1.2. Frame of Reference

All physical phenomena take place in space and time. It is impossible to investigate physical phenomena without introducing a frame of reference or reference of bodies relative to which observation will be done. *A frame of reference is a system of coordinate axes which is considered fixed for a given problem and with respect to which the position of a particle is specified.*

There are two types of frame of references.

- **Inertial frame** **A frame of reference which is either at rest or in motion with a uniform velocity is called inertial frame.** Newton's laws of motion hold good on inertial frame.

- **Non-inertial frame** **The frame of reference which is moving with acceleration in a straight line or rotating is known as non-inertial frame.**

1.3. Space and Time Frame

When only the three coordinates are specified in a reference frame, we consider it as a '**point**' in space, but when *we specify the place of occurrence of a phenomenon as well as the time of occurrence, we call* it an **'event'**. So, to determine the exact location and the exact time of occurrence of an event, we need another coordinate 'time (t)'

perpendicular to the other three spatial coordinates x, y and z. Thus, every physical phenomenon takes place in space and time.

A reference frame which has such four coordinates x, y, z and t is called a space-time frame. The four axes defining a four dimensional continuum is called space-time.

1.4 Galilean Transformation

Newton's first two laws of motion do not hold good in each and every frame of reference. The frame of reference where the two laws of motion hold good is called Newtonian frame of reference.

Galilean transformation is a method of transference of observation of an event from one inertial frame to another.

Let us consider two inertial frames of reference S and S' [1]. S is at rest and the frame S' is moving with a constant velocity 'v' along the common positive direction of X-axis.

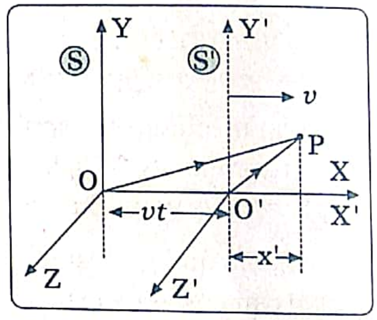

Fig. 1 ▷ S' frame is moving with constant velocity v along the common direction of positive X-axis

Let an event occurs at a point P. The coordinates of point P *w.r.t.* the frames S and S' are (x, y, z, t) and (x', y', z', t') respectively.

At $t = 0$, the origins of S and S' are coincided at O. As the frame S' is moving with constant velocity v *w.r.t.* S the origin O' of S' is shifted from O of S at time t.

Now, during the motion of frame S', the X-axis of frames S and S' remain coinciding. So y coordinates and z coordinates do not change. Only there is a change in x coordinates.

From the figure, it is obvious that

$$x' = x - vt \qquad \cdots 1.1(a) \qquad\qquad y' = y \qquad \cdots 1.1(b)$$

$$z' = z \qquad \cdots 1.1(c) \qquad\qquad t' = t \qquad \cdots 1.1(d)$$

These equations are called Galilean transformation equations.

Inverse Galilean transformation The inverse Galilean transformation equations will be obtained by changing $v \to -v$ and hence

$$\left.\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}\right\} \qquad \cdots(1.2$$

[1] The origin of the frame of reference may not always coincide with the position of th observer even if the frame of reference is at rest *w.r.t.* observer.

② *The speed of light in free space is invariant in all inertial frames of reference irrespective of the motion of the source or the observer.*

On account of this negative result, a critical revision of the absoluteness of space and time was needed. The result gave the birth of special theory of relativity.

Problem 1.

In a Michelson-Morley experiment, the effective length of each plate was 11 m and the wavelength of light used was 6000 Å. Assuming the ratio of the velocity of the earth to the velocity of light in stationary ether medium to be 10^{-4}, calculate the expected fringe shift.

Solution The number of fringes moving through the cross-wire of the telescope

$$n = \frac{c\Delta t}{\lambda}\left(= \frac{\text{total path difference}}{\text{wavelength of light used}}\right) = \frac{2d\frac{v^2}{c^2}}{\lambda}$$

Here, d = path length = 11 m, λ = 6000 Å = 6000×10^{-10} m, $\frac{v}{c} = 10^{-4}$

$$\therefore \quad n = \frac{2 \times 11 \times (10^{-4})^2}{6000 \times 10^{-10}} = 0{\cdot}37$$

So, the expected fringe shift will be 0·37 of a fringe width.

1.6. Einstein's Special Theory of Relativity

Due to invariance of the velocity of light in all inertial frames as observed from Michelson-Morley's experiment, Einstein claimed that time and space can not be absolute though the speed of light in vacuum is absolute.

In 1905, Einstein developed the special theory of relativity. **This theory is called 'special' as it is restricted with the special problems involving in inertial (non-accelerated) frames of reference.** In 1915, he proposed the **general theory of relativity** that includes the problems involving frames of reference accelerated with respect to an inertial frame. Einstein deals with the physics of objects moving with very high velocity comparable to the velocity of light.

