Example 3 *Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$. Determine the following:*

(1) $|A \times B|$.

(2) *Number of relations from A to B.*

(3) *Number of binary relations on A.*

(4) *Number of relations from A to B that contain $(1, 2)$ and $(1, 5)$.*

(5) *Number of relations from A, B that contain exactly five ordered pairs.*

(6) *Number of binary relations on A that contain at least seven ordered pairs.*

▶ We have $|A| = m = 3$, $|B| = n = 3$. Therefore:

(1) $|A \times B| = mn = 9$.

(2) No. of relations from A to B is $2^{mn} = 2^9 = 512$.

(3) No. of binary relations on A is $2^{mm} = 2^{m^2} = 2^9 = 512$.

(4) Let $R_1 = \{(1, 2), (1, 5)\}$. We note that every relation from A to B that contains the elements $(1, 2)$ and $(1, 5)$ is of the form $R_1 \cup R_2$, where R_2 is a subset of $\overline{R_1}$ in $A \times B$. Therefore, the number of such relations is equal to the number of subsets of $\overline{R_1}$. Since $|\overline{R_1}| = |A \times B| - |R_1| = 9 - 2 = 7$, the number of subsets of $\overline{R_1}$ is $2^7 = 128$. Thus, there are $2^7 = 128$ number of relations from A to B that contain the elements $(1, 2)$ and $(1, 5)$.

(5) Since $A \times B$ contains 9 ordered pairs, the number of relations from A to B that contain exactly five ordered pairs is precisely the number of ways of choosing five ordered pairs from nine ordered pairs. This number is ${}^9C_5 = 126$.

(6) Similarly, the number of binary relations on A that contains at least seven elements (ordered pairs) is ${}^9C_7 + {}^9C_8 + {}^9C_9 = 46$. ■

Example 5 *Let a function $f : \mathsf{R} \to \mathsf{R}$ be defined by $f(x) = x^2 + 1$. Determine the images of the following subsets of* R*:*

(i) $A_1 = \{2, 3\}$ (ii) $A_2 = \{-2, 0, 3\}$

(iii) $A_3 = (0, 1)$ (iv) $A_4 = [-6, 3]$

► (i) We have $f(2) = 5$, $f(3) = 10$. Therefore, $f(A_1) = \{5, 10\}$.

(ii) We have $f(-2) = 5$, $f(0) = 1$, $f(3) = 10$. Therefore, $f(A_2) = \{5, 1, 10\}$.

(iii) Here,

$$A_3 = \{\, x \in R \mid 0 < x < 1 \,\}.$$

Therefore, $$f(A_3) = \{\, f(x) \mid 0 < x < 1 \,\} = \left\{ \left(x^2 + 1\right) \mid 0 < x < 1 \right\}$$

(iv) $f(A_4) = \{\, f(x) \mid -6 \le x \le 3 \,\} = \left\{ \left(x^2 + 1\right) \mid -6 \le x \le 3 \right\}$. ∎

(i) *Determine* $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$.

(ii) *Find* $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$.

(iii) *What are* $f^{-1}([-5,5])$ *and* $f^{-1}([-6,5])$?

► (i) By using the definition of the given f, we find that

$$f(0) = (-3\times 0)+1 = 1, \qquad f(-1) = \{-3\times(-1)\}+1 = 4,$$
$$f(5/3) = (3\times 5/3)-5 = 0, \qquad f(-5/3) = \{-3\times(-5/3)\}+1 = 6.$$

(ii) From the definition of the given f, we find that $f(x) = 0$ only when $x = 5/3$. (Observe that $f(x) \neq 0$ for $x \leq 0$). Therefore,

$$f^{-1}\{0\} = \{5/3\}.$$

Similarly,

$$f^{-1}(1) = \{\, x \in \mathsf{R} \mid f(x) = 1 \,\} = \{2, 0\}.$$

$f^{-1}(-1) = \{\, x \in \mathsf{R} \mid f(x) = -1 \,\} = \{4/3\}$; observe that $f(x) \neq -1$ when $x \leq 0$.

$$f^{-1}(3) = \{\, x \in \mathsf{R} \mid f(x) = 3 \,\} = \{8/3, -2/3\}.$$

$$f^{-1}(-3) = \{2/3\}.$$

$f^{-1}(-6) = \Phi$, because $f(x) \neq -6$ for *any* $x \in \mathsf{R}$.

