ME231B/EE220C: HW 5 – Due: 12:00 (midday) on Monday, 16 March

Instructions: You are free to collaborate in how to approach the problems, but the solution you submit must be yours alone. Be explicit in your solutions – points are awarded for clearly defining variables, etc. If your solutions are hard to read (e.g. illegible), or hard to understand, we reserve the right to assign penalties, up to discarding the solution. This classification is at the instructor's discretion, and non-negotiable.
If a solution relies on computer code, you must provide the full listing of the code that you used. You may use any programming language you like. Any code must be sufficiently well documented (comments) that it can be easily understood.
Submitting: submit your solution using gradescope. You must clearly mark up your solutions. Failure to submit according to these instructions will be penalized, at our discretion.
Late submissions: you will lose 20 percentage points if your homework is late by less than eight hours. For each additional hour, you lose an additional 10 percentage points.
Academic conduct: Please see `http://sa.berkeley.edu/conduct/integrity/definition`. We have no patience for misconduct.
Last update: 2020-02-26

Problem 1
(12P) Consider the following system:

$$\begin{aligned} x(k) &= x(k-1) + v(k-1) \\ z(k) &= x(k) + w(k) \end{aligned}$$

where $x(0)$, $\{v(\cdot)\}$, $\{w(\cdot)\}$ are independent, and each is drawn from a uniform distribution in the range $[-1, 1]$.
At $k = 1$, you make the observation $z(1) = 1$.

a) **(8P)** Solve the PDF of the state $x(1)$ conditioned on $z(1) = 1$.

b) **(2P)** Compute the Minimum Mean Squared Error estimate for $x(1)$.

c) **(2P)** Compute the Maximum A Posteriori estimate for $x(1)$.

Problem 2
(19P) Consider the following system:

$$\begin{aligned} x(k) &= Ax(k-1) + v(k-1) \\ z(k) &= Hx(k) + w(k) \end{aligned}$$

where

$$A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, v(k) \sim \mathcal{N}(0, I), \quad w(k) \sim \mathcal{N}(0, I), \quad x(0) \sim \mathcal{N}\left(0, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

and $x(0)$, $\{v(\cdot)\}$, $\{w(\cdot)\}$ are independent.
Define $y(k) := \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$.

a) **(7P)** Compute the PDF of $y(1)$ given the observation $z(1) = 2.22$.

b) **(4P)** At which time step k (for $k \leq 10$) do you have least knowledge about $y(k)$? For which do you have the most knowledge? *Hint:* interpret "little knowledge of *something*" as "our estimate of *something* has large variance".

Define the estimation error $e(k) := x(k) - \hat{x}(k)$, where $\hat{x}(k)$ is the Kalman filter estimate of the true state $x(k)$.

c) **(4P)**Compute the PDF of $e(10)$.

d) **(4P)** Implement a simulation of the system and a Kalman filter that produces an estimate $\hat{x}(k)$. Execute this 10,000 times and plot a histogram of the resulting values for $e(10)$ (make two histograms, one for each component of $e(10)$). Comment on how this compares to your answer in c).

Problem 3

(6P) Design a Kalman filter for the system given in Problem 1. What is the Kalman filter estimate at time $k = 1$?
Comment on how this compares to the solution you got for Problem 1.

Problem 4

(10P) Consider the following system:

$$x(k) = Ax(k-1) + v(k-1)$$
$$z(k) = Hx(k) + w(k)$$

where

$$A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathrm{E}\left[v(k)\right] = 0, \mathrm{Var}\left[v(k)\right] = I$$
$$\mathrm{E}\left[w(k)\right] = 0, \mathrm{Var}\left[w(k)\right] = I, \mathrm{E}\left[x(0)\right] = 0, \mathrm{Var}\left[x(0)\right] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

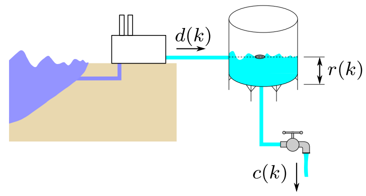
and $x(0)$, $\{v(\cdot)\}$, $\{w(\cdot)\}$ are independent.
Design an optimal linear estimator for this system (a Kalman filter), and implement a simulation of the system and a Kalman filter that produces an estimate $\hat{x}(k)$. For the simulation, draw all of the random variables from uniform distributions that match the required mean/variance.
Execute this 10,000 times and plot a histogram of the resulting values for $e(10)$ (make two histograms, one for each component of $e(10)$). Comment on how this compares to your answers in Problem 2c) and Problem 2d).
Also give the ensemble average and variance for each component of $e(10)$ (that is, the average over the samples from the simulations for each component). Compare that to the estimator's output, and briefly discuss.

