Lecture #1 & Lecture #2

The geometry of linear equations

The fundamental problem of linear algebra is to solve n linear equations in n unknowns; for example:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

In this first lecture on linear algebra we view this problem in three ways.

The system above is two dimensional ($n = 2$). By adding a third variable z we could expand it to three dimensions.

Row Picture

Plot the points that satisfy each equation. The intersection of the plots (if they do intersect) represents the solution to the system of equations. Looking at Figure 1 we see that the solution to this system of equations is $x = 1, y = 2$.

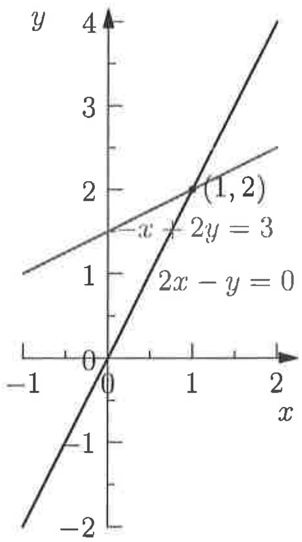


Figure 1: The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point $(1, 2)$.

We plug this solution in to the original system of equations to check our work:

$$\begin{aligned} 2 \cdot 1 - 2 &= 0 \\ -1 + 2 \cdot 2 &= 3. \end{aligned}$$

The solution to a three dimensional system of equations is the common point of intersection of three planes (if there is one).

Column Picture

In the column picture we rewrite the system of linear equations as a single equation by turning the coefficients in the columns of the system into vectors:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Given two vectors $\mathbf{c}$ and $\mathbf{d}$ and scalars x and y, the sum $x\mathbf{c} + y\mathbf{d}$ is called a *linear combination* of $\mathbf{c}$ and $\mathbf{d}$. Linear combinations are important throughout this course.

Geometrically, we want to find numbers x and y so that x copies of vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ added to y copies of vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. As we see from Figure 2, $x = 1$ and $y = 2$, agreeing with the row picture in Figure 2.

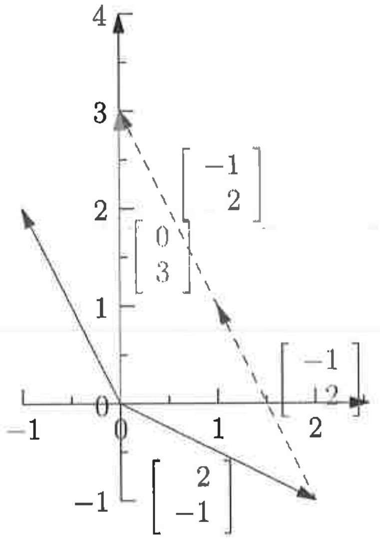


Figure 2: A linear combination of the column vectors equals the vector $\mathbf{b}$.

In three dimensions, the column picture requires us to find a linear combination of three 3-dimensional vectors that equals the vector $\mathbf{b}$.

Matrix Picture

We write the system of equations

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

as a single equation by using matrices and vectors:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is called the *coefficient matrix*. The vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector of unknowns. The values on the right hand side of the equations form the vector $\mathbf{b}$:

$$A\mathbf{x} = \mathbf{b}.$$

The three dimensional matrix picture is very like the two dimensional one, except that the vectors and matrices increase in size.

Matrix Multiplication

How do we multiply a matrix A by a vector $\mathbf{x}$?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \ ?$$

One method is to think of the entries of $\mathbf{x}$ as the coefficients of a linear combination of the column vectors of the matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This technique shows that $A\mathbf{x}$ is a linear combination of the columns of A.

You may also calculate the product $A\mathbf{x}$ by taking the dot product of each row of A with the vector $\mathbf{x}$:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Linear Independence

In the column and matrix pictures, the right hand side of the equation is a vector $\mathbf{b}$. Given a matrix A, can we solve:

$$A\mathbf{x} = \mathbf{b}$$

for every possible vector $\mathbf{b}$? In other words, do the linear combinations of the column vectors fill the xy-plane (or space, in the three dimensional case)?

If the answer is "no", we say that A is a *singular matrix*. In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in two dimensions) or on a point, line or plane (in three dimensions). The combinations don't fill the whole space.

An overview of key ideas

This is an overview of linear algebra given at the start of a course on the mathematics of engineering.

Linear algebra progresses from vectors to matrices to subspaces.

Vectors

What do you do with vectors? Take combinations.

We can multiply vectors by scalars, add, and subtract. Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can form the *linear combination* $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$.

An example in $\mathbb{R}^3$ would be:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The collection of all multiples of $\mathbf{u}$ forms a line through the origin. The collection of all multiples of $\mathbf{v}$ forms another line. The collection of all combinations of $\mathbf{u}$ and $\mathbf{v}$ forms a plane. Taking *all combinations* of some vectors creates a *subspace*.

We could continue like this, or we can use a matrix to add in all multiples of $\mathbf{w}$.

Matrices

Create a matrix A with vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in its columns:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The product:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

equals the sum $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$. The product of a matrix and a vector is a combination of the columns of the matrix. (This particular matrix A is a *difference matrix* because the components of $A\mathbf{x}$ are differences of the components of that vector.)

When we say $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$ we're thinking about multiplying numbers by vectors; when we say $A\mathbf{x} = \mathbf{b}$ we're thinking about multiplying a matrix (whose columns are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$) by the numbers. The calculations are the same, but our perspective has changed.

For any input vector $\mathbf{x}$, the output of the operation "multiplication by A" is some vector $\mathbf{b}$:

$$A\begin{bmatrix}1\\4\\9\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}.$$

A deeper question is to start with a vector $\mathbf{b}$ and ask "for what vectors $\mathbf{x}$ does $A\mathbf{x}=\mathbf{b}$?" In our example, this means solving three equations in three unknowns. Solving:

$$A\mathbf{x}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

is equivalent to solving:

$$\begin{aligned}x_1&=b_1\\x_2-x_1&=b_2\\x_3-x_2&=b_3.\end{aligned}$$

We see that $x_1=b_1$ and so x_2 must equal b_1+b_2. In vector form, the solution is:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix}.$$

But this just says:

$$\mathbf{x}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},$$

or $\mathbf{x}=A^{-1}\mathbf{b}$. If the matrix A is invertible, we can multiply on both sides by A^{-1} to find the unique solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$. We might say that A represents a transform $\mathbf{x}\to\mathbf{b}$ that has an inverse transform $\mathbf{b}\to\mathbf{x}$.

In particular, if $\mathbf{b}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ then $\mathbf{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The second example has the same columns $\mathbf{u}$ and $\mathbf{v}$ and replaces column vector $\mathbf{w}$:

$$C=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}.$$

Then:

$$C\mathbf{x}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1-x_3\\x_2-x_1\\x_3-x_2\end{bmatrix}$$

and our system of three equations in three unknowns becomes circular.

Where before $A\mathbf{x} = \mathbf{0}$ implied $\mathbf{x} = \mathbf{0}$, there are non-zero vectors $\mathbf{x}$ for which $C\mathbf{x} = \mathbf{0}$. For any vector $\mathbf{x}$ with $x_1 = x_2 = x_3$, $C\mathbf{x} = \mathbf{0}$. This is a significant difference; we can't multiply both sides of $C\mathbf{x} = \mathbf{0}$ by an inverse to find a non-zero solution $\mathbf{x}$.

The system of equations encoded in $C\mathbf{x} = \mathbf{b}$ is:

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3. \end{aligned}$$

If we add these three equations together, we get:

$$0 = b_1 +_2 + b_3.$$

This tells us that $C\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ only when the components of $\mathbf{b}$ sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

Subspaces

Geometrically, the columns of C lie in the same plane (they are *dependent*; the columns of A are *independent*). There are many vectors in $\mathbb{R}^3$ which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of C and so correspond to values of $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$. The linear combinations of the columns of C form a two dimensional *subspace* of $\mathbb{R}^3$.

This plane of combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be described as "all vectors $C\mathbf{x}$". But we know that the vectors $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ satisfy the condition $b_1 + b_2 + b_3 = 0$. So the plane of all combinations of $\mathbf{u}$ and $\mathbf{v}$ consists of all vectors whose components sum to 0.

If we take all combinations of:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the entire space $\mathbb{R}^3$; the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$. We say that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a *basis* for $\mathbb{R}^3$.

A *basis* for $\mathbb{R}^n$ is a collection of n independent vectors in $\mathbb{R}^n$. Equivalently, a basis is a collection of n vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

A *vector space* is a collection of vectors that is closed under linear combinations. A *subspace* is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

When you look at a matrix, try to see "what is it doing?"

Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix A^TA that often appears when studying rectangular matrices may be invertible.

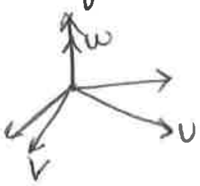
* Linear Algebra goes from vectors → matrices → subspaces.

Vectors u, v, w : linear combination of u, v, w

$$\underbrace{x_1}_{\text{scalar}} u + \underbrace{x_2}_{\text{scalar}} v + \underbrace{x_3}_{\text{scalar}} w = b$$

Let $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ The linear combination of u, v will fill plane in 3D

Now $A = \begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$



$$\Rightarrow Ax = \underbrace{\begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}}_{\text{Difference matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}}_{\text{combination of columns}} = b$$

$$A \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

will be same as $= x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$

Now if we want to find x in $Ax = b$ then we know

$$\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Sol

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$

OR $x = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}^{S} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $Ax = b$ then $x = A^{-1} b$

The inverse of ~~sum~~ different matrix is Sum matrix & we have seen that thing: $S = A^{-1}$

Another example

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
u v w

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Can we solve
$x_1 - x_3 = b_1$ — (i)
$x_2 - x_1 = b_2$ — (ii)
$x_3 - x_2 = b_3$ — (iii)

If $[b_1, b_2, b_3] = [0, 0, 0]$ then all x's are zero.

~~there are~~ Now for $Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \dots, \begin{bmatrix} c \\ c \\ c \end{bmatrix}$ i.e we have a whole line through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which can give $Cx = 0$ i.e $Cx = 0$ have multiple solution

If we add (i), (ii), (iii) then we get $\boxed{b_1 + b_2 + b_3 = 0}$

This mean $Cx = b$ will have solution only when $\boxed{b_1 + b_2 + b_3 = 0}$. ~~& not always~~

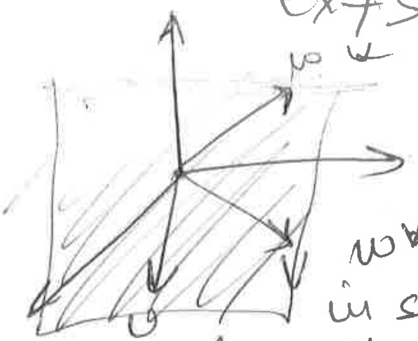
$Ax = b$, A^{-1} exist

U, V, w are independent & all combination of U, V, w fills the whole 3D space.

* (U, V, w) are basis & U, V, w are independent & fills $\mathbb{R}^3$ space
* If we create matrix with basis U, V, w as columns then that matrix will be invertible.

* Rectangular matrices ~~does~~ do not have inverse so watch for $\boxed{A^T A}$ because then it will be symmetric, (maybe) invertible as well.

$Cx = 0$, C^{-1} doesn't exist



w^* is in same plane thats why C^{-1} doesn't exist

U, V, w^* dependent

* all combinat of U, V, w^* gives plane = all vectors Cx

$$= \begin{bmatrix} U & V & w^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1(U) + x_2(V) + x_3(w^*)$$

* If $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ & $Cx = b$

which b's we do get? with U, V, w^*? (dependent vector)

thus those b's where $\boxed{b_1 + b_2 + b_3 = 0}$

& $b_1 + b_2 + b_3 = 0$ is a plane in 3D.

Lecture 3: Multiplication and inverse matrices

Matrix Multiplication

We discuss four different ways of thinking about the product $AB = C$ of two matrices. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C is an $m \times p$ matrix. We use c_{ij} to denote the entry in row i and column j of matrix C.

Standard (row times column)

The standard way of describing a matrix product is to say that c_{ij} equals the dot product of row i of matrix A and column j of matrix B. In other words, $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Columns

The product of matrix A and column j of matrix B equals column j of matrix C. This tells us that the columns of C are combinations of columns of A.

Rows

The product of row i of matrix A and matrix B equals row i of matrix C. So the rows of C are combinations of rows of B.

Column times row

A column of A is an $m \times 1$ vector and a row of B is a $1 \times p$ vector. Their product is a matrix:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}.$$

The columns of this matrix are multiples of the column of A and the rows are multiples of the row of B. If we think of the entries in these rows as the coordinates $(2,12)$ or $(3,18)$ or $(4,24)$, all these points lie on the same line; similarly for the two column vectors. Later we'll see that this is equivalent to saying that the *row space* of this matrix is a single line, as is the *column space*.