(iii) We note that

$$\begin{aligned} f^{-1}([-5,5]) &= \{\, x \in \mathsf{R} \mid f(x) \in [-5,5] \,\} \\ &= \{\, x \in \mathsf{R} \mid -5 \leq f(x) \leq 5 \,\} \end{aligned}$$

When $x > 0$, we have $f(x) = 3x-5$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (3x-5) \leq 5$, or $0 \leq 3x \leq 10$, or $0 < x \leq 10/3$.

When $x \leq 0$, we have $f(x) = -3x+1$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (-3x+1) \leq 5$, or $-6 \leq -3x \leq 4$, or $2 \geq x \geq -4/3$, or $-4/3 \leq x \leq 2$. Thus,

$$\begin{aligned} f^{-1}([-5,5]) &= \{\, x \in \mathsf{R} \mid -4/3 \leq x \leq 2 \text{ or } 0 < x \leq 10/3 \,\} \\ &= \{\, x \in \mathsf{R} \mid -4/3 \leq x \leq 10/3 \,\} \\ &= [-4/3, 10/3] \end{aligned}$$

Similarly, we find that

$$f^{-1}([-6,5]) = [-4/3, 10/3]$$

■

Example 3 *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{1, 2, 3, 4, 5, 6\}$.

(a) *Find how many functions are there from A to B. How many of these are one-to-one? How many are onto?*

(b) *Find how many functions are there from B to A. How many of these are one-to-one? How many are onto?*

▶ Here, $|A| = m = 4$ and $|B| = n = 6$. Therefore:

(a) The number of functions possible from A to B is $n^m = 6^4 = 1296$.

The number of one-to-one functions possible from A to B is

$$\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360.$$

There is no onto function from A to B.

(b) The number of functions possible from B to A is $m^n = 4^6 = 4096$.

There is no one-to-one function from B to A.

The number of onto functions from B to A is

$$p(6,4) = \sum_{k=0}^{4} (-1)^k ({}^4C_{4-k})(4-k)^6$$

$$= 4^6 - 4 \times 3^6 + 6 \times 2^6 - 4 = 1560.$$

■

Example 1 *ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than 1/2 cm.*

▶ Consider the triangle DEF formed by the mid-points of the sides BC, CA and AB of the given triangle ABC; see Figure 5.16. Then the triangle ABC is partitioned into four small equilateral triangles (portions), each of which has sides equal to 1/2 cm. Treating each of these four portions as a pigeonhole and five points chosen inside the triangle as pigeons, we find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently, the distance between such points is less than 1/2 cm.

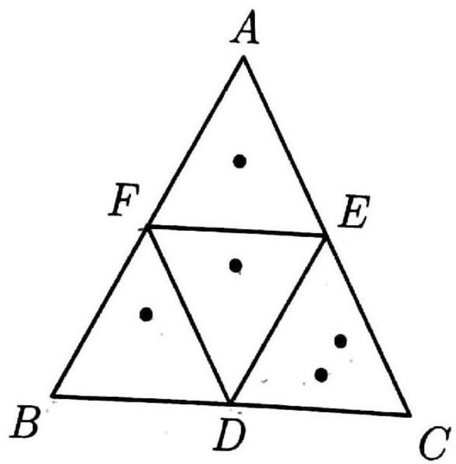

Figure 5.16

Example 5 *If 5 colours are used to paint 26 doors, prove that at least 6 doors will have the same colour.*

▶ Treating 26 doors as pigeons and 5 colours as pigeonholes, we find by using the generalized pigeonhole principle that at least one of the colours must be assigned to $\left(\frac{26-1}{5}\right) + 1 = 6$ or more doors.

This proves the required result. ■

Example 6 *Let f, g, h be functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by*

$$f(x) = x - 1, \quad g(x) = 3x,$$

$$h(x) = \begin{cases} 0, & \textit{if } x \textit{ is even} \\ 1, & \textit{if } x \textit{ is odd.} \end{cases}$$

Determine $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ and verify that $f \circ (g \circ h) = (f \circ g) \circ h$.

