Module - 2

Chapter 3

Mathematical Induction

In Section 2.4, we outlined some standard methods of proving and disproving mathematical statements. In this chapter, we consider one more method of proof called the *Method of Mathematical Induction*.

3.1 Well-Ordering Principle; Induction Principle

The method of mathematical induction is based on a Principle called the *Induction Principle**. This principle can be proved by using another principle known as the *Well-ordering Principle*. These two principles highlight some important ***Properties of Integers***.

Throughout our discussions here, Z^+ denotes the set of all positive integers.

Well-ordering Principle

The Well-ordering Principle states as follows:

Every nonempty subset of Z^+ *contains a smallest (least) element.*

An alternative way of stating this principle is that the set of all positive integers is *well-ordered*.

Induction Principle

The Induction Principle states as follows:

Let $S(n)$ denote an open statement that involves a positive integer n. Suppose that the following conditions hold:

(1) *$S(1)$ is true.*

(2) *If whenever $S(k)$ is true for some particular, but arbitrarily chosen $k \in Z^+$, then $S(k+1)$ is true.*

Then $S(n)$ is true for all $n \in Z^+$.

*Also known as the *Finite Induction Principle*.

The following is a proof of this statement.

Proof: Suppose $S(n)$ is a statement for which the conditions (1) and (2) hold.

Let F denote the set of all those positive integers for which $S(n)$ is false; that is,

$$F = \{m \in \mathrm{Z}^+ \mid S(m) \text{ is false}\}.$$

Assume that $F \neq \Phi$. Then, by the well-ordering principle, F contains a least element, say k. Since $S(1)$ is true, $1 \notin F$ and so $k \neq 1$. As such, $k > 1$. and consequently, $k - 1 \in \mathrm{Z}^+$. Since $(k-1) < k$ and k is a least element of F, $k-1 \notin F$. Therefore, $S(k-1)$ is true. By the condition (2), it follows that $S(k-1+1) = S(k)$ is true. This means that $k \notin F$. This contradicts the fact that F contains k. This contradiction arose from the assumption that $F \neq \Phi$. Hence $F = \Phi$. This means that there is no positive integer m for which $S(m)$ is false. This is equivalent to saying that $S(n)$ is true for all positive integers n.

This completes the proof of the Induction Principle. •

The following is an alternative form of the Induction Principle which is referred to as Strong Induction Principle.

Strong Induction Principle

Statement: Let $S(n)$ denote a statement that involves a positive integer n. Suppose that

(i) for some positive integers n_0, $n_0 + 1$, $n_0 + 2, \ldots, n_0 + p$, the statements $S(n_0)$, $S(n_0 + 1)$, $S(n_0 + 2), \ldots S(n_0 + p)$ are true, and

(ii) if whenever $S(n_0)$, $S(n_0 + 1), \ldots S(k)$, where $k \geq n_0 + p$ are true, then $S(k+1)$ is true.

Then $S(n)$ is true for all positive integers $n \geq n_0$.

Proof: Let $F = \{m \in \mathrm{Z}^+ \mid m \geq n_0 \text{ and } S(m) \text{ is false}\}$. Then, since $S(n_0), S(n_0+1), S(n_0+2), \ldots, S(n_0+p)$ are supposed to be true, it follows that $n_0, n_0+1, n_0+2, \ldots n_0+p \notin F$.

Now suppose that $F \neq \Phi$. Then, by the well-ordering principle, it follows that F has a least element, say r. Then $r > n_0 + p$ and $r - 1 \notin F$. Thus, $S(n_0)$, $S(n_0+1), \ldots S(r-1)$ are true. Therefore, by the second of the suppositions, it follows that $S(r)$ is true. Hence $r \notin F$. This is a contradiction. This contradiction arose because of the assumption that $F \neq \Phi$. Hence $F = \Phi$. This means that there is no $m \in \mathrm{Z}^+$ with $m \geq n_0$ for which $S(m)$ is false. In other words, $S(n)$ is true for all positive integers $n \geq n_0$. This completes the proof. •

3.2 Method of Mathematical Induction

Suppose we wish to prove that a certain statement $S(n)$ is true for all integers $n \geq 1$. The method of proving such a statement on the basis of the Induction Principle is called *the method of mathematical induction.** This method consists of the following two steps, respectively called the *basis step* and the *induction step*.