The special theory of relativity is thus based on the following two postulates :

- **Principle of relativity** All the basic laws of physics are the same in all inertial frames moving with constant velocity relative to each other.
- **Principle of constancy of speed of light** The speed of light in vacuum is the same in all inertial frames of reference regardless of the motion of the source relative to the observer.

From the first postulate, we can point out that there is no meaning in the idea of absolute velocity, (*i.e.* universal frame of reference), such as that with reference to an imaginary 'ether' (that Michelson had tried to measure). Only, the relative motion of one body with respect to another has some significance.

The postulate 'principle of constancy of speed of light' follows directly from the null results of Michelson-Morley experiment.

1.7. Lorentz Transformations

The transformation equations that will satisfy the postulates of the special theory of relativity are known as **Lorentz transformation** equations. It provides us with a method of obtaining the coordinates of a point in a certain frame of reference moving with a constant velocity relative to a frame in which the coordinates of the point are known and vice-versa.

Let us consider, two inertial frames S and S'. Here, S is at rest and S' is moving with uniform velocity v *w.r.t.* S along the common positive direction of X-axis.

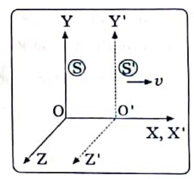

Fig. 4 ▷ S' frame is moving with velocity v *w.r.t.* frame 'S' along common positive direction of X-axis

Let the origin O and O' of frames S and S' coincide at an instant $t = t' = 0$, when a signal of light is generated at O (*i.e.* O'). Now, consider the situation when the pulse reaches at point P whose position is described by (x, y, z, t) and (x', y', z', t') in S and S' frames respectively.

The signal (or wavefront) propagates in two systems, satisfy the equations

$$x^2 + y^2 + z^2 = c^2t^2 \quad \cdots(1.10)$$

and
$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \cdots(1.11)$$

in S and S' frames respectively.

[$\because$ The velocity of light in inertial frames S and S' is

$$c = \frac{\sqrt{x^2 + y^2 + z^2}}{t} = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{t'}\,]$$

So, a spherical wavefront of light will appear to propagate outwards from the origins of the respective system S or S' with the velocity c to the observers of both frames.

To simplify the problem, we assume that S' frame is moving with uniform velocity v *w.r.t.* S along the common positive direction of X-axis. As there is no motion along Y-axis and Z-axis, , y and z-coordinates transformations remain unchanged *i.e.* $y' = y$ and $z' = z$.

But x' must transform linearly into x and t. Similarly t' must transform linearly into x and t.

This is needed for homogenity of space. Thus, the transformation equations relating to x and x' and t and t' can be written as,

$$\left.\begin{aligned} x' &= k(x-vt) \\ t' &= \alpha t+\gamma x \end{aligned}\right\} \quad \cdots(1.12)$$

where k, α, γ are three constants which are to be determined.

Putting the values of x', t' in equation (1.11), we get

$$k^2(x-vt)^2+y^2+z^2 = c^2(\alpha t+\gamma x)^2$$

or, $$x^2(k^2-\gamma^2c^2)+y^2+z^2-2xt(k^2v+\alpha\gamma c^2) = t^2(\alpha^2c^2-k^2v^2) \quad \cdots(1.13)$$

Comparing coefficients of different terms in the above equation with corresponding terms in equation (1.10)

$$k^2-\gamma^2c^2 = 1 \quad \cdots(1.14) \quad \text{or, } \gamma^2c^2 = k^2-1 \quad \cdots(1.14a)$$

$$k^2v+\alpha\gamma c^2 = 0 \quad \cdots(1.15) \quad \text{or, } \alpha^2\gamma^2c^4 = k^4v^2 \quad \cdots(1.15a)$$

and $$\alpha^2c^2-k^2v^2 = c^2 \quad \cdots(1.16) \quad \text{or, } \alpha^2c^2 = k^2v^2+c^2 \quad \cdots(1.16a)$$

Now, we have from equation (1.15a)

$$\alpha^2\gamma^2c^4 = k^4v^2 \quad \text{or, } \alpha^2c^2(\gamma^2c^2) = k^4v^2 \quad \text{or } \alpha^2c^2 = \frac{k^4v^2}{k^2-1}$$

[$\because$ from equation (1.14a) $\gamma^2c^2 = k^2-1$]

or, $$k^2v^2+c^2 = \frac{k^4v^2}{k^2-1}$$ [$\because$ from equation (1.16a) $\alpha^2c^2 = k^2v^2+c^2$]

or, $$k^4v^2 = k^4v^2+k^2c^2-k^2v^2-c^2$$

or, $$k^2(v^2-c^2) = -c^2 \quad \text{or, } k^2 = \frac{1}{1-\frac{v^2}{c^2}} \quad \cdots(1.17)$$

Again, we have from equation 1.16

$$\alpha^2c^2-k^2v^2 = c^2 \quad \text{or, } (\alpha^2-1)c^2 = k^2v^2$$

or, $$\alpha^2-1 = \frac{v^2}{c^2}\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \quad \text{or, } \alpha^2 = 1+\frac{v^2}{c^2-v^2}$$

or, $$\alpha^2 = \frac{c^2}{c^2-v^2} = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\therefore \quad k = \alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \cdots(1.18)$$

Now from equation (1.15)

$$k^2v+\alpha\gamma c^2 = 0 \quad \text{or, } \gamma = -\frac{k^2v}{\alpha c^2} \quad \text{or, } \gamma = -\frac{k^2v}{kc^2}$$

or, $$\gamma = -\frac{kv}{c^2} = \frac{-\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \cdots(1.19)$$

Thus, the transformation equations (1.12) will be

$$\left.\begin{aligned} x' &= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right\} \quad \cdots(1.20)$$

These equations are known as Lorentz transformation equations.

