BACHELOR OF COMPUTER SCIENCE & ENGINEERING EXAMINATION, 2015

(1st Year, 2nd Semester)

MATHEMATICS - III

Full Marks : 100 — Time : Three Hours

All questions carry equal marks

Answer any **ten** questions

1. (a) Define a group. Let $\mathbb{R}$ be the set of real numbers and $G = \mathbb{R} \setminus \{-1\}$. Define $a \star b = a + b + ab$ for all $a, b \in G$. Verify that whether $(G, \star)$ is a group.

   (b) In a group G, if $a^5 = e$ and $aba^{-1} = b^2$ for some $a, b \in G$. Show that $b^{31} = e$.

2. (a) Define the permutation group. Show that in the permutation group S_n $(n \geqslant 3)$, every even permutation is a product of 3-cycles.

   (b) Define a subgroup of a group. Let G be a commutative group. Prove that $H = \{a \in G \mid \circ(a) \text{ divides } 10\}$ is a subgroup of G.

3. (a) Define a cyclic group. Let G be an infinite cyclic group. Show that G is isomorphic to the group $(\mathbb{Z}, +)$.

   (b) Let G be a group of order p^n (where p is prime and n is a natural number). Then show that G has an element of order p.

4. (a) Define homomorphisms of groups. Prove that the map $f : G \longrightarrow G$ defined by $f(a) = a^{-1}$ is a homomorphism if and only if G is commmutative.

   (b) Define the groups $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$. Prove that $\frac{GL_n(\mathbb{R})}{SL_n(\mathbb{R})} \cong (\mathbb{R}^*, \cdot)$, where $(\mathbb{R}^*, \cdot)$ denotes the group of non-zero real numbers under usual multiplication.

5. (a) Define a ring. Define a Boolean ring. Show that a ring R with identity is a Boolean ring if and only if $a(a + b)b = 0$ for all $a, b \in R$.

   (b) Define a field. Let R be a commutative ring with identity $1 \neq 0$. Prove that R is a field if and only if R has only two ideals, namely, $\{0\}$ and R itself.

6. (a) Define a prime ideal of a commutative ring. Let R be a commutative ring with identity $1 \neq 0$. Then show that an ideal P of R is prime if and only if R/P is an integral domain.

   (b) Define a maximal ideal of a ring. Let R be a commutative ring with identity $1 \neq 0$. Show that every maximal ideal of R is prime.

7. (a) Let $F[x]$ be the polynomial ring over a field F. Show that $f(x) \in F[x]$ is irreducible if and only if the ideal $\langle f(x) \rangle$ generated by $f(x)$ is a maximal ideal in $F[x]$.

   (b) Let F be a finite field. Show that the number of elements of F is a power of a prime number. Give an example of a field with 4 elements.

8. (a) Define a row-reduced echelon matrix. Find a row-reduced echelon matrix that is row-equivalent to the matrix $\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$

   (b) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ using the elementary row operations.

9. (a) Define a subspace of a vector space. Show that intersection of any collection of a subspaces of a vector space forms a subspace of it.

   (b) Suppose W_1, W_2 are two subspaces of a vector space V over a field F such that $W_1 \cup W_2$ is a subspace of V. Show that either $W_1 \subset W_2$ or $W_2 \subset W_1$.

10. (a) Define a linearly independent subset of a vector space over a field F. Show that if S is a linearly independent subset of a vector space V over a field F and β is a vector in V which is not in the subspace spanned by S, then the set $S \cup \{\beta\}$ is linearly independent over F.

    (b) Define a basis and the dimension of a vector space over a field. What is the dimension of the vector space of all 2×2 matrices over the field of real numbers? Find a basis for it.

11. (a) Define the row space of a matrix over a field. Consider the matrix $A = \begin{pmatrix} 3 & 21 & 0 \\ 1 & 7 & -1 \\ 2 & 14 & 0 \end{pmatrix}$.

    Find a basis for the row space of A over $\mathbb{R}$.

    (b) Show that the set of real numbers forms a vector space (with usual operations) over the field of rational numbers which is not finite dimensional.

12. (a) Define the characteristic polynomial and eigenvalues of a matrix. Consider the matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$ over the field of real numbers, $\mathbb{R}$. Find the characteristic polynomial and eigenvalues of A.

(b) Find one eigenvector for each distinct eigenvalues of the above matrix A. Also find the matrix $A^3 - 5A^2 + 8A - 3I_3$ with the help of Cayley-Hamilton theorem, where I_3 is the identity matrix of order 3×3.

13. (a) Find the Jordan form of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Find A^{101}.