The product of A and B is the sum of these "column times row" matrices:

$$AB = \sum_{k=1}^{n} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kn} \end{bmatrix}.$$

$$\begin{array}{c} \\ \text{Row3} \\ \\ \end{array}\begin{bmatrix} & & \\ \hline & & \\ & & \end{bmatrix} \begin{array}{c}\text{Col 4}\\ \begin{bmatrix} & \vert & \\ & \vert & \\ & \vert & \end{bmatrix}\end{array} = \begin{bmatrix} C_{34} & \\ \searrow \cdot & \\ & \end{bmatrix}$$

$$A\,(m\times n) \qquad B\,(n\times p) \qquad C = AB\;(m\times p)$$

$$C_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{column } 4 \text{ of } B)$$
$$= a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}b_{k4}$$

To do this thing again: (2nd way to multiply matrix)

$$\underset{m\times n}{A}\begin{bmatrix} & & \\ & & \end{bmatrix} \quad \underset{n\times p}{B}\begin{bmatrix} C_{B1} & & \\ & & \end{bmatrix} = \underset{m\times p}{C}\begin{bmatrix} C_1 & C_2 & \cdots & C_p \\ \vert & \vert & & \vert \end{bmatrix}$$

← Columns of C are the linear combination of columns of A.

i.e each column of C is obtained by multiplying A with respective column of B. i.e $C_1 = A \times C_{B1}$

Another way to multiply a matrix (Row multiplication)

$$\underset{m\times n}{A}\begin{bmatrix} \text{---} \end{bmatrix} \quad \underset{n\times p}{B}\begin{bmatrix} \equiv \end{bmatrix} = \underset{m\times p}{C}\begin{bmatrix} \text{---} \end{bmatrix}$$

rows of C are combinations of row of B

4th Way:

$$\boxed{\underset{(m\times 1)}{\text{col of } A} \times \underset{(1\times p)}{\text{row of } B} = m\times p \text{ matrix}}$$

$$\underset{(3\times 1)}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \underset{(1\times 2)}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \underset{(3\times 2)}{\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}}$$

i) each column of (3x2) is combination of col of (1x2) i.e if we draw picture of columns of (3x2) & (3x1), they are in same direction or these columns are along one line.

$$AB =$$

$$\underset{3\times 2}{\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

Blocks

If we subdivide A and B into blocks that match properly, we can write the product $AB = C$ in terms of products of the blocks:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

Here $C_1 = A_1B_1 + A_2B_3$.

Inverses

Square matrices

If A is a square matrix, the most important question you can ask about it is whether it has an inverse A^{-1}. If it does, then $A^{-1}A = I = AA^{-1}$ and we say that A is *invertible* or *nonsingular*.

If A is *singular* – i.e. A does not have an inverse – its determinant is zero and we can find some non-zero vector $\mathbf{x}$ for which $A\mathbf{x} = 0$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this example, three times the first column minus one times the second column equals the zero vector; the two column vectors lie on the same line.

Finding the inverse of a matrix is closely related to solving systems of linear equations:

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

can be read as saying "A times column j of A^{-1} equals column j of the identity matrix". This is just a special form of the equation $A\mathbf{x} = \mathbf{b}$.

Gauss-Jordan Elimination

We can use the method of elimination to solve two or more linear equations at the same time. Just augment the matrix with the whole identity matrix I:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

(Once we have used Gauss' elimination method to convert the original matrix to upper triangular form, we go on to use Jordan's idea of eliminating entries in the upper right portion of the matrix.)

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

OR in $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$

A B C

i) Row space of "C" (all rows of C) ~~are~~ is a line. It is in the direction of $[1 \; 6]$.

ii) Also column space is also a "line". It is in the direction of $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. All the column lies on the vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Block (Another way to multiply matrix)

10x10

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & \\ & \end{bmatrix}$$

A (20x20) B (20x20)

So there are 5 ways to multiply matrix.

i) Conventional (dot product)
ii) linear combination of columns
iii) linear combination of rows
iv) column x row multiplication
v) Block multiplication

INVERSE (Square Matrix)

$$A^{-1}A = I = AA^{-1}$$

If it exist? Yes then A is non-singular, invertible

[The left inverse is the same as right inverse b/c we are dealing with square matrix. This will not be the case with rectangular matrix]

Singular Case (No inverse)

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ This has no inverse because

i) $\det A = 0$

ii) $AA^{-1} = I$, the columns of I should be combination of columns of A. Hence it is ~~diff~~ impossible to find such combination in A^{-1}.

iii) You can find a vector X with $AX = 0$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\Rightarrow AX = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus if $AX = 0$ then $A^{-1}AX = A^{-1}(0)$
$\Rightarrow X = 0$ (This is not true as $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$)
(i.e. we can not recover X)

Conclusion: singular matrices can yield zero-vector of column with some combination of their columns. ~~To we a~~

As in the last lecture, we can write the results of the elimination method as the product of a number of elimination matrices E_{ij} with the matrix A. Letting E be the product of all the E_{ij}, we write the result of this Gauss-Jordan elimination using block matrices: $E[A \mid I] = [I \mid E]$. But if $EA = I$, then $E = A^{-1}$.

Let take $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$. It is invertible as its columns are pointing in different direction.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}\underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

i.e i) $A \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ii) $A \times \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ } i.e finding an inverse is solving 2-system

OR $A \times$ column j of A^{-1} = column j of I

Gauss Jordan (Solve 2 equations at once):

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\qquad A \quad I \qquad\qquad\qquad\qquad\qquad\qquad I \quad A^{-1}$$

Gauss-Jordan: ~~multiplier~~ $E[A \; I] = [I \; A^{-1}]$

thus $EA = I$ tells us $E = A^{-1}$

lecture - 4

Factorization into $A = LU$

One goal of today's lecture is to understand Gaussian elimination in terms of matrices; to find a matrix L such that $A = LU$. We start with some useful facts about matrix multiplication.

Inverse of a product

The inverse of a matrix product AB is $B^{-1}A^{-1}$.

Transpose of a product

We obtain the *transpose* of a matrix by exchanging its rows and columns. In other words, the entry in row i column j of A is the entry in row j column i of A^T.

The transpose of a matrix product AB is B^TA^T. For any invertible matrix A, the inverse of A^T is $(A^{-1})^T$.

$A = LU$

We've seen how to use elimination to convert a suitable matrix A into an upper triangular matrix U. This leads to the factorization $A = LU$, which is very helpful in understanding the matrix A.

Recall that (when there are no row exchanges) we can describe the elimination of the entries of matrix A in terms of multiplication by a succession of elimination matrices E_{ij}, so that $A \to E_{21}A \to E_{31}E_{21}A \to \cdots \to U$. In the two by two case this looks like:

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}.$$

We can convert this to a factorization $A = LU$ by "canceling" the matrix E_{21}; multiply by its inverse to get $E_{21}^{-1}E_{21}A = E_{21}^{-1}U$.

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}.$$

The matrix U is upper triangular with pivots on the diagonal. The matrix L is *lower triangular* and has ones on the diagonal. Sometimes we will also want to factor out a diagonal matrix whose entries are the pivots:

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U'}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}.$$

BASIC FACTS

$(AB)(B^{-1}A^{-1}) = I$ As $AA^{-1} = I = A^{-1}A$

$B^{-1}A^{-1}AB = I$

$AA^{-1} = I$ As $I^{T} = I$

$\Rightarrow \underbrace{(A^{-1})^T}_{\text{This is the inverse of } A^T \text{ i.e } (A^T)^{-1}} A^T = I$

Let $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

Now $\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$

$A = L U$ (L → lower triangle, U → upper triangle)

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

(more balanced form)

For 3x3 matrix:

$E_{21}A$ → 1st pivot

$E_{31}E_{21}A$ → 2nd pivot

$E_{32}E_{31}E_{21}A = U$ (no row exchange) → 3rd pivot

we want $A = LU$

$\Rightarrow A = \underline{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}\, U$

$A = LU$

here $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ is better than $E_{32}E_{31}E_{21}$. How?

Suppose $\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E\left(\text{left of } A\right) : EA = U$

inverses (reverse order)

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L\left(\text{left of } U\right) : A = LU$

In the three dimensional case, if $E_{32}E_{31}E_{21}A = U$ then $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$.

For example, suppose E_{31} is the identity matrix and E_{32} and E_{21} are as shown below:

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}}.$$

The 10 in the lower left corner arises because we subtracted twice the first row from the second row, then subtracted five times the new second row from the third.

The factorization $A = LU$ is preferable to the statement $EA = U$ because the combination of row subtractions does not have the effect on L that it did on E. Here $L = E^{-1} = E_{21}^{-1}E_{32}^{-1}$:

$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}.$$

Notice the 0 in row three column one of $L = E^{-1}$, where E had a 10. If there are no row exchanges, the multipliers from the elimination matrices are copied directly into L.

How expensive is elimination?

Some applications require inverting very large matrices. This is done using a computer, of course. How hard will the computer have to work? How long will it take?

When using elimination to find the factorization $A = LU$ we just saw that we can build L as we go by keeping track of row subtractions. We have to remember L and (the matrix which will become) U; we don't have to store A or E_{ij} in the computer's memory.

How many operations does the computer perform during the elimination process for an $n \times n$ matrix? A typical operation is to multiply one row and then subtract it from another, which requires on the order of n operations. There are n rows, so the total number of operations used in eliminating entries in the first column is about n^2. The second row and column are shorter; that product costs about $(n-1)^2$ operations, and so on. The total number of operations needed to factor A into LU is on the order of n^3:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2 \approx \int_0^n x^2\,dx = \frac{1}{3}n^3.$$

While we're factoring A we're also operating on $\mathbf{b}$. That costs about n^2 operations, which is hardly worth counting compared to $\frac{1}{3}n^3$.

$A=LU$

If norow exchanges, the multiplier goes directly into L

How many operations on an $n \times n$ matrix A? (For elimination)

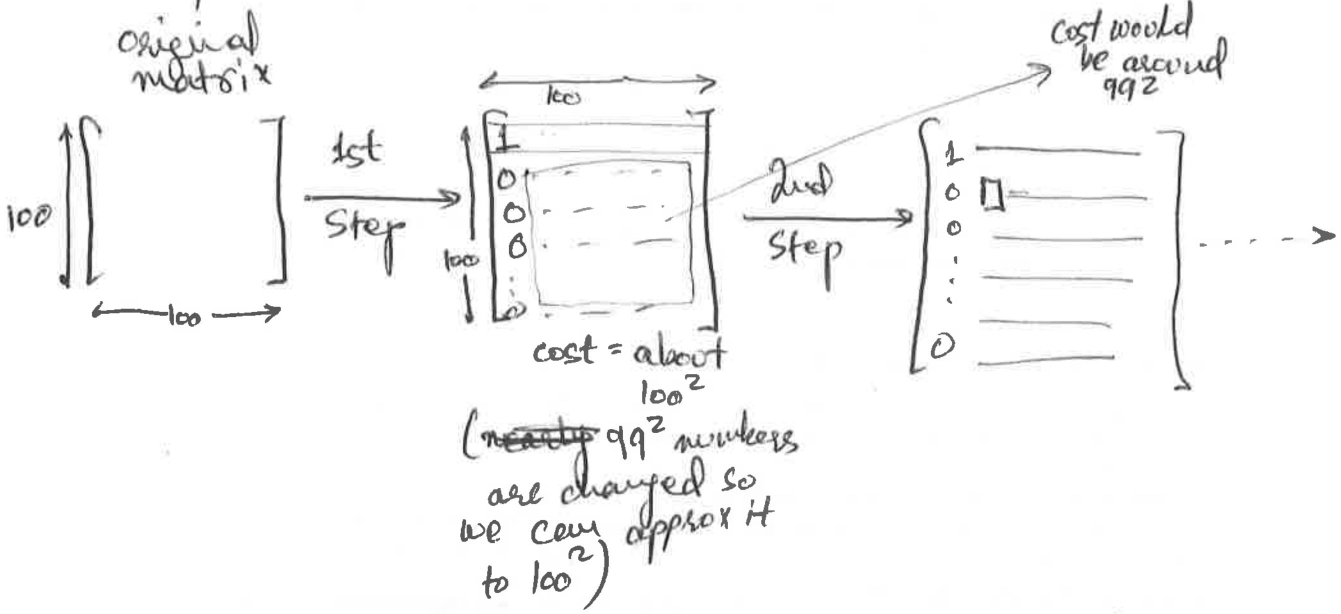
Say $n=100$

We have to get 99 zeros. after 1st step. What is the cost of 1st step.

what is the meaning of operation? It is addition, sub, mult, add.

