* Half range series
→ We have 2-types of Half range series
i) Fourier Cosine series :- Defined as,
$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx \quad \text{over } (0, \pi) \text{ where}$$
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \; ; \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$
ii) Fourier Sine series :- Defined as,
$$f(x) = \sum b_n \sin nx \quad \text{over } (0, \pi) \text{ where}$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$
→ Half range series of $f(x)$ over $(0, l)$ defined as
Fourier cosine series over $(0, l)$
$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l} \quad \text{where,}$$
$$a_0 = 2/l\int_0^{l} f(x)\,dx \; ; \; a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$
Fourier sine series in $(0, l)$
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where,}$$
$$b_n = 2/l\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
Sums:-
1. Expand $f(x) = \begin{cases} 1/4 - x, & 0 < x < 1/2 \\ x - 3/4, & 1/2 < x < 1 \end{cases}$ as a fourier sine series

→ given $l = 1$, $\frac{n\pi x}{l} = n\pi x$

Fourier sine series in $(0,1)$, $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \rightarrow (1)$

where,

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\left[\int_0^{1/2}\left(\tfrac{1}{4}-x\right)\sin(n\pi x)\,dx + \int_{1/2}^{1}\left(x-\tfrac{3}{4}\right)\sin(n\pi x)\,dx\right]$$

$$= 2\left[\left[\left(\tfrac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2} + \left[\left(x-\tfrac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right]$$

$$= 2\left[\left[\left(-\tfrac{1}{4}\right)\left(-\frac{\cos n\pi/2}{n\pi}\right) - \frac{\sin n\pi/2}{n^2\pi^2}\right] - \left[-\frac{1}{4}\frac{\cos 0}{n\pi}\right]\right] +$$

$$\left[\tfrac{1}{4}\left(-\frac{\cos n\pi}{n\pi}\right)\right] - \left[\left(-\frac{1}{4}\right)\left(-\frac{\cos n\pi/2}{n\pi}\right) + \frac{\sin n\pi/2}{n^2\pi^2}\right]\Big].$$

$$= 2\left[\cancel{\frac{\cos\frac{n\pi}{2}}{4n\pi}} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{1}{4n\pi} - \frac{\cos n\pi}{4n\pi} - \cancel{\frac{\cos n\pi/2}{4n\pi}} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= 2\left[\frac{-2\sin\frac{n\pi}{2}}{n^2\pi^2} - \frac{(-1)^n}{4n\pi} + \frac{1}{4n\pi}\right]$$

$$= \frac{-4\sin n\pi/2}{n^2\pi^2} - \frac{(-1)^n}{2n\pi} + \frac{1}{2n\pi}$$

Substituting in (1)

$$f(x) = \sum\left[-\frac{4\sin\frac{n\pi}{2}}{n^2\pi^2} - \frac{(-1)^n}{2n\pi} + \frac{1}{2n\pi}\right]\sin(n\pi x)$$

2) obtain the fourier cosine series of $f(x) = x\sin x$ in $(0,\pi)$ (2)

→ Fourier cosine series in $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx \rightarrow ① \text{ where,}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad ; \quad a_n = 2/\pi \int_0^\pi f(x)\,\cancel{dx}\cos nx\,dx$$

$$a_0 = 2/\pi \int_0^\pi x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^\pi$$

$$= 2/\pi\Big[(-\pi\cos\pi)\Big] = 2/\cancel{\pi} \times (-\cancel{\pi}) \times (-1) = 2$$

$$a_n = 2/\pi \int_0^\pi x\sin x\cos nx\,dx = 2/\pi \int_0^\pi x(\cos nx\sin x)\,dx$$

$$= 2/\pi \int_0^\pi x \cdot 1/2\Big[\sin(nx+x) - \sin(nx-x)\Big]dx$$

$$= \frac{1}{\pi}\int_0^\pi \Big[x\sin(n+1)x - x\sin(n-1)x\Big]dx$$

$$= \frac{1}{\pi}\left[\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) - 1\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right] - \left[x\left(-\frac{\cos(n-1)x}{n-1}\right) - 1\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]\right]_0^\pi$$

$$= 1/\cancel{\pi}\left[\left[\frac{-\cancel{\pi}\cos(n+1)\pi}{n+1} + \frac{\cancel{\pi}\cos(n-1)\pi}{n-1}\right] - 0\right]$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$-(-1)^{n+1} = -(-1)(-1)^n = (-1)^n$$

$$(-1)^{n-1} = (-1)^n(-1)^{-1} = (-1)^n \cdot \frac{1}{-1} = -(-1)^n$$

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$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}$$

$$= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1}\right] = (-1)^n \left[\frac{n-1-n-1}{(n+1)(n-1)}\right]$$

$$= \frac{-2(-1)^n}{n^2-1} \quad \text{for } n \neq 1$$

Since a_n is infinity for $n=1$

$\therefore$ we need to find a_1

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{2}{\pi}\int_0^{\pi} x \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1 \cdot \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi \cdot \cos 2\pi}{2}\right] = \frac{-1}{2}$$

