$$

$$v^2 = 4(u+1)$$

which represents a parabola with vertex (–1,0) and opening to the right of the vertex. The image of the line $y = 1$ is the parabola $v^2 = 4(u+1)$.

(iii) When $x + y = 1$

$$(x+y)^2 = 1$$

$$x^2 + y^2 + 2xy = 1$$

$$\left(x^2 + y^2\right)^2 = (1 - 2xy)^2$$

$$\left(x^2 - y^2\right)^2 + 4x^2y^2 = 1 + 4x^2y^2 - 4xy$$

$$\left(x^2 - y^2\right)^2 = 1 - 4xy$$

Substituting $u = \left(x^2 - y^2\right)$ and $v = 2xy$,

$$u^2 = 1 - 2v = -2\left(v - \frac{1}{2}\right)$$

which represents a parabola with vertex $\left(0, \frac{1}{2}\right)$ opening below the vertex.

The image of the line $x + y = 1$ is the parabola $u^2 = 1 - 2v$.
Hence, the image of the triangular region ABC in the z-plane is the curvilinear triangular region $A'B'C'$ bounded by the parabolas in the w-plane (Fig. 14.8).

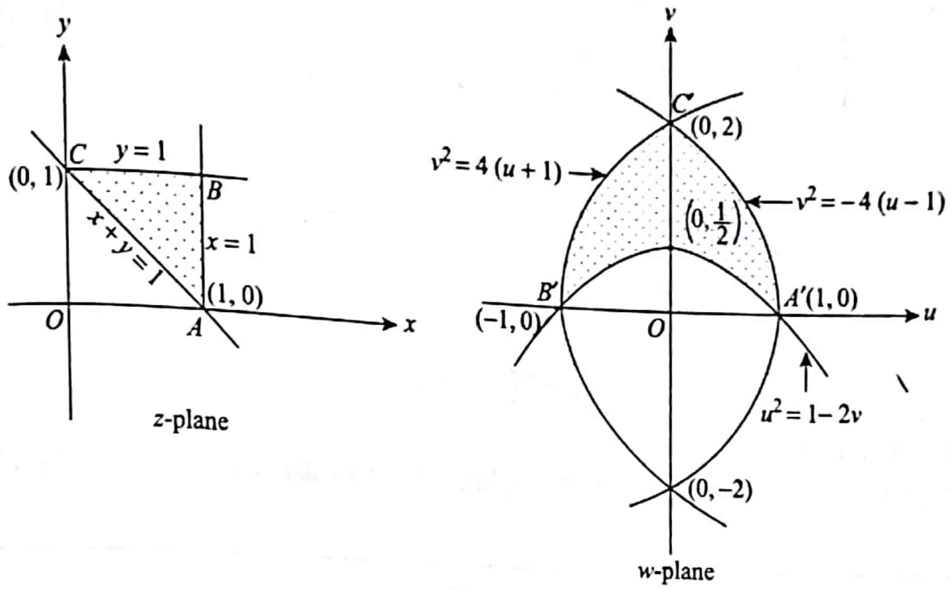

Fig. 14.8

EXAMPLE 14.27 *Find the image of $|z-1|=1$ under the transformation $w = z^2$.*

Solution:

$$w = z^2$$

$$Re^{i\phi} = \left(re^{i\theta}\right)^2 \qquad \text{[Considering polar form]}$$

$$= r^2 e^{2i\theta}$$

$$\therefore \quad R = r^2, \ \phi = 2\theta$$

Given

$$|z-1| = 1$$

$$\left|re^{i\theta} - 1\right| = 1$$

$$\left|r(\cos\theta + i\sin\theta) - 1\right| = 1$$

$$\left|(r\cos\theta - 1) + ir\sin\theta\right|^2 = 1$$

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

$$r^2 = 4\cos^2\theta$$

$$= 2(1 + \cos 2\theta) \qquad [\because \cos 2\theta = 2\cos^2\theta - 1]$$

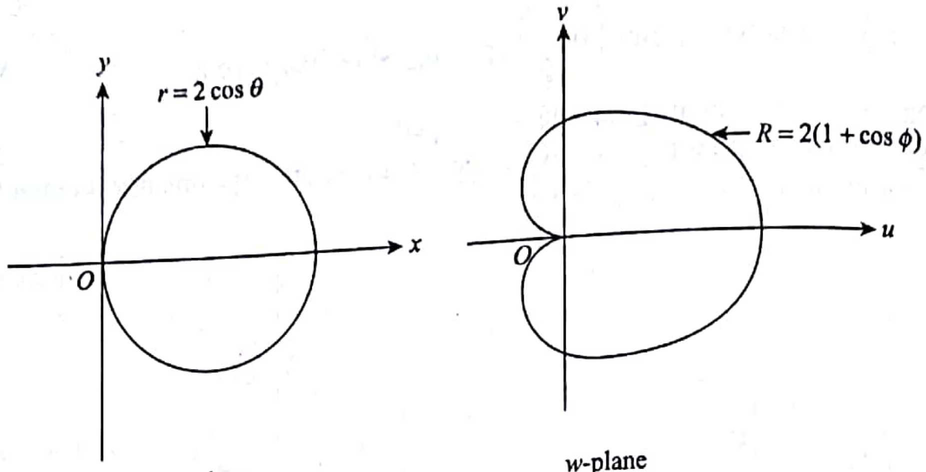

Fig. 14.9

Substituting r^2 and 2θ,

$$R = 2(1+\cos\phi)$$

which represents a cardioid.
Hence, the image of the circle $|z-1| = 1$ in the z-plane is the cardioid $R = 2(1+\cos\phi)$ in the w-plane (Fig. 14.9).

---

2. $w = e^z$

$$u + iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

(i) Let x = constant = a, say

Substituting $x = a$ in u and v,

$$u = e^a \cos y,\ v = e^a \sin y$$

$$\therefore u^2 + v^2 = e^{2a}$$

which represents a circle with centre (0, 0) and radius e^a.

For $a > 0$, $e^a > 1$, the radius of the circle is greater than 1.

For $a < 0$, $e^a < 1$, the radius of the circle is less than 1.

For $a = 0$, $e^a = 1$, the radius of the circle is 1.

Hence, the lines parallel to y-axis in the z-plane are mapped onto the circles $|w| > 1$ or $|w| < 1$ or $|w| = 1$ in the w-plane according as $a > 0$ or $a < 0$ or $a = 0$ respectively.

(ii) Let y = constant = b, say

Substituting $y = b$ in u and v,

$$u = e^x \cos b,\ v = e^x \sin b$$

$$\frac{v}{u} = \frac{e^x \sin b}{e^x \cos b} = \tan b$$

$$v = u \tan b$$

Hence, the lines parallel to the x-axis in the z-plane are mapped onto radial lines in the w-plane.

EXAMPLE 14.28 *Prove that the transformation $w = e^z$ transforms the region between the real axis and a line parallel to the real axis $y = \pi$ into the upper half of the w-plane.*

Solution:

$$w = e^z$$

$$u + iv = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

The given region lies between the real axis i.e. $y = 0$ and the line $y = \pi$.
In the region $0 < y < \pi$,

(i) $\sin y > 0$

$e^x \sin y > 0$ $\quad [\because e^x \text{ is always positive}]$

$v > 0$

(ii) $\cos y > 0,\ 0 < y < \frac{\pi}{2}$

and $\cos y < 0,\ \frac{\pi}{2} < y < \pi$

$\therefore$ u may be negative or positive.

Hence, the region between the real axis and the line $y = \pi$ in the z-plane is transformed into the upper half of the w-plane (i.e., $v > 0$).

EXAMPLE 14.29 *Prove that the image of the straight line $y = mx$ is an equiangular spiral under the transformation $w = e^z$.*

Solution:

$$w = e^z$$

$$\text{Re}^{i\phi} = e^{x+iy} \qquad [\text{Considering polar form of } w]$$

$$= e^x e^{iy}$$

$$R = e^x, \phi = y$$
$$x = \log R$$

Given
$$y = mx.$$

Substituting x and y
$$\phi = m \log R$$
$$\log R = \frac{\phi}{m}$$
$$R = e^{\frac{\phi}{m}}$$

which represents an equiangular spiral.

Hence, the image of the straight line $y = mx$ in the z-plane is an equiangular spiral $R = e^{\frac{\phi}{m}}$ in the w-plane.

EXERCISE 14.4

1. Find the image of the circle $|z| = 1$ under the transformation $w = z + 2 + 4i$.

   $\left[\textbf{Ans.:} (u-2)^2 + (v-4)^2 = 1\right]$

2. Find the image of the circle $|z| = a$ under the transformation $w = (1+i)z + 2 - i$.

   $\left[\textbf{Ans.:} (u-2)^2 + (v+1)^2 = 2a^2\right]$

3. Find the image of the region bounded by the lines $x = 0$, $y = 0$, $x = 2$, $y = 1$ under the transformation $w = z + 1 - 2i$.

   [**Ans.:** Rectangle bounded by the lines $u = 1, v = -2, u = 3, v = -1$]

4. Find the image of the line $2x + y - 3 = 0$ under the transformation $w = z + 2i$.

   $\left[\textbf{Ans.:} 2u + v - 5 = 0\right]$

5. Find the image of the circle $|z| = 2$ under the transformation $w = 3z$.

   $\left[\textbf{Ans.:} \frac{u^2}{9} + \frac{v^2}{9} = 4\right]$

6. Find the image of the triangular region bounded by the lines $x = 0, y = 0, x + y = 1$ under the transformation $w = ze^{\frac{-i\pi}{4}}$.

   [**Ans.:** Triangle bounded by the lines $v = u, v = -u, v = \frac{1}{\sqrt{2}}$]

7. Find the image of the strip $x > 0$, $0 < y < 2$ under the transformation $w = iz + 1$.

   [**Ans.:** Strip $-1 < u < 1, \quad v > 0$]

8. Find the image of the region $y > 1$ under the transformation $w = (1-i)z$.

   $\left[\textbf{Ans.:} u + v > 2\right]$

9. Find the image of the circle $|z-1| = 1$ under the transformation $w = \frac{1}{z}$.

   $\left[\textbf{Ans.:} 2u - 1 = 0\right]$

10. Find the image of $|z-2i|=3$ under the transformation $w=\frac{1}{z}$.

$$\left[\textbf{Ans.}: u^2+v^2-\frac{4}{5}v-\frac{1}{5}=0\right]$$

11. What will be the image of a circle passing through the origin in the xy-plane under the transformation $w=\frac{1}{z}$.

[**Ans.:** Straight line $2gu-2fv+1=0$ if centre of the circle is $(-g,-f)$]

12. Show that the transformation $w=\frac{1}{z}$ maps the circle $|z-3|=5$ into a circle $\left|w+\frac{3}{16}\right|=\frac{5}{16}$ in the w-plane.

14.8 BILINEAR TRANSFORMATION

The transformation $w=\frac{az+b}{cz+d}$, where a, b, c, d are complex or real constants such that $ad-bc\neq 0$ is known as bilinear transformation. Bilinear transformation is also known as Mobius transformation.

HISTORICAL DATA

August Ferdinand Möbius (1790–1868) was a German mathematician and theoretical astronomer.

He is best known for his discovery of the Möbius strip, a non-orientable two-dimensional surface with only one side when embedded in three-dimensional Euclidean space. It was independently discovered by Johann Benedict Listing around the same time. The Möbius configuration, formed by two mutually inscribed tetrahedra, is also named after him. Möbius was the first to introduce homogeneous coordinates into projective geometry.

Many mathematical concepts are named after him, including the Möbius transformations, important in projective geometry, and the Möbius transform of number theory. His interest in number theory led to the important Möbius function $\mu(n)$ and the Möbius inversion formula. In Euclidean geometry, he systematically developed the use of signed angles and line segments as a way of simplifying and unifying results.

Möbius was born in Schulpforta, Saxony-Anhalt, and was descended on his mother's side from religious reformer Martin Luther. He studied mathematics under Carl Friedrich Gauss and Johann Pfaff.

Note $\frac{dw}{dz}=\frac{a(cz+d)-c(az+b)}{(cz+d)^2}=\frac{ad-bc}{(cz+d)^2}\neq 0 \qquad [\because ad-bc\neq 0]$

Thus, bilinear transformation is conformal.

Special Cases

(i) If $c = 0, a = d \neq 0$,

$$w = z + \frac{b}{d}$$

which represents translation.

(ii) If $c = 0,\ b = 0, d \neq 0$,

$$w = \frac{a}{d} z$$

which represents rotation and magnification.

(iii) If $a = 0, d = 0, b = c \neq 0$,

$$w = \frac{1}{z}$$

which represents inversion and reflection.

14.8.1 Fixed (Invariant) Points

A fixed point is a point $z = x + iy$ which maps into itself in the w-plane $(i.e.\, w = z)$ under the bilinear transformation $w = \dfrac{az+b}{cz+d}$.

$$z = \frac{az+b}{cz+d}$$

$$cz^2 + z(d-a) - b = 0$$

This is a quadratic equation if $c \neq 0$. The roots of this equation are the fixed points of the bilinear transformation.

Notes

(i) If both the roots are equal then the bilinear transformation is said to be parabolic.

(ii) A bilinear transformation with two finite fixed points α, β can be written as

$\dfrac{w-\alpha}{w-\beta} = \lambda\left(\dfrac{z-\alpha}{z-\beta}\right)$, where λ is a complex constant.

(iii) If a bilinear transformation has only one fixed point α then it can be written as $\dfrac{1}{w-\alpha} = k + \dfrac{1}{z-\alpha}$,

where $k = \dfrac{c}{a - \alpha c} \neq 0$

This form is known as *normal form* or *canonical form* of a bilinear transformation.

14.8.2 Cross Ratio

The cross ratio of four points z_1, z_2, z_3, z_4 is defined as $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ and is denoted by (z_1, z_2, z_3, z_4).

Theorem A bilinear transformation preserves the cross ratio of four points, i.e., cross ratio of four points is invariant under a bilinear transformation.

$$\text{i.e. } \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Proof Let $w=\dfrac{az+b}{cz+d}$ be the bilinear transformation.

Let z, z_1, z_2, z_3 be four points in the z-plane which are mapped to the points w, w_1, w_2, w_3 in the w-plane respectively.

$$w_1=\frac{az_1+b}{cz_1+d}$$

$$w-w_1=\frac{az+b}{cz+d}-\frac{az_1+b}{cz_1+d}=\frac{(az+b)(cz_1+d)-(cz+d)(az_1+b)}{(cz+d)(cz_1+d)}$$

$$=\frac{aczz_1+adz+bcz_1+bd-cazz_1-cbz-daz_1-db}{(cz+d)(cz_1+d)}$$

$$=\frac{z(ad-bc)-z_1(ad-bc)}{(cz+d)(cz_1+d)}=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

Similarly,

$$w_2-w_3=\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$(w-w_1)(w_2-w_3)=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\cdot\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

Similarly,

$$(w-w_3)(w_2-w_1)=\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\cdot\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}$$

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Thus, bilinear transformation preserves the cross ratio.

EXAMPLE 14.30 *Find the invariant points of the transformation* $w=\dfrac{1+z}{1-z}$.

Solution: The invariant points are obtained by putting $w=z$.

$$z=\frac{1+z}{1-z}$$

$$z-z^2=1+z$$

$$z^2=-1$$

$$z=\pm i$$

EXAMPLE 14.31 *Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.*

Solution: Let $z_1 = 0, z_2 = -i, z_3 = -1$ and $w_1 = i, w_2 = 1, w_3 = 0$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w(1-i)} = \frac{z(-i+1)}{(z+1)(-i)}$$

$$\frac{w-i}{w-iw} = \frac{-iz+z}{-iz-i}$$

$$-iwz - iw + i^2z + i^2 = -iwz + i^2wz + wz - iwz$$

$$-iwz - iw - z - 1 = -iwz - wz + wz - iwz$$

$$-iw + iwz = z + 1$$

$$iw(-1+z) = z + 1$$

$$w = \frac{1}{i} \cdot \frac{z+1}{z-1} = -i\frac{z+1}{z-1}$$

EXAMPLE 14.32 *Find the bilinear transformation which maps the points 1, i, −1 onto the points 0, 1, ∞. Also, show that the transformation maps interior of the unit circle of the z-plane onto upper half of the w-plane.*

Solution:

(i) Let $z_1 = 1$, $z_2 = i, z_3 = -1$ and $w_1 = 0, w_2 = 1, w_3 = \infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \quad \left[\because w_3 = \infty, \frac{w_2}{w_3} = 0, \frac{w}{w_3} = 0\right]$$

$$\frac{-w}{-1} = -\frac{(z-1)(1+i)}{(z+1)(1-i)} = -\frac{(z-1)(1+i)(1+i)}{(z+1)(1-i)(1+i)}$$

$$= -\frac{(z-1)(1+i)^2}{(z+1)(2)} = -\frac{(z-1)(2i)}{(z+1)(2)} = -i\frac{z-1}{z+1}$$

(ii) $(z+1)w = -i(z-1)$

$wz + w = -iz + i$

$(w+i)z = i - w$

$$z = \frac{i-w}{w+i} = -\left(\frac{w-i}{w+i}\right)$$

Interior of the unit circle is given by

$$|z| < 1$$

$$\left|-\left(\frac{w-i}{w+i}\right)\right| < 1$$

$$\left|\frac{w-i}{w+i}\right| < 1$$

$$\frac{|u+iv-i|}{|u+iv+i|} < 1$$

$$|u+iv-i| < |u+iv+i|$$

$$|u+i(v-1)| < |u+i(v+1)|$$

$$u^2 + (v-1)^2 < u^2 + (v+1)^2$$

$$(v-1)^2 < (v+1)^2$$

$$v^2 - 2v + 1 < v^2 + 2v + 1$$

$$-4v < 0$$

$$v > 0$$

Hence, the interior of the unit circle of the z-plane is mapped onto the upper half of the w-plane (Fig. 14.10).

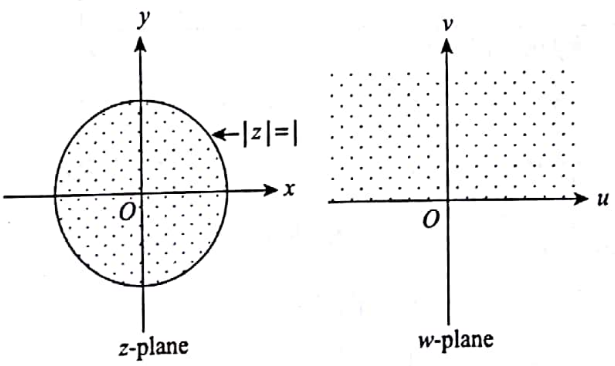

Fig. 14.10

EXAMPLE 14.33 *Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of $|z| = 1$ under this transformation?*

Solution:

$$w = \frac{z}{1-z}$$

$$w(1-z) = z$$

$$w - wz = z$$

$$w = z + wz = z(1+w)$$

$$z = \frac{w}{1+w}$$

$$x+iy = \frac{u+iv}{1+u+iv} = \frac{(u+iv)[(1+u)-iv]}{[(1+u)+iv][(1+u)-iv]}$$

$$= \frac{[u(1+u)+v^2]+i[v(1+u)-uv]}{(1+u)^2+v^2} = \frac{(u^2+v^2+u)+iv}{(1+u)^2+v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u^2+v^2+u}{(1+u)^2+v^2}, \quad y = \frac{v}{(1+u)^2+v^2}$$

(i) Upper half of the z-plane is given by

$$y > 0$$

$$\frac{v}{(1+u)^2+v^2} > 0$$

$$v > 0$$

which represents upper half of the w-plane.

Hence, the upper half of the z-plane is mapped onto the upper half of the w-plane (Fig. 14.11).