A typical operation = multiply + subtract.

Cost of elimination:

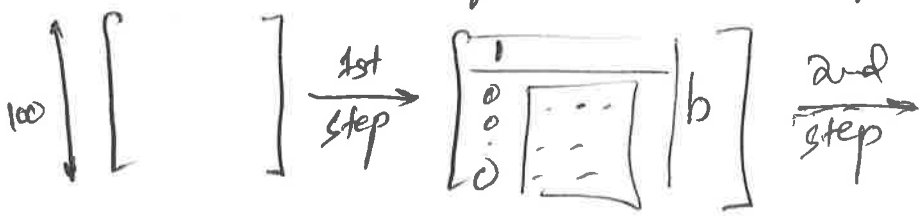


$$\text{Total cost} \approx 100^2 + 99^2 + 98^2 + \cdots + 1^2 = (n)^2 + (n-1)^2 + (n-2)^2 + \cdots + (1)^2$$

} operation on A to get to U

> Cost formula
>
> $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3}$$
>
> This is on A

Now whats the cost of b which is ~~inp~~ inside Elimination matrix? Its n^2.



Row exchanges

What if there are row exchanges? In other words, what happens if there's a zero in a pivot position?

To swap two rows, we multiply on the left by a permutation matrix. For example,

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

swaps the first and second rows of a 3×3 matrix. The inverse of any permutation matrix P is $P^{-1} = P^T$.

There are $n!$ different ways to permute the rows of an $n \times n$ matrix (including the permutation that leaves all rows fixed) so there are $n!$ permutation matrices. These matrices form a *multiplicative group*.

Permutations: (Used for row exchange)

P_{12} = Permutation matrix exchanges row 1 & row 2 $= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

How many 3x3 permutation matrices are there? 6 Ps

Identity: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, P_{12}: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, P_{13}: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

P_{23}: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

About permutation matrix

$$P^{-1} = P^T$$

For 4x4: How many Ps are there? 24 Ps

$(4)! = 4 \times 3 \times 2 = 24$

Lecture-5 (The beginning of real linear Algebra)

Permutations P: execute row exchanges.

$$A = LU = \begin{bmatrix} 1 & & 0 \\ \cdots & 1 & \\ \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \ast & \cdots & \cdots \\ & \ddots & \cdots \\ 0 & & \ast \end{bmatrix}$$

$A = LU$ become $\underbrace{PA = LU}_{\text{For any invertible "A"}}$ (P is used for row-exchange — when pivot appears to be zero)

Permutations:

* P is the identity matrix with reordered rows.
* $n! = n(n-1)\dots(3)(2)(1)$ counts the possible reordering. counts all $n \times n$ permutations
* P^{-1} exist And $P^{-1} = P^T$ And $P^T P = I$

TRANSPOSES $(A^T)_{ij} = A_{ji}$

Symmetric Matrices: $A^T = A$

If "R" is rectangular matrix the $R^T R$ is always symmetric.

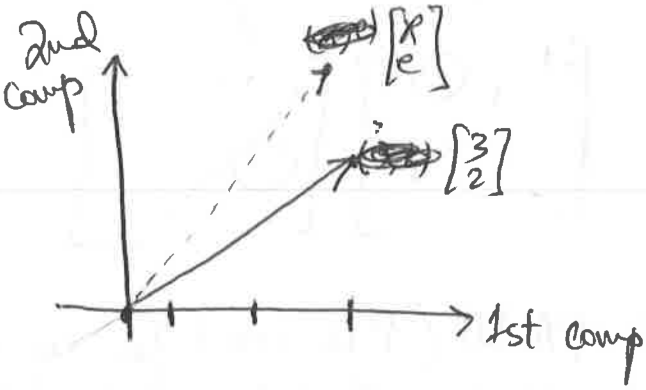
Let $R = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ then $R^T R = \begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}$

How $R^T R$ is symmetric? Why is this symmetric. Take transpose of $R^T R$.

$$\Rightarrow (R^T R)^T = R^T R^{TT}$$
$$= R^T R \quad \text{Hence symmetric}$$

VECTOR SPACES (space means bunch of vectors or space of vectors)

Examples: R^2 = all 2-D vectors / 2D real vectors $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}x\\e\end{bmatrix}, \ldots$



(0,0) is the important vector

$\mathbb{R}^2$ = XY plane or "x-y" plane. Its a vector space.

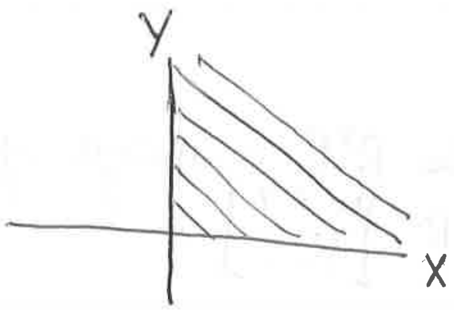
$\mathbb{R}^3$ = all vectors with 3 real components

For instance $\begin{bmatrix}3\\2\\0\end{bmatrix}$

$\mathbb{R}^N$ = all column vectors with n-components (n-real numbers)

Not a Vector-space

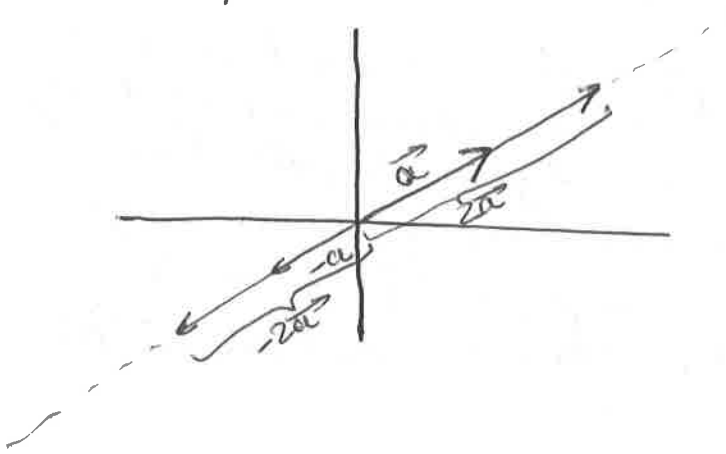
Consider R^2:



If we consider the shaded portion as vector-space we will conclude that its not a valid vector-space because it is not closed under multiplication. For e.g if we multiply any vector by negative real number, it will take us into 3rd-quadrant. Hence our assumption to consider XY's 1st quadrant as vector space is not correct.

Again:

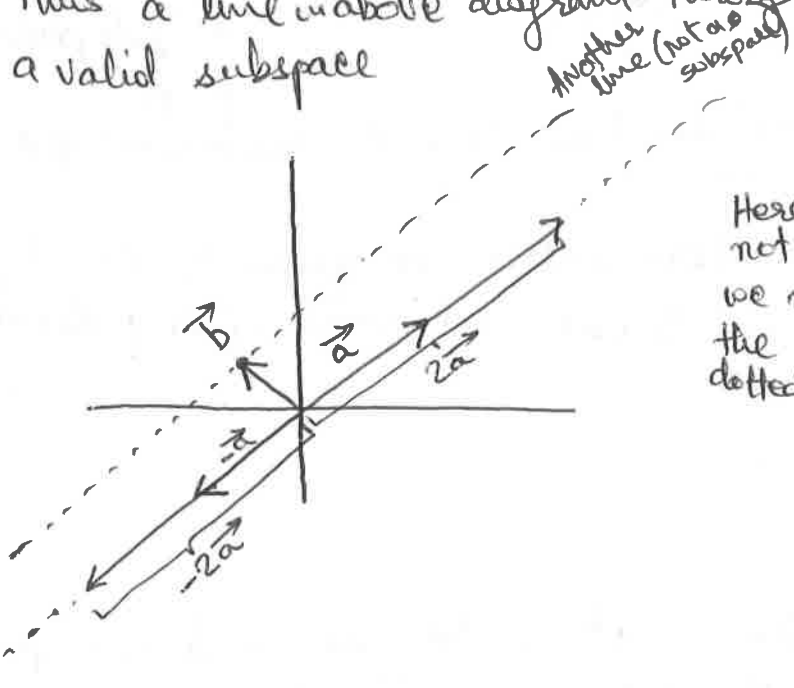
A Vector space inside R^2 (Subspace of R^2)



To be a Subspace
A line ~~through~~ in R^2 (not any line) must go through zero vector.

To be a subspace, we need to check whether
i) addition of ~~any~~ vector by itself remains in subspace
ii) multiplication by any real no., do we still remain in subspace.

Thus a line in above diagram through origin is representing a valid subspace



Here the dotted line is not a subspace b/c if we multiply $\vec{b}$ by $\vec{0}$ vector, the result will not be on dotted line.

Subspaces of R^2:
① All of $\mathbb{R}^2$
② Any line through $\begin{bmatrix}0\\0\end{bmatrix}$ (L)
③ Zero Vector only (Z)

Subspaces of R^3:
① All of R^3
② line through origin / or plane through origin
③ Zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$

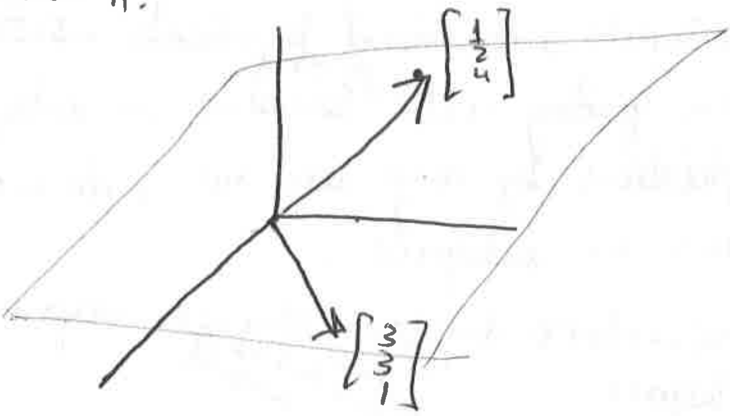
Now, how do these subspaces come out of matrix?

Consider a matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

i) Columns are in $\mathbb{R}^3$. I want these columns (vectors) to be in subspace. We can't say these are our subspace. So if all their linear combination form a subspace then columns in $\mathbb{R}^3$ form a subspace called column space $C(A)$

Lets draw A:



linear combination of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ fills a plane through origin which is a "subspace" of columns of A.

If columns of A would have shared same line then "column space" of A would be a line. Here it is a plane

Thus to get subspace from matrix ~~is~~ requires us to take columns of matrix, get their linear combination (all of them) and we get column subspace.

Recitation Video

Linear Space: For a set to be in a linear space, these are the following conditions:

i) You take any element from that set, take their sum. Sum should be in the same set.

ii) Take any multiple of any element from that set, the result should be in the same set.

<u>Subspace</u>: If in linear space, we find a space which satisfies two conditions (already stated), we call that space a subspace.

Linear Combination of x, y : $ax+by$; (a,b) are constants & lies in the space of $ax+by$.

Lecture 6

Transposes, permutations, spaces R^n

In this lecture we introduce vector spaces and their subspaces.

Permutations

Multiplication by a permutation matrix P swaps the rows of a matrix; when applying the method of elimination we use permutation matrices to move zeros out of pivot positions. Our factorization $A = LU$ then becomes $PA = LU$, where P is a permutation matrix which reorders any number of rows of A. Recall that $P^{-1} = P^T$, i.e. that $P^TP = I$.

Transposes

When we take the transpose of a matrix, its rows become columns and its columns become rows. If we denote the entry in row i column j of matrix A by A_{ij}, then we can describe A^T by: $\left(A^T\right)_{ij} = A_{ji}$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}.$$

A matrix A is *symmetric* if $A^T = A$. Given any matrix R (not necessarily square) the product R^TR is always symmetric, because $\left(R^TR\right)^T = R^T\left(R^T\right)^T = R^TR$. (Note that $\left(R^T\right)^T = R$.)

Vector spaces

We can add vectors and multiply them by numbers, which means we can discuss *linear combinations* of vectors. These combinations follow the rules of a *vector space*.

One such vector space is $\mathbb{R}^2$, the set of all vectors with exactly two real number components. We depict the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ by drawing an arrow from the origin to the point (a, b) which is a units to the right of the origin and b units above it, and we call $\mathbb{R}^2$ the "$x - y$ plane".

Another example of a space is $\mathbb{R}^n$, the set of (column) vectors with n real number components.

Closure

The collection of vectors with exactly two *positive* real valued components is *not* a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, -5, gives a vector that's not in the collection. We say that this collection of positive vectors is *closed* under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example, take any non-zero vector $\mathbf{v}$ in $\mathbb{R}^2$. Then the set of all vectors $c\mathbf{v}$, where c is a real number, forms a subspace of $\mathbb{R}^2$. This collection of vectors describes a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$ and is closed under addition.