Inverse Lorentz transformation **The inverse Lorentz transformation equations** will be obtained by changing $v \to -v$ and hence

$$\left.\begin{aligned} x &= \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right\} \quad \cdots(1.21)$$

1.7.1. Condition for which Lorentz Transformation is Converted to Galilean Transformation

If the relative velocity v between two inertial frames S and S' is small (*i.e.* $v \ll c$), then $\frac{v}{c} \to 0$.

Hence, Lorentz transformation equations will be changed to

$$x' = x-vt; \qquad y' = y;$$
$$z' = z; \qquad t' = t$$

These equations are Galilean transformation equations. Hence, Lorentz transformation reduces to the Galilean transformation for $v \ll c$.

Special Note :

Why $t' = \alpha t+\gamma x$?

As the equations of physics must have the same form in both S and S', the corresponding equation of x in terms of x' and t' will be obtained by changing v to $-v$ on the equation

$$x' = (x-vt)k$$

So, $$x = k(x'+vt') = k[k(x-vt)+vt'] \quad [\because x' = k(x-vt)]$$

or, $$t' = kt+\frac{1-k^2}{kv}x = \alpha t+\gamma x, \text{ where } \alpha = k \text{ and } \gamma = \frac{1-k^2}{kv} \text{ (say)}$$

Problem 1

In a frame S, the following two events occur. Event 1 : $x_1 = x_0$; $t_1 = x_0/c$ and $y_1 = z_1 = 0$; Event 2 : $x_2 = 2x_0$, $t_2 = x_0/2c$ and $y_2 = z_2 = 0$. Find the velocity of S' frame (relative to S) in which the events occur at the same time and what is the value of corresponding time? [C.U. (Hons)]

Solution Since, the two events occur at the same time in S' frame

$$\therefore \quad t_1' = t_2' \quad \text{or,} \quad \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{or,} \quad t_1 - \frac{vx_1}{c^2} = t_2 - \frac{vx_2}{c^2}$$

$$\text{or,} \quad \frac{x_0}{c} - \frac{vx_0}{c^2} = \frac{x_0}{2c} - \frac{2vx_0}{c^2} \quad \text{or,} \quad \frac{x_0}{2c} = -\frac{vx_0}{c^2}; \quad \therefore \ v = -\frac{c}{2}$$

The events are simultaneous in S' frame for

$$t_2'(=t_1') = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{x_0}{c} - \left(-\frac{c}{2}\right)\frac{x_0}{c^2}}{\sqrt{1-\frac{1}{4}}} \quad \left[\because t_1 = \frac{x_0}{c} \text{ and } v = -\frac{c}{2}\right] = \frac{\sqrt{3}x_0}{c}$$

Problem 2

Show that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$ under Lorentz transformation.

Solution $x^2 + y^2 + z^2 - c^2t^2$

$$= \left(\frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + y'^2 + z'^2 - \left(\frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 c^2$$

$$= y'^2 + z'^2 + \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[x'^2 + 2x'vt' + v^2t'^2 - \left(t'c + \frac{vx'}{c}\right)^2\right]$$

$$= y'^2 + z'^2 + \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[x'^2 + 2x'vt' + v^2t'^2 - c^2t'^2 - \frac{v^2x'^2}{c^2} - 2vx't'\right]$$

$$= y'^2 + z'^2 + \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[-c^2t'^2\left(1-\frac{v^2}{c^2}\right) + x'^2\left(1-\frac{v^2}{c^2}\right)\right]$$

$$= y'^2 + z'^2 + x'^2 - c^2t'^2 = ds'(\text{say})$$

So, the above equation is invariant under Lorentz transformation.

If $ds = x^2 + y^2 + z^2 - c^2t^2$, then we can say *$ds$ is invariant under Lorentz transformation.*

1.8. Consequences of Lorentz Transformation Equations

The concept of length contraction, time dilation etc. can be explained from Lorentz transformation equations.

1.8.1. Length Contraction or The Lorentz-Fitzgerald Length Contraction

Let, a rod be placed in inertial frame 'S' along X-axis. So, an observer of frame S is at rest with respect to the rod. If x_1 and x_2 are the coordinates of its two end points, then the length of the rod observed by him

$$L_S = x_2 - x_1 \quad \cdots(1.22)$$

Here, L_S is known as **proper length** *i.e.* the length of the rod in the frame of reference in which the rod is at rest.