Substitute a_0, a_1 and a_n in eqn (1).

$$f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos nx$$

Additional problems:

1) Obtain the fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$

2) Express $f(x)$ as fourier cosine series in $(0, \pi)$ if

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$$

3) Find fourier series for $f(x) = |x|$ in $(-\pi, \pi)$,

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hence deduce $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+ = \pi^2/8$

4) $f(x) = |\cos x|$ for $-\pi \le x \le \pi$ (2)

5) $f(x) = \begin{cases} \pi x^2 & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$ obtain fourier series.

6. obtain cosine series for $f(x) = (x-1)^2$ in $(0,1)$.

7) obtain cosine series for $f(x) = \begin{cases} kx , & 0 \le x \le l/2 \\ k(l-x) , & l/2 \le x \le l. \end{cases}$

* PRACTICAL HARMONIC ANALYSIS:

It is the numerical method of finding fourier series over the interval $(0, 2\pi)$.

consider a fourier series $y = f(x)$ over $(0, 2\pi)$ i.e

$$y = \frac{a_0}{2} + (a_1\cos x + a_2\cos 2x + \dots) + (b_1\sin x + b_2\sin 2x + \dots)$$

where $\frac{a_0}{2}$ is the constant term & we compute

a_0 using formula, $a_0 = 2 \times \frac{\Sigma y}{n}$.

~~a~~ &) $a_1\cos x + ~~a~~ b_1\sin x$ is called the 1st harmonic of y. Illy $a_2\cos 2x + b_2\sin 2x$ is called the 2nd harmonic of y.

We compute these co-efficients using formula,

$$a_n = 2 \times \frac{\Sigma y \cos nx}{n}$$

$$b_n = 2 \times \frac{\Sigma y \sin nx}{n}$$

IIIly we compute fourier sine series & fourier cosine series upto nth harmonic using suitable formula.

Prob:- 1) Express y as a fourier series upto 2nd harmonic.

x:	0	60°	120°	180°	240°	300°	360°
y:	1	1.4	1.9	1.7	1.5	1.2	1

→ Fourier series of 'y' upto 2nd harmonic.

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) \rightarrow (1)$$

where $a_0 = 2 \times \frac{\Sigma y}{n}$, $a_1 = 2 \times \frac{\Sigma y \cos x}{n}$.

$b_1 = 2 \times \frac{\Sigma y \sin x}{n}$

$a_2 = 2 \times \frac{\Sigma y \cos 2x}{n}$; $b_2 = 2 \times \frac{\Sigma y \sin 2x}{n}$.

we compute fourier co-eff using following table.

x	y	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	1	1	0	1	0
60	1.4	0.7	1.21	-0.70	1.21
120	1.9	-0.95	1.65	-0.95	-1.65

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180	1.7	-1.7	0	+1.7. ~~(0.85)~~.	0
240	1.5	-0.7	-1.29	-0.75.	1.3
300	1.2	0.6	-1.04	-0.6.	-1.04
	$\Sigma y = 8.7$	$\Sigma y\cos x = -1.05$	$\Sigma y \sin x = 0.5$	$\Sigma y\cos 2x = -0.3$	$\Sigma y \sin 2x = -0.18$

(2)

$n = 6$

$a_0 = \frac{2 \times 8.7}{6} = \underline{2.9}$

$a_1 = \frac{2 \times (-1.1)}{6} = \underline{-0.37}$

$b_1 = \frac{2 \times 0.5}{6} = \underline{0.17}$

$a_2 = \frac{2 \times (-0.3)}{6} = \underline{-0.1}$

$b_2 = \frac{2 \times (-0.18)}{6} = \underline{-0.06}.$

$\frac{a_0}{2} = \frac{2.9}{2} = \underline{1.45}.$

Substitute in (1)

$y = 1.45 - 0.37\cos x + 0.17 \sin x - 0.1 \cos 2x - 0.06 \times \sin 2x.$

3) The following table give variation of periodic current over a period

(x)	t(sec) :	0	T/6	T/3	T/2	2T/3	5T/6	T
(y)	A (amp) :	1.98	1.3	1.05	1.3	−0.88	−0.25	1.98

Show that there is a direct current path of 0.75 amp, In the variable current & obtaining amplitude of first harmonic.

→ given $2l = T$ $\therefore l = T/2$ $\therefore \frac{\pi x}{l} = \frac{\pi t}{T/2} = \frac{2\pi t}{T}$.

