1.1

1. $||a+b||^2 + ||a-b||^2$

$= (a+b)^T (a+b) + (a-b)^T (a-b)$

$[\, v^T v = ||v||^2 \,]$

$= a^T a + \cancel{a^T b} + \cancel{b^T a} + b^T b \cancel{+ a^T b}$
$+ a^T a - \cancel{a^T b} - \cancel{b^T a} + b^T b$

$= 2a^T a + 2b^T b$

$= 2(a^T a + b^T b)$

$= 2(||a||^2 + ||b||^2)$

$[\because v^T v = ||v||^2]$

Verified

2. $(a+b)^T (a-b)$

$= (a^T + b^T)(a-b)$

$[\because (A+B)^T = A^T + B^T$
property of transposes$]$

$= a^T a - a^T b + b^T a - b^T b$

$= a^T a - a^T b + a^T b - b^T b$

$[\because b^T a = \sum_i b_i a_i = \sum_i a_i b_i = a^T b]$

$= ||a||^2 - ||b||^2$

$[\because a^T a = ||a||^2]$

Verified

1.2

$(B+B^T)^T = B^T + (B^T)^T \quad [\because (A+B)^T = A^T + B^T]$

$= B^T + B \quad [\because (X^T)^T = X]$

$= B + B^T \quad [\because A+B = B+A]$

Since $(B+B^T)^T = B+B^T$,

$B+B^T$ is a symmetric matrix.

If A is invertible,

$A \cdot A^{-1} = I$

$(A \cdot A^{-1})^T = I$

$\Rightarrow (A^{-1})^T \cdot A^T = I \quad [\because (AB)^T = B^T A^T]$

$\Rightarrow$ The inverse of A^T is $(A^{-1})^T$

$\Rightarrow \boxed{(A^T)^{-1} = (A^{-1})^T}$

Hence proved.

1.3 We know that

$(x_1 + x_2 + x_3 \dots)^2 = x_1^2 + x_2^2 + \dots + 2x_1 x_2 \dots$

Hence

$(|x_1| + |x_2| + |x_3| + \dots)^2 = \sum_i |x_i|^2 + 2\sum_j\sum_i |x_i x_j|$

$\Rightarrow (\sum_i |x_i|)^2 \geq \sum |x_i|^2$

$\Rightarrow \sum_i |x_i| \geq \sqrt{\sum |x_i|^2} \quad —(1)$

$+\sum_j \sum_i |x_i x_j|$

Moreover,

$(|x_i| - |x_j|)^2 \geq 0,$

$\Rightarrow |x_i|^2 + |x_j|^2 \geq 2|x_i x_j|$

$\Rightarrow \sum\sum |x_i x_j| \leq \sum_i |x_i|^2 + \sum_j |x_j|^2$

$\Rightarrow \sum_i |x_i|^2$

$+ 2\sum\sum |x_i x_j| \leq 3\sum_i |x_i|^2$

$\Rightarrow \left(\sum |x_i|\right)^2 \leq 3\sum |x_i|^2$

$\Rightarrow \sum |x_i| \leq \sqrt{3}\sqrt{\sum |x_i|^2}$ — (2)

From (1) & (2),

$\|x_2\| \leq \|x_1\| \leq \sqrt{3}\,\|x_2\|$

$C_1 = 1,\ C_2 = \sqrt{3},$

∴ L1 & L2 norms are equivalent.

(Continued on Last Page)

