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Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown frunction of two or more independent variables is known as a *partial differential equation*.

For examples of partial differential equations we list the following:

$\partial z/\partial x + \partial z/\partial y = z + xy$ ... (1) $\quad (\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x)$ ... (2)

$z(\partial z/\partial x) + \partial z/\partial y = x$ ... (3) $\quad \partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz$ ... (4)

$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2}$ ... (5) $\quad y\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\} = z(\partial z/\partial y)$ ... (6)

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$p = \partial z/\partial x$, $\quad q = \partial z/\partial y$, $\quad r = \partial^2 z/\partial x^2$, $\quad s = \partial^2 z/\partial x \partial y$ $\quad$ and $\quad t = \partial^2 z/\partial y^2$

In case there are n independent variables, we take them to be $x_1, x_2, \ldots, x_n$ and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z/\partial x_1$, $\quad p_2 = \partial z/\partial x_2$, $\quad p_3 = \partial z/\partial x_3$, and $\quad p_n = \partial z/\partial x_n$.

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x\partial y$ and so on.

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 2001; 2004]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $\quad yx^2 p + xy^2 q = xyz + x^2 y^3 \quad$ and $\quad p + q = z + xy$

are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $\quad P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $\quad xyp + x^2yq = x^2y^2z^2 \quad$ and $\quad yp + xq = (x^2z^2/y^2)$

are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $\quad x^2zp + y^2zp = xy \quad$ and $\quad (x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $\quad p^2 + q^2 = 1, \quad p\, q = z \quad$ and $\quad x^2 p^2 + y^2 q^2 = z^2$

are all non-linear partial differential equations.

1.8 Origin of partial differential equations. We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $\quad F(x, y, z, a, b) = 0, \quad$ ...(1)

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\partial F/\partial x + p(\partial F/\partial z) = 0 \quad \text{and} \quad \partial F/\partial y + q(\partial F/\partial z) = 0 \quad \text{...(2)}$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \quad \text{...(3)}$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0, \qquad \partial F/\partial x = 0 \qquad \text{and} \qquad \partial F/\partial y = 0.$$

The equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider

$$z = ax + y, \qquad \ldots (1)$$

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $\qquad \partial z/\partial x = a \qquad \ldots (2)$

Differentiating (1) partially w.r.t. 'y', we get $\qquad \partial z/\partial y = 1 \qquad \ldots (3)$

Eliminating a between (1) and (2) yields $\qquad z = x(\partial z/\partial x) + y \qquad \ldots (4)$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $\qquad az + b = a^2x + y \qquad \ldots (1)$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$a(\partial z/\partial x) = a^2 \quad \ldots (2) \qquad \text{and} \qquad a(\partial z/\partial y) = 1 \quad \ldots (3)$$

Eliminating a from (2) and (3), we have $\qquad (\partial z/\partial x)(\partial z/\partial y) = 1,$

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $\qquad z = ax + by + cxy \qquad \ldots (1)$

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$$\partial z/\partial x = a + cy \quad \ldots (2) \qquad \partial z/\partial y = b + cx \quad \ldots (3)$$

From (2) and (3), $\qquad \partial^2 z/\partial x^2 = 0, \qquad \partial^2 z/\partial y^2 = 0 \qquad \ldots (4)$

and $\qquad \partial^2 z/\partial x\partial y = c \qquad \ldots (5)$

Now, (2) and (3) $\Rightarrow \qquad x(\partial z/\partial x) = ax + cxy \qquad \text{and} \qquad y(\partial z/\partial y) = by + cxy$

$\therefore \qquad x(\partial z/\partial x) + y(\partial z/\partial y) = ax + by + cxy + cxy$

or $\qquad x(\partial z/\partial x) + y(\partial z/\partial y) = z + xy(\partial^2 z/\partial x\partial y)$, using (1) and (5). $\qquad \ldots (6)$

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

1.10 SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $\qquad z = ax + by + a^2 + b^2. \qquad \ldots (1)$

Differentiating (1) partially with respect to x and y, we get

NIT $\quad \partial z/\partial x = a \qquad \text{and} \qquad \partial z/\partial y = b.$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from $z = (x - a)^2 + (y - b)^2$ to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $\quad z = (x - a)^2 + (y - b)^2.$ ...(1)

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x - a)$ and $\partial z/\partial y = 2(y - b).$

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4[(x - a)^2 + (y - b)^2]$$

or $\quad (\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z,$ using (1).

