4. Let s_{max} be the state that achieves the maximum loss i.e.

$\forall s \in S, s \neq s_{max}$ $\quad L^{\pi'}(s_{max}) \geq L^{\pi'}(s)$

Let $a = \pi^*(s_{max})$ and $b = \pi'(s_{max})$

Since $\pi'(s)$ takes a maximizing greedy action then we have that

$$R^a_{s_{max}} + \gamma \sum_{s' \in S} P^a_{s_{max}s'} V_{apr}(s') \leq R^b_{s_{max}} + \gamma \sum_{s' \in S} P^b_{s_{max}s'} V_{apr}(s')$$

Since $|v^*(s) - V_{apr}(s)| \leq \varepsilon$ by assumption

$$V_{apr}(s) - \varepsilon \leq v^*(s) \leq V_{apr}(s) + \varepsilon$$

$$\gamma \sum_{s' \in S} P^a_{s_{max}s'} (V^*(s') - \varepsilon) \leq \gamma \sum_{s' \in S} P^a_{s_{max}s'} V_{apr}(s')$$

$$\gamma \sum_{s' \in S} P^a_{s_{max}s'} v^*(s') - \varepsilon \gamma \sum_{s' \in S} P^a_{s_{max}s'}$$

$$\gamma \sum_{s' \in S} P^b_{s_{max}s'} (v^*(s') + \varepsilon) \geq \gamma \sum_{s' \in S} P^b_{s_{max}s'} V_{apr}(s')$$

$$\gamma \sum_{s' \in S} P^b_{s_{max}s'} v^*(s') + \varepsilon \gamma \sum_{s' \in S} P^b_{s_{max}s'} V_{apr}(s')$$

Rewriting new inequalities

$$R^a_{s_{max}} + \gamma \sum_{s' \in S} P^a_{s_{max}s'} V^*(s') - \epsilon\gamma$$

$$\leq$$

$$R^b_{s_{max}} + \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^*(s') + \epsilon\gamma$$

① $$R^a_{s_{max}} - R^b_{s_{max}} \leq 2\epsilon\gamma + \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^*(s') - P^a_{s_{max}s'} V^*(s')$$

$$L^{\pi'}(s_{max}) = V^*(s_{max}) - V^{\pi'}(s_{max})$$

$$= R^a_{s_{max}} + \gamma \sum_{s' \in S} P^a_{s_{max}s'} V^*(s') - R^b_{s_{max}} - \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^{\pi'}(s')$$

$$L^{\pi'}(s_{max}) = R^a_{s_{max}} + \gamma \sum_{s' \in S} P^a_{s_{max}s'} V^*(s') - R^b_{s_{max}} \; \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^{\pi'}(s')$$

$$\leq 2\epsilon\gamma + \cancel{\gamma \sum_{s' \in S} P^a_{s_{max}s'} V^*(s')} - \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^{\pi'}(s)$$

$$+ \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^*(s') - \cancel{\gamma \sum_{s' \in S} P^a_{s_{max}s'} V^*(s')}$$

substitution

$$\leq 2\epsilon\gamma + \gamma \sum_{s' \in S} P^b_{s_{max}s'} V^*(s') - P^b_{s_{max}s'} V^{\pi'}(s')$$

$$\leq 2\epsilon\gamma + \gamma \sum_{s' \in S} P^b_{s_{max}s'} \left(V^*(s') - V^{\pi'}(s')\right)$$

$$\leq 2\epsilon\gamma + \gamma \sum_{s' \in S} P^b_{s_{max}s'} L^{\pi'}(s')$$

$$\leq 2\epsilon\gamma + \gamma \sum_{s' \in S} P^b_{s_{max}s'} L^{\pi'}(s_{max}) = 2\epsilon\gamma + \gamma L^{\pi'}(s_{max})$$

$$L^{\pi'}(s_{max}) \leq 2\epsilon\gamma + \gamma L^{\pi'}(s_{max})$$

$$L^{\pi'}(s_{max}) - \gamma L^{\pi'}(s_{max}) \leq 2\epsilon\gamma$$

$$L^{\pi'}(s_{max}) \leq \frac{2\epsilon\gamma}{1-\gamma}$$

Thus

$$\max_{s \in S} \{L^{\pi'}(s)\} \leq \frac{2\epsilon\gamma}{1-\gamma}$$