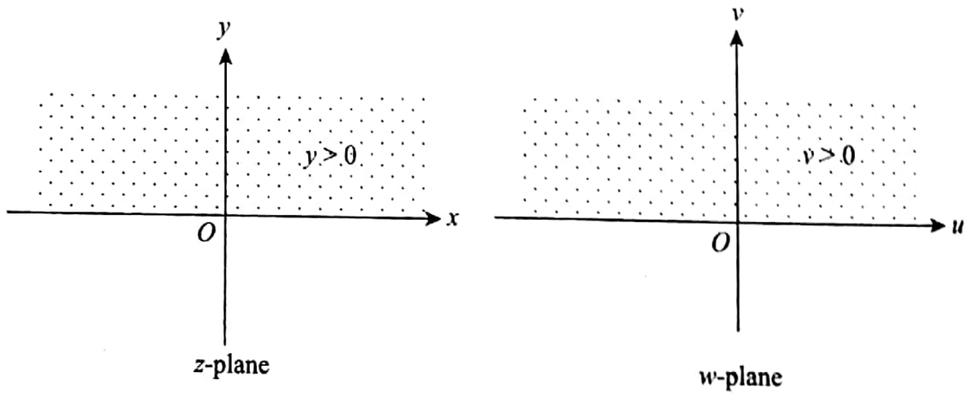

Fig. 14.11

(ii)

$$|z| = 1$$

$$\left|\frac{w}{1+w}\right| = 1$$

$$\frac{|w|}{|1+w|}=1$$

$$\frac{|u+iv|}{|1+u+iv|}=1$$

$$\frac{|u+iv|^2}{|(1+u)+iv|^2}=1$$

$$\frac{u^2+v^2}{(1+u)^2+v^2}=1$$

$$u^2+v^2=1+2u+u^2+v^2$$

$$0=1+2u$$

$$u=-\frac{1}{2}$$

which represents a straight line.

Hence, the image of $|z|=1$ in the z-plane is the line $u=-\frac{1}{2}$ in the w-plane (Fig. 14.12).

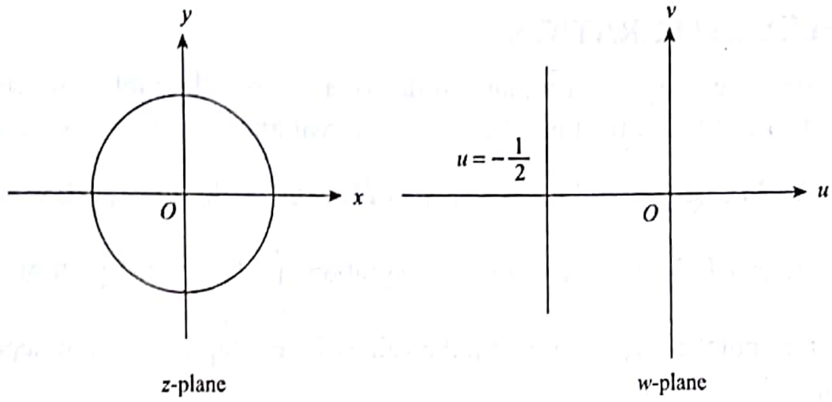

Fig. 14.12

EXERCISE 14.5

1. Find the invariant point of the transformation $w=\frac{1}{z-2i}$.

   [Ans.: i]

2. Find the invariant points of the bilinear transformation $\frac{z-1}{z+1}$.

   [Ans.: $\pm i$]

3. Find the fixed points of $w=\frac{3z-4}{z-1}$.

   [Ans.: 2, 2]

4. Find the fixed points under the transformation $w=\frac{2z-5}{z+4}$.

   [Ans.:$-1\pm 2i$]

5. Find the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in the w-plane.

   $$\left[\textbf{Ans.: } -3i\frac{(z+1)}{(z-3)}\right]$$

6. Find the bilinear transformation that maps $z = 0, 1, \infty$ onto the points $w = -5, -1, 3$ respectively. What are the fixed points?

$$\left[\textbf{Ans.:} \frac{3z-5}{z+1}; 1 \pm 2i\right]$$

7. Find the bilinear transformation mapping the points $z = 1, i, -1$ into the points $w = 2, i, -2$.

$$\left[\textbf{Ans.:} -\frac{6z-2i}{iz-3}\right]$$

8. Find the bilinear transformation which maps points $i, -1, 1$ of the z-plane into the points $0, 1, \infty$ of the w-plane respectively.

$$\left[\textbf{Ans.:} \frac{2z-2i}{(1+i)(z-1)}\right]$$

9. Find the bilinear transformation which maps the points $\infty, i, 0$ of the z-plane into the points $0, -i, \infty$ of the w-plane.

$$\left[\textbf{Ans.:} \frac{1}{z}\right]$$

10. Obtain the bilinear transformation which maps $z = 0$ onto $w = -i$ and has $-1, 1$ as fixed points. Also, show that the upper half plane is mapped onto the interior of the unit circle in the w-plane.

$$\left[\textbf{Ans.:} \frac{z-1}{1-iz}\right]$$

14.9 COMPLEX INTEGRATION

The concept of a real line integral is extended to that of a complex line integral. Complex integration plays an important role in the evaluation of complicated real integrals. It is a powerful tool in evaluating certain integrals. In case of real integration, $\int_a^b f(x)\,dx$, the path of integration is always along the x-axis from $x = a$ to $x = b$. In case of complex integration, $\int_{z_1}^{z_2} f(z)\,dz$, the path of integration can be along any curve from point $z = z_1$ to $z = z_2$ but the value of the integral does not depend upon the path if $f(z)$ is analytic.

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve C (Fig. 14.13) whose end points are A and B. Divide the curve C into n parts at the points $A = P_0(z_0), P_1(z_1), P_2(z_2) \cdots P_{i-1}(z_{i-1}), P_i(z_i), \cdots, P_n(z_n) = B$. Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^{n} f(\xi_i)\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the line integral of $f(z)$ along the path C and is denoted by $\int_C f(z)\,dz$. If C is a closed curve i.e., if P_0 and P_n coincide, the integral is called the contour integral and is denoted by $\oint_C f(z)\,dz$.

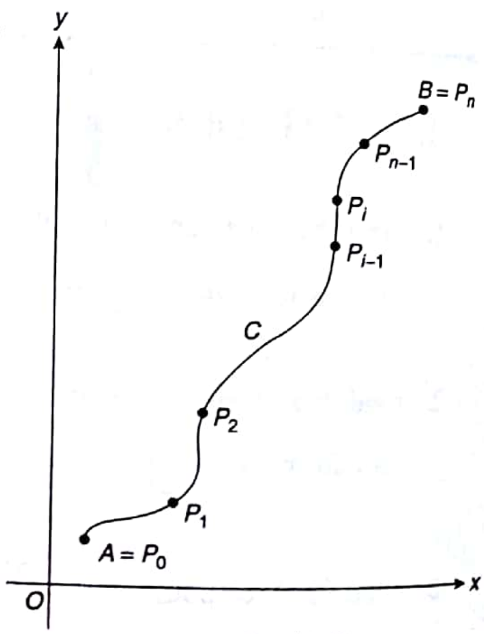

Fig. 14.13

Evaluation of Line Integrals

Let $f(z) = u + iv$

$$z = x + iy, \qquad dz = dx + idy$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

Hence, the evaluation of the line integral of a complex function can be converted to the evaluation of two line integrals of real functions.

EXAMPLE 14.34 *Evaluate* $\int_0^{1+i} (x - y + ix^2)\,dz$ *along the line from* $z = 0$ *to* $z = 1 + i$.

Solution: Let

$$z = x + iy$$
$$dz = dx + idy$$

$$\int_0^{1+i} (x - y + ix^2)\,dz = \int_0^{1+i} (x - y + ix^2)(dx + idy) = \int_{OA} (x - y + ix^2)(dx + idy)$$

Along OA, $y = x$, $dx = dy$, x varies from 0 to 1 (Fig. 14.14).

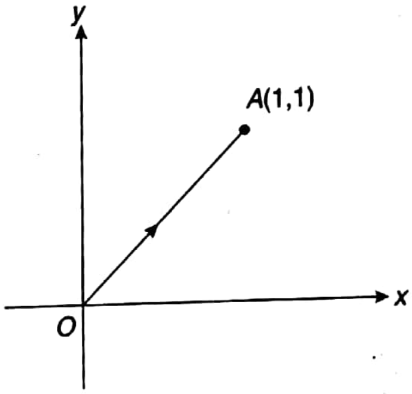

Fig. 14.14

$$\int_0^{1+i} (x - y + ix^2)\,dz = \int_0^1 ix^2(dx + idx) = \int_0^1 ix^2(1+i)\,dx$$

$$= (-1+i)\int_0^1 x^2 dx = (-1+i)\left|\frac{x^3}{3}\right|_0^1$$

$$= \frac{-1+i}{3} = -\frac{1}{3} + \frac{1}{3}i$$

EXAMPLE 14.35 *Evaluate* $\int_C |z|\,dz$, *where C is the left half of the unit circle* $|z| = 1$ *from* $z = -i$ *to* $z = i$.

Solution: Let $z = re^{i\theta} = e^{i\theta}$ $\quad [\because r = 1 \text{ for } |z| = 1]$

$$dz = ie^{i\theta}d\theta$$

In the left half of the unit circle, θ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$ (Fig. 14.15).

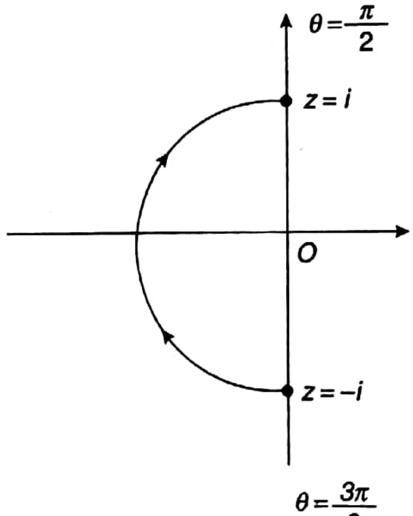

Fig. 14.15

$$\int_C |z|\,dz = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} 1.\, ie^{i\theta}d\theta = i\left|\frac{e^{i\theta}}{i}\right|_{\frac{3\pi}{2}}^{\frac{\pi}{2}} = e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} = 0 + i - 0 - i(-1) = 2i$$

EXAMPLE 14.36 *Evaluate* $\int_C (3z^2+2z+1)\,dz$, *where C is the arc of the cycloid* $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$, *between* $\theta=0$ *to* $\theta=2\pi$.

Solution: Let $f(z)=3z^2+2z+1$

Since $f(z)$ is a polynomial, it is analytic everywhere. Hence, the integral is independent of the path.

$$z=x+iy=a(\theta+\sin\theta)+ia(1-\cos\theta)$$

At $\theta=0,\quad z=0$

At $\theta=2\pi,\ z=2a\pi$

$$\int_C f(z)\,dz=\int_0^{2a\pi}(3z^2+2z+1)\,dz=\left|z^3+z^2+z\right|_0^{2a\pi}=2a\pi(4a^2\pi^2+2\pi a+1)$$

EXAMPLE 14.37 *Evaluate* $\int_0^{1+i}(x^2+iy)\,dz$ *along the path (i)* $y=x$, *(ii)* $y=x^2$. *Is the line integral independent of the path?*

Solution:

(i) Along the path $y=x$

$dy=dx$, x varies from 0 to 1 (Fig. 14.16).

$dz=dx+idy=dx+idx=(1+i)\,dx$

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^{1+i}(x^2+iy)(dx+idy)=\int_0^1(x^2+ix)(1+i)\,dx$$

$$=(1+i)\left|\frac{x^3}{3}+i\frac{x^2}{2}\right|_0^1=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)$$

$$=(1+i)\frac{(2+3i)}{6}=\frac{2+2i+3i-3}{6}=\frac{-1+5i}{6}$$

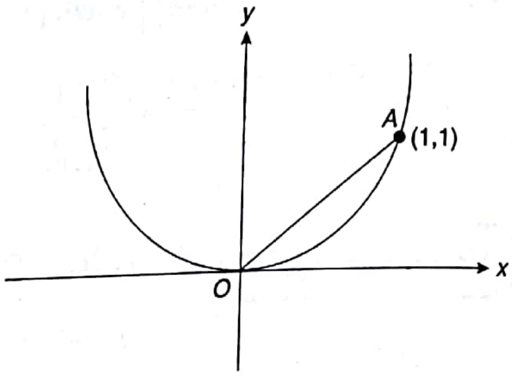

Fig. 14.16

(ii) Along the path $y=x^2, dy=2xdx$, x varies from 0 to 1.

$$dz=dx+idy=dx+i2x\,dx$$

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^1(x^2+iy)(dx+2ix\,dx)$$

$$=\int_0^1(x^2+ix^2)(1+2ix)\,dx=(1+i)\int_0^1(x^2+2ix^3)\,dx$$

$$=(1+i)\left|\frac{x^3}{3}+2i\frac{x^4}{4}\right|_0^1=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=\frac{-1+5i}{6}$$

Hence, the given line integral has same value along both the paths.

Now,
$$f(z) = x^2 + iy$$
$$u + iv = x^2 + iy$$

Comparing real and imaginary parts,

$$u = x^2, \quad v = y$$
$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 1$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, C–R equations are not satisfied

Therefore, $f(z)$ is not analytic.

Hence, the line integral is not independent of the path.

EXAMPLE 14.38 *Integrate $f(z) = x^2 + ixy$ from (1,1) to (2,4) along the curve $x = t, y = t^2$.*

Solution:
$$f(z) = x^2 + ixy$$

Along the curve
$$x = t, y = t^2$$
$$f(z) = x^2 + ixy = t^2 + i(t)(t^2) = t^2 + it^3$$
$$dz = dx + idy = dt + i2tdt = (1 + 2it)dt$$

When $x = 1, y = 1, \quad t = 1$

When $x = 2, y = 4, \quad t = 2$

Thus, t varies from 1 to 2.

$$\int_c f(z)dz = \int_1^2 (t^2 + it^3)(1 + 2it)dt = \int_1^2 (t^2 + 2it^3 + it^3 + 2i^2t^4)dt$$
$$= \int_1^2 [t^2 - 2t^4 + 3it^3]dt = \left|\frac{t^3}{3} - 2\frac{t^5}{5} + 3i\frac{t^4}{4}\right|_1^2$$
$$= \left(\frac{8}{3} - \frac{64}{5} + 3i\frac{16}{4}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{3i}{4}\right) = -\frac{151}{15} + i\frac{45}{4}$$

EXERCISE 14.6

1. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)dz$ along (i) the straight line joing $(1-i)$ to $(2+i)$ (ii) $x=t+1, y=2t^2-1$

$$\left[\textbf{Ans.:} \text{ (i) } 4(1+2i) \quad \text{(ii) } 4+\frac{25}{3}i\right]$$

2. Evaluate $\int_C \frac{2z+3}{z}dz$, where C is (i) the upper half of the circle $|z|=2$ (ii) the lower half of the circle $|z|=2$ (iii) the whole circle in anti-clockwise direction.

[**Ans.:** (i) $2(3\pi i-4)$ (ii) $2(4-3\pi i)$ (iii) 12]

3. Show that $\int_C \log z dz = 2\pi i$, where C is the unit circle in the z-plane

4. Evaluate $\int_C (z-z^2)dz$, where C is the upper half of the circle $|z|=1$

$$\left[\textbf{Ans.:} \frac{2}{3}\right]$$

5. Evaluate $\int_C |z|^2 dz$, where C is the boundary of the square C with vertices (0,0), (1,0), (1,1), (0,1)

[**Ans.:** $-1+i$]

6. Evaluate $\int_C (2z^3+8z+2)dz$, where C is the arc of the cycloid $x=a(\theta-\sin\theta)$ $y=a(1-\cos\theta)$ between the points (0,0) and $(2\pi a, 0)$

[**Ans.:** $4\pi a(2\pi^3a^3+4\pi a+1)$]

7. Evaluate $\int_C \bar{z}dz$ from $z=0$ to $z=4+2i$, where C is the curve given by $z=t^2+it$.

$$\left[\textbf{Ans.:} 10-\frac{8}{3}i\right]$$

8. Prove that $\int_{-2}^{-2+i} (2+z)^2 dz = -\frac{i}{3}$.

9. Evaluate $\int_C (z^2-2\bar{z}+1)dz$, where C is the circle $x^2+y^2=2$.

[**Ans.:** $-8\pi i$]

10. Evaluate $\int_C z^2 dz$ from $P(1,1)$ to $Q(2,4)$, where C is the curve $x=t, y=t^2$

$$\left[\textbf{Ans.:} -\frac{86}{3}-6i\right]$$

14.10 SIMPLY CONNECTED AND MULTIPLY CONNECTED REGIONS

1. Simply Connected Region A simply connected region R is the region enclosed by a simple curve, e.g., interior of a circle, rectangle, triangle, ellipse, etc. (Fig. 14.17)

2. Multiply Connected Region A multiply connected region is the region enclosed by more than one simple curve, e.g., annulus region, regions with holes, etc.
A multiply connected region can be converted into a simply connected region by introducing cross-cuts. (Fig. 14.18)

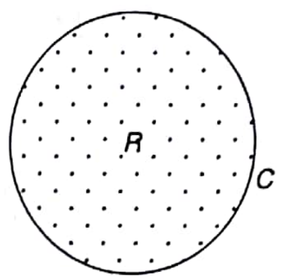

Fig. 14.17 Simply Connected Region

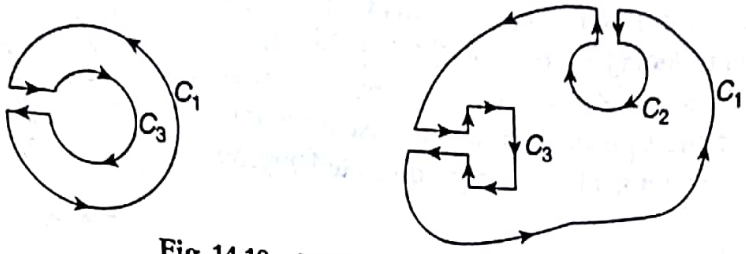

Fig. 14.18 Multiply Connected Regions

3. Independence of Path If $f(z)$ is analytic in a simply connected region then the line integral is independent of the path.

14.11 CAUCHY'S INTEGRAL THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point inside and on a closed curve C then

$$\oint_C f(z)\mathrm{d}z = 0$$

Proof Let $\quad f(z) = u + iv, \qquad z = x + iy$

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$$

$$\oint_C f(z)\mathrm{d}z = \oint_C (u+iv)(\mathrm{d}x + i\,\mathrm{d}y)$$

$$= \oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\oint_C (v\mathrm{d}x + u\mathrm{d}y)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \quad \text{[Using Green's theorem]}$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y \quad \text{[Using C–R equations]}$$

$$= 0$$

14.11.1 Cauchy-Goursat Theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C contained in a simply connected domain D then

$$\oint_C f(z)\mathrm{d}z = 0$$

14.11.2 Extension of Cauchy's Integral Theorem to a Multiply Connected Region

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_2} f(z)\mathrm{d}z$$

Proof The multiply connected region is made a simply connected region by introducing a cross-cut AB (Fig. 14.19). The path of integration along AB and C_2 is in the clockwise direction, and along BA and C_1 is in the anticlockwise direction. By Cauchy's integral theorem, in a simply connected region C_1ABC_2,

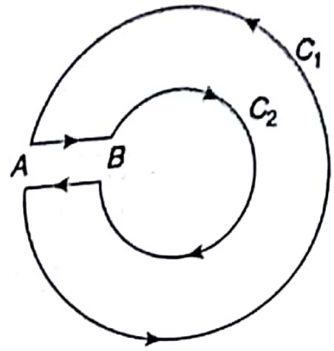

Fig. 14.19 Multiply connected region converted to simply connected region

$$\oint f(z)dz = 0$$

$$\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$$

$$\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0 \quad \left[\because \int_{AB} f(z)dz = -\int_{BA} f(z)dz\right]$$

Reversing the direction of the integral around C_2,

$$-\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

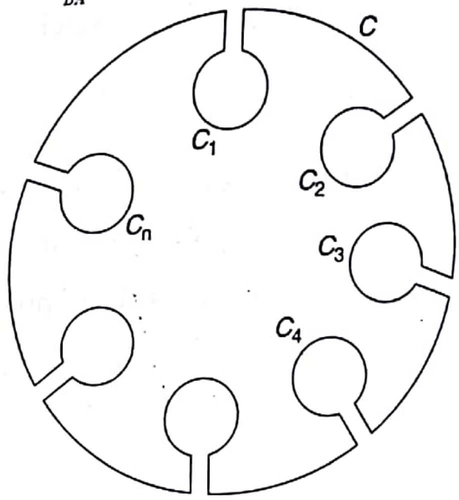

Fig. 14.20 Closed Curves with in C

Note If $C_1, C_2, C_3, \ldots, C_n$ be n number of closed curves within C (Fig. 14.20) then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots + \int_{C_n} f(z)dz$$

EXAMPLE 14.39 Evaluate $\int_C e^{\sin z^2} dz$, where C is $|z| = 1$.