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(*a*) $z = a(x + y) + b.$ (*b*) $z = ax + by + ab.$ **[Bhopal 2010, Rewa 1996**

(*c*) $z = ax + a^2y^2 + b.$ **[Agra 2010]** (*d*) $z = (x + a)(y + b).$ **[Lucknow 2011; Madras 2012**

Sol. (*a*) Given $\quad z = a(x + y) + b$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = a.$

Eliminating a between these, we get $\quad \partial z/\partial x = \partial z/\partial y,$

which is the required partial differential equation.

(*b*) Given $\quad z = ax + by + ab.$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$ ...(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/dx)(\partial z/\partial y),$$

which is the required partial differential equation.

(*c*) Try yourself. **Ans.** $\partial z/\partial y = 2y(\partial z/\partial x)^2.$

(*d*) Try yourself. **Ans.** $z = (\partial z/\partial y)(\partial z/\partial x).$

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b.$ **[Meerut 2006]**

Sol. Given $\quad z = axe^y + (1/2) \times a^2e^y + b.$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \qquad ...(2)$$

and $\quad \partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2.$ ...(3)

Substituting the value of ae^y from (2) in (3), we get $\quad \partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2.$

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$ **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $\quad (x - h)^2 + (y - k)^2 + z^2 = \lambda^2.$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$2(x - h) + 2z(\partial z/\partial x) = 0$ or $(x - h) = -z(\partial z/\partial x)$ ...(2)

and $2(y - k) + 2z(\partial z/\partial y) = 0$ or $(y - k) = -z(\partial z/\partial y).$ ...(3)

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2$ or $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2,$

which is the required partial differential equation.

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Ex. 5(b). *Find the differential equation of all spheres of radius λ, having centre in the xy-plane.* **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$$(x-h)^2+(y-k)^2+(z-0)^2=\lambda^2 \quad \text{or} \quad (x-h)^2+(y-k)^2+z^2=\lambda^2 \qquad \text{...(1)}$$

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z=(x^2+a)(y^2+b)$.

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[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $z=(x^2+a)(y^2+b)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2x(y^2+b) \quad \text{or} \quad (y^2+b)=(1/2x)\times(\partial z/\partial x) \qquad \text{...(2)}$$

and

$$\partial z/\partial y = 2y(x^2+a) \quad \text{or} \quad (x^2+a)=(1/2y)\times(\partial z/\partial y). \qquad \text{...(3)}$$

Substituting the values of (y^2+b) and (x^2+a) from (2) and (3) in (1) gives

$$z=(1/2y)\times(\partial z/\partial y)\times(1/2x)\times(\partial z/\partial x) \quad \text{or} \quad 4xyz=(\partial z/\partial x)(\partial z/\partial y),$$

which the required partial differential equation.

Ex. 7. *Form differential equation by eliminating constants A and p from* $z=A\,e^{pt}\sin px$.

