ME 193B / 292B: Feedback Control of Legged Robots
HW #1

Background: The Three-Link Robot

In this homework, we will derive the kinematics and dynamics of a simple, multi-link legged robot - one with two legs (without knees), a torso and constrained to move in the vertical plane (see Figure 1). Similar to the bouncing ball problem illustrated in class, the robot has a continuous-time dynamics (that we derive in Problem 1) and undergoes a discrete change in its velocities upon a rigid impact (that you will derive in HW 2). It is important to note that the methods used to derive the dynamics in this homework also apply to higher dimensional robots such as *Cassie* and *Atlas*.

Robot parameters

Table 1 presents the various mechanical parameters of the robot links. Assume the center of mass of each link to be located at the geometric center of the link.

Link	Mass (kg)	Moment of Inertia about CoM ($kg - m^2$)	Link Length (m)
Torso	10	1	0.5
Leg	5	0.5	1

Table 1: Model Parameters for the Three-Link Robot

Configuration Variables

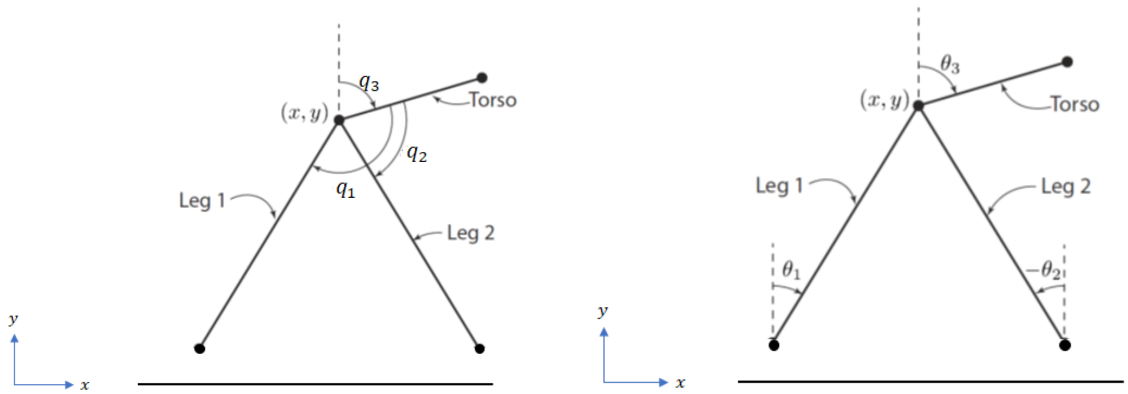
There are several ways to represent the configuration of the robot. *Configuration variables* are the minimum number of variables required to completely define the configuration (i.e. the position and orientation of the various links) of the robot. Figure 1 illustrates two different (but equivalent) representations - Figure 1a illustrates the configuration variables in terms of relative angles, where as, Figure 1b illustrates the configuration variables in terms of absolute angles.

Throughout this homework, unless otherwise stated, we will assume the order of the configuration variable $q \in \mathbb{R}^5$ as:

$$q = \begin{bmatrix} x \\ y \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{1}$$

Problems

Problem 1 Lagrangian Dynamics



(a) Relative Coordinates (b) Absolute Coordinates

Figure 1: Two configuration variable representations of the Three-Link Robot. Here, x, y represent the Cartesian position of the hip joint with respect to the inertial frame, q_3 or θ_3 are the absolute torso angle with respect to the vertical, while q_1, q_2 are the two leg angles relative to the torso and θ_1, θ_2 are the absolute leg angles. Note that each link has distributed mass with the center-of-mass (CoM) at the center of the link. The link length, link mass and link inertia about the link CoM are provided in Table 1.

We will derive the continuous-time dynamics model for the three-link robot. We will use the configuration variables represented in Figure 1a. We will numerically compute various quantities for the following two configurations and velocities $(q, \dot{q})$:

$$\text{(i)} \quad (q, \dot{q}) = \left(\begin{bmatrix} 0.5\ m \\ \frac{\sqrt{3}}{2}\ m \\ 150^\circ \\ 120^\circ \\ 30^\circ \end{bmatrix}, \begin{bmatrix} -0.8049\ m \cdot s^{-1} \\ -0.4430\ m \cdot s^{-1} \\ 0.0938\ rad \cdot s^{-1} \\ 0.9150\ rad \cdot s^{-1} \\ 0.9298\ rad \cdot s^{-1} \end{bmatrix} \right), \text{ and}$$

$$\text{(ii)} \quad (q, \dot{q}) = \left(\begin{bmatrix} 0.3420\ m \\ 0.9397\ m \\ 170^\circ \\ 20^\circ \\ 30^\circ \end{bmatrix}, \begin{bmatrix} -0.1225\ m \cdot s^{-1} \\ -0.2369\ m \cdot s^{-1} \\ 0.5310\ rad \cdot s^{-1} \\ 0.5904\ rad \cdot s^{-1} \\ 0.6263\ rad \cdot s^{-1} \end{bmatrix} \right).$$

(a) **Position of Link Center-of-Mass**

Symbolically compute the position of the center of mass of each of the links as a function of the configuration variables q.

For the given two numerical configurations, provide the position of the center-of-mass of the three links as a matrix:

$$P := \begin{bmatrix} \text{Position of CoM of Link 1}, & \text{Position of CoM of Link 2}, & \text{Position of CoM of Link 3} \end{bmatrix}.$$

(b) **Velocity of Link Center-of-Mass**

Symbolically compute the velocities of center-of-mass of the three links as a function of the configuration variables q and their velocities $\dot{q}$.