Solution:

(i) Let $f(z) = e^{\sin z^2}$

(ii) Since $f(z)$ is differentiable, it is analytic inside and on C.
$f'(z)$ is continuous inside and on C.

(iii) By Cauchy's integral theorem,

$$\int_C e^{\sin z^2} dz = 0$$

EXAMPLE 14.40 Evaluate $\int_C \frac{e^{2z}}{z^2+1} dz$, where C is $|z| = \frac{1}{2}$.

Solution:

(i) Let $f(z) = \frac{e^{2z}}{z^2+1} = \frac{e^{2z}}{(z+i)(z-i)}$

$f(z)$ is not analytic at $z = \pm i$.

(ii) C is the circle $|z|=\frac{1}{2}$ with centre $(0, 0)$ and radius $\frac{1}{2}$ (Fig. 14.21).

(iii) For $z=-i, |z|=|-i|=1>\frac{1}{2}$

Hence, $z=-i$ lies outside C.

For $z=i, |z|=|i|=1>\frac{1}{2}$

Hence, $z=i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside and on C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{e^{2z}}{z^2+1}\,dz=0$$

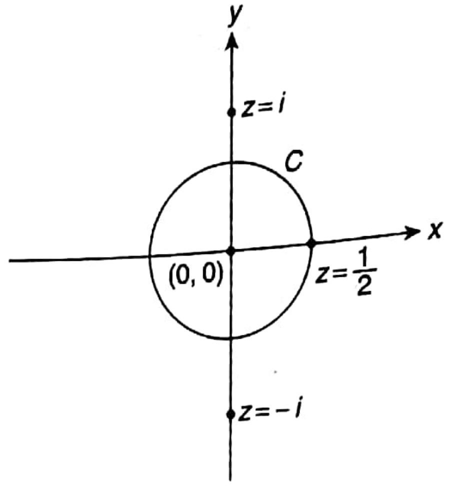

Fig. 14.21

EXAMPLE 14.41 *Evaluate* $\int_C \frac{z+1}{\left(z^2+2z+4\right)^2}\,dz$, *where C is* $|z-1+i|=2$.

Solution:

(i) Let $f(z)=\frac{z+1}{\left(z^2+2z+4\right)^2}$

$$=\frac{z+1}{\left[\left(z+1-\sqrt{3}i\right)\left(z+1+\sqrt{3}i\right)\right]^2}$$

$f(z)$ is not analytic at $z=-1+\sqrt{3}i$ and $z=-1-\sqrt{3}i$.

(ii) C is the circle $|z-1+i|=2$ with centre $(1,-1)$ and radius 2 (Fig. 14.22).

(iii) For $z=-1+\sqrt{3}i$, $|z-1+i|=|-1+\sqrt{3}i-1+i|=3.39>2$

Hence, $z=-1+\sqrt{3}i$ lies outside C.

For $z=-1-\sqrt{3}i$, $|z-1+i|=|-1-\sqrt{3}i-1+i|=2.13>2$

Hence, $z=-1-\sqrt{3}i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside and on C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{z+1}{(z^2+2z+4)^2}\,dz=0$$

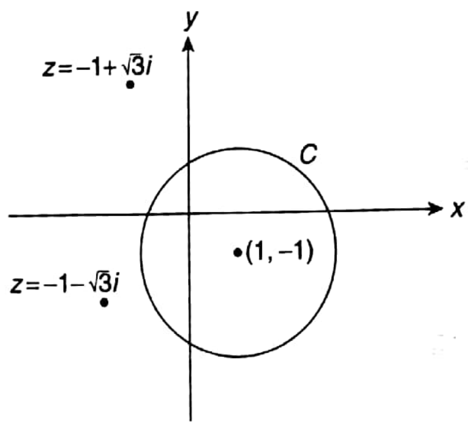

Fig. 14.22

EXAMPLE 14.42 *Evaluate* $\int_C \tan z\,dz$, *where* C *is* $|z|=1$.

Solution:

(i) Let $f(z)=\tan z=\dfrac{\sin z}{\cos z}$

$f(z)$ is not analytic at $\cos z=0$ *i.e.* $z=\pm\dfrac{\pi}{2},\pm\dfrac{3\pi}{2},\ldots$

(ii) C is the circle $|z|=1$ with centre $(0,0)$ and radius 1 (Fig.14.23).

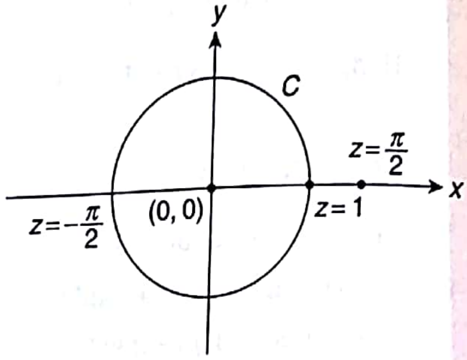

Fig. 14.23

(iii) For $z=\pm\dfrac{\pi}{2},\pm\dfrac{3\pi}{2},\pm\dfrac{5\pi}{2}\ldots\ldots|z|>1$

Hence, all these points lie outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside and on C.

(v) By Cauchy's integral theorem,

$$\int_C \tan z\,dz=0$$

EXERCISE 14.7

Evaluate the following integrals using Cauchy's integral theorem:

1. $\displaystyle\int_C \frac{z+3}{z^2-2z+5}dz$, where C is $|z-1|=1$ **[Ans.: 0]**

2. $\displaystyle\int_C \frac{z}{z-2}dz$, where C is $|z|=1$ **[Ans.: 0]**

3. $\displaystyle\int_C \frac{1}{2z-3}dz$, where C is $|z|=1$ **[Ans.: 0]**

4. $\displaystyle\int_C \frac{e^{2z}}{z^2+1}dz$, where C is $|z|=\frac{1}{2}$ **[Ans.: 0]**

5. $\displaystyle\int_C \frac{3z-1}{z^3-z}dz$, where C is $|z|=2$ **[Ans.: 0]**

6. $\displaystyle\int_C (x^2-y^2+2ixy)dz$, where C is $|z|=2$ **[Ans.: 0]**

7. $\displaystyle\int_C \frac{e^{2z}}{z-1}dz$, where C is $|z|=\frac{1}{2}$ **[Ans.: 0]**

8. $\displaystyle\int_C \cot z\,dz$, where C is $\left|z+\frac{1}{2}\right|=\frac{1}{3}$ **[Ans.: 0]**

14.12 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic inside and on a closed curve C and if a is any point inside C then

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

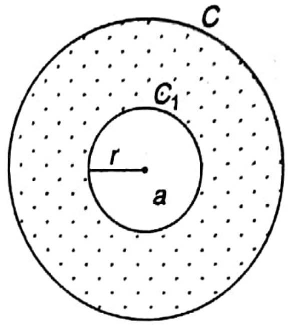

Fig. 14.24 Illustration of Cauchy's integral formula

Proof Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C except at $z=a$.

Draw a small circle C_1 with centre at $z=a$ and radius r lying entirely inside C (Fig. 14.24).

Now, $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

By Cauchy's integral theorem for multiply connected region,

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_{C_1} \frac{f(z)}{z-a}\,dz$$

For the circle C_1, let

$$z-a=re^{i\theta},\ dz=rie^{i\theta}\,d\theta$$

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_0^{2\pi} \frac{f\left(a+re^{i\theta}\right)}{re^{i\theta}}ire^{i\theta}\,d\theta$$

$$=i\oint_0^{2\pi} f\left(a+re^{i\theta}\right)d\theta \qquad \text{...(14.15)}$$

In the limiting position, as $r\to 0$, the circle C_1 shrinks to the point a. Hence, Eq. (14.15) reduces to

$$\oint_C \frac{f(z)}{z-a}\,dz=i\oint_0^{2\pi} f(a)\,d\theta=i\,f(a)\left|\theta\right|_0^{2\pi}$$

$$=i\,f(a)(2\pi-0)=2\pi i\,f(a)$$

$$\therefore f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Cauchy's Integral Formula for the Derivative of an Analytic Function

If a function $f(z)$ is analytic in a region R then its derivative at any point $z=a$ of R is also analytic in R and is given by

$$f'(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

where C is any closed curve in R surrounding the point $z=a$.

Proof By Cauchy's integral formula,

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Differentiating w.r.t. a using Differentiation Under Integral Sign (DUIS),

$$f'(a)=\frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz=\frac{1}{2\pi i}\oint_C f(z)\left[-\frac{1}{(z-a)^2}(-1)\right]dz$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

Differentiating again w.r.t. a,

$$f''(a)=\frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$$

Similarly,

$$f'''(a)=\frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}\,dz$$

In general,

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

EXAMPLE 14.43 *Evaluate* $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$, *where C is* $|z|=4$.

Solution:

(i) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz=2\pi i\, f(a)$$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz=f(a) \qquad ...(1)$$

Given $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$ ...(2)

Comparing integrand of Eqs (1) and (2),

$$f(z)=z^2+5$$

(ii) C is the circle $|z| = 4$ with centre (0, 0) and radius 4 (Fig. 14.25).

(iii) For $z = 3, |z| = |3| = 3 < 4$

Hence, $z = 3$ lies inside C.

(iv) $f(z)$ is analytic inside and on C.

(v) $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = f(3)$

$$= \left[z^2+5\right]_{z=3} = 3^2+5 = 14$$

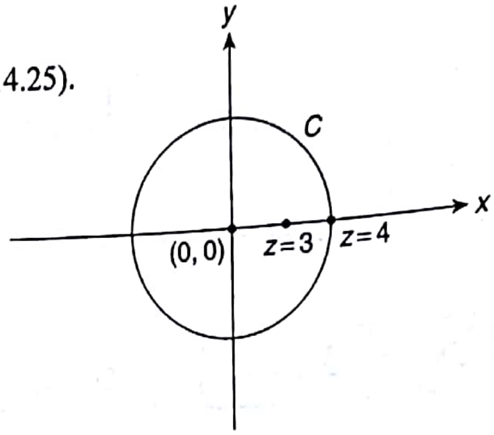

Fig. 14.25

EXAMPLE 14.44 *Evaluate* $\int_C \frac{z+4}{z^2+2z+5}\,dz$*, where C is the circle* $|z+1+i| = 2$.

Solution:

(i) Let $I = \int_C \frac{z+4}{z^2+2z+5}\,dz$

$$= \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz$$

(ii) C is the circle $|z+1+i| = 2$ with centre $(-1,-1)$ and radius 2 (Fig.14.26).

(iii) For $z = -1-2i$, $|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$

Hence, $z = -1-2i$ lies inside C.

For $z = -1+2i, |z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$

Hence, $z = -1+2i$ lies outside C.

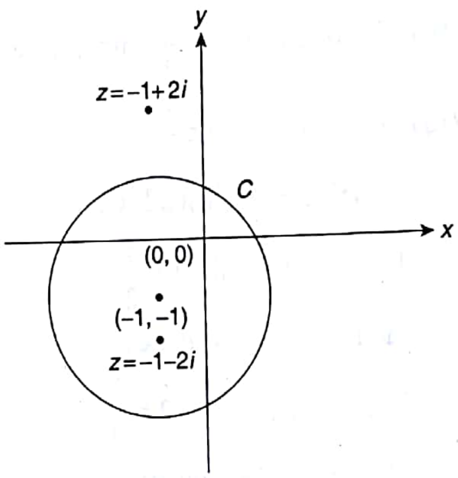

Fig. 14.26

(iv) Let $f(z) = \frac{z+4}{z+1-2i}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{\frac{z+4}{z+1-2i}}{(z+1+2i)}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\left(\frac{z+4}{z+1-2i}\right)}{z+1+2i}\,dz = 2\pi i\left[\frac{z+4}{z+1-2i}\right]_{z=-1-2i}$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left(\frac{-1-2i+4}{-1-2i+1-2i}\right)$$

$$= 2\pi i\left(\frac{3-2i}{-4i}\right) = \frac{\pi}{2}(2i-3)$$

EXAMPLE 14.45 Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, where C is the circle $|z| = \frac{3}{2}$.

Solution:

(i) Let $I = \int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$

(ii) C is the circle $|z| = \frac{3}{2}$ with centre $(0,0)$ and radius $\frac{3}{2}$ (Fig.14.27).

(iii) For $z = 0, |z| = 0 < \frac{3}{2}$

Hence, $z = 0$ lies inside C.

For $z = 1, |z| = |1| = 1 < \frac{3}{2}$

Hence, $z = 1$ lies inside C.

For $z = 2, |z| = |2| = 2 > \frac{3}{2}$

Hence, $z = 2$ lies outside C.

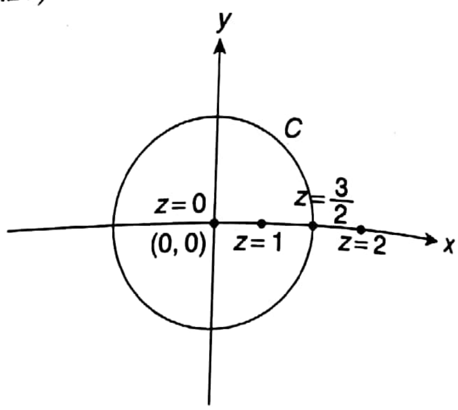

Fig. 14.27

(iv) Let $f(z) = \frac{4-3z}{z-2}$

$f(z)$ is analytic inside and on C.

$$\frac{4-3z}{z(z-1)(z-2)} = \frac{\left(\frac{4-3z}{z-2}\right)}{z(z-1)} = \left(\frac{4-3z}{z-2}\right)\left[\frac{z-(z-1)}{z(z-1)}\right] = \frac{4-3z}{z-2}\left[\frac{1}{z-1} - \frac{1}{z}\right]$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \left[\frac{\left(\frac{4-3z}{z-2}\right)}{z-1} - \frac{\left(\frac{4-3z}{z-2}\right)}{z}\right]dz = \int_C \frac{\left(\frac{4-3z}{z-2}\right)}{z-1}\,dz - \int_C \frac{\left(\frac{4-3z}{z-2}\right)}{z}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\left(\frac{4-3z}{z-2}\right)}{z-1}dz - \int_C \frac{\left(\frac{4-3z}{z-2}\right)}{z}dz = 2\pi i\, f(1) - 2\pi i\, f(0)$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i\left(\frac{4-3}{1-2}\right) - 2\pi i\left(\frac{4-0}{0-2}\right) = -2\pi i + 4\pi i = 2\pi i$$

EXAMPLE 14.46 *Evaluate* $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$, *where C is* $|z| = 3$.

Solution:

(i) Let $I = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$

(ii) C is the circle $|z| = 3$ with centre $(0,0)$ and radius 3 (Fig. 14.28).

(iii) For $z = 1, |z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2, |z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

(iv) Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

$f(z)$ is analytic inside and on C.

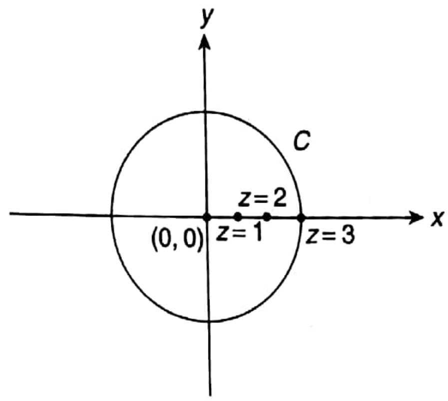

Fig. 14.28

$$\frac{1}{(z-1)(z-2)} = \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz = 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i(\sin \pi + \cos \pi)$$

$$= 2\pi i(0+1) - 2\pi i(0-1) = 2\pi i + 2\pi i = 4\pi i$$

EXAMPLE 14.47 *If* $f(a) = \int_C \frac{3z^2 + 7z + 1}{z-a}dz$, *where C is the circle* $|z| = 2$, *find the values of* $f(3), f'(1+i)$ *and* $f''(1+i)$.

Solution:

(i) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz \qquad ...(1)$$

Given $f(a) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$ ...(2)

Comparing the integrand of Eqs (1) and (2),

$$\frac{f(z)}{2\pi i} = 3z^2+7z+1$$

$$f(z) = 2\pi i\left(3z^2+7z+1\right)$$

(ii) C is the circle $|z| = 2$ with centre $(0,0)$ and radius 2 (Fig. 14.29).

(iii) For $z = 3, |z| = |3| = 3 > 2$

Hence, $z = 3$ lies outside C.

(iv) $\frac{f(z)}{z-3}$ is analytic inside and on C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{f(z)}{z-3}\,dz = 0$$

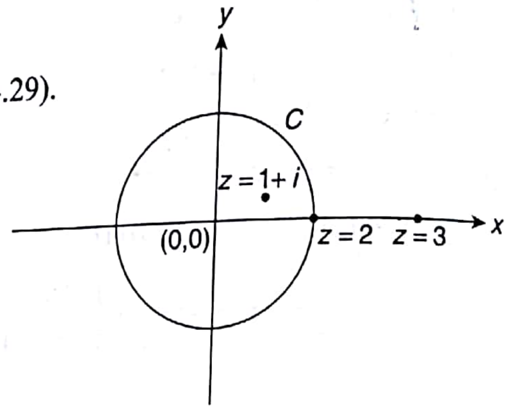

Fig. 14.29

$$\int_C \frac{3z^2+7z+1}{z-3}\,dz = 0$$

$$f(3) = 0 \qquad [\because a = 3]$$

(vi) $f'(z) = 2\pi i(6z+7)$

$f''(z) = 2\pi i(6) = 12\pi i$

For $z = 1+i, |z| = |1+i| = \sqrt{2} < 2$

Hence, $z = 1+i$ lies inside C.

$$f'(1+i) = 2\pi i\left[6(1+i)+7\right] = 2\pi i(13+6i) = 2\pi(13i+6i^2) = 2\pi(-6+13i)$$

$$f''(1+i) = 12\pi i$$

EXAMPLE 14.48 *Evaluate* $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, *where C is* $|z| = 1$.

Solution:

(i) Let $I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$

(ii) C is the circle $|z| = 1$ with centre $(0,0)$ and radius 1 (Fig. 14.30).

(iii) For $z = \frac{\pi}{6}, |z| = \left|\frac{\pi}{6}\right| = 0.52 < 1$

Hence, $z = \frac{\pi}{6}$ lies inside C.

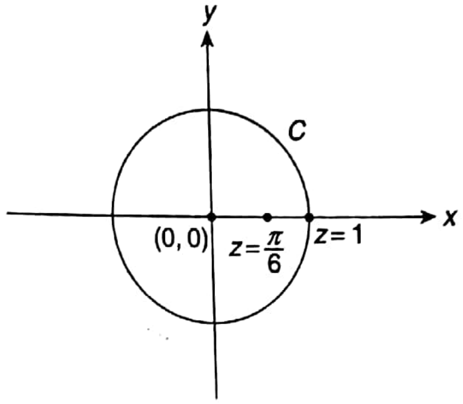

Fig. 14.30

(iv) Let $f(z) = \sin^6 z$

$f(z)$ is analytic inside and on C.