Sol. Given $z=A\,e^{pt}\sin px$. ...(1)

Differentiating (1) partially with respect to x and t, we get

$$\partial z/\partial x = Ap\,e^{pt}\cos px \quad \text{...(2)} \qquad \partial z/\partial t = Ap\,e^{pt}\sin px. \quad \text{...(3)}$$

Differentiating (2) and (3) partially with respect to x and t respectively gives

$$\partial^2 z/\partial x^2 = -Ap^2e^{pt}\sin px. \quad \text{...(4)} \qquad \partial^2 z/\partial t^2 = Ap^2e^{pt}\sin px. \quad \text{...(5)}$$

Adding (4) and (5), $\partial^2 z/\partial x^2+\partial^2 z/\partial t^2=0$,

which is the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2+y^2=(z-c)^2\tan^2\alpha, \qquad \text{... (1)}$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$$2x=2(z-c)(\partial z/\partial x)\tan^2\alpha \quad \text{and} \quad 2y=2(z-c)(\partial z/\partial y)\tan^2\alpha$$

$$\Rightarrow\ y(z-c)(\partial z/\partial x)\tan^2\alpha=xy \quad \text{and} \quad x(z-c)(\partial z/\partial y)\tan^2\alpha=xy$$

$$\Rightarrow\ y(z-c)(\partial z/\partial x)\tan^2\alpha=x(z-c)(\partial z/\partial y)\tan^2\alpha$$

Thus, $y(\partial z/\partial x)=x(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is* $px+qy=z$. *Verify that* $yz+zx+xy=0$ *is a surface satisfying the above equation.* **[I.A.S. 1979, 2009]**

Sol. The equation of any cone with vertex at origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \qquad \text{...(1)}$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p=\partial z/\partial x$ and $q=\partial z/\partial y$)

$$2ax+2czp+2fyp+2g(px+z)+2hy=0 \quad \text{or} \quad ax+gz+hy+p(cz+gx+fy)=0 \qquad \text{...(2)}$$

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $a = \dfrac{1 + (\partial z/\partial y)}{z}$ ... (4)

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1+(\partial z/\partial y)}{z(\partial z/\partial y)} \cdot \frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

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****Ex. 13.** *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2013, Delhi B.A. (Prog.) II, 2014; Ravishanker 2010]

Sol. Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad \text{or} \quad c^2x + a^2z\frac{dz}{dx} = 0 \qquad ... (2)$$

and

$$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2z\frac{\partial z}{\partial y} = 0. \qquad ... (3)$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \quad ... (4) \qquad c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0. \quad ... (5)$$

From (2), $c^2 = -(a^2z/x) \times (\partial z/\partial x)$ ... (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0. \qquad ... (7)$$

Similarly, from (3) and (5), $zy\dfrac{\partial^2 z}{\partial y^2} + y\left(\dfrac{\partial z}{\partial y}\right)^2 - z\dfrac{\partial z}{\partial y} = 0.$ ... (8)

Differentiating (2) partially w.r.t. y, $0 + a^2\left\{(\partial z/\partial y)(\partial z/\partial x) + z(\partial^2 z/\partial x\partial y)\right\} = 0$

r $(\partial z/\partial x)(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0$ ... (9)

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 14. *Find the partial differential equation of all planes which are at a constant distance* *'from the origin.*

Sol. Let $lx + my + nz = a$ ... (1)

the equation of the given plane where l, m, n are direction colines of the normal to the plane so at $l^2 + m^2 + n^2 = 1$, l, m, n being parameters ... (2)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$l + np = 0$ ...(3) $m + nq = 0$, ... (4)

here $p = \partial z/\partial x$ and $q = \partial z/\partial y$. From (3) and (4), $l = -np$ and $m = -nq$. Substituting these lues in (2), we have

$n^2(p^2 + q^2 + 1) = 1$ so that $n = (p^2 + q^2 + 1)^{-1/2}$ ... (5)

$\therefore\ l = -np = -p(p^2 + q^2 + 1)^{-1/2}$ and $m = -nq = -q(p^2 + q^2 + 1)^{-1/2}$ ... (6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

WORKED OUT EXAMPLES

Elimination of arbitrary constants

Form (obtain) partial differential equation by eliminating the arbitrary constants/functions:

Example 1: $z = ax^2 + by^2$

Solution: Differentiating partially w.r.t. x and y, we get

$$z_x = 2ax, z_y = 2by \quad \text{or} \quad a = \frac{z_x}{2x}, b = \frac{z_y}{2y}.$$

Eliminating the two arbitrary constants a and b

$z=\frac{z_x}{2x}\cdot x^2+\frac{z_y}{2y}\cdot y^2$ or $2z=xz_x+yz_y=xp+yq$

Example 2: $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$ where α is a parameter.