Problem 5

(34P) In this problem we estimate the behavior of a city's water network, where fresh water is supplied by a desalination plant, and consumed at various points in the network. The network contains various consumers, and various reservoirs where water is locally stored.
Our objective is to use noisy information about production and consumption levels, and noisy measurements of the amount of water in the reservoirs, to estimate the state of the network. We will consider 3 networks of increasing complexity.

Let k be a time index, and let $d(k)$ be the fresh water produced by the desalination plant at time k. Each city district i has a reservoir with current level $r_i(k)$, and the district consumes a quantity of water $c_i(k)$. At each time step, we receive a noisy measurement $z_i(k) = r_i(k) + w_i(k)$ of the reservoir level $r_i(k)$, where $w_i(k)$ is the measurement error (with w zero-mean, white, and independent of all other quantities).

a. We will first investigate a very simplified system, with a single reservoir and single consumer (i.e. we only have one tank level r to keep track of). We will model our consumers as $c(k) = m + v(k)$, where m is the typical consumption, and $v(k)$ is a zero-mean uncertainty, assumed white and independent of all other quantities.



(a-i) **(2P)** Write down the model equations for this problem. Make explicit what is the system state, the measurement, the process noise, and the measurement noise.

(a-ii) **(6P)** Design a Kalman filter to estimate the level of the tank. Run the Kalman filter for ten steps, with the following problem data:

At time $k = 0$, we know $\mathrm{E}[r(0)] = 20$, with uncertainty $\mathrm{Var}[r(0)] = 25$. The desalination plant delivers a constant supply of water, so that $d(k) = 10$ for all $k \geq 0$. The consumer is predicted to use a supply $m = 7$, and our process uncertainty is $\mathrm{Var}[v(k)] = 9$. Our sensor uncertainty is $\mathrm{Var}[w(k)] = 25$.

We receive the following sequence of measurements $z(k)$:

time k	1	2	3	4	5	6	7	8	9	10
measurement $z(k)$	32.1	39.8	45.9	52.0	51.0	63.4	58.1	60.3	76.6	73.0

Using the Kalman filter, estimate the actual volume of water r for $k \in \{1, 2, \ldots, 10\}$. Also provide the associated uncertainty for your estimate of r. Provide this using graphs.

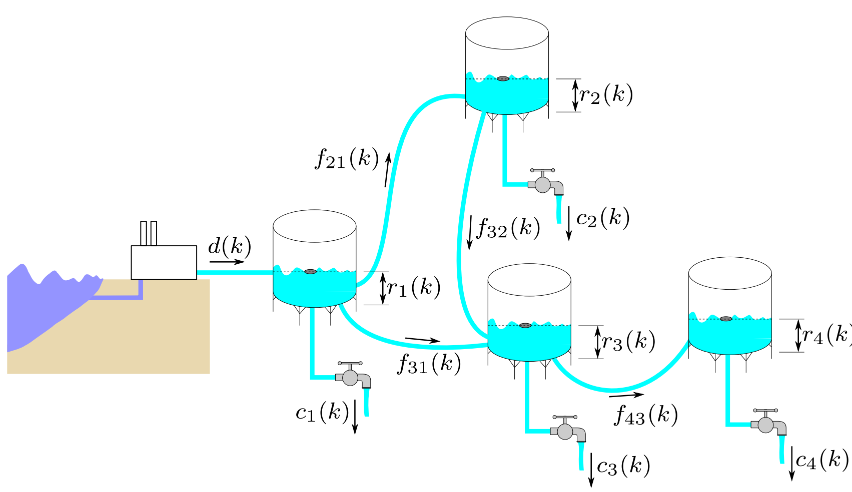
b. We now extend the previous part by also estimating the consumption, $c(k)$. We model the consumption as evolving as $c(k) = c(k-1) + n(k-1)$, where $n(k-1)$ is a zero-mean random variable, which is white and independent of all other quantities.

(b-i) **(4P)** Write down the model equations for this, noting that now your state has dimension 2. Make explicit what is the system state, the measurement, the process noise, and the measurement noise.