$f'(z) = 6\sin^5 z \cos z$

$f''(z) = 6\left(5\sin^4 z\cos^2 z - \sin^6 z\right)$

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

$$\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!}\left[6\left(5\sin^4 z\cos^2 z - \sin^6 z\right)\right]_{z=\frac{\pi}{6}}$$

$$= 6\pi i\left(5\sin^4\frac{\pi}{6}\cos^2\frac{\pi}{6} - \sin^6\frac{\pi}{6}\right) = 6\pi i\left[5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right]$$

EXAMPLE 14.49 *Evaluate* $\int_C \frac{z+1}{z^4 - 4z^3 + 4z^2} dz$, *where C is the circle* $|z-2-i| = 2$.

Solution:

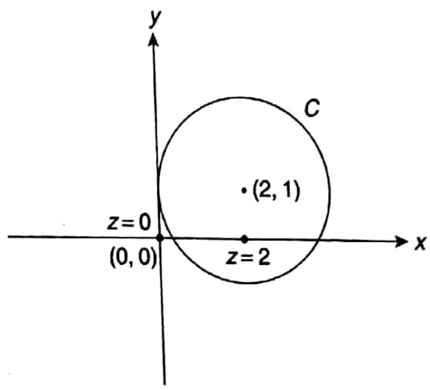

Fig. 14.31

(i) Let $I = \int_C \frac{z+1}{z^4 - 4z^3 + 4z^2} dz = \int_C \frac{z+1}{z^2\left(z^2 - 4z + 4\right)} dz$

$$= \int_C \frac{z+1}{z^2(z-2)^2} dz$$

(ii) C is the circle $|z-2-i|=2$ with centre $(2,1)$ and radius 2 (Fig. 14.31).

(iii) For $z=0, |z-2-i|=|0-2-i|=\sqrt{5}>2$

Hence, $z=0$ lies outside C.

For $z=2, |z-2-i|=|2-2-i|=1<2$

Hence, $z=2$ lies inside C.

(iv) Let $f(z)=\dfrac{z+1}{z^2}$

$f(z)$ is analytic inside and on C.

$$f'(z)=\frac{z^2(1)-(z+1)2z}{z^4}=\frac{-z^2-2z}{z^4}=-\frac{z+2}{z^3}$$

$$\int_C \frac{z+1}{z^4-4z^3+4z^2}\,dz=\int_C \frac{z+1}{z^2(z-2)^2}=\int_C \frac{\left(\frac{z+1}{z^2}\right)}{(z-2)^2}\,dz$$

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz=2\pi i\,f'(a)$$

$$\int_C \frac{\left(\frac{z+1}{z^2}\right)}{(z-2)^2}\,dz=2\pi i\left[-\frac{z+2}{z^3}\right]_{z=2}$$

$$\int_C \frac{z+1}{z^4-4z^3+4z^2}\,dz=2\pi i\left[-\frac{2+2}{2^3}\right]=2\pi i\left(-\frac{1}{2}\right)=-\pi i$$

EXERCISE 14.8

Evaluate the following integrals using Cauchy's integral formula:

1. $\displaystyle\int_C \frac{z\,dz}{(z-1)(z-2)}$, where C is $|z-2|=\frac{1}{2}$

[Ans.: $4\pi i$]

2. $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}\,dz$, where C is $|z|=4$

[Ans.: $-4\pi i$]

3. $\displaystyle\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z-i|=2$

$\left[\textbf{Ans.:}\ \dfrac{\pi}{16}\right]$

4. $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}\,dz$, where C is $|z|=3$

[Ans.: $4\pi(\pi+1)i$]

5. $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz$, where C is $|z|=3$

$\left[\text{Ans.: } \frac{2\pi i}{e^2}\right]$

6. $\int_C \frac{z+1}{z^2+2z+4}\,dz$, where C is $|z+1-i|=2$

[Ans.: πi]

7. $\int_C \frac{z^3-z}{(z-2)^3}\,dz$, where C is $|z|=3$

[Ans.: $12\pi i$]

8. $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is $|z+1-i|=2$

$\left[\text{Ans.: } \frac{\pi}{2}(3+2i)\right]$

9. $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is $|z|=3$

[Ans.: $4\pi i$]

10. $\int_C \frac{dz}{z^3(z+4)}$, where C is $|z|=2$

$\left[\text{Ans.: } \frac{2\pi i}{27}\right]$

11. $\int_C \frac{e^z}{(z-1)(z-4)}\,dz$, where C is $|z|=2$

$\left[\text{Ans.: } \frac{2\pi ie}{3}\right]$

12. $\int_C \frac{z^2+4}{(z-2)(z+3i)}\,dz$, where C is $|z-2|=2$

$\left[\text{Ans.: } \frac{16\pi i}{2+3i}\right]$

13. $\int_C \frac{z+1}{z^3-2z}\,dz$, where C is $|z|=1$

$\left[\text{Ans.: } -\frac{3\pi i}{2}\right]$

14. $\int_C \frac{ze^{2z}}{(z-1)^3}\,dz$, where C is $|z+i|=2$

[Ans.: $8\pi ie^2$]

15. $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where C is $|z-i|=2$

$\left[\text{Ans.: } -\frac{2\pi i}{9}\right]$

14.13 TAYLOR'S SERIES

If $f(z)$ is analytic inside a circle C with centre at $z=a$ then for each z inside C, $f(z)$ can be expanded as a power series about $z=a$ as,

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots+\frac{(z-a)^n}{n!}f^{(n)}(a)+\ldots$$

Proof Let z be any point inside the circle C. Draw a circle C_1 inside C with centre at $z=a$, enclosing the point z (Fig. 14.32).

Let w be any point on C_1.

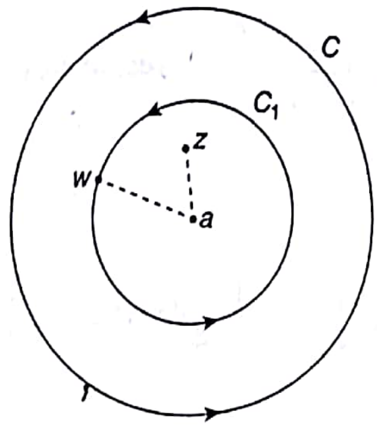

Fig. 14.32 Illustration of Taylor's series

$$|z-a| < |w-a|$$

$$\frac{|z-a|}{|w-a|} < 1$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw \qquad ...(14.16)$$

Consider $\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\ldots+\left(\frac{z-a}{w-a}\right)^n+\ldots\ldots\right] \quad \text{[Using binomial expansion]}$$

$$= \frac{1}{(w-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (14.16),

$$f(z) = \frac{1}{2\pi i}\int_{C_1} f(w)\left[\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right]dw$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n\left[\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw\right]$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n\left[2\pi i\,\frac{f^{(n)}(a)}{n!}\right] \quad \text{[Using Cauchy's integral formula for derivatives]}$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^{(n)}(a)$$

$$= f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots+\frac{(z-a)^n}{n!}f^{(n)}(a)+\ldots$$

EXAMPLE 14.50 *Expand $\frac{1}{z+2}$ at $z=1$ in Taylor's series.*

Solution: Let $f(z)=\frac{1}{z+2}$, $a=1$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots \quad \ldots(1)$$

$$f(z)=\frac{1}{z+2} \qquad f(1)=\frac{1}{1+2}=\frac{1}{3}$$

$$f'(z)=-\frac{1}{(z+2)^2} \qquad f'(1)=-\frac{1}{(1+2)^2}=-\frac{1}{9}$$

$$f''(z)=\frac{2}{(z+2)^3} \qquad f''(1)=\frac{2}{(1+2)^3}=\frac{2}{27}$$

$$f'''(z)=-\frac{6}{(z+2)^4} \qquad f'''(1)=-\frac{6}{(1+2)^4}=-\frac{6}{81}=-\frac{2}{27}$$

Substituting in Eq. (1),

$$f(z)=f(1)+f'(1)(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(z-1)^3+\ldots$$

$$=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{2!}\frac{2}{27}(z-1)^2+\frac{1}{3!}\left(-\frac{2}{27}\right)(z-1)^3+\ldots$$

$$=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\ldots$$

$$f(z)=\frac{1}{z+2}$$

Let $\quad z-1=t$ then $z=t+1$.

$$f(z)=\frac{1}{t+1+2}=\frac{1}{3+t}$$

$$=\frac{1}{3\left(1+\frac{t}{3}\right)}=\frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}=\frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{27}+\ldots\right)$$

$$=\frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\ldots=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\ldots$$

EXAMPLE 14.51 *Expand* $f(z)=\sin z$ about $z=\dfrac{\pi}{4}$.

Solution:

$$f(z)=\sin z, \qquad a=\frac{\pi}{4}$$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots \qquad \ldots(1)$$

$$f(z)=\sin z \qquad f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f'(z)=\cos z \qquad f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''(z)=-\sin z \qquad f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f'''(z)=-\cos z \qquad f'''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

Substituting in Eq. (1),

$$\begin{aligned} f(z)&=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)+\frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z-\frac{\pi}{4}\right)^3+\ldots \\ &=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2-\frac{1}{3!}\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3+\ldots \\ &=\frac{1}{\sqrt{2}}\left[1+\left(z-\frac{\pi}{4}\right)-\frac{1}{2}\left(z-\frac{\pi}{4}\right)^2-\frac{1}{6}\left(z-\frac{\pi}{4}\right)^3+\ldots\right] \end{aligned}$$

EXAMPLE 14.52 *Expand* $f(z)=\dfrac{z+1}{(z-3)(z-4)}$ *as a Taylor's series about* $z=2$.

Solution:

$$f(z)=\frac{z+1}{(z-3)(z-4)}=\frac{A}{z-3}+\frac{B}{z-4}$$

$$z+1=A(z-4)+B(z-3)$$

Putting $z=3$, $$A=-4$$

Putting $z=4$, $$B=5$$

$$\therefore f(z)=-\frac{4}{z-3}+\frac{5}{z-4}$$

Let $z-2=t$ then $z=t+2$.

$$f(z)=-\frac{4}{t+2-3}+\frac{5}{t+2-4}=-\frac{4}{t-1}+\frac{5}{t-2}$$

$$=\frac{4}{1-t}-\frac{5}{2\left(1-\frac{t}{2}\right)}=4(1-t)^{-1}-\frac{5}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$=4(1+t+t^2+\cdots)-\frac{5}{2}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$=\frac{3}{2}+\frac{11}{4}t+\frac{27}{8}t^2+\cdots$$

$$=\frac{3}{2}+\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+\cdots$$

EXAMPLE 14.53 *Find the Taylor series to represent the function* $\frac{z^2-1}{(z+2)(z+3)}$ *in* $|z|<2$.

Solution: Let $f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of numerator,

$$f(z)=1+\frac{-5z-7}{(z+2)(z+3)}$$

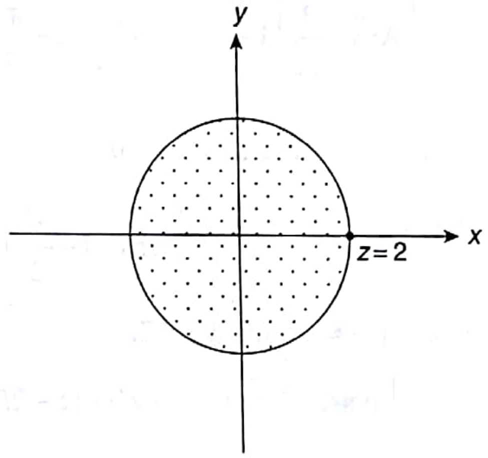

Fig. 14.33

Let $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Putting $z=-2,\quad A=3$

Putting $z=-3,\quad B=-8$

$$\therefore f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

$f(z)$ is not analytic at $z=-2$ and $z=-3$. But $f(z)$ is analytic inside the region $|z|<2$ (Fig. 14.33).

$$|z|<2,\ \left|\frac{z}{2}\right|<1,\ \left|\frac{z}{3}\right|<1$$

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\ldots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\ldots\right]$$

$$=-\frac{1}{6}+\frac{5}{36}z+\frac{17}{216}z^2+\ldots$$

EXERCISE 14.9

Find the Taylor's series for the following functions about the indicated points:

1. $f(z)=\cos z$ about $z=\frac{\pi}{4}$

$$\left[\textbf{Ans}:\frac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)+\frac{1}{2}\left(z-\frac{\pi}{4}\right)^2-\cdots\right]\right]$$

2. $f(z)=\cos z$ about $z=0$

$$\left[\textbf{Ans.}:1-\frac{z^2}{2}+\frac{z^4}{24}-\cdots\right]$$

3. $f(z)=e^{2z}$ about $z=2i$

$$\left[\textbf{Ans.}:e^{4i}\left\{1+2(z-2i)-(z-2i)^2+\cdots\right\}\right]$$

4. $f(z)=\tanh z$ about $z=0$

$$\left[\text{Ans.}:z-\frac{z^3}{3}+\cdots\right]$$

5. $f(z)=\frac{z-1}{z+1}$ in the region $|z|<1$

$$[\textbf{Ans.}:1-2(1+z+z^2+z^3+\cdots)]$$

6. $f(z)=\frac{1}{z^2+4}$ about $z=-i$

$$\left[\textbf{Ans.}:\frac{1}{3}+\frac{1}{18}(z+i).i+\frac{7}{27}(z+1)^2-\cdots\right]$$

7. $f(z)=\frac{z-1}{z^2}$ about $z=1$.

$$[\textbf{Ans.}:(z-1)-2(z-1)^2+3(z-1)^3-\cdots]$$

8. $f(z)=\frac{1}{z^2-4z+3}$ about $z=4$

$$\left[\textbf{Ans.}:\frac{1}{3}-\frac{4}{9}(z-4)+\frac{13}{27}(z-4)^2-\cdots\right]$$

14.14 LAURENT'S SERIES

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with centre at $z = a$ and radii $r_1, r_2\,(r_2 < r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw$$

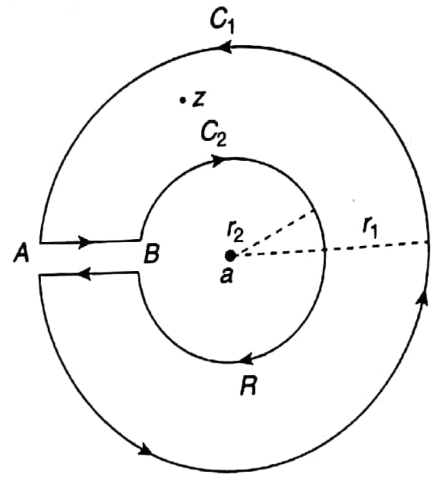

Fig. 14.34 Illustration of Laurent's series

Proof Let C_1, C_2 be two concentric circles with centre at $z = a$ and radii r_1 and $r_2\,(r_2 < r_1)$ respectively (Fig. 14.34).

The annular region R between C_1 and C_2 is a multiply connected region. This multiply connected region is converted to simply connected region by introducing a cross-cut AB.

$f(z)$ is analytic in this simply connected region.

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{AB} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{BA} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw \quad [\because \text{integrals around } AB \text{ and } BA \text{ cancel each other}]$$

$$= f_1(z) - f_2(z)$$

(i) $$f_1(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw \qquad \ldots(14.17)$$

z is any point in R and for this integral, w lies on C_1.

$$\therefore |z-a| < |w-a|$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

Consider
$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$=\frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\ldots+\left(\frac{z-a}{w-a}\right)^n+\ldots\right]$$

$$=\frac{1}{(w-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n=\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (14.17),

$$f_1(z)=\frac{1}{2\pi i}\int_{C_1}f(w)\left[\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right]dw$$

$$=\sum_{n=0}^{\infty}\left[(z-a)^n\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw\right\}\right]=\sum_{n=0}^{\infty}(z-a)^n a_n$$

where $$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

(ii) $$f_2(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw \qquad \ldots(14.18)$$

z is any point in R and for this integral w lies on C_2.

$$\therefore |z-a|>|w-a|$$

$$1>\left|\frac{w-a}{z-a}\right|$$

$$\left|\frac{w-a}{z-a}\right|<1$$

Consider $$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}=\frac{1}{(z-a)\left(\frac{w-a}{z-a}-1\right)}$$

$$=-\frac{1}{(z-a)}\cdot\frac{1}{\left(1-\frac{w-a}{z-a}\right)}=-\frac{1}{(z-a)}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$=-\frac{1}{(z-a)}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\ldots+\left(\frac{w-a}{z-a}\right)^{n-1}+\ldots\right]$$

$$=-\frac{1}{(z-a)}\sum_{n=1}^{\infty}\left(\frac{w-a}{z-a}\right)^{n-1}=\sum_{n=1}^{\infty}-\frac{(w-a)^{n-1}}{(z-a)^n}$$

Substituting $\frac{1}{w-z}$ in Eq. (14.18),

$$f_2(z)=\frac{1}{2\pi i}\int_{C_2} f(w)\left[\sum_{n=1}^{\infty}-\frac{(w-a)^{n-1}}{(z-a)^n}\right]dw$$

$$=\sum_{n=1}^{\infty}\left[\frac{-1}{(z-a)^n}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right\}\right]$$

$$=\sum_{n=1}^{\infty}-\frac{1}{(z-a)^n}b_n$$

where $$b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

$$\therefore f(z)=f_1(z)-f_2(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw \text{ and } b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

The second term $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ is known as the principal part of Laurent's series.

EXAMPLE 14.54 *Find all possible Laurent's expansion of* $f(z)=\frac{4-3z}{z(1-z)(2-z)}$ *about* $z=0$. *Indicate the region of convergence in each case.*

Solution:

$$f(z)=\frac{4-3z}{z(1-z)(2-z)}=\frac{A}{z}+\frac{B}{1-z}+\frac{C}{2-z}$$

$$4-3z=A(1-z)(2-z)+Bz(2-z)+Cz(1-z)$$

Putting $z=0$, $A=2$

Putting $z=1$, $B=1$

Putting $z=2$, $C=1$

$$\therefore f(z)=\frac{2}{z}+\frac{1}{1-z}+\frac{1}{2-z}$$

$f(z)$ is not analytic at $z=0, z=1$ and $z=2$.

(i) $|z|<1$

$f(z)$ is analytic in the region $|z|<1$ about $z=0$ (Fig. 14.35).

Since $|z|<1$, $|z|<2$,

$$\left|\frac{z}{2}\right|<1$$

$$f(z)=\frac{2}{z}+\frac{1}{1-z}+\frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}=\frac{2}{z}+(1-z)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{2}{z}+\left(1+z+z^2+\ldots\right)+\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right]$$

$$=\left(\frac{3}{2}+\frac{5}{4}z+\frac{9}{8}z^2+\ldots\right)+\frac{2}{z}$$

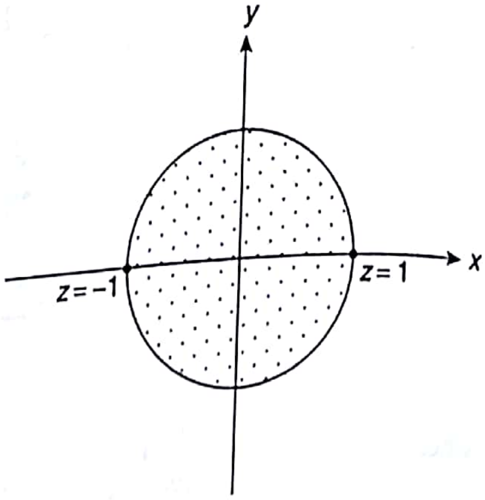

Fig. 14.35

(ii) $1<|z|<2$

$f(z)$ is analytic in the annular region $1<|z|<2$ about $z=0$ (Fig. 14.36).