Solution: Differentiating partially w.r.t. x and y, we get

$$2(x-a)+0=2z\cdot z_x\cdot\cot^2\alpha$$

$$2\cdot 0+2(y-b)=2z\,z_y\cot^2\alpha$$

Substituting $(zz_x\cot^2\alpha)^2+(zz_y\cot^2\alpha)^2=z^2\cot^2\alpha$

$$p^2+q^2=\tan^2\alpha$$

Example 3: Find the differential equation of all spheres whose centres lie on the z-axis.

Solution: Equation $x^2+y^2+(z-a)^2=b^2$ where a and b are arbitrary constants.

Differentiating

$$2x+0+2(z-a)z_x=0 \quad (1)$$

$$2y+2(z-a)z_y=0 \quad (2)$$

From (2),

$$(z-a)=-\frac{y}{z_y} \quad (3)$$

Substituting (3) in (1)

$$x+\left(-\frac{y}{z_y}\right)\cdot z_x=0$$

$$xz_y-yz_x=0$$

or $$xq-yp=0.$$

Example 4: $ax+by+cz=1$

Solution: Differentiating w.r.t. x, $a+0+cz_x=0$. Differentiating again w.r.t. x, $0+cz_{xx}=0$, since $c\neq 0$, $z_{xx}=0$. Similarly by differentiating w.r.t. y and z twice, we get $z_{yy}=0$, $z_{xy}=0$ so $r=0$ or $s=0$ or $t=0$. Thus we get 3 PDE.

Elimination of one arbitrary functions

Example 5:

$$z=(x+y)\,\phi\,(x^2-y^2) \quad (1)$$

Solution: Differentiating

$$z_x=1\cdot\phi+(x+y)2x\cdot\phi' \quad (2)$$

$$z_y=1\cdot\phi+(x+y)(-2y)\phi' \quad (3)$$

From (3),

$$\phi'=\frac{\phi-z_y}{2y(x+y)} \quad (4)$$

Substituting (4) in (2)

$$z_x=\phi+2x(x+y)\cdot\left[\frac{\phi-z_y}{2y(x+y)}\right]$$

$$p=\phi+\frac{x}{y}(\phi-q)$$

$$p=\left(\frac{x+y}{y}\right)\phi-\frac{x}{y}q$$

From the given Equation (1), $\phi=\frac{z}{(x+y)}$
Substituting ϕ,

$$p=\frac{(x+y)}{y}\cdot\frac{z}{x+y}-\frac{x}{y}q=\frac{z}{y}-\frac{x}{y}q$$

or $$yp+xq=z$$

Example 6: $z=x^n f\left(\frac{y}{x}\right)$

Solution: By differentiation,

$$z_x=nx^{n-1}\cdot f+x^n\cdot\left(-\frac{y}{x^2}\right)\cdot f'$$

$$z_y=x^n\cdot\frac{1}{x}\cdot f' \quad \text{or } f'=\frac{z_y}{x^{n-1}}$$

Eliminating f',

$$z_x=nx^{n-1}f-x^{n-2}\cdot y\cdot\frac{z_y}{x^{n-1}}$$

$$xp=nx^n f-yq$$

or $$xp=nz-yq$$

Example 7: $xyz=f(x+y+z)$

Solution: Differentiating w.r.t. x and y

$$yz+xy\,z_x=1\cdot f'+f'\cdot z_x \quad (1)$$

$$xz+xy\,z_y=1\cdot f'+f'\cdot z_y \quad (2)$$

From (2),

$$f'=\frac{xz+xy\,z_y}{1+z_y}=\frac{xz+xyq}{1+q} \quad (3)$$

Put (3) in (1)