Now let us find the length (L_S') of the rod in frame S' moving with velocity v with respect to 'S'. If x_1' and x_2' are the coordinates of its two end points, then the length of the rod

$$L_S' = x_2' - x_1' \quad \cdots(1.23)$$

Now, from inverse Lorentz transformation, we have,

$$x_2 = \frac{x_2' + vt_2'}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad x_1 = \frac{x_1' + vt_1'}{\sqrt{1-\frac{v^2}{c^2}}} \quad \cdots(1.24)$$

Here $t_2' = t_1' = t'(\text{say})$ as the observer in S' frame measures the two ends of the rod simultaneously.

Hence, using equation (1.24) we get from equation (1.22),

$$L_S = x_2 - x_1 \quad \text{or,} \quad L_S = \frac{x_2' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad [\because t_2' = t_1' = t'(\text{say})]$$

$$\text{or,} \quad L_S = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{L_S'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{or,} \quad \boxed{L_S' = L_S\sqrt{1-\frac{v^2}{c^2}}} \quad \cdots(1.25)$$

So, L_S' is shorter than L_S by a factor $\left(\sqrt{1-\frac{v^2}{c^2}}\right)$ in the direction of motion.

The phenomenon by virtue of which the length of a rod in uniform motion relative to an observer appears to be shortened (than when it is at rest with respect to the observer) is called ***length contraction***. This phenomenon is also known as Lorentz-Fitzgerald contraction.

▶ ***Special Note :***

Since, equation (1.25) involves factor v^2, the opinion of the observer will be reciprocal (*i.e.* the same effect will be seen if the rod is placed in S' frame). So, when the rod is placed in moving frame S',

$$L_S = L'_S\sqrt{1-\frac{v^2}{c^2}}$$

i.e. $L'_S > L_S$

Hence, the length of an object is maximum in a reference frame in which it is at rest.

Now, we note the following cases from equation (1.25)

Case 1 If $v \ll c$, $\frac{v^2}{c^2} \ll 1$, so $L'_S = L_S$ *i.e.* the length of the moving object same as that when it was at rest.

Case 2 If $v > c$, $\frac{v^2}{c^2}$ is greater than unity. So L'_S is an imaginary quantity which is impossible.

Case 3 If $v = c$, $\frac{v^2}{c^2} = 1$. So $L'_S = 0$ which is also impossible.

Hence, we can conclude from case (2) and case (3) that material bodies can never attain velocities which are equal to or greater than the velocity of light.

Problem 1

A rod of length 1 m placed in a satellite moving with a velocity of 0˙8c relative to laboratory. Find the length of the rod as measured by an observer

I in the satellite II in the laboratory

Solution

I Since, the rod is placed in moving frame S', the relative velocity of the observer *w.r.t.* rod is zero. Hence the length of the rod measured by the observer in satellite,

L'_S = the proper length of the rod = 1 m

II Let, L_S be the length of the rod as it appears to an observer in the laboratory (*i.e.* on the earth) 'S' frame

$$L_S = L'_S\sqrt{1-\frac{v^2}{c^2}}$$

$$= 1\sqrt{1-\left(\frac{0˙8c}{c}\right)^2}$$

$$= \sqrt{1-0˙64} = \sqrt{0˙36} = 0˙6 \text{ m}$$

Problem 2

A rocket ship is 100 m long on the ground. When this ship is in flight, its length is appeared 99 m to an observer on the ground. Find the speed of the flight.

Solution Here, the proper length of the rocket ship placed in inertial frame S = 100 m.

Let the length of the rocket ship in frame S' moving with relative velocity v *w.r.t* an observer in S frame be L'_S.

Thus, $L'_S = 99$ m

$$\therefore \quad L'_S = L_S\sqrt{1-\frac{v^2}{c^2}} \quad \text{or,} \quad 99 = 100\sqrt{1-\frac{v^2}{c^2}}$$

$$\text{or,} \quad 1-\frac{v^2}{c^2} = \frac{(99)^2}{(100)^2} \quad \text{or,} \quad \frac{v^2}{c^2} = 1-\frac{(99)^2}{(100)^2} \quad \text{or,} \quad v = c\frac{\sqrt{199}}{100}$$

$$\text{or,} \quad v = 0˙42 \times 10^8 \text{ m s}^{-1}$$

1.8.2. Time Dilation

Let a clock be situated in frame S at position x. Suppose that t_1 and t_2 are the two instants recorded by the observer in frame S. Then the time interval between two events in frame 'S' is given by

$$\tau_S = t_2 - t_1 \quad \cdots(1.26)$$

Here, τ_S is called **proper time.**

The observer in frame S' which is moving with uniform velocity v with respect to frame S, however, measure these instants as

$$\left.t'_1 = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad t'_2 = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right\} \quad \cdots 1.27$$

Since the clock of frame S must be at rest during the time of observation by an observer of S' frame *i.e.* $x_2 = x_1 = x$ (say).