A line in $\mathbb{R}^2$ that does not pass through the origin is *not* a subspace of $\mathbb{R}^2$. Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of $\mathbb{R}^2$ are:

1. all of $\mathbb{R}^2$,
2. any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and
3. the zero vector alone (Z).

The subspaces of $\mathbb{R}^3$ are:

1. all of $\mathbb{R}^3$,
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

Column space

Given a matrix A with columns in $\mathbb{R}^3$, these columns and all their linear combinations form a subspace of $\mathbb{R}^3$. This is the *column space* $C(A)$. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$, the column space of A is the plane through the origin in $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

Our next task will be to understand the equation $A\mathbf{x} = b$ in terms of subspaces and the column space of A.

lecture 7

The column space of a matrix A tells us when the equation Ax=b will have solution. The null space of X will tells us which values of x solve Ax=0.

Column space and nullspace

In this lecture we continue to study subspaces, particularly the column space and nullspace of a matrix.

Review of subspaces

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in the space and any two real numbers c and d, the vector $c\mathbf{v} + d\mathbf{w}$ is also in the vector space. A subspace is a vector space contained inside a vector space.

A plane P containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and a line L containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are both subspaces of $\mathbb{R}^3$. The union $P \cup L$ of those two subspaces is generally not a subspace, because the sum of a vector in P and a vector in L is probably not contained in $P \cup L$. The intersection $S \cap T$ of two subspaces S and T is a subspace. To prove this, use the fact that both S and T are closed under linear combinations to show that their intersection is closed under linear combinations.

Column space of A

The *column space* of a matrix A is the vector space made up of all linear combinations of the columns of A.

Solving $A\mathbf{x} = \mathbf{b}$

Given a matrix A, for what vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{b}$ does not have a solution for every choice of $\mathbf{b}$ because solving $A\mathbf{x} = \mathbf{b}$ is equivalent to solving four linear equations in three unknowns. If there is a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ must be a linear combination of the columns of A. Only three columns cannot fill the entire four dimensional vector space – some vectors $\mathbf{b}$ cannot be expressed as linear combinations of columns of A.

Big question: what $\mathbf{b}$'s allow $A\mathbf{x} = \mathbf{b}$ to be solved?

A useful approach is to choose $\mathbf{x}$ and find the vector $\mathbf{b} = A\mathbf{x}$ corresponding to that solution. The components of $\mathbf{x}$ are just the coefficients in a linear combination of columns of A.

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is *solvable* exactly when $\mathbf{b}$ is a vector in the *column space* of A.

For our example matrix A, what can we say about the column space of A? Are the columns of A *independent*? In other words, does each column contribute something new to the subspace?

The third column of A is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix A is a two dimensional subspace of $\mathbb{R}^4$.

Nullspace of A

The *nullspace* of a matrix A is the collection of all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $A\mathbf{x} = 0$.

The column space of the matrix in our example was a subspace of $\mathbb{R}^4$. The nullspace of A is a subspace of $\mathbb{R}^3$. To see that it's a vector space, check that any sum or multiple of solutions to $A\mathbf{x} = \mathbf{0}$ is also a solution: $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0} + \mathbf{0}$ and $A(c\mathbf{x}) = cA\mathbf{x} = c(\mathbf{0})$.

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the nullspace $N(A)$ consists of all multiples of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$; column 1 plus column 2 minus column 3 equals the zero vector. This nullspace is a line in $\mathbb{R}^3$.

Other values of b

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in $\mathbb{R}^3$ that passes through the points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ but not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

* Vector Space : i) $V+W$ and cV are in the space
  ii) all combinations $cV+dW$ are in the space

* Subspace : Any vector space inside vector space is subspace.
  For instance i) all planes passing through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ } are subspaces of $\mathbb{R}^3$
  ii) all lines passing through $\begin{bmatrix}0\\0\\0\end{bmatrix}$

* Two Important Question :
  lets imagine 2 subspaces P and L (let "P" be a plane passing through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ in $\mathbb{R}^3$ and "L" be a line passing through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ in $\mathbb{R}^3$)

  $P \cup L$ = is it a subspace? (No b/c $\underset{\text{on } P}{V} + \underset{\text{on } L}{V}$ may be outside subspace)
  = all vectors in P or all vectors in L or both.

  $P \cap L$ = is it a subspace? (Yes)
  = all vectors in P & L

* General Question :
  Subspaces S and T. Intersection $S \cap T$ is a subspace.

Column Space of A is a subspace of R^4 because A ~~...~~ 4×3

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \Rightarrow C(A) = \text{all linear combinations of columns}$$

column space of A

Q1: Do the linear combination of column vectors fill 4-Dimension space completely? ~~(Yes)~~ No in above example

Q2 Does $Ax=b$ always ~~solvable~~ have a solution for every b? (No)

Q3 For which "b" does $Ax=b$ is not solvable and solvable?

For Q2

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{4\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Answer is No in this particular case as there are 3 vars & 4 unknowns. ~~...~~ Also, abstractly the linear combination of columns doesn't fill $\mathbb{R}^4$ completely.

Q4: Which "b" allow the above system to be solved?

Ans: i) If $b_1=b_2=b_3=b_4=0$ then above system becomes solvable.

ii) Can we solve it for $\begin{bmatrix} b_1=1 \\ b_2=2 \\ b_3=3 \\ b_4=4 \end{bmatrix}$, Yes because $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Similarly $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

And $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Central Idea

I can solve $Ax=b$ exactly when b in in $C(A)$. This is because column space contains every Ax.

If "b" is the combination of columns of A, then we can solve $Ax=b$. If "b" is not the combination of column of "A", then we can not solve $Ax=b$.

In $A\mathbf{x} = \begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \end{array}$. Can we throw any vector from A and still we have retained the same $C(A)$. Yes we can remove C_3 b/c $C_3 = C_1 + C_2$ i.e vector defined by C_3 is in the subspace of C_1 & C_2. Thus the column space of this matrix "A" is 2-dimensional subspace of $\mathbb{R}^4$.

NULL SPACE

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

Nullspace of A = all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ (Nullspace is in $\mathbb{R}^3$)

while $C(A)$ is in $\mathbb{R}^4$

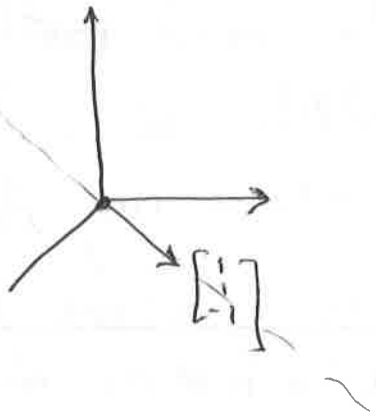
Nullspace $N(A)$:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i) one solution is $\vec{0}$ i.e $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$ OR $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

In this example, $N(A)$ is line in $\mathbb{R}^3$



Now check that solution to $Ax=0$ always give a subspace.

Proof If $Ax=0$ and also $Ax^*=0$

or $AV=0$ and $Aw=0$ then $A(v+w)=0$ for being qualified as a subspace.

$$\left.\begin{aligned} A(v+w) &= AV + Aw \\ &= 0+0 \\ &= 0 \end{aligned}\right\} \text{distribution law}$$

when $Ax=b$:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Q1: Do I get or solution $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ form vector space or subspace? b/c here X can't be $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\vdots \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

There are solutions $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ but they are not forming subspace because they don't contain $x=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Considering the case of subspace:

$$Ax = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The case of Null space

Lecture 8

Solving $Ax = 0$: pivot variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

Computing the nullspace

The *nullspace* of a matrix A is made up of the vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$.

Suppose:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = 0$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = 0$

(Note that the columns of this matrix A are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that A is not invertible. We don't need to use an augmented matrix because the right side (the vector $\mathbf{b}$) is $\mathbf{0}$ in this computation.

The row operations used in the method of elimination don't change the solution to $A\mathbf{x} = \mathbf{b}$ so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

We don't find a pivot in the second column, so our next pivot is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The matrix U is in *echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated.

The *rank* of a matrix A equals the number of pivots it has. In this example, the rank of A (and of U) is 2.

Special solutions

Once we've found U we can use back-substitution to find the solutions $\mathbf{x}$ to the equation $U\mathbf{x} = \mathbf{0}$. In our example, columns 1 and 3 are *pivot columns* containing pivots, and columns 2 and 4 are *free columns*. We can assign any value to x_2 and x_4; we call these *free variables*. Suppose $x_2 = 1$ and $x_4 = 0$. Then:

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$x_3 + x_4 = 0$

$$2x_3 + 4x_4 = 0 \quad \Longrightarrow \quad x_3 = 0$$

and:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \implies x_1 = -2.$$

So one solution is $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (because the second column is just twice the first column). Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has $x_4 = 1$ and $x_2 = 0$. The nullspace of A is the collection of all linear combinations of these "special solution" vectors.

The rank r of A equals the number of pivot columns, so the number of free columns is $n - r$: the number of columns (variables) minus the number of pivot columns. This equals the number of special solution vectors and the dimension of the nullspace.

Reduced row echelon form

By continuing to use the method of elimination we can convert U to a matrix R in *reduced row echelon form* (rref form), with pivots equal to 1 and zeros above and below the pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = R.$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

F = free cols

$$R = \begin{array}{c} \begin{matrix} x_1 & x_3 & x_2 & x_4 \end{matrix} \\ \left[\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{Pivot colms}}^{I} \underbrace{\begin{bmatrix} 2 & -2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{\text{free columns}}^{F} \right] \end{array}$$

(Here I is an r by r square matrix.)

If N is the *nullspace matrix* $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ then $RN = 0$. (Here I is an $n - r$ by $n - r$ square matrix and 0 is an m by $n - r$ matrix.) The columns of N are the special solutions.

For Nullspace we know that $RX = 0$

or $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} X = \vec{0}$

$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} N = \vec{0}$

$(n-r)\times(n-r)$ $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -F \\ I \end{bmatrix}}_{\vec{N}} = \vec{0}\ (m \times n-r)$

We Apply the method of elimination to all matrices, invertible or not. Counting the pivot gives us the rank of the matrix. further simplifying the matrix puts it in reduced echelon form R & improves our description of the null space.

we will talk about rectangular matrix in this lect.

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = 0$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = 0$$

i) C_2 & R_3 are not independent

we will apply elimination. Elimination does not change nullspace but alters column space.

$$A \rightarrow \text{1st Pivot} \xrightarrow{1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & \boxed{0} & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

can't pivot E_{22} &

Here the number of pivots is 2

thus $\boxed{\text{rank of } A = \# \text{ of Pivots} = 2}$

We were solving $Ax=0$. Now we will solve $UX=0$.

$$Ax = 0 = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns   Pivot columns

Free columns   free columns

Rough work

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$x_3 + 4x_4 = 0$$

Pivot free pivot free

let solution $X = \begin{bmatrix} - \\ 1 \\ - \\ 0 \end{bmatrix}$ Assign 1 & 0 to free columns

As $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$2x_3 + 4x_4 = 0$
As $x_2 = 1$
$x_4 = 0$
$\Rightarrow x_3 = 0$
$x_1 = -2$

$\Rightarrow X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ This is sol to $Ax = 0$

Another sol

$\boxed{\begin{matrix} x_2 = 0 \\ x_4 = 1 \end{matrix}}$ $\Rightarrow x_1 = 4 + 2 \Rightarrow \boxed{x_1 = 6}$
$\Rightarrow \boxed{x_3 = -2}$

Another $X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ for which $UX = 0$

This is the vector in Null space. we can get series of x which forms null space.

$$\Rightarrow X = C \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ for all } C$$

Now assign free variables different value

$X = \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$ By backsubstitution $X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

This x ~~says~~ satisfies $Ax = 0$, $\Rightarrow X = d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Thus special solution $= c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

i.e the nullspace contains linear combination of special solution.
And there is one special sol. for each free variable $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

If the matrix is $m \times n$, and rank "r". this means:
$r \rightarrow$ # of pivots, or # of pivot vars
$n-r \rightarrow$ # of free variables which can produce $(n-r)$ special sol.