$1<|z|$, $\left|\frac{1}{z}\right|<1$

Since $|z|<2$, $\left|\frac{z}{2}\right|<1$,

$$f(z)=\frac{2}{z}-\frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)}+\frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}$$

$$=\frac{2}{z}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{2}{z}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)+\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\ldots\right]$$

$$=\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right)+\left(\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\ldots\right)$$

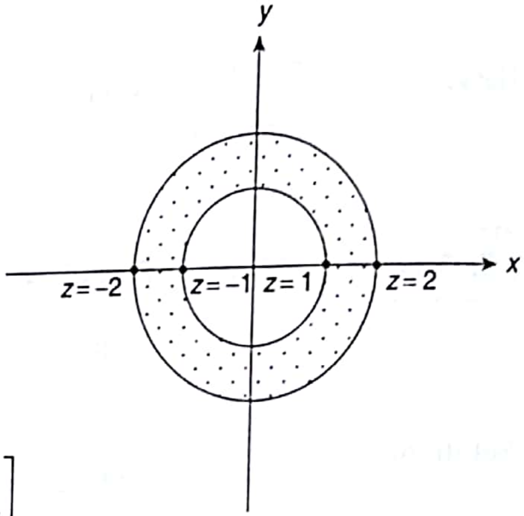

Fig. 14.36

(iii) $|z|>2$

$f(z)$ is analytic in the region $|z|>2$ about $z=0$ (Fig. 14.37).

Since $|z|>2$, $2<|z|$,

$$\left|\frac{2}{z}\right|<1 \text{ and } \left|\frac{1}{z}\right|<\frac{1}{2}<1$$

$$f(z)=\frac{2}{z}-\frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)}-\frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)}$$

$$=\frac{2}{z}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=\frac{2}{z}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)-\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots\right]$$

$$=-\frac{3}{z^2}-\frac{5}{z^3}-\dots$$

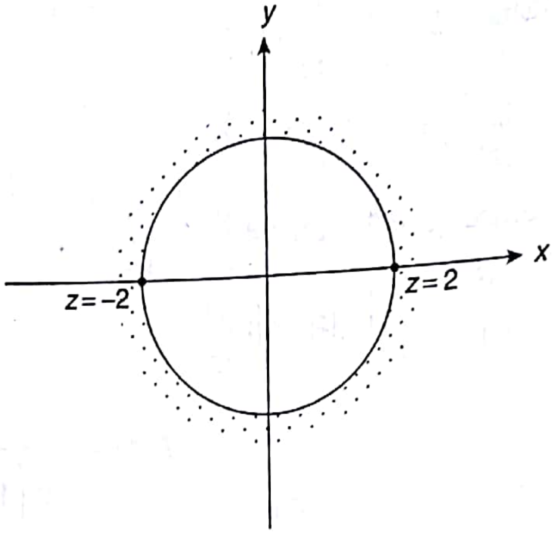

Fig. 14.37

EXAMPLE 14.55 *Find the Laurent's series of* $f(z)=\dfrac{z}{(z^2+1)(z^2+4)}$ *in* $1<|z|<2$.

Solution: $f(z)=\dfrac{z}{(z^2+1)(z^2+4)}$

$$\frac{f(z)}{z}=\frac{1}{(z^2+1)(z^2+4)}$$

Putting $z^2=x$,

$$\frac{1}{(z^2+1)(z^2+4)}=\frac{1}{(x+1)(x+4)}$$

$$=\frac{A}{x+1}+\frac{B}{x+4}$$

$$1=A(x+4)+B(x+1)$$

Putting $x=-1$, $A=\frac{1}{3}$

Putting $x=-4$, $B=-\frac{1}{3}$

$$\frac{f(z)}{z}=\frac{1}{3}\frac{1}{x+1}-\frac{1}{3}\frac{1}{x+4}$$

$$=\frac{1}{3}\frac{1}{z^2+1}-\frac{1}{3}\frac{1}{z^2+4} \quad \text{[Resubstituting } x\text{]}$$

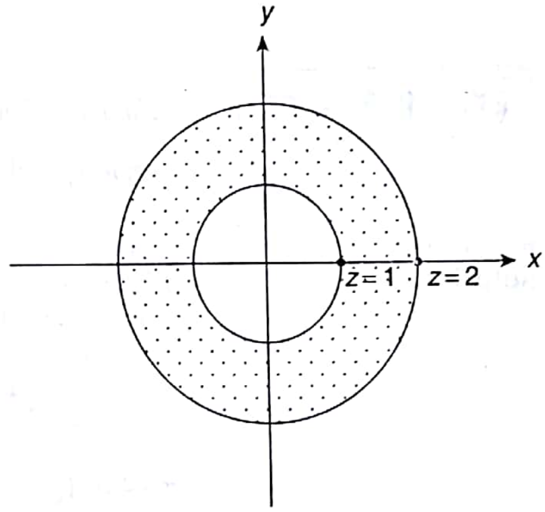

Fig. 14.38

$f(z)$ is analytic in the annular region $1<|z|<2$ about $z=0$ (Fig. 14.38).

Since $1<|z|$, $\left|\dfrac{1}{z}\right|<1$,

$$\left|\frac{1}{z^2}\right|<1$$

Since $|z|<2$, $|z^2|<4$,

$$\left|\frac{z^2}{4}\right|<1 \quad \left[\because |z|^2=|z^2|\right]$$

$$f(z)=\frac{z}{3}\frac{1}{z^2\left(1+\frac{1}{z^2}\right)}-\frac{z}{3}\frac{1}{4\left(1+\frac{z^2}{4}\right)}$$

$$=\frac{1}{3z}\left(1+\frac{1}{z^2}\right)^{-1}-\frac{z}{12}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$=\frac{1}{3z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\ldots\right)-\frac{z}{12}\left[1-\frac{z^2}{4}+\left(\frac{z^2}{4}\right)^2-\ldots\right]$$

$$=\left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\ldots\right)-\left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\ldots\right)$$

$$=-\left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\ldots\right)+\left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\ldots\right)$$

EXAMPLE 14.56 *Find the Laurent's series expansion of* $f(z)=\dfrac{z+4}{(z+3)(z-1)^2}$ *in the region* $|z-1|>4$.

Solution:

$$f(z)=\frac{z+4}{(z+3)(z-1)^2}$$

$$=\frac{A}{z+3}+\frac{B}{z-1}+\frac{C}{(z-1)^2}$$

$$z+4=A(z-1)^2+B(z+3)(z-1)+C(z+3)$$

Putting $z=-3$, $A=\dfrac{1}{16}$

Putting $z=1$, $C=\dfrac{5}{4}$

Putting $z = 0$,

$$4 = A - 3B + 3C$$

$$4 = \frac{1}{16} - 3B + \frac{15}{4}$$

$$B = -\frac{1}{16}$$

$$\therefore f(z) = \frac{1}{16}\cdot\frac{1}{z+3} - \frac{1}{16}\cdot\frac{1}{z-1} + \frac{5}{4}\frac{1}{(z-1)^2}$$

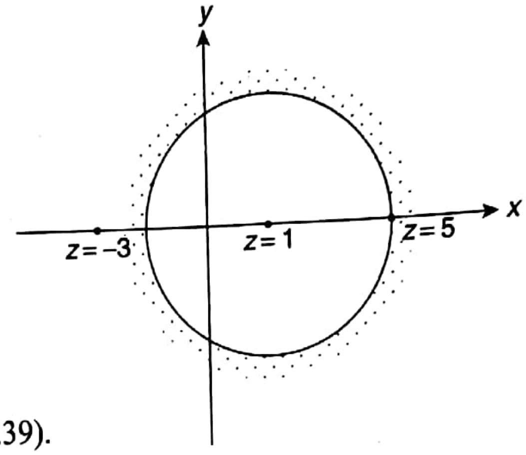

Fig. 14.39

$f(z)$ is analytic in the region $|z-1| > 4$ about $z = 1$ (Fig. 14.39).

Let $z - 1 = t$, $z = t + 1$.

Since $|z-1| > 4$, $|t| > 4$,

$\left|\frac{4}{t}\right| < 1$ and $\left|\frac{1}{t}\right| < \frac{1}{4} < 1$

$$f(z) = \frac{1}{16}\cdot\frac{1}{t+4} - \frac{1}{16}\frac{1}{t} + \frac{5}{4}\frac{1}{t^2} = \frac{1}{16}\frac{1}{t\left(1+\frac{4}{t}\right)} - \frac{1}{16}\frac{1}{t} + \frac{5}{4}\cdot\frac{1}{t^2}$$

$$= \frac{1}{16t}\left(1+\frac{4}{t}\right)^{-1} - \frac{1}{16t} + \frac{5}{4t^2} = \frac{1}{16t}\left[1 - \frac{4}{t} + \left(\frac{4}{t}\right)^2 - \ldots\right] - \frac{1}{16t} + \frac{5}{4t^2}$$

$$= \frac{1}{16t}\left(-\frac{4}{t} + \frac{16}{t^2} - \ldots\right) + \frac{5}{4t^2} = \frac{1}{16(z-1)}\left[-\frac{4}{(z-1)} + \frac{16}{(z-1)^2} - \ldots\right] + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \ldots$$

EXAMPLE 14.57 *Find the Laurent's series of* $f(z) = \dfrac{1}{z(1-z)}$ *valid in the region* (i) $|z+1| < 1$ (ii) $|z+1| > 2$. (iii) $1 < |z+1| < 2$.

Solution: $$f(z) = \frac{1}{z(1-z)} = \frac{1-z+z}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

$f(z)$ is not analytic at $z = 0$ and $z = 1$.

(i) $|z+1| < 1$

$f(z)$ is analytic in the region $|z+1| < 1$ about $z = -1$ (Fig. 14.40).

Let $z + 1 = t$ then $z = t - 1$.

$$\therefore |t| < 1, \ \left|\frac{t}{2}\right| < \frac{1}{2} < 1$$

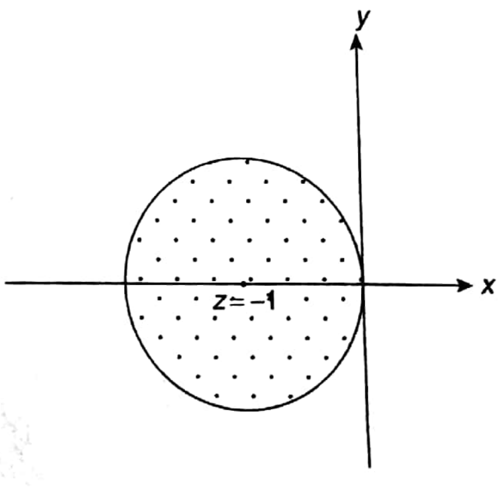

Fig. 14.40

$$f(z) = \frac{1}{t-1} + \frac{1}{1-(t-1)} = \frac{1}{t-1} + \frac{1}{2-t}$$

$$= -\frac{1}{1-t} + \frac{1}{2\left(1-\frac{t}{2}\right)} = -(1-t)^{-1} + \frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$= -\left(1+t+t^2+\cdots\right) + \frac{1}{2}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$= -\frac{1}{2}\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\cdots\right)$$

$$= -\frac{1}{2}\left[1+\frac{3}{2}(z+1)+\frac{7}{4}(z+1)^2+\cdots\right]$$

(ii) $|z+1| > 2$

$f(z)$ is analytic in the region $|z+1| > 2$ about $z = -1$ (Fig. 14.41).

Let $z+1 = t$ then $z = t-1$.

$$\therefore |t| > 2, \quad \left|\frac{2}{t}\right| < 1, \quad \left|\frac{1}{t}\right| < \frac{1}{2} < 1$$

$$f(z) = \frac{1}{t-1} + \frac{1}{2-t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{1}{t\left(1-\frac{2}{t}\right)}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{1}{t}\left(1-\frac{2}{t}\right)^{-1}$$

$$= \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\ldots\right)$$

$$-\frac{1}{t}\left[1+\frac{2}{t}+\left(\frac{2}{t}\right)^2+\ldots\right]$$

$$= \frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots-\left(\frac{1}{t}+\frac{2}{t^2}+\frac{4}{t^3}+\ldots\right)$$

$$= -\frac{1}{t^2}-\frac{3}{t^3}-\frac{7}{t^4}-\ldots$$

$$= -\frac{1}{(z+1)^2}-\frac{3}{(z+1)^3}-\frac{7}{(z+1)^4}-\ldots$$

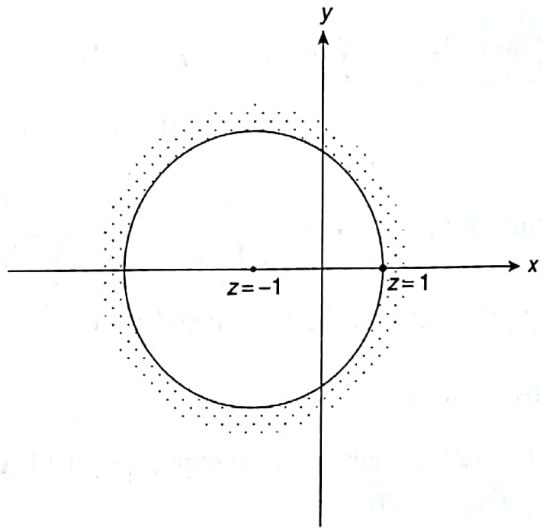

Fig. 14.41

(iii) $1<|z+1|<2$

$f(z)$ is analytic in the annular region $1<|z+1|<2$ about $z=-1$ (Fig. 14.42).

Let $z+1=t$ then $z=t-1$.

$\therefore 1<|t|<2$

$1<|t|,\ \left|\frac{1}{t}\right|<1$

$|t|<2,\ \left|\frac{t}{2}\right|<1$

$$f(z)=\frac{1}{t-1}+\frac{1}{2-t}$$

$$=\frac{1}{t\left(1-\frac{1}{t}\right)}+\frac{1}{2\left(1-\frac{t}{2}\right)}$$

$$=\frac{1}{t}\left(1-\frac{1}{t}\right)^{-1}+\frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$=\frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\ldots\right)+\frac{1}{2}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\ldots\right]$$

$$=\left(\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots\right)+\left(\frac{1}{2}+\frac{t}{4}+\frac{t^2}{8}+\ldots\right)$$

$$=\left[\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\right]+\left[\frac{1}{2}+\frac{z+1}{4}+\frac{(z+1)^2}{8}+\ldots\right]$$

$$=\left[\frac{1}{2}+\frac{z+1}{4}+\frac{(z+1)^2}{8}+\ldots\right]+\left[\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\right]$$

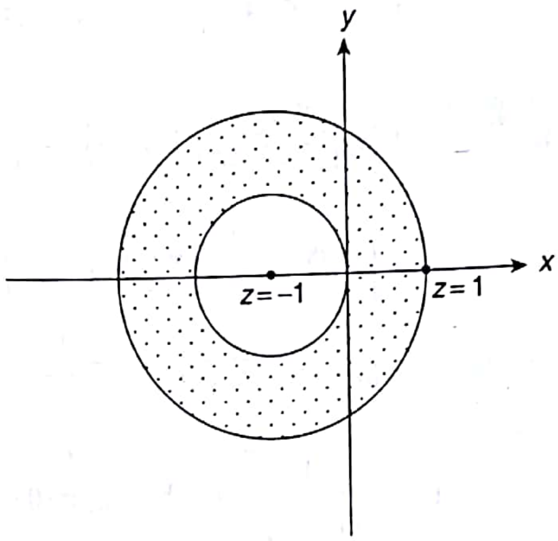

Fig. 14.42

EXERCISE 14.10

Expand the following functions in Laurent's series:

1. $f(z)=\frac{z-1}{(z+2)(z+3)},\quad 2<|z|<3$

$\left[\textbf{Ans.: }-\frac{3}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}\ldots\right]+\frac{4}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}\ldots\right]\right]$

2. $f(z)=\frac{1}{z(z-1)}$ for $0<|z|<1$ and $0<|z-1|<1$

$\left[\textbf{Ans.: }-\frac{1}{z}(1+z+z^2+\ldots)\text{ and } \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-\ldots\right]\right]$

3. $f(z)=\frac{z-1}{z^2}$ in $|z-1|>1$

$\left[\textbf{Ans.:} \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\ldots\right]$

4. $f(z)=\frac{z+3}{z(z^2-z-2)}$ in $1<|z|<2$

$\left[\textbf{Ans.:} -\frac{5}{12}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right)-\frac{3}{2z} + \frac{2}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\ldots\right)\right]$

5. $f(z)=\frac{1}{z(1-z)^2}$ in the region $0<|z|<1$ and $0<|z-1|<1$

$\left[\textbf{Ans.:} \text{(i) } \frac{1}{z}+2+3z+4z^2+\ldots \quad \text{(ii) } \frac{1}{(z-1)^2}-\frac{1}{z-1}+1-(z-1)+(z-1)^2-\ldots\right]$

6. $f(z)=\frac{1}{z(1-z)^2}$ in the region $|z|>1$

$\left[\textbf{Ans.:} \frac{1}{z^3}+\frac{2}{z^4}+\frac{3}{z^5}+\ldots\right]$

7. $f(z)=\frac{1}{z^3-3z^2+2z}$ about $z=0$ for (i) $1<|z|<2$ (ii) $|z|>2$

$\left[\textbf{Ans.:} \text{(i) } -\frac{1}{4}-\frac{1}{8}z-\frac{1}{16}z^2-\ldots -\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\ldots \quad \text{(ii) } \frac{1}{z^3}+\frac{3}{z^4}+\frac{7}{z^5}+\ldots\right]$

8. $f(z)=\frac{z-1}{z^2-2z-3}$ for (i) $1<|z|<3$ (ii) $|z|>3$

$\left[\textbf{Ans.:} \text{(i) } \frac{1}{2}\left(-\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\ldots-\frac{1}{z^2}+\frac{1}{z^3}-\ldots\right) \quad \text{(ii) } \frac{1}{z}+\frac{1}{z^2}+\frac{5}{z^3}+\ldots\right]$

9. $f(z)=\frac{1}{z^2(z-2)}$; $0<|z|<2$

$\left[\textbf{Ans.:} -\frac{1}{2z^2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\ldots\right)\right]$

10. $f(z)=\frac{1+2z}{z+z^2}$; $0<|z|<1$

$\left[\textbf{Ans.:} \frac{1}{z}\left(1+z+z^2+\ldots\right)\right]$

11. $f(z)=\frac{z}{(z+1)(z+2)}$; $0<|z+2|<1$

$\left[\textbf{Ans.:} \frac{2}{z+2}+1+(z+2)+(z+2)^2+\ldots\right]$

12. $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}$ in (i) $1<|z|<4$ (ii) $|z|>4$

$\left[\textbf{Ans.:} \text{(i) } 1-\left(\frac{1}{z}-\frac{1}{z^2}+\ldots\ldots\right) -\left(1-\frac{z}{4}+\frac{z^2}{16}-\ldots\ldots\right) \quad \text{(ii) } 1-\left(\frac{1}{z}-\frac{1}{z^2}+\ldots\right)-4\left(\frac{1}{z}-\frac{4}{z^2}+\ldots\right)\right]$

14.15 SINGULAR POINTS

A point at which the function $f(z)$ is not analytic is known as singular point or singularity of the function.

Types of Singularities

1. ***Isolated Singularity*** A singular point $z = a$ is known as isolated singularity if there is no other singularity within a small circle surrounding the point $z = a$, otherwise it is called non-isolated singularity.

   For example, $f(z) = \dfrac{z^2}{z-2}$ has isolated singularity at $z = 2$.

