ME231B/EE220C: HW 7 – Due: 12:00 (midday) on Friday, 3 April – for 41 P.

Instructions: You are free to collaborate in how to approach the problems, but the solution you submit must be yours alone. Be explicit in your solutions – points are awarded for clearly defining variables, etc. If your solutions are hard to read (e.g. illegible), or hard to understand, we reserve the right to assign penalties, up to discarding the solution. This classification is at the instructor's discretion, and is not negotiable.
If a solution relies on computer code, you must provide the full listing of the code that you used. You may use any programming language you like. Any code must be sufficiently well documented (comments) that it can be easily understood.
Submitting: submit your solution using gradescope. You must clearly mark up your solutions. Failure to submit according to these instructions will be penalized, at our discretion.
Late submissions: you will lose 20 percentage points if your homework is late by less than eight hours. For each additional hour, you lose an additional 10 percentage points.
Academic conduct: Please see `http://sa.berkeley.edu/conduct/integrity/definition`. We have no patience for misconduct.
Last update: 2020-03-14

Problem 1 (10 P.)

Design an Extended Kalman Filter (EKF) for the following 2-state nonlinear system and compute its posteriori estimate $\hat{x}_m(1)$, given $z(1) = 0.5$.

$$x(k) = q(x(k-1), v(k-1))$$
$$z(k) = h(x(k), w(k))$$

where

$$q(x(k)), v(k)) = \begin{bmatrix} \sin x_1(k) + \cos x_2(k) + v_1(k) \\ \cos x_1(k) - \sin x_2(k) + v_2(k) \end{bmatrix}$$
$$h(x(k), w(k)) = x_1(k)x_2(k) + w(k)$$
$$x(0) \sim \mathcal{N}\left(\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\right), \quad v(k) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}\right), \quad w(k) \sim \mathcal{N}(0, 0.2),$$

and our usual independence assumptions apply.
In your solution, show clearly the following steps:

a) Initialization.

b) Prior update: Jacobian matrices A and L as defined in lectures.

c) Measurement update: Jacobian matrices H and M as defined in lectures.

d) Highlight the final answer $\hat{x}_m(1)$.

Problem 2 (8 P.)

Let x be a scalar random variable, with a symmetric PDF (that is, $f_x(\mu_x + \bar{x}) = f_x(\mu_x - \bar{x})$ for $\mu_x = \mathrm{E}[x]$ and any $\bar{x}$). Let $y = g(x)$ for an analytic, scalar-valued, function g.

a) **(6P.)** Show that the unscented transform correctly predicts the mean of y up to third order in the function g.

b) **(2P.)** Up to which order is the unscented transform's variance prediction correct?

Problem 3 (9 P.)

This problem will look at three different systems with deterministic dynamics, but uncertain initial state. In each case, $\mathrm{E}[x(0)] = 1$ and $\mathrm{Var}[x(0)] = 4$, and we are interested in $x(1) = q(x(0))$. We will consider three different approaches to predict $\mathrm{E}[x(1)]$ and $\mathrm{Var}[x(1)]$, described below.

a. **(3P.)** Consider $q(x) = -x + 2|x|$

- (a-i) Use techniques from the EKF (i.e. linearization).
- (a-ii) Use the unscented transform.
- (a-iii) Numerically approximate the statistics by repeated sampling. Generate 10^6 samples, and do this for both $x(0)$ normally distributed, and uniformly distributed.

b. **(3P.)** Consider $q(x) = (x-1)^3$

- (b-i) Use techniques from the EKF (i.e. linearization).
- (b-ii) Use the unscented transform.
- (b-iii) Numerically approximate the statistics by repeated sampling. Generate 10^6 samples, and do this for both $x(0)$ normally distributed, and uniformly distributed.

c. **(3P.)** Consider $q(x) = 3x$

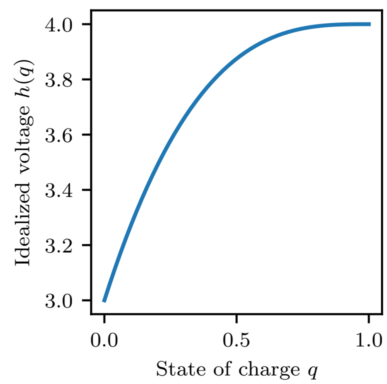
- (c-i) Use techniques from the EKF (i.e. linearization).
- (c-ii) Use the unscented transform.
- (c-iii) Numerically approximate the statistics by repeated sampling. Generate 10^6 samples, and do this for both $x(0)$ normally distributed, and uniformly distributed.

Ungraded: how do the different estimates agree / differ? What makes them sometimes work well, or not well?

Problem 4 (14 P.)

For the battery system of Homework 4's Problem 3, we now add a voltage sensor which gives us a noisy reading of the battery voltage after each time cycle.

The measurement is $z(k) = h(q(k)) + w(k)$, with $w(k) \sim \mathcal{N}\left(0, \sigma_w^2\right)$, and where $h(q)$ is a nonlinear function mapping from current state of charge to voltage, plotted alongside, and with



$$h(q) = 4 + (q-1)^3.$$

The battery is subject to the same discharging process as before, with

$$q(k) = q(k-1) - j(k-1)$$

where $q(0) = 1$ with no uncertainty; and $j(k) = j_0 + v(k)$ with $v(k)$ zero mean with variance σ_v^2.

For our system, we set $j_0 = 0.1$, $\sigma_v = 0.05$, and $\sigma_w = 0.1$.

a. **(2P.)** Design an extended Kalman filter (EKF) to estimate the state of charge.

We note that the amount of information that the EKF gets from the measurement is determined by the function h, and specifically its slope with respect to the state, $H(k) = \frac{\partial h}{\partial q}$. Because our system and measurements are scalar, all quantities are also scalar. We can thus easily reason about how useful a measurement is, by comparing the state variance after getting the measurement $P_m(k)$ to the variance before the measurement, $P_p(k)$.

b. **(2P.)** Show that the reduction in variance due to a measurement (i.e. a measure of its usefulness) can be described as below:

$$\frac{P_p(k) - P_m(k)}{P_p(k)} = \frac{H(k)^2 P_p(k)}{W + H(k)^2 P_p(k)}$$

where a value of 1 means that the measurement removed all uncertainty, and a value of 0 means that the measurement made no difference to the uncertainty.

c. **(4P.)** Using this metric, make a plot of the usefulness of the voltage measurement as a function of the estimated state of charge, for $\hat{q} \in [0, 1]$ (with usefulness as defined in the subproblem b). Set $P_p(k) = 0.1$, and $W = 0.1$. Where is the measurement most informative? Where is it least informative?

For the remainder of the problem, given is the following sequence of measurements:

time k	1	2	3	4	5	6	7	8	9
measurement $z(k)$	4.04	3.81	3.95	3.90	3.88	3.88	3.90	3.55	3.18

d. **(4P.)** Run your extended Kalman filter with this data, and generate two plots: the estimated state of charge, and the variance of this estimate, across k.

e. **(2P.)** After 9 steps, what would the mean and variance be if you did not have the voltage measurements (use your results from the previous). How does this compare to your EKF output?