R = Reduced row echelon form (It has zeros above & below the pivot)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Making pivot vars = 1

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

indicating it is a combination of above rows

In Matlab R = rref(A)

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ setting in the pivot rows and colums

Now $Rx = 0$

$$x_1 + 2x_2 - 2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$Ax=0,\ Ux=0,\ Rx=0 \rightarrow$ all have same solutions

Pivot cols

$$I \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

free Cols

$$\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \rightarrow \text{Free part of matrix}$$

Rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \rightarrow \text{pivot rows}$$

free columns

33:34

Now let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ This $A = A^T_{old}$

$$A \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow{\text{Row exchg}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

Rank = 2 (# of pivots)

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot cols

free cols = 3 - 2 = 1

Fact:
# of pivots for matrix S & S' is always same

eq(i) $x_1 + 2x_2 + 3x_3 = 0$
$2x_2 + 2x_3 = 0$

If $x_3 = 1$ (free var)
$x_2 = -1$
$x_1 = -1$

Now for special sol $X = \begin{bmatrix} \\ \\ 1 \end{bmatrix}$ Give free var a convenient value

$$\Rightarrow X = \begin{bmatrix} -1 \\ -1 \\ +1 \end{bmatrix} \Rightarrow -1(C_1 \text{ of } U) - 1(C_2 \text{ of } U) + 1(C_3 \text{ of } U) = 0$$

$$\Rightarrow X = C \begin{bmatrix} -1 \\ -1 \\ +1 \end{bmatrix}$$

Basis for nullspace = $\begin{bmatrix} -1 \\ -1 \\ +1 \end{bmatrix}$

$$\text{Now } U \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Reduced Row}} \begin{array}{ccc} I & & F \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow X = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = C \begin{bmatrix} -F \\ I \end{bmatrix}$$

N

lecture 9

Solving $A\mathbf{x} = \mathbf{b}$: row reduced form R

When does $A\mathbf{x} = \mathbf{b}$ have solutions $\mathbf{x}$, and how can we describe those solutions?

Solvability conditions on b

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

The third row of A is the sum of its first and second rows, so we know that if $A\mathbf{x} = \mathbf{b}$ the third component of $\mathbf{b}$ equals the sum of its first and second components. If $\mathbf{b}$ does not satisfy $b_3 = b_1 + b_2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of $\mathbf{b}$ must equal zero.

One way to find out whether $A\mathbf{x} = \mathbf{b}$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in $\mathbf{b}$. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.$$

If $A\mathbf{x} = \mathbf{b}$ has a solution, then $b_3 - b_2 - b_1 = 0$. For example, we could choose $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $A\mathbf{x} = \mathbf{b}$ is solvable exactly when $\mathbf{b}$ is in the column space $C(A)$. We have these two conditions on $\mathbf{b}$; in fact they are equivalent.

Complete solution

In order to find all solutions to $A\mathbf{x} = \mathbf{b}$ we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ is to set all free variables to zero, then solve for the pivot variables.

For our example matrix A, we let $x_2 = x_4 = 0$ to get the system of equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our particular solution is:

$$\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}.$$

Combined with the nullspace

The general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is a generic vector in the nullspace. To see this, we add $A\mathbf{x}_p = \mathbf{b}$ to $A\mathbf{x}_n = \mathbf{0}$ and get $A\left(\mathbf{x}_p + \mathbf{x}_n\right) = \mathbf{b}$ for every vector $\mathbf{x}_n$ in the nullspace.

Last lecture we learned that the nullspace of A is the collection of all combinations of the special solutions $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are real numbers.

The nullspace of A is a two dimensional subspace of $\mathbb{R}^4$, and the solutions to the equation $A\mathbf{x} = \mathbf{b}$ form a plane parallel to that through $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of pivots of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then from the previous lecture we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$. For any vector $\mathbf{b}$ in $\mathbb{R}^m$ that's not a linear combination of the columns of A, there is no solution to $A\mathbf{x} = \mathbf{b}$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of $\mathbf{b}$ to satisfy. The equation $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$. There are $n - r = n - m$ free variables, so there are $n - m$ special solutions to $A\mathbf{x} = \mathbf{0}$.

Full row and column rank

If $r = m = n$ is the number of pivots of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Summary

If R is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many

GOAL: $Ax=b$

$$x_1+2x_2+2x_3+2x_4=b_1$$
$$2x_1+4x_2+6x_3+8x_4=b_2$$
$$3x_1+6x_2+8x_3+10x_4=b_3$$

1st Pivot

2nd Pivot

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \xrightarrow[\text{Pivot}]{\text{1st}} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

Augmented matrix $[A\ b]$

Pivot columns

$$\xrightarrow[\text{Pivot}]{\text{2nd}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

$\Rightarrow$ The condition of solvability: $b_3-b_2-b_1=0$
i.e we can only solve $Ax=b$ for those b's where $b_2+b_1=b_3$

Suppose $b=\begin{bmatrix}1\\5\\6\end{bmatrix} \Rightarrow$ Above matrix would be $\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & \boxed{0} \end{array}\right]$ This means $b=\begin{bmatrix}1\\5\\6\end{bmatrix}$ is OK

Solvability of $Ax=b$ [Condition on b]

$Ax=b$ is solvable when b is in $C(A)$ or "b" has to be the linear combination of columns of A.

or in other words, if a combination of rows of A gives zero row, then same combination of entries of "b" must give "0".

To find Complete Solution to $Ax=b$

① $X_{particular}$: Set all free variables to zero. Solve $Ax=b$ for pivot variables.

$x_2=x_4=0$

when $x_2=x_4=0 \Rightarrow \begin{matrix} x_1+2x_3=1 \\ 2x_3=3 \end{matrix} \Rightarrow X_p=\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

② $X_{nullspace}$: we can find all solutions ~~out of~~ in nullspace

---

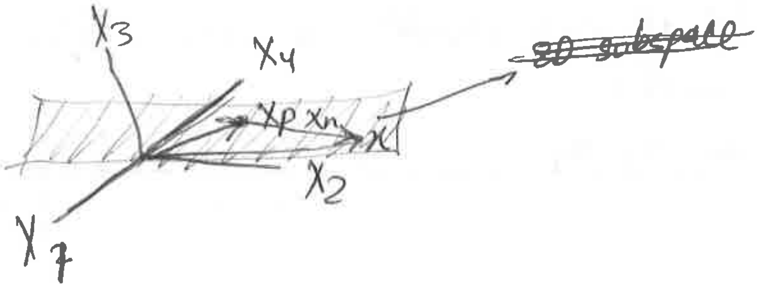
$$\text{Complete solution } X = X_p + X_n$$

$$\begin{aligned} Ax_p &= b \\ Ax_n &= 0 \\ \hline A(x_p+x_n) &= b \end{aligned}$$

Thus $X_{Complete}$ for above examples are $=$

$$\underbrace{\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{c_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\substack{x_{special} \\ \text{(from previous lectures)}}} + c_2 \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\substack{x_{special} \\ \text{(from previous lectures)}}}}_{X_N \text{ (All combination of special solutions)}}$$

Plot all solutions X in R^4:



m by n matrix A of rank "r":-
We know ($r \le m$ & $r \le n$).

Full Column Rank means $r = n$: There will be "n" pivots thus there will be "No free variables". Hence $N(A) = \{$zero vector$\}$ because we can't assign free values as there are no free variables.

Solution to $Ax = b$: $x = x_p$ (i.e unique solution if it exist or 0 or 1 solution)

For example, $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ After elimination, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Full row rank $\boxed{r = m}$ (every row has a pivot) I can solve $Ax = b$ for every "b" or every RHS. Thus solution exists.

left with $(n-r)$ OR $(n-m)$ free variables. Here $m < n$

$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \longrightarrow$ ~~R = [~~

$R = \begin{bmatrix} 1 & 0 & \_ & F & \_ \\ 0 & 1 & \_ & & \_ \end{bmatrix}$

Now $r = m = n$ (Square matrix. Its Full rank as columns & rows are equal. Here A will be invertible.)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Reduced Row echelon form $= I$

Q. Whats the nullspace $N(A)$??

$N(A) = \{zero\}$

Q. If we want to solve $Ax=b$ with $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, which R.H.S are OK? $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We can solve for every "b".

Full rank	Full row rank	Full column rank
$r = m = n$	$r = m < n$	$r = m < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [I \quad F]$ → F could be partly in I
1 sol to $Ax=b$	((0 or 1 solution to $Ax=b$))	((∞ solution b/c we have null space))

$r < m$, $r < n$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

((0 or ∞ solution))

* RANK tells everything about the "number" of solution.

Find sol to $\begin{array}{l} x-2y-2z=b_1 \\ 2x-5y-4z=b_2 \\ 4x-9y-8z=b_3 \end{array}$ } we will see when can we find solution depending on the value of b

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1+b_2 \\ 0 & -1 & 0 & -4b_1+b_3 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1+b_2 \\ 0 & 0 & 0 & -2b_1-b_2+b_3 \end{array}\right]$$

* If $-2b_1-b_2+b_3 \neq 0$
  → we will have no solution

* If $-2b_1-b_2+b_3 = 0$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & b_1+(-2)(2b_1-b_2) \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

multiply by -1 & adding R_1: $R_1 = R_1 + (-2R_2)$

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1+b_2 \\ 0 & 0 & 0 & -2b_1-b_2+b_3=0 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & 5b_1-2b_2 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot vars x — Pivot vars y — Free variable z

$$\Rightarrow \boxed{\begin{array}{l} x - 2z = 5b_1 - 2b_2 \\ y = 2b_1 - b_2 \end{array}}$$

* We have two type of solution → Particular Solution / Special Solution

PARTICOLAR SOLUTION (It solves $Ax=b$)
- $Ax=b$
- $z=0$

$$\Rightarrow X_p = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix}$$

Special Solution
- It solves $Ax=0$, There are as many solutions
- as free variables. Here its one
- Set $z=1$ (fix one free var to 1 & other free vars to zero)

$$X_S = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

All solutions $\vec{X} : \vec{X_p} + C\,\vec{X_S}$

lecture 10

Session Overview A basis is a set of vectors, as few as possible, whose combination produce all vectors in the space. The number of basis vector for a space equals the dimension of that space.

Independence, basis, and dimension

What does it mean for vectors to be independent? How does the idea of independence help us describe subspaces like the nullspace?

Linear independence

Suppose A is an m by n matrix with $m < n$ (so $A\mathbf{x} = \mathbf{b}$ has more unknowns than equations). A has at least one free variable, so there are nonzero solutions to $A\mathbf{x} = \mathbf{0}$. A combination of the columns is zero, so the columns of this A are *dependent*.

We say vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ are *linearly independent* (or just *independent*) if $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ only when $c_1, c_2, \ldots, c_n$ are all 0. When those vectors are the columns of A, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Two vectors are independent if they do not lie on the same line. Three vectors are independent if they do not lie in the same plane. Thinking of $A\mathbf{x}$ as a linear combination of the column vectors of A, we see that the column vectors of A are independent exactly when the nullspace of A contains only the zero vector.

If the columns of A are independent then all columns are pivot columns, the rank of A is n, and there are no free variables. If the columns of A are dependent then the rank of A is less than n and there are free variables.

Spanning a space

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ *span* a space when the space consists of all combinations of those vectors. For example, the column vectors of A span the column space of A.

If vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ span a space S, then S is the smallest space containing those vectors.

Basis and dimension

A *basis* for a vector space is a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_d$ with two properties:

- $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_d$ are independent
- $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_d$ span the vector space.

The basis of a space tells us everything we need to know about that space.

Example: $\mathbb{R}^3$

One basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. These are independent because:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is only possible when $c_1 = c_2 = c_3 = 0$. These vectors span $\mathbb{R}^3$.

As discussed at the start of Lecture 10, the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ do not form a basis for $\mathbb{R}^3$ because these are the column vectors of a matrix that has two identical rows. The three vectors are not linearly independent.

In general, n vectors in $\mathbb{R}^n$ form a basis if they are the column vectors of an invertible matrix.

Basis for a subspace

The vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ span a plane in $\mathbb{R}^3$ but they cannot form a basis for $\mathbb{R}^3$. Given a space, every basis for that space has the same number of vectors; that number is the *dimension* of the space. So there are exactly n vectors in every basis for $\mathbb{R}^n$.

Bases of a column space and nullspace

Suppose:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}.$$

By definition, the four column vectors of A span the column space of A. The third and fourth column vectors are dependent on the first and second, and the first two columns are independent. Therefore, the first two column vectors are the pivot columns. They form a basis for the column space $C(A)$. The matrix has rank 2. In fact, for any matrix A we can say:

$$\text{rank}(A) = \text{number of pivot columns of } A = \text{dimension of } C(A).$$

(Note that matrices have a rank but not a dimension. Subspaces have a dimension but not a rank.)

The column vectors of this A are not independent, so the nullspace $N(A)$ contains more than just the zero vector. Because the third column is the sum of the first two, we know that the vector $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace. Similarly, $\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is also in $N(A)$. These are the two special solutions to $A\mathbf{x} = \mathbf{0}$. We'll see that:

$$\text{dimension of } N(A) = \text{number of free variables} = n - r,$$

so we know that the dimension of $N(A)$ is $4 - 2 = 2$. These two special solutions form a basis for the nullspace.