► We have

$$(g \circ h)(x) = g\{h(x)\} = 3h(x)$$

Therefore,

$$\begin{aligned} f \circ (g \circ h)(x) &= f\{(g \circ h)(x)\} \\ &= f\{3h(x)\} = 3h(x) - 1 \\ &= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd.} \end{cases} \end{aligned} \tag{i}$$

On the other hand, $(f \circ g)(x) = f\{g(x)\} = g(x) - 1 = 3x - 1$. Therefore,

$$\begin{aligned}\{(f \circ g) \circ h\}(x) &= (f \circ g)\{h(x)\} \\ &= 3h(x) - 1 \\ &= \begin{cases} -1 \text{ if } x \text{ is even} \\ 2 \text{ if } x \text{ is odd.} \end{cases} \end{aligned} \tag{ii}$$

From expression (i) and (ii), it follows that

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

■

Theorem 1. *If a function $f : A \to B$ is invertible then it has a unique inverse. Further, if $f(a) = b$, then $f^{-1}(b) = a$.*

Proof: Suppose $f : A \to B$ is invertible and it has g and h as inverses. Then g and h are functions from B to A such that

$$g \circ f = I_A, \quad h \circ f = I_A,$$

$$f \circ g = I_B, \quad h \circ g = I_B,$$

Then, we find that

$h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g.$

This proves that h and g are not different. Thus, f has a unique inverse (when it is invertible).

Now, suppose that $f(a) = b$. Then, if g is the inverse of f, we have

$$a = I_A(a) = (g \circ f)(a) = g\{f(a)\} = g(b).$$

Since $g = f^{-1}$, this proves that $f^{-1}(b) = a$. •

Remarks

(1) If f is invertible, the statements $f(a) = b$ and $a = f^{-1}(b)$ are equivalent.

(2) If $f = \{(a, b) \mid a \in A, b \in B\}$ is invertible, then $f^{-1} = \{(b, a) \mid b \in B, a \in A\}$ and conversely.

(3) If f is invertible then f^{-1} is invertible, and $\left(f^{-1}\right)^{-1} = f$.

Theorem 2. *A function $f : A \to B$ is invertible if and only if it is one-to-one and onto.*

Proof: First suppose that f is invertible. Then there exists a unique function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

Take any $a_1, a_2 \in A$. Then

$$\begin{aligned} f(a_1) = f(a_2) &\Rightarrow g\{f(a_1)\} = g\{f(a_2)\} \\ &\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2) \\ &\Rightarrow I_A(a_1) = I_A(a_2) \\ &\Rightarrow a_1 = a_2. \end{aligned}$$

This proves that f is one-to-one.

Next, take any $b \in B$. Then $g(b) \in A$, and $b = I_B(b) = (f \circ g)(b) = f\{g(b)\}$. Thus, b is the image of an element $g(b) \in A$ under f. Therefore, f is onto as well.

Conversely, suppose that f is one-to-one and onto. Then for each $b \in B$ there is a unique $a \in A$ such that $b = f(a)$. Now, consider the function $g : B \to A$ defined by $g(b) = a$. Then

$$(g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a), \text{ and}$$

$$(f \circ g)(b) = f\{g(b)\} = f(a) = b = I_B(b).$$

These show that f is invertible with g as the inverse.

This completes the proof of the theorem. •

Theorem 3. *Let A and B be finite sets with $|A| = |B|$ and f be a function from A to B. Then the following statements are equivalent.*

(1) *f is one-to-one* (2) *f is onto* (3) *f is invertible.*

Proof: Suppose $f : A \to B$ is one-to-one. Since A and B are finite sets with $|A| = |B|$, it follows that f is onto[§]. Consequently, f is invertible[¶].

Conversely, suppose f is invertible, then f is one-to-one and onto[**].

Thus, each of the three statements of the theorem implies the other two. The three statements are therefore equivalent. •

Remark: In view of the above theorem, we note that a function $f : A \to B$ where A and B are finite with $|A| = |B|$ is invertible if and only if f is one-to-one *or* onto.

Theorem 4. *If $f : A \to B$ and $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof: Since f and g are invertible functions, they are both one-to-one and onto. Consequently, $g \circ f$ is both one-to-one and onto. Therefore, $g \circ f$ is invertible.

