BACHELOR OF COMPUTER SCIENCE & ENGINEERING EXAMINATION, 2016

(1st Year, 2nd Semester)

MATHEMATICS - III

Full Marks : 100 | Time : Three Hours

The figures in the margin indicate full marks

Answer Q. No. 10 and any **six** questions from Q. Nos. 1 - 9.

1. (a) Define a group. Let G be a group and $a, b \in G$ such that $ab^3a^{-1} = b^2$ and $b^{-1}a^2b = a^3$. Show that $a = b = e$, where e is the identity of G. 8

   (b) If $\beta = (1\ 2\ 3)(1\ 4\ 5) \in S_5$, write β^{99} in disjoint cycle notation. 8

2. (a) Define a subgroup of a group. Let G be a group, $a \in G$ and $C(a) = \{x \in G \mid ax = xa\}$. Prove that $C(a)$ is a subgroup of G. 8

   (b) Define a cyclic group. Find the number of generators of a finite cyclic group of order n. 8

3. (a) State and prove Lagrange's theorem for finite groups. 8

   (b) Define homomorphisms of groups and kernels of homomorphisms. Let $f : G \longrightarrow G_1$ be an onto homomorphism of groups with the kernel, $\ker f$. Show that $G/\ker f \cong G_1$. 8

4. (a) Define a Boolean ring. Let R be a Boolean ring. Then show that $2x = 0$ and $xy = yx$ for all $x, y \in R$. 8

   (b) Define an ideal of a ring. Show that the set

$$I = \left\{a + b\sqrt{3} \in \mathbb{Z}[\sqrt{3}] \mid a - b \text{ is an even integer}\right\}$$

   is an ideal of the ring $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. 8

5. (a) Define a maximal ideal of a ring. Let R be a commutative ring with identity $1 \neq 0$. Then show that an ideal M of R is maximal if and only if R/M is a field. 8

   (b) Let $F[x]$ be the polynomial ring over a field F. Define an irreducible polynomial $f(x) \in F[x]$. Show that there is an extension field K of F such that $f(x)$ has a root in K. 8

6. (a) Find the nature of solutions of the following system of linear equations: 8

$$\begin{aligned} 3x - 2y + 3z &= 8 \\ x + 3y + 6z &= -3 \\ 5x + 4y + 15z &= 2 \end{aligned}$$

(b) Define eigen values and corresponding eigen spaces of a square matrix over a field. Find eigen values and corresponding eigen spaces of A. Also find the dimensions of the eigen spaces, where 8

$$A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

7. (a) Define similar matrices. Define the characteristic polynomial of a square matrix. Prove that similar matrices have the same characteristic polynomial. 8

(b) Define the rank of a matrix. Find the rank of the following matrix: 8

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 4 \\ 5 & -2 & 9 \end{pmatrix}$$

8. (a) Define a basis of a vector space over a field. Find a basis of $\mathbb{R}^3$ containing the vectors $(1, -2, 0)$ and $(2, 3, -7)$. 8

(b) Define a linearly independent and linearly dependent subsets of a vector space over a field F. Determine whether the following subsets of $\mathbb{R}^3$ are linearly independent or linearly dependent: (i) $\emptyset$; (ii) $\{(0, 0, 0), (3, 4, 0), (0, 0, 1)\}$. 8

9. (a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\theta_V\}$. Prove that for each vector $u \in V$, there exist unique vectors $u_1 \in W_1$ and $u_2 \in W_2$ such that $u = u_1 + u_2$. 8

(b) Define a subspace of a vector space. Show that $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 3y + 4z = 0\}$ is a subspace of $\mathbb{R}^3$. Find a basis and the dimension of W over $\mathbb{R}$. 8

10. Construct a finite field of order 27. 4