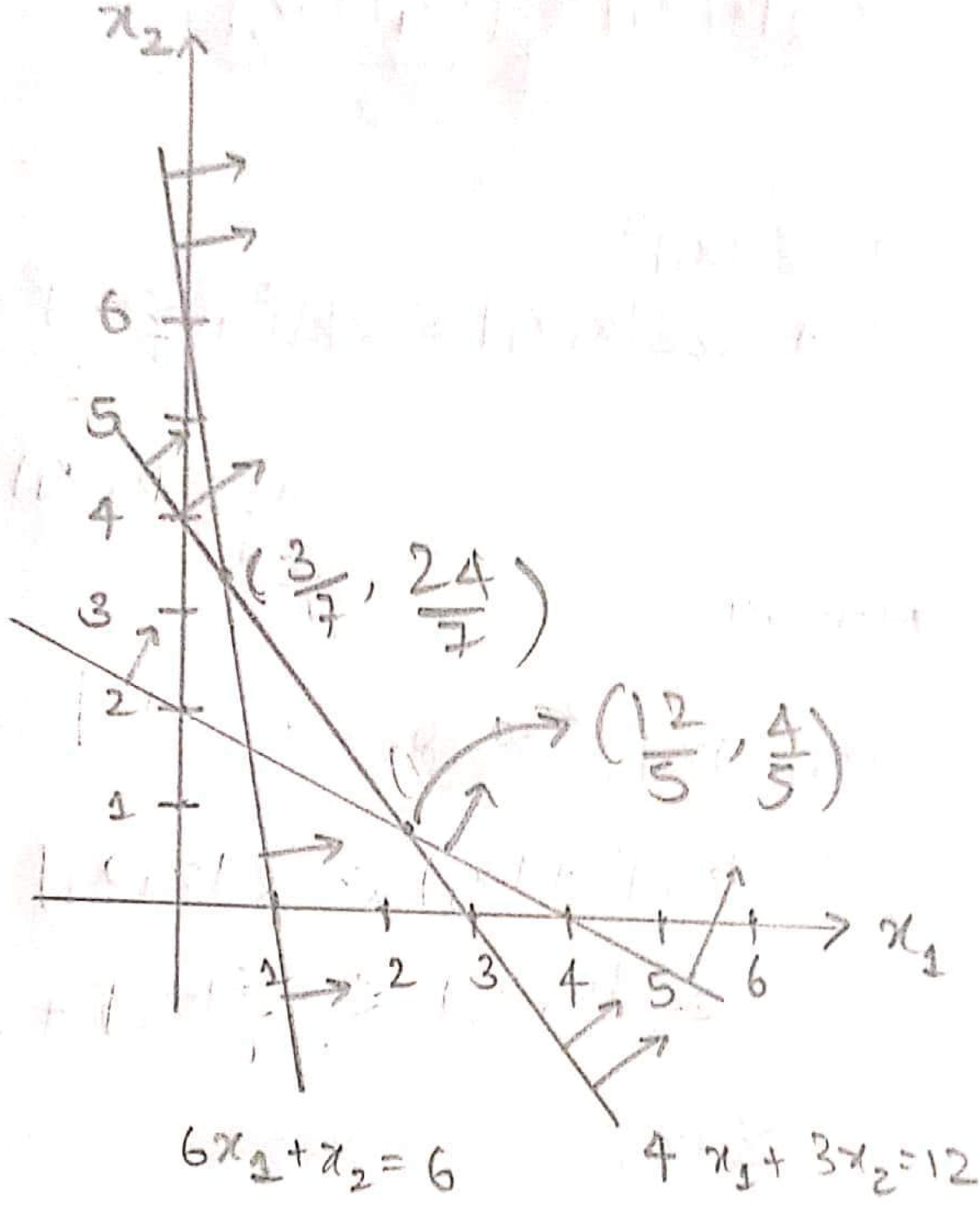
21.

$$Z = 5x_1 + 2x_2$$

s.t. $6x_1 + x_2 \geq 6$

$4x_1 + 3x_2 \geq 12$

$x_1 + 2x_2 \geq 4.$



$$Z_{\left(3/7, \frac{24}{7}\right)} = \frac{15}{7} + \frac{48}{7} = 9$$

$$Z_{\left(\frac{12}{5}, \frac{4}{5}\right)} = 12 + 8/5 = \frac{68}{5}$$

$$= 13.6.$$

$\cancel{(x_1, x_2) \equiv \left(\frac{12}{5}, \frac{4}{5}\right)}$

$$\boxed{(x_1, x_2)^* = \left(3/7, \frac{24}{7}\right), \; Z^* = 9}$$

Q.2.