(b-ii) **(6P)** Again, design a Kalman filter to estimate the level of the tank. Use the same problem data as before (including as given in subproblem aa-ii), except that now $\mathrm{Var}[n(k)] = 0.1$. Also use $\mathrm{E}[c(0)] = 7$ with $\mathrm{Var}[c(0)] = 1$.

Using the Kalman filter, estimate the actual volume of water r for $k \in \{1, 2, \ldots, 10\}$, and the consumption rate $c(k)$. Also provide the associated uncertainty for both. Provide this using graphs. How do your answers for r compare to the previous case?

c. Now, we extend the system to a more complex network with four reservoirs, and four consumers, as shown below.



The reservoirs are connected in a network, and an automatic balancing system is in place that pumps water between the reservoirs. If two reservoirs i and j are connected, there is a balancing flow between them that is proportional to the difference in volume between the two, so that $f_{ij}(k) = \alpha\,(r_j(k) - r_i(k))$. For example, for reservoirs #1 and #2 in our network, we have the following dynamics:

$$r_1(k) = r_1(k-1) + d(k-1) - c_1(k-1) + \alpha\,(r_2(k-1) - r_1(k-1)) + \alpha\,(r_3(k-1) - r_1(k-1))$$
$$r_2(k) = r_2(k-1) - c_2(k-1) + \alpha\,(r_1(k-1) - r_2(k-1)) + \alpha\,(r_3(k-1) - r_2(k-1))$$

where, as before, $c_i(k) = c_i(k-1) + n_i(k-1)$. Note that the balancing does not change the *total* amount of water in the system, it just moves it around. Our sensor uncertainty is $\mathrm{Var}\left[w_i(k)\right] = 25$ for all tanks, and the consumption uncertainty is $\mathrm{Var}\left[n_i(k)\right] = 0.1$ for all consumers. Model the consumers as independent but identically distributed; also model the sensors as independent but identically distributed.

(c-i) **(4P)** Write down the model equations for this problem. Make explicit what is the system state, the measurement, the process noise, and the measurement noise. Write the solution as a linear problem, and clearly give terms of the matrices A, H, etc.

(c-ii) **(6P)** Design a Kalman filter to estimate the level of all the tanks, and the consumption levels. Run the Kalman filter for ten steps, with the following problem data:

At time $k = 0$, we know the following about the tanks and their consumers:

tank number i	1	2	3	4
$\mathrm{E}\left[r_i(0)\right]$	20	40	60	20
$\mathrm{Var}\left[r_i(0)\right]$	20	20	20	20
$\mathrm{E}\left[c_i(0)\right]$	7	7	7	7
$\mathrm{Var}\left[c_i(0)\right]$	1	1	1	1

The desalination plant delivers a constant supply of water, so that $d(k) = 30$ for all $k \geq 0$. The balancing is done with $\alpha = 0.3$.

We receive the following sequence of measurements $z_i(k)$:

time k	1	2	3	4	5	6	7	8	9	10
meas. $z_1(k)$	62.6	70.3	73.5	77.2	73.2	94.2	87.4	89.7	90.4	94.2
meas. $z_2(k)$	29.4	44.8	37.3	40.1	44.1	43.8	53.8	49.9	51.9	52.1
meas. $z_3(k)$	35.9	38.8	25.9	39.	31.2	46.9	39.6	44.	42.5	54.2
meas. $z_4(k)$	40.9	21.9	18.	8.8	23.9	17.2	18.6	22.	22.9	17.2

Using the Kalman filter, estimate the actual volume of water r_i for $k \in \{1, 2, \ldots, 10\}$ for all tanks. Also provide the associated uncertainty. Provide this using graphs. Notice that the estimate uncertainty for tank 1 is *lower* than the solution you had for the simpler network – why is this?

(c-iii) **(6P)** We will now repeat the previous problem, except that now the sensor of tank 3 has failed, and thus no longer provides any measurements. Modify your Kalman filter from before (reduce your sensor model to remove this), and run this using the same data as before (except that you remove the measurements $z_3(k)$). Estimate the actual volume of water r_i for $k \in \{1, 2, \ldots, 10\}$ for all tanks, and also provide the associated uncertainty. How does this compare to before? That is, how much has losing the sensor affected your ability to estimate the tank levels?