2. ***Removable Singularity*** An isolated singular point $z = a$ is known as removable singularity if $\lim_{z \to a} f(z)$ exists or there is no negative power term of $(z-a)$ in Laurent's series of $f(z)$, i.e. $b_n = 0$.

   For example, $f(z) = \dfrac{\sin z}{z}$

   $z = 0$ is a singularity of $f(z)$.

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

   $\therefore$ $f(z)$ has removable singularity at $z = 0$.

3. ***Essential Singularity*** A singular point $z = a$ is known as essential singularity if the number of negative power terms of $(z-a)$ in Laurent's series is infinite.

   For example, $f(z) = e^{\frac{1}{z-2}}$

$$= 1 + \frac{1}{z-2} + \frac{1}{2!}\frac{1}{(z-2)^2} + \ldots\ldots$$

   Since the number of negative power terms of $(z-2)$ is infinite, $z = 2$ is an essential singularity.

4. ***Pole of order m*** An isolated singularity $z = a$ is known as a pole of order m if in the Laurent's series, all the negative powers of $(z-a)$ after the m^{th} power are zero, i.e., the highest power of $\dfrac{1}{z-a}$ is m.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} \frac{b_n}{(z-a)^n}$$

   For example,

$$f(z) = \frac{1}{(z-1)(z-3)^2}$$

$z=1$ is a pole of order 1 and $z=3$ is a pole of order 2.

A pole of order one is also known as simple pole.

Meromorphic Function A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

EXAMPLE 14.58 *Identify the type of singularities of the function* $f(z)=e^{\frac{1}{z-1}}$.

Solution:

$$f(z)=e^{\frac{1}{z-1}}=1+\frac{1}{(z-1)}+\frac{1}{2!(z-1)^2}+\ldots\ldots\infty$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z=1$ is an essential singularity.

EXAMPLE 14.59 *What is the nature of the singularity at* $z=0$ *of the function* $f(z)=\frac{\sin z-z}{z^3}$.

Solution:

$$f(z)=\frac{\sin z-z}{z^3}$$

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{\sin z-z}{z^3} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{z\to 0}\frac{\cos z-1}{3z^2} \qquad [\text{Using L'Hospital's rule}]$$

$$=\lim_{z\to 0}\frac{-\sin z}{6z} \qquad [\text{Using L'Hospital's rule}]$$

$$=-\frac{1}{6} \qquad \left[\because \lim_{z\to 0}\frac{\sin z}{z}=1\right]$$

Since the limit is finite, $z=0$ is a removable singularity.

EXAMPLE 14.60 *Write the singularity of the function* $\frac{1}{1-e^z}$.

Solution: Let $f(z)=\frac{1}{1-e^z}$

$$=\frac{1}{1-\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots\right)}=\frac{1}{-\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots\right)}$$

$$=\frac{1}{-z\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\right)}=-\frac{1}{z}\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\dots\right)\right]^{-1}$$

$$= -\frac{1}{z}\left[1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\dots\right)+\frac{\left(\frac{z}{2}+\frac{z^2}{6}+\dots\right)^2}{2!}-\dots\right]$$

$$= -\frac{1}{z}+\frac{1}{2}+\frac{z}{24}-\frac{z^2}{24}+\dots$$

Since the highest power of $\frac{1}{z}$ is 1, $z=0$ is a pole of order 1.

EXAMPLE 14.61 *Find the singularities of the function* $f(z)=\frac{\cot \pi z}{(z-a)^3}$.

Solution:

$$f(z)=\frac{\cot \pi z}{(z-a)^3}=\frac{\cos \pi z}{(z-a)^3 \sin \pi z}$$

Singularities are at $(z-a)^3 \sin \pi z = 0$.

$$z-a=0, \qquad \sin \pi z = 0$$

$$z=a, \qquad \pi z = n\pi, n=0,\pm 1,\pm 2,\dots$$

$$z=n$$

$$=0,\pm 1,\pm 2,\pm 3,\dots$$

Since in the neighbourhood of $z=a$, there are infinite singularities, $z=a$, is a non-isolated singularity.

EXAMPLE 14.62 *Identify the type of singularity of* $f(z)=\frac{z-2}{4}\sin\left(\frac{1}{z-1}\right)$.

Solution:

$$f(z)=\frac{z-2}{4}\sin\left(\frac{1}{z-1}\right)$$

Let $z-1=t$ then $z=t+1$.

$$f(z)=\frac{t+1-2}{4}\sin\left(\frac{1}{t}\right)$$

$$=\frac{t-1}{4}\left[\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\dots\right]$$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots\right) - \left(\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots\right)\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{t} - \frac{1}{3!t^2} + \frac{1}{3!t^3} + \dots\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{z-1} - \frac{1}{3!(z-1)^2} + \frac{1}{3!(z-1)^3} + \dots\right]$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z = 1$ is an essential singularity.

EXAMPLE 14.63 *Find the singularities of the function* $f(z) = \dfrac{z^2+1}{e^z}$.

Solution:

$$f(z) = \frac{z^2+1}{e^z} = (z^2+1)e^{-z}$$

$$= (z^2+1)\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots\right)$$

$f(z)$ is analytic everywhere.
Hence, there is no singularity.

EXERCISE 14.11

Determine the nature of singularities of the following functions:

1. $z^2 e^{\frac{1}{z}}$

   [**Ans.:** $z = 0$ is an essential singularity.]

2. $\dfrac{\sin 3z}{z}$

   [**Ans.:** $z = 0$ is a removable singularity.]

3. $e^{-\frac{1}{z^2}}$

   [**Ans.:** No singularity]

4. $\dfrac{2-e^z}{z^3}$

   [**Ans.:** $z = 0$ is a pole of order 3.]

5. $z^3 e^{\frac{1}{z-1}}$

   [**Ans.:** $z = 1$ is an essential singularity.]

6. $\dfrac{1}{(z-5)^2(z^2-4)}$

   [**Ans.:** $z = 2, -2$ are simple poles and $z = 5$ is a pole of order 2.]

7. $\sin\dfrac{1}{z-2}$

   [**Ans.:** $z = 2$ is an essential singularity.]

8. $z e^{\frac{1}{z^2}}$

   [**Ans.:** $z = 0$ is an essential singularity.]

14.16 RESIDUES

The coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ is known as residue of $f(z)$ at $z=a$

Laurent's series expansion about $z=a$ is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

$$\text{Residue}(\text{at } z=a)=\text{Coefficient of }\frac{1}{z-a}=b_1$$

$$=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-a)^{-1+1}}dz=\frac{1}{2\pi i}\int_C f(z)dz$$

Evaluation of Residues

1. *Residue at a Simple Pole*

(i) If $f(z)$ has a simple pole at $z=a$ then

$$\text{Res}\left[f(z);z=a\right]=\lim_{z\to a}(z-a)f(z)$$

(ii) If $f(z)$ is of the form $f(z)=\frac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic at $z=a$.

If $h(a)=0$ but $h'(a)\neq 0$ then $z=a$ is a simple pole.

If $h(a)=0$ but $g(a)\neq 0$ then

$$\text{Res}\left[f(z);z=a\right]=\lim_{z\to a}\frac{g(z)}{h'(z)}$$

2. *Residue at a Pole of Order m*

If $f(z)$ has a pole of order m at $z=a$ then

$$\text{Res}\left[f(z);z=a\right]=\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]$$

EXAMPLE 14.64 *If* $f(z)=\frac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\ldots\right]$, *find the residue of* $f(z)$ *at* $z=1$.

Solution: The residue of $f(z)$ at $z=1$ is equal to the coefficient of $\frac{1}{z-1}$ in the Laurent series of $f(z)$ about $z=1$, which is equal to -1.

Hence, $\text{Res}\left[f(z);z=1\right]=-1$

EXAMPLE 14.65 *Test for singularity of* $\frac{1}{z^2+1}$ *and hence, find the corresponding residues.*

Solution: Let
$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$$

$z=-i$ and $z=i$ are simple poles.

$$\text{Res}\left[f(z);z=-i\right]=\lim_{z\to -i}(z+i)f(z)=\lim_{z\to -i}\frac{1}{z-i}=\frac{1}{-i-i}=-\frac{1}{2i}$$
$$\text{Res}\left[f(z);z=i\right]=\lim_{z\to i}(z-i)f(z)=\lim_{z\to i}\frac{1}{z+i}=\frac{1}{i+i}=\frac{1}{2i}$$

EXAMPLE 14.66 *Find the residue of* $\tan z$ *at* $z=\frac{\pi}{2}$.

Solution: Let $f(z)=\tan z$

$z=\frac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z);z=\frac{\pi}{2}\right]=\lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\tan z=\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)}{\cot z}\qquad\left[\frac{0}{0}\text{ form}\right]$$
$$=\lim_{z\to\frac{\pi}{2}}\frac{1}{-\text{cosec}^2 z}\qquad\text{[Using L'Hospital's rule]}$$
$$=-\frac{1}{\text{cosec}^2\frac{\pi}{2}}=-1$$

EXAMPLE 14.67 *Determine poles and their orders for the function* $\frac{z+2}{(z+1)^2(z-2)}$. *Find the residues at the poles.*

Solution: Let
$$f(z)=\frac{z+2}{(z+1)^2(z-2)}$$

The poles are given by

$$(z+1)^2(z-2)=0$$
$$z=-1,\,-1,\,2$$

$z=-1$ is a pole of order 2.

$$\text{Res}[f(z); z=-1] = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2 f(z)\right] = \lim_{z\to -1}\frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = \lim_{z\to -1}\left[\frac{-4}{(z-2)^2}\right]$$

$$= \frac{-4}{(-1-2)^2} = -\frac{4}{9}$$

$z=2$ is a simple pole.

$$\text{Res}[f(z); z=2] = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z+2}{(z+1)^2} = \frac{2+2}{(2+1)^2} = \frac{4}{9}$$

EXERCISE 14.12

Find the residue of $f(z)$ at the singular points:

1. $\dfrac{z}{(z-1)^2}$ $\left[\textbf{Ans.: } -\dfrac{1}{2i}\right]$

2. $\dfrac{z+2}{(z+1)^2}$ [**Ans.:** 1]

3. $\dfrac{z^2-z}{(z+1)^2(z^2+4)}$ $\left[\textbf{Ans.: } -\dfrac{11}{25}, \dfrac{11\pm 2i}{50}\right]$

4. $\dfrac{e^z}{(z-1)^3}$ $\left[\textbf{Ans.: } \dfrac{e}{2}\right]$

5. $\dfrac{\sin^2 z}{z^3}$ $\left[\textbf{Ans.: } \dfrac{1}{4}\right]$

6. $\dfrac{z+3}{z(z-1)(z+2)}$ $\left[\textbf{Ans.: } -\dfrac{3}{2}, \dfrac{4}{3}, \dfrac{1}{6}\right]$

7. $\dfrac{e^{2z}}{z^2+\pi^2}$ $\left[\textbf{Ans.: } \dfrac{1}{2\pi i}, -\dfrac{1}{2\pi i}\right]$

8. $\dfrac{z^3}{(z+a)^2}$ [**Ans.:** $3a^2$]

9. $\dfrac{z^2+1}{z(z-2)}$ $\left[\textbf{Ans.: } -\dfrac{1}{2}, \dfrac{5}{2}\right]$

10. $z\cos\dfrac{1}{z}$ [**Ans.:** -4]

14.17 CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points inside C then

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

where the integral is taken in the anticlockwise direction around C.

Proof Let $z_1, z_2, \ldots, z_n$ be finite numbers of singular points inside C (Fig. 14.43). Enclose each of the singular point in a small circle such that no other singular point lies inside this circle. The curve C along with these small circles $C_1, C_2, C_3, \ldots, C_n$ form a *multiply* connected region and $f(z)$ is analytic in this region.

By Cauchy's integral theorem for a *multiply* connected region,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i[\text{Res. at } z_1] + 2\pi i[\text{Res. at } z_2] + \ldots$$

$$+ 2\pi i[\text{Res. at } z_n]$$

$$= 2\pi i \text{ (sum of residues)}$$

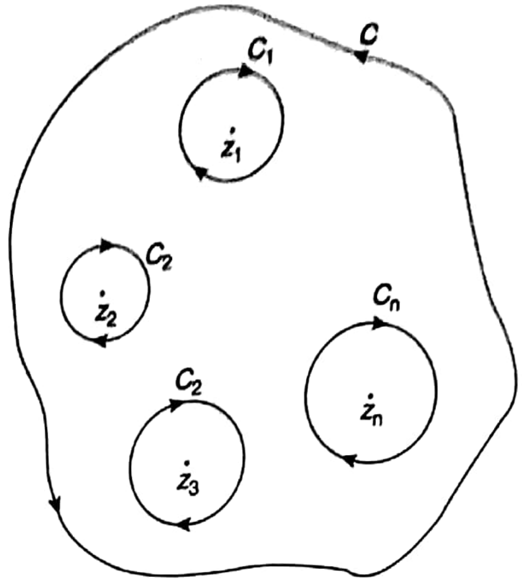

Fig. 14.43 Illustration of Cauchy's residue theorem

EXAMPLE 14.68 *Evaluate* $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, *where* C *is* $|z| = 3$.

Solution:

(i) Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

(ii) C is a circle $|z| = 3$ with centre (0, 0) and radius 3 (Fig. 14.44).

(iii) For $z = 1$, $|z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

(iv) $z = 1$ is a simple pole.

$$\text{Res}[f(z); z = 1] = \lim_{z \to 1} (z-1)\, f(z)$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$= \frac{\sin \pi + \cos \pi}{-1} = 1$$

$z = 2$ is a simple pole.

$$\text{Res}[f(z); z = 2] = \lim_{z \to 2} (z-2)\, f(z)$$

$$= \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1} = 1$$

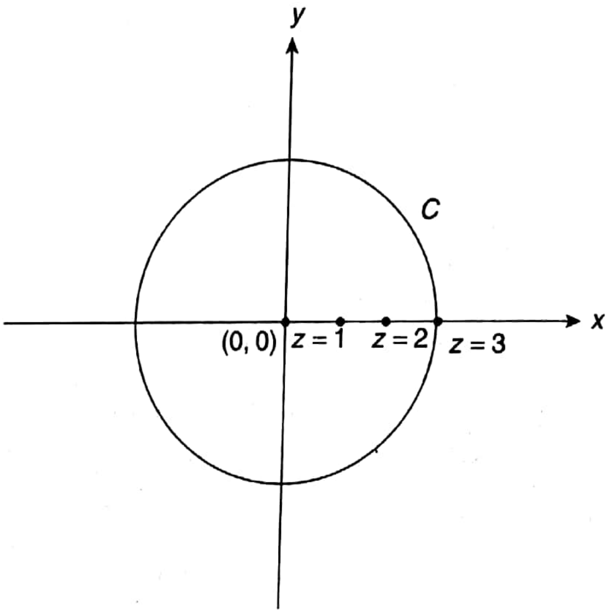

Fig. 14.44

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i(1+1) = 4\pi i$$

EXAMPLE 14.69 *Evaluate* $\int_C \frac{e^{2z}}{(z-\pi i)^3}\,dz$, *where C is* $|z-2i| = 2$.

Solution:

(i) Let $f(z) = \frac{e^{2z}}{(z-\pi i)^3}$

(ii) C is a circle $|z-2i| = 2$ with centre (0, 2) and radius 2 (Fig. 14.45).

(iii) For $z = \pi i, |z-2i| = |\pi i - 2i| = \pi - 2 = 1.14 < 2$

Hence, $z = \pi i$ lies inside C.

(iv) $z = \pi i$ is a pole of order 3.

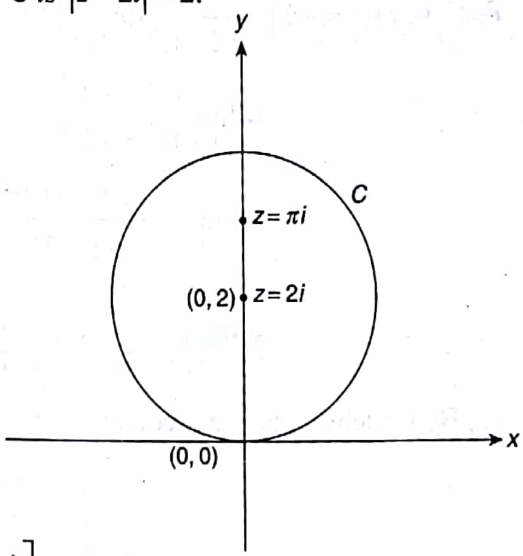

Fig. 14.45

$$\text{Res}\left[f(z); z = \pi i\right] = \frac{1}{(3-1)!}\lim_{z\to\pi i}\frac{d^2}{dz^2}\left[(z-\pi i)^3 f(z)\right]$$

$$= \frac{1}{2!}\lim_{z\to\pi i}\frac{d^2}{dz^2}\left(e^{2z}\right) = \frac{1}{2}\lim_{z\to\pi i}\frac{d}{dz}\left(2e^{2z}\right)$$

$$= \frac{1}{2}\lim_{z\to\pi i} 4e^{2z} = 2e^{2\pi i} = 2(\cos 2\pi + i\sin 2\pi) = 2$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^{2z}}{(z-\pi i)^3}\,dz = 2\pi i(2) = 4\pi i$$

EXAMPLE 14.70 *Evaluate* $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, *where C is* $|z-i| = 2$.

Solution:

(i) Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

(ii) C is a circle $|z-i|=2$ with centre (0, 1) and radius 2 (Fig. 14.46).

(iii) For $z=-1, |z-i|=|-1-i|=\sqrt{2}<2$

Hence, $z=-1$ lies inside C.

For $z=2, |z-i|=|2-i|=\sqrt{5}>2$

Hence, $z=2$ lies outside C.

(iv) $z=-1$ is a pole of order 2.

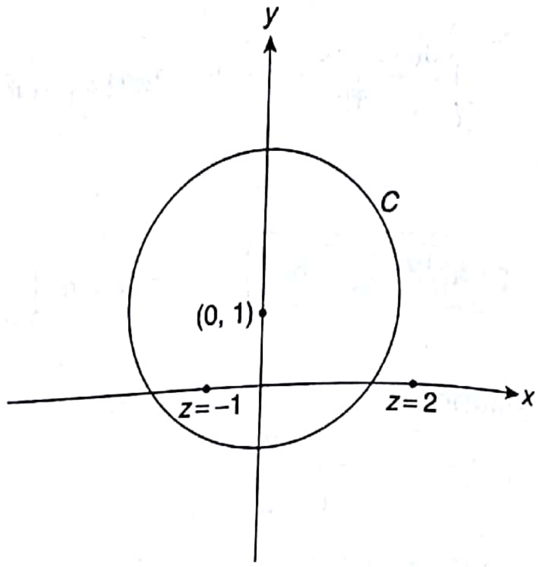

Fig. 14.46

$$\text{Res}[f(z); z=-1] = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z=-1}\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}$$

$$= \lim_{z=-1}\frac{-1}{(z-2)^2} = -\frac{1}{(-1-2)^2} = -\frac{1}{9}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

EXAMPLE 14.71 *Evaluate* $\int_C z^2 e^{\frac{1}{z}}\,dz$, *where* C *is* $|z|=1$.

Solution: Let

$$f(z) = z^2 e^{\frac{1}{z}} = z^2\left(1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\ldots\right)$$

$$= z^2+z+\frac{1}{2}+\frac{1}{6z}+\frac{1}{24z^2}+\ldots$$

$$= z^2+z+\frac{1}{2}+\frac{1}{6}z^{-1}+\frac{1}{24}z^{-2}+\ldots$$

Since number of terms of negative powers of z are infinite, $z=0$ is an essential singularity.

$$\text{Res}[f(z); z=0] = \text{Coefficient of } \frac{1}{z} = \frac{1}{6}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C z^2 e^{\frac{1}{z}}\,dz = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

EXAMPLE 14.72 Evaluate $\int_C \frac{dz}{z \sin z}$, where C is $|z| = 1$.

