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Design and Analysis of Algorithms (COEN 279)
Homework 5 – 150 points

Note: For all algorithm design questions, you must (i) explain your algorithm and correctness in English, (ii) provide pseudocode (if modifications to algorithms discussed in the class are required), and (iii) explain and provide running time complexity to receive full credit.

Question 1 (30 points)**.** *A* ***path cover*** *of a directed graph $G = (V, E)$ is a set P of vertex-disjoint paths such that every vertex in V is included in exactly one path in P. Paths may start and end anywhere, and may be of any length, including 0. A* ***minimum path cover*** *in G is a path cover containing the fewest possible paths.*

(a) *Give an efficient algorithm to find the minimum path cover of DAG $G = (V, E)$. (Hint: Assuming that $V = \{1, 2, \dots, n\}$, construct a flow network based on the graph $G' = (V', E')$, where $V' = \{x_0, x_1, \dots, x_n\} \cup \{y_0, y_1, \dots, y_n\}$ and $E' = \{(x_0, x_i) : i \in V\} \cup \{(y_i, y_0) : i \in V\} \cup \{(x_i, y_j) : (i, j) \in E\}$, and run a maximum flow algorithm.*

(b) *Does your algorithm work on directed graphs that contain cycles? Explain.*

Question 2 (30 points)**.** *In a public building such as a movie theater, it is important to have a plan of exit in the event of a fire. We will design such an emergency exit plan in this question using max-flows. Suppose a movie theater is represented by a graph $G = (V, E)$, where each room, landing, or other location is represented by a vertex and each corridor or stairway is represented by an edge. Each corridor has an associated capacity c, meaning that at most c people can pass through the corridor at once. Traversing a corridor from one end to the other takes one timestep. (Traversing a room takes zero time.)*

(a) *Suppose all people are initially in a single room s, and there is a single exit t. Show how to use maximum flow to find a fastest way to get everyone out of the building. (Hint: create another graph G' that has vertices to represent each room at each time step.)*

(b) *Show how the same idea can be used when people are initially in multiple locations and there are multiple exits.*

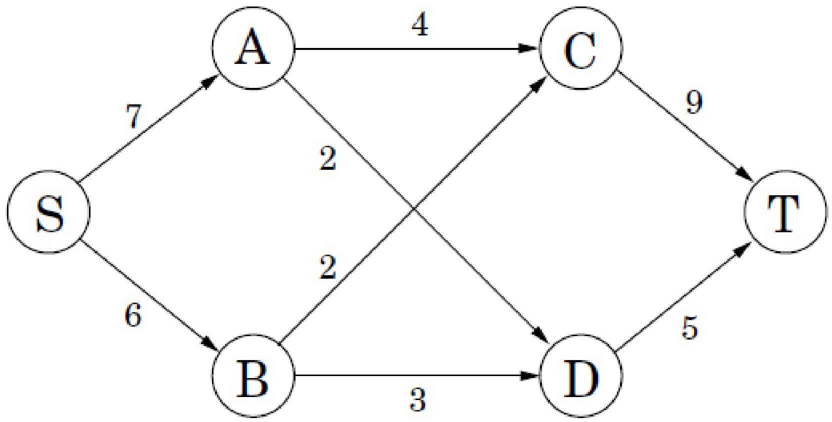
(c) *Finally, suppose that it takes different (but integer) amounts of time to cross different corridors or stairways, and that for each such corridor or stairway e, you are also given an integer $t(e)$ which is the number of seconds required to cross e. Now show how to transform your algorithm in (a) to find a fastest way to get everyone out of the building.*

Question 3 (20 points)**.** *The Canine Products company offers two dog foods, Frisky Pup and Husky Hound, that are made from a blend of cereal and meat. A package of Frisky Pup requires 1 pound of cereal and 1.5 pounds of meat, and sells for \$7. A package of Husky Hound uses 2 pounds of cereal and 1 pound of meat, and sells for \$6. Raw cereal costs \$1 per pound and raw meat costs \$2 per pound. It also costs \$1.40 to package the Frisky Pup and \$0.60 to package the Husky Hound. A total of 240,000 pounds of cereal and 180,000 pounds of meat are available each month. The only production bottleneck is that the factory can only package 110,000 bags of Frisky Pup per month. Needless to say, management would like to maximize profit.*

(a) *Formulate the problem as a linear program in two variables.*

(b) Graph the feasible region, give the coordinates of every vertex, and circle the vertex maximizing profit. What is the maximum profit possible?

Question 4 (40 points). *Consider the following network (the numbers are edge capacities).*



(a) Find the maximum flow f and a minimum cut.

(b) Draw the residual graph G_f (along with its edge capacities). In this residual network, mark the vertices reachable from S and the vertices from which T is reachable.

(c) An edge of a network is called a ***bottleneck edge*** *if increasing its capacity results in an increase in the maximum flow. List all bottleneck edges in the above network.*

(d) Give a very simple example (containing at most four nodes) of a network which has no bottleneck edges.

(e) Give an efficient algorithm to identify all bottleneck edges in a network. (Hint: Start by running the usual network flow algorithm, and then examine the residual graph.)

Question 5 (30 points). *There are many variants of Rudrata's problem, depending on whether the graph is undirected or directed, and whether a cycle or path is sought. Reduce the DIRECTED RUDRATA PATH problem to each of the following.*

(a) The (undirected) RUDRATA PATH problem.

(b) The undirected RUDRATA (s, t)-PATH problem, which is just like RUDRATA PATH except that the endpoints of the path are specified in the input.