So, the time interval according to the observer in frame 'S' is

$$\tau'_S = t'_2 - t'_1$$

After substituting the values of t'_1 and t'_2 from equation (1.27), we get

$$\tau'_S = \frac{t_2 - t_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{or,} \quad \boxed{\tau'_S = \frac{\tau_S}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \cdots(1.28)$$

1.10. Variation of Mass with Velocity

In non-relativistic (classical) mechanics, the mass of a body is constant, independent of velocity and is denoted as rest mass (m_0) of a body. But in relativistic mechanics, the mass of a body will increase with increasing velocity. ***This variable mass which is a function of the velocity of the moving body is called relativistic mass of the body.***

To prove this, we shall consider collision of two bodies assuming that,

1. the law of conservation of momentum
2. the law of conservation of relativistic masses of the particles.

Let us consider two bodies A and B each of mass 'm'are moving in opposite directions along the X-axis with velocities u' and $-u'$ respectively in the inertial frame S' moving with velocity v *w.r.t.* to inertial frame S [Fig. 6]. Let the two bodies collide with each other elastically and coalesce into one body. *As the two bodies are* *moving with equal velocity in opposite directions, according to the law of conservation of momentum they will be at rest in S' frame after collision.*

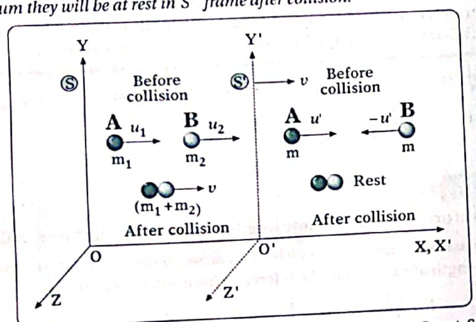

Fig. 6 ▷ Collision between two bodies as observed from frames S and S'

Let us assume the above collision is observed *w.r.t.* an observer from frame S. *The velocities of the two bodies* A and B observed from S by using the law of addition of velocities are given by

$$u_1 = \frac{u'+v}{1+\frac{u'v}{c^2}} \quad \cdots(1.33) \quad \text{and} \quad u_2 = \frac{-u'+v}{1-\frac{u'v}{c^2}} \quad \cdots(1.34)$$

respectively along the X-axes.

Let m_1 and m_2 be the masses of the two bodies A and B *w.r.t.* frame 'S'. When the two bodies coalesce into one body, then according to the law of conservation of mass, the corresponding mass of the body is $(m_1 + m_2)$. It moves with a velocity v along the X-axis with respect to S frame. At that moment this body is at rest *w.r.t.* S' frame.

Now, applying the conservation of momentum in S frame, we can write

$$m_1u_1 + m_2u_2 = (m_1+m_2)v \quad \cdots(1.35)$$

or, $$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}\right] + m_2\left[\frac{-u'+v}{1-\frac{u'v}{c^2}}\right] = (m_1+m_2)v$$

or, $$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}} - v\right] = m_2\left[v - \frac{-u'+v}{1-\frac{u'v}{c^2}}\right]$$

or, $$m_1\left[\frac{u'+v-v-\frac{u'v^2}{c^2}}{1+\frac{u'v}{c^2}}\right] = m_2\left[\frac{v-\frac{u'v^2}{c^2}+u'-v}{1-\frac{u'v}{c^2}}\right]$$

or, $$\frac{m_1}{m_2} = \frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} \quad \cdots(1.36)$$

We can write equation (1.33) as,

$$\frac{u_1^2}{c^2} = \frac{(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

or, $$1-\frac{u_1^2}{c^2} = 1-\frac{(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2} = \frac{c^2\left(1+\frac{u'v}{c^2}\right)^2-(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

$$= \frac{c^2+\frac{v^2u'^2}{c^2}+2u'v-u'^2-v^2-2u'v}{c^2\left(1+\frac{vu'}{c^2}\right)^2}$$

$$= \frac{\left(c^2-u'^2-v^2+\frac{v^2u'^2}{c^2}\right)}{\left[c^2\left(1+\frac{u'v}{c^2}\right)^2\right]} \quad \cdots(1.37)$$

Similarly from equation (1.34), we get,

$$1-\frac{u_2^2}{c^2} = \frac{\left[c^2-u'^2-v^2+\frac{v^2u'^2}{c^2}\right]}{\left[c^2\left(1-\frac{vu'}{c^2}\right)^2\right]} \quad \cdots(1.38)$$

Hence, $$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} = \frac{\left(1+\frac{vu'}{c^2}\right)^2}{\left(1-\frac{vu'}{c^2}\right)^2} \quad \cdots(1.39)$$

or, $$\frac{1+\frac{vu'}{c^2}}{1-\frac{vu'}{c^2}} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

Now, we can write from equation (1.36)

$$\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} \quad \cdots(1.40)$$

Now, we consider the body B is at rest *w.r.t.* frame S *i.e.* $u_2 = 0$. Since, mass is a **function of velocity, the mass of the body B can be called rest mass m_0. When, the** two bodies A and B are identical, the rest mass of the body A is also m_0. Hence, the above equation can be applicable to a single body whose rest mass is m_0 and which is moving with velocity $u_1 (= v, \text{say})$ with variable mass $m_1 (= m, \text{say})$ w.r.t. frame 'S'.