*The student is already familiar with this method of proof; he may recall the proof of the binomial theorem for a positive integral index.

(1) **Basis step:** Verify that the statement $S(1)$ is true; that is, verify that $S(n)$ is true for $n = 1$.

(2) **Induction step:** Assuming that $S(k)$ is true, where k is an integer ≥ 1, show that $S(k+1)$ is true.

Remarks:

(1) In the method of induction, the basis step is of fundamental importance, and the induction step must follow the basis step; the induction step without the basis step does not constitute a proof by induction.

(2) Some times, we will be required to prove a statement $S(n)$ for $n \geq n_0$, where n_0 is a fixed integer (which may be zero, less than zero or greater than zero). In such a situation, the verification of the truthness of $S(n_0)$ forms the basis step for the method of induction. Then, in the induction step, we take $k \geq n_0$.

(3) In some situations, the verification of the truthness of $S(n)$ for $n = n_0, n_0 + 1, n_0 + 2, \ldots, n_0 + p$, for some particular p, will be required for the induction step. Then, in the induction step, the truthness of $S(n_0), S(n_0 + 1), S(n_0 + 2), \ldots S(k - 1), S(k)$ for $k \geq n_0 + p$ will be assumed and the truthness of $S(k + 1)$ will be established. The Strong Induction Principle forms the basis for this modified method.

Example 1 *Prove by mathematical induction that, for all positive integers $n \geq 1$,*

$$1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n + 1).$$

▶ Here, we have to prove the statement

$$S(n) : 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$$

for all integers $n \geq 1$.

Basis step: We note that $S(1)$ is the statement

$$1 = \frac{1}{2} \cdot 1 \cdot (1 + 1)$$

which is clearly true. Thus, the statement $S(n)$ is verified for $n = 1$.

Induction step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k) : 1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k + 1).$$

Using this, we find that (by adding $k+1$ to both sides)

$$\begin{aligned}(1+2+3+\cdots k)+(k+1) &= \frac{1}{2}k(k+1)+(k+1)\\ &= (k+1)\left\{\frac{1}{2}k+1\right\} = \frac{1}{2}(k+1)(k+2)\end{aligned}$$

This is precisely the statement $S(k+1)$.

Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truthness of $S(n)$ for $n = k+1$ is established.

The proof of the desired result by the method of induction is complete. ■

Example 2 *Prove that, for each* $n \in \mathbb{Z}^+$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

▶ Let $S(n)$ denote the given statement.

Basis step: We note that $S(1)$ is the statement

$$1^2 = \frac{1}{6} \times 1 \times 2 \times 3$$

which is clearly true.

Induction step: We assume that $S(n)$ is true for $n = k$ where $k \geq 1$; that is, we assume that the following statement is true:

$$S(k): 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

Adding $(k+1)^2$ to both sides of this, we obtain

$$\begin{aligned}&1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2\\ &\quad = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2\\ &\quad = (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]\\ &\quad = (k+1) \times \left(\frac{2k^2 + k + 6k + 6}{6}\right)\\ &\quad = \frac{1}{6}(k+1)(k+2)(2k+3).\end{aligned}$$

This is precisely the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true whenever the statement $S(k)$ is true for $k \geq 1$.

This completes the proof of the required result by the method of induction. ■

Example 3 *Prove, by mathematical induction, that*

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

for all integers $n \geq 1$.

► Let $S(n)$ denote the given statement.

Basis step: We note that $S(1)$ is the statement

$$1^2 = \frac{1}{3} \times 1 \times 3$$

which is clearly true.

Induction step: We assume that $S(n)$ is true for $n = k$, where $k \geq 1$. Then

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Adding $(2k+1)^2$ to both sides of this, we obtain

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots &+ (2k-1)^2 + (2k+1)^2 \\
&= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2 \\
&= \frac{1}{3}(2k+1)\{k(2k-1) + 3(2k+1)\} \\
&= \frac{1}{3}(2k+1)\left(2k^2 + 5k + 3\right) \\
&= \frac{1}{3}(2k+1)(k+1)(2k+3)
\end{aligned}$$

This is the precisely the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true whenever the statement $S(k)$, where $k \geq 1$, is true. Hence, by mathematical induction, it follows that $S(n)$ is true for all integers $n \geq 1$. ∎

Example 4 *If n is any positive integer, prove that*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2),$$

using mathematical induction.