~~Minimize $z = c_1 x_1 + c_2 x_2 +$~~

(i) $c = (-1, 0, 1)$

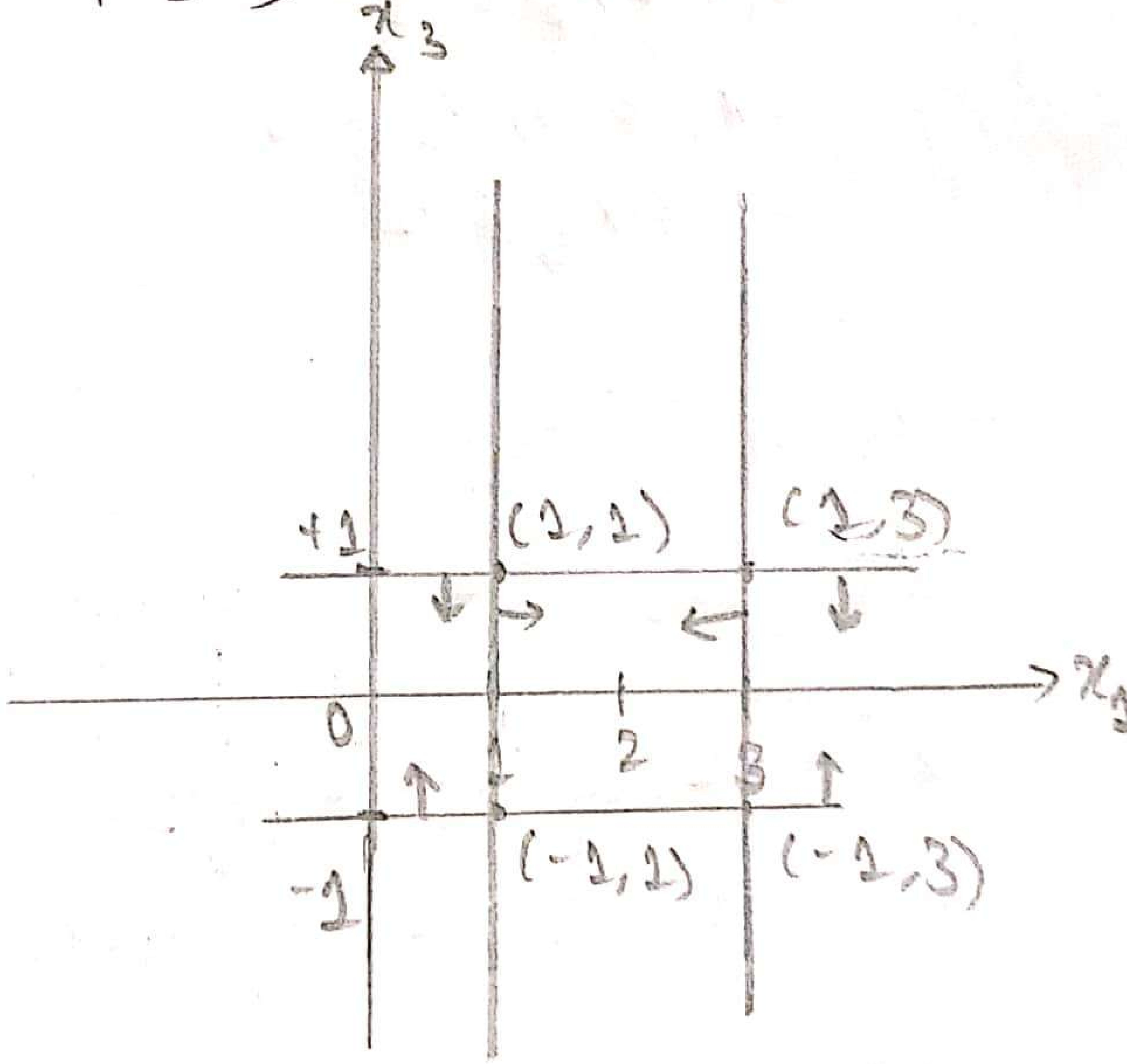
$$z = -x_1 + x_3$$

s.t. $x_1 \geq 1$

$x_1 \leq 3$

$x_1 \geq 0$, ~~$-1 \leq x_3 \leq +1$~~

$-1 \leq x_3 \leq +1$



$Z_{(1,1)} = 0$ $\quad Z_{(-1,1)} = 2$

$Z_{(1,3)} = 2$ $\quad Z_{(-1,3)} = 4$

$(x_1, x_3)^* = (1,1)$ $\quad Z^* = 0$

~~$0 \leq x_2 \leq 0$~~ $\quad x_2 = 0$

(ii)

$$Z = x_2$$

s.t.