$$yz + xyp = (1+p)f' = (1+p)\left(\frac{xz+xyq}{1+q}\right)$$

$$(1+q)(yz+xyp) = (1+p)(xz+xyq)$$

or $$x(y-z)p + y(z-x)q = z(x-y)$$

Elimination of two arbitrary functions

Example 8: $z = f(x)\, g(y)$

Solution: Differentiating w.r.t. x and y, we get $z_x = f'g,\ z_y = fg'$

$$z_x \cdot z_y = f'g\, fg' = fg\, f'g' = z\, f'g'$$

But

$$z_{xy} = f'g' \quad \text{so}$$

$$z_x \cdot z_y = z \cdot z_{xy}$$

or $$pq = z \cdot s.$$

Example 9: $z = f(x+y)\cdot g(x-y)$

Solution: Differentiating partially w.r.t. x and y, we get

$$p = z_x = f'\cdot 1\cdot g + f\cdot 1\cdot g' \quad (1)$$

$$q = z_y = f'\cdot 1\cdot g + f\cdot(-1)g' \quad (2)$$

$$r = z_{xx} = f''g + f'g' + f'g' + fg''$$

$$= f''g + 2f'g' + fg'' \quad (3)$$

$$t = z_{yy} = f''g + f'g'(-1)$$

$$-f'g' - fg''(-1)$$

$$= f''g - 2f'g' + fg'' \quad (4)$$

$$s = z_{xy} = f''g + f'g'(-1)$$

$$+f'g' + fg''(-1)$$

$$= f''g - fg'' \quad (5)$$

Adding (1) and (2),

$$f' = \frac{p+q}{2g} \quad (6)$$

Subtracting (2) from (1),

$$g' = \frac{p-q}{2f} \quad (7)$$

Adding (3) and (4)

$$r + t = 2(f''g + fg'')$$

From (5), $$2s = 2(f''g - fg'')$$

Adding $$r + t + 2s = 4f''g \quad (8)$$

Subtracting $$r + t - 2s = 4fg'' \quad (9)$$

Substituting (6), (7), (8), (9) in (3)

$$r = \left(\frac{r+t+2s}{4}\right) + 2\cdot\frac{(p+q)}{2g}\cdot\frac{(p-q)}{2f}$$

$$+\left(\frac{r+t-2s}{4}\right)$$

$$(r-t)z = (p+q)(p-q)$$

Example 10: $z = xf(ax+by) + g(ax+by)$

Solution: Differentiating w.r.t. x and y, we get

$$z_x = f + xaf' + ag'$$

$$z_{xx} = af' + af' + ax\, f''a + a^2g''$$

$$= a[2f' + axf'' + ag''] \quad (1)$$

$$z_y = bx\, f' + bg'$$

$$z_{yy} = b^2xf'' + b^2g'' = b^2[xf'' + g''] \quad (2)$$

$$z_{yx} = bf' + bx\, af'' + ba\, g''$$

$$= b[f' + a(xf'' + g'')] \quad (3)$$

Substituting (2) in (3)

$$\frac{z_{yx}}{b} = f' + a\cdot\frac{z_{yy}}{b^2}$$

Solving

$$f' = \frac{z_{yx}}{b} - \frac{a}{b^2}z_{yy} \quad (4)$$

Substituting (2) and (4) in (1)

$$z_{xx} = 2af' + a^2[xf'' + g'']$$

$$= 2a\cdot\left[\frac{z_{yx}}{b} - \frac{a}{b^2}z_{yy}\right] + a^2\left[\frac{z_{yy}}{b^2}\right]$$

$$b^2z_{xx} + a^2z_{yy} = 2ab\, z_{xy}$$

$$b^2r + a^2t = 2abs$$

Elimination of arbitrary function of specific functions

$F(u, v) = 0$ where u and v are given.

