ME231B/EE220C: HW 6 – Due: 12:00 (midday) on Friday, 20 March

Instructions: You are free to collaborate in how to approach the problems, but the solution you submit must be yours alone. Be explicit in your solutions – points are awarded for clearly defining variables, etc. If your solutions are hard to read (e.g. illegible), or hard to understand, we reserve the right to assign penalties, up to discarding the solution. This classification is at the instructor's discretion, and is not negotiable.
If a solution relies on computer code, you must provide the full listing of the code that you used. You may use any programming language you like. Any code must be sufficiently well documented (comments) that it can be easily understood.
Submitting: submit your solution using gradescope. You must clearly mark up your solutions. Failure to submit according to these instructions will be penalized, at our discretion.
Late submissions: you will lose 20 percentage points if your homework is late by less than eight hours. For each additional hour, you lose an additional 10 percentage points.
Academic conduct: Please see `http://sa.berkeley.edu/conduct/integrity/definition`. We have no patience for misconduct.
Last update: 2020-03-03

Problem 1 (12 P.)
Consider the following system:

$$x(k) = Ax(k-1) + v(k-1), \qquad z(k) = Hx(k) + w(k)$$
$$A = 1.2, \quad H = 1, v(k) \sim \mathcal{N}(0, 2), \quad w(k) \sim \mathcal{N}(0, 1), \quad x(0) \sim \mathcal{N}(0, 3)$$

and $x(0)$, $\{v(\cdot)\}$, $\{w(\cdot)\}$ are independent. You will compare the estimation error when estimating the system state with the (time varying) Kalman filter, and the steady-state Kalman filter.

a. **(4P.)** Using the standard Kalman filter, at $k \in \{1, 2, 10, 1000\}$, what is the variance of the posterior estimation error?

b. **(2P.)** What is the steady-state Kalman filter for this system? Provide the gain, and the steady-state posterior variance.

c. **(6P.)** Using now the steady-state Kalman filter to estimate the state, at $k \in \{1, 2, 10, 1000\}$, what is the variance of the posterior estimation error? Comment on how this compares to the standard Kalman filter of part a.

Problem 2 (12 P.)
In this problem, we will investigate a problem with model mismatch, and specifically look at stability of the estimation error. You are to design a state estimator for a scalar system with state $x(k)$ and measurement $z(k)$. You are given these dynamic equations to use:

$$x(k) = x(k-1) + v(k-1) \text{ with } v(k-1) \sim \mathcal{N}(0, 1), x(0) \sim \mathcal{N}(0, 1),$$
$$z(k) = x(k) + w(k) \text{ with } w(k) \sim \mathcal{N}(0, 6).$$

The values $x(0)$, $\{v(\cdot)\}$, and $\{w(\cdot)\}$ are mutually independent.

a. **(4P.)** Compute the steady-state Kalman filter gain K_∞.

For part b): the true system dynamics differ from the ones used to derive the estimator, and are given by

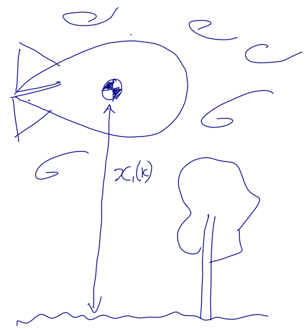
$$x_{\text{true}}(k) = (1 + \delta)x_{\text{true}}(k-1) + v(k-1) \text{ with } v(k-1) \sim \mathcal{N}(0, 1)$$
$$z(k) = x_{\text{true}}(k) + w(k) \text{ with } w(k) \sim \mathcal{N}(0, 6)$$

where $\delta \in \mathbb{R}$ is a scalar constant. Define the estimation error as $e(k) := x_{\text{true}}(k) - \hat{x}(k)$, where $\hat{x}$ is the estimate at time k of your steady-state Kalman filter, as computed in part a).

b. **(8P.)** For what values δ will the error $e(k)$ remain bounded as k tends to infinity?

Problem 3 (18 P.)

You are investigating various trade-offs in the design of an autonomous blimp. The system dynamics are given as below, with the system state $x(k) \in \mathbb{R}^2$ comprising the height (x_1 in units of m) and vertical velocity (x_2 in units of m/s) of the blimp, driven by a random acceleration:



$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x(k{-}1) + v(k{-}1)$$

where Δt is the sampling time, and $v(k{-}1)$ is zero-mean white noise, with variance parametrized through the scalar σ_a^2:

$$\mathrm{Var}\left[v(k{-}1)\right] = \sigma_a^2 \begin{bmatrix} \frac{1}{4}\Delta t^4 & \frac{1}{2}\Delta t^3 \\ \frac{1}{2}\Delta t^3 & \Delta t^2 \end{bmatrix}$$

You may assume all random quantities in this problem are normally distributed.
(Aside, ungraded – the physics model here is an exact discretization of a continuous time double integrator driven by random acceleration, where the acceleration is assumed to be constant over the sampling time Δt. Can you verify this?)
The original system is equipped with a GPS sensor, so that:

$$z_1(k) = x_1(k) + w_1(k)$$

where $w_1(k)$ is a zero-mean, white noise sequence with variance $\mathrm{Var}\left[w_1(k)\right] = m_1^2$.
We are given $\Delta t = \frac{1}{10}$s, $\sigma_a = 5$m/s, and $m_1 = 10$m.
The robot's mission relies on accurate knowledge of its position, which we will quantify with the metric $J_p := \sqrt{\lim_{k\to\infty} \mathrm{Var}\left[x_1(k)|z(1{:}k)\right]}$.

a. **(6P.)** Write a program that computes J_p as a function of the usual Kalman filter matrices A, V, H, and W. Use this program to confirm that the original system has $J_p \approx 3.085$m.

We are considering various ways to improve the system's estimation performance, enumerated below. Each modification is an independent modification of the original system.
For each of the following suggested modifications to the system, compute the performance metric J_p.
(Before you compute anything, guess what the answer will be for each case (no need to report your guesses). Try to predict the order of performance, from best to worst.)

b. **(2P.)** Replace the GPS sensor with that of brand A, which has $m_1 = 5$m (the noise standard deviation is cut in half).

c. **(2P.)** Replace the GPS sensor with that of brand B, which has $\Delta t = \frac{1}{20}$s, and $m_1 = 10$m (i.e. the sensor returns equally noisy measurements, but twice as frequently).

d. **(2P.)** Retain the original GPS sensor, and add an airspeed sensor which gives the additional measurement $z_2(k) = x_2(k) + w_2(k)$, where $w_2(k)$ is a zero-mean, white noise sequence, independent of all quantities, and with $\mathrm{Var}\left[w_2(k)\right] = m_2^2$ with $m_2 = 1$m/s. The sensors z_1 and z_2 return data at the same instants of time.

e. **(2P.)** Retain the original GPS sensor, and add a second, independent barometric height sensor $z_2(k) = x_1(k) + w_2(k)$, where $w_2(k)$ is a zero-mean, white noise sequence, independent of all quantities, and with $\mathrm{Var}\left[w_2(k)\right] = m_2^2$ with $m_2 = 10$m. The sensors z_1 and z_2 return data at the same instants of time.

f. **(2P.)** Retain the original GPS sensor, and add a second identical GPS sensor from the same manufacturer, so that $z_2(k) = x_1(k) + w_2(k)$, where $w_2(k) = w_1(k)$.

g. **(2P.)** Retain the original sensor, but modify the system design by making it more aerodynamic and thereby reducing the effect of aerodynamic disturbances, so that $\sigma_a = 1$m/s.

(Revisit your guesses – how did you do? How good is your estimation intuition?)