► Let $S(n)$ denote the given statement.

Basis step: We note that $S(1)$ is the statement

$$1 \cdot 2 = \frac{1}{3}(1 \times 2 \times 3)$$

which is clearly true.

Induction step: We assume that $S(n)$ is true for $n = k$, where $k \geq 1$. Then

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2).$$

Consequently, we get [by adding $(k+1)(k+2)$ to both sides]

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)(k+2) = \left\{\frac{1}{3}k(k+1)(k+2)\right\} + (k+1)(k+2)$$
$$= (k+1)(k+2)\left\{\frac{1}{3}k + 1\right\} = \frac{1}{3}(k+1)(k+2)(k+3)$$

This is precisely the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true whenever the statement $S(k)$ is true for $k \geq 1$.

Hence by mathemetical induction, it follows that the statement $S(n)$ is true for all integers $n \geq 1$.

■

Example 5 *By mathematical induction, prove that* $(n!) \geq 2^{n-1}$ *for all integers* $n \geq 1$.

▶ Here, we have to prove that

$$S(n) : n! \geq 2^{n-1} \quad \text{is true for all integers } n \geq 1 .$$

Basis step: For $n = 1$, $S(n)$ reads $1! \geq 2^{1-1}$, which is obviously true. Thus, $S(n)$ is verified for $n = 1$.

Induction step: We assume that $S(n)$ is true for $n = k$, where $k \geq 1$; that is, we assume that

$$k! \geq 2^{k-1}, \quad \text{or} \quad 2^{k-1} \leq k!$$

is true. This yields

$$\begin{aligned} 2^k = 2 \cdot 2^{k-1} &\leq 2 \cdot k! \\ &< (k+1) \cdot k!, \text{ because } 2 < k+1 \text{ for } k \geq 1. \\ &= (k+1)! \\ \textit{i.e.,} \quad (k+1)! &\geq 2^k. \end{aligned}$$

This is precisely the statement $S(n)$ for $n = k+1$. Thus, on the assumption that $S(n)$ is true for $n = k \geq 1$, we have proved that $S(n)$ is true for $n = k+1$.

Hence, by mathematical induction, $S(n)$ is true for all integers $n \geq 1$. ■

Example 6 *Prove that* $4n < (n^2 - 7)$ *for all positive integers* $n \geq 6$.

▸ Here, we have to prove that the statement

$$S(n) : 4n < \left(n^2 - 7\right)$$

is true for all positive integers $n \geq n_0$, where $n_0 = 6$.

Basis step: We observe that

$$S(6) : (4 \times 6) < \left(6^2 - 7\right)$$

is true. Thus, $S(n)$ is true for $n = n_0 = 6$.

Induction step: We assume that $S(n)$ is true for $n = k$ where $k \geq 6$; i.e., we assume that

$$4k < \left(k^2 - 7\right) \quad \text{for} \quad k \geq 6.$$

Then,

$$\begin{aligned} 4(k+1) &= 4k + 4 \\ &< \left(k^2 - 7\right) + 4 \\ &< \left(k^2 - 7\right) + (2k+1) \quad \text{because when } k \geq 6, \text{ we have } 2k + 1 \geq 13 > 4, \\ &= (k+1)^2 - 7 \end{aligned}$$

This shows that $S(k+1)$ is true.

By mathematical induction, it now follows that $S(n)$ is true for all positive integers $n \geq 6$. This proves the required result. ∎

Example 7 *Show that* $2^n > n^2$ *for all positive integers n greater than 4.*

▸ Here, we have to prove that the statement $S(n) : 2^n > n^2$ is true for all integers $n \geq 5$.

Basis step: For $n = 5$, the statement $S(n)$ reads

$$S(5) : 2^5 > 5^2 \quad \text{(i.e., } 32 > 25)$$

which is clearly true.