$m < n$

$$\underset{2\times 3}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}} \underset{3\times 1}{\overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \underset{2\times 1}{\overset{B}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}} \Rightarrow \begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{matrix}$$

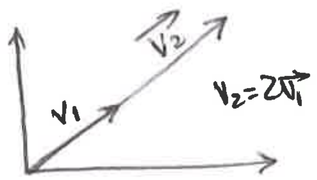
Suppose A is m by n with $m < n$ (more unknowns than equations). Then, there are non-zero solutions to $Ax = 0$ (i.e nullspace of A will have vectors other than zero-vector)

Reason: Suppose we perform elimination to $\underset{\substack{m\times n \\ m<n}}{A} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$ then we will have atleast one free variable. We can assign non-zero value to free-var & can get solution to $Ax = 0$

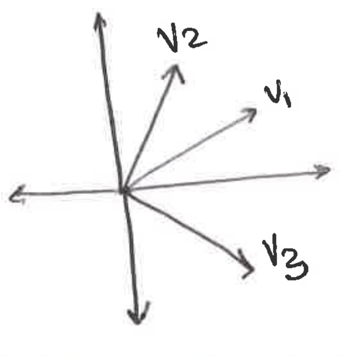
Independence Vectors $x_1, x_2, x_3 \dots x_n$ are independent if no (linear) combination gives zero vector ~~i.e~~ except the zero combination when $c_1, c_2, \dots c_N = 0$

$$\boxed{c_1 x_1 + c_2 x_2 + \dots + c_N x_N \neq 0}$$ For linear independence of vector

Example Are these vectors dependent or not?



v_1 & v_2 are dependendent b/c $-2v_1 + v_2 = 0$



v_1, v_2 & v_3 are dependent b/c now lets assume v_1, v_2, v_3 are columns of A.

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array}$$

Here $A = 2\times 3$ hence we will have atleast one free variable. Thus we will have non-zero solution to $Ax = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Repeating: when $V_1, \dots, V_n$ are columns of A.

i) They are independent if nullspace of A is zero vector → OR here the rank = n b/c there will be no free variable & $N(A) = \{0\}$

ii) They are dependent if $Ac = 0$ for some non-zero c. Here the rank < n b/c there will be n-r free variables

Q: what's the meaning of span?

Ans: Vectors $V_1, \dots, V_l$ <u>span</u> a space means: the space consist of all combinations of those vectors (exactly what we did with column space OR we can say that the column of matrix spans a column space).

Now rather than saying that ~~span~~ linear combination of vectors form space, now we will say that few vectors <u>spans</u> space.

Now if we have matrix $A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ with a bunch of vectors in it. We might want to know whether they are dependent or independent. There will be some vectors (column vectors) which will ~~form~~ be independent & spans column space while other columns might be the linear combination of previous independent column vectors.

<u>BASIS</u> for a vector space is a sequence of vectors $V_1, V_2 \dots V_d$ with two properties

i) They are independent

ii) They span the space

<u>Example</u>

Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Another bases

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

Are these two vectors basis?

i) They are independent

ii) They don't space space in $\mathbb{R}^3$

Now if we add another vector

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$

At least it should not be present in plane formed by previous 2 vec

How to see if $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ forms basis. We will stack them inside matrix A. We will apply elimination & get row-reduced form. If there are free variables then this means that ~~some~~.

$\underbrace{c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\2\\5\end{bmatrix} + c_3\begin{bmatrix}3\\3\\8\end{bmatrix} = 0}_{AC=0}$ hence these vectors are not basis.

If there are no free variables then $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ would be independent b/c there $AC=0$ if & only if $C=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

OR In $\mathbb{R}^3$, 3 vectors gives basis if 3×3 matrix, with those 3 vectors as columns, is invertible i.e A^{-1} exist when $A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix} \end{array}$

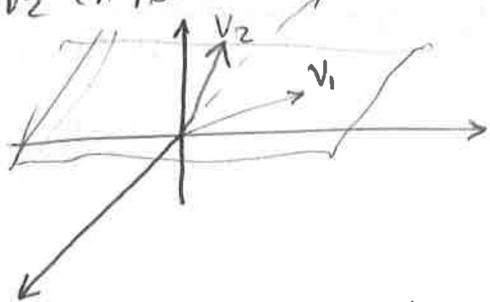
IN $\mathbb{R}^N$, n-vectors gives basis if $n\times n$ matrix (square matrix) is invertible ($n\times n$ matrix have n-vectors as its column)

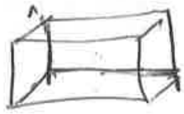
<u>For instance</u> Is there a space for which $v_1=\begin{bmatrix}1\\1\\2\end{bmatrix}, v_2=\begin{bmatrix}2\\2\\5\end{bmatrix}$ ~~...~~ is basis?

Yes, they are basis ~~for~~ of a plane inside $\mathbb{R}^3$ b/c v_1 & v_2

i) Are independent

ii) spans space formed by linear combination of v_1 & v_2 (A plane in $\mathbb{R}^3$)





If we stick v_3 in above picture, then v_1, v_2, v_3 won't be a basis b/c they will be dependent ($v_3 = av_1 + bv_2$)

Basis are not unique. There are zillions of basis. For example if we take $A_{3\times3}$ & if it is inveertible then its column vectors are basis. There are many many basis. But there is one fact which is :

Given a space

Every basis for the space has the same number of vectors [R^3 will have basis & its basis will have 3 vectors]

Def "D" This "D" is the dimension of space which is equal to the number of vectors forming basis of that space.

Examples Space is $C(A)$

$$A = \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Q1: ~~Are~~ Do v_1, v_2, v_3, v_4 span the column space of that matrix?

A: YES

Q2: Are they (v_1, v_2, v_3, v_4) a basis of $C(A)$

A: No. b/c they are not independent b/c there is something in $N(A)$

$$N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$ (Not independent)

Q3: whats the basis of the $C(A)$?

Ans: C_1 & C_2 (v_1, v_2). They will also be the pivot columns. Hence the rank of matrix $= 2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$
$$\begin{array}{cc} \uparrow & \uparrow \\ PC & PC \end{array}$$

Now we have new definition:

$2 = \text{rank}(A) = \#$ of pivot columns = dimensions of $C(A)$

USE Right words "Rank" is used for matrix

"dimensions" is used for column space of A NOT OF MATRIX "A".

Now tell me another basis of column space (A)?

We can take C_1 & C_3, C_2 & C_4, ...

Another basis for $C(A)$: $\underbrace{\begin{bmatrix}2\\2\\2\end{bmatrix}}_{2V_1}$, $\underbrace{\begin{bmatrix}7\\5\\7\end{bmatrix}}_{V_1+V_2+V_3+V_4}$ (Both are independent & spans column space And since we know that the dimension of $C(A)$ is 2, hence we will only with 2 vectors).

$$\boxed{\textbf{FACT} \quad \dim C(A) = r}$$

Now what about $N(A)$. We know that

$N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ are ~~but~~ these other vectors in $N(A)$? Yes, this is not a basis b/c it doesn't span. we have got more in $N(A)$.

$N(A) = \underbrace{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}_{\text{two special solutions}}$

with $x_3=1$ $x_4=0$
& $x_4=1$ $x_3=0$

These vectors in $N(A)$ are ~~telling~~ revealing us the combination with which we can discover that columns of A are dependent $[Ax=0]$. Have we got enough vectors? These are the two vectors in $N(A)$ $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ & they are independent. Are they forming a basis of $N(A)$? Whats the dimension of Nullspace? Ans: These two special solution $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ forms basis of Nullspace b/c $N(A)$ consists of all the combinations of $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$. Hence $N(A)$ is of two-dimension here.

$$\boxed{\begin{aligned}\dim N(A) &= \#\text{ of free variable}\\ &= n - r \quad \begin{pmatrix} m \rightarrow \text{rows of matrix} \\ n \rightarrow \text{col of matrix} \\ r \rightarrow \text{rank of matrix} \\ (\#\text{ of pivot columns}) \end{pmatrix}\end{aligned}}$$

Lecture 11

* For some ~~equations~~ vectors "b" the equation $Ax=b$ has solutions & for others it does not. Some vectors "x" are solutions to the equation $Ax=0$ and some are not. To understand these equations we study the column space, nullspace, rowspace and left nullspace of matrix A.

The four fundamental subspaces

In this lecture we discuss the four fundamental spaces associated with a matrix and the relations between them.

Four subspaces

Any m by n matrix A determines four subspaces (possibly containing only the zero vector):

Column space, $C(A)$

$C(A)$ consists of all combinations of the columns of A and is a vector space in $\mathbb{R}^m$.

Nullspace, $N(A)$

This consists of all solutions $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{0}$ and lies in $\mathbb{R}^n$.

Row space, $C(A^T)$

The combinations of the row vectors of A form a subspace of R^n. We equate this with $C(A^T)$, the column space of the transpose of A.

Left nullspace, $N(A^T)$

We call the nullspace of A^T the *left nullspace* of A. This is a subspace of $\mathbb{R}^m$.

Basis and Dimension

Column space

The r pivot columns form a basis for $C(A)$

$$\dim C(A) = r.$$

Nullspace

The special solutions to $A\mathbf{x} = \mathbf{0}$ correspond to free variables and form a basis for $N(A)$. An m by n matrix has $n - r$ free variables:

$$\dim N(A) = n - r.$$

Row space

We could perform row reduction on A^T, but instead we make use of R, the row reduced echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

Although the column spaces of A and R are different, the row space of R is the same as the row space of A. The rows of R are combinations of the rows of A, and because reduction is reversible the rows of A are combinations of the rows of R.

The first r rows of R are the "echelon" basis for the row space of A:

$$\dim C(A^T) = r.$$

Left nullspace

The matrix A^T has m columns. We just saw that r is the rank of A^T, so the number of free columns of A^T must be $m - r$:

$$\dim N(A^T) = m - r.$$

The left nullspace is the collection of vectors y for which $A^T y = 0$. Equivalently, $y^T A = 0$; here y and 0 are row vectors. We say "left nullspace" because y^T is on the left of A in this equation.

To find a basis for the left nullspace we reduce an augmented version of A:

$$\begin{bmatrix} A_{m \times n} & I_{m \times n} \end{bmatrix} \longrightarrow \begin{bmatrix} R_{m \times n} & E_{m \times n} \end{bmatrix}.$$

From this we get the matrix E for which $EA = R$. (If A is a square, invertible matrix then $E = A^{-1}$.) In our example,

$$EA = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

The bottom $m - r$ rows of E describe linear dependencies of rows of A, because the bottom $m - r$ rows of R are zero. Here $m - r = 1$ (one zero row in R).

The bottom $m - r$ rows of E satisfy the equation $\mathbf{y}^T A = \mathbf{0}$ and form a basis for the left nullspace of A.

New vector space

The collection of all 3×3 matrices forms a vector space; call it M. We can add matrices and multiply them by scalars and there's a zero matrix (additive identity). If we ignore the fact that we can multiply matrices by each other, they behave just like vectors.

Some subspaces of M include:

- all upper triangular matrices
- all symmetric matrices
- D, all diagonal matrices

D is the intersection of the first two spaces. Its dimension is 3; one basis for D is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Mistake in previous lecture:

Example: Basis for $\mathbb{R}^3$ could be

$$\overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}}$$

: This is what we have seen in previous lecture. Now we can observe the $A = [v_1\ v_2\ v_3]$ is invertible b/c R_1 & R_2 are same thus these vectors would'nt be the basis of $\mathbb{R}^3$. Thats why we will have a look on row spaces as well.