Solution:

(i) Let $f(z) = \frac{1}{z \sin z}$

The poles are given by

$$z \sin z = 0$$

$$z = 0, \quad \sin z = 0$$

$$z = 0, \ \pm\pi, \pm 2\pi, \ldots$$

(ii) C is a circle $|z| = 1$ with centre (0, 0) and radius 1 (Fig. 14.47).

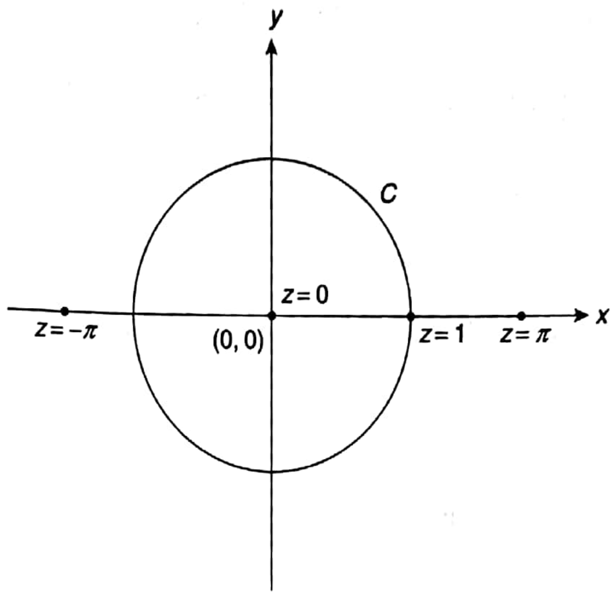

Fig. 14.47

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

Other poles $z = \pm\pi, \pm 2\pi, \ldots$ lies outside C.

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}[f(z); z = 0] = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z \to 0} \frac{d}{dz}\left(\frac{z}{\sin z}\right)$$

$$= \lim_{z \to 0} \frac{(1)\sin z - z(\cos z)}{(\sin z)^2} \qquad \left[\frac{0}{0} \text{form}\right]$$

$$= \lim_{z \to 0} \frac{\cos z - (\cos z - z \sin z)}{2 \sin z \cos z} \qquad \text{[Using L'Hospital's rule]}$$

$$= \lim_{z \to 0} \frac{z \sin z}{2 \sin z \cos z} = \lim_{z \to 0} \frac{z}{2 \cos z} = 0$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{z \sin z} = 2\pi i(0) = 0$$

EXAMPLE 14.73 Evaluate $\int \frac{\cos \pi z}{z^2 - 1} dz$, where C is the rectangle whose vertices are $2 \pm i$, $-2 \pm i$.

Solution:

(i) Let $f(z)=\dfrac{\cos \pi z}{z^2-1}=\dfrac{\cos \pi z}{(z+1)(z-1)}$

(ii) C is a rectangle with vertices $2\pm i$ and $-2\pm i$. (Fig. 14.48).

(iii) The poles $z=\pm 1$ lie inside C.

(iv) $z=1$ is a simple pole.

$$\text{Res}[f(z); z=1]=\lim_{z\to 1}(z-1)\,f(z)$$
$$=\lim_{z\to 1}\frac{\cos \pi z}{z+1}=-\frac{1}{2}$$

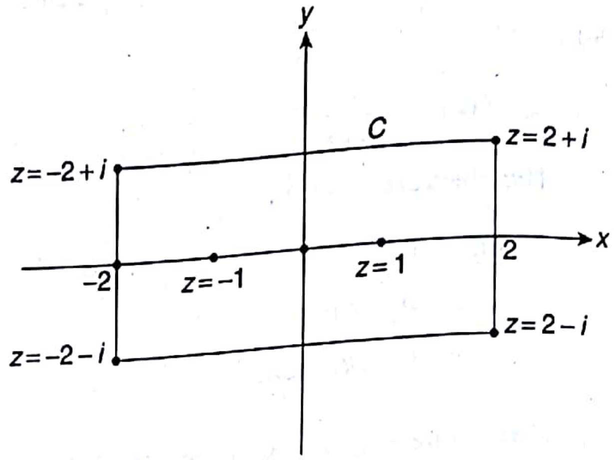

Fig. 14.48

$z=-1$ is a simple pole.

$$\text{Res}[f(z); z=-1]=\lim_{z\to -1}(z+1)\,f(z)=\lim_{z\to -1}\frac{\cos \pi z}{z-1}=\frac{\cos(-\pi)}{-2}$$
$$=\frac{-1}{-2}=\frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz=2\pi i\,(\text{sum of residues})$$
$$\int_C \frac{\cos \pi z}{z^2-1}\,dz=2\pi i\left(-\frac{1}{2}+\frac{1}{2}\right)=0$$

EXERCISE 14.13

Evaluate the following integrals using Cauchy's residue theorem:

1. $\int_C \dfrac{\sin z}{z^6}\,dz$, where C is $|z|=1$

$\left[\textbf{Ans.:}\ \dfrac{\pi i}{60}\right]$

2. $\int_C z\,e^{\frac{1}{z}}\,dz$, where C is $|z|=1$

[**Ans.:** πi]

3. $\int_C \dfrac{z}{(z-1)^2(z+1)}\,dz$, where C is $|z|=\dfrac{3}{4}$

[**Ans.:** 0]

4. $\int_C \dfrac{z}{(z+1)^2(z-2)}\,dz$, where C is $|z-i|=2$

$\left[\textbf{Ans.:}\ -\dfrac{4\pi i}{9}\right]$

5. $\displaystyle\int_C \frac{dz}{(z^2+1)^2}$, where C is $|z-i|=1$

$\left[\textbf{Ans.: } \dfrac{\pi}{2}\right]$

6. $\displaystyle\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}$, where C is $|z|=1$

$\left[\textbf{Ans.: } -\dfrac{16\pi i}{9}\right]$

7. $\displaystyle\int_C \frac{3z^2+2z-4}{z^3-4z}$, where C is $|z-i|=3$

[**Ans.:** $6\pi i$]

8. $\displaystyle\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is $|z+1-i|=2$

[**Ans.:** $\pi(2+i)$]

9. $\displaystyle\int_C \frac{z^2+3}{z^2-1}$, where C is $|z-1|=1$

[**Ans.:** $4\pi i$]

10. $\displaystyle\int_C \frac{15z+9}{z^3-9z}$, where C is $|z-1|=3$

[**Ans.:** 7]

14.18 APPLICATIONS OF RESIDUE THEOREM TO EVALUATE REAL INTEGRALS

The residue theorem can be applied to evaluate the real definite integrals. These real integrals are evaluated by expressing them in terms of complex functions over a suitable contour. The process of evaluation of these integrals is called contour integration.

Type I (Evaluation of Real Definite Integral of Rational Function of cos θ and sin θ)

$$\text{Let } I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$$

where $f(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.
To evaluate the integral putting $z = e^{i\theta}$,

$$dz = i\, e^{i\theta}\, d\theta = i\, z\, d\theta.$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

$$|z| = |e^{i\theta}| = 1$$

Substituting these values in the integral,

$$I = \oint_C f(z)\, dz$$

where C is a unit circle $|z|=1$
By Cauchy's residue theorem,

$$I = 2\pi i \text{ (sum of residues at all poles lying inside the circle } |z|=1)$$

EXAMPLE 14.74 *Evaluate* $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$ *using contour integration.*

Solution:

(i) Let $I = \int_0^{2\pi} \frac{\cos 3\theta}{5--4\cos\theta} d\theta$

Consider the contour C as the unit circle $|z|=1$ (Fig. 14.49).

$z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$z^3 = e^{3i\theta} = \cos 3\theta + i\cos 3\theta$

$\cos 3\theta = \text{R.P } z^3$

(ii) $I = \text{RP}\int_C \frac{z^3 \frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)} = \text{RP}\frac{1}{i}\int_C \frac{z^3 dz}{5z-2z^2-2}$

$= \text{RP}\left(-\frac{1}{i}\right)\int_C \frac{z^3 dz}{2z^2-5z+2}$ ...(1)

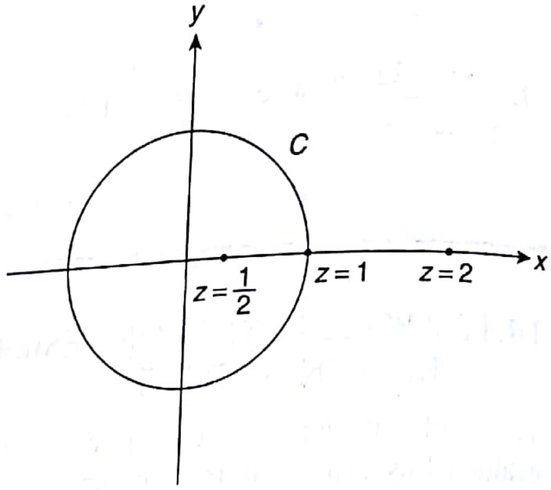

Fig. 14.49

(iii) Let $f(z) = \frac{z^3}{2z^2-5z+2}$

The poles are given by

$$2z^2-5z+2=0$$
$$2z^2-4z-z+2=0$$
$$2z(z-2)-1(z-2)=0$$
$$(z-2)(2z-1)=0$$
$$z=2, \frac{1}{2}$$

$\therefore$ $f(z) = \frac{z^3}{2(z-2)\left(z-\frac{1}{2}\right)}$ [$\because ax^2+bx+c = a(x-\alpha)(x-\beta)$]

(iv) For $z=2, |z|=|2|=2>1$

Hence, $z=2$ lies outside C.

For $z=\frac{1}{2}, |z|=\left|\frac{1}{2}\right|=\frac{1}{2}<1$

Hence, $z=\frac{1}{2}$ lies inside C.

(v) Res $\left[f(z); z=\frac{1}{2}\right] = \lim_{z\to\frac{1}{2}} \left[\left(z-\frac{1}{2}\right)f(z)\right] = \lim_{z\to\frac{1}{2}} \frac{z^3}{2(z-2)} = \frac{\frac{1}{8}}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24}$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i(\text{sum of residues}) = 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

Substituting in Eq. (1),

$$I = \text{R.P}\left(-\frac{1}{i}\right)\left(\frac{-\pi i}{12}\right) = \frac{\pi}{12}$$

EXAMPLE 14.75 *Evaluate* $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ *using contour integration.*

Solution:

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

Consider the contour C as the unit circle $|z|=1$ (Fig. 14.50).

$z=e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $\sin\theta = \frac{z^2-1}{2iz}$,

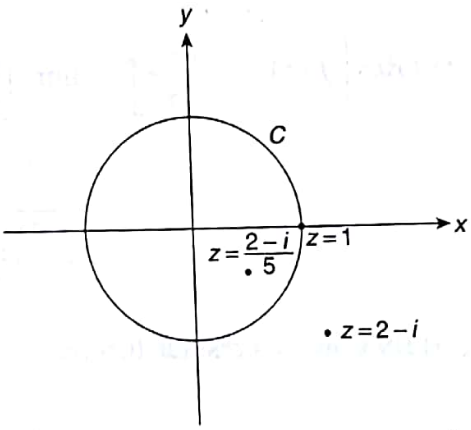

Fig. 14.50

(ii) $I = \int_C \frac{\frac{dz}{iz}}{3-2\left(\frac{z^2+1}{2z}\right)+\left(\frac{z^2-1}{2iz}\right)} = \int_C \frac{\frac{dz}{iz}}{3-\frac{2z^2+2}{2z}+\frac{z^2-1}{2iz}}$

$$= \int_C \frac{2dz}{6iz-(2z^2+2)i+(z^2-1)} = 2\int \frac{dz}{(1-2i)z^2+6iz-(2i+1)} \quad \ldots(1)$$

(iii) Let $f(z) = \frac{1}{(1-2i)z^2+6iz-(2i+1)}$

The poles are given by

$$(1-2i)z^2+6iz-(2i+1)=0$$

$$z = \frac{-6i\pm\sqrt{(6i)^2+4(1-2i)(2i+1)}}{2(1-2i)} = \frac{-6i\pm\sqrt{-36+4(2i+1+4-2i)}}{2(1-2i)}$$

$$= \frac{-6i\pm\sqrt{-36+20}}{2(1-2i)} = \frac{-6i\pm 4i}{2(1-2i)}$$

$$= \frac{-i}{1-2i}, \frac{-5i}{1-2i} = \frac{2-i}{5}, \ 2-i$$

$$\therefore f(z) = \frac{1}{(1-2i)[z-(2-i)]\left[z-\left(\frac{2-i}{5}\right)\right]}$$

(iv) For $z = 2-i, |z| = |2-i| = \sqrt{5} > 1$

Hence, $z = 2-i$ lies outside C.

For $z = \frac{2-i}{5}, |z| = \left|\frac{2-i}{5}\right| = \frac{\sqrt{5}}{5} = 0.45 < 1$

Hence, $z = \frac{2-i}{5}$ lies inside C.

(v) $\text{Res}\left[f(z); z = \frac{2-i}{5}\right] = \lim_{z \to \frac{2-i}{5}} \left[\left(z - \frac{2-i}{5}\right) f(z)\right] = \lim_{z \to \frac{2-i}{5}} \frac{1}{(1-2i)(z-2+i)}$

$$= \frac{1}{(1-2i)\left(\frac{2-i}{5} - 2 + i\right)} = \frac{1+2i}{5\left(\frac{-8+4i}{5}\right)} = \frac{1+2i}{4i(2i+1)} = \frac{1}{4i}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, \mathrm{d}z = 2\pi i \text{ (sum of residues)} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

Substituting in Eq.(1),

$$I = 2\left(\frac{\pi}{2}\right) = \pi$$

EXAMPLE 14.76 *Evaluate* $\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2a\cos\theta + a^2}, (0 < a < 1)$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2a\cos\theta + a^2}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 14.51).

Let $z = e^{i\theta}$, $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$,

(ii) $I = \int_C \frac{\frac{dz}{iz}}{1 - 2a\left(\frac{z^2+1}{2z}\right) + a^2} = \frac{1}{i}\int_C \frac{dz}{z - az^2 - a + a^2 z}$

$$= \frac{1}{i}\int_C \frac{dz}{-az^2 + (a^2+1)z - a}$$

$$= -\frac{1}{i}\int_C \frac{dz}{az^2 - (a^2+1)z + a} \quad \ldots(1)$$

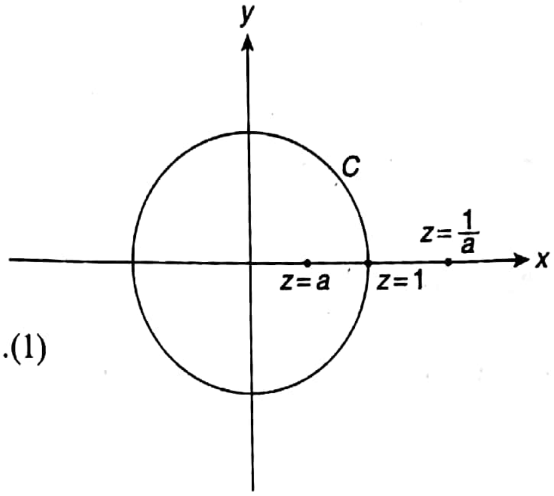

Fig. 14.51

(iii) Let $f(z) = \frac{1}{az^2 - (a^2+1)z + a}$

The poles are given by

$$z^2 - \left(\frac{a^2+1}{a}\right)z + 1 = 0$$

$$z = \frac{a^2+1 \pm \sqrt{(a^2+1)^2 - 4a^2}}{2a} = \frac{a^2+1 \pm \sqrt{a^2 + 2a^2 + 1 - 4a^2}}{2a}$$

$$= \frac{a^2+1 \pm \sqrt{(a^2-1)^2}}{2a} = \frac{a^2+1 \pm (a^2-1)}{2a} = a, \frac{1}{a}$$

$$\therefore f(z) = \frac{1}{a(z-a)\left(z - \frac{1}{a}\right)}$$

(iv) For $z = a$,

$|z| = |a| = a < 1$ [Given $0 < a < 1$]

Hence, $z = a$ lies inside C.

For $z = \frac{1}{a}$,

$$|z| = \left|\frac{1}{a}\right| = \frac{1}{a} > 1$$

Hence, $z = \frac{1}{a}$ lies outside C.

(v) $\text{Res}\left[f(z); z = a\right] = \lim_{z \to a} \left[(z-a) f(z)\right] = \lim_{z \to a} \frac{1}{a\left(z - \frac{1}{a}\right)} = \frac{1}{a\left(a - \frac{1}{a}\right)} = \frac{1}{a^2 - 1}$

(vi) By Cauchy's residue theorem,

$$\int_c f(z)\, dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left(\frac{1}{a^2-1}\right)$$

Substituting in Eq.(1),

$$I = -\frac{1}{i}2\pi i\left(\frac{1}{a^2-1}\right) = -\frac{2\pi}{a^2-1} = \frac{2\pi}{1-a^2}, 0 < a < 1$$

EXERCISE 14.14

Using contour integration, evaluate the following integrals:

1. $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ $\left[\textbf{Ans.:} \frac{2\pi}{3}\right]$

2. $\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta} d\theta$ $[\textbf{Ans.:} 0]$

3. $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$ $\left[\textbf{Ans.:} \frac{\pi}{6}\right]$

4. $\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta} d\theta$ $\left[\textbf{Ans.:} \frac{2\pi}{3}\right]$

5. $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$ $\left[\textbf{Ans.:} \frac{\pi}{2}\right]$

6. $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, \quad a > b > 0$ $\left[\textbf{Ans.:} \frac{2\pi}{\sqrt{a^2-b^2}}\right]$

7. $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}, \quad |a| < 1$ $\left[\textbf{Ans.:} \frac{2\pi}{\sqrt{1-a^2}}\right]$

8. $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, \quad a > b > 0$ $\left[\textbf{Ans.:} \frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}\right]$

Type II (Evaluation of Improper Real Integral of Rational Function)

(a) Let $I = \int_{-\infty}^{\infty} f(x)dx$

where $f(x) = \frac{P(x)}{Q(x)}$ and the degree of $Q(x)$ is greater than degree of $P(x)$ by at least 2 and $Q(x)$ has no real roots.

To evaluate the integral, consider $\int_C f(z)dz$, where C is the contour consisting of the upper semicircle C_1 of radius R with centre at the origin and the part of real axis from $-R$ to R (Fig. 14.52).

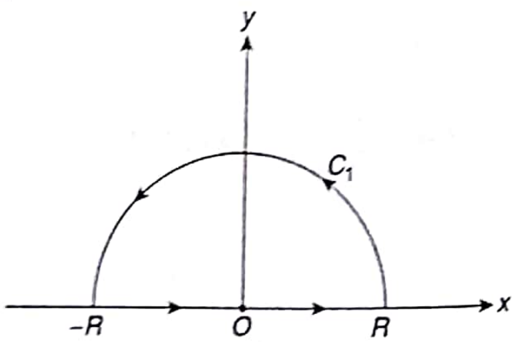

Fig. 14.52

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{-R}^{R} f(x)dx$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

$$\int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

Taking limit $R \to \infty$,

$$\lim_{R\to\infty} \int_{C_1} f(z)\,dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues at poles inside } C\text{)} \quad \left[\because \text{ by Cauchy's lemma, } \lim_{R\to\infty} \int_{C_1} f(z)\,dz = 0\right]$$

(b) Let $I = \int_{-\infty}^{\infty} \frac{\cos mx}{Q(x)}\,dx$ or $\int_{-\infty}^{\infty} \frac{\sin mx}{Q(x)}\,dx$

where $Q(x)$ is a polynomial in x.