Induction step: We assume that $S(n)$ is true for $n = k$, where $k \geq 5$; that is, we assume that

$$2^k > k^2, \quad \text{for} \quad k \geq 5.$$

This yields (on multiplying both sides by 2)

$$2^{k+1} > 2\,k^2 \tag{i}$$

Now, since $k > 4$, we find that

$$\begin{aligned} 2k^2 = k^2 + k^2 &= k^2 + (k \times k) \\ &> k^2 + 4k = k^2 + 2k + 2k \\ &> k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

Thus, for $k > 4$, we have $2k^2 > (k+1)^2$, and expression (i) yields $2^{k+1} > (k+1)^2$. This is the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true when $S(k)$ is strue for $k \geq 5$.

Hence, by mathematical induction, the given statement $S(n)$ is true for all integers greater than 4. ■

Example 8 *Prove by mathematical induction that, for every positive integer n, 5 divides* $n^5 - n$.

▶ Let $S(n)$ be the given statement.

Basis step: We note that $S(1)$ is the statement

$$5 \text{ divides } 1^5 - 1.$$

Since $1^5 - 1 = 0$, this statement is true.

Induction step: We assume that $S(n)$ is true for $n = k$, where $k \geq 1$. That is, we assume that, for $k \geq 1$,

$$5 \text{ divides } k^5 - k.$$

This means that $k^5 - k$ is a multiple of 5; that is

$$k^5 - k = 5m, \text{ for some positive integer } m.$$

Consequently, we find that

$$\begin{aligned} (k+1)^5 - (k+1) &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1) \\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k) \\ &= 5m + 5(k^4 + 2k^3 + 2k^2 + k) \\ &= 5(m + k^4 + 2k^3 + 2k^2 + k) \end{aligned}$$

This shows that $(k+1)^5 - (k+1)$ is a multiple of 5; that is, 5 divides $(k+1)^5 - (k+1)$. This is precisely the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true whenever the statement $S(k)$ is true for $k \geq 1$. Hence, by mathematical inductin, the statement $S(n)$ is true for every positive integer n. ■

Example 9 *Prove that every positive integer $n \geq 24$ can be written as a sum of 5′s and/or 7′s.*

► Here, we have to prove that the statement

$S(n)$: n can be written as a sum of 5′s and/or 7′s

is true for all integers $n \geq 24$.

Basis step: We note that

$$24 = (7+7) + (5+5).$$

This shows that $S(24)$ is true.

Induction step: We assume that $S(n)$ is true for $n = k$ where $k \geq 24$. Then

$$k = (7+7+\cdots) + (5+5+\cdots).$$

Suppose this representation of k has r number of 7's and s number of 5's. Since $k \geq 24$, we should have $r \geq 2$ and $s \geq 2$.

Using this representation of k, we find that

$$\begin{aligned} k+1 &= \{\underbrace{(7+7+\cdots)}_{r} + \underbrace{(5+5+\cdots)}_{s}\} + 1 \\ &= \underbrace{(7+7+\cdots)}_{(r-2)} + (7+7) + \underbrace{(5+5+\cdots)}_{s} + 1 \\ &= \underbrace{(7+7+\cdots)}_{(r-2)} + \underbrace{(5+5+\cdots)}_{s+3} \end{aligned}$$

This shows that $(k+1)$ is a sum of 7's and 5's. Thus, $S(k+1)$ is true.

Hence, by mathematical induction, $S(n)$ is true for all positive integers $n \geq 24$.

<u>***Aliter:***</u> The above result can also be proved with the use of the alternative form of the Principle of Induction, as described below.

Basis step: We first note that

$$24 = 7+7+5+5, \quad 25 = 5+5+5+5+5,$$
$$26 = 7+7+7+5, \quad 27 = 7+5+5+5+5,$$
$$28 = 7+7+7+7.$$

Thus, $S(n)$ is true for $n_0 = 24$, $n_0 + 1 = 25$, $n_0 + 2 = 26$, $n_0 + 3 = 27$, $n_0 + 4 = 28$.

Induction step: We assume that $S(n)$ is true for $n_0, n_0+1, n_0+2, n_0+3, n_0+4, \ldots k$, where $k \geq (n_0 + 4) = 28$. Then, in particular, $S(k-4)$ is true; that is, $k-4$ is a sum of 7′s and/or 5′s. Consequently,

$k+1 = (k-4) + 5$ is also a sum of 7′s and/or 5′s. That is, $S(k+1)$ is true.