Example 11:

$$F(xy + z^2, x + y + z) = 0 \quad (1)$$

Solution:

$$\text{Let } u(x, y, z) = xy + z^2 \quad (2)$$

$$v(x, y, z) = x + y + z \quad (3)$$

Differentiating (1) partially w.r.t. x by chain rule

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

i.e., $$F_u \cdot (y + 2z\, z_x) + F_v \cdot (1 + z_x) = 0$$

Differentiating w.r.t. 'y', we get

$$F_u(x + 2z\, z_y) + F_v(1 + z_y) = 0$$

Eliminating F_u and F_v (i.e., the coefficient matrix should be singular)

$$\begin{vmatrix} y + 2z\, z_x & 1 + z_x \\ x + 2z\, z_y & 1 + z_y \end{vmatrix} = 0$$

or $$(1 + q)[y + 2zp] - (1 + p)[x + 2zq] = 0$$

$$(2z - x)p + (y - 2z)q = x - y.$$

Example 12: $xyz = f(x + y + z)$

Solution: Put $u = x + y + z$, $v = xyz$ so that the given equation may be written as $F(u, v) = 0$

Differentiating w.r.t. x and y, we get

$$F_u \cdot (1 + z_x) + F_v(yz + xy\, z_x) = 0$$

$$F_u(1 + z_y) + F_v(xz + xy\, z_y) = 0$$

Eliminating F_u, F_v, we have

$$\begin{vmatrix} 1 + p & yz + xyp \\ 1 + q & xz + xy\, q \end{vmatrix} = 0$$

or $$(xz + xy\, q)(1 + p) - (1 + q)(yz + xyp) = 0$$

$$x(z - y)p + (x - z)y\, q + z(x - y) = 0.$$

Example 13: $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

Solution: Let $u = x^2 + y^2 + z^2$, $v = z^2 - 2xy$

Differentiating F partially w.r.t. x, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$F_u \cdot 2x + F_u \cdot 2z \cdot p + F_v(-2y) + F_v\, 2z \cdot p = 0$$

Similarly w.r.t. y, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$F_u \cdot 2y + F_u\, 2z \cdot q + F_v(-2x) + F_v \cdot 2z \cdot q = 0$$

Solving

$$\begin{vmatrix} x + zp & -y + zp \\ y + zq & -x + zq \end{vmatrix} = 0$$

$$(x + zp)(zq - x) - (zp - y)(y + zq) = 0$$

$$xz(q - p) + yz(q - p) + (y^2 - x^2) = 0$$

$$(x + y)[z(q - p) + (y - x)] = 0.$$

EXERCISE

Form (obtain) partial differential equation by eliminating the arbitrary constants/functions:

Elimination of arbitrary constants

1. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ans. $2z = xp + yq$

2. $z = ax + by + a^2 + b^2$

Ans. $z = px + qy + p^2 + q^2$

3. $z = (x^2 + a^2)(y^2 + b^2)$

Ans. $4xyz = pq$

4. $z = axy + b$

Ans. $px = qy$

5. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

Ans. $q = px + p^2$

6. $z = (x - a)^2 + (y - b)^2 + 1$

Ans. $4z = p^2 + q^2 + 4$

7. $z = a(x + y) + b(x - y) + abt + c$

Hint: Number of independent variables x, y, t are $3 =$ number of arbitrary constants a, b, c. So P.D.E. is of 1st order.

Ans. $z_x^2 - z_y^2 = 4z_t$

8. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Ans. $pz = xp^2 + xz\, r$

Chapter 18

Partial Differential Equations

INTRODUCTION

Real world problems in general involve functions of several (independent) variables giving rise to partial differential equations more often than ordinary differential equations. Thus most problems in engineering and science abound with first and second order linear non homogeneous partial differential equations. In this chapter, we consider methods of obtaining solutions by Lagrange's and Charpits method for first order. The general solution of non homogeneous second order linear P.D.E. with constant coefficients is obtained as the sum of complementary function and particular integral. Monge's method is also considered for solving nonlinear second order P.D.E.