Hence, the equation (1.40) becomes

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

or, $$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$ where 'm' is called relativistic mass ...(1.41)

The variation of mass with velocity can be expressed in a graph between $\frac{m}{m_0}$ against $\frac{v}{c}$. [Fig. 7].

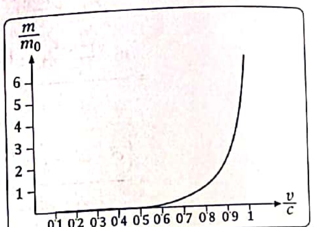

Fig. 7 ▷ Variation of mass with velocity

Discussion

Now we consider the following cases from the equation (1.41)

Case 1 If velocity (v) of the body is very smaller than the velocity of light i.e. $v \ll c$, $\frac{v^2}{c^2}$ is negligible in comparison to unity.

$$\therefore \quad m = m_0$$

Case 2 If v is comparable to c, $\sqrt{1-\frac{v^2}{c^2}} < 1 \quad \therefore \quad m > m_0$

So, ***the mass of the moving object appears to be greater than when it is at rest.***

Case 3 If $v = c$, $m = \frac{m_0}{0} = \infty$ (infinity). Again if $v > c$, m = imaginary as $\frac{v^2}{c^2}$ is greater than unity.

But, the mass of the body can not be zero or imaginary.

So material body can never attain velocities which are equal to or greater than the velocity of light.

1.11. Relativistic Energy : Mass Energy Relation

Let a particle with rest mass m_0 be initially at rest in an inertial reference frame S. If a force component F accelerated it through a distance x from rest to a velocity v, the kinetic energy of the particle

$$T = \int_0^x F\,dx, \; F \text{ is along the displacement } dx \qquad \text{...(1.42)}$$

Now, $F = \frac{d}{dt}(mv)$, m is the mass of the moving particle

Hence, we get from equation (1.42),

$$T = \int_0^x \frac{d}{dt}(mv)\,dx = \int_0^{mv} v\,d(mv) \quad \left[\because \; v = \frac{dx}{dt}\right]$$

$$= \int_0^v v\, d\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}v\right)$$

Integrating by parts ($\int y\,dx = xy - \int y\,dx$)

$$T = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 \int_0^v \frac{v\,dv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\left[-c^2\sqrt{1-\frac{v^2}{c^2}}\right]_0^v$$

$$\left[\text{By putting } \sqrt{1-\frac{v^2}{c^2}} = z, \text{ or, } \frac{1}{2}\frac{-\frac{2v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}dv = dz \quad \text{or, } \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}dv = -c^2 dz\right]$$

$$= \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0 c^2\sqrt{1-\frac{v^2}{c^2}} - m_0 c^2 = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v^2}{c^2}\right) - m_0 c^2$$

$$= \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2$$

$$= \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 = mc^2 - m_0 c^2 = (m - m_0)c^2$$

or, $\boxed{T = \Delta mc^2}$ ($\Delta m = m - m_0$ = increase in mass of the particle) ...(1.43)

This is the **expression of relativistic kinetic energy**. Here we see that *T is not equal to the classical expression $\frac{1}{2}mv^2$.*

Since, **the quantity $m_0 c^2$ does not depend on the velocity v, it is called rest mass energy (or rest energy) of the particle. So, the total energy (E) of the particle** can be written from equation (1.43)

$$E = \text{K.E.}(= T) + \text{Rest mass energy } (= m_0 c^2)$$

or, $$E = (m - m_0)c^2 + m_0 c^2$$

or, $$\boxed{E = mc^2} \qquad \text{...(1.44)}$$

This equation (1.44) gives universal equivalence between mass and energy or Einstein's **mass energy relation** or **mass energy equivalence of relativity.**

Special Note :

① If the particle velocity v is much less than the velocity of light *i.e.* $v \ll c$, the expression of relativistic kinetic energy (T) becomes

$$T = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 = m_0 c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

Since $v \ll c$, $\frac{v^2}{c^2}$ is much less than unity, then we can write,

$$T = m_0 c^2\left[\left(1+\frac{v^2}{2c^2}+\cdots\right)-1\right] = \frac{1}{2}m_0 v^2 \quad \cdots(1)$$

which is the **same as that obtained from classical mechanics.** Thus, when particle moves with a velocity much less compared to the velocity of light in free space, the relativistic expression of kinetic energy approaches the classical limit.

② **Principle of conservation of mass and energy:**

We can say from equation (1.44) that mass and energy are related to each other. **Mass can be created or destroyed provided that an equivalent amount of energy simultaneously disappeared or created.** This is known as **principle of conservation of mass and energy.** It has a great importance in generation of energy in nuclear reaction, e.g., nuclear fusion, nuclear fission etc.