Hence, by the alternative form of the principle of induction, it follows that $S(n)$ is true for all integers ≥ 24. ∎

Example 10 *By mathematical induction, prove that, for every positive integer n, the number* $A_n = 5^n + 2 \cdot 3^{n-1} + 1$ *is a multiple of 8.*

▶**Basis step:** We note that

$$A_1 = 5^1 + 2 \cdot 3^0 + 1 = 8$$

Thus, for $n = 1$, the number A_n is a multiple of 8.

Induction step: Assume that A_n is a multiple of 8 for $n = k \geq 1$.

Using the given definition of A_n, we find that

$$\begin{aligned} A_{k+1} - A_k &= (5^{k+1} + 2.3^k + 1) - (5^k + 2 \cdot 3^{k-1} + 1) \\ &= (5-1)5^k + 2(3-1)3^{k-1} \\ &= 4\left(5^k + 3^{k-1}\right). \end{aligned}$$

Since 5 and 3 are odd, 5^k and 3^{k-1} are also odd. Consequently, $5^k + 3^{k-1}$ is even. Hence, $4\left(5^k + 3^{k-1}\right)$ is a multiple of 8. That is, $(A_{k+1} - A_k)$ is a multiple of 8. Since A_k is a multiple of 8 by assumption, it follows that A_{k+1} is also a multiple of 8. Thus, A_n is a multiple of 8 for $n = k + 1$ if A_n is a multiple of 8 for $n = k$.

This completes the proof of the required result, by induction. ■

Example 11 *Prove by mathematical induction that, for any positive integer n, the number* $11^{n+2} + 12^{2n+1}$ *is divisible by 133.*

▶ Let $A_n = 11^{n+2} + 12^{2n+1}$.

Basis step: We note that

$$A_1 = 11^{1+2} + 12^{2+1} = 11^3 + 12^3 = 1331 + 1728 = 3059.$$

We readily check that $3059 = 23 \times 133$, so that 133 divides 3059.

Thus, A_n is divisible by 133 for $n = 1$.

Induction step: Assume that A_n is divisible by 133 for $n = k \geq 1$.

Now, we find that

$$\begin{aligned} A_{k+1} &= 11^{k+3} + 12^{2(k+1)+1} \quad \text{(using the definition of } A_n) \\ &= \left(11^{k+2} \times 11\right) + \left(12^{2k+1} \times 12^2\right) \\ &= \left(11^{k+2} \times 11\right) + \left(12^{2k+1} \times 144\right) \\ &= \left\{11^{k+2} \times 11\right\} + \left\{12^{2k+1} \times (11 + 133)\right\} \\ &= \left(11^{k+2} + 12^{2k+1}\right) \times 11 + (12^{2k+1} \times 133). \\ &= (A_k \times 11) + (12^{2k+1} \times 133) \end{aligned}$$

This representation shows that A_{k+1} is divisible by 133 when A_k is divisible by 133.

This completes the proof of the required result by induction. ■

Example 12 *Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ for all positive integers n.*

▶ Consider the statement $S(n) : a_n \leq 3^n$.

Basis step: We observe that

$$a_0 = 1 \leq 3^0, \quad a_1 = 2 \leq 3^1, \quad a_2 = 3 \leq 3^2$$

Thus, $S(n)$ is true for $n = 0, 1, 2$.

Induction step: Now, assume that $S(n)$ is true for $n = 0, 1, 2, \ldots, k$, where $k \geq 2$. Then, we note that

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} + a_{k-2} && \text{(by the definition of } a_n \text{ given)} \\ &\leq 3^k + 3^{k-1} + 3^{k-2}, && \text{because } S(k), S(k-1), \\ & && S(k-2) \text{ and } S(k-3) \text{ are true by assumption} \\ &\leq 3^k + 3^k + 3^k, && \text{because } 3^{k-1} \leq 3^k \text{ and } 3^{k-2} \leq 3^k \\ &= 3 \times 3^k = 3^{k+1} \end{aligned}$$

Thus, $S(k+1)$ is true.

Therefore, by the principle of mathematical induction (alternative form), $S(n)$ is true for all positive integers n. This proves the required result. ∎

Example 13 *Let $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}, \cdots, H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that*

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n$$

for all positive integers $n \geq 1$.