Now, the inverse f^{-1} of f is a function from B to A and the inverse g^{-1} of g is a function from C to B. Therefore, if $h = f^{-1} \circ g^{-1}$ then h is a function from C to A.

We find that

$$\begin{aligned}(g \circ f) \circ h &= (g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ \left(f \circ f^{-1}\right) \circ g^{-1} = g \circ I_B \circ g^{-1} \\ &= g \circ g^{-1} = I_C\end{aligned}$$

and

$$\begin{aligned}h \circ (g \circ f) &= \left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) \\ &= f^{-1} \circ \left(g^{-1} \circ g\right) \circ f = f^{-1} \circ I_B \circ f \\ &= f^{-1} \circ f = I_A\end{aligned}$$

The above expressions show that h is the inverse of $g \circ f$; that is $h = (g \circ f)^{-1}$. Thus,

$$(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}.$$

This completes the proof of the theorem. •

[§]See Theorem 2 of Section 5.3.2.
[¶]See Theorem 2 above.

Example 6 *Let A = B = R, the set of all real numbers, and the functions $f : A \to B$ and $g : B \to A$ be defined by*

$$f(x) = 2x^3 - 1,\ \forall x \in A;\ g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3},\ \forall y \in B.$$

Show that each of f and g is the inverse of the other.

▶ We find that, for any $x \in A$,

$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g(y) &= \left\{\frac{1}{2}(y+1)\right\}^{1/3}, \text{ where } y = f(x)\\ &= \left\{\frac{1}{2}(2x^3 - 1 + 1)\right\}^{1/3}, \text{ because } y = f(x) = 2x^3 - 1\\ &= x.\end{aligned}$$

Thus, $g \circ f = I_A$.

Next, for any $y \in B$,

$$\begin{aligned}(f \circ g)(y) = f(g(y)) &= f\left(\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right)\\ &= 2\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]^3 - 1\\ &= 2\left[\frac{1}{2}(y+1)\right] - 1 = y\end{aligned}$$

Thus, $f \circ g = I_B$.

Accordingly, each of f and g is an invertible function, and further more each is the inverse of the other. ■

Example 15 *Let $A = \{1,2,3,4,5\}$. Define a relation R on $A \times A$ by $(x_1,y_1)R(x_2,y_2)$ if and only if $x_1 + y_1 = x_2 + y_2$.*

(i) *Verify that R is an equivalence relation on $A \times A$.*

(ii) *Determine the equivalence classes $[(1,3)], [(2,4)]$ and $[(1,1)]$.*

(iii) *Determine the partition of $A \times A$ induced by R.*

► (i) For all $(x,y) \in A\times A$, we have $x+y = x+y$; that is, $(x,y)R(x,y)$. Therefore, R is reflexive.

Next, take any $(x_1,y_1), (x_2,y_2) \in A\times A$ and suppose that $(x_1,y_1)R(x_2,y_2)$ Then $x_1+y_1 = x_2+y_2$. This gives $x_2+y_2 = x_1+y_1$ which means that $(x_2,y_2)R(x_1,y_1)$. Therefore, R is symmetric.

Next, take any $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in A \times A$ and suppose that $(x_1,y_1)R(x_2,y_2)$ and $(x_2,y_2)R(x_3,y_3)$. Then $x_1+y_1 = x_2+y_2$ and $x_2+y_2 = x_3+y_3$. This gives $x_1+y_1 = x_3+y_3$; that is, $(x_1,y_1)R(x_3,y_3)$. Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive. Therefore, R is an equivalence relation.

(ii) We note that

$$\begin{aligned}
[(1,3)] &= \{(x,y) \in A \times A \mid (x,y)R(1,3)\} \\
&= \{(x,y) \in A \times A \mid x+y = 1+3\} \\
&= \{(1,3),(2,2),(3,1)\}, \quad \text{because } A = \{1,2,3,4,5\}
\end{aligned}$$

Similarly, $[(2,4)] = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$[(1,1)] = \{(1,1)\}$.