For the given two numerical configurations and velocities, provide the velocity of the center-of-mass of the three links as a matrix:

$$V := \begin{bmatrix} \text{Velocity of CoM of Link 1}, & \text{Velocity of CoM of Link 2}, & \text{Velocity of CoM of Link 3} \end{bmatrix}.$$

(c) **Total Kinetic Energy**
Symbolically compute the total kinetic energy of the system. For the given two numerical configurations, provide the kinetic energy.

(d) **Total Potential Energy**
Symbolically compute the total potential energy of the system. For the given two numerical configurations and velocities, provide the potential energy.

(e) **Dynamical Model**
The dynamics of the three-link robot can be computed in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u, \tag{2}$$

where $q \in \mathbb{R}^5$ is the vector of configuration variables, $u \in \mathbb{R}^2$ is a vector of motor torques representing the two actuators that actuate the relative leg angles for Leg 1 and Leg 2 respectively (i.e., the two motors actuate θ_1 and θ_2 in Figure 1a), $D \in \mathbb{R}^{5\times5}$ is the *Inertia Matrix*, $C \in \mathbb{R}^{5\times5}$ is the *Coriolis Matrix*, $G \in \mathbb{R}^{5\times1}$ is the *Gravity Vector*, and $B \in \mathbb{R}^{5\times2}$ is the *Input Mapping Matrix* that maps the control inputs to the configuration variables.

Use the Matlab function `LagrangianDynamics.m` (uploaded on bcourses) to symbolically compute the dynamics by computing the terms $D(q), C(q,\dot{q}), G(q), B(q)$. For the give two numerical configurations, provide the numerical values of the matrix $D(q)$, the vectors $C(q,\dot{q})\dot{q}$, $G(q)$ and the matrix $B(q)$.

Problem 2 Change of Coordinates
Consider the configuration variable $q \in \mathbb{R}^5$ specified in (1) for the configuration variables illustrated in Figure 1a. Suppose the configuration variable $\tilde{q} \in \mathbb{R}^5$ specified in (3) represents the configuration variables illustrated in Figure 1b.

$$\tilde{q} = \begin{bmatrix} x \\ y \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}. \tag{3}$$

(a) Show that the two sets of configuration variables are equivalent, i.e. show that there exists an invertible function that maps the configuration variables q to the configuration variables $\tilde{q}$.
Hint*: In this simple case, the transformation relating the two sets of configuration variables is affine, i.e. there exists a constant matrix $T \in \mathbb{R}^{5\times5}$ and constant vector $d \in \mathbb{R}^5$ such that $\tilde{q} = Tq + d$, with T invertible and $q = T^{-1}\tilde{q} - T^{-1}d$.*

(b) What is the mapping between the velocities $\dot{q}$ and $\dot{\tilde{q}}$? What is the mapping between the accelerations $\ddot{q}$ and $\ddot{\tilde{q}}$?

(c) For the two numerical configuration and velocities for $(q,\dot{q})$ given in Problem 1, compute the corresponding configuration and velocities for $(\tilde{q},\dot{\tilde{q}})$.

(d) For the absolute coordinates $(\tilde{q},\dot{\tilde{q}})$, recompute the dynamics of the system resulting in:

$$\tilde{D}(\bar{q})\ddot{\tilde{q}} + \tilde{C}(\tilde{q},\dot{\tilde{q}})\dot{\tilde{q}} + \tilde{G}(\tilde{q}) = \tilde{B}(\tilde{q})u, \tag{4}$$

For the two numerical configurations and velocities $(\tilde{q},\dot{\tilde{q}})$ you computed in part (c) above, numerically compute the values of the matrix $\tilde{D}(\tilde{q})$, the vectors $\tilde{C}(\tilde{q},\dot{\tilde{q}})\dot{\tilde{q}}$, $\tilde{G}(\tilde{q})$ and the matrix $\tilde{B}(\tilde{q})$.

(e) Compute the inertial matrix $\tilde{D}(\tilde{q})$ in terms of the inertial matrix $D(q)$ without recomputing the dynamics (as we did in part (d)).
Hint*: Note that the kinetic energy computed for either coordinates are equivalent, i.e.,*

$$\frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\dot{\tilde{q}}^T \tilde{D}(\tilde{q})\dot{\tilde{q}}.$$

(f) Compute the Coriolis matrix $\bar{C}(\bar{q}, \dot{\bar{q}})$, Gravity vector $\bar{G}(\bar{q})$, and input mapping matrix $\bar{B}(\bar{q})$ in terms of $C(q,]\dot{q}), G(q), B(q)$ without recomputing the dynamics.
Hint*: Write* (4) *in the form of* (2) *and use results from part (e). You can evaluate your result by numerically evaluating what you obtain and comparing with the dynamics you derived in part (d).*

Concluding Remarks

In this homework, we looked at deriving equations of motions of legged robots that are modeled as kinematic chains of rigid bodies. In fact, we derived the continous-time component of a *hybrid* dynamical model for a legged robot. Later in the course, we will use these models to develop optimal trajectories and feedback controllers for locomotion tasks such walking, running and jumping!

Instructions

1. Please submit a single pdf of your HW. (If typset on a computer, please save to pdf. If handwritten, please scan to pdf.)
2. You may choose to use a symbolic math package such as the `Symbolic Math Toolbox` (`https://www.mathworks.com/help/symbolic/index.html`) in `MATLAB` or `Mathematica`.
3. Do include all your code, if any.
4. You will be uploading your pdf solution on Gradescope. As part of this, you will be marking out regions of your solution for each question / subquestion.
5. **Honor Code.** You are to do your own work. Discussing the homework with a friend is fine. Sharing results or MATLAB code is not.