▶ Here, we have to prove that the statement

$$S(n) : \sum_{i=1}^{n} H_i = (n+1)H_n - n$$

is true for all positive integers $n \geq 1$.

Basis step: We note that $S(1)$ is the statement

$$H_1 = 2H_1 - 1$$

$$\text{i.e,} \quad 1 = 2 \times 1 - 1 \quad (\text{because } H_1 = 1)$$

which is clearly true.

Induction step: We assume that $S(n)$ is true for $n = k \geq 1$. Then

$$\sum_{i=1}^{k} H_i = (k+1)H_k - k$$

so that

$$\sum_{i=1}^{k+1} H_i = \sum_{i=1}^{k} H_i + H_{k+1}$$

$$= \{(k+1)H_k - k\} + H_{k+1}$$

$$= (k+1)\left(H_{k+1} - \frac{1}{k+1}\right) - k + H_{k+1}$$

because $H_k = H_{k+1} - \dfrac{1}{k+1}$, by the definition of H_n

$$= (k+2)H_{k+1} - (k+1)$$

This shows that $S(k+1)$ is true.

Hence, by mathematical induction, it follows that $S(n)$ is true for all positive integers $n \geq 1$. This proves the required result.

Remark: The numbers $H_1, H_2 \cdots H_n$ considered in this Example are called ***Harmonic numbers***. The sequence formed by these is the *Harmonic sequence*. ■

Example 14 *Prove the following statement by mathematical induction:*

"If a set has n elements, then its power set has 2^n elements".

► Here, we have to prove the statement

$S(n)$: For any finite set A, if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Basis step: For $n = 0$, we have $A = \Phi$, the null set. Then $\mathcal{P}(A) = \{\Phi\}$, so that $|\mathcal{P}(A)| = 1 = 2^0$. This verifies the truthness of the statement $S(n)$ for $n = 0$.

Induction step: Assume that the statement $S(n)$ is true for $n = k \geq 0$; that is, assume that $|\mathcal{P}(A)| = 2^k$ when $|A| = k$ for any set A.

Now, consider a set B with $|B| = k + 1$. From this set, let us keep aside one particular element, say x. Then $C = B - \{x\}$ is a set with k elements. By the assumption made, $|\mathcal{P}(C)| = 2^k$; that is, C has 2^k subsets. We note that all these subsets are also subsets of B. We can form another 2^k subsets of B by taking the union of each subset of C with $\{x\}$. None of these newly formed subsets belong to $\mathcal{P}(C)$. As such, the number of subsets of B is $2^k + 2^k = 2^{k+1}$; that is, $|\mathcal{P}(B)| = 2^{k+1}$. This is precisely the statement $S(n)$ for $n = k + 1$.

Thus, on the assumption that $S(n)$ is true for $n = k$, we have proved that $S(n)$ is true for $n = k + 1$. This completes the proof of the given statement, by mathematical induction. ■

Exercises

1. Prove the following statements by mathematical induction. (Here, n is an integer ≥ 1).

(1) $2 + 4 + 6 + \cdots + 2n = n(n+1)$

(2) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

(3) $1 + 4 + 7 + \cdots + (3n-2) = (n/2)(3n-1)$

(4) $1 + 5 + 9 + \cdots + (4n-3) = n(2n-1)$

(5) $1^3 + 2^3 + 3^3 + \cdots n^3 = [n(n+1)/2]^2$

(6) $1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

(7) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

(8) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2n+1}$

(9) $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

2. Prove the following (by mathematical induction):

(1) $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$ (2) $\sum_{i=1}^{n} i2^i = 2 + (n-1)2^{n+1}$

(3) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ (4) $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

3. Prove by mathematical induction that

(1) $n < 2^n$ for all integers $n \geq 1$. (2) $3^n > n^3$ for all integers $n \geq 1$.

4. Prove that, for every integer $n \geq 1$, $n^3 + 2n$ is divisible by 3.

5. Prove that 3 divides $\left(n^3 - n\right)$ for every integer $n \geq 2$.

6. By mathematical induction, prove that 6 divides $n\left(n^2 + 5\right)$ for each positive integer n.

7. Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

8. Prove that every positive integer greater than or equal to 14 may be written as a sum of 3's and/or 8's.

9. Let $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Prove that $a_n < (7/4)^n$ for all integers $n \geq 1$.