Problem 1

If an electron and a positron annihilate, calculate the energy released, given that mass of an electron is $9{\cdot}1 \times 10^{-31}$ kg. [W.B.U.T. 2003]

Solution The mass (m_e) of the electron = the mass of the positron. So, when an electron and positron annihilate, the released energy

$$E = 2m_e c^2$$

$$= 2 \times (9{\cdot}1 \times 10^{-31}) \times (3 \times 10^8)^2 \text{ joule}$$

$$= 163{\cdot}8 \times 10^{-15} \text{ joule}$$

$$= \frac{163{\cdot}8 \times 10^{-15}}{1{\cdot}6 \times 10^{-19}} \text{ eV} = 102{\cdot}4 \times 10^4 \text{ eV}$$

1.12. Total Relativistic Energy and Momentum

According to the special theory of relativity, the momentum (p) of a moving particle of mass 'm' is given by

$$p = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \quad [\because m \text{ is a function of its velocity } v] \quad \cdots(1.45)$$

The total relativistic energy

$$E = mc^2 = \frac{m_0}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}c^2 \quad \cdots(1.46)$$

So, using the above two equations, we have,

$$\therefore \quad E^2 - p^2c^2 = \frac{m_0^2 c^4}{1-\frac{v^2}{c^2}} - \frac{m_0^2 v^2 c^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2 c^4\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}}$$

or, $$E^2 - p^2c^2 = m_0^2 c^4 \quad \cdots(1.47a)$$

or, $$\boxed{E = \sqrt{p^2c^2 + m_0^2 c^4}} \quad \cdots(1.47b)$$

This equation gives **the relation between the total relativistic energy (E) and the momentum (p) of the particle.** This equation is known as **Dirac equation in quantum mechanics.** The equation (1.47a) **shows that $E^2 - p^2c^2$ is invariant.**

Special Note :

① For a massless particle (*i.e.* a particle with no rest mass, $m_0 = 0$), we have from equation (1.47)

$$E = pc$$

So, for the **massless particle (such as photon or neutrino, etc) can still have a momentum.**

② For a photon of energy $E = h\nu$, its momentum is $\frac{h\nu}{c}$ and relativistic mass is $\frac{h\nu}{c^2}$. Now, we can write from equation (1.45), the rest mass of a photon

$$m_0 = \frac{p}{vc}\sqrt{c^2 - v^2}$$

So, for a photon moving with velocity of light (*i.e.* $v = c$), we get from the above equation the rest mass (m_0) of a photon is zero.

Problem 1

Find the speed of the particle when its mass is five times of its rest mass.

Solution We know, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $$5m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad [\because m = 5m_0]$$

or, $$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{5} \quad \text{or,} \quad \frac{v^2}{c^2} = 1 - \frac{1}{25}$$

or, $$\frac{v^2}{c^2} = \frac{24}{25} \quad \text{or,} \quad \frac{v}{c} = \frac{\sqrt{24}}{5} \quad \text{or,} \quad v = 0{\cdot}98\,c$$

1.5. Concept of Ether

According to electromagnetic theory, light may propagate even in vacuum in contrast to other waves. The velocity of light does not depend on the medium. But this contradicts conventional theory of physics, specially Newtonian mechanics. So, physicist thought that **the space is filled with a massless, invisible, perfectly transparent and non-resistive medium where the velocity of light is constant.** They called it as **"ether"**. This medium was assumed to be at absolute rest and is present everywhere even in free space. It is also assumed that earth rotates through the ether medium without producing any disturbance within it. Thus, if this ether hypothesis is correct, then we measure the velocity of earth *w.r.t.* the stationary ether medium. In 1887, Michelson in collaboration with Morley performed an experiment using Michelson interferometer for this purpose.

1.5.1. Michelson-Morley Experiment and Emergence of Special Theory of Relativity

Michelson and Morley designed a high precision experiment **to demonstrate the existence of ether frame and to determine the velocity of earth with respect to the stationary ether medium by using Michelson interferometer.**

The apparatus consists of two plane mirrors M_1 and M_2 (Fig. 2). The planes of two mirrors are mutually perpendicular and they are at a distance 'd' from the half silvered glass plate P[2]. The light from a monochromatic source 'S' is incident at

[2] With the help of half silvered glass plate, we can produce coherent sources by division of amplitude of a source. In that case, a single beam of light is divided by means of partial reflection in this half silvered glass plate and the separated subsequently to produce interference pattern.

an angle 45° on plate P that divides it into two coherent beams— a transmitted beam PB and a reflected beam PA. There is a compensating glass plate 'C' placed between P and M_2 to make the equal optical distance of the mirrors M_1 and M_2 from plate P.