(iii) To determine the partition induced by R, we have to find the equivalence classes of all elements (x,y), of $A \times A$, w.r.t. R. From what has been found above, we note that

$$\begin{aligned}
&[(1,1)] = \{(1,1)\}, \\
&[(1,3)] = [(2,2)] = [(3,1)], \\
&[(2,4)] = [(1,5)] = [(3,3)] = [(4,2)] = [(5,1)].
\end{aligned}$$

The other equivalence classes are

$$\begin{aligned}
&[(1,2)] = \{(1,2),(2,1)\} = [(2,1)] \\
&[(1,4)] = \{(1,4),(2,3),(3,2),(4,1)\} = [(2,3)] = [(3,2)] = [(4,1)] \\
&[(2,5)] = \{(2,5),(3,4),(4,3),(5,2)\} = [(3,4)] = [(4,3)] = [(5,2)] \\
&[(3,5)] = \{(3,5),(4,4),(5,3)\} = [(4,4)] = [(5,3)]
\end{aligned}$$

$$[(4,5)] = \{(4,5),(5,4)\} = [(5,4)]$$
$$[(5,5)] = \{(5,5)\}$$

Thus, $[(1,1)]$, $[(1,2)]$, $[(1,3)]$, $[(1,4)]$, $[(1,5)]$, $[(2,5)]$, $[(3,5)]$, $[(4,5)]$ and $[(5,5)]$ are the only distinct equivalence classes of $A \times A$ w.r.t. R. Hence the partition of $A \times A$ induced by R is represented by

$$A \times A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)] \cup [(4,5)] \cup [(5,5)]. \blacksquare$$

Example 3 *Let $A = \{1, 2, 3, 4, 6, 12\}$. On A, define the relation R by aRb if and only if a divides b. Prove that R is a partial order on A. Draw the Hasse digram for this relation.*

► From the definition of R, we note that

$$R = \{(a,b) \mid a, b \in A \text{ and } a \text{ divides } b\}$$
$$= \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,12),(2,2),(2,4),(2,6)$$
$$(2,12),(3,3),(3,6),(3,12),(4,4),(4,12),(6,6),(6,12),(12,12)\}$$

Evidently, $(a, a) \in R$ for all $a \in A$. Therefore, R is reflexive.

We check that the elements of R are such that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. Therefore, R is transitive.

Further, for all $a, b \in A$, if a divides b and b divides a, then $a = b$. Hence R is antisymmetric.

Therefore, R is a partial order on A. The Hasse diagram for R is shown below.

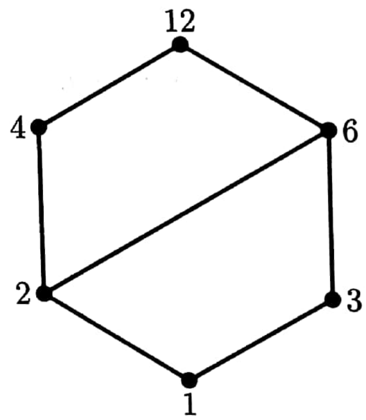

Figure 6.26

■

Example 1 *Consider the Hasse diagram of a poset (A, R) given below.*

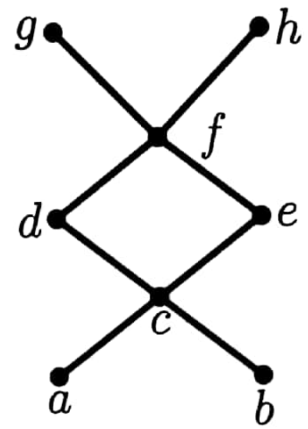

Figure 6.56

If $B = \{c, d, e\}$, find (if they exist)

(i) *all upper bounds of B,* (ii) *all lower bounds of B,*

(iii) *the least upper bound of B,* (iv) *the greatest lower bound of B.*

► By examining the given Hasse diagram, we note the following:

(i) All of c, d, e which are in B are related to f, g, h. Therefore, f, g, h are upper bounds of B.

(ii) The elements a, b and c are related to all of c, d, e which are in B. Therefore, a, b and c are lower bounds of B.

(iii) The upper bound f of B is related to the other upper bounds g and h of B. Therefore, f is the LUB of B.

(iv) The lower bounds a and b of B are related to the lower bound c of B. Therefore, c is the GLB of B. ■