To evaluate the integral I, consider $\int_C e^{imz} f(z)\,dz$.

where $f(z) = \frac{1}{Q(z)}$ and C is the contour consisting the upper semicircle C_1 of radius R with centre at the origin and the part of the real axis from $-R$ to R (Fig. 14.53).

$$\int_C e^{imz} f(z)\,dz = \int_{-R}^{R} e^{imx} f(x)\,dx + \int_{C_1} e^{imz} f(z)\,dz$$

By Cauchy's residue theorem,

$$\int_C e^{imz} f(z)\,dz = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

$$\int_{-R}^{R} e^{imx} f(x)\,dx + \int_{C_1} e^{imz} f(z)\,dz = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

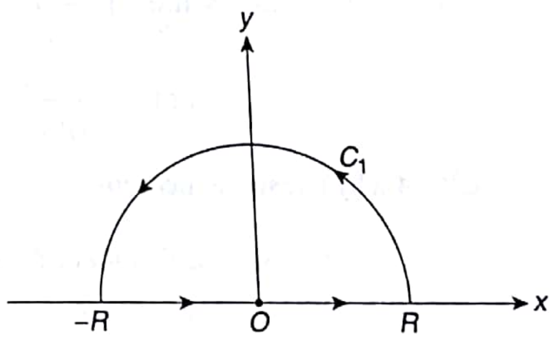

Fig. 14.53

Taking limit $R \to \infty$,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x)\,dx + \lim_{R\to\infty} \int_{C_1} e^{imz} f(z)\,dz = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

$$\int_{-\infty}^{\infty} e^{imx} f(x)\,dx = 2\pi i \text{ (sum of residues at poles inside } C\text{)}$$

$$\left[\because \text{ by Cauchy s lemma, } \lim_{R\to\infty} \int_{C_1} e^{imz} f(z)\,dz = 0\right]$$

EXAMPLE 14.77 *Evaluate* $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$ *using contour integration.*

Solution:

(i) Let $\int_C f(z)\,dz = \int_C \frac{z^2}{(z^2+1)(z^2+4)}dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and part of the real axis from $-R$ to R (Fig. 14.54).

(ii) Let $f(z)=\frac{z^2}{(z^2+1)(z^2+4)}$

The poles are given by

$(z^2+1)(z^2+4)=0$

$$z=\pm i, \pm 2i,$$

$$\therefore f(z)=\frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$$

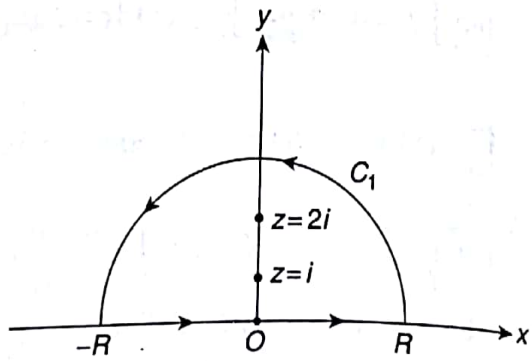

Fig. 14.54

(iii) The poles $z=i$ and $z=2i$ lie inside C.

(iv) $\text{Res}\left[f(z); z=i\right] = \lim_{z\to i}\left[(z-i)f(z)\right] = \lim_{z\to i}\frac{z^2}{(z+i)(z+2i)(z-2i)} = \frac{i^2}{(2i)(3i)(-i)} = -\frac{1}{6i}$

$$\text{Res}\left[f(z); z=2i\right] = \lim_{z\to 2i}\left[(z-2i)f(z)\right]$$

$$= \lim_{z\to 2i}\frac{z^2}{(z+i)(z-i)(z+2i)} = \frac{4i^2}{(3i)(i)(4i)} = \frac{1}{3i}$$

(v) By Cauchy's residue theorem,

$$\int_c f(z)\,dz = 2\pi i\,(\text{sum of residues}) = 2\pi i\left(-\frac{1}{6i}+\frac{1}{3i}\right) = 2\pi\left(\frac{1}{6}\right) = \frac{\pi}{3}$$

(vi) $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi}{3}$

Taking limit $R\to\infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{C_1} f(z)dz = \frac{\pi}{3}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{3} \qquad \left[\because \lim_{R\to\infty}\int_{C_1} f(z)dz = 0\right]$$

$$\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$

EXAMPLE 14.78 Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^2}$ using contour integration.

Solution:

(i) Let $\int_C f(z)dz = \int_C \frac{dz}{(1+z^2)^2}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of real axis from $-R$ to R (Fig. 14.55).

(ii) Let $f(z) = \frac{1}{(1+z^2)^2}$

The poles are given by

$$(1+z^2)^2 = 0$$
$$1+z^2 = 0$$
$$z^2 = -1$$
$$z = \pm i$$

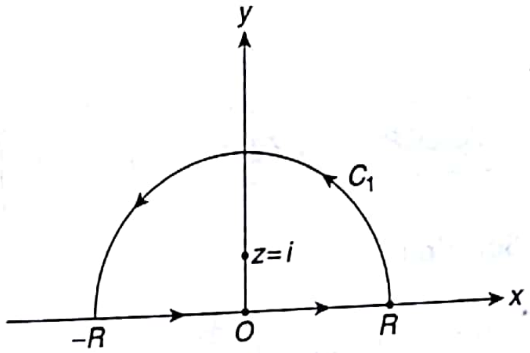

Fig. 14.55

$$\therefore f(z) = \frac{1}{(z+i)^2(z-i)^2}$$

(iii) The pole $z = i$ of order 2 lies inside C.

(iv) $\text{Res}[f(z); z=i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)] = \lim_{z\to i}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$

$$= \lim_{z\to i}-\left[\frac{2}{(z+i)^3}\right] = -\frac{2}{(i+i)^3} = \frac{2}{8i} = \frac{1}{4i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues}) = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

(vi) $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi}{2}$

Taking limit $R \to \infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{C_1} f(z)dz = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2} \qquad \left[\because \lim_{R\to\infty}\int_{C_1} f(z)dz = 0\right]$$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{2}$$

$$2\int_{0}^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{2} \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{4}$$

EXAMPLE 14.79 *Show that* $\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Solution:

(i) Let $\int_C f(z)\,dz = \int_C \frac{dz}{1+z^4}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of real axis from $-R$ to R (Fig. 14.56).

(ii) Let $f(z) = \frac{1}{1+z^4}$

The poles are given by

$$1+z^4 = 0$$

$$z^4 = -1$$

$$z = (-1)^{\frac{1}{4}}$$

$$= e^{\frac{i\pi}{4}}, e^{-\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{-\frac{i3\pi}{4}}$$

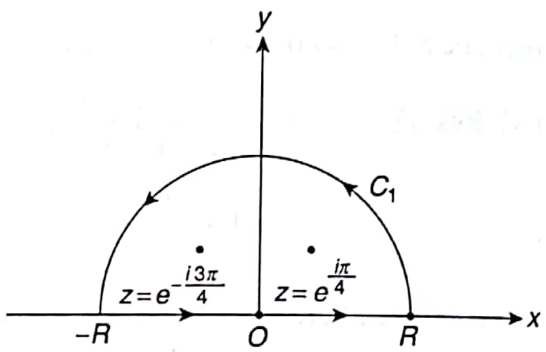

Fig. 14.56

(iii) The poles $z = e^{\frac{i\pi}{4}}$ and $z = e^{\frac{i3\pi}{4}}$ lie inside C.

(iv) $\text{Res}\,[f(z); z = e^{i\frac{\pi}{4}}] = \lim_{z\to e^{i\frac{\pi}{4}}} (z - e^{i\frac{\pi}{4}}) f(z) = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z - e^{i\frac{\pi}{4}}}{1+z^4} \quad \left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{4z^3} = \frac{1}{4e^{3i\frac{\pi}{4}}} = \frac{1}{4}e^{-i3\frac{\pi}{4}} = \frac{1}{4}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\frac{(1+i)}{4\sqrt{2}}$$

$$\text{Res}\left[f(z); z = e^{\frac{3i\pi}{4}}\right] = \lim_{z \to e^{\frac{i\pi}{4}}} \left(z - e^{\frac{3i\pi}{4}}\right) f(z) = \lim_{z \to e^{\frac{3i\pi}{4}}} \frac{z - e^{i\frac{3\pi}{4}}}{1 + z^4} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{4z^3} \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}} = \frac{1}{4} e^{-i\frac{9\pi}{4}} = \frac{1}{4}\left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{1-i}{4\sqrt{2}}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left[-\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right] = 2\pi i\left(-\frac{2i}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

(vi) $\int_C f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz = \frac{\pi}{\sqrt{2}}$

Taking limit R $\to \infty$,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_1} f(z)\,dz = \frac{\pi}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{2}} \qquad \left[\because \lim_{R\to\infty} \int_{C_1} f(z)\,dz = 0\right]$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

EXAMPLE 14.80 *Evaluate* $\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\,dx$, $a > 0$, $m > 0$ *using contour integration.*

Solution:

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{imz}}{z^2 + a^2}\,dz$

Consider the contour C consisting of upper semicircle C_1 of radius R and the part of real axis from $-R$ to R (Fig. 14.57).

(ii) Let $f(z)=\dfrac{e^{imz}}{z^2+a^2}$

The poles are given by

$$z^2+a^2=0$$
$$z=\pm ai$$
$$\therefore f(z)=\frac{e^{imz}}{(z+ai)(z-ai)}$$

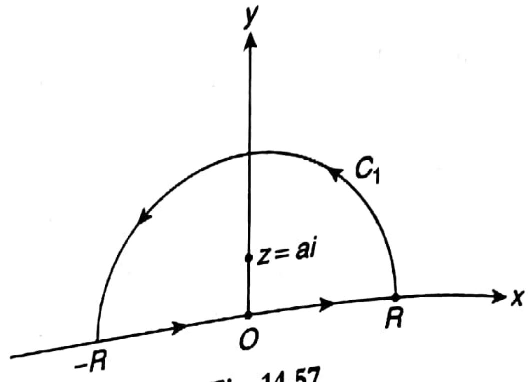

Fig. 14.57

(iii) The pole $z=ai$ lies inside C.

(iv) $\text{Res}\,[f(z); z=ai]=\lim\limits_{z\to ai}(z-ai)f(z)=\lim\limits_{z\to ai}\dfrac{e^{imz}}{(z+ai)}=\dfrac{e^{im(ai)}}{(ai+ai)}=\dfrac{e^{-ma}}{2ai}$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz=2\pi i\text{ (sum of residues)}=2\pi i\left(\frac{e^{-ma}}{2ai}\right)=\frac{\pi e^{-ma}}{a}$$

(vi) $\displaystyle\int_C f(z)\,dz=\int_{-R}^{R} f(x)\,dx+\int_{C_1} f(z)\,dz=\frac{\pi e^{-ma}}{a}$

Taking limit $R\to\infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx+\lim_{R\to\infty}\int_{C_1} f(z)\,dz=\frac{\pi e^{-ma}}{a}$$

$$\int_{-\infty}^{\infty} f(x)\,dx=\frac{\pi e^{-ma}}{a}\qquad\left[\because \lim_{R\to\infty}\int_{C_1} f(z)\,dz=0\right]$$

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}=\frac{\pi e^{-ma}}{a}$$

$$\int_{-\infty}^{\infty}\frac{(\cos mx+i\sin mx)}{x^2+a^2}\,dx=\frac{\pi e^{-ma}}{a}$$

Comparing real parts,

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}\,dx=\frac{\pi e^{-ma}}{a}$$

$$2\int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx=\frac{\pi e^{-ma}}{a}\qquad\left[\because \int_{-a}^{a} f(x)\,dx=2\int_0^{a} f(x)\,dx \text{ if } f(-x)=f(x)\right]$$

$$\int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx=\frac{\pi e^{-ma}}{2a}$$

EXAMPLE 14.81 Evaluate $\int_0^\infty \frac{x \sin mx}{x^2 + a^2} \mathrm{d}x, \quad a > 0, m > 0.$

Solution:

(i) Let $\int_C f(z)\mathrm{d}z = \int_C \frac{ze^{imz}}{z^2 + a^2}\mathrm{d}z$

Consider the contour C consisting of upper semicircle C_1 of radius R and the part of real axis from $-R$ to R (Fig. 14.58).

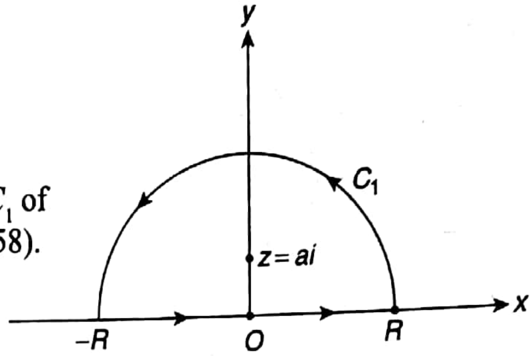

Fig. 14.58

(ii) Let $f(z) = \frac{ze^{imz}}{z^2 + a^2}$

The poles are given by

$$z^2 + a^2 = 0$$

$$z = \pm ai$$

$$\therefore f(z) = \frac{ze^{imz}}{(z+ai)(z-ai)}$$

(iii) The pole $z = ai$ lies inside C.

(iv) $\operatorname{Res}\left[f(z); z = ai\right] = \lim_{z \to ai}(z - ai)f(z) = \lim_{z \to ai}\frac{ze^{imz}}{z + ai} = \frac{aie^{i^2 ma}}{ai + ai} = \frac{aie^{-ma}}{2ai} = \frac{e^{-ma}}{2}$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i(\text{sum of residues}) = 2\pi i\left(\frac{e^{-ma}}{2}\right) = \pi i e^{-ma}$$

(vi) $\int_C f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\mathrm{d}x + \int_{C_1} f(z)\mathrm{d}z = \pi i e^{-ma}$

Taking limit $R \to \infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\mathrm{d}x + \lim_{R\to\infty}\int_{C_1} f(z)\mathrm{d}z = \pi i e^{-ma}$$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \pi i e^{-ma} \qquad \left[\because \lim_{R\to\infty}\int_{C_1} f(z)\mathrm{d}z = 0\right]$$

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2}\mathrm{d}x = \pi i e^{-ma}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos mx + i\sin mx)}{x^2 + a^2}\mathrm{d}x = \pi i e^{-ma}$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2 + a^2}\mathrm{d}x = \pi e^{-ma}$$

$$2\int_0^\infty \frac{x\sin mx}{x^2 + a^2}\mathrm{d}x = \pi e^{-ma} \qquad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x) \text{ if } f(-x) = f(x)\right]$$

$$\int_0^\infty \frac{x\sin mx}{x^2 + a^2}\mathrm{d}x = \frac{\pi e^{-ma}}{2}$$

EXAMPLE 14.82 Evaluate $\int_0^\infty \frac{\cos mx}{(1+x^2)^2}dx,\ m>0$ using contour integration.

Solution:

(i) Let $\int_C f(z)dz = \int_C \frac{e^{imz}}{(1+z^2)^2}dz$

Consider the contour C consisting of the upper semicircle of radius R and the part of the real axis from $-R$ to R (Fig. 14.59).

(ii) $f(z) = \frac{e^{imz}}{(1+z^2)^2}$

The poles are given by

$$(1+z^2)^2 = 0$$
$$1+z^2 = 0$$
$$z = \pm i$$

$$\therefore f(z) = \frac{e^{imz}}{(z+i)^2(z-i)^2}$$

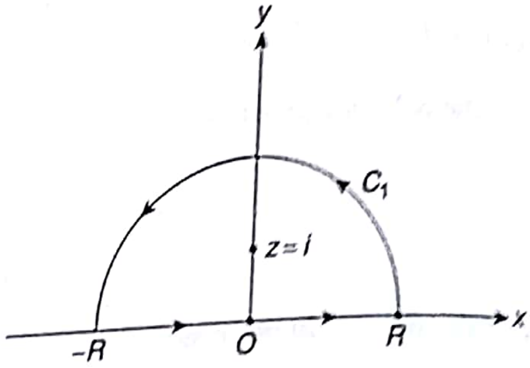

Fig. 14.59

(iii) The pole $z=i$ of order 2 lies inside C.

(iv) $\text{Res}[f(z); z=i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)] = \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{imz}}{(z+i)^2}\right]$

$$= \lim_{z\to i}\frac{(z+i)^2 e^{imz}\cdot im - e^{imz}2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{[im(z+i)-2]e^{imz}}{(z+i)^3} = \frac{[(i+i)im-2]e^{i^2m}}{(i+i)^3} = \frac{(-2m-2)e^{-m}}{-8i} = \frac{(m+1)e^{-m}}{4i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i(\text{sum of residues}) = 2\pi i\frac{(m+1)e^{-m}}{4i} = \frac{\pi(m+1)e^{-m}}{2}$$

(vi) $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi(m+1)e^{-m}}{2}$

Taking limit $R\to\infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{C_1} f(z)dz = \frac{\pi(m+1)e^{-m}}{2}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi(m+1)e^{-m}}{2} \qquad \left[\because \lim_{R\to\infty}\int_{C_1} f(z)dz = 0\right]$$

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{(1+x^2)^2}dx = \frac{\pi(m+1)e^{-m}}{2}$$

$$\int_{-\infty}^{\infty}\frac{\cos mx + i\sin mx}{(1+x^2)^2}dx = \frac{\pi(m+1)e^{-m}}{2}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2} \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{4}$$

EXERCISE 14.15

1. Evaluate the following integrals using contour integration:

(i) $\displaystyle\int_{0}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)}$ $\left[\textbf{Ans.: } \dfrac{\pi}{10}\right]$

(ii) $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a>b>0$ $\left[\textbf{Ans.: } \dfrac{\pi}{2ab(a+b)}\right]$

(iii) $\displaystyle\int_{0}^{\infty} \frac{dx}{x^4+10x^2+9}$ $\left[\textbf{Ans.: } \dfrac{\pi}{24}\right]$

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ $\left[\textbf{Ans.: } \dfrac{3\pi}{8}\right]$

(v) $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2}, a>0$ $\left[\textbf{Ans.: } \dfrac{\pi}{4a^3}\right]$

(vi) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4}\,dx$ $\left[\textbf{Ans.: } \dfrac{5\pi}{6}\right]$

(vii) $\displaystyle\int_{0}^{\infty} \frac{x^2}{(x^2+1)^3}\,dx$ $\left[\textbf{Ans.: } \dfrac{\pi}{6}\right]$

(viii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx,\ a>0$ $\left[\textbf{Ans.: } \dfrac{3\pi}{2a}\right]$

(ix) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)}\,dx$ $\left[\textbf{Ans.: } \dfrac{3\pi}{5}\right]$

(x) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+25)}$ $\left[\textbf{Ans.: } \dfrac{\pi}{8}\right]$

2. Evaluate the following integrals using contour integration:

(i) $\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2} dx, a>0, m>0$

$\left[\textbf{Ans.: } \frac{\pi}{4a^3}(1+ma)e^{-ma}\right]$

(ii) $\int_0^\infty \frac{\cos x}{1+x^2} dx$

$\left[\textbf{Ans.: } \frac{\pi}{e}\right]$

(iii) $\int_{-\infty}^\infty \frac{x \sin x}{x^2+2x+2} dx$

$\left[\textbf{Ans.: } \frac{\pi}{e}(\sin 1 + 1)\right]$

(iv) $\int_{-\infty}^\infty \frac{\cos x}{x^4+5x^2+4} dx$

$\left[\textbf{Ans.: } \pi\left(\frac{1}{3e} - \frac{1}{6e^2}\right)\right]$

(v) $\int_{-\infty}^\infty \frac{\sin x}{x^2+4x+5} dx$

$\left[\textbf{Ans.: } -\frac{\pi}{e}\sin 2\right]$

(vi) $\int_{-\infty}^\infty \frac{\cos ax}{x^4+10x^2+9} dx, a>0$

$\left[\textbf{Ans.: } \frac{\pi}{8}\left(e^{-a} - \frac{e^{-3a}}{3}\right)\right]$

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