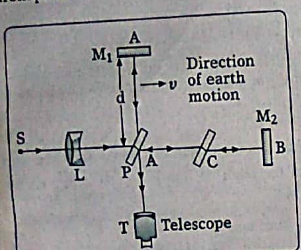

Fig. 2 ▷ Michelson Interferometer

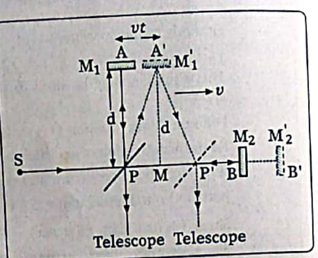

Fig. 3 ▷ The Michelson-Morley experiment

Let 'v' be the velocity of earth in space through ether medium along the direction of incident ray. Hence, the interferometer is also moving with a same velocity v and is parallel to the direction of earth's motion. The reflected and transmitted beams fall normally on mirrors M_1 and M_2. Those two rays each travel a distance 'd' and are then reflected back from mirrors M_1 and M_2. On returning back, the beam A is partially transmitted and beam B is partially reflected by plate P.

These two beams reunite at a point of the half silvered face of plate P to produce **interference pattern** which are observed through a telescope T.

The relative velocity of transmitted ray *w.r.t.* the apparatus towards M_2 will be $c-v$ in moving from P to B and $c+v$ for the return journey from B to P. Hence the time taken by this wave for its roundtrip is

$$t_1 = \frac{d}{c-v}+\frac{d}{c+v} = \frac{2dc}{c^2-v^2} = \frac{2dc}{c^2\left(1-\frac{v^2}{c^2}\right)} \simeq \frac{2d}{c}\left(1+\frac{v^2}{c^2}\right) \quad \left[\text{since } \frac{v^2}{c^2} \ll 1\right] \quad \cdots(1.3)$$

Again, due to earth's motion, the reflected beam PA towards M_1 takes the path PA' to reach the mirror M_1' and returns via $A'P'$ from the mirror [Fig. 3]. The time taken by the reflected ray for its roundtrip is

$$t_2 = 2t' \quad \cdots(1.4)$$

where t' is the time taken by the wave to travel from P to A'.

Now from $\Delta PA'M$

$$c^2t'^2 = v^2t'^2 + d^2 \quad \text{or,} \quad t'^2 = \frac{d^2}{c^2-v^2}$$

$$[\because \text{ from the law of reflection } PP' = 2PM = 2AA']$$

or $$t' = \frac{d}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \simeq \frac{d}{c}\left(1+\frac{v^2}{2c^2}\right)$$

Hence, $$t_2 = 2t' = \frac{2d}{c}\left(1+\frac{v^2}{2c^2}\right) \quad \cdots(1.5)$$

The time difference between the two light beams to arrive at the back of plate P is

$$(\Delta t)_1 = t_1 - t_2 = \frac{dv^2}{c^3} \quad \cdots(1.6)$$

To remove any effect due to the failure of two path lengths (PM_1 and PM_2) to be exactly equal, in each measurement apparatus should be rotated by 90° to interchange the path lengths and introduce the same path difference in opposite direction. Therefore, the time difference for the two beams, after rotation is

$$(\Delta t)_2 = \frac{2d}{c}\left(1+\frac{v^2}{2c^2}\right)-\frac{2d}{c}\left(1+\frac{v^2}{c^2}\right) \quad \cdots(1.7)$$

Hence the time difference caused by rotation of the apparatus is

$$\Delta t = (\Delta t)_1-(\Delta t)_2 = \frac{dv^2}{c^3}-\left[\frac{2d}{c}\left(1+\frac{v^2}{2c^2}\right)-\frac{2d}{c}\left(1+\frac{v^2}{c^2}\right)\right]$$

$$= \frac{dv^2}{c^3}+\frac{dv^2}{c^3} = \frac{2d}{c}\left(\frac{v^2}{c^2}\right) \quad \cdots(1.8)$$

A non-zero value of Δt indicates a fringe shift for the change of phase relation between the two beams. Again, we know that a change in optical path difference by λ corresponds to a shift of one fringe. So, the expected fringe shift n, across the cross-wire of the telescope due to rotation of 90° of the apparatus is

$$n = \frac{c\Delta t}{\lambda} = \frac{2d}{\lambda}\frac{v^2}{c^2} \quad \cdots(1.9)$$

where, λ is the wavelength of the light used.

In Michelson-Morley experiment,

$$d = 11 \text{ m}, v = 3\times10^4 \text{ ms}^{-1}, c = 3\times10^8 \text{ ms}^{-1}, \lambda = 5{\cdot}9\times10^{-7} \text{ m}$$

$$\therefore \quad n = \frac{2\times11\times(3\times10^4)^2}{5{\cdot}9\times10^{-7}\times(3\times10^8)^2} = 0{\cdot}37$$

Thus, 0·37 *fringe will be displaced across the cross-wire.*

1.5.2. Result of the Experiment

The experimentally observed displacement of the fringe was 0·01 of a fringe. This was extremely small as compared to the theoretically predicted value. It is also noted that these results were not consistent. They repeated the experiment at different places on the earth's surface and at different seasons of the year with their sophisticated apparatus but they could not detect any measurable shift. So, practically it was a **null or negative result** which indicates that no fringe shift whatsoever was observed.

1.5.3. Significance of the Negative Result

This negative or null result suggests that

① There is no ether wind. So, *the concept of a absolute rest frame must be decarded and it is impossible to measure the speed of the earth relative to ether.*