4 Subspaces (Heart of LA)

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

1. Column space $C(A)$
2. Nullspace $N(A)$
3. Rowspace (Rows spans row space) = All combination of rows
   = all combinations of the columns of A^T
   = $C(A^T)$ (since our vectors are columns in A thus we would like to stick to those vectors)
4. Nullspace of A^T = $N(A^T)$ = left nullspace of A

If A is $m \times n$ then

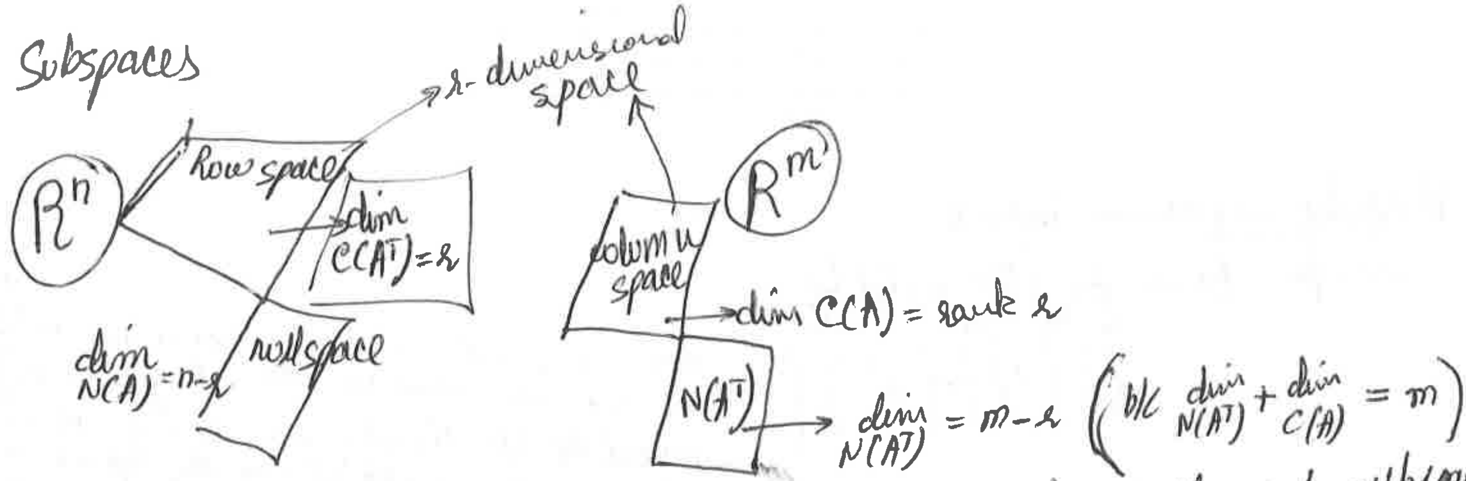
i) $N(A)$ is in $\mathbb{R}^n$ (sol to $Ax = b$)

ii) $C(A)$ is in $\mathbb{R}^m$

iii) $C(A^T)$ is in $\mathbb{R}^n$

iv) $N(\underset{n \times m}{A^T})$ is in $\mathbb{R}^m$

4 - Subspaces



What is the systematic way to construct basis of these 4 subspaces
And whats their dimension?

For C(A): * basis is pivot columns
* Dimension of C(A) is r

For $C(A^T)$ row-space:
* Fact: dim $C(A^T)$ = dim $C(A)$ = rank
* dimension of row-space = r

For N(A): $\underset{m \times n}{A}\underset{n \times 1}{x} = 0$
Recall, to compute N(A) we had n-r free variables & thus n-r ~~free~~ special solution. These special solution will be the basis of N(A).
That's why D[N(A)] = dim N(A) = n-r (n-r basis vectors)
* dim N(A) = n-r

Row space

$$A=\begin{bmatrix}\boxed{1}&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}\xrightarrow{\text{Row reduction}}\begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix}\xrightarrow{-R_2\times(-1)}\begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\longrightarrow\begin{array}{c}\begin{array}{cc}I & F\end{array}\\\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\end{array}=R$$

~~The~~ ~~column~~ Here $C(A) \neq C(A)$ Different column spaces b/c we did row operations

Same row space as I did row operations i.e vectors represented in rows of A & rows of R are same (These are vectors ~~are~~ of 4-components)

Now whats the basis of $C(A^T)$? Basis for Rowspace of A or of R is $\boxed{\text{first "r" rows of R "not" of "A"}}$

Basis for row-space

* Dimension of row-space $= r = \#$ of pivot rows of matrix R.
* $[1\,0\,1\,1]$ & $[0\,1\,1\,0]$ are the basis of 2D row-space

4th Space: $N(A^T)$

$A^T y = 0$ Then "y" is in the $N(A^T)$

$$[\quad]\, y = [0]$$

Taking transpose of $A^T y = 0$

$$y^T A = [0 \cdots 0]$$

y^T = row vector

$$\underbrace{[\cdots y^T \cdots]\,[\;A\;]}= [\cdots 0 \cdots]$$

Since "y" is on left that why we call it as left nullspace of A. Due to our convention, we will stay with $A^T y = 0$

* Now how to compute a basis of left nullspace?

~~Follow~~ $\boxed{\text{Next page}}$ for example

$$\text{rref}\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} \longrightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m}\end{bmatrix}$$

$\Big\}$ The steps for row-reduction are stored in E

(Remember Gauss Jordan)

$\downarrow$

$$E\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} \longrightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m}\end{bmatrix}$$

$$EA = R$$

In chap 2, $R = I$ (reduced echelon form of invertible matrix)
Then $E = A^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E}$$

I believe $EA = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

<u>Check</u> if $EA = R$ And it is equal to R.

* We want EA because we would like to know left nullspace.

$$\underset{E}{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* Dim of $N(A^T) = m - r$
$= 3 - 2$
$= 1$

* basis of $N(A^T) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ b/c combination of $-1(R_1 \text{ of } A) + 0(R_2 \text{ of } A) + (-1)(R_3 \text{ of } A) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

New Vector Space !

All 3x3 matrices !!

{ We are interested in $A + B$, CA. (not AB for now)

M = ~~Vector~~ All 3x3 matrices

Q: Tell me the subspace of M.

A: Subspaces of M || All upper triangular matrices || all Symmetric matrices || diagonal matrices

We will compute the dimension of all upper triangular matrices, all symmetric matrices, diagonal matrices. Thus we can get basis as well.

diagonal matrices (3x3) = all symmetric matrices $\cap$ All upper triangular matrices

* The $\underbrace{\text{dimension}(D) \text{ of all diagonal matrices}}_{\text{subspace of } M}$ is: 3

because:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$$

3 - diagonal ~~bas~~ matrices and we believe that they are basis b/c :

1) They are independent
2) Any diagonal matrix is a combination of above 3 matrices i.e the above 3 - matrices span the subspace of diagonal matrices

Suppose $B = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ find a basis for & compute the dimension of each of the 4 fundamental subspaces

$B = \quad L \qquad U$

$\left\{ B = \begin{bmatrix} 5 & 0 & 3 \\ 10 & 1 & 7 \\ -5 & 0 & -3 \end{bmatrix} \right\}$

- $\dim C(B) = 2$

A basis for $C(B)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

- $\dim N(B) = 1 \; (n-r)$

A basis for $N(B)$ is $\begin{pmatrix} -3/5 \\ -1 \\ 1 \end{pmatrix}$

- $\dim C(B^T) =$ dimension of $C(B)$

$= 2$

A Basis for $C(B^T)$ is $\underbrace{\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}}_{\text{Pivot Rows of } U}$

- $\dim N(B^T) = 3 - 2$

$= 1$

$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 1 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix}$

corresponds to free row

Free - Vars

A basis for this $N(B^T)$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(See the picture)

Lecture 12

Vectors don't have to be list of numbers. In this session, we explore important new vector spaces while practicing the skills we learned. Then we begin the application of matrices to the study of networks.

Matrix spaces; rank 1; small world graphs

We've talked a lot about $\mathbb{R}^n$, but we can think about vector spaces made up of any sort of "vectors" that allow addition and scalar multiplication.

New vector spaces

3 by 3 matrices

We were looking at the space M of all 3 by 3 matrices. We identified some subspaces; the symmetric 3 by 3 matrices S, the upper triangular 3 by 3 matrices U, and the intersection D of these two spaces – the space of diagonal 3 by 3 matrices.

The dimension of M is 9; we must choose 9 numbers to specify an element of M. The space M is very similar to $\mathbb{R}^9$. A good choice of basis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The subspace of symmetric matrices S has dimension 6. When choosing an element of S we pick three numbers on the diagonal and three in the upper right, which tell us what must appear in the lower left of the matrix. One basis for S is the collection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The dimension of U is again 6; we have the same amount of freedom in selecting the entries of an upper triangular matrix as we did in choosing a symmetric matrix. A basis for U is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This happens to be a subset of the basis we chose for M, but there is no basis for S that is a subset of the basis we chose for M.

The subspace $D = S \cap U$ of diagonal 3 by 3 matrices has dimension 3. Because of the way we chose bases for U and S, a good basis for D is the intersection of those bases.

Is $S \cup U$, the set of 3 by 3 matrices which are either symmetric or upper triangular, a subspace of M? No. This is like taking two lines in $\mathbb{R}^2$ and asking if together they form a subspace; we have to fill in between them. If we take all possible sums of elements of S and elements of U we get what we call the *sum* $S + U$. This is a subspace of M. In fact, $S + U = M$. For unions and sums, dimensions follow this rule:

$$\dim S + \dim U = \dim S \cup U + \dim S \cap U.$$

Differential equations

Another example of a vector space that's not $\mathbb{R}^n$ appears in differential equations.

We can think of the solutions y to $\frac{d^2y}{dx^2} + y = 0$ as the elements of a nullspace. Some solutions are:

$$y = \cos x, \quad y = \sin x, \quad \text{and} \quad y = e^{ix}.$$

The complete solution is:

$$y = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 can be any complex numbers. This solution space is a two dimensional vector space with basis vectors $\cos x$ and $\sin x$. (Even though these don't "look like" vectors, we can build a vector space from them because they can be added and multiplied by a constant.)

Rank 4 matrices

Now let M be the space of 5×17 matrices. The subset of M containing all rank 4 matrices is not a subspace, even if we include the zero matrix, because the sum of two rank 4 matrices may not have rank 4.

In $\mathbb{R}^4$, the set of all vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ for which $v_1 + v_2 + v_3 + v_4 = 0$ is a subspace. It contains the zero vector and is closed under addition and scalar multiplication. It is the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Because A has rank 1, the dimension of this nullspace is $n - r = 3$. The subspace has the basis of special solutions:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The column space of A is $\mathbb{R}^1$. The left nullspace contains only the zero vector, has dimension zero, and its basis is the empty set. The row space of A also has dimension 1.

Rank one matrices

The rank of a matrix is the dimension of its column (or row) space. The matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

has rank 1 because each of its columns is a multiple of the first column.

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}.$$

Every rank 1 matrix A can be written $A = \mathbf{U}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are column vectors. We'll use rank 1 matrices as building blocks for more complex matrices.

Small world graphs

In this class, a *graph* G is a collection of nodes joined by edges:

$$G = \{\text{nodes}, \text{edges}\}.$$

A typical graph appears in Figure 1. Another example of a graph is one in

Figure 1: A graph with 5 nodes and 6 edges.

which each node is a person. Two nodes are connected by an edge if the people are friends. We can ask how close two people are to each other in the graph – what's the smallest number of friend to friend connections joining them? The question "what's the farthest distance between two people in the graph?" lies behind phrases like "six degrees of separation" and "it's a small world".

Another graph is the world wide web: its nodes are web sites and its edges are links.

We'll describe graphs in terms of matrices, which will make it easy to answer questions about distances between nodes.

GOAL

i) Bases of new vector spaces
ii) Rank one matrices
iii) Small world graphs

New Vector Spaces

M = all 3 by 3 matrices

We can add them, we can multiply them by scalars. We can also multiply them together but we don't do that because we want to stay in 3x3 matrix space.

Subspaces of M: Symmetric matrices (3x3) (S)
Its a subspace because
i) If we add two symmetric matrices, we are still in that space
(we won't multiply matrix here)

Upper triangular 3x3
Its a subspace b/c
i) If we add upper triangular matrix, we are still in that space
(· same)

Q. whats the basis of this subspace & what is its dimension → (... same)

Basis for M = all 3x3's [9 numbers, may be 9 dimensions]

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{9 \text{ matrices}}$$

$\dim(M) = 9$ → dimension of matrix space

$\dim(S) = 6$ → dimension of $\underbrace{\text{symmetric matrices}}_{3\times 3}$ because we need 6 ~~numbers~~ entries to define symmetric matrices.

$\dim(U) = 6$ → dimension of upper triangular matrices

Another Subspace: $S \cap U$ = symmetric & upper triangular
= diagonal 3x3's $\dim(S \cap U) = 3$

Why are we not interested in $(S \cup U)$

$S \cup U$ = Because its not in subspace

6D 6D

To make it subspace $S \cup U$ will be $S+U$

$S+U$ = combination of S & U

= any element of S + any element of U = $\boxed{\text{all } 3\times3 \text{ matrices}}$

$\dim(S+U) = 9$ (As we got all 3×3)

$\dim(S)=6$

$\dim(U)=6$

Formula $\dim(S) + \dim(U) = \dim(S+U) + \dim(S \cap U)$

ONE More Vector Space (where we don't have vectors like M)

$\frac{d^2y}{dx^2} + y = 0 \rightarrow$ 2nd order differential equation

Its solution is $y = \cos x, \sin x, e^{ix}$

Complete solution

$y = c_1 \cos x + c_2 \sin x$ (linear combination of solution &

its a vector space. whats the basis of this vector space?