18.1 PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is an equation involving two (or more) independent variables x, y and a dependent variable z and its partial derivatives such as $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$, etc.,

$$\text{i.e., } F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \ldots\right) = 0$$

Standard notation

$$p = \frac{\partial z}{\partial x} = z_x, \quad q = \frac{\partial z}{\partial y} = z_y, \quad r = \frac{\partial^2 z}{\partial x^2} = z_{xx},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}, \quad t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

Order of a partial differential equation (P.D.E.) is the order of the highest ordered derivative appearing in the P.D.E.

Formation of Partial Differential Equation

By elimination of arbitrary constants

Let

$$f(x, y, z, a, b) = 0 \tag{1}$$

be an equation involving two arbitrary constants a and b. Differentiating this equation partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \tag{2}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \tag{3}$$

By eliminating a, b from (1), (2), (3), we get an equation of the form

$$F(x, y, z, p, q) = 0 \tag{4}$$

which is a partial differential equation of first order.

Note 1: If the number of arbitrary constants equals to the number of independent variables in (1), then the P.D.E. obtained by elimination is of first order.

Note 2: If the number of arbitrary constants is more than the number of independent variables then the P.D.E. obtained is of 2nd or higher orders.

By elimination of arbitrary functions of specific functions

a. *One arbitrary function* (resulting in first order P.D.E.):

Consider

$$z = f(u) \tag{5}$$

where $f(u)$ is an arbitrary function of u and u is a given (known) function of x, y, z i.e., $u = u(x, y, z)$.

Differentiating (5) partially w.r.t. x and y by chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \tag{6}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \tag{7}$$

By eliminating the arbitrary function f from (5), (6), (7) we get a P.D.E. of first order.

b. Two arbitrary functions:

Differentiating twice or more, the elimination process results in a P.D.E. of 2nd or higher order.

Note: When n is the number of arbitrary functions, one may get several P.D. equations. But generally the one with the least order is chosen.

Example: For $z = x\,f(y) + y\,g(x)$ involving two arbitrary functions f and g, $\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$ is also a P.D.E. obtained by elimination. The other P.D.E. $xys = xp + yq - z$ of second order obtained by elimination may be chosen.

Elimination of Arbitrary Function F

from the equation

$$F(u, v) = 0 \tag{8}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are given functions of x, y, z.

Differentiating the Equation (8) partially w.r.t. x by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \tag{9}$$

Similarly, differentiating Equation (8) partially w.r.t. y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0 \tag{10}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (9) and (10), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix} = 0$$

Rewriting

$$\begin{bmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{bmatrix} = 0$$

Expansion of this determinant results in a P.D.E. which is free of the arbitrary function F as

$$(u_x + pu_z)(v_y + qv_z) - (u_y + qu_z)(v_x + pv_z) = 0$$

or

$$Pq + Qq = R$$

which is a first order linear P.D.E. Here

$$P = u_z v_y - u_y v_z = \frac{\partial(u, v)}{\partial(y, z)}$$

$$Q = u_x v_z - u_z v_x = \frac{\partial(u, v)}{\partial(z, x)}$$

$$R = u_y v_x - u_x v_y = \frac{\partial(u, v)}{\partial(x, y)}.$$

WORKED OUT EXAMPLES

Elimination of arbitrary constants

Form (obtain) partial differential equation by eliminating the arbitrary constants/functions:

Example 1: $z = ax^2 + by^2$

Solution: Differentiating partially w.r.t. x and y, we get

$$z_x = 2ax, z_y = 2by \quad \text{or} \quad a = \frac{z_x}{2x}, b = \frac{z_y}{2y}.$$