The fourier series upto 1st harmonic of $y = A$.

$$y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)$$

i.e

$$A = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right) \longrightarrow (1)$$

where $a_0 = \frac{2 \times \Sigma y}{n} = \frac{2 \times \Sigma A}{n}$.

$$a_1 = \frac{2 \times \Sigma A \cos\left(\frac{2\pi t}{T}\right)}{n} ; \quad b_1 = \frac{2 \times \Sigma A \sin\left(\frac{2\pi t}{T}\right)}{n}$$

We compare fourier coefficient as follows.

t	$2\pi t/T$	A	$A\cos(2\pi t/T)$	$A\sin(2\pi t/T)$
0	0	1.98	1.98	0
T/6	60°	1.3	0.65	1.13

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$\pi/3$	120°	1.06	-0.53	0.9
$\pi/2$	180°	1.3	-1.30	0
$2\pi/3$	240°	-0.88	0.44	0.76
$5\pi/6$	300°	-0.25	-0.2513	0.22

$\Sigma A = 4.5$ $\quad \Sigma A\cos(2\pi t/T)$ $= 1.11$ $\quad \Sigma A\sin(2\pi t/T)$ $= 3.01$

$n = 6$.

$a_0 = 2 \times \frac{4.5}{6} = \underline{1.5}$. $\quad b_1 = 2 \times \frac{3.01}{6} = \underline{1}$

$a_1 = 2 \times \frac{1.11}{6} = \underline{0.37}$ $\quad \frac{a_0}{2} = \frac{1.5}{2} = \underline{0.75}$.

Substitute in ①.

$A = 0.75 + 0.37\cos\left(2\pi t/T\right) + \sin\left(2\pi t/T\right)$

Here the constant term means direct current path of a is 0.75.

The amplitude of first harmonic $= \sqrt{a_1^2 + b_1^2}$

$= \sqrt{0.37^2 + 1^2} = \underline{1.07}$

4) Obtain the 1st three co-efficient in the fourier cosine series of y where y is given below.

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

given $2l = 6$ (because there is no same term in 1st & last terms of y)

$\therefore l = 3$; $\frac{\pi x}{l} = \frac{\pi x}{3}$.

The fourier cosine series upto 3rd harmonic.

$$y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + a_3 \cos\left(\frac{3\pi x}{l}\right)$$

$$y = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta \longrightarrow (1)$$

where $a_0 = 2 \times \frac{\sum y}{n}$, $a_1 = 2 \times \frac{\sum y \cos\theta}{n}$

$a_2 = 2 \times \frac{\sum y \cos\theta}{n}$, $a_3 = \frac{2 \times \sum y \cos 3\theta}{n}$

we compute fourier co-eff as follows :-

x	$\theta = \pi x/3$	y	$y\cos\theta$	$y \cos 2\theta$	$y \cos 3\theta$
0	$\frac{\pi \times 0}{3} = 0$	4	4	4	4
1	$\pi/3 \times 1 = 60^\circ$	8	4	-4	-8
2	$\pi/3 \times 2 = 120^\circ$	15	-7.5	-7.5	15
3	180°	7	-7	7	-7
4	240°	6	-3	-3	6
5	300°	2	1	-1	-2
		$\sum y = 42$	$\sum y \cos 2\theta = -8.5$	$\sum y \cos 2\theta = -4.5$	$\sum y \cos 3\theta = 8$

$n = 6$.

$a_0 = 14$, $a_1 = -2.8$ $a_2 = -1.5$ $a_3 = 2.7$

$$y = 14/2 + (-2.8)\cos\theta + (-1.5)\cos 2\theta + 2.7 \cos 3\theta.$$

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5) The turning moment given for a series of value of

θ :	0°	30°	60°	90°	120°	150°	180°
T :	0	5224	8097	7850	5499	2626	0

(2).

Obtain the 1st four terms of series of sine's to represent T & calculate T for $\theta = 75^\circ$.

→ The fourier sine series upto 4^{th} harmonic.

$y = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x$

i.e $T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta$ → (1).

where $b_1 = 2 \times \frac{\Sigma y \sin x}{n} = 2\times \frac{\Sigma T \sin\theta}{n}$.

$b_2 = 2\times \frac{\Sigma T \times \sin 2\theta}{n}$; $b_3 = 2\times \frac{\Sigma T \times \sin 3\theta}{n}$.

$b_4 = 2\times \frac{\Sigma T \times \sin 4\theta}{n}$

θ	T	$T\sin\theta$	$T\sin 2\theta$	$T\sin 3\theta$	$T\sin 4\theta$
0	0	0	0	0	0
30	5224				
60	8097				
90	7850				
120	5499				
150	2626.				

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$b_1 = 7850$.

$b_2 = 1500$.

$b_3 = 0$

$b_4 = 0$

Substitute in (1)

$T = 7850 \sin\theta + 1500 \sin 2\theta$

when $\theta = 75^{0}$.

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