BASIS $\cos x$, $\sin x$ (because all guys in solution space ~~are~~ comes from $\cos x, \sin x$)

* dim(solution space) = 2 (which is highlighting a fact that we have second order differential equation)

Why are we talking about this example? ~~Be~~ Because we can see that y = cosx, sinx & they don't look like ~~numbers~~ vectors instead they look like function. But we can add them & thus we can say that their combination fills solution space.

RANK ONE MATRICES: (Building block of all matrices)

$$\underset{2\times3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

$\underline{r = 1}$

* Basis for row space = $[1 \ 4 \ 5]$
* A Basis for column space.

Dim C(A) = 1 = rank

dim C(A) = rank = dim C(A^T)

$$\underset{2\times3}{A} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{2\times1}^{\text{Pivot col}} \underbrace{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}_{1\times3}^{\text{Basis for row space}}$$

Thus Rank-1 Matrix

$A = UV^T$ U, V } both are column vectors

U → Col Vector

Let

M = all 5x17 matrices

i) Subset of rank 4 matrices → is it a subspace?

Q: If I add two Rank-4 matrices, is the sum be rank-4?

Ans: Not probably

ii) Subset of rank 1 matric →

Q: If I add rank 1 matrix, is that a subspace

Not a subspace

Q: Suppose we are in $\mathbb{R}^4$. $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

Suppose S = all V in $\mathbb{R}^4$ with $\underbrace{V_1 + V_2 + V_3 + V_4 = 0}_{AV=0}$? Is it a subspace?

Q1. Is it subspace
A: Yes

Q2: whats the basis of subspaces & its dimension
A: Dimension is 3

This S = nullspace of $A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}}$

$\boxed{\begin{aligned} &\text{Rank of } A = 1 \\ &\dim N(A) = n - r \\ &= 4 - 1 \\ &= 3 \end{aligned}}$

Four Fundamental Subspaces of A

i) $C(A^T) = 1\text{-D}$

ii) $N(A) = 3\text{-D}$

BASIS = special sol (I look for free vars)

$\begin{bmatrix} \boxed{1} & \underbrace{1 \quad 1 \quad 1}_{\text{free var}} \end{bmatrix}$ → 3 vectors for bases of S.

$V_1 + V_2 + V_3 + V_4 = 0$

Special Sol: $V_2 = 1$, $V_3 = 0$, $V_4 = 0$, then $V_1 = -1$

Basis for the subspace S

free var $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right.$

iii) $C(A)$ = subspace of $\mathbb{R}^1$
$= \mathbb{R}^1$

iv) $N(A^T) = \{0\}$ Dim $N(A^T) = 0$ ~~then~~ and basis is empty set.

$A^T = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$

Graph = {nodes, edges} = bunch of nodes & edges

~~Do its~~ Recalation video ?

This session explores the linear algebra of electrical networks and the internet, and sheds light on important results in graph theory.

Lecture 12

Graphs, networks, incidence matrices

When we use linear algebra to understand physical systems, we often find more structure in the matrices and vectors than appears in the examples we make up in class. There are many applications of linear algebra; for example, chemists might use row reduction to get a clearer picture of what elements go into a complicated reaction. In this lecture we explore the linear algebra associated with electrical networks.

Graphs and networks

A *graph* is a collection of nodes joined by edges; Figure 1 shows one small graph.

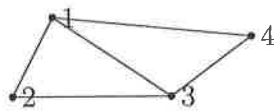


Figure 1: A graph with $n = 4$ nodes and $m = 5$ edges.

We put an arrow on each edge to indicate the positive direction for currents running through the graph.

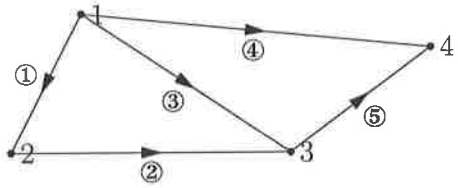


Figure 2: The graph of Figure 1 with a direction on each edge.

Incidence matrices

The *incidence matrix* of this directed graph has one column for each node of the graph and one row for each edge of the graph:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

If an edge runs from node a to node b, the row corresponding to that edge has -1 in column a and 1 in column b; all other entries in that row are 0. If we were studying a larger graph we would get a larger matrix but it would be *sparse*; most of the entries in that matrix would be 0. This is one of the ways matrices arising from applications might have extra structure.

Note that nodes 1, 2 and 3 and edges ①, ② and ③ form a loop. The matrix describing just those nodes and edges looks like:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Note that the third row is the sum of the first two rows; loops in the graph correspond to linearly dependent rows of the matrix.

To find the nullspace of A, we solve $A\mathbf{x} = \mathbf{0}$:

$$A\mathbf{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If the components x_i of the vector $\mathbf{x}$ describe the electrical potential at the nodes i of the graph, then $A\mathbf{x}$ is a vector describing the *difference* in potential across each edge of the graph. We see $A\mathbf{x} = \mathbf{0}$ when $x_1 = x_2 = x_3 = x_4$, so the nullspace has dimension 1. In terms of an electrical network, the potential difference is zero on each edge if each node has the same potential. We can't tell what that potential is by observing the flow of electricity through the network, but if one node of the network is grounded then its potential is zero. From that we can determine the potential of all other nodes of the graph.

The matrix has 4 columns and a 1 dimensional nullspace, so its rank is 3. The first, second and fourth columns are its pivot columns; these edges connect all the nodes of the graph without forming a loop – a graph with no loops is called a *tree*.

The left nullspace of A consists of the solutions $\mathbf{y}$ to the equation: $A^T\mathbf{y} = \mathbf{0}$. Since A^T has 5 columns and rank 3 we know that the dimension of $N(A^T)$ is $m - r = 2$. Note that 2 is the number of loops in the graph and m is the number of edges. The rank r is $n - 1$, one less than the number of nodes. This gives us # loops = # edges − (# nodes − 1), or:

$$\text{number of nodes} - \text{number of edges} + \text{number of loops} = 1.$$

This is Euler's formula for connected graphs.

Kirchhoff's law

In our example of an electrical network, we started with the potentials x_i of the nodes. The matrix A then told us something about potential differences. An engineer could create a matrix C using Ohm's law and information about

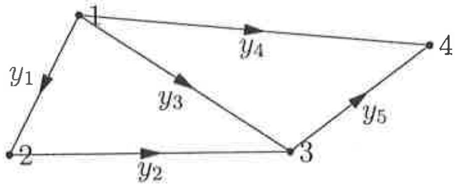


Figure 3: The currents in our graph.

the conductance of the edges and use that matrix to determine the current y_i on each edge. Kirchhoff's Current Law then says that $A^T\mathbf{y} = \mathbf{0}$, where $\mathbf{y}$ is the vector with components y_1, y_2, y_3, y_4, y_5. Vectors in the nullspace of A^T correspond to collections of currents that satisfy Kirchhoff's law.

$\mathbf{x} = x_1, x_2, x_3, x_4$ potentials at nodes		$A^T\mathbf{y} = 0$ Kirchhoff's Current Law
$\mathbf{e} = A\mathbf{x} \downarrow$		$\uparrow A^T\mathbf{y}$
$x_2 - x_1$, etc. potential differences	$\mathbf{y} = C\mathbf{e}$ $\longrightarrow$ Ohm's Law	y_1, y_2, y_3, y_4, y_5 currents on edges

Written out, $A^T\mathbf{y} = \mathbf{0}$ looks like:

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying the first row by the column vector $\mathbf{y}$ we get $-y_1 - y_3 - y_4 = 0$. This tells us that the total current flowing out of node 1 is zero – it's a balance equation, or a conservation law. Multiplying the second row by $\mathbf{y}$ tells us $y_1 - y_2 = 0$; the current coming into node 2 is balanced with the current going out. Multiplying the bottom rows, we get $y_2 + y_3 - y_5 = 0$ and $y_4 + y_5 = 0$.

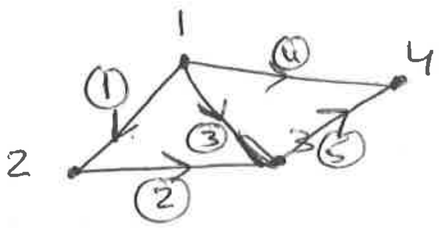
We could use the method of elimination on A^T to find its column space, but we already know the rank. To get a basis for $N(A^T)$ we just need to find two independent vectors in this space. Looking at the equations $y_1 - y_2 = 0$ we might guess $y_1 = y_2 = 1$. Then we could use the conservation laws for node 3 to guess $y_3 = -1$ and $y_5 = 0$. We satisfy the conservation conditions on node 4 with $y_4 = 0$, giving us a basis vector $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. This vector represents one unit of current flowing around the loop joining nodes 1, 2 and 3; a multiple of this vector represents a different amount of current around the same loop.

We find a second basis vector for $N(A^T)$ by looking at the loop formed by nodes 1, 3 and 4: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ that represents a current around the outer loop is also in the nullspace, but it is the sum of the first two vectors we found.

We've almost completely covered the mathematics of simple circuits. More complex circuits might have batteries in the edges, or current sources between nodes. Adding current sources changes the $A^T\mathbf{y} = \mathbf{0}$ in Kirchhoff's current law to $A^T\mathbf{y} = \mathbf{f}$. Combining the equations $\mathbf{e} = A\mathbf{x}$, $\mathbf{y} = C\mathbf{e}$ and $A^T\mathbf{y} = \mathbf{f}$ gives us:

$$A^T C A \mathbf{x} = \mathbf{f}.$$

Graph: Nodes, edges



$n=4$ (nodes)

$m=5$ edges (# of rows)

Incidence Matrix

$$A=\begin{array}{c}\begin{array}{cccc}\text{node }1 & \text{node }2 & \text{node }3 & \text{node }4\end{array}\\ \begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}\end{array}\begin{array}{l}\\ \text{edge }1\\ \text{edge }2\\ 3\\ 4\\ 5\end{array}$$

} loop: (corresponds to dependent rows)

Real matrices have structure in them. Now:

Q. what's the nullspace of A? OR Are the columns of A linearly independent? (Nullspace space ~~tells~~ of any matrix contains zero only, this means that ~~matrix~~ columns of matrix are independent.

$$Ax=0$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}x_2-x_1\\x_3-x_2\\x_3-x_1\\x_4-x_1\\x_4-x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$X=x_1,x_2,x_3,x_4$
Potential at nodes

$\downarrow \times A$

$\underbrace{x_2-x_1}_{}$, e.t
potential differences across edges

$x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = N(A)$

$N(A) = c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ Basis of $N(A) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Dim $N(A) = 1$

* Rank of $\underset{5\times4}{A} = 3$ (3 independent columns)

$A^T y = 0 \longrightarrow N(A^T) = ??$ dim $N(A^T) = m - r = 5 - 3 = 2$ } It will have 2-bases

4×5

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Q Why are we interesting in $N(A^T)$? (KCL in our examples)

$$\boxed{A^T C A x = f}$$

BASIC equation of applied maths (in equilibrium)

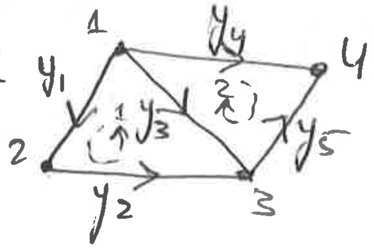
$x = x_1, x_2, x_3, x_4$
Potential at nodes

$e = Ax$

$x_2 - x_1$ etc
potential difference $\xrightarrow[\text{OHMs law}]{y = Ce}$ currents y_1, y_2, y_3, y_4, y_5 on edges

$\xrightarrow{A^T}$ $A^T y = 0$ Kirchoff's current law

will become $A^T y = f$ if we add source



$A^T y$: $-y_1 - y_3 - y_4 = 0$ (Row 1) KCL at 1

$y_1 - y_2 = 0$

$y_2 + y_3 - y_5 = 0$

$y_4 + y_5 = 0$

To get nullspace of $N(A^T)$, we can do elemination.

<u>BASIS for $N(A^T)$</u> $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

satisfies KCL

basis for $N(A^T)$ can be obtained by having loop 1 & loop 2

Q: $C(A^T) = 3 = \text{rank}$ (

Column 1, 2, 4 A^T are pivot columns.

$\boxed{\text{Tree: Its a graph with no loop}}$

$\dim N(A^T) = m - r$

$\boxed{\#\text{ loops} = \#\text{ edges} - (\#\text{ of nodes} - 1)}$ $\underbrace{\#\text{ nodes} - \#\text{ edges} + \#\text{ loops} = 1}_{\text{Euler's formula for any graph topology.}}$

$(\text{rank} = n - 1)$

* Eulers formula is proven by linear algebra

Final Question

In A^TCAx, A^TA will be a symmetric matrix.

$\boxed{\text{Recitation Videos ??}}$