$$x_1 + x_2 \geq 1$$
$$2x_2 \leq 3$$
$$x_2 \leq 0.$$

$$\boxed{Z^* = 0 \quad \text{at } x_2 = 0, x_1 \geq 1, -1 \leq x_3 \leq 1}$$

(iii)

$$Z = -x_3$$

$$x_1 + x_2 \geq 1$$
$$x_1 + 2x_2 \leq 3$$
$$x_1 \geq 0, x_2 \leq 0,$$
$$-1 \leq x_3 \leq 1$$

$$Z_{(x_3 = 1)} = -1,$$

$$Z_{(x_3 = -1)} = 1$$

$$\boxed{\begin{array}{l} Z^* = -1, \text{ at } x_3 = 1, \forall (x_1, x_2) \text{ s.t.} \\ x_1 + x_2 \geq 1, x_1 \geq 0, x_2 \leq 0 \end{array}}$$

2.3

If x_1 = # containers shipped by 1 to a

x_2 = " " " " to b

x_3 = " " " " c.

$200 - x_1$ = # containers shipped by 2 to a

$200 - x_2$ = " " " 2 to b

$200 - x_3$ = # " " " 2 to c

$$x_1 + x_2 + x_3 \leq 250,$$

$$200 - x_1 + 200 - x_2 + 200 - x_3 \leq 450$$

$$\Rightarrow \quad x_1 + x_2 + x_3 \geq 150$$

Min

$$Z = 3.4x_1 + 2.2x_2 + 2.9x_3$$
$$+ 3.4(200 - x_1) + 2.4(200 - x_2)$$
$$+ 2.5(200 - x_3)$$

$$\text{Min } Z = -0.2x_2 + 2.4x_3 + 680$$
$$+ 480 + 500$$

$\Rightarrow$ Min $Z = -0.2\,x_2 + 2.4\,x_3$

Subjet to $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

and the two constraints circled above.

Q 3

3.1. Adjacency matrix for graph:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

walks of length 5 from 4 to 7

$= A_{47}$ in A^5.

The element in 4^{th} row, 7^{th} column

in $A^5 = 1111$

$\Rightarrow$ $\boxed{\text{\# 5 length walks from vertex 4 to vertex 7} = 1111}$

3.2

$f \in O(g)$

$\Rightarrow f \leq c.g \quad \forall x \geq x_0$

$\Rightarrow \log(f) \leq \log c + \log(g)$

$\Rightarrow$ ~~log f~~

Since g is a mono increasing function

$\exists\ x \geq x_1$ s.t.

$g(x) \geq c.$

$\Rightarrow \log(g) \geq \log c$

$\Rightarrow \log f \leq \log(g) + \log(g)$

$\log f \leq 2\log(g) \quad \forall x \geq x_1$

$\boxed{\text{Hence } \log f \in O(\log g)}$

3.3

$\log(n!) = \log(n) + \log(n-1) + \dots \log(2)$

$\leq \log(n) + \log(n) + \dots \log(n) + \log(n)$

$\log(n!) \leq n\log n$

$\log(n!) = \log(n) + \dots \log(1)$

$\geq \log(\frac{n}{2}) + \log(\frac{n}{2}) + \dots \log(\frac{n}{2}) + \log(\frac{n}{2}-1) + \log(\frac{n}{2}-2) + \dots \log(1)$

$\log(n!) \geq \frac{n}{2}\log(\frac{n}{2}) + \log(\frac{n}{2}-1) + \dots \log(1)$

$$\Rightarrow \log(n!) \geq \frac{n}{2}\log\left(\frac{n}{2}\right) + \log(2) + \dots \log(2)$$

$$\geq \frac{n}{2}\log(n) - \cancel{\frac{n}{2}\log 2} + \cancel{\frac{n}{2}\log 2}$$

$$\geq \frac{n}{2}\log(n)$$

$$\Rightarrow \frac{n}{2}\log n \leq \log(n!) \leq n\log n$$

$$\Rightarrow \boxed{\log(n!) \in \Theta(n\log n)}$$

Hence proved.

Hence proved.

1.3. contd:

$$\sum_i |x_i|^2 \geq |x_i|^2$$

$$\sum_i \left(\sqrt{\sum_i |x_i|^2}\right) \geq \sum_i |x_i|$$

$$\Rightarrow \sqrt{\sum_i |x_i|^2 \cdot n} \geq \sum_i |x_i|$$

$$\Rightarrow \text{LHS} \; \|x_1\| \leq n \cdot \|x_2\|$$

$$\Rightarrow \|x_2\| \leq \|x_1\| \leq n \